1

Wave propagation of Cellular Neural Network model of tsunamis ⋆

MayaMarkova[1], *AngelaSlavova*[2], *PietroZecca*[3]

[1] *Department of Informatics, University of Russe, Russe 7000, e-mail:maya.markova@gmail.com*
[2] *Institute of Mathematics, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria, e-mail: slavova@math. bas.bg*
[3] *Dipartimento di Energetica, Universita di Firenze, via S.Marta 3, Firenze 50139, Italy, e-mail:zecca@unifi.it*

Summary. In this paper two models of tsunami waves are considered. First model is for the long water waves with nonlinear vorticity. For this model Cellular Neural Network (CNN) approach is applied. The dynamics of the CNN model is studied by means of describing function method. Travelling wave solutions are obtained for this model and the simulations illustrate the theoretical results. Second model is the two component Camssa-Holm type equation. For this model CNN is constructed and traveling wave solutions are obtained theoretically and via simulations.

1 INTRODUCTION

The study of propagation of tsunami from their small disturbance at the sea level to the size they reach approaching the coast has involved the interest of several scientists. It is clear that in order to predict accurately the appearance of a tsunami it is

⋆ This paper is partially supported by the NATO Grant NRCLR.9837 16 and project DID 02/15

fundamental to built up a good model. From this point of view the most important tool in the context of water waves is soliton theory [7]. Frequently in the literature it is stated that a tsunami is produced by a large enough soliton. Solitons arise as special solutions of a widespread class weakly nonlinear dispersive PDEs modeling water waves, such as the KdV or Camassa-Holm equation [2], representing to various degrees of accuracy approximations to the governing equations for water waves in the shallow water regime. How the tsunami is initiated? The thrust of a mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature. We aim to describe how an initial disturbance gives rise to a tsunami wave. Let h is the average depth of the water, λ is the typical wavelength of the wave and a is a typical amplitude. There are two important parameters: $\varepsilon = \frac{a}{h}$, called amplitude parameter, and the shallowness parameter $\delta = \frac{h}{\lambda}$. According to these parameters a rigorous validity ranges are obtained [1] for the main physical regimes encountered in modelling of two-dimensional water waves:

1. shallow-water, large amplitude ($\delta \ll 1$, $\varepsilon \sim 1$), leading at first order to the shallow-water equations [10] and at second order to the Green-Naghdi model [9];

2. shallow-water, medium amplitude regime ($\delta \ll 1$, $\varepsilon \sim \delta$) leading to the Serre equations [10] and to the Camassa-Holm equation [2];

3. shallow-water, small amplitude or long-wave regime ($\delta \ll$, $\varepsilon \sim \delta^2$) leading at first order to the linear wave equation

$$\varphi_{tt} - \varphi_{xx} = 0 \tag{1}$$

with general solution

$$\varphi(x,t) = \varphi_+(x-t) + \varphi_-(x+t), \tag{2}$$

where the sign $\pm$ refers to a wave profile $\varphi_\pm$ moving with unchanged shape to the right/left at constant unit speed. The small effects that were ignored at first order (small amplitude, long wave) build up on longer time/spatial scales to have a significant cumulative nonlinear effect so that on a longer time scale each of the waves that make up the solution (2) to (1) satisfies the KdV equation [7].

In Constantin and Johnson [5] the model of the motion of the water before arrival of a tsunami wave is proposed. They require that a flat free surface for the background state excludes linear vorticity functions, unless the flow is trivial. So, nonlinear vorticity distributions are introduced in order to admit nontrivial flows with a flat free surface. Consider the following dynamical system

$$\varphi_{tt} + \varphi_{xx} = -f(\varphi), \tag{3}$$

$$f(\varphi) = \begin{cases} \varphi - \varphi|\varphi|^{-1/2} & \text{if } \varphi \neq 0 \\ 0 & \text{if } \varphi = 0. \end{cases} \tag{4}$$

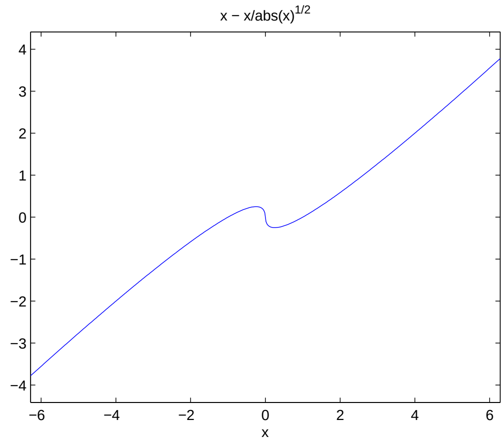

Fig.1. Bifurcation diagram of the nonlinear vorticity distribution.

By applying Cellular Neural Networks (CNN) approach we shall study the wave propagation of the model (3), (4).

In Section 2 we give brief introduction to CNN phenomena and we construct the CNN model of (3), (4). Section 3 deals with traveling wave solutions for the tsunami model. In section 4 we study traveling waves in two-component Camassa-Holm type system with applications to tsunamis.

2 CNN MODEL AND ITS DYNAMICS

Cellular Neural Networks (CNNs) are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets. It was shown that as the cell self-feedback coefficients are changed to a critical value, a CNN with opposite-sign template may change from stable to unstable. Namely speaking, this phenomenon arises as the loss of stability and the birth of a limit cycles.

CNN [3] is simply an analogue dynamic processor array, made of cells, which contain linear capacitors, linear resistors, linear and nonlinear controlled sources. Let us consider a two-dimensional grid with 3×3 neighbourhood system as it is shown on Fig.2.

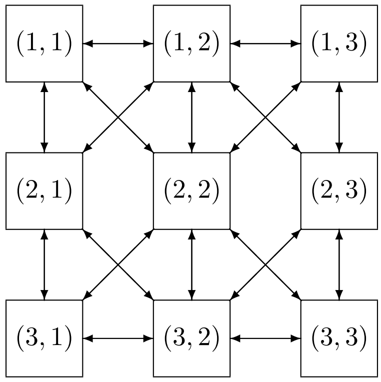

Fig.2. 3×3 CNN.

The squares are the circuit units - cells, and the links between the cells indicate that there are interactions between linked cells. One of the key features of a CNN is that the individual cells are nonlinear dynamical systems, but that the coupling between them is linear. Roughly speaking, one could say that these arrays are nonlinear but have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We propose below the general definition of a CNN which follows the original one [3]:

Definition 1. *The CNN is a*

a). 2-, 3-, or n- dimensional array of

b). mainly identical dynamical systems, called cells, which satisfies two properties:

c). most interactions are local within a finite radius r, and

d). all state variables are continuous valued signals.

Definition 2. *An $M \times M$ cellular neural network is defined mathematically by four specifications:*

1). CNN cell dynamics;

2). CNN synaptic law which represents the interactions (spatial coupling) within the neighbour cells;

3). Boundary conditions;

4). Initial conditions.

Now in terms of Definition 2 we can present the dynamical systems describing CNNs. For a general CNN whose cells are made of time-invariant circuit elements, each cell $C(ij)$ is characterized by its CNN cell dynamics :

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I^s_{ij}), \tag{5}$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbour cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$\begin{aligned} I^s_{ij} = {} & A_{ij,kl} x_{i+k,j+l} + \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \\ & + \tilde{B}_{ij,kl} * u_{i+k,j+l}(t). \end{aligned} \tag{6}$$

The first term $A_{ij,kl} x_{i+k,j+l}$ of (6) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

It is known [4,14] that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs). The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate . An one-dimensional discretized Laplacian template will be in the following form:

$$A_1 = (1, -2, 1),$$

This is the two-dimensional discretized Laplacian A_2 template:

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 3. *For any cloning template A which defines the dynamic rule of the cell circuit, we define the convolution operator $*$ by the formula*

$$A * z_{ij} = \sum_{C(k,l)\in N_r(i,j)} A(k-i, l-j) z_{kl},$$

where $A(m,n)$ denotes the entry in the pth row and rth column of the cloning template, $p = -1, 0, 1$, and $r = -1, 0, 1$, respectively.

CNN model of our system (3), (4) will be the following:

$$\begin{aligned} \frac{dv_j}{dt} &= A_1 * u_j + f(u_j), \\ \frac{du_j}{dt} &= v_j, 1 \leq j \leq N \end{aligned} \tag{7}$$

We shall study the dynamics of the above model (7) by applying the describing function method [12]. Applying the double Fourier transform:

$$F(s, z) = \sum_{k=-\infty}^{k=\infty} z^{-k} \int_{-\infty}^{\infty} f_k(t) exp(-st) dt,$$

to the CNN model (7) we obtain:

$$\begin{aligned} sV(s, z) = (z^{-1}U(s, z) - 2U(s, z) + \\ +zU(s, z)) + F(U(s, z)) \\ sU(s, z) = V(s, z). \end{aligned} \tag{8}$$

Without loss of generality we denote $F(U) = U(s, z) - U(s, z)|U(s, z)|^{-1/2}$. In the double Fourier transform we suppose that $s = i\omega_0$, and $z = exp(i\Omega_0)$, where ω_0 is a temporal frequency, Ω_0 is a spatial frequency.

According to the describing function method [12], $H(s, z) = \frac{1}{s-(z^{-1}-2z+z)}$ is the transform function, which can be presented in terms of ω_0 and Ω_0, i.e. $H(s, z) = H_{\Omega_0}(\omega_0)$.

We are looking for possible periodic state solutions of system (7) of the form:

$$X_{\Omega_0}(\omega_0) = X_{m_0} sin(\omega_0 t + j\Omega_0), \tag{9}$$

where $X = (U, V)$. According to the describing function method we take the first harmonics, i.e. $j = 0 \Rightarrow$

$$X_{\Omega_0}(\omega_0) = X_{m_0} sin\omega_0 t,$$

On the other side if we substitute $s = i\omega_0$ and $z = exp(i\Omega_0)$ in the transfer function $H(s, z)$ we obtain:

$$H_{\Omega_0}(\omega_0) = \frac{1}{i\omega_0^2 - (2cos\Omega_0 - 2)}. \tag{10}$$

According to (10) the following constraints hold:

$$\begin{aligned} \Re(H_{\Omega_0}(\omega_0)) &= \frac{U_{m_0}}{V_{m_0}} \\ \Im(H_{\Omega_0}(\omega_0)) &= 0. \end{aligned} \tag{11}$$

Since our CNN model (7) is a finite circular array of $L = N.N$ cells we have finite set of spatial frequencies:

$$\Omega_0 = \frac{2\pi k}{L}, 0 \leq k \leq L - 1. \tag{12}$$

Thus, (10), (11) and (12) give us necessary set of equations for finding the unknowns U_{m_0}, V_{m_0}, ω_0, Ω_0. As we mentioned above we are looking for a periodic wave solution of (7), therefore U_{m_0} and V_{m_0} will determine approximate amplitudes of the waves, and $T_0 = 2\pi/\omega_0$ will determine the wave speed.

Based on the above considerations the following proposition hold:

Proposition 1. *CNN model (7) of circular array of L identical cells has periodic state solution $u_j(t)$, $v_j(t)$ with a finite set of spatial frequencies $\Omega_0 = 2\pi k/L$, $0 \leq k \leq L - 1$.*

After simulating our CNN model (7) we obtain the following results:

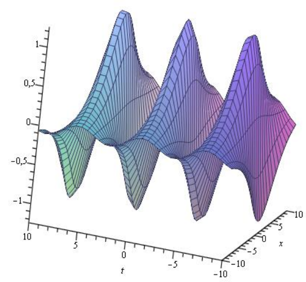

Fig.3. Periodic wave solution of CNN model (7).

Remark 1. We consider a CNN programmable realization allowing the calculation of all necessary processing steps in real time. The network parameter values of CNN model, described by the dynamical system (7), are determined in a supervised optimization process. During the optimization process the mean square error is minimized using Powell method and Simulated Annealing [17]. The results are obtained by the CNN simulation system MATCNN applying 4th order Runge-Kutta integration.

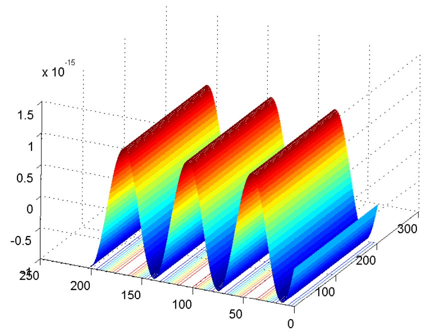

Fig.4. Simulation of CNN model (7) after the optimization process.

3 TRAVELING WAVES IN CNN MODEL

Our objective in this paper is to study the structure of the travelling wave solutions of the CNN model (7). There has been many studies on the travelling wave solutions of spatially discrete or both spatially and time discrete systems [14,16]. The study of travelling wave solutions can proceed as follows. Consider solutions of (7) of the form:

$$z_j = \Phi(j - ct), \tag{13}$$

$z_j = col(u_j, v_j)$ for some continuous functions $\Phi : \mathbf{R}^1 \to \mathbf{R}^1$ and for some unknown real number c. Denote $s = j - ct$. Let us substitute (13) in our CNN model (7). Then $\Phi(s)$ and c satisfies the system of the form:

$$\begin{aligned} -c\Phi^{'}(s) = G(\Phi(s + r_0), \Phi(s + r_1), \dots, \Phi(s + r_n) + \\ + F(\Phi(s + r_0) = 0, \end{aligned} \tag{14}$$

here $r_0 = 0$, r_i are real numbers for $i = 1$ to n. Equation (14) is called bistable because it has three spatially homogeneous solutions $\Phi(s) \equiv z^-, z^0, z^+$ satisfying $z^- < z^0 < z^+$, and

$$G(z, z, \dots, z) > 0 \ \text{ for } \ z \in (-\infty, z^-) \cup (z^0, z^+),$$

$$G(z, z, \dots, z) < 0 \ \text{ for } \ z \in (z^-, z^0) \cup (z^+, \infty),$$

Recently, Mallet-Paret [11] showed that (14) has a unique monotone solutions satisfying the boundary conditions:

$$lim_{s\to-\infty}\Phi(s)=z^- \quad \text{and} \quad lim_{s\to\infty}\Phi(s)=z^+. \tag{15}$$

More precisely, it is proved that under some assumptions, there is a unique c^* such that (14) has a monotone solutions satisfying (15) iff $c=c^*$, and such solution is also unique up to a phase shift if $c=c^*\neq 0$. Indeed, the solution $\Phi(s)$ of (14) and $lim_{s\to\infty}\Phi(s)=z^+$ can be represented as

$$\Phi(s)=z^+-\gamma e^{\sigma s}-\tilde{\Phi}(s)e^{2\sigma s},$$

for $s\gg 1$, $\sigma^+<0$, $\gamma>0$, $\tilde{\Phi}(s)$ is a bounded and C^1-function.

Suppose that our CNN model (7) is a finite circular array of $L=N.N$ cells. For this case we have finite set of frequences [16]:

$$\Omega=\frac{2\pi k}{L},\ \ 0\leq k\leq L-1. \tag{16}$$

The following proposition then hold:

Proposition 2. *Suppose that $z_j(t)=\Phi(j-ct)$ is a travelling wave solutions of the CNN model (7) with $\Phi\in C^1(\mathbf{R^1},\mathbf{R^1})$ and $\Omega=\frac{2\pi k}{L}$, $0\leq k\leq L-1$. Then there exist constants $c_*<c^*<0$ such that*

(i) if $c\leq c_*$ then $\Phi(s;c)$ is nondecreasing and satisfies

$$lim_{s\to-\infty}\Phi(s)=z^0 \quad \text{and} \quad lim_{s\to\infty}\Phi(s)=z^+; \tag{17}$$

(ii) if $c=c^*>c_*$, then $\Phi(s;c)$ is nondecreasing and satisfy (15);

(iii) if $c^*<c<0$, then $\Phi(s;c)$ is nondecreasing and unbounded.

Let us introduce the following energy function for our CNN model (7):

$$E(u_j,v_j)=\frac{1}{2}u_j^2+\frac{1}{2}v_j^2-\frac{2}{3}|v_j|^{3/2}. \tag{18}$$

We obtain the following simulation results in the plane (u,v)which present tow closed curves representing the solution set of the equation $u_j^2=\frac{4}{3}|v_j|^{3/2}-v_j^2$:

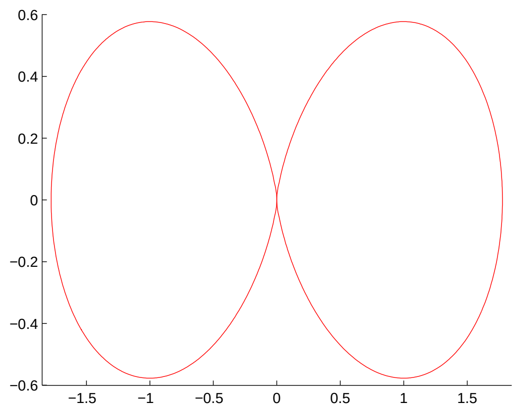

Fig.5.

The stationary points of (7) are $(\pm 1, 0)$, or $(0, 0)$. If the energy function $E < 0$ the set of solutions consists of the interiors of the two closed curves given on Fig.5. By virtue of (18), once a solution of (7) intersects the boundary of these two closed curves at a point other than $(0, 0)$, it will remain in that set having the asymptotic limit $(-1, 0)$, or, correspondingly $(1, 0)$. Notice that on the boundary of the two closed curves (Fig.5) we have $E = 0$, while within these sets $E < 0$, with the minimum attained at $(\pm 1, 0)$ where $E = -\frac{1}{6}$.

Let us consider the following initial conditions for our CNN model (7):

$$\left| \begin{array}{l} u_j(0) = a \\ v_j(0) = 0. \end{array} \right. \tag{19}$$

If $z^+ \to a$, for $s \to \infty$, the number of intersections approaches infinity. Denote by Ω_+, Ω_- the sets of points $(a, 0)$, where $a \in (-\infty, 0) \cup (0, \infty)$ and suppose that the corresponding solution of (7) has asymptotic limit $(1, 0)$, $(-1, 0)$ respectively. It is easy to prove that all intersections being transversal to the horizontal axis are stable under small perturbations [5]. Therefore, for any integer $M \geq 1$ by continuous dependence on the initial data it might be proved that as $a \to \infty$ the number M of intersections approaches infinity.

In our case $\Phi : [0, \infty) \to [0, \infty)$ is defined implicitly by

$$r = \int_{\Phi}^{1/4} \frac{dt}{\sqrt{\frac{4}{3}|t|^{3/2} - t^2}}, r \in [0, I]$$

where

$$I = \int_{0}^{1/4} \frac{dt}{\sqrt{\frac{4}{3}|t|^{3/2} - t^2}}$$

and extend to $s \geq 1$ by setting Φ equal to zero here. Simulations of the implicit solution Φ are given below:

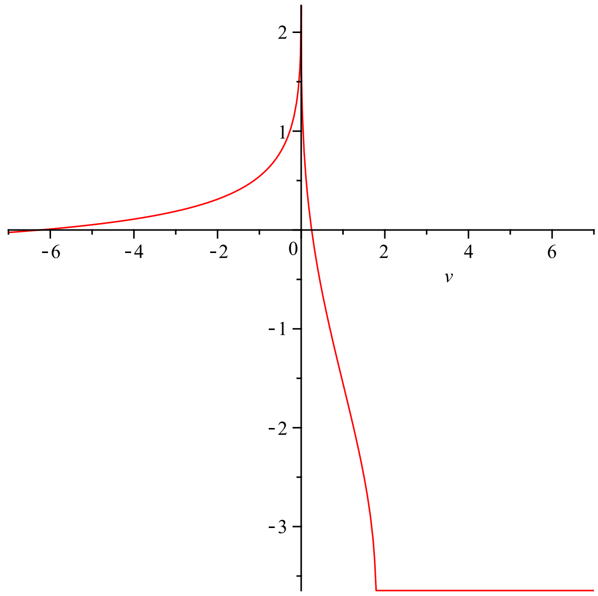

Fig.6. Real part of the solution Φ.

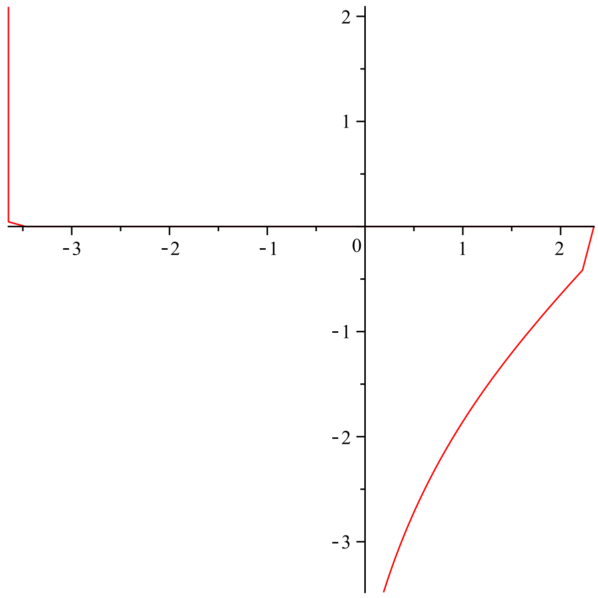

Fig.7. Implicitly solution Φ.

Remark 2. Notice that

$$\Phi_+^{''} + \Phi_+ - \Phi_+|\Phi_+|^{-1/2} = 0,$$

with

$$\Phi_+(0) = \frac{1}{4}, \Phi_+^{'}(0) = -\frac{1}{4}\sqrt{\frac{5}{3}}, \Phi_+(I) = \Phi_+^{'}(I) = 0.$$

Since $lim_{s\to\infty}\Phi_0(s) = 0$, there exists some $s_0 > 0$ such that $|\Phi_0(s)| < \frac{1}{4}$ for $s \geq s_0$. So we claim

$$-\Phi_+(s - s_0) \leq \Phi_0(s) \leq \Phi_+(s - s_0), s \geq s_0.$$

4 TRAVELING WAVE SOLUTIONS FOR THE TWO COMPONENT CAMASSA-HOLM TYPE SYSTEM

Our aim in this section is to find model equations which admit peaked travelling waves: waves that are smooth except at their crest and which capture therefore the main features of the waves of greatest height encountered as solutions to governing equations for water waves [7].

The motion of inviscid fluid with a constant density is described by the well-known Euler's equations. On the other hand, the motion of a shallow water over a flat bottom is described by a 3×3 semilinear system of first order partial differential equations. Because of the lack of space we do not give the exact expressions of the above mentioned systems. Starting from the semilinear 3×3 system one can obtain the so-called Green-Naghdi system considered for the first time in 1976. The latter system can be related to the following two component Camassa-Holm type system:

$$\left| \begin{array}{l} m_t + 2u_x m + u m_x + \rho\rho_x = 0 \\ \rho_t + (u\rho)_x = 0, \text{where} \quad m = u - u_{xx}. \end{array} \right. \tag{20}$$

We are looking for travelling wave solutions of (20), i. e. $u(t,x) = u(\xi)$, $m(t,x) = m(\xi)$, $\xi = x - ct$, $c = const$. Substituting $u(\xi)$, $m(\xi)$ in (20) we get from the second equation that $-c\rho^{'} + (u\rho)^{'} = 0 \Rightarrow c\rho(\xi) = u(\xi)\rho(\xi) - \alpha$, $\alpha = const \Rightarrow \rho = \frac{\alpha}{u-c}$; $m = u - u^{''}$. Therefore, the first equation in (20) implies:

$$-cm^{'} + (um)^{'} + u^{'}m + \rho\rho^{'} = 0 \Rightarrow$$

$$-c(u^{'} - u^{'''}) + (um)^{'} + u^{'}(u - u^{''}) + \rho\rho^{''} = 0$$

and after an integration with respect to ξ we get that

$$-cu + cu^{''} + u^2 - uu^{''} + \frac{1}{2}u^2 - \frac{1}{2}(u^{'})^2 + \frac{1}{2}\rho^2 = \frac{\beta}{2} = const,$$

i. e.

$$-cu + cu^{''} + \frac{3}{2}u^2 - uu^{''} - \frac{1}{2}(u^{'})^2 + \frac{\alpha^2}{(u-c)^2} = \frac{\beta}{2}. \tag{21}$$

Assuming $c > 0$ we make in (21) the change $u = c(1 + z)$. In the case $c < 0$ we shall make the change $u = c(1 - z)$. We shall confine ourselves to the case when $c > 0$ and $z = \frac{u}{c} - 1$ as the other case is treated similarly. Consequently, $\rho = \frac{\alpha}{cz}$, $z \neq 0$ and then (21) can be written as:

$$-c^2(1+z)+c^2z^{''}+\frac{3}{2}c^2(1+z)^2-c^2(1+z)z^{''}-\frac{1}{2}c^2(z^{'})^2+\frac{1}{2}\frac{\alpha^2}{c^2z^2}=\frac{\beta}{2}\Rightarrow$$

$$zz^{''}+\frac{1}{2}(z^{'})^2-\frac{3}{2}z^2-2z-\frac{\alpha^2}{2c^4z^2}+\frac{\beta-c^2}{2c^2}=0. \tag{22}$$

We multiply (22) by $z^{'}$ then integrate in ξ and having in mind that $zz^{'}z^{''}+\frac{1}{2}(z^{'})^3=\frac{1}{2}(z(z^{'})^2)'$ we obtain:

$$\frac{1}{2}z(z^{'})^2-\frac{z^3}{2}-z^2+\frac{\alpha^2}{2c^4z}+\frac{\beta-c^2}{2c^2}z=\frac{\gamma}{2}=const,$$

i. e.

$$z^2(z^{'})^2=z^4+2z^3+\frac{c^2-\beta}{c^2}z^2+\gamma z-\frac{\alpha^2}{c^4}. \tag{23}$$

Put $\frac{c^2-\beta}{c^2}=\beta_1\Leftrightarrow\beta=c^2(1-\beta_1)$, $-\frac{\alpha^2}{c^4}=\alpha_1<0\Leftrightarrow\alpha=\pm c^2\sqrt{-\alpha_1}$, i. e. $\alpha\to 0\Leftrightarrow\alpha_1\to 0$.

Let us fix $c>0$. Then $\beta_1\in\mathbf{R}$, $\gamma\in\mathbf{R}$ and $\alpha_1<0$ are arbitrary real constants.

Put $P_4(z)=z^4+2z^3+\beta_1z^2+\gamma z+\alpha_1$.

If $\gamma=0$ we write $\tilde{P}_4(z)=z^4+2z^3+\beta_1z^2+\alpha_1$. Define now $w=\tilde{\tilde{P}}_4(z)=z^2(z^2+2z+\beta_1)$. Evidently, $\tilde{\tilde{P}}_4(z)=0\Leftrightarrow z_{1,2}=0$, $z_{3,4}=-1\pm\sqrt{1-\beta_1}$. We shall assume further on that

$$0<\beta_1<1\Leftrightarrow 0<1-\beta_1<1\Leftrightarrow -1<z_3<0,\ -2<z_4<-1. \tag{24}$$

Consider now the cross points of the biquadratic parabola $w=\tilde{\tilde{P}}_4(z)$ and the straight line $w=-\alpha_1>0$, $0<-\alpha_1\ll 1\Leftrightarrow 0<|\alpha|\ll 1$. Geometrically we have the Fig.8.

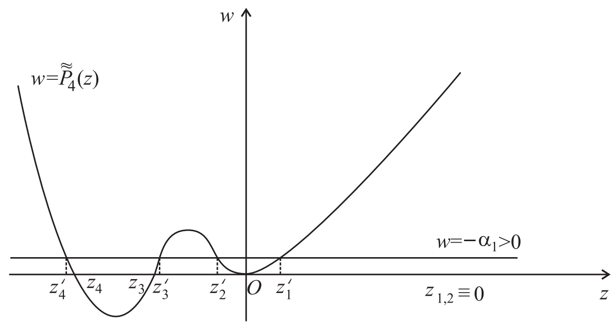

Fig.8

Evidently, the curve $w=\tilde{\tilde{P}}_4(z)$ crosses the line $w=-\alpha_1$, $|\alpha_1|\ll 1$ at the points $z_4^{'}<z_4<z_3<z_3^{'}<z_2^{'}<0<z_1^{'}$. Therefore $\tilde{P}_4(z)_{=}0\Leftrightarrow\tilde{P}_4(z_j^{'})=0$, $z_{j,\,1\le j\le 4}$ being 4 simple roots of the algebraic equation $\tilde{P}_4(z)=0$. As it is known from Analysis, the simple roots of the algebraic equations depend continuously on the coefficients of the corresponding polynomials. This way we come to the Proposition 3.

Proposition 3. *Consider the 4^{th} order polynomial $P_4(z)$, fix the constant $c > 0$ and suppose that $0 < \beta_1 < 1$. Then one can find some $0 < \varepsilon_0$ such that if $|\gamma| \leq \varepsilon_0$, $|\alpha| \leq \varepsilon_0$, then $P_4(z) = 0$ has 4 simple roots $z_4^{'} < z_3^{'} < z_2^{'} < 0 < z_1^{'}$.*

Remark 3. Consider the ODE

$$\left| \begin{array}{l} (z^{'})^2 = \frac{P_4(z)}{z^2} \geq 0 \\ z(0) = z_0 \in [z_3^{'}, z_2^{'}] \end{array} \right. , \tag{25}$$

where $P_4(z) = z^4 + 2z^3 + \beta_1 z^2 + \gamma z + \alpha_1$, β_1, γ, $\alpha_1 < 0$ being arbitrary constants. Certainly, $P_4(z) = 0$ possesses at least two real roots as $P_4(0) = \alpha_1 < 0$. We assume that $k_1 = k_2 < 0$ is a double root of $P_4(z)$, while $k_1 < k_3 < 0 < k_4$ are simple roots. Let us suppose that $k_3 < 0 < k_4$ as $k_1^2 k_3 k_4 = \alpha_1 < 0$. Therefore, $k_1 < z < k_3 \Rightarrow P_4(z) > 0$. Then the Cauchy problem

$$\left| \begin{array}{l} z^2 z'^2 = P_4(z) = (z - k_1)^2 (z - k_3)(z - k_4) \\ z(0) = k_3 \end{array} \right. \tag{26}$$

possesses a smooth solution $z(\xi)$, $\xi \in \mathbf{R}^1$, such that $z(-\xi) = z(\xi)$, $\forall \xi \in \mathbf{R}^1$, $z'(0) = 0$, $z'(\xi) < 0$ for $\xi > 0$. This is a soliton, of course (see Fig.9). Moreover, we can give an explicit formula for the solution as the corresponding integral $\xi = \displaystyle\int_{k_3}^{z} \frac{\lambda d\lambda}{(\lambda - k_1)\sqrt{(\lambda - k_3)(\lambda - k_4)}} \equiv H_1(z) > 0$, $k_1 < z < k_3$, $H_1'(z) < 0$, $H_1(k_3) = 0$, $\lim\limits_{z \to k_1} H_1(z) = +\infty$, $H_1'(k_3) = -\infty$ can be calculated by using the standard Euler's substitutions.

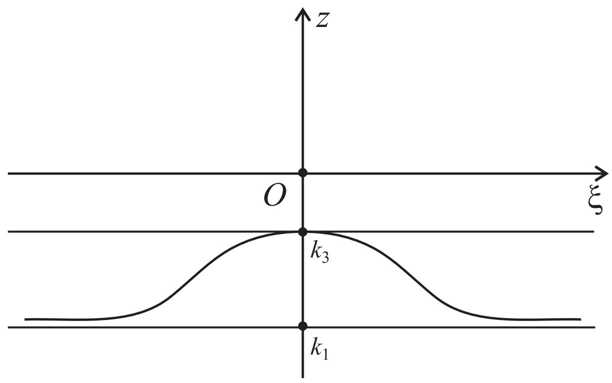

Fig.9. Soliton solution

Our next step is to construct CNN model of the two-component Camassa-Holm type system (20). In our case we have the following $N \times N$ CNN system:

$$\left| \begin{array}{l} \frac{du_{ij}}{dt} - \frac{d}{dt}(A_2 * u_{ij}) + 2A_1 * u_{ij}(u_{ij} - A_2 * u_{ij}) + \rho_{ij} A_1 * \rho_{ij} = 0 \\ \frac{d\rho_{ij}}{dt} + \rho_{ij} A_1 * u_{ij} + u_{ij} A_1 * \rho_{ij} = 0 \end{array} \right. , \tag{27}$$

where $1 \leq i, j \leq N$.

Our objective in this paper is to study the structure of the travelling wave solutions of the CNN model of two component Camassa-Holm type system (27). We shall study the travelling wave solutions of the CNN model (27) of the form:

$$\left| \begin{array}{l} u_{ij} = \Phi(icos\Theta + jsin\Theta - ct), \\ \rho_{ij} = \Psi(icos\Theta + jsin\Theta - ct), \end{array} \right. \tag{28}$$

for some continuous functions $\Phi, \Psi : \mathbf{R}^1 \to \mathbf{R}^1$ and for some unknown real number c. As we mentioned above $s = icos\Theta + jsin\Theta - ct$. Let us substitute (28) in our CNN model (27). Therefore we consider solution $\Phi(s; c)$, $\Psi(s; c)$ of:

$$\left| \begin{array}{l} -c\Phi^{'}(s; c) + G_1(\Phi(s; c), \Psi(s; c)) = 0, \\ -c\Psi^{'}(s; c) + G_2(\Psi(s; c), \Psi(s; c)) = 0, \end{array} \right. \tag{29}$$

where $G_1(\Phi, \Psi)$, $G_2(\Phi, \Psi) \in \mathbf{R}^1$ are satisfying

$$\left| \begin{array}{l} lim_{s \to \pm\infty} \Phi(s; c) = 0, \\ lim_{s \to \pm\infty} \Psi(s; c) = 0, \end{array} \right. \tag{30}$$

for some $c > 0$. We shall investigate the basic properties of the solutions of (27).

Suppose that our CNN model (27) are finite circular arrays of $L = N.N$ cells. For this case we have finite set of frequences [104]:

$$\Omega = \frac{2\pi k}{L}, \;\; 0 \leq k \leq L - 1. \tag{31}$$

The following proposition then hold:

Proposition 4. *Suppose that $u_{ij}(t) = \Phi(icos\Theta + jsin\Theta - ct)$, $\rho_{ij} = \Psi(icos\Theta + jsin\Theta - ct)$ are the travelling wave solutions of the CNN model (27) of the two component Camassa-Holm type system (20) with $\Phi, \Psi \in C^1(\mathbf{R^1}, \mathbf{R^1})$ and $\Omega = \frac{2\pi k}{L}$, $0 \leq k \leq L - 1$. Then there exist constants $c > 0$ and $s_0 > 0$ such that*

(i) for $s < s_0$ the solutions $\Phi(s; c)$, $\Psi(s; c)$ of (29) satisfying (30) is increasing;

(ii) for $s > s_0$ the solutions $\Phi(s; c)$, $\Psi(s; c)$ of (29) satisfying (30) is decreasing;

(iii) for $s = s_0$ the solutions $\Phi(s; c)$, $\Psi(s; c)$ of (29) have maximum of cusp type or maximum of angle type with positive opening (peakon-cuspon, respectively peakon).

Moreover, the solutions $\Phi(s; c)$, $\Psi(s; c)$ are either non vanishing everywhere or compactly supported, i.e. $\Phi(s; c) = 0$, $\Psi(s; c) = 0$ for $|s - s_0| \geq d$, d being an appropriate positive constant.

Remark 4. There has been many studies on travelling wave solutions of spatially and time discrete systems [11]. However, as far as we know travelling wave solutions of peakon type have been hardly studied in such discrete systems. For this reason we apply CNN approach and the numerical simulations of our CNN model (26) confirm the proposed results.

The simulations of the CNN model (27) give us the following Figures 10 and 11:

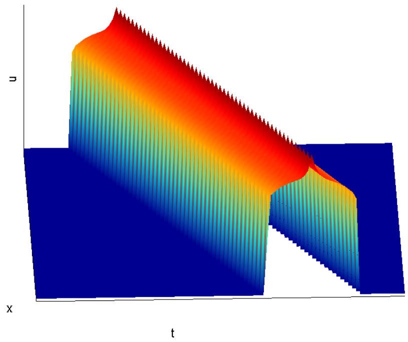

Fig.10 The wave solution of (27) of the type peakon-cuspon.

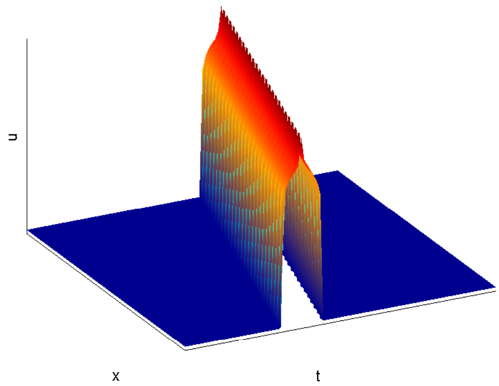

Fig.11.Peakon wave solution of (27)

References

1. B.Alvarez-Samaniego, D.Lannes, Large time exitence for 3D water-waves and asymptotics, *Invent.Math.*, 171, pp. 485-541, 2008.
2. R.Camassa, D. Holm, An integrable shallow water equation with peaked solutions, *Phys. Rev. Letters*, vol. 71, pp. 1661–1664, 1993
3. L.O.Chua, L.Yang, Cellular Neural Network: Theory and Applications, *IEEE Trans. CAS*, vol. 35, pp. 1257-1290, Oct. 1988.
4. L.O.Chua, M.Hasler, G.S.Moschytz, J.Neirynsk, Autonomous cellular neural networks: a unified paradigm for pattern formation and active wave propagation, *IEEE Trans. CAS-I*, vol. 42, N 10, pp. 559-577, Oct. 1995.
5. A.Constantin, R.S.Johnson, Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis, *Fluid Dyn.Res.*, 40, pp. 175-211, 2008.
6. A.Constantin, W.Strauss, Stability of the Camassa-Holm solitons, *J.Nonlinear Sci.*, 12, pp.415-422, 2002.

7. P.G.Drazin, R.S.Johnson, *Solitons:An Introduction*, Cambridge University Press, Cambridge, 1989.
8. A. Forsyth, A.,*Theory of differential equations*, vol. 6, Dover Publications, NY, 1959
9. A.E.Green, P.M. Naghdi, A derivation of equations for wave propagation in water of variable deph, *J.Fluid Mech.*, 78, pp.237-246, 1976.
10. R.S.Johnson, *A Modern Introduction to the Mathematical Theory of Water Waves*, Cambridge University Press, Cambridge, 1997.
11. J. Mallet-Paret, Spatial patterns, spatial chaos and travelling waves in lattice of differential equations, in: *Stochastic and Spatial Structure of Dynamical Systems*, editors van Strien, S. and Verduyn Lunel, S., North-Holland, pp. 105–129, 1996.
12. A.I.Mees, *Dynamics of Feedback Systems*, London, England:Wiley, 1981.
13. L.Perko, *Differential equations and dynamical systems*, Springer, NY, 1991.
14. T.Roska, L.O.Chua, D.Wolf, T.Kozek, R.Tetzlaff, F.Puffer, Simulating nonlinear waves and partial differential equations via CNN- Part I: Basic techniques, *IEEE Trans. CAS-I*, vol. 42, N 10, pp. 807-815, Oct. 1995.
15. F. Serre, Contribution á l'étude des écoulements permanents et variables dans les canaux, *La Houille Blanche*, 3, pp. 374-388, 1953.
16. A.Slavova,*Cellular Neural Networks: Dynamics and Modelling*, Kluwer Academic Publishers, 2003.
17. R. Tetzlaff, F. Gollas, Modeling complex systems by reaction-diffusion Cellular Nonlinear Networks with polynomial Weight-Functions,*Proc. IEEE CNNA2005*,Taiwan, 2005.