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MATHEMATIQUES

Mathématiques appliquées

DYNAMIC BEHAVIOUR OF CRACKED FUNCTIONALLY GRADED MAGNETOELECTROELASTIC SOLIDS

Tsviatko Rangelov, Yonko Stoynov*

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Abstract

Exponentially inhomogeneous magnetoelectroelastic (MEE) continuum with a finite crack, subjected to an incident time-harmonic anti-plane mechanical and in-plane electrical and magnetic load is considered. Fundamental solutions for the coupled system of equations are implemented in the non-hypersingular traction based Boundary Integral Equations Method (BIEM).

Numerical examples show the sensitivity of the dynamic generalized stress intensity factor (SIF) to the type of the magnetoelectroelastic continua, to the characteristics of the applied generalized dynamic load and to the magnitude and direction of the material inhomogeneity in functionally graded magnetoelectroelastic material.

Key words: functionally graded magnetoelectroelastic solid, anti-plane cracks, BIEM, SIF

2000 Mathematics Subject Classification: 35E05, 74S15, 74F99

1. Introduction. Magnetoelectroelastic (MEE) materials, possessing simultaneously piezoelectric, piezomagnetic and magnetoelectric properties, have drawn the interest of the researchers in the recent years. The multilayered solid made of different multifunctional materials is a base component in almost all products of the modern smart structure technology. These structures accumulate stress between the layers due to the discontinuous material properties and thus they can cause failures in the devices made of such materials (see Chue and Hsu [1]). To

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overcome the sharp interface and reduce stress concentration functionally graded materials (FGM) with properties that vary continuously in the spatial domain are developed (see MA and LEE [2]). With the wide use of graded materials in the new hi-tech industry a number of research results are devoted to the various crack problems. For instance, FENG and SU [3] have presented an analysis of a FGMEE strip, containing an internal crack, using integral transforms. MA et al. [4] have investigated embedded and edge antiplane cracks in a functionally graded MEE strip reducing the problem to singular integral equations and employing integral transforms. STOYNOV and RANGELOV [5] have considered classes of cracked inhomogeneous MEE materials and derived in a closed form time-harmonic fundamental solutions. Hypersingular BIEM is used by ROJAS DIAZ et al. [6], who have studied cracks interactions in homogeneous MEE materials, subjected to a time-harmonic in-plane load. A time-domain analysis of a homogeneous MEE medium with a crack is presented in LI [7], who has used integral transforms to reduce the problem to a Fredholm integral equation.

The aim of this paper is to investigate the dynamic anti-plane crack problem in an exponentially inhomogeneous MEE plane subjected to an incident SH time-harmonic type wave by BIEM. The crack is oriented arbitrarily to the material gradient. Numerical results are presented to show the effects of the material graded index and the characteristics of the external loading on the stress concentration field near the crack tip.

2. Statement of the problem. In a Cartesian coordinate system is considered a linear MEE medium poled in Ox_3 direction and subjected to a time-harmonic anti-plane mechanical load on Ox_3 axis and in-plane electrical and magnetic loads in the plane Ox_1x_2. The only non-vanishing fields are the anti-plane mechanical displacement u_3, the in-plane electrical displacement D_i and the in-plane magnetic induction B_i. The constitutive relations in the plane Ox_1x_2 are (see SOH and LIU [8])

$$\sigma_{iK} = C_{iKJl} u_{J,l}, \quad x \in R^2 \setminus \Gamma, \tag{1}$$

where $x = (x_1, x_2)$, $\Gamma = \Gamma^+ \cup \Gamma^-$ is a crack – an open arc. Here the comma denotes partial differentiation, the small indexes $i, l = 1, 2$, the capital indexes $K, J = 3, 4, 5$, and a summation in the repeating indexes is assumed. The generalized displacement is $u_J = (u_3, \phi, \varphi)$, where ϕ is the electric potential and φ is the magnetic potential. The generalized stress tensor is $\sigma_{iJ} = (\sigma_{i3}, D_i, B_i)$, where σ_{i3} is the stress and C_{iJKl} is the generalized elasticity tensor defined as

$$C_{i33l} = \begin{cases} c_{44} & i = l \\ 0, & i \neq l \end{cases}, C_{i34l} = C_{i43l} = \begin{cases} e_{15}, & i = l \\ 0, & i \neq l \end{cases}, C_{i35l} = C_{i53l} = \begin{cases} h_{15}, & i = l \\ 0, & i \neq l \end{cases},$$

$$C_{i44l} = \begin{cases} -\varepsilon_{11}, & i = l \\ 0, & i \neq l \end{cases}, C_{i45l} = C_{i54l} = \begin{cases} -d_{11}, & i = l \\ 0, & i \neq l \end{cases}, \quad C_{i55l} = \begin{cases} -\mu_{11}, & i = l \\ 0, & i \neq l \end{cases}.$$

Functions $c_{44}(x)$, $e_{15}(x)$, $\varepsilon_{11}(x)$ are: elastic stiffness, piezoelectric coupled coefficient and dielectric permittivity, while $h_{15}(x)$, $d_{11}(x)$, $\mu_{11}(x)$ are piezomagnetic and magnetoelectric coefficients and magnetic permeability correspondingly. It is assumed that $c_{44}(x)$, $\varepsilon_{11}(x)$ and $\mu_{11}(x)$ are positive that correspond to a stable material (see [8]).

Suppose that the material parameters C_{iJKl} and the density ρ depend in the same manner exponentially on x

$$C_{iKJl}(x) = C^0_{iKJl}e^{2\langle a,x\rangle}, \quad \rho(x) = \rho^0 e^{2\langle a,x\rangle}, \tag{2}$$

where $\langle,\rangle$ means the scalar product in R^2, $a = (a_1, a_2)$ and we use the notations $a_1 = r\cos\alpha$, $a_2 = r\sin\alpha$, $r = |a|$ is the magnitude and α is the direction of the material inhomogeneity.

Assuming the quasistatic approximation of MEE material in the absence of body forces, electric charges and magnetic current densities the balance equation is

$$\sigma_{iK,i} + \rho_{KJ}\omega^2 u_J = 0, \tag{3}$$

where $\rho_{QJ} = \begin{cases} \rho, & Q = J = 3 \\ 0, & Q, J = 4 \text{ or } 5 \end{cases}$ and ω is the frequency of the applied time-harmonic load.

The boundary condition on the crack is

$$t_J|_\Gamma = 0, \tag{4}$$

where $t_J = \sigma_{iJ}n_i$ is the generalized traction and $n = (n_1, n_2)$ is the normal vector to Γ. That means the crack is impermeable, i.e. the crack line is free of mechanical traction, electric charge and magnetic current.

Using the superposition principle the displacement and the traction are represented as $u_J = u^{\rm in}_J + u^{\rm sc}_J$, $t_J = t^{\rm in}_J + t^{\rm sc}_J$, where $u^{\rm in}_J$, $t^{\rm in}_J$ is the free-field solution of equation (3) and its traction, $u^{\rm sc}_J$, $t^{\rm sc}_J$ is the scattering field due to the crack Γ.

Denote $\omega_0 = \sqrt{\dfrac{\det M}{(\varepsilon^0_{11}\mu^0_{11} - d^{02}_{11}\rho^0)}}|a|$, where $M = \begin{pmatrix} c^0_{44} & e^0_{15} & q^0_{15} \\ e^0_{15} & -\varepsilon^0_{11} & -d^0_{11} \\ q^0_{15} & -d^0_{11} & -\mu^0_{11} \end{pmatrix}$.

In the following we will study the case

$$\omega > \omega_0 \tag{5}$$

when the dynamic behaviour of the MEE material is characterized by a wave propagation phenomena.

Let us introduce the smooth change of functions

$$u_J(x,\omega) = e^{-\langle a,x\rangle}U_J(x,\omega) \tag{6}$$

and suppose that $U_J(x,\omega)$ satisfies Sommerfeld-type condition on infinity, more specifically

$$(7) \qquad U_3 = o(|x|^{-1}), \quad U_4 = o(e^{-|a||x|}), \quad U_5 = o(e^{-|a||x|}) \quad \text{for} \quad |x| \to \infty.$$

Condition (7) ensures uniqueness of the scattering field $u_J^{\rm sc}$ for a given incident field $u_J^{\rm in}$. Following Akamatsu and Nakamura [9] for the piezoelectric case it can be proved that the boundary value problem (BVP) (3), (4) admits continuous differentiable solutions.

In order to solve problem (3), (4) numerically we will use BIEM, i.e. we transform the problem to equivalent integrodifferential equation on the crack Γ.

3. Non-hypersingular BIEM and SIF evaluation. The non-hypersingular traction BIE is derived following Wang and Zhang [10] for the homogeneous and [11] for the inhomogeneous piezoelectric case and Stoynov and Rangelov [5,12] for the inhomogeneous and homogeneous MEE materials.

For u_J, u^*_{JK}, where u^*_{JK} is the fundamental solution of (3), we apply the Green formula in the domain $\Omega_R \setminus \Omega_\varepsilon$, Ω_R is a circular domain with large radius R, and Ω_ε is a small neighbourhood of Γ. Applying the representation formulae for the generalized displacement gradient $u_{K,l}$ (see [10]) an integro-differential equation on $\partial\Omega_R \cup \partial\Omega_\varepsilon$ is obtained (see [5]). Using condition (7) integrals over $\partial\Omega_R$ go to 0 for $R \to \infty$. Taking the limit $\varepsilon \to 0$, i.e. $x \to \Gamma$, and using the boundary condition (4), i.e. $t_J^{\rm sc} = -t_J^{\rm in}$ on Γ the following system of BIE is equivalent to the BVP (3), (4)

$$(8) \quad -t_J^{\rm in}(x,\omega) = C_{iJKl}(x)n_i(x)\int_\Gamma [(\sigma^*_{\eta PK}(x,y,\omega)\triangle u_{P,\eta}(y,\omega)$$
$$- \rho_{QP}(y)\omega^2 u^*_{QK}(x,y,\omega)\triangle u_P(y,\omega))\delta_{\lambda l} - \sigma^*_{\lambda PK}(x,y,\omega)\triangle u_{P,l}(y,\omega)]n_\lambda(y)\, d\Gamma,$$

$x \in \Gamma$, where $\sigma^*_{iJQ} = C_{iJMl}u^*_{MQ,l}$ is the stress of the fundamental solution, $\Delta u_J = u_J|_{\Gamma+} - u_J|_{\Gamma-}$ is the generalized crack opening displacement (COD), $x = (x_1, x_2)$, $y = (y_1, y_2)$ denote the field point and the source point, respectively. Equation (8) is a traction non-hypersingular BIE on the crack line Γ for the unknown Δu_J. From its solution the generalized displacement u_J can be obtained at every point in $R^2 \setminus \Gamma$ by using the corresponding representation formulae (see [12]).

A fundamental solution u^*_{JK} of (3) is defined as a solution of equation

$$(9) \qquad \sigma^*_{iJM,i} + \rho_{KJ}\omega^2 u^*_{KM} = -\delta_{JM}\delta(x,\xi),$$

where $\delta(x,\xi)$ is the Dirak's function and δ_{JM} is the Kroneker symbol. For the considered inhomogeneity the fundamental solution is obtained in [5] and we will shortly list the solution method. First, the smooth transform $u^*_{KM} = e^{-\langle a,x\rangle}U^*_{KM}$ applied to (9) gives

$$(10) \qquad C^0_{iJKi}U^*_{KM,ii} + [\rho^0_{JK}\omega^2 - C^0_{iJKi}a_i^2]U^*_{KM} = e^{-\langle a,\xi\rangle}\delta_{JM}\delta(x,\xi).$$

Second, applying the Radon transform

$$R(f) = \widehat{f}(s,m) = \int_{-\infty}^{+\infty} f(x)\delta(s - \langle x, m\rangle)\, dx$$

(see ZAYED [13]) to both sides of (10), having in mind that only m, $|m| = 1$ are used for the inverse Radon transform and solving the corresponding system of ordinary differential equations for $\widehat{U}^* = R(U^*)$ we get

$$
\begin{aligned}
&\widehat{U}^*_{33} = -e^{-\langle a,\xi\rangle}\frac{1}{2ika_0}e^{ik|s-\tau|}, \qquad \widehat{U}^*_{34} = \widehat{U}^*_{43} = -\frac{A}{2ika_0}e^{ik|s-\tau|},\\
&\widehat{U}^*_{44} = -e^{-\langle a,\xi\rangle}\left(\frac{A^2}{2ika_0}e^{ik|s-\tau|} + \frac{1}{2\overline{\varepsilon}^0_{11}|a|}e^{|a||s-\tau|}\right),\\
&\widehat{U}^*_{35} = \widehat{U}^*_{53} = -e^{-\langle a,\xi\rangle}\frac{B}{2ika_0}e^{ik|s-\tau|},\\
&\widehat{U}^*_{45} = \widehat{U}^*_{54} = -e^{-\langle a,\xi\rangle}\left(\frac{AB}{2ika_0}e^{ik|s-\tau|} - \frac{d^0_{11}}{2\overline{\varepsilon}^0_{11}\mu^0_{11}|a|}e^{|a||s-\tau|}\right),\\
&\widehat{U}^*_{55} = -e^{-\langle a,\xi\rangle}\left(\frac{B^2}{2ika_0}e^{ik|s-\tau|} + \frac{1}{2|a|}\Big(\frac{d^{02}_{11}}{\overline{\varepsilon}^0_{11}\mu^{02}_{11}} + \frac{1}{\mu^0_{11}}\Big)e^{|a||s-\tau|}\right),
\end{aligned} \tag{11}
$$

where the following notations are used: $\tau = \langle \xi, m\rangle$,

$$a_0 = \overline{c}^0_{44} + \frac{\overline{e}^{02}_{15}}{\overline{\varepsilon}^0_{11}}, \quad k = \sqrt{\frac{\rho^0\omega^2}{a_0} - |a|^2}, \quad \overline{c}^0_{44} = c^0_{44} + \frac{q^{02}_{15}}{\mu^0_{11}}, \quad \overline{e}^0_{15} = e^0_{15} - \frac{d^0_{11}q^0_{15}}{\mu^0_{11}},$$
$$\overline{\varepsilon}^0_{11} = \varepsilon^0_{11} - \frac{d^{02}_{11}}{\mu^0_{11}}, \quad A = \frac{\mu^0_{11}e^0_{15} - q^0_{15}d^0_{11}}{\mu^0_{11}\varepsilon^0_{11} - d^{02}_{11}}, \quad B = \frac{q^0_{15}\varepsilon^0_{11} - d^0_{11}e^0_{15}}{\mu^0_{11}\varepsilon^0_{11} - d^{02}_{11}}.$$

The third step is to apply the inverse Radon transform and to obtain functions U^*_{JK} and correspondingly u^*_{JK}.

The free-field solution u^{in}_J and its traction t^{in}_J on the crack Γ is obtained using the wave decomposition method. With $\eta = (\eta_1, \eta_2)$, $|\eta| = 1$ – the wave propagation direction, p_J – the polarization vector, and k – the wave number, we get

$$u^{\text{in}}_J = p_J e^{\langle x, -a+ik\eta\rangle}, \quad p_1 = 1, \quad p_2 = A, \quad p_3 = B, \tag{12}$$

$$t^{\text{in}}_3 = \frac{\det M}{\varepsilon^0_{11}\mu^0_{11} - d^{02}_{11}}\langle a + ik\eta, n\rangle e^{\langle x, a+ik\eta\rangle}, \quad t_4 = 0, \quad t_5 = 0. \tag{13}$$

The dynamic fracture state of MEE is characterized by the leading term of the asymptotic of the displacement and traction near the crack tip – generalized stress intensity factor. For the considered dynamic problem of MEE media SIFs are: mechanical K_{III}, electrical displacement K_D and magnetic displacement K_B.

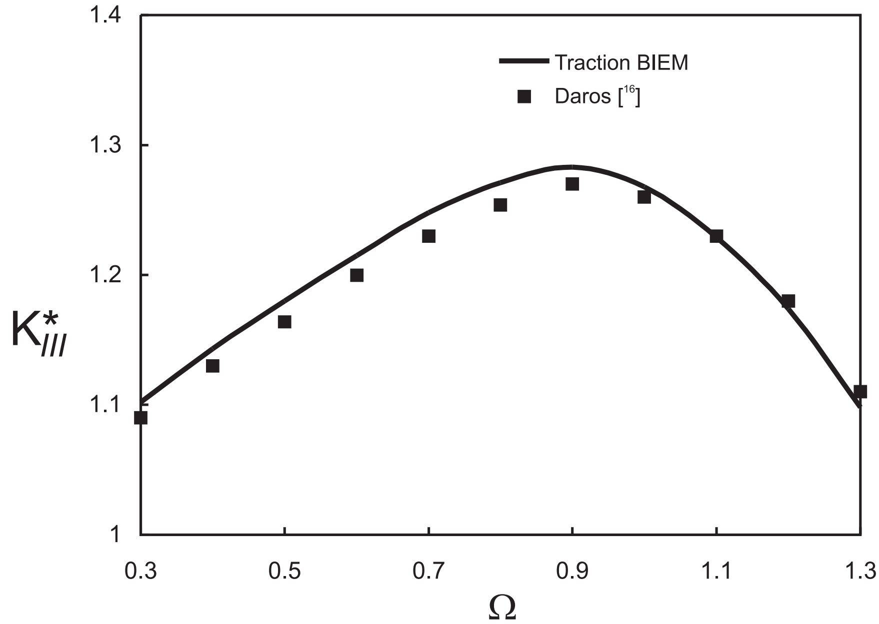

Fig. 1. Dynamic normalized SIF K^*_{III} versus normalized frequency Ω of normal incident SH type wave propagating in exponentially inhomogeneous MEE plane with material inhomogeneity magnitude $\beta = 0.4$ and material gradient $\alpha = \pi/2$

They are obtained directly from the traction values ahead the crack-tip (see Suo et al. [14]) for the piezoelectric case. In the case of the straight crack along the axis Ox_1, $\Gamma = (-c, c)$, SIFs are computed by the formulae

$$(14)\qquad \begin{aligned} K_{III} &= \lim_{x_1 \to \pm c} t_3 \sqrt{2\pi(x_1 \mp c)}, \\ K_D &= \lim_{x_1 \to \pm c} t_4 \sqrt{2\pi(x_1 \mp c)}, \\ K_B &= \lim_{x_1 \to \pm c} t_5 \sqrt{2\pi(x_1 \mp c)}, \end{aligned}$$

where t_J is calculated at the point $(x_1, 0)$ close to the crack-tip.

4. Numerical realization. The numerical procedure for the solution of the boundary value problem follows the numerical algorithm developed and validated in [11] for the inhomogeneous piezoelectric case and in [12] for the homogeneous MEE case. The crack Γ is discretized by quadratic boundary elements (BE) away from the crack-tips and special crack-tip quarter-point BE near the crack-tips to model the asymptotic behaviour of the displacement and the traction.

In the numerical examples the crack Γ with a half-length $c = 5$ mm, occupying an interval $(-c, c)$ along Ox_1 axis is considered. The crack is divided into 7 BE with lengths correspondingly: $l_1 = l_7 = 0.15c$, $l_2 = \cdots = l_6 = 0.34c$. First BE is a left-quarter point BE, 7th BE is a right-quarter point BE, and the rest BE are

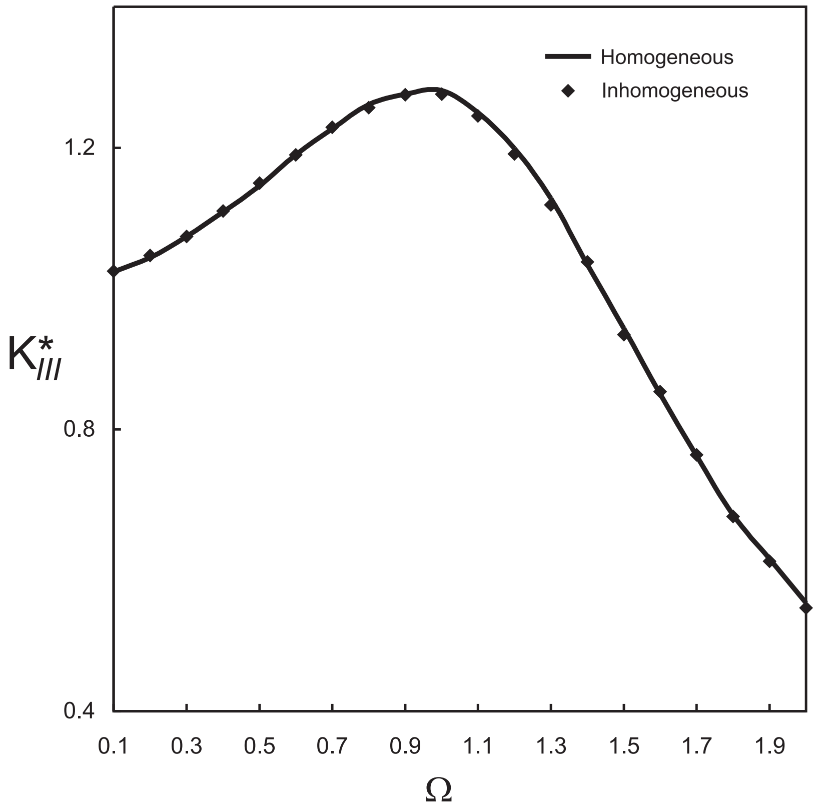

Fig. 2. Dynamic normalized SIF K^*_{III} versus normalized frequency Ω of normal incident SH wave propagating in homogeneous MEE plane (for material inhomogeneity magnitude $\beta = 0$)

ordinary BE. The reference properties C^0_{iJKl} of the used materials: piezoelectric material $BaTiO_3$, piezomagnetic material $CoFe_2O_4$, and magnetoelectroelastic composite $BaTiO_3/CoFe_2O_4$ can be found in [12] and in Song and Sih [15].

The proposed method is validated with the results of Daros [16], who used BIEM to study propagation of SH wave in anisotropic materials with reference elastic properties and density equal to the properties of the piezoceramic PZT 6B. The absolute values of the normalized SIF $K^*_{III} = \frac{K_{III}}{t^{\text{in}}_3\sqrt{\pi c}}$ versus the normalized frequency $\Omega = ck_0$, where $k_0 = \sqrt{\dfrac{\rho^0\omega^2}{c^0_{44}} - |a|^2}$ are presented in Fig. 1, where the magnitude and the direction of the exponentially inhomogeneous piezoceramic are correspondingly $\beta = 0.4$ and $\alpha = \pi/2$. The comparison shows a very close coincidence of the results obtained by the authors and those in [16]. Figure 2 reveals that when the magnitude of the material inhomogeneity is zero, then the solution for the inhomogeneous case recovers the solution of the boundary-value problem for the pure homogeneous material.

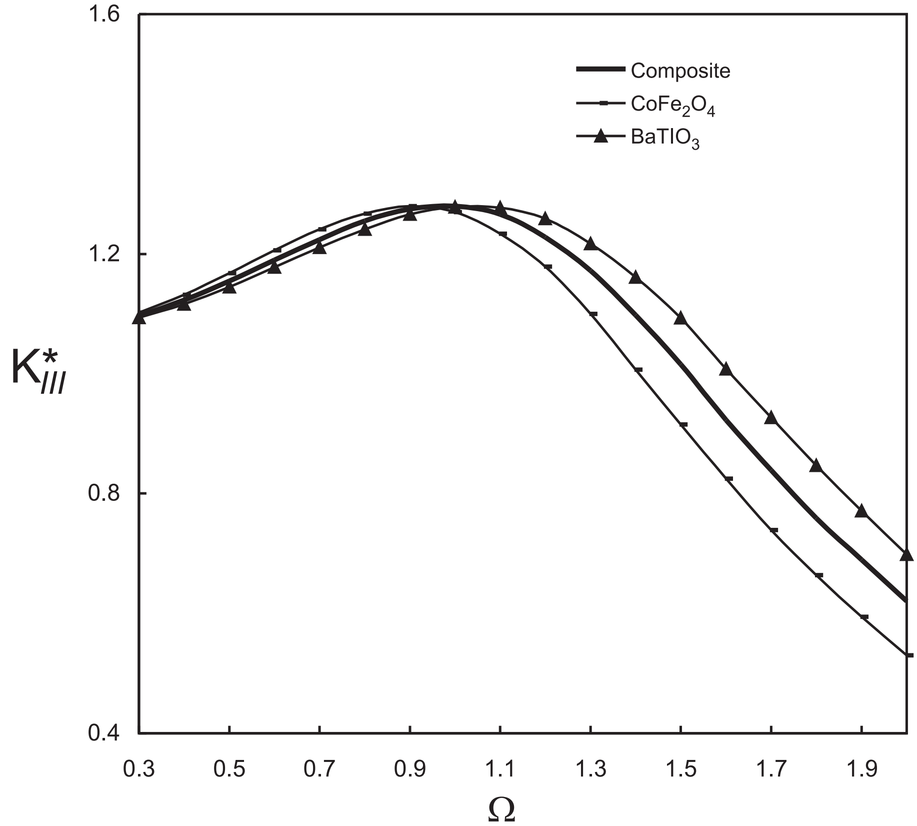

Fig. 3. Dynamic normalized SIF K^*_{III} at the left crack tip versus normalized frequency Ω of normal incident SH type wave propagating in exponentially inhomogeneous MEE plane made of different materials with material inhomogeneity magnitude $\beta = 0.4$ and inhomogeneity direction $\alpha = \pi/2$

The aim of the simulation examples is to show the sensitivity of the dynamic SIF K^*_{III} on the type of the material, the characteristics of the material inhomogeneity and on the properties of the propagating time-harmonic SH type wave. In Figure 3 the normalized SIFs K^*_{III} is plotted versus the normalized frequency $\Omega = c\omega\sqrt{\rho^0/c^0_{44}}$ for fixed inhomogeneity characteristics $\beta = 0.4$ and $\alpha = \pi/2$, i.e. the material constants vary continuously in direction perpendicular to the crack. The results show dependence of the SIF on the type of the reference material in the prescribed frequency interval.

The normalized SIF K^*_{III} versus different values of $m = 0.0, 0.1, \ldots, 1.0$, where $\alpha = m\pi/2$ are plotted in Fig. 4. The normalized frequency $\Omega = c\omega\sqrt{\rho^0/a_0}$ is 1.0 and $\beta = 0.2, 0.4, 0.6$. It can be seen that for fixed β SIF K^*_{III} is increasing when α is increasing. If α is fixed, K^*_{III} is the smallest for the largest value of β and vice versa. The results show dependence of the SIF on the inhomogeneity characteristics of the MEE material.

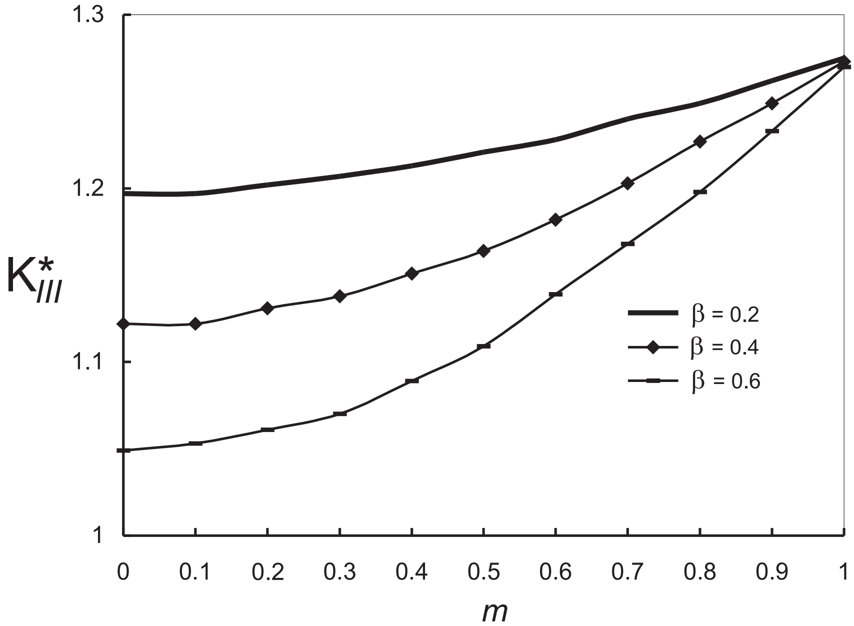

Fig. 4. Dynamic normalized SIF K^*_{III} at the left crack tip versus direction of the material gradient $\alpha = m\pi/2$, $m = 0.0, 0.1, \ldots, 1.0$ for normalized frequency $\Omega = 1.0$ of the incident SH wave and for different inhomogeneity magnitude $\beta = 0.2, 0.4, 0.6$

5. Conclusion. A dynamic fracture analysis of exponentially inhomogeneous MEE materials subjected to time-harmonic anti-plane mechanical and in-plane electromagnetic loads is presented in this study. It is developed, validated and used in simulations an effective non-hypersingular traction based BIEM for the solution of the posed problem. The method can be successfully used for more complex problems of crack interactions, cracks with arbitrary shapes and composites with different combinations of piezoelectric and piezomagnetic constituents.

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
e-mail: `rangelov@math.bas.bg`

**Faculty of Applied Mathematics*
and Informatics
Technical University of Sofia
e-mail: `ids@tu-sofia.bg`