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CONVECTION DIFFUSION MODEL FOR IMAGE PROCESSING

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Abstract

Recently many mathematical models for image processing have been widely applied in computer visualization. The nonlinear diffusion PDE has been broadly applied in image processing. In this paper we propose a convection-diffusion filter by adding a convection term in the modified diffusion equation as a physical interpretation for removing the noise. We study the dynamics of such equations by the discretization of this convection-diffusion model. Numerical experiments show that our method is reasonably better in removing noise.

Key words: convection-diffusion equations, Cellular Neural Networks (CNN), algorithms, filters, image processing

2000 Mathematics Subject Classification: 92B20, 37M05, 37N30, 35Q68, 35K67, 35K57

1. Introduction. Recently many mathematical models for image processing have been widely applied in computer visualization. The nonlinear diffusion partial differential equations have been broadly applied in image processing since the first model was introduced in 1987 [4]. Through the time evolution, diffusion can effectively remove the noise as well to have edge enhancement simultaneously. Since then various nonlinear diffusion filters have been widely proposed in implementing the image denoising/enhancement, edge detection and flow filed visualization. A common feature for nonlinear diffusion model is that the diffusion coefficient is small while the gradient of image is large. However, the diffusion coefficient is a function of the convolution of the Gaussian kernel and solution such that this requires an extra cost in computing the nonlinear diffusion coefficient. In the numerical experiments we find that when using the nonlinear diffusion model in denoising the noise is not quite good [5]. Hence, we propose a convection-diffusion filter by adding a convection term in the modified diffu-

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sion equation as a physical interpretation for removing the noise. The aim of this paper is to focus on the noise removal algorithm for extracting the target information (image) precisely. The main idea of our model algorithm is to diffuse the noise by following the convection direction during time evolution. To prevent the numerical layer in the discontinuities on the relative coarse grids we use the Cellular Neural Network (CNN) approach [1, 2, 6].

Here we propose a popular nonlinear convection-diffusion model for image diminishing as well as image compression. We shall modify nonlinear isotropic equation [4] and we shall construct both convection and diffusion terms based on the gradient of image intensity; thus the direction of image smoothing is normalized to the gradient of image intensity. Consider our filter: the convection-diffusion problem (CD) with manipulation in both diffusion and convection terms for controlling the smoothing process:

$$
(1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon(|\nabla u|)\Delta u + \beta(|\nabla u|).\nabla u = 0 & \text{in} \quad \Omega \times I, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega \times I, \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases}
$$

where the diffusion coefficient is denoted by $\varepsilon(|\nabla u|) \equiv \dfrac{1}{1+|\nabla u|^2}$ and the convection vector $\beta(|\nabla u|) \equiv \gamma \dfrac{\nabla u^{\perp}}{|\nabla u|_{\epsilon}}$ for positive constant γ (the size of convection vector) with the Evans-Spruck regularization [6] $|\nabla u|_{\epsilon} \equiv \sqrt{u_x^2 + u_y^2 + \epsilon}$ for $0 < \varepsilon \ll 1$ to avoid the singularity. As the gradient of image intensity is big the diffusion coefficient ε is inhibited. CD preserves the edges of image and protects the brightness of the image simultaneously.

It is well known that if the diffusion coefficient ε is sufficiently small in comparison with the quantity $|\beta|h$ where the solution is discontinuous then the Galerkin finite element scheme leads to a severe oscillation [5]. The streamline diffusion finite element method resolves the oscillation problem; however, it causes some artifacts (overshooting/ down shooting) around the edge of discontinuities.

In this paper we shall study the following convection-diffusion problem:

$$
(2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = b\nabla u + cu, & \text{in} \quad D \equiv (0,1)^2, \\ u = 0 & \text{on} \quad \partial D, \end{cases}
$$

where $b(x,t) \geq \beta$, $c(x,t) \geq 0$ and $c_0^2(x,t) \equiv (c - b_x/2) \geq \gamma$, $x, t \in \bar{D}$, β and γ are some positive constants. It is known [5] that for this assumptions there exists a unique solution of the CD problem (2).

In the next section we shall construct the CNN algorithm for studying the dynamics of the CD equation (2). Section 2 deals also with the CNN model of the convection-diffusion equation and its dynamics. Two examples are given in order to demonstrate the simulation results.

2. CNN approach for studying the dynamics of the convection-diffusion model. It is known that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs). The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. Consider the following well-known PDE, generally referred to us in the literature as a reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = f(u) + D\nabla^2 u,$$

where $u \in \mathbf{R}^N$, $f \in \mathbf{R}^N$, D is a matrix with the diffusion coefficients, and $\nabla^2 u$ is the Laplacian operator in $\mathbf{R}^2$. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate A-template.

In our case the CNN model of CD equation (2) is

$$\dot{u}_j(t) = b * A_1 * u_j + c * u_j, 1 \le j \le N, \tag{3}$$

where $A_1 = (1, -2, 1)$ is one-dimensional discretized Laplacian CNN template.

In this section we will introduce an approximative method for studying the dynamics of CNN model (3), based on a special Fourier transform. The idea of using Fourier expansion for finding the solutions of PDEs is well known in physics. It is used to predict what spatial frequencies or modes will dominate in nonlinear PDEs. In CNN literature this approach has been developed for analyzing the dynamics of CNNs with symmetric templates [1].

We shall investigate the dynamic behaviour of CNN model (3) using the Harmonic Balance Method which is well-known in the control theory and in the study of electronic oscillators [3] as a describing function method. The method is based on the fact that all cells in CNN are identical [1], and therefore by introducing a suitable double transform, the network can be reduced to a scalar Lur's scheme [3].

We shall present the algorithm briefly:

1. Apply the double Fourier transform

$$F(s, z) = \sum_{k=-\infty}^{k=\infty} z^{-k} \int_{-\infty}^{\infty} f_k(t) \exp(-st)\, dt,$$

to CNN equation (3).

2. Find the transform function $H(s, z) =$, where $s = i\omega_0$, $z = \exp(i\Omega_0$, $i = \sqrt{-1}$, ω_0 is a temporal frequency, Ω_0 is a spatial frequency.

3. Look for possible solutions of (3) in the form

$$u_j = U_{m_0} \sin(\omega_0 t + j\Omega_0).$$

4. The amplitude U_{m_0}, the temporal frequency ω_0 and the spatial frequency Ω_0 are unknowns to be determined.

5. Our CNN model (3) is a finite circular array of N cells, so we have a finite set of spatial frequencies

$$\Omega_0 = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1. \tag{4}$$

Based on the above considerations the following proposition holds:

Proposition 1. *CNN model* (3) *of the convection-diffusion problem* (2)*, comprised of circular array of N cells, has state solution $u_j(t)$ with a finite set of spatial frequencies $\Omega_0 = 2\pi k/N$, $0 \leq k \leq N-1$.*

We obtain the following simulation results for different values of the parameters β and γ.

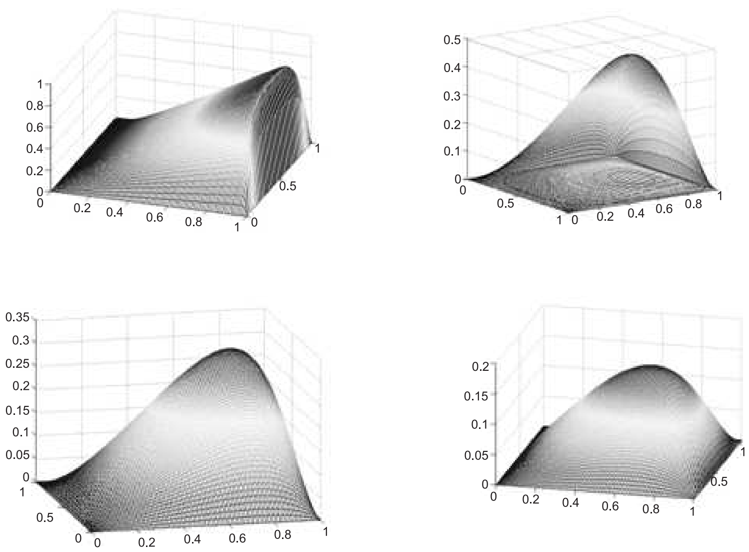

Fig. 1. Simulations of CNN algorithm for CD problem (2)

3. Examples. Example 1. Consider the following singularly perturbed boundary value problem

$$\begin{aligned} -\varepsilon\Delta u + bu_x + cu = 0, &\quad \text{in} \quad \Omega \equiv (0,1)^2 \\ u = 0, &\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{5}$$

In order to construct a robust numerical method for the considered problem, it is of key interest to have information on a behaviour of the solution. The state equation of the CNN model of (5) is

$$-\varepsilon A_1 * u_i + b * A_1 * u_i + c * u_i = 0. \tag{6}$$

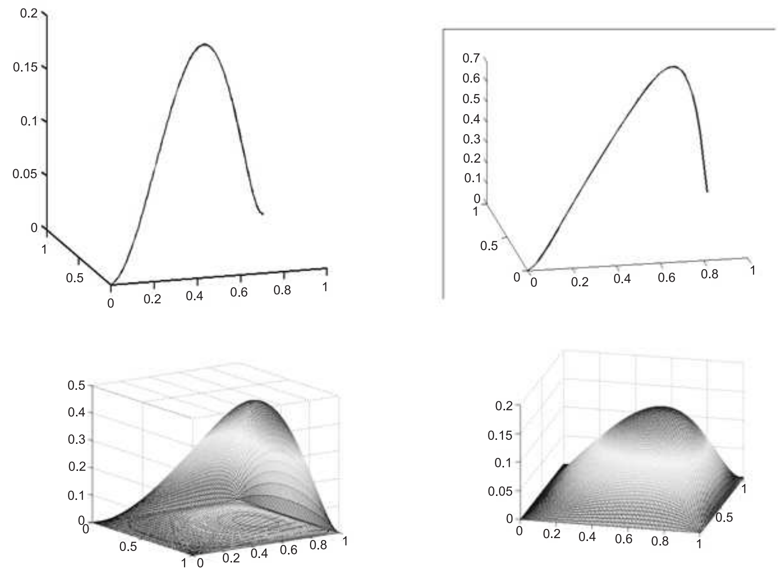

Fig. 2. Simulation of CNN algorithm for problem (5)

Applying CNN algorithm we obtain simulation results shown in Fig. 2.

Example 2. We consider the following example of (2) on the layer-adapted

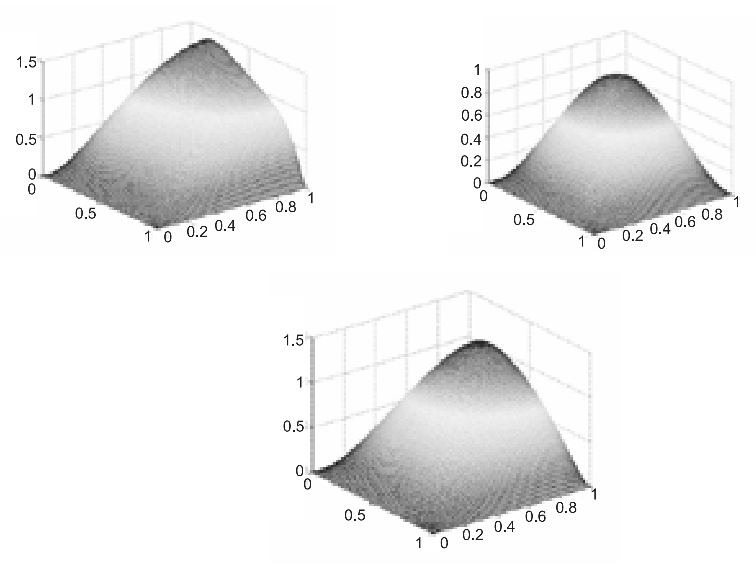

Fig. 3. Simulation of CNN algorithm for problem (7)

mesh

$$-\varepsilon\Delta u + (1+x^3)u_x + (1+xy)u = 0, \text{ in } \Omega \tag{7}$$

$$u = 0, \text{ on } \partial\Omega.$$

CNN model of the above system is

$$-\varepsilon A_1 * u_i + (1+x^3)A_1 * u_i + (1+xy)u_i = 0 \tag{8}$$

$$1 \le i \le N.$$

We obtain simulation results of (8) in Fig. 3.

Remark 1. We consider a CNN programmable realization allowing the calculation of all necessary processing steps in real time. The network parameter values of CNN models are determined in a supervised optimization process. During the optimization process the mean square error is minimized using Powell method and Simulated Annealing [7]. The results are obtained by the CNN simulation system MATCNN applying 4th order Runge-Kutta integration.

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