Almost Complex Structures on quaternion-Kähler manifolds of positive type

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based on:

Other reference:
Quatetion-Kähler manifolds

A \textit{quaternion-Kähler} — qK for short — manifold is a Riemannian manifold \((M, g)\) of dimension \(n = 4k \geq 8\), whose holonomy group is contained in \(\text{Sp}(k) \text{Sp}(1) \subset \text{SO}(4k)\).

Remark 1

1. \(\text{Sp}(k) \text{Sp}(1) = (\text{Sp}(k) \times \text{Sp}(1))/\pm 1\)

2. \(\text{Sp}(k) \subset \text{U}(2k)\): preserves the \textit{quaternionic structure} \(j\) of \(\mathbb{C}^{2k}\) (\(j\) anti-\(\mathbb{C}\)-linear, \(j^2 = -1\)), or, equivalently, the \textit{complex symplectic 2-form} \(\Omega = \langle \cdot, j \cdot \rangle\)

3. \(\text{Sp}(k) \text{Sp}(1) \subset \text{SO}(4k)\), via

\[
(A, p) \cdot u = A u p^{-1},
\]

for any \(A\) in \(\text{Sp}(k)\), any \(p\) in \(\text{Sp}(1)\), any \(u\) in \(\mathbb{R}^{4k} = \mathbb{H}^k\).
A **quaternion-Kähler manifold** is a (complete) oriented Riemannian manifold \((M, g)\) of dimension \(n = 4k \geq 8\) equipped with a rank 3 subbundle \(Q \subset A(M)\) of the bundle \(A(M)\) of anti-symmetric endomorphims of the tangent bundle \(TM\) such that:

(i) \(Q\) is preserved by the Levi-Civita connection \(\nabla\) of \(g\), and

(ii) \(Q\) is locally generated by positively oriented almost-complex structures \(J_1, J_2, J_3\) such that \(J_1J_2J_3 = -\text{Id}_{TM}\).
Classical examples

1. Quaternionic projective space: \( M = \mathbb{HP}^k = \frac{\text{Sp}(k+1)}{\text{Sp}(k) \times \text{Sp}(1)} \)

2. Grassmannian of complex 2-planes in \( \mathbb{C}^{2+k} \):
   \( M = \text{Gr}_2(2 + k, \mathbb{C}) = \frac{\text{SU}(k+2)}{\text{S}(\text{U}(k) \times \text{U}(2))} \)

3. Grassmannian of real oriented 4-planes in \( \mathbb{R}^{4+k} \):
   \( M = \text{Gr}_4(4 + k, \mathbb{R}) = \frac{\text{SO}(k+4)}{\text{SO}(k) \times \text{SO}(4)} \)

In each case

\[ T_x M = \text{Hom}_K(x, x^\perp) , \]

for any \( x \) in \( M \), with \( K = \mathbb{H}, \mathbb{C}, \mathbb{R} \) respectively, and the defining bundle \( Q \) is then:

\[ Q_x = \{ \Phi \in A_x(M) \text{ of the form } \Phi(X) = X \circ \phi \} \]

for any \( X \) in \( T_x M \), where \( \phi \) is either

1. a quaternionic skew-Hermitian endomorphism, or
2. a trace-free, skew-hermitian endomorphism, or
3. a self-dual skew-symmetric endomorphism

of \( x \), i.e. an element of \( \text{sp}(x), \text{su}(x) \) or \( \text{so}(x) \), respectively.
Wolf spaces

The above classical examples, together with

\[
\begin{array}{ccc}
G_2 & F_4 & E_6 \\
SO(4)' & Sp(3)Sp(1)' & SU(6)Sp(1)' \\
E_7 & E_8 & E_7Sp(1)'
\end{array}
\]

are so-called Wolf spaces, constructed, as well as their twistor spaces, by J. A. Wolf in 1965.

For any compact simple Lie group \( G \), the corresponding twistor space is realized as

1. the (co-)adjoint orbit in the Lie algebra \( \mathfrak{g} \) of the highest coroot, J. A. Wolf 1965,

2. the projectivization \( \mathbb{P}(\mathcal{N}_{\text{min}}) \) of the minimal (co-)adjoint orbit \( \mathcal{N}_{\text{min}} \) of \( G^C \) in \( \mathfrak{g}^C \), A. Beauville 1998.
Quaternion-Kähler manifolds of positive type


*Quaternion-Kähler manifolds are Einstein.*

Three types, according to the sign of the scalar curvature $s$:

1. $s > 0$ \quad **qK of positive type** (compact)
2. $s = 0$ \quad **locally hyperkähler**: $\text{Hol}^0(M, g) \subset \text{Sp}(k)$
3. $s < 0$ \quad **qK of negative type**

The talk is mainly devoted to qK manifolds of positive type, which we simply call **positive qK manifolds**.
The twistor fibration

To any qK manifold \((M, g, Q)\) is associated its **twistor space**, \(Z = Z(M)\), introduced and worked out independently by S. Salamon and L. Bérard Bergery in 1982, defined by

\[
Z(M) = S(Q) = \text{(sphere bundle of } Q\text{)} = \text{bundle of compatible complex structures on } TM.
\]

**Canonical almost complex structure** \(\mathcal{J}\) on \(Z\): obtained by combining the natural complex structure on the fibers with the tautological complex structure on the horizontal distribution, \(\nabla\), determined by the Levi-Civita connection, \(\nabla\), of \(g\).

The canonical almost complex structure \(\mathcal{J}\) is **integrable**, making \(Z\) into a complex manifold of (complex) dimension \(2k + 1\). Moreover, \(Z\) admits a natural **real structure**, \(\kappa\), with no fixed point, namely \(\kappa(J) = -J\), for any \(J\) in \(Z\).
Main properties of the twistor space
We henceforth assume that \((M, g, Q)\) is a **positive** qK manifold.

Then,

1. The pull-back of \(g\) on \(Z\), together with an appropriate scaling of the round metric on the fibers of \(Z\), determines a Riemannian metric, \(\tilde{g}\), which is **Kähler-Einstein**, with positive scalar curvature, making \(Z\) into a **Fano manifold** (= the anti-canonical line bundle \(K_{Z}^{-1}\) of \((Z, \mathcal{J})\) is **ample**).

2. The horizontal distribution \(H^\nabla\) is a \(\mathcal{J}\)-**holomorphic subbundle** of the (holomorphic) tangent bundle \(TZ\).

3. The vertical projection, \(\theta : TZ \to T^V\), regarded as 1-form on \(Z\) with values in the holomorphic line bundle

\[
L = TZ/H^\nabla,
\]  

is a **holomorphic contact form**, meaning that \(\theta \wedge (d\theta)^k\) is a non-vanishing holomorphic section of \(K_Z \otimes L^{k+1}\).
Inverse construction

It follows that $Z$ is a contact Fano manifold, with

$$K_Z^{−1} = L^{k+1}$$

(in particular, the index of $Z$, as a Fano manifold, is at least equal to $k + 1 = \frac{\dim C Z + 1}{2}$.)

Conversely,

Theorem 2 (C. LeBrun 1995, A. Moroianu 1998 if $M$ is spin)

Any contact Fano manifold, admitting a Kähler-Einstein metric and a suitable real structure, is the twistor space of a positive qK manifold.
Four-dimensional qK manifolds

The above definitions for qK manifolds are void for 4-dimensional manifolds: indeed, $Sp(1)Sp(1) = SO(4)$, whereas $Q = A^+ M$, bundle of self-dual elements of $A(M)$, always satisfies required properties for any oriented 4-dimensional riemannian manifold.

Definition 1

A 4-dimensional oriented Riemannian manifold $(M, g)$ is called quaternion-Kähler if

1. $g$ is Einstein, and
2. $W^+ = 0 \iff$ the canonical almost complex structure of the twistor space is integrable (Atiyah–Hitchin–Singer)

Positive 4-dimensional qK manifolds: $\mathbb{HP}^1 = S^4$ and $Gr_2(3, \mathbb{C}) = \mathbb{CP}^2$. 
Facts and conjectures

At the moment, only known examples of qK manifolds of positive type are Wolf spaces.


**Known fact:** For any $k$, finitely many $4k$-dimensional qK manifolds of positive type (LeBrun-Salamon 1994)

**Theorem 3 (C. LeBrun–S. Salamon)**

*Let $(M, g)$ be any $4k$-dimensional qK manifold of positive type. Then, $b_2(M) \leq 1$, with equality iff $M = \mathbb{G}r_2(2 + k, \mathbb{C})$.*

Uses works on Fano manifolds, in the framework of the Mori program, in particular works of S. Mori, J. Kollár, Y. Miyaoka, J. A. Wiśniewski...
Almost complex structures on positive qK manifolds


Positive qK manifolds admit no compatible almost complex structure. Equivalently, the twistor fibration of a positive qK manifold has no global section.

Notice that the natural complex structure $J$ of $M = \mathbb{Gr}_2(2 + k, \mathbb{C})$ is defined by $JX = X \circ i$, for $X$ in $T_xM = \text{Hom}_\mathbb{C}(x, x^\perp)$, hence is not a compatible complex structure.

The main goal of this talk is to give a proof of

Theorem 5 (P. Gauduchon–A. Moroianu–U. Semmelmann 2011)

Let $(M, g, Q)$ be any positive qK manifold, of dimension $4k \geq 8$, different from $\mathbb{Gr}_2(2 + k, \mathbb{C})$. Then, $M$ admits no almost complex structure, even in the stable sense (= a complex structure on $TM \oplus \mathbb{R}^\ell$, for some $\ell$).
1. $\mathbb{HP}^1 = S^4$ is stably complex, since $TS^4 \oplus \mathbb{R}$ is trivial, but has no almost complex structure: otherwise, $\chi + \tau$ would be divisible by 4, but $\chi = 2$ and $\tau = 0$. Similarly, $\mathbb{CP}^2$ has no almost complex structure compatible with the reversed orientation, as $\chi = 3$ and $\tau = -1$.

2. The non-existence of (stable) almost complex structures on $\mathbb{H}^k$ was established by F. Hirzebruch (1953 for $k \neq 2, 3$, 1958 for any $k$). Alternative argument by W. S. Massey in 1962.

3. The non-existence of (stable) almost complex structures on most real Grassmannians $\mathcal{G}r_4(4 + k, \mathbb{R})$ was established by P. Sankaran in 1991 and Z.-Z. Tang in 1994 (only $\mathcal{G}r_4(8, \mathbb{R})$ and $\mathcal{G}r_4(10, \mathbb{R})$ are not covered).
The Atiyah–Singer index theorem

The proof of Theorem 5 relies on the **Atiyah–Singer theorem** in the following form: Let

1. $M$ be any compact, oriented, **spin**, even-dimensional Riemannian manifold,
2. $\Sigma$ be its **spinor bundle**, 
3. $V$ be any auxiliary complex vector bundle over $M$,
4. $D_V : \Sigma^+ \otimes V \to \Sigma^- \otimes V$ be the corresponding **twisted Dirac operator**,

then, the index of $D_V$ is given by

$$ind_V = \left( \text{ch}(V) \hat{A}(TM) \right)[M].$$

Still holds if $M$ is **not spin** — so that $\Sigma$ is only locally defined — and, similarly, $V$ is only **locally defined**, provided that $\Sigma \otimes V$ is **globally** defined.
The canonical vector bundles $H, E$

For any positive qK manifold $(M, g, Q)$ of dimension $n = 4k$, we have

$$T^C M = H \otimes E,$$

where:

1. $T^C M = TM \otimes \mathbb{C}$: complexified tangent bundle,
2. $H$: rank 2 complex vector bundle induced by the standard representation of $Sp(1) = SU(2)$, and
3. $E$: rank $2k$ complex vector bundle induced by the standard representation of $Sp(k) \subset SU(2k)$.

**Beware:** $E, H$ are only globally defined if the $Sp(k)Sp(1)$-structure of $(M, g)$ lifts to a $Sp(k) \times Sp(1)$-structure. This happens iff $w_2(Q) = 0$, iff $M = \mathbb{HP}^k$ ((S. Marchiafava–G. Romani 1976, S. Salamon 1982). For $M \neq \mathbb{HP}^k$, $E, H$ are only **locally defined** but any **even** tensor combination of them is **globally defined**.
Spinor bundles on qK manifolds

A $4k$-dimensional positive qK manifold $(M, g, Q)$ is spin iff: $k$ is even, or $M = \mathbb{HP}^k$ (S. Salamon 1974). In both cases, the spinor bundle is

$$\Sigma = \bigoplus_{r+s=k} R^{r,s},$$

with

$$R^{r,s} = \text{Sym}^r H \otimes \Lambda_0^s E$$

where:

1. $\text{Sym}^r H$: $r$-th symmetric tensor power of $H$, and
2. $\Lambda_0^s H$: primitive part with respect to the (complex) symplectic form of $E$ of the $s$-th exterior power of $E$.

(Kramer–Semmelmann-Weingart 1999)
Index of twisted Dirac operators

Even if $M$ is not spin, the twisted spinor bundle $\Sigma \otimes R^{p,q}$ is globally defined, as well as the corresponding twisted Dirac operator $D_{R^{p,q}}$, whenever $p + q + k$ is even.

Theorem 6 (C. LeBrun-S. Salamon 1994)

The index of $D_{R^{p,q}}$ is then given by

$$
\text{ind } D_{R^{p,q}} = \begin{cases}
0 & \text{if } p + q < k \\
(-1)^q (b_{2q-2}(M) + b_{2q}(M)) & \text{if } p + q = k
\end{cases}
$$
Proof of Theorem 6 (Penrose correspondence)

Todd class \( td(Z) \) of \( Z \) versus \( \hat{A} \)-class \( \hat{A}(M) \):

\[
\text{td}(Z) = \frac{c_1(L) e^{k c_1(L)}}{1 - e^{-c_1(L)}} \pi^* \hat{A}(M). \tag{2}
\]

from which we infer:

\[
\text{ind } D_{R^p,q} = \chi_q(Z, L^{-r}) - \chi_{q-2}(Z, L^{-(r+1)}), \tag{3}
\]

with \( r = \frac{k-p-q}{2} \), whereas:

\[
\chi_q(Z, L^{-r}) = 0, \quad \text{if} \quad r > 0, \tag{4}
\]

\[
\chi_q(Z, \mathcal{O}) = (-1)^q b_{2q}(Z) = (-1)^q (b_{2q}(M) + b_{2q-2}(M)).
\]
Proof of Theorem 5, Step 1

Let $M$ be any positive $qK$ manifold, of dimension $4k$, $k > 1$, and assume that $M \neq \text{Gr}_2(2 + k, \mathbb{C})$, so that, by Theorem 3:

$$b_2(M) = 0$$

Apply the Atiyah–Singer index theorem for twisted dirac operator with auxiliary bundle

$V = T^C M \otimes \text{Sym}^{k-2} H \simeq H \otimes \text{Sym}^{k-2} H \otimes E$. By Clebsch–Gordan:

1. $V = R^{1,1}$ if $k = 2$, or
2. $V = R^{k-1,1} \oplus R^{k-3,1}$ if $k \geq 3$.

In both cases, the twisted spinor bundle $\Sigma \otimes V$ is well defined, and:

1. $\text{ind}_V = \text{ind} D_{R^{1,1}}$ if $k = 2$, and
2. $\text{ind} D_V = \text{ind} D_{R^{k-1,1}} + \text{ind} D_{R^{k-3,1}}$ if $k \geq 3$.

In both cases, we get (Theorem 6):

$$\text{ind}_V = -1.$$
On the other hand, by the Atiyah–Singer index theorem:

\[
\text{ind } D_V = (\text{ch}(T^C M) \text{ch}((\text{Sym}^{k-2} H) \hat{A}(TM)))[M].
\]

Notice that the Chern character \( \text{ch}(\text{Sym}^{k-2} H) \) is well-defined in \( H^*(MK, \mathbb{Q}) \), even if \( k \) is odd, by

\[
\text{ch}(\text{Sym}^{k-2} H) = (\text{ch}(\text{Sym}^{k-2} H) \otimes 2)^{1/2}.
\]

We now assume, for a contradiction, that \( M \) admits an almost complex structure, so that \( T^C M = T \oplus \bar{T} = T \oplus T^* \), where \( T = TM \) is a complex vector bundle. We thus have

\[
\text{ind } D_V = \left( (\text{ch}(T) + \text{ch}(T^*)) \text{ch}((\text{Sym}^{k-2} H) \hat{A}(TM)) \right)[M].
\]
Proof of Theorem 5, Step 3

Observe:

1. $H, E$ are (complex) **symplectic** bundle, hence **self-dual**, as well as their tensor powers: $\text{ch}(\text{Sym}^{k-2}H) \hat{A}(TM)$ then only involves elements of $H^{4\ell}(M, \mathbb{Q})$.

2. Only components in $H^{4\ell}(M, \mathbb{Q})$ in $\text{ch}(T) + \text{ch}(T^*)$ contribute non-trivially in the index formula. Since $\text{ch}(T^*) = \sum (-1)^i \text{ch}_i(T)$, we can replace $T^*$ by $T$, to get

$$\text{ind } D_V = 2 \left( \text{ch}(T) \text{ch}(\text{Sym}^{k-2}H) \hat{A}(TM) \right)[M].$$

3. $\left( \text{ch}(T) \text{ch}(\text{Sym}^{k-2}H) \hat{A}(TM) \right)[M]$ is the index of the twisted Dirac operator $D_{\text{Sym}^{k-2}H \otimes T} = D_{R^{k-2,0} \otimes T}$, acting on the twisted spinor bundle $\Sigma \otimes R^{k-2,0} \otimes T$ (globally defined, since $k - 2 + k = 2k - 2$ is even).
Proof of Theorem 5: Final step

Putting all this together, we get:

\[ \text{ind } D_V = -1 = 2 \text{ ind } D_{R^{k-2,0} \otimes T}, \]

which is clearly a contradiction, as both indices are integers. It follows that \( TM \) \textbf{cannot be} a complex vector bundle.

To complete the proof of Theorem 5, we have to consider the \textbf{stable case}, when, for a contradiction, \( TM \oplus \mathbb{R}^{2\ell} \) is supposed to be a complex vector bundle for some integer \( \ell \), so that

\[ T^C M \oplus \mathbb{C}^{2\ell} = T \oplus T^*, \quad (5) \]

where \( T = TM \oplus \mathbb{R}^{2\ell} \) is a complex vector. We thus get

\[ \text{ind } D_V = -1 = 2 \text{ ind } D_{R^{k-2,0} \otimes T} - 2\ell \text{ ind } D_{R^{k-2,0}}. \]

Contradiction again.
In the same paper, a similar argument was used to establish the non-existence of (stable) almost complex structures on any compact inner symmetric space, namely compact symmetric space $G/H$, with $\text{rank } G = \text{rank } H$, with the possible exception of $E_7/SU(8)/\mathbb{Z}_2$.

Notice that Wolf spaces are inner symmetric spaces.

Later, Andrei Moroianu and Uwe Semmelmann established a similar non-existence theorem of (stable) almost complex structures on a larger class of homogeneous spaces (including the missing $E_7/SU(8)/\mathbb{Z}_2$).