Introduction to tropical mathematics
Usual algebra
Usual algebra

Operations on numbers:
Usual algebra

Operations on numbers: addition +
Usual algebra

Operations on numbers: addition + subtraction -
Usual algebra

Operations on numbers:  
addition  +  
subtraction  -  
multiplication  ×
Usual algebra

Operations on numbers:  
addition \ (+)  
subtraction \ (−)  
multiplication \ (×)  
division \ (÷)
Usual algebra

Operations on numbers:

- addition +
- subtraction -
- multiplication ×
- division ÷

1 ÷ 0 = ?
Usual algebra

Operations on numbers:  
addition  +
subtraction  -
multiplication  ×
Usual algebra

Operations on numbers:  
- addition  
- subtraction  
- multiplication

Rules (axioms):
Usual algebra

Operations on numbers:
- addition $+$
- subtraction $-$
- multiplication $\times$

Rules (axioms):
- $a + b = b + a$
- $a + (b + c) = (a + b) + c$
- $a + (b - a) = b$
Usual algebra

Operations on numbers:
- addition +
- subtraction -
- multiplication ×

Rules (axioms):
- \( a + b = b + a \)
- \( a + (b + c) = (a + b) + c \)
- \( a + (b - a) = b \)

- \( a \times b = b \times a \)
- \( a \times (b \times c) = (a \times b) \times c \)
Usual algebra

Operations on numbers:

- addition \(+\)
- subtraction \(-\)
- multiplication \(\times\)

Rules (axioms):

\[ a+b=b+a \]
\[ a+(b+c)=(a+b)+c \]
\[ a+(b-a)=b \]

\[ a\times b=b\times a \]
\[ a\times (b\times c)=(a\times b)\times c \]
\[ a\times (b+c)=(a\times b)+(a\times c) \]
Usual algebra

Operations on numbers:  
addition  \(+\)  
subtraction  \(-\)  
multiplication  \(\times\)  

Rules (axioms):  
\[ a + b = b + a \]
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\[ a \times (b + c) = (a \times b) + (a \times c) \]
Tropical algebra
Tropical algebra

Operations on "numbers":
- addition +
- subtraction -
- multiplication ×
**Tropical algebra**

Operations on "numbers":
- **addition** +
- **subtraction** −
- **multiplication** ×

Axioms:

\[ a + b = b + a \]
\[ a + (b + c) = (a + b) + c \]
\[ a + a = a \]
\[ a \times b = b \times a \]
\[ a \times (b \times c) = (a \times b) \times c \]
\[ a \times (b + c) = (a \times b) + (a \times c) \]
\[ a + 0 = a \]
\[ a \times 0 = 0 \]
\[ a \times 1 = a \]
**Usual algebra**

Commutative associative ring

**Tropical algebra**

Idempotent

Commutative associative semiring
Usual algebra

comutative associative ring

The simplest ring is the ring of integers

\[ \ldots -2, -1, 0, 1, 2, 3 \ldots \]

Tropical algebra

idempotent

commutative associative semiring
**Usual algebra**

- commutative associative ring

The simplest ring is the ring of integers

... -2, -1, 0, 1, 2, 3 ...

---

**Tropical algebra**

- idempotent
- commutative associative

**semiring**

The simplest idempotent semiring consists only of

0, 1
**Usual algebra**

commutative associative ring

The simplest ring is the ring of integers

\[ \ldots -2, -1, 0, 1, 2, 3 \ldots \]

**Tropical algebra**

idempotent

commutative associative

semiring

The simplest idempotent semiring consists only of

\[ 0, 1 \]

**Powers of \((1+A)\)**

\[
(1+A) = (1+A) \times (1+A) = 1 + 2 \times A + A^2
\]

\[
(1+A)^3 = 1 + 3 \times A + 3 \times A^2 + A^3
\]

\[
(1+A)^4 = 1 + 4 \times A + 6 \times A^2 + 4 \times A^3 + A^4
\]

\[ \ldots \]

\[
(1+A)^2 = (1+A) \times (1+A) = 1 + A + A^2
\]

\[
(1+A)^3 = 1 + A + A^2 + A^3
\]

\[
(1+A)^4 = 1 + A + A^2 + A^3 + A^4
\]

\[ \ldots \]
An important example of a tropical ring
An important example of a tropical ring

0
1
10
100
1000
10000
100000
1000000
10000000
100000000
1000000000
10000000000
100000000000
...
An important example of a tropical ring

tropical multiplication = usual multiplication

1000 \times 100000 = 100000000
An important example of a tropical ring

tropical multiplication = usual multiplication

\[ 1000 \times 100000 = 100000000 \]

tropical addition = usual maximum

\[ 1000 + 100000 = 100000 \]
An important example of a tropical ring

tropical multiplication = usual multiplication

\[
1000 \times 100000 = 100000000
\]

tropical addition = usual maximum

\[
1000 + 100000 = 100000
\]

Not very far from the usual sum!

\[
100000 \sim 101000
\]
It is convenient to replace powers of 10 by logarithms:

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- tropical addition $\rightarrow$ usual maximum
- tropical multiplication $\rightarrow$ usual sum
It is convenient to replace powers of 10 by logarithms:

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tropical addition  →  usual maximum

tropical multiplication  →  usual sum

Tropical ring  =  (max,+) algebra
Tropical algebra is a very useful tool when one deals with very BIG or very small numbers.
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Example: polynomial equation with big and small coefficients

\[ .001 + 1000 \times X + 100 \times X^2 + X^3 = 0 \]
Tropical algebra is a very useful tool when one deals with very **BIG**
or very **small** numbers

Example: polynomial equation with big and small coefficients

\[ 0.001 + 1000 \times X + 100 \times X^2 + X^3 = 0 \]

has three solutions

\[ X_1 = -88.72983360757139270781600622... \sim -10^2 \]
\[ X_2 = -11.27016539242850729216499378... \sim -10^1 \]
\[ X_3 = -0.00000100000010000001900000... \sim -10^{-6} \]
Tropical algebra is a very useful tool when one deals with very **BIG** or very **small** numbers

Example: polynomial equation with big and small coefficients

\[ .001 + 1000 \times X + 100 \times X^2 + X^3 = 0 \]

has tropical version:

\[ \max (-3, 3+x, 2+2x, 3x) \text{ is achieved at least TWICE} \]
Graph of function \( f(x) = \max ( -3, 3+x, 2+2x, 3x ) \)
breaks at points \( x = -6, 1, 2 \)
Graph of function $f(x) = \max (-3, 3+x, 2+2x, 3x)$
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Equations of planar curves
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<td>Tropical geometry</td>
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Equations of planar curves

Usual geometry

\[ A \times X + B \times Y + 1 = 0 \]

Tropical geometry

Diagram showing a line on a coordinate plane.
Equations of planar curves

Usual geometry

\[ A \times X + B \times Y + 1 = 0 \]

Tropical geometry

\[ \max(a+x, b+y, 0) \text{ is achieved twice} \]
Two tropical lines intersect at one point
Two tropical lines intersect at one point
Two tropical lines intersect at one point
Two tropical lines intersect at one point, but not always.....
Pappus Theorem
Pappus Theorem
Pappus Theorem

has a tropical version ...
Equations of planar curves

Usual geometry

Quadrics

Tropical geometry
Equations of planar curves

Usual geometry

Quadrics

Tropical geometry

intersect lines at two points
Equations of planar curves

Usual geometry

Quadrics

Tropical geometry

intersect lines at two points
Equations of planar curves

Usual geometry

Quadrics

intersect lines at two points not always...

Tropical geometry

but ALWAYS in tropical geometry!
Quadrics intersect lines at two points in COMPLEX algebraic geometry.

Tropical geometry is closely related to the complex world.
Equations of planar curves

Usual geometry

Cubics
\[ Y^2 = A \times X^3 + X + B \]

Tropical geometry
Complex picture of a line is real plane with coordinates (real part, imaginary part)

\[ z = x + iy \]

which is sphere minus one point.

Similarly, complex affine quadric is topologically a cylinder, which is a sphere minus two points.

Complex affine cubic is a torus minus three points.

Tropical line is sphere minus three points, tropical quadric is sphere minus six points, tropical cubic is torus minus nine points.
Planar tropical curves are piece-wise linear curves satisfying at each vertex BALANCE CONSTRAINT

\[
\text{Sum of vectors} = 0
\]
Higher-dimensional tropical objects:

piece-wise linear gadgets describing the "shape" of a complex algebraic variety which depends on a small parameter $t$ and degenerates badly as $t \rightarrow 0$
Higher-dimensional tropical objects:

table-wise linear gadgets describing the "shape" of a complex algebraic variety which depends on a small parameter $t$ and degenerates badly as $t \to 0$.

Closely related with non-archimedean geometry and so-called Berkovich spectrum.

Example of a non-archimedean field: $p$-adic numbers.

$$|x+y| \leq \max(|x|, |y|)$$
Tropical varieties outside of the locus of singularities look locally as open domains in coordinate spaces.

Coordinate changes are integral affine transformations, for example in the case of two coordinates \((x,y)\) they look as

\[(x, y) \rightarrow (ax+by+e, cx+dy+f)\]

where \(a,b,c,d,e,f\) are integer numbers and \(ad - bc = 1\).
Seeds in sunflower:

If we go around the center, the coordinate change will be

\[(x, y) \rightarrow (x+y, x+2y)\]

Closely related to Fibonacci numbers

\[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\]
Tropical varieties outside of the locus of singularities look locally as open domains in coordinate spaces.

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where \(a, b, c, d, e, f\) are integer numbers and \(ad - bc = 1\).

Fibers of the map from complex variety to tropical one are smooth tori outside the locus of singularities. The degeneration of fibers over singular points is similar to those in integrable systems!
Tropical geometry in mirror symmetry
In string theory space-time is 10-dimensional manifold which (if applicable to real world) is the product of the usual physical 4-dimensional space-time and of 6-dimensional compact manifold whose Ricci curvature is zero.

This auxiliary 6-dimensional manifold is in fact complex algebraic 3-dimensional variety, called Calabi-Yau variety.

Its metric depends on two types of parameters, called complex and Kähler parameters. Mirror symmetry interchanges these two groups of parameters, leading to the same physical theory.
The simplest example of 3-dimensional Calabi-Yau variety is quintic 3-fold, a hypersurface in 4-dimensional complex projective space given by a homogenous equation of degree 5 in 5 complex variables.

This Calabi-Yau variety depends on 101 complex parameters and 1 Kähler parameter.

Mirror dual Calabi-Yau variety depends on 1 complex parameter and 101 Kähler parameters.
In fact, mirror symmetry appears when both dual varieties depend on small parameter and are degenerating in a maximal possible manner.

Hence we obtain two tropical varieties, both of real dimension 3.

Mirror symmetry is nothing but a new duality between tropical varieties.
Even the condition that Ricci curvature vanishes, can be formulated for metrics on tropical varieties.
True picture of quintic 3-fold
True picture of quintic 3-fold

First step: choose 4 points in 3-dimensional space, and then the 5th point inside the tetrahedron spanned by first 4 points.

Connect by intervals all 5 points.

We obtain 10 nonintersecting triangles
True picture of quintic 3-fold

Step 2. Draw in each triangle already familiar picture of a planar quintic.
True picture of quintic 3-fold

We obtain a complicated graph in 3-dimensional space, with 200 vertices each of degree 2, and 300 edges. This graph is the locus of singularities of the integral affine structure on two tropical varieties, associated with degenerating quintic 3-folds and with degenerating dual Calabi-Yau 3-folds.
We obtain a complicated graph in 3-dimensional space, with 200 vertices each of degree 2, and 300 edges. This graph is the locus of singularities of the integral affine structure on two tropical varieties, associated with degenerating quintic 3-folds and with degenerating dual Calabi-Yau 3-folds.

The fundamental group of the complement to the graph maps to $\text{SL}(3,\mathbb{Z})$, dual varieties gives conjugate representations.
Towards a counterexample to the Hodge conjecture
Towards a counterexample to the Hodge conjecture

A. Weil: expected that the Hodge conjecture is wrong!
Potential counterexample:
generic complex abelian variety $A$ of dimension $= 2n \geq 4$
with an endomorphism $\delta$ such that

$$\delta^2 = -d \times \text{identity endomorphism of } A, \quad d=1,2,... \text{ square-free}$$

$A$ has only one Hodge class in $H^2(A), \text{ and 3 linearly independent Hodge classes in } H^n(A)$. Hence 2 classes can not be represented as an intersection of divisors.
Towards a counterexample to the Hodge conjecture

Tropical version:
generic real 2n-dimensional torus $T$ with integral affine structure together with an endomorphism $\delta$ such that

$$\delta^2 = -d \times \text{identity endomorphism of } T, \quad d=1,2,\ldots$$

Tropical Hodge conjecture: $T$ has tropical n-dimensional cycles representing 3 linearly independent tropical Hodge classes.
Towards a counterexample to the Hodge conjecture

Tropical Hodge conjecture fails

⇒ the usual (complex) Hodge conjecture fails,
also Tate conjecture (about etale cohomology) fails ...
Towards a counterexample to the Hodge conjecture

Tropical Hodge conjecture fails

⇒ the usual (complex) Hodge conjecture fails, also Tate conjecture (about etale cohomology) fails ...

But one can show that (for given n,d) that the failure of the tropical Hodge conjecture is

≡ existence of a solution of certain explicit countable system of linear equations with a countable set of variables. Hence we may hope to find a closed formula for the solution!