Introduction to tropical mathematics

Usual algebra

Usual algebra

Operations on numbers:

Usual algebra

Operations on numbers: addition $+$

Usual algebra

Operations on numbers:	addition	+
	subtraction	-

Usual algebra

Operations on numbers:	addition	$+$
	subtraction	-
	multiplication	$\times$

Usual algebra

Operations on numbers:	addition	$+$
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	multiplication	$\times$
	division	$\div$

Usual algebra

Operations on numbers:	addition	$+$
	subtraction	-
	multiplication	$\times$
	~~division~~	$\div$

$1 \div 0 = ?$

Usual algebra

Operations on numbers:	addition	$+$
	subtraction	$-$
	multiplication	$\times$

Usual algebra

Operations on numbers:	addition	$+$
	subtraction	-
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Rules (axioms):

Usual algebra

Operations on numbers:	addition	$+$
	subtraction	$-$
	multiplication	$\times$

Rules (axioms):

$a+b=b+a$

$a+(b+c)=(a+b)+c$

$a+(b-a)=b$

Usual algebra

Operations on numbers:	addition	$+$
	subtraction	$-$
	multiplication	$\times$

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$a+(b-a)=b$

$a\times b=b\times a$

$a\times(b\times c)=(a\times b)\times c$

Usual algebra

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	multiplication	$\times$

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$a\times(b+c)=(a\times b)+(a\times c)$

Usual algebra

Operations on numbers: addition $+$

subtraction $-$

multiplication $\times$

Rules (axioms):

$a+b=b+a$

$a+(b+c)=(a+b)+c$ $\quad$ $a+0=a$

$a+(b-a)=b$

$a\times b=b\times a$

$a\times(b\times c)=(a\times b)\times c$ $\quad$ $a\times 1=a$

$a\times(b+c)=(a\times b)+(a\times c)$

Tropical algebra

Tropical algebra

Operations on "numbers":	addition	$+$
	~~subtraction~~	$-$
	multiplication	$\times$

Tropical algebra

Operations on "numbers": addition $+$

~~subtraction~~

multiplication $\times$

Axioms:

$a+b=b+a$ $\qquad$ $a+0=a$

$a+(b+c)=(a+b)+c$

$a+a=a$

$a\times b=b\times a$ $\qquad$ $a\times 0=0$

$a\times(b\times c)=(a\times b)\times c$ $\qquad$ $a\times 1=a$

$a\times(b+c)=(a\times b)+(a\times c)$

Usual algebra

commutative associative ring

Tropical algebra

idempotent

commutative associative

semiring

Usual algebra

commutative associative ring

The simplest ring is the ring of integers

. . . -2, -1, 0, 1, 2, 3 . . .

Tropical algebra

idempotent

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Tropical algebra

idempotent commutative associative semiring

The simplest idempotent semiring consists only of

0, 1

Usual algebra

commutative associative ring

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Tropical algebra

idempotent

commutative associative

semiring

The simplest idempotent semiring consists only of

0, 1

Powers of (1+A)

$(1+A)^2=(1+A)\times(1+A)=1+2\times A+A^2$

$(1+A)^3= 1+3\times A+3\times A^2+A^3$

$(1+A)^4= 1+4\times A+6\times A^2+4\times A^3+A^4$

. . .

$(1+A)^2= (1+A)\times(1+A)=1+A+A^2$

$(1+A)^3= 1+A+A^2+A^3$

$(1+A)^4= 1+A+A^2+A^3+A^4$

. . .

An important example of a tropical ring

An important example of a tropical ring

0

1

10

100

1000

10000

100000

1000000

10000000

100000000

. . .

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tropical multiplication = usual multiplication

$$1000 \times 100000 = 100000000$$

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tropical multiplication = usual multiplication

$$1000 \times 100000 = 100000000$$

tropical addition = usual maximum

$$1000 + 100000 = 100000$$

An important example of a tropical ring

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. . .

tropical multiplication = usual multiplication

$1000 \times 100000 = 100000000$

tropical addition = usual maximum

$1000 + 100000 = 100000$

Not very far from the usual sum!

$100000 \sim 101000$

It is convenient to replace powers of 10 by logarithms:

1		0
10		1
100	⇒	2
1000		3
10000		4
100000		5

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It is convenient to replace powers of 10 by logarithms:

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tropical addition $\Rightarrow$ usual maximum

tropical multiplication $\Rightarrow$ usual sum

Tropical ring = (max,+) algebra

Tropical algebra is a very useful tool when one deals with

very BIG

or very small numbers

Tropical algebra is a very useful tool when one deals with

very ***BIG***

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Example: polynomial equation with big and small coefficients

$$.001 + 1000\times X + 100\times X^2 + X^3 = 0$$

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very ***BIG***

or very *small* *numbers*

Example: polynomial equation with big and small coefficients

$$.001 + 1000\times X + 100\times X^2 + X^3 = 0$$

has three solutions

$$X_1 = -88.72983360757139270781600622... \sim -10^2$$

$$X_2 = -11.27016539242850729216499378... \sim -10^1$$

$$X_3 = -.000000100000001000000190000... \sim -10^{-6}$$

Tropical algebra is a very useful tool when one deals with

very ***BIG***

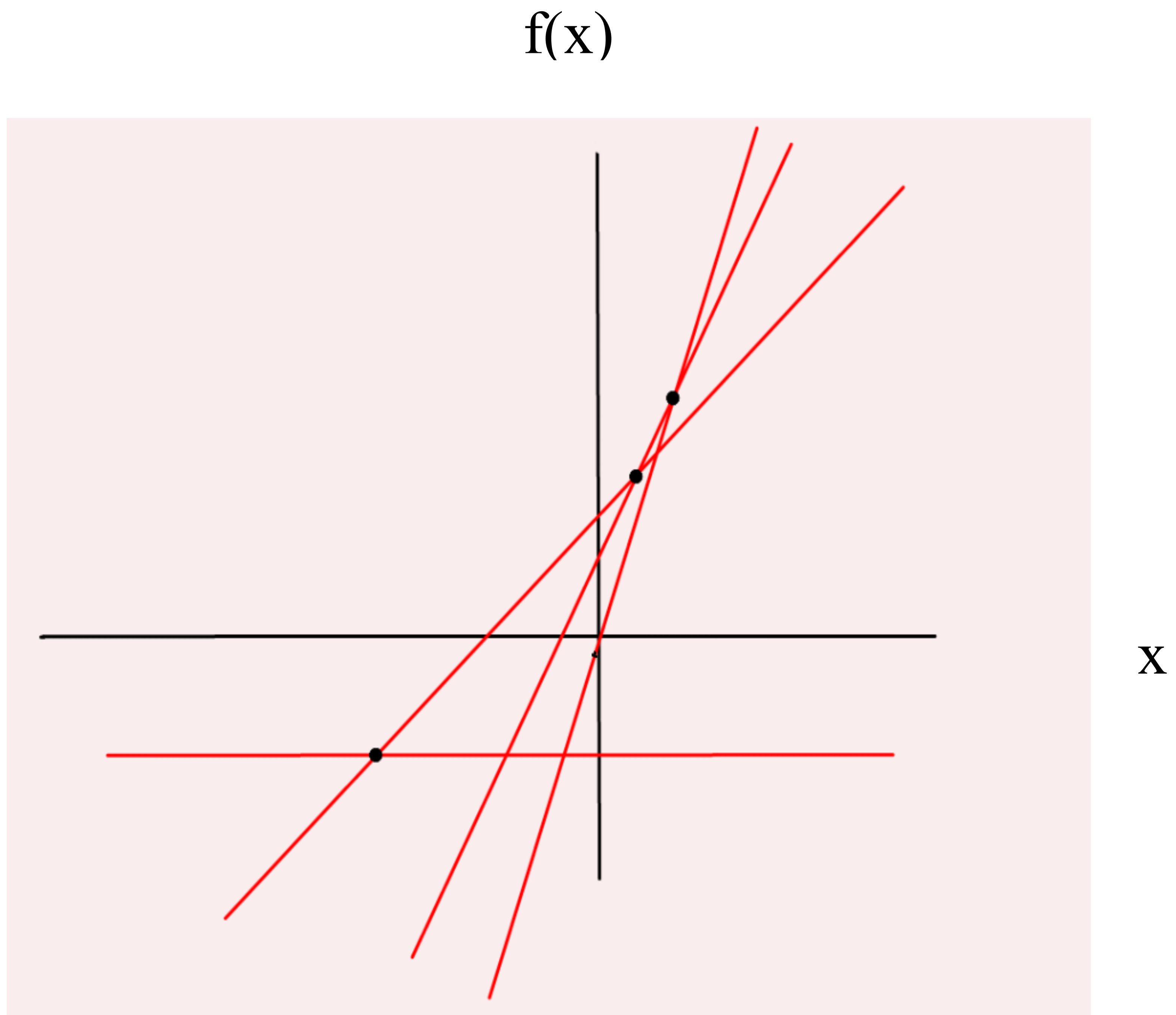
or very small numbers

Example: polynomial equation with big and small coefficients

$$.001 + 1000 \times X + 100 \times X^2 + X^3 = 0$$

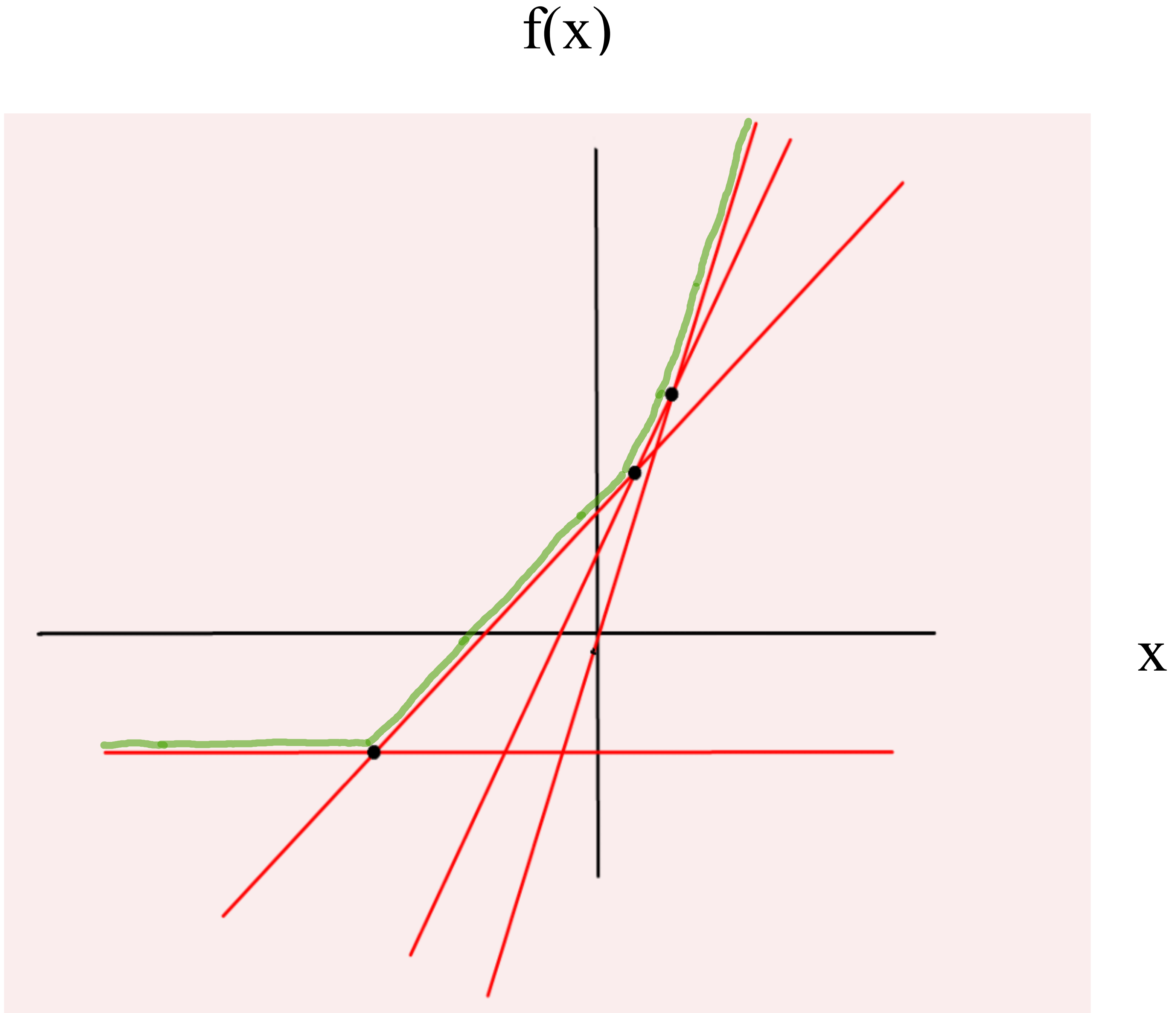
has tropical version:

max (-3, 3+x, 2+2x, 3x) is achieved at least TWICE



Graph of function $f(x)=\max(-3, 3+x, 2+2x, 3x)$

breaks at points $x = -6, 1, 2$



Graph of function $f(x)=\max(-3, 3+x, 2+2x, 3x)$

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Equations of planar curves

Equations of planar curves

Usual geometry

Tropical geometry

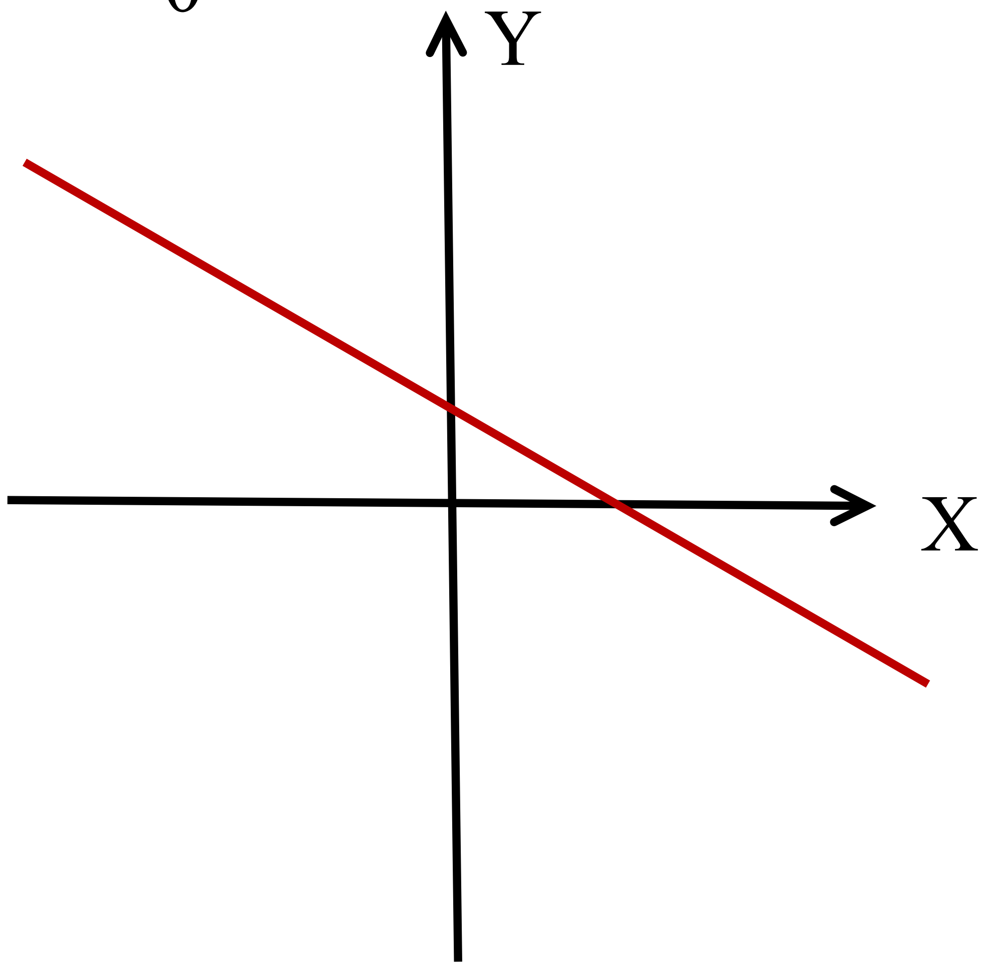
Equations of planar curves

Usual geometry

Tropical geometry

A×X+B×Y+1=

0

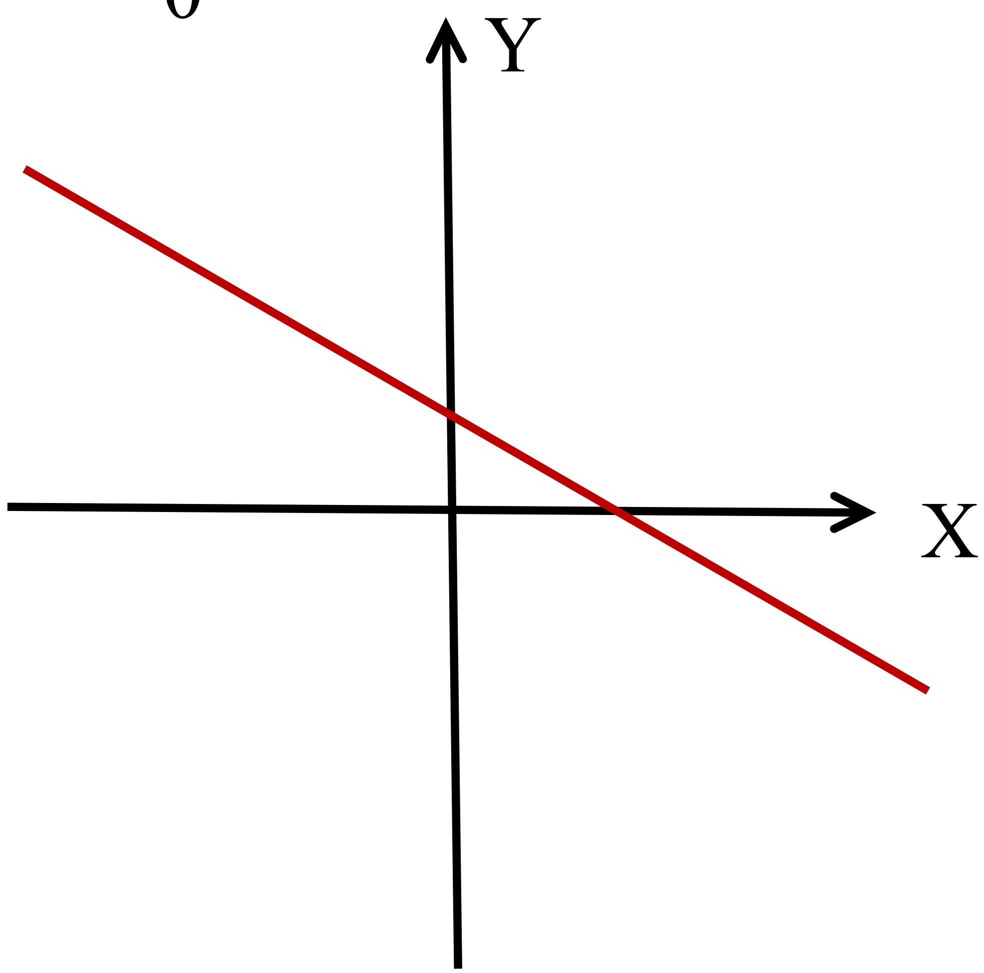


Equations of planar curves

Usual geometry

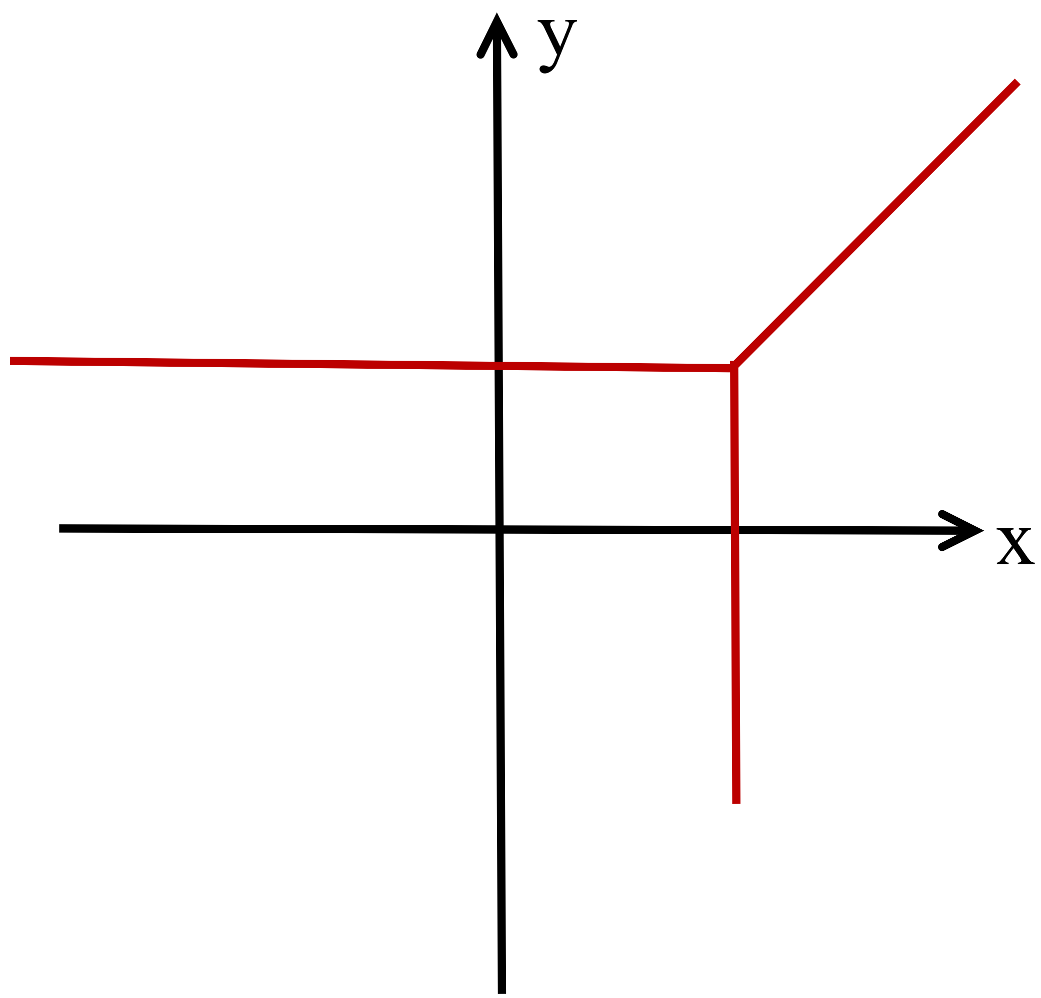
$A\times X+B\times Y+1=$

0

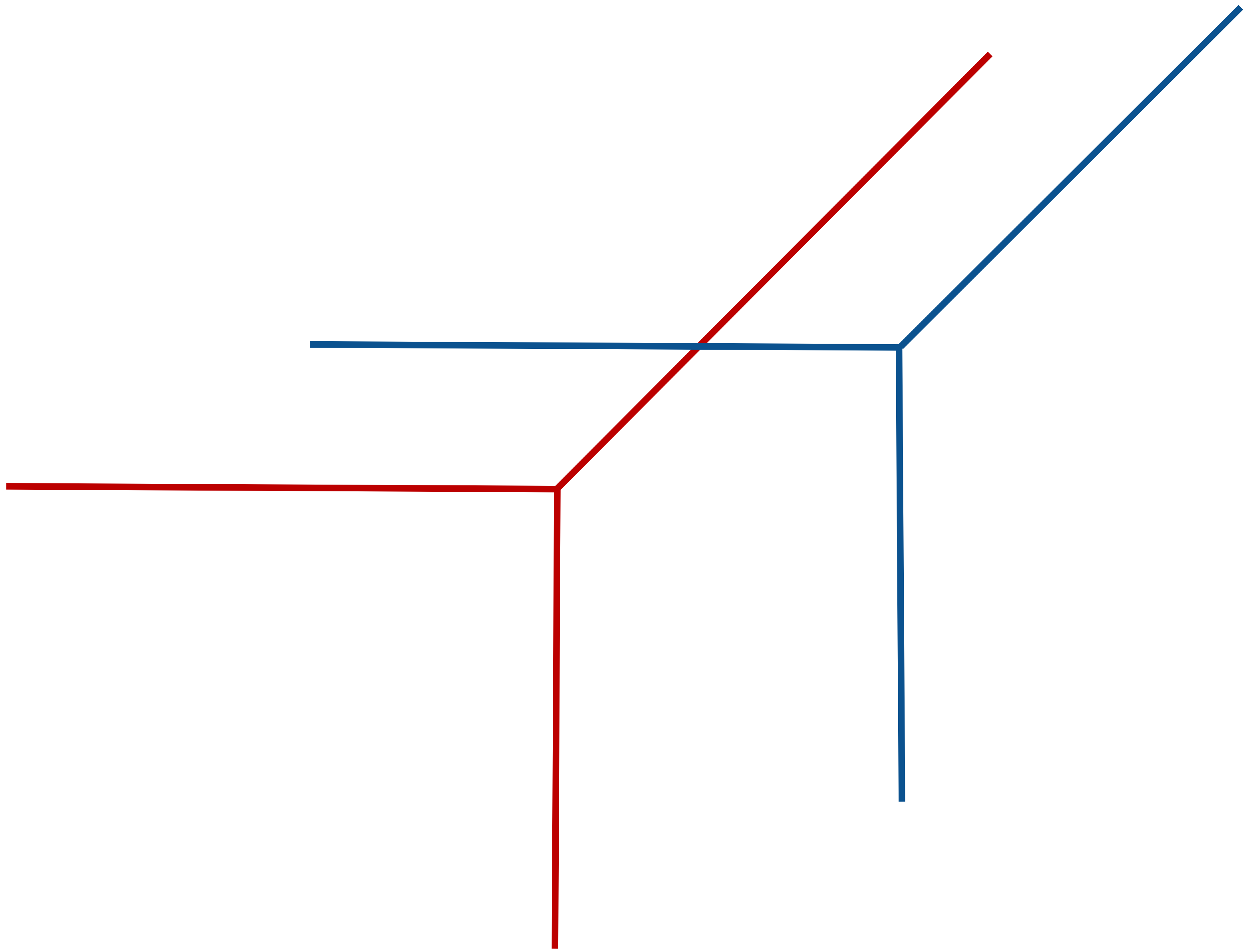


Tropical geometry

$\max(a+x, b+y, 0)$ is achieved twice



Two tropical lines intersect at one point



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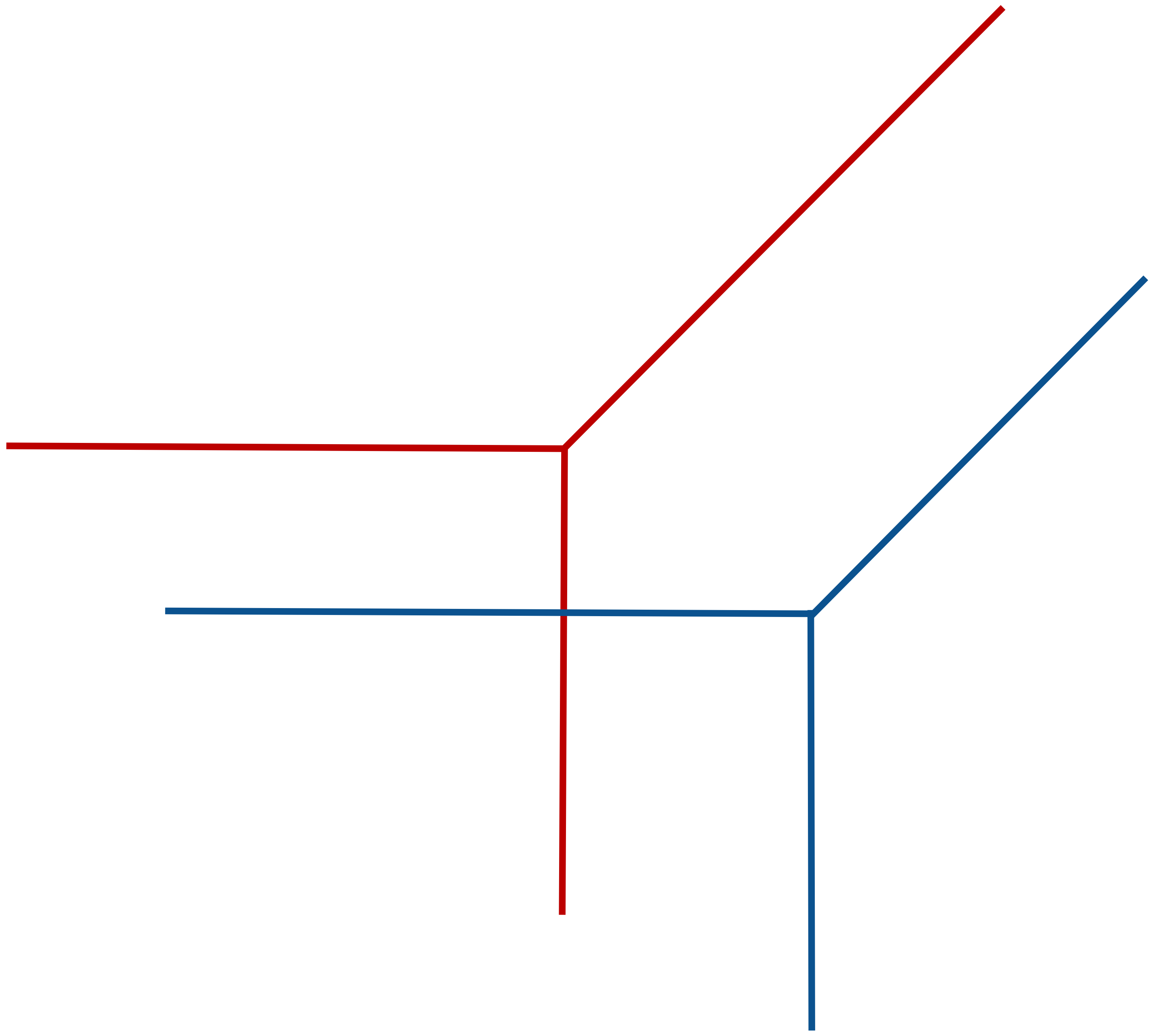

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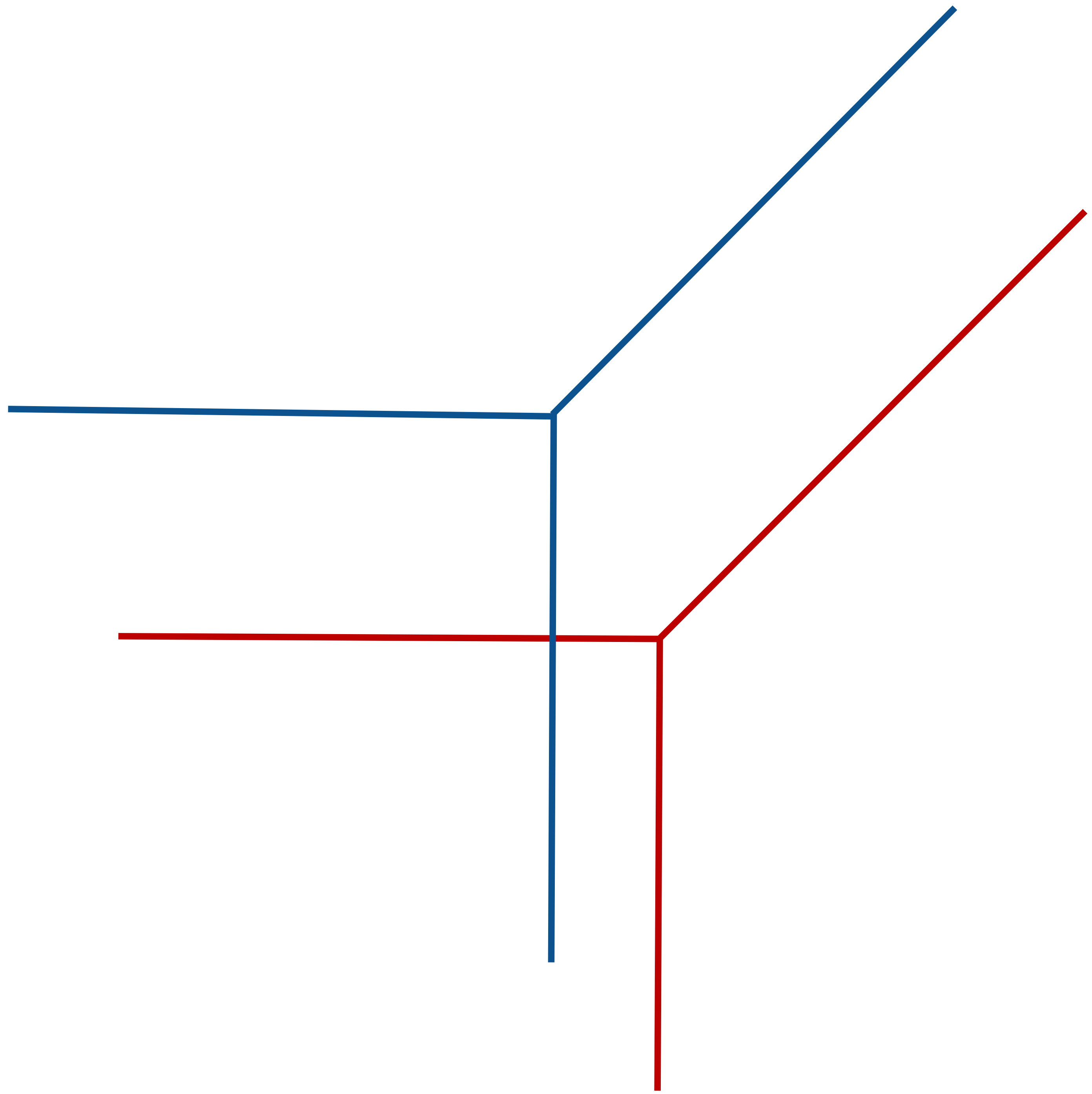
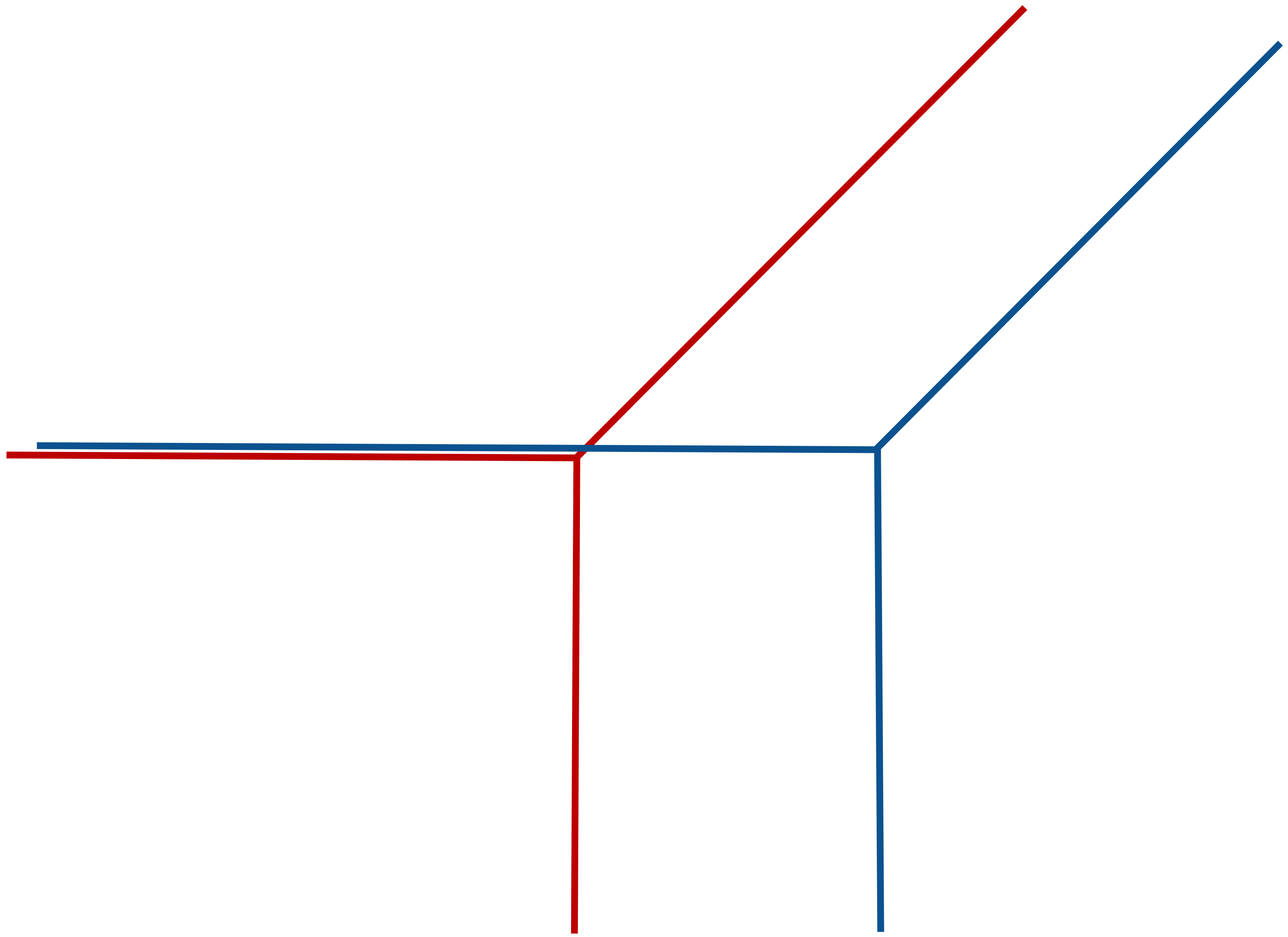
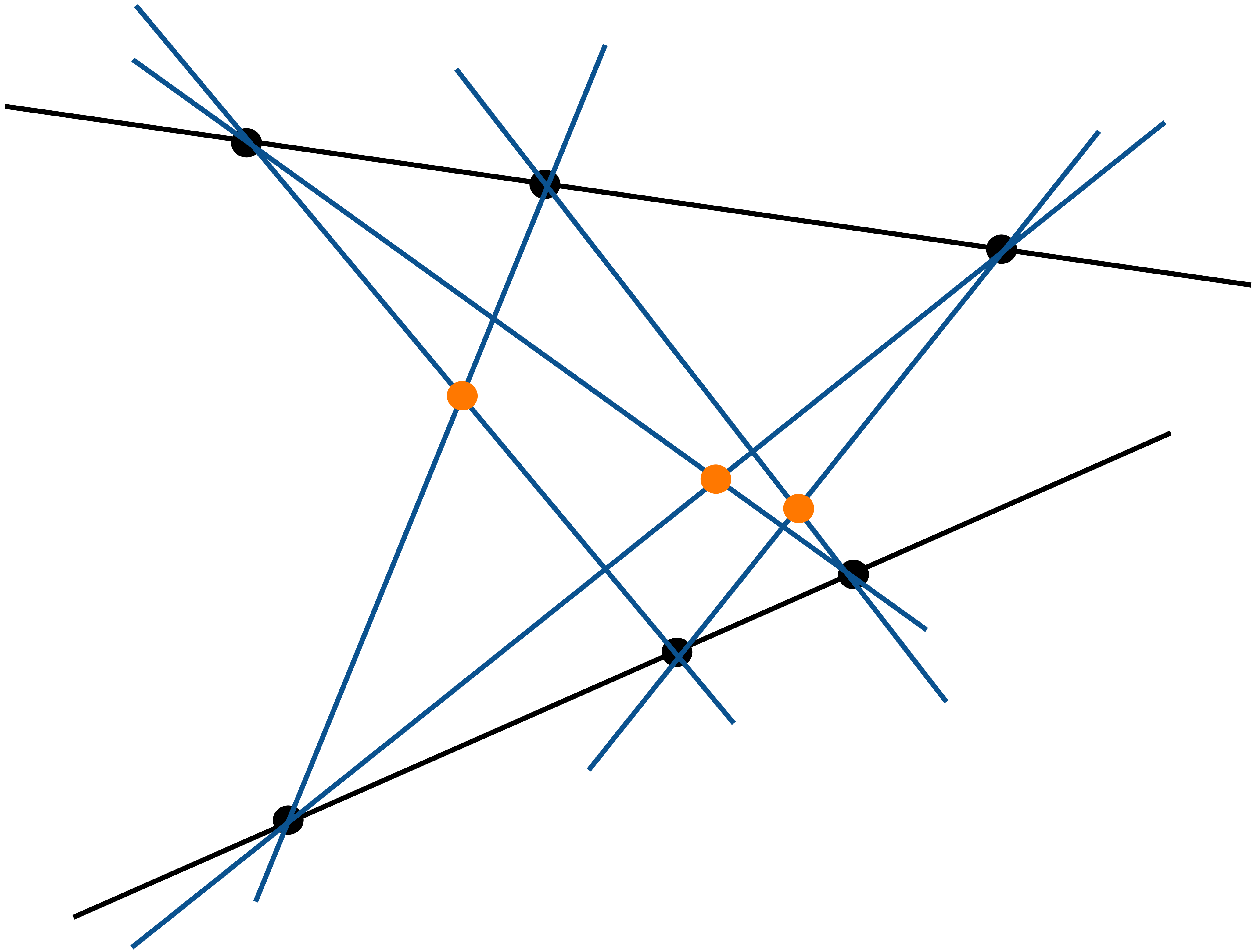


Two tropical lines intersect at one point,
but not always.....



Pappus Theorem



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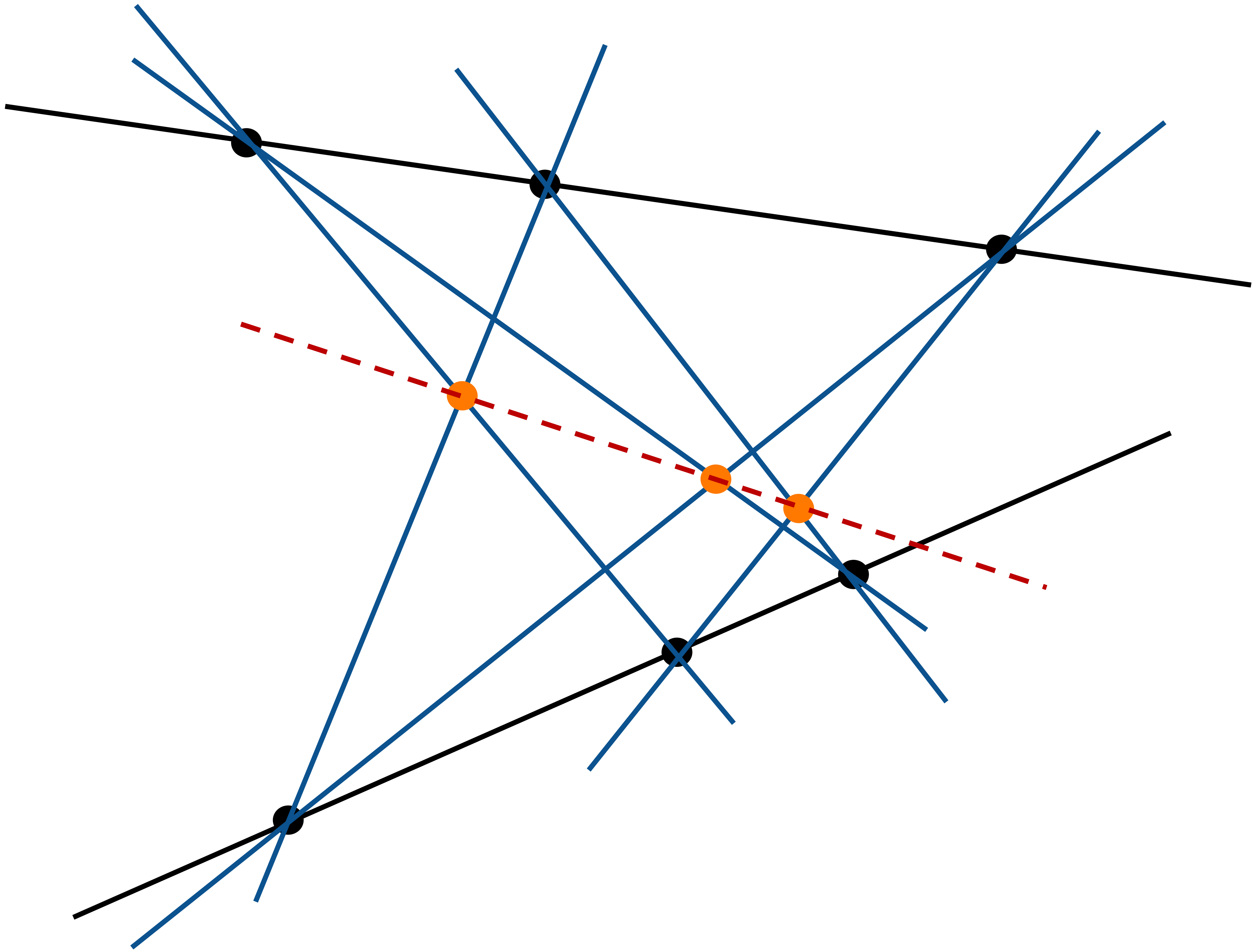
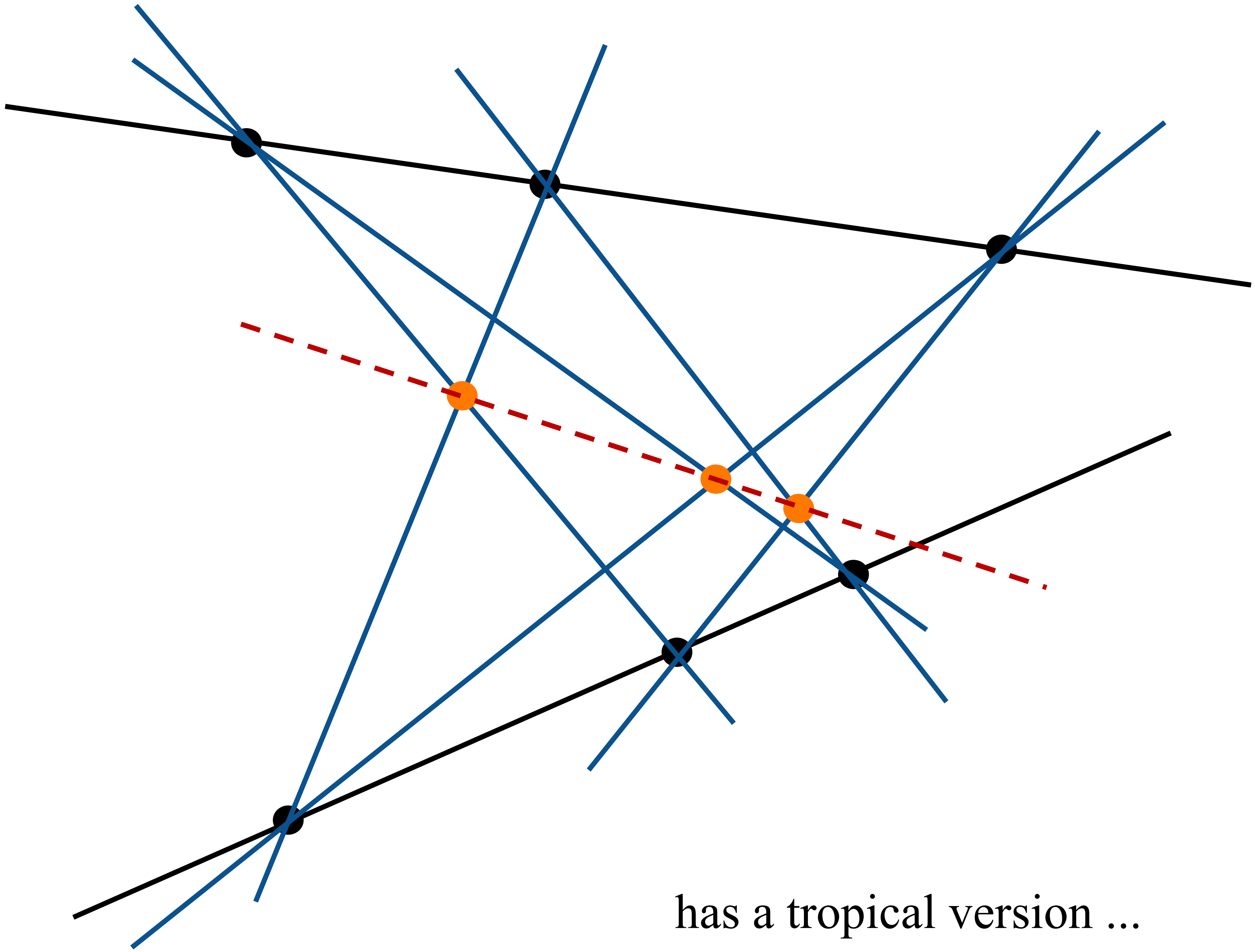


Pappus Theorem

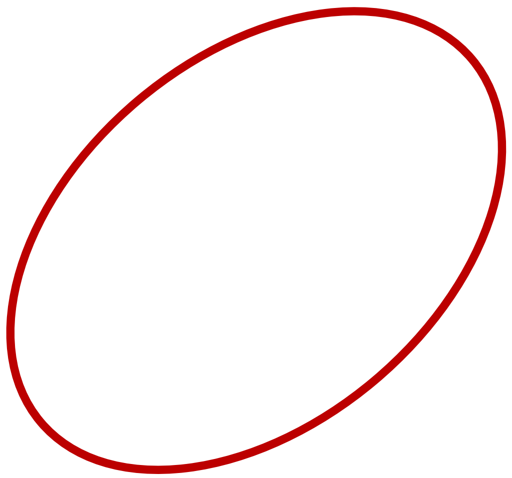


has a tropical version ...

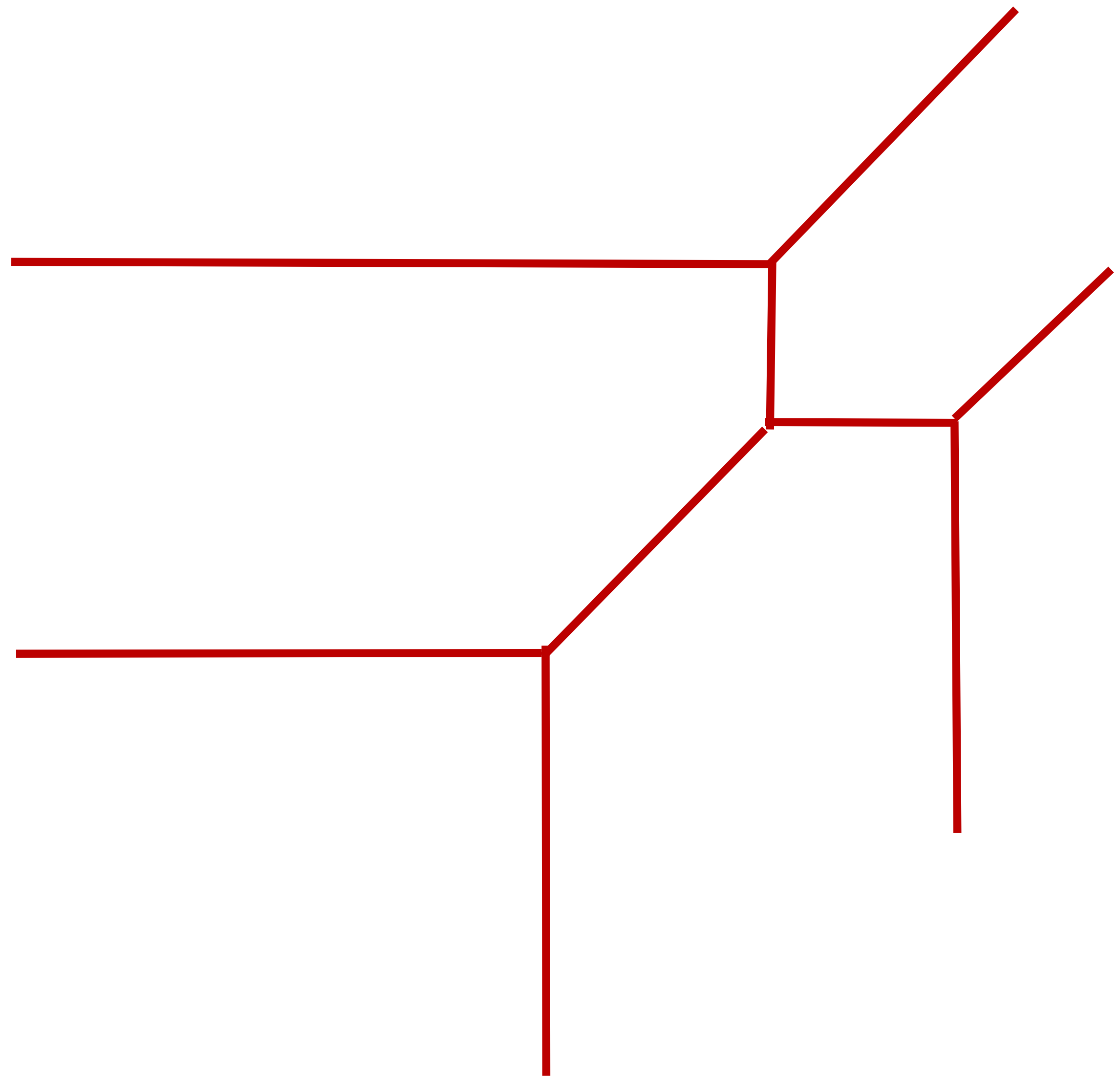
Equations of planar curves

Usual geometry

Quadrics



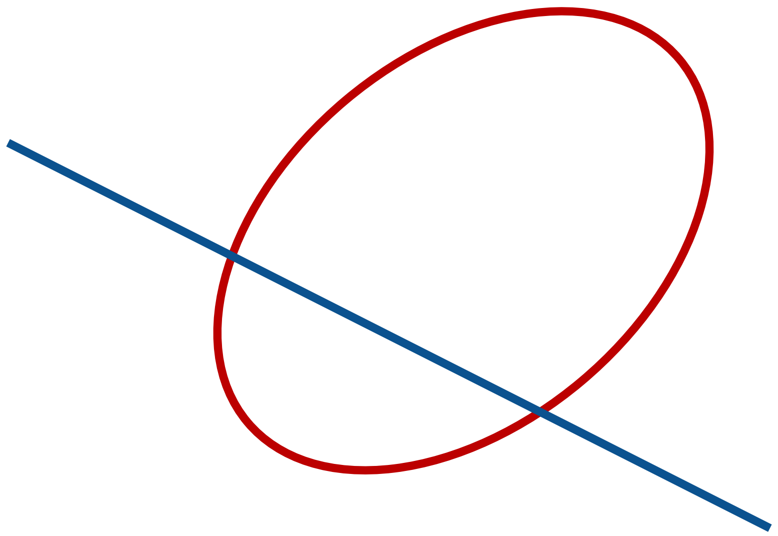
Tropical geometry



Equations of planar curves

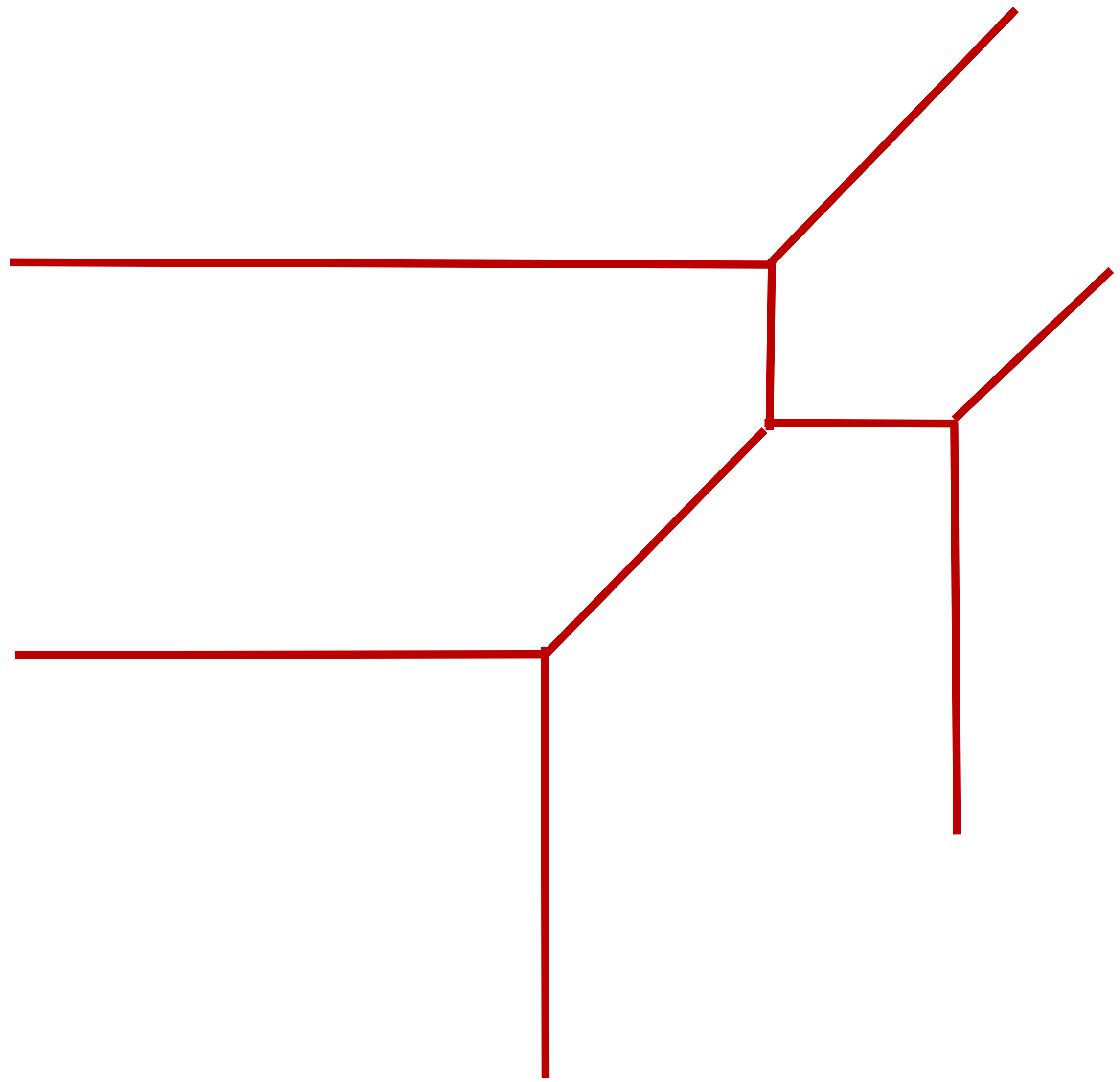
Usual geometry

Quadrics



intersect lines at two points

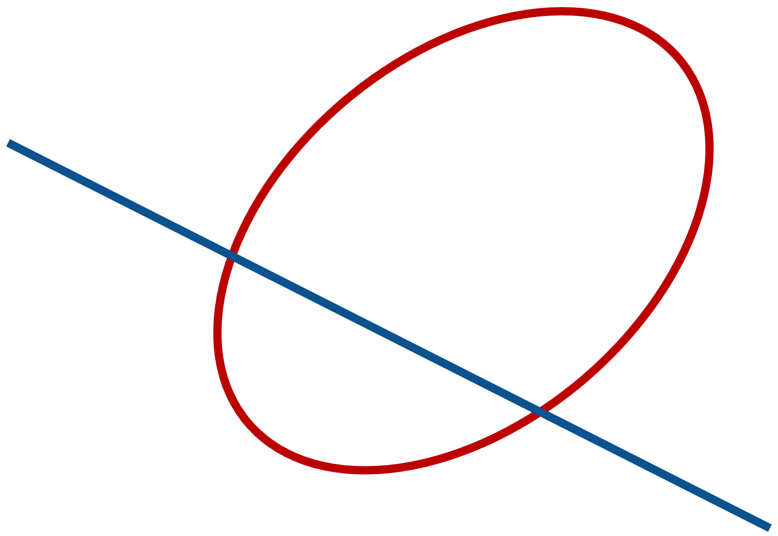
Tropical geometry



Equations of planar curves

Usual geometry

Quadrics



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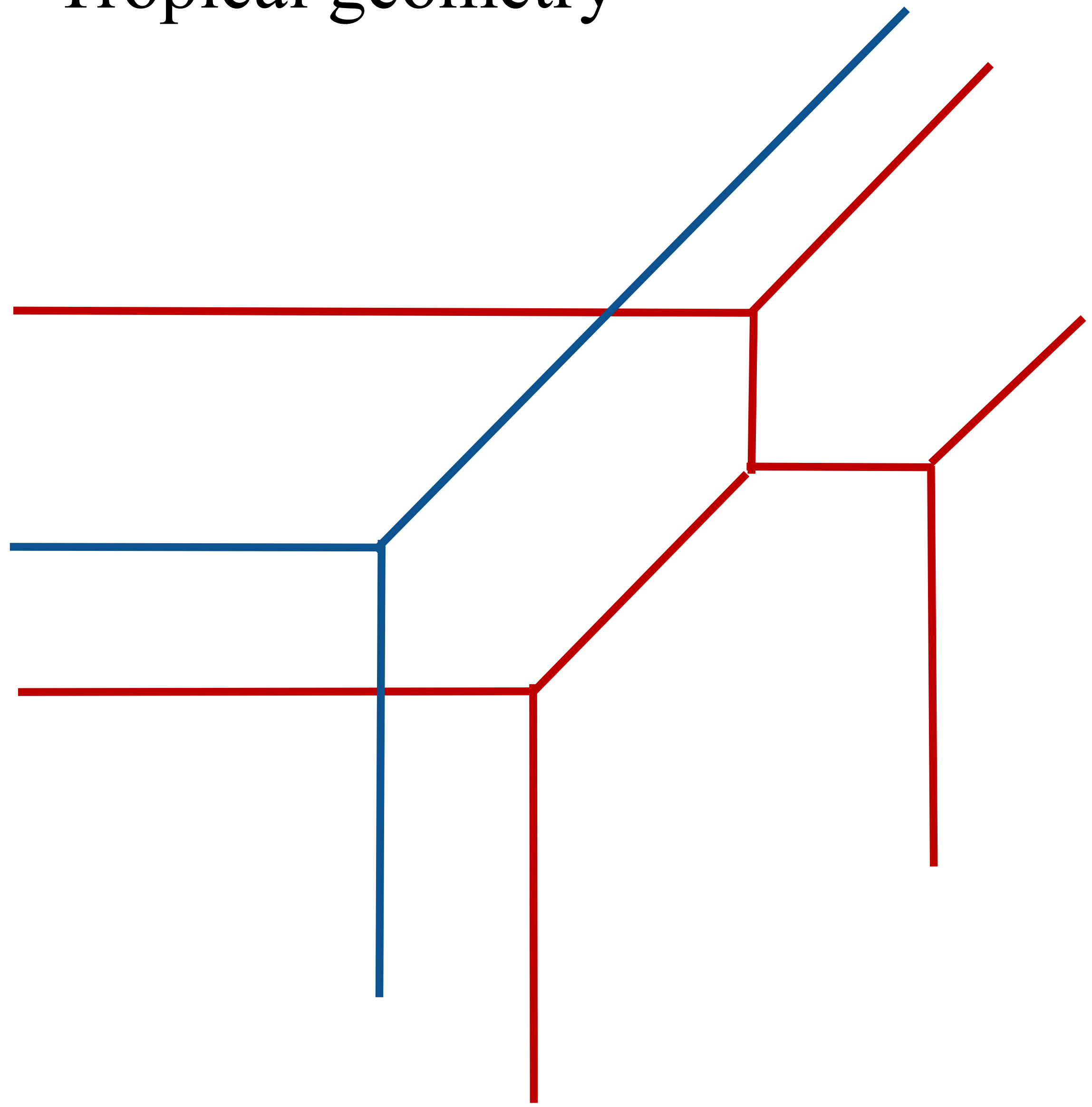
Tropical geometry



Equations of planar curves

Usual geometry

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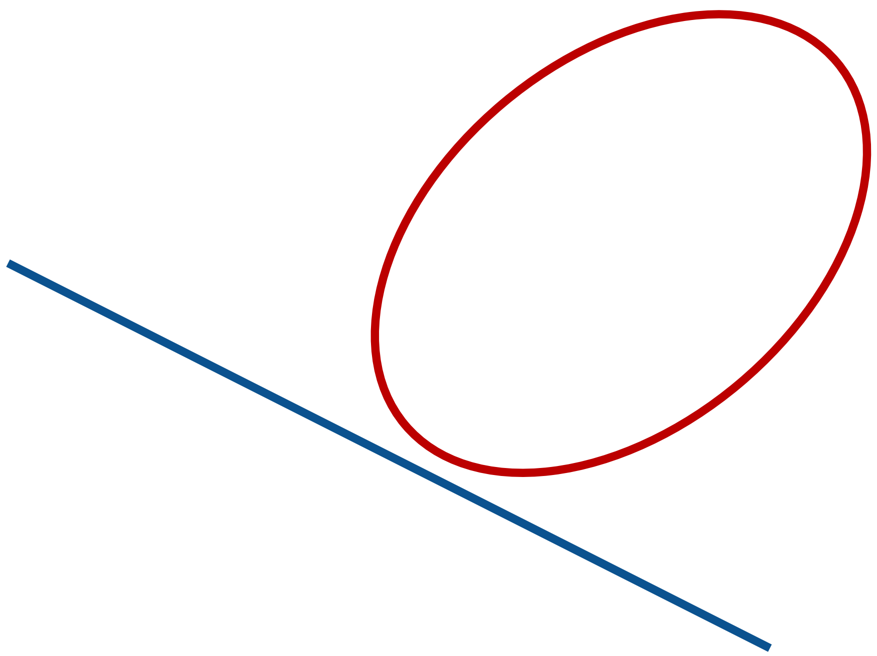
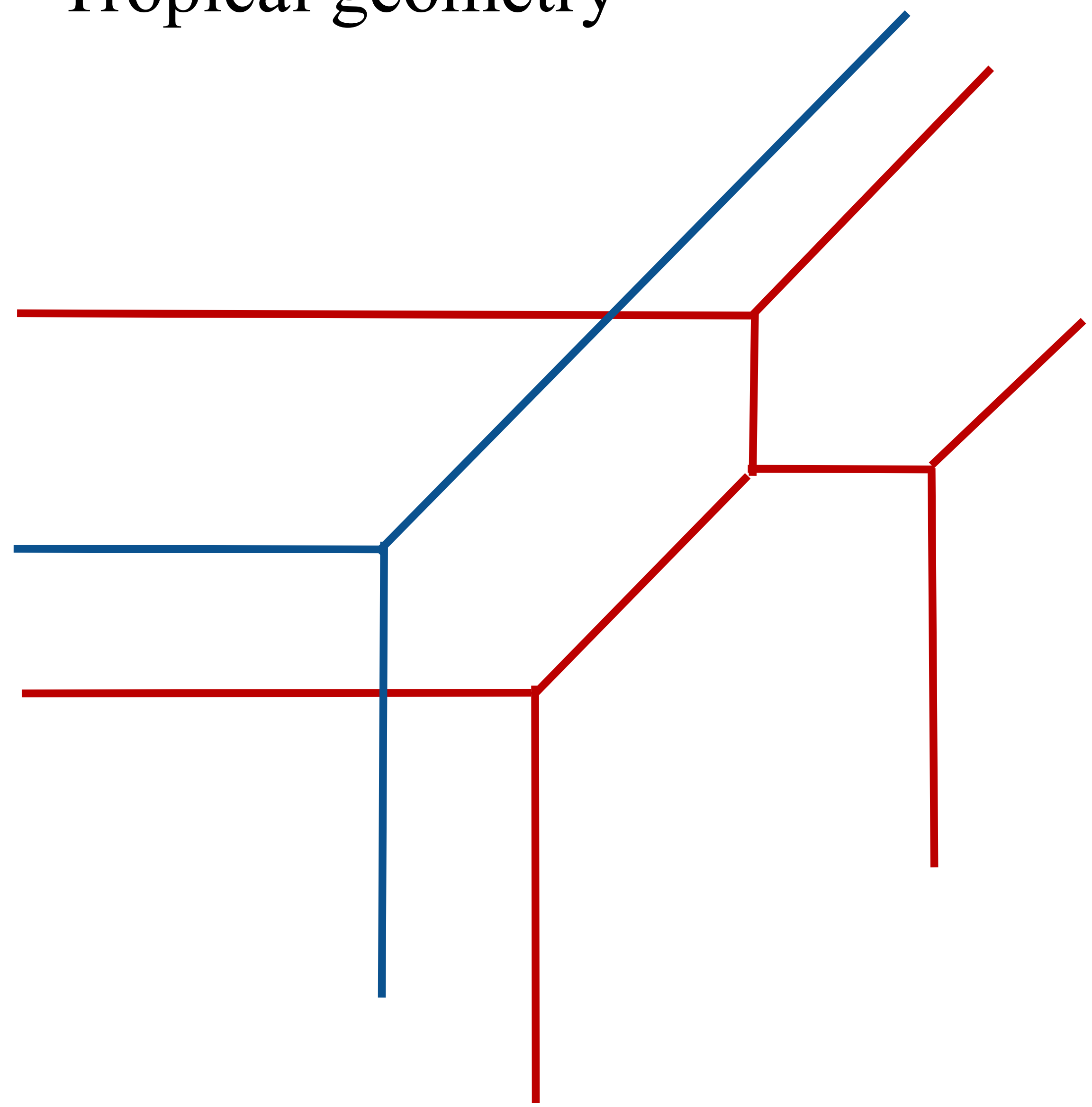


intersect lines at two points

not always...

Tropical geometry



but ALWAYS in tropical geometry!

Quadrics intersect lines at two points in COMPLEX algebraic geomerty.

Tropical geometry is closely related to the complex world.

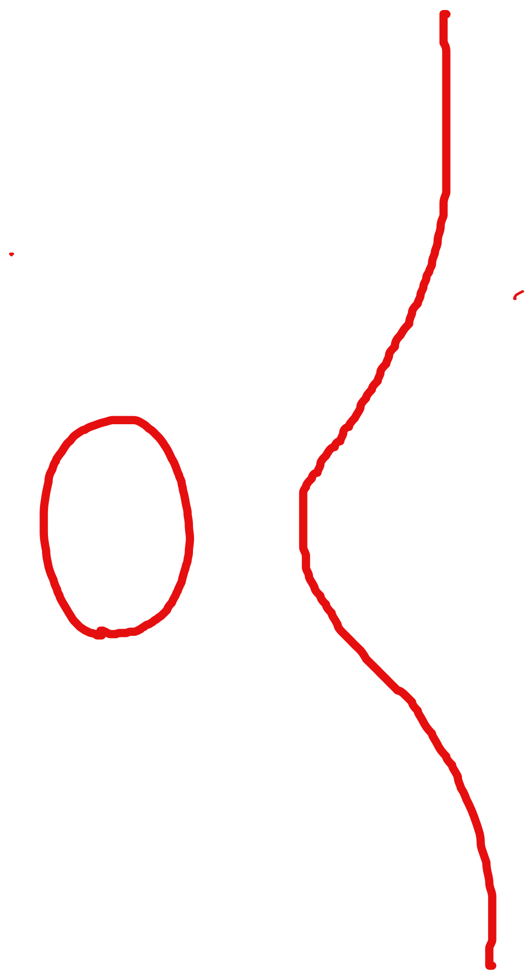
Equations of planar curves

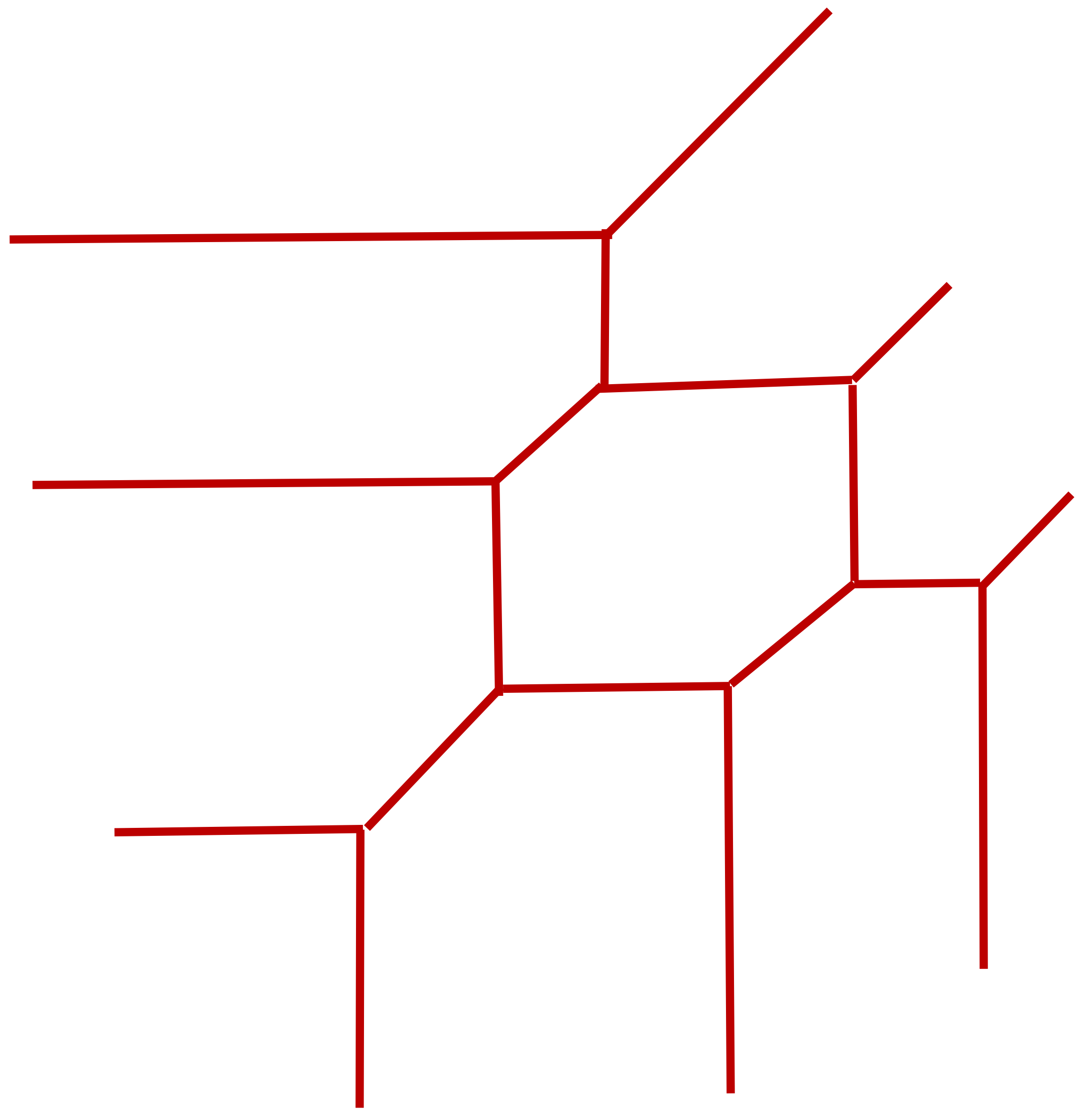
Usual geometry

Tropical geometry

Cubics

$Y^2 = A \times X^3 + X + B$





Complex picture of a line is real plane with coordinates

(real part, imaginary part)

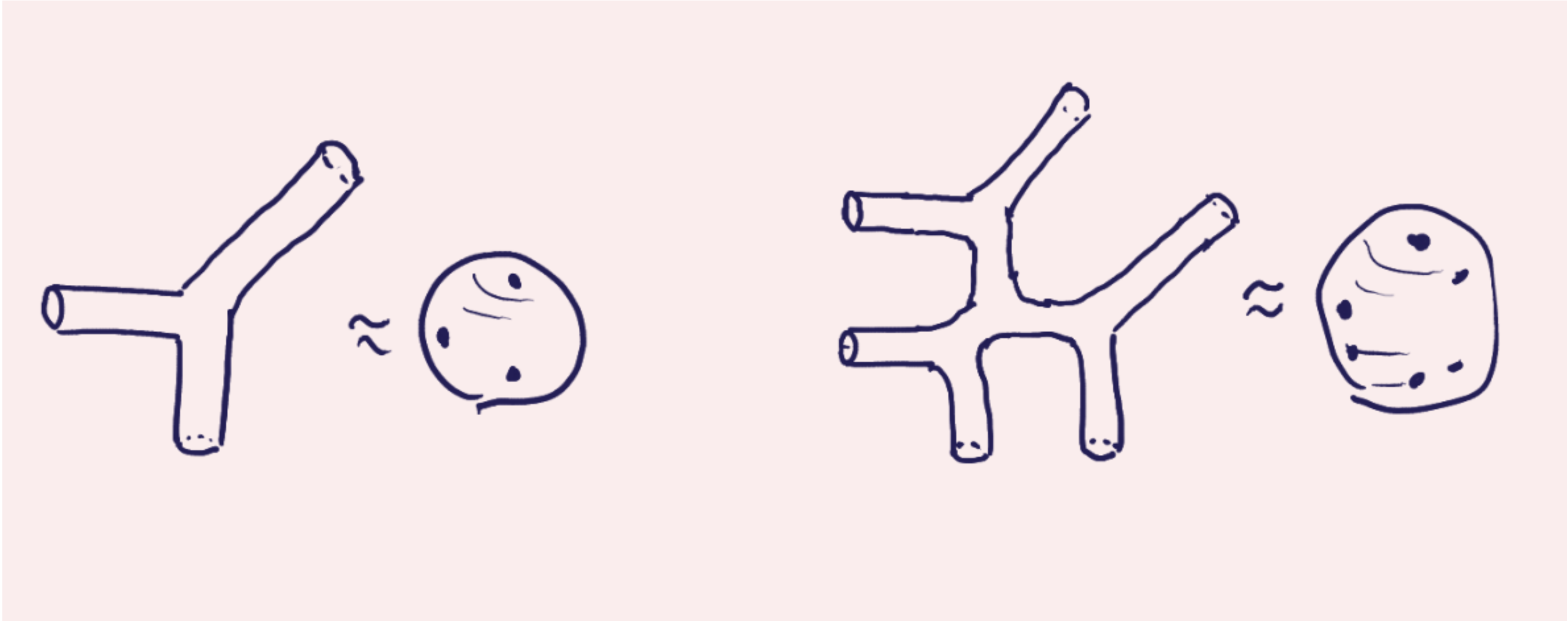
$z=x+iy$

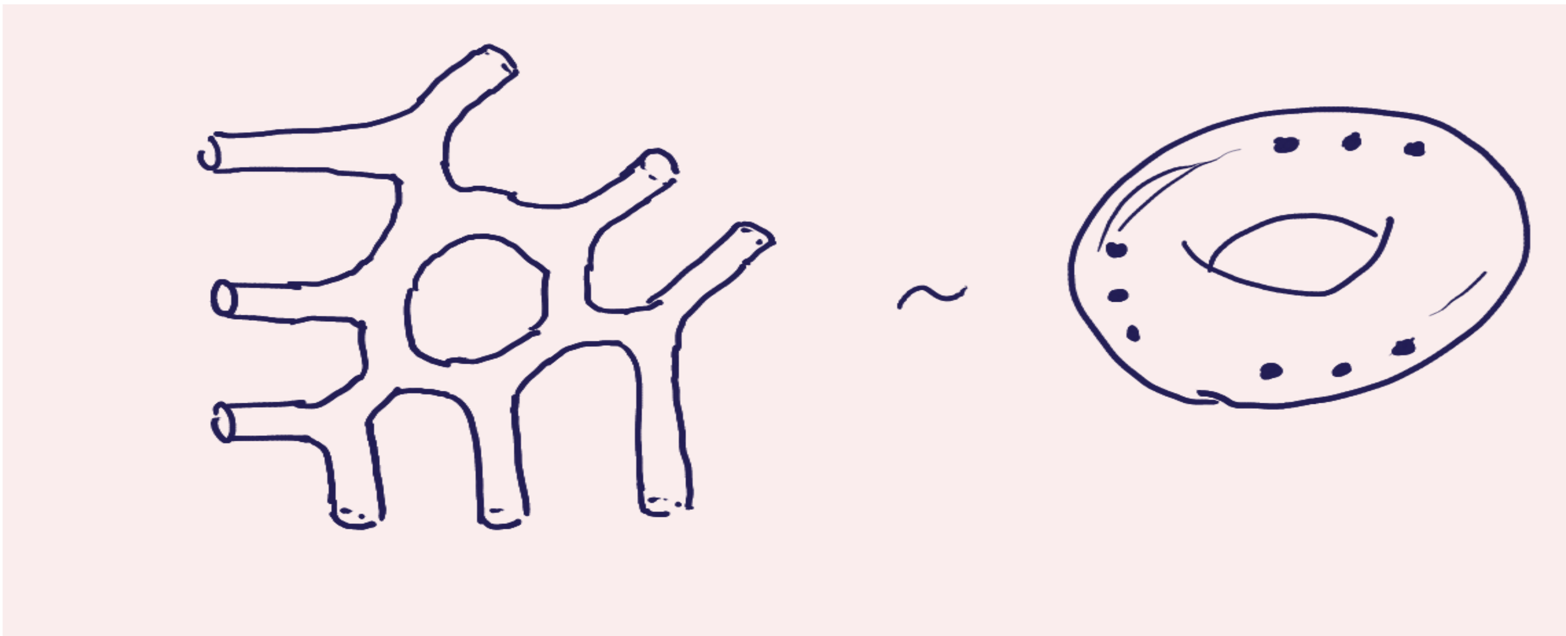
which is sphere minus one point.

Similarly, complex affine quadric is topologically a cylinder, which is a sphere minus two points.

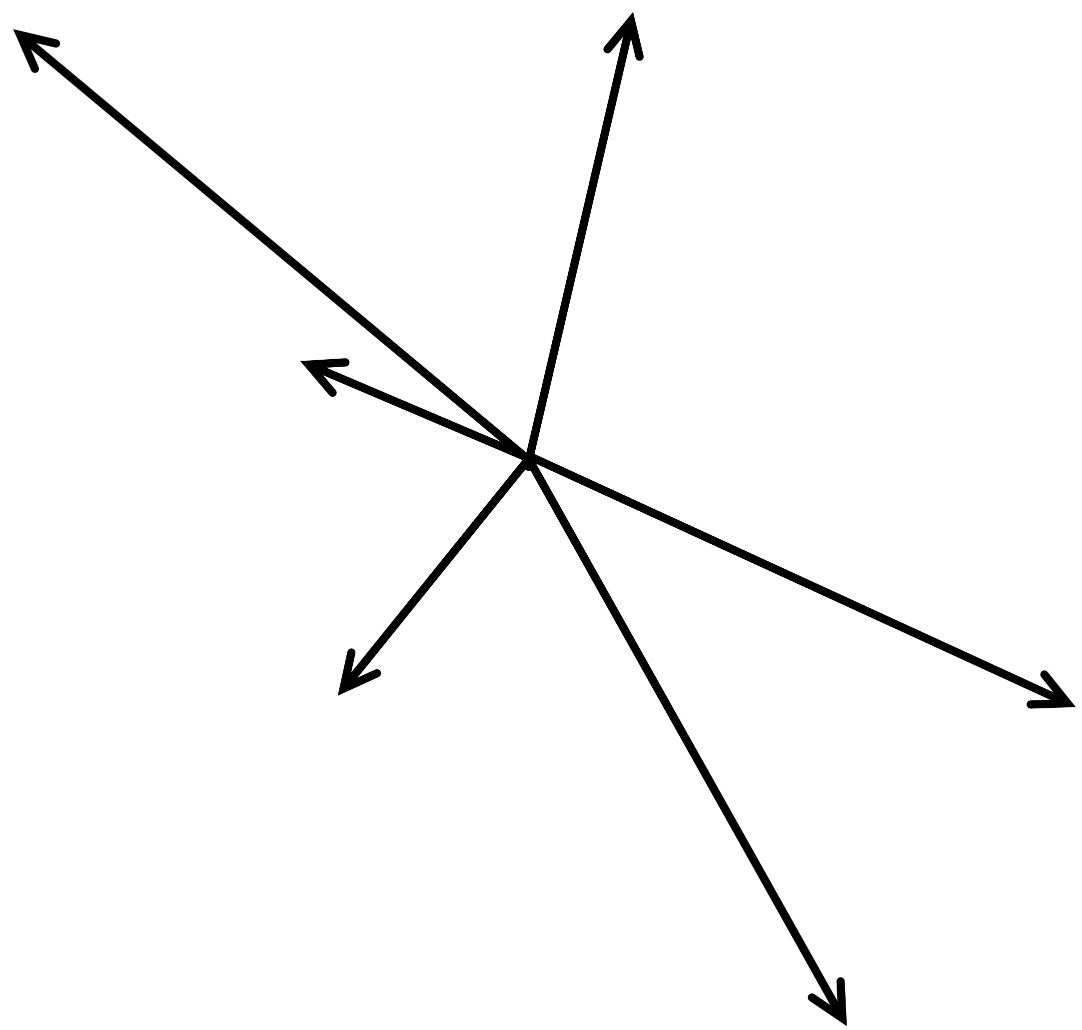
Complex affine cubic is a torus minus three points.

Tropical line is sphere minus three points,
tropical quadric is sphere minus six points,
tropical cubic is torus minus nine points.





Planar tropical curves are piece-wise linear curves satisfying at each vertex BALANCE CONSTRAINT



Sum of vectors = 0

Higher-dimensional tropical objects:

piece-wise linear gadgets describing the "shape" of a complex algebraic variety which depends on a small parameter t and degenerates badly as t---->0

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piece-wise linear gadgets describing the "shape" of a complex algebraic variety which depends on a small parameter t and degenerates badly as t---->0

Closely related wtih non-acrhimedean geometry and so called Berkovich spectrum.

Example of a non-archimedean field: p-adic numbers.

$|x+y|$ < or = maximum($|x|$, $|y|$)

Tropical varieties outside of the locus of singularities look locally as open domains in coordinate spaces.

Coordinate changes are integral affine transformations,
for example in the case of two coordinates (x,y) they look as

$$(x , y) \dashrightarrow (ax+by+e, cx+dy+f)$$

where a,b,c,d,e,f are integer numbers and $ad - bc = 1$.

Seeds in sunflower:

If we go around the center, the coordinate change will be

$$(x,y) \dashrightarrow (x+y,\ x+2y)$$

Closely related to Fibonacci numbers

1,1,2,3,5,8,13,21,34,55,...

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Fibers of the map from complex variety to tropical one are smooth tori outside the locus of singularities. The degeneration of fibers over singular points is similar to those in integrable systems!

Tropical geometry in mirror symmetry

Tropical geometry in mirror symmetry

In string theory space-time is 10-dimensional manifold which (if applicable to real world) is the profuct of the usual physical 4-dimensional space-time and of 6-dimensional compact manifold whose Ricci curvature is zero.

This auxilary 6-dimensional manifold is in fact complex algebraic 3-dimensional variety, called Calabi-Yau variety.

Its metric depends on two types of parameters, called complex and Kähler parameters.
Mirror symmetry interchanges these two groups of parameters, leading to the same physical theory.

The simplest example of 3-dimensional Calabi-Yau variety is quintic 3-fold, a hypersurface in 4-dimensional complex projective space given by a homogenous equation of degree 5 in 5 complex variables.

This Calabi-Yau variety depends on 101 complex prameters and 1 Kähler parameter.

Mirror dual Calabi-Yau variety depends on 1 complex parameter and 101 Kähler parameters.

In fact, mirror symmetry appears when both dual varieties depend on small parameter and are degenerating in a maximal possible manner.

Hence we obtain two tropical varieties, both of real dimension 3.

Mirror symmetry is nothing but a new duality between tropical varieties.
Even the condition that Ricci curvature vanishes, can be formulated for metrics on tropical varieties.

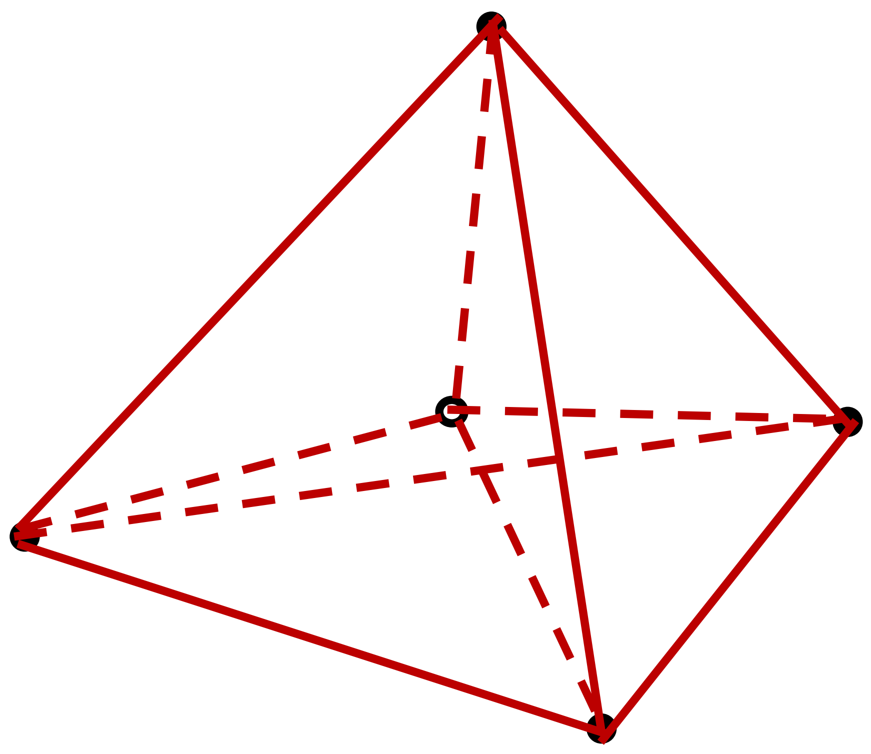
True picture of quintic 3-fold

True picture of quintic 3-fold

First step: choose 4 points in 3-dimensional space, and then the 5th point inside the tetrahedron spanned by first 4 points.

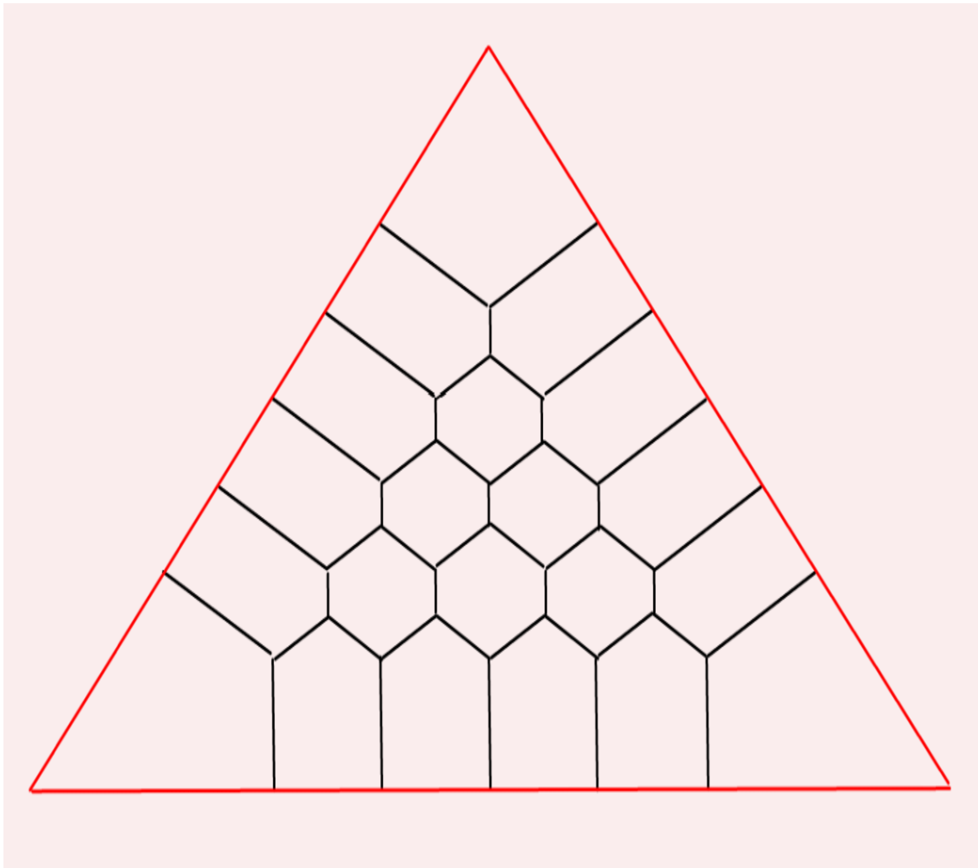
Connect by intervals all 5 points.

We obtain 10 nonintersecting triangles



True picture of quintic 3-fold

Step 2. Draw in each triangle already familar picture of a planar quintic.



True picture of quintic 3-fold

We obtain a complicated graph in 3-dimensional space, with 200 vertices each of degree 2, and 300 edges. This graph is the locus of singularities of the integral affine structure on two tropical varieties,
associated with degenerating quintic 3-folds and with degenerating dual Calabi-Yau 3-folds.

True picture of quintic 3-fold

We obtain a complicated graph in 3-dimensional space, with 200 vertices each of degree 2, and 300 edges. This graph is the locus of singularities of the integral affine structure on two tropical varieties,
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The fundamental group of the complement to the graph maps to SL(3,Z),
dual varieties gives conjugate representations.

Towards a counterexample to the Hodge conjecture

Towards a counterexample to the Hodge conjecture

A.Weil: expected that the Hodge conjecture is wrong!
Potential counterexample:
generic complex abelian variety A of dimension $= 2n \geq 4$
with an endomorphism δ such that

$$\delta^2 = -d \times \text{identity endomorphism of } A, \quad d=1,2,\ldots \text{ square-free}$$

A has only one Hodge class in $H^2(A)$, and 3 linearly independent Hodge classes in $H^n(A)$. Hence 2 classes can not be represented as an intersection of divisors.

Towards a counterexample to the Hodge conjecture

Tropical version:

generic real 2n-dimensional torus T with integral affine structure together with an endomorphism δ such that

$$\delta^2 = -d \times \text{identity endomorphism of } T, \quad d=1,2,....$$

Tropical Hodge conjecture: T has tropical n-dimensional cycles representing 3 lineraly independent tropical Hodeg classes.

Towards a counterexample to the Hodge conjecture

Tropical Hodge conjecture fails

$\Rightarrow$ the usual (complex) Hodge conjecture fails,
also Tate conjecture (about etale cohomology) fails ...

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But one can show that (for given n,d) that
the failure of the tropical Hodge conjecture is

$\Leftrightarrow$ existence of a solution of certain explicit <u>countable</u> system
of linear equations with a countable set of variables.
Hence we may hope to find a closed formula for the solution!