Some LP bounds for covering radius of spherical designs

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Outline of the talk

- Spherical designs (SphD)
- Covering radius of spherical designs
- Gegenbauer polynomials; another characterization of SphD
- LP bounds (obtained by Fazekas and Levenshtein) for covering radius of spherical designs
- Some notations for the structure of spherical designs
- A signed measures and corresponding orthogonal polynomials
- Properties of the polynomials $P_i^{0,\ell}(t)$
- Improving the Fazekas-Levenshtein (FL) bound for even strengths
- Case: $\ell \leq t_1(y)$; Case: $t_1(y) \in [-1, \ell]$
- A procedure for finding new lower bounds
- Upper bounds
- Examples for 4-designs
- Future work
- References

Spherical designs (1)

Spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [3].

Definition (1)

A *spherical* τ*-design* $C \subset \mathbb{S}^{n-1}$ is a finite subset of $\mathbb{S}^{n-1}$ such that

$$\frac{1}{\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} f(x)d\mu(x) = \frac{1}{|C|}\sum_{x\in C} f(x)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of degree at most τ (i.e. the average of f over the set is equal to the average of f over $\mathbb{S}^{n-1}$).

The maximal possible $\tau = \tau(C)$ is called strength of C.

Covering radius of spherical designs

Definition (2)

Let $C \subset \mathbb{S}^{n-1}$ be a finite set (spherical design in our applications). For a fixed point $y \in \mathbb{S}^{n-1}$ the *distance between* y *and* C is defined in the usual way by

$$d(y, C) := \min\{d(y, x) : x \in C\}.$$

Then the covering radius of C is

$$r(C) := \max\{d(y, C) : y \in \mathbb{S}^{n-1}\}.$$

We consider the equivalent quantity

$$\rho(C) := 1 - \frac{r^2(C)}{2} = \min_{y \in \mathbb{S}^{n-1}} \max_{x \in C}\{\langle x, y \rangle\}.$$

Gegenbauer polynomials

Definition (3)

By $P_i^{(n)}(t)$, $i = 0, 1, \ldots$, we denote the *Gegenbauer polynomials* normalized by $P_i^{(n)}(1) = 1$, which satisfy the following three-term recurrence relation

$$(i+n-2)\, P_{i+1}^{(n)}(t) = (2i+n-2)\, t\, P_i^{(n)}(t) - i\, P_{i-1}^{(n)}(t) \text{ for } i \geq 1,$$

where $P_0^{(n)}(t) := 1$ and $P_1^{(n)}(t) := t$.
In the standard Jacobi polynomial notation, we have that

$$P_i^{(n)}(t) = \frac{P_i^{((n-3)/2,(n-3)/2)}(t)}{P_i^{((n-3)/2,(n-3)/2)}(1)}.$$

Spherical designs (2)

Theorem (4)

[3] A code $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design if and only if for any point $y \in \mathbb{S}^{n-1}$ and any real polynomial $f(t)$ of degree at most τ, the equality

$$\sum_{x \in C} f(\langle x, y \rangle) = f_0 |C| \qquad (1)$$

holds, where f_0 is the first coefficient in the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$.

Linear programming (LP) bounds - Fazekas-Levenshtein bounds for covering radius of spherical designs

Linear programming bounds for covering radius of spherical designs were obtained by Fazekas and Levenshtein [2, Theorem 2] in the more general setting of polynomial metric spaces.
They prove that if C is a $(2k-1+e)$-design, $e \in \{0,1\}$, then

$$\rho(C) \geq t_{FL} = t_k^{0,e}, \tag{2}$$

where $t_k^{0,e}$ is the largest zero of the Jacobi polynomial $P_k^{(\alpha,\beta)}(t)$, $\alpha = \frac{n-3}{2}$, $\beta = \frac{n-3}{2} + e$.
For example, (2) gives $\rho(C) \geq (1+\sqrt{n+3})/(n+2)$ for every spherical 4-design of $M \geq n(n+3)/2$ points.
Note that the Fazekas-Levenshtein bound does not depend on the cardinality of the designs under consideration.

Problems

Problem (1)

To improve the lower bounds on the covering radius $\rho(C)$ of spherical designs with fixed dimension n, even strength $2k$, and cardinality $|C|$.

Problem (2)

To obtain the upper bounds on the covering radius $\rho(C)$ of spherical designs with fixed dimension n, even strength $2k$, and cardinality $|C|$.

Delsarte-Goethals-Seidel bound

For fixed dimension $n \geq 2$ and strength $\tau \geq 1$ the minimum cardinality of a spherical τ-design $C \subset \mathbb{S}^{n-1}$ is bounded from below by Delsarte-Goethals-Seidel as follows:

$$|C| \geq D(n,\tau) := \binom{n+k-2+e}{n-1} + \binom{n+k-2}{n-1}, \qquad (3)$$

where $\tau = 2k-1+e$, $e \in \{0,1\}$. This bound (3) is rarely attained. Improvements (Boyvalenkov, Yudin, Nikova-Nikov);
It is still unknown, if there exist spherical 4-designs of 10 points on $\mathbb{S}^2$.
(Bondarenko, Radchenko and Viazovska) For fixed dimension n and strength τ there exist spherical τ-designs on $\mathbb{S}^{n-1}$ for any cardinality $N \geq C_n \tau^{n-1}$, where the constant C_n depends on the dimension n only.

Some notations for the structure of spherical designs

Let $C \in \mathbb{S}^{n-1}$ be a spherical design. For an arbitrary point $y \in \mathbb{S}^{n-1}$, consider the (multi)set

$$I(y) = \{\langle x, y\rangle : x \in C\} = \{t_1(y), t_2(y), \ldots, t_{|C|}(y)\},$$

where we assume that $I(y)$ is ordered by
$-1 \leq t_1(y) \leq t_2(y) \leq \ldots \leq t_{|C|}(y) \leq 1$, $(t_{|C|}(y) = 1 \Leftrightarrow y \in C)$.
In what follows we always assume that y is a point on $\mathbb{S}^{n-1}$ where the covering radius is realized, in particular $t_{|C|}(y) = \rho(C)$.

Lemma (5)

If y is a point on $\mathbb{S}^{n-1}$ where the covering radius is realized, then

$$t_{|C|}(y) = t_{|C|-1}(y) = \cdots = t_{|C|-n+1}(y) = \rho(C).$$

A signed measure $\Rightarrow$ orthogonal polynomials (1)

The measure of orthogonality of Gegenbauer polynomials $P_i^{(n)}(t)$ is

$$d\mu(t) := c_n(1-t^2)^{\frac{n-3}{2}}\,dt, \quad t \in [-1,1], \quad c_n := \Gamma(\frac{n}{2})/\sqrt{\pi}\Gamma(\frac{n-1}{2}).$$

We need also measures which are positive definite up to certain degree considered by Cohn and Kumar (2006).

Definition (6)

A signed Borel measure ν on $\mathbb{R}$ for which all polynomials are integrable is called *positive definite up to degree m* if $\int p^2(t)d\nu(t) > 0$ for all real nonzero polynomials $p(t)$ of degree at most m.

A signed measure ⇒ orthogonal polynomials (2)

It was proved (B.-Dragnev-Hardin-Saff-S., 2019) that the signed measure

$$d\mu_\ell(t) := c_{n,\ell}(t-\ell)d\mu(t), \quad t \in [-1,1], \quad c_{n,\ell} := -1/\ell,$$

is positive definite up to degree $k-1$ provided that $\ell < t_{k,1}$, where $t_{k,1}$ is the smallest zero of the Gegenbauer polynomial $P_k^{(n)}(t)$.

This implies the existence of a finite sequence of polynomials $\{P_i^{0,\ell}(t)\}_{i=0}^k$ which are orthogonal with respect to $d\mu_\ell(t)$ and normalized by $P_i^{0,\ell}(1) = 1$. The uniqueness of these polynomials allows us to write explicitly

$$P_i^{0,\ell}(t) = \frac{T_i(t,\ell)}{T_i(1,\ell)} = \frac{(1-\ell)\left(P_{i+1}^{(n)}(t) - P_i^{(n)}(t)P_{i+1}^{(n)}(\ell)/P_i^{(n)}(\ell)\right)}{(t-\ell)\left(1 - P_{i+1}^{(n)}(\ell)/P_i^{(n)}(\ell)\right)} \qquad (4)$$

The boundary case $\ell = -1$ leads to polynomials which Levenshtein denoted by $P_i^{0,1}(t)$ - the (normalized) Jacobi polynomials with parameters $(\alpha, \beta) = ((n-3)/2, (n-1)/2)$.

Properties of the polynomials $P_i^{0,\ell}(t)$ - interlacing of roots

Let $\tau = 2k$. Denote by $t_{i,1} < t_{i,2} < \ldots < t_{i,i}$ the zeros of $P_i^{(n)}(t)$ and $t_{i,1}^{0,\ell} < t_{i,2}^{0,\ell} < \ldots < t_{i,i}^{0,\ell}$ the zeros of $P_i^{0,\ell}(t)$.

Theorem (7)

Let ℓ and k be such that $t_{k+1,1} < \ell < t_{k,1}$ and $P_{k+1}^{(n)}(\ell)/P_k^{(n)}(\ell) < 1$. Then

$$P_i^{0,\ell}(t) = \frac{T_i(t,\ell)}{T_i(1,\ell)}, \quad i = 0, 1, \ldots, k, \tag{5}$$

where the leading coefficient of $P_i^{0,\ell}$ is $m_i^{0,\ell} > 0$. All zeros $\{t_{i,j}^{0,\ell}\}_{j=1}^{i}$ of $P_i^{0,\ell}(t)$ are in the interval $[\ell, 1]$ and the interlacing rules

$$\begin{aligned} &t_{i,j}^{0,\ell} \in (t_{i,j}, t_{i+1,j+1}),\ i = 1, \ldots, k-1, j = 1, \ldots, i; \\ &t_{k,j}^{0,\ell} \in (t_{k+1,j+1}, t_{k,j+1}),\ j = 1, \ldots, k-1,\ t_{k,k}^{0,\ell} \in (t_{k+1,k+1}, 1), \end{aligned} \tag{6}$$

hold.

Properties of the polynomials $P_i^{0,\ell}(t)$ - a quadrature formula

We denote by $L_i(t)$, $i = 0, 1, \ldots, k$, the Lagrange basic polynomials generated by the nodes $\ell < t_{k,1}^{0,\ell} < t_{k,2}^{0,\ell} < \cdots < t_{k,k}^{0,\ell}$ and set

$$\theta_i := \int_{-1}^{1} L_i(t) d\mu(t), \quad i = 0, 1, \ldots, k.$$

Theorem (8)

Let $t_{k,1}^{0,\ell} < t_{k,2}^{0,\ell} < \cdots < t_{k,k}^{0,\ell}$ be the zeros of the polynomial $P_k^{0,\ell}(t)$. Then the quadrature formula

$$f_0 = \int_{-1}^{1} f(t) d\mu(t) = \theta_0 f(\ell) + \sum_{i=1}^{k} \theta_i f(t_{k,i}^{0,\ell}) \tag{7}$$

is exact for all polynomials of degree at most $2k$ and has positive weights $\theta_i > 0$, $i = 0, 1, \ldots, k$.

Improving the FL bound for even strengths
Case: $-1 < \ell \leq t_1(y)$ (1)

Let $C \subset \mathbb{S}^{n-1}$ be a spherical $2k$-design of cardinality $|C| > D(n, 2k)$. Let $y \in \mathbb{S}^{n-1}$ be a point which realizes the covering radius of C.
Lower bounds $-1 < \ell \leq t_1(y)$ imply improvements of the t_{FL}.

Theorem (9)

Let $C \subset \mathbb{S}^{n-1}$ be a spherical $2k$-design and $y \in \mathbb{S}^{n-1}$ be a point which realizes the covering radius of C. If $\ell \leq t_1(y)$, then

$$\rho(C) \geq t_{k,k}^{0,\ell}.$$

In the boundary case $\ell = -1$ this theorem gives the Fasekas-Levenshtein bound $\rho(C) \geq t_{FL} = t_k^{0,1}$. Therefore, we have improvement of the Fasekas-Levenshtein bound whenever it is known (or it is presumed) that $\ell \leq t_1(y)$ for a point y where the covering radius is realized.

Case: $-1 < \ell \leq t_1(y)$ Some lower bounds $t_{k,k}^{0,\ell}$ (2)

Dimension n	Cardinality $\lvert C \rvert$	Strength $\tau = 2k$	ℓ	FL-lower bound $\rho(C) \geq t_k^{0,1}$	New lower bound if $\ell \leq t_1(y)$ $\rho(C) \geq t_{k,k}^{0,\ell}$
3	10	4	-0.97	0.689897	0.694892
3	10	4	-0.95	0.689897	0.698664
3	10	4	-0.9	0.689897	0.710257
4	15	4	-0.97	0.607625	0.611772
4	15	4	-0.95	0.607625	0.614815
4	15	4	-0.9	0.607625	0.623682
3	17	6	-0.97	0.822824	0.825859
3	17	6	-0.95	0.822824	0.828450
3	17	6	-0.9	0.822824	0.839165

Improving the FL bound for even strengths
Case: $t_1(y) \in [-1, \ell]$ (1)

This case is more subtle.
We will have to optimize in two classes of real polynomials. We consider

$A(n,k,\ell) := \{f(t) = A^2(t) : \deg(f) = 2k, A(t)$ has k real zeros in $[\ell, t_{FL}]\}$.

Similarly, we use polynomials from the set

$$B(n,k,s) := \{g(t) = (t+1)B^2(t)(t-s) : \deg(g) = 2k,$$

$$B(t) \text{ has } k-1 \text{ real zeros in } [-1,s]\},$$

where the parameter s (close to t_{FL}) will be chosen in advance.

Case: $t_1(y) \in [-1, \ell]$ (2)

The next lemma sets an auxiliary parameter $m(C)$ after optimization in the set $A(n, \tau, \ell)$.

Lemma (10)

Let $f(t) \in A(n, k, \ell)$ and the positive integer m be such that

$$f_0|C| < f(\ell) + (m+1)f(t_{FL}). \quad (8)$$

Then $t_{|C|-m}(y) < t_{FL}$.

We define

$$m(C) := \min\{m : \exists f \in A(n, k, \ell) \text{ such that } f_0|C| < f(\ell) + (m+1)f(t_{FL})\}.$$

This lemma implies that

$$m(C) \geq n.$$

Case: $t_1(y) \in [-1, \ell]$ (3)

Example (11)

We have $m(C) = n = 3$ for $(n, \tau, |C|) = (3, 4, 10)$, the first case, where the existence/nonexistence of spherical 4-designs is undecided. Similarly, for $(n, \tau, |C|) = (4, 4, 15)$ we have $m(C) = n + 1 = 5$.

Lemma (12)

Let $f(t) \in A(n, k, \ell)$ be such that $f_0|C| < f(\ell) + (m(C) + 1)f(t_{FL})$. Then $t_{|C|-m(C)}(y) \leq s$, where s is the largest root of the equation

$$f_0|C| - f(\ell) = (m(C) + 1)f(t).$$

Case: $t_1(y) \in [-1, \ell]$ (4)

The previous two Lemmas 10 and 12 imply that $t_{|C|-m(C)}(y) \leq s < t_{FL}$. We utilize this in a second optimization dealing with the location of t_{FL} between two inner products from $I(y)$.

We have

$$t_{|C|-m(C)}(y) \leq s < t_{FL} \leq t_{|C|-n+1}(y) = \rho(C).$$

Therefore, there exist $j \in \{0, 1, \dots, m(C) - n\}$ such that

$$t_{|C|-m(C)+j}(y) < t_{FL} \leq t_{|C|-m(C)+j+1}(y). \tag{9}$$

This clarification of the location of t_{FL} with respect to the points of $I(y)$ allows more precise estimations.

Case: $t_1(y) \in [-1, \ell]$ (5)

Lemma (13)

If $g(t) \in B(n,k,s)$*, then* $\rho(C) \geq m_{\ell,s}^{(j)}$*, where* $m_{\ell,s}^{(j)}$ *is the largest root of the equation*

$$jg(t_{FL}) + (m(C)-j)g(t) = g_0|C|. \qquad (10)$$

Lemma (14)

Let $f(t) \in A(n,k,\ell)$ *be such that* $f_0|C| < f(\ell) + (m(C)+1)f(t_{FL})$*. Then* $t_{|C|-m(C)}(y) \leq s^{(j)}$*, where* $s^{(j)}$ *is the largest root of the equation*

$$f_0|C| = (j+1)f(t) + f(\ell) + (m(C)-j)f(t_{FL}). \qquad (11)$$

A procedure for finding new lower bounds (1)

In the case $t_1(y) \in [-1, \ell]$, for each fixed $j \in \{0, 1, \ldots, m(C) - n\}$, we can start an iterative procedure with previous Lemmas 13 and 14 for obtaining consecutive improvements of $s^{(j)}$ and $m_{\ell,s}^{(j)}$.

This procedure may converge to some bounds or may be divergent which will mean nonexistence of designs with the corresponding parameters (dimension, strength, and cardinality).

The better bound $\rho(C) \geq t_k^{0,\ell}$ when $t_1(y) \geq \ell$ allows starting a similar procedure with analogs of the above Lemmas 13 and 14 as the only difference will be the absence of ℓ.

A procedure for finding new lower bounds (2)

Example (15)

Considering again $(n, \tau, |C|) = (3, 4, 10)$
(recall that $m(C) = n = 3$ in this case, i.e. $j = 0$ only),
we obtain for $\ell = -0.97$ that $\rho(C) \geq 0.724753$ if $t_1(y) \in [-1, -0.97]$
and $\rho(C) \geq 0.728787$ if $t_1(y) \geq -0.97$.
Therefore, we have $\rho(C) \geq 0.724753$ in the worst case.

Similarly, for $(n, \tau, |C|) = (4, 4, 15)$
(note that now $m(C) = 5$, i.e. $j = 0, 1$),
we obtain for $\ell = -0.97$ that $\rho(C) \geq 0.625572$ if $t_1(y) \in [-1, -0.97]$
and $\rho(C) \geq 0.627354$ if $t_1(y) \geq -0.97$ for $j = 0$;
$\rho(C) \geq 0.616854$ if $t_1(y) \in [-1, -0.97]$
and $\rho(C) \geq 0.619259$ if $t_1(y) \geq -0.97$ for $j = 1$.
Summarizing, we conclude that $\rho(C) \geq 0.616854$ in the worst case.

General upper bounds (1)

We write (1) for y, C and $f(t)$, $\deg(f) \leq \tau(C)$, as

$$nf(\rho(C)) + \sum_{i=1}^{|C|-n} f(t_i(y)) = f_0|C|. \qquad (12)$$

The identity (12) provides upper bounds for $\rho(C)$ as follows.

Theorem (16)

(Linear programming upper bounds of the covering radius of spherical designs) *Let $f(t)$, $\deg(f) \leq \tau$, be a real polynomial which is nonnegative in $[-1, t_{FL}]$ and increasing in $[t_{FL}, 1]$. Then for every τ-design $C \subset \mathbb{S}^{n-1}$ we have*

$$\rho(C) \leq m_u,$$

where m_u is the largest root of the equation $nf(t) = f_0|C|$.

General upper bounds (2)

The following theorem shows which kind of extremal polynomials should be investigated.

Theorem (17)

The best polynomials for use in Theorem 16 are $f(t) = (t+1)^e A^2(t)$*, where* $\tau = 2k - e$*,* $e \in \{0,1\}$*,* $\deg(A) = k - e$ *and* $A(t)$ *has* $k - e$ *zeros in* $[-1, t_{FL}]$*.*

Upper bounds for spherical 4-designs

We now find the optimal polynomials in the above Theorem for $\tau = 4$.

Theorem (18)

If $C \subset \mathbb{S}^{n-1}$ is a spherical 4-design, then

$$\rho(C) \le u(a_0, b_0),$$

where the function $u(a,b)$ and the (optimal) parameters a_0 and b_0 are defined in the proof.

Example (19)

Looking again in the first open case $(n, \tau, |C|) = (3, 4, 10)$, we obtain (for $b_0 = \frac{\sqrt{69}-7}{30}$ and $a_0 = \frac{(3+\sqrt{69})\sqrt{45+10\sqrt{69}}}{150}$) the upper bound $\rho(C) \le u(a_0, b_0) \approx 0.754443$.

Examples for 4-designs

Dimension n	Cardinality $\|C\|$	$m(C)$	FL-lower bound $\rho(C) \geq t_2^{0,1}$	New lower bound $\ell = -0.97$	New upper bound
3	10	3	0.689897	0.724753	0.7545
3	11	4	0.689897	0.694717	0.7794
4	15	5	0.607625	0.616854	0.6918
4	16	5	0.607625	0.610537	0.7072
5	21	7	0.546918	0.550012	0.6503
5	22	8	0.546918	0.548132	0.6604
6	28	10	0.500000	0.501717	0.6198
6	29	10	0.500000	0.501288	0.6269
7	36	13	0.462475	0.463455	0.5960
7	37	13	0.462475	0.462961	0.6012

Upper bounds for antipodal 3- and 5-designs (1)

A spherical design C is called antipodal if $C = -C$.
The set $I(y)$ is symmetric for antipodal designs;
We have

$$t_i = t_{|C|-i+1}, \ i = 1, 2, \ldots, n.$$

In Lemma 5 and the equation in Theorem (16) becomes

$$2nf(t) = f_0|C|.$$

The upper bounds for antipodal designs are easier for $\tau = 3$ and $\tau = 5$.
We are able to obtain an explicit bound in these cases for all dimensions and cardinalities.

Upper bounds for antipodal 3- and 5-designs (2)

Theorem (20)

If C is an antipodal 3-design, then

$$t_{FL} = \frac{1}{\sqrt{n}} \leq t_c \leq \frac{1}{n}\sqrt{\frac{|C|}{2}}.$$

Theorem (21)

If C is an antipodal 5-design, then

$$t_{FL} = \left(\frac{3}{n+2}\right)^{1/2} \leq t_c \leq \left(\frac{1}{n} + \frac{1}{n}\sqrt{\frac{(n-1)(|C|-2n)}{n(n+2)}}\right)^{1/2}.$$

Future work

- To obtain an explicit bounds for covering radius of spherical designs with $\tau = 6, 8$, etc.
- To obtain bounds for covering radius of spherical designs with odd strengths.

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