On the Spherical code $(4, \rho, 9)$ [1]

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Abstract. A well known spherical code $(4, \rho, 9)$ of dimension 4 and size 9 is considered. The square of the Euclidean minimal distance is equal to $\rho = 1.67596...$ and is a solution to a cubic equation. It is shown that this code is optimal and unique.

1 Introduction

Let $X : (n, \rho, M)$ be a finite set (called *spherical code*) of M points on the Euclidean sphere $S^{n-1} \subset R^n$ of radius 1, which satisfies the following property: for an arbitrary two points $x = (x_1, ..., x_n)$ and $y = (y_1, ...y_n)$ from X the following inequality holds:

$$\rho(x, y) \ = \ \sum_{i=1}^{n} (x_i - y_i)^2 \geq \rho.$$

In the current paper, by distance it is always assumed the square of Euclidean distance. Moreover, a *sphere* S^{n-1} in space R^n is always assumed to be a sphere of radius 1 centered at the origin of R^n.

According to the Rankin [1,2] for ρ in the interval $[2, 4]$, all optimal spherical codes are known. But it is not the case for $\rho < 2$. There is only one infinite family of optimal codes with $\rho < 2$. In particular, the following result (see section 9.4 in [3]) takes place: an optimal spherical code with parameters

$$n \ = \ q(q^2 - q + 1), \ \ \rho \ = \ 2 - \frac{2}{q^2}, \ \ M \ = \ (q + 1)(q^3 + 1),$$

exists for any prime power q [4 - 6]. In addition to this infinite family, there exist a few other optimal spherical codes in dimension $n \leq 24$, or for $n = 3$ obtained from the strongly regular graphs or lattices (see [3]).

There also exist two codes (whose optimality is not mentined in [3], since it was shown rather recently). One is the code $(4, 1, 24)$, whose optimality was proved by Musin [7] and the code $(4, 5/3, 10)$ whose optimality and uniqueness were established by Bachoc and Vallentin [8].

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In [3], codes with parameters $(n, \rho, 2n+1)$, where ρ is a root of some cubic equation have been constructed. The optimality of the first non-trivial code of this family with parametrs $(3, \rho, 7)$ and $\rho \approx 1.57972...$, has been proved by Schütte and van der Waerden [9]. The goal of the present paper is to show that the next representative of this family, i.e. the $(4, \rho, 9)$ code is unique and optimal.

2 The construction of code

Recall that in the space R^n, there are known two infinite families of regular polytopes whose vertices lie on the sphere S^{n-1}: simplices and cross-polytopes. The codes of our interest $(n, \rho, 2n+1)$ have an extra point compared to codes formed by the vertices of cross-polytopes, which are the well-known biorthogonal codes with parameters $(n, \rho, M) = (n, 2, 2n)$. According to the Rankin bound [2] they are optimal. The vertices of a simplex form another well known infinite family of the optimal spherical codes with parameters $(n, 2+2/n, n+1)$ that lie on the Rankin bound [1]. Denote by P_n the simplex code, i.e. the spherical code with parameters $(n, 2+2/n, n+1)$.

For a spherical code X and any real number $b \in R$ denote by $b\,X$ the following set of points of R^n, obtained from X:

$$b\,X = \{b\,\mathbf{x} = (b\,x_1, b\,x_2, ..., b\,x_n) : \ \mathbf{x} \in X\}.$$

For a given point $\mathbf{x} = (x_1, \ldots, x_n)$ denote by $\bar{\mathbf{x}} = -\,\mathbf{x} = (-\,x_1, \ldots, -\,x_n)$ the *antipodal* point (on the same sphere). This point is known to be at the maximal distance 4 from $\mathbf{x}$. A pair of points $\mathbf{x}$ and $\bar{\mathbf{x}}$ is called the *antipadal pair*. Similarly, for a set X on the sphere, denote by $\bar{X}$ the corresponding *antipodal set*

$$\bar{X} = \{\bar{\mathbf{x}} : \ \mathbf{x} \in X\}.$$

For a given set of points $Z \subset R^n$ and an arbitrary real a, such that $0 < a \leq 1$ denote by $Z\,|\,a$ the set of points in R^{n+1}, obtained from Z by adding a coordinate position with value a:

$$Z\,|\,a \ = \ \{(z\,|\,a) = (z_1, ..., z_n, a) : \ z \in Z\}.$$

For a pair of spherical codes X and Y denote by $\rho\,(X, Y)$ the minimal distance between their points:

$$\rho\,(X, Y) \ = \ \min\{\rho\,(x, y) : \ x \in X, \ y \in Y\}.$$

Let $s(x, y)$ be the inner product of $\mathbf{x}$ and $\mathbf{y}$, where for $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ we have

$$s(x, y) \ = \ \sum_{i=1}^{n} x_i y_i.$$

For a pair of spherical codes X and Y let

$$s\,(X,Y) \;=\; \max\{s\,(x,y):\; x\in X,\; y\in Y\}.$$

Present a construction of the code (see Example 5.10.4 in [3]). For a given Simplex code P_3 (with parameters $(3, 8/3, 4)$), construct a new spherical code $X : (4, \rho, 9)$ as a union of three spherical codes:

$$X \;=\; X_1 \cup X_2 \cup X_3,$$

where

$$\begin{array}{rcrcl} X_1 & = & b_1\, P_3 & | & a_1, \\ X_2 & = & b_2\, \bar{P}_3 & | & -\,a_2, \\ X_3 & = & (0,\ldots,0 & | & 1). \end{array}$$

i.e. the set X_3 contains a single point. Choose the rational numbers a_i and b_i so that the following enequality is satisfied:

$$a_i^2 + b_i^2 \;=\; 1, \quad i=1,2 \;\text{ and }\; a_2 > a_1 > 0. \tag{1}$$

Lemma 1. *There exist real numbers a_i, $i=1,2$, so that the given code X is the spherical code $(4,\;\rho_m,\;M=9)$, where*

$$\rho_m = 2(1-a_1) \approx 1.6759696,$$

and a_1 is the minimal positive root to the following equation:

$$16v^3 + 16v^2 - 4v - 1 = 0. \tag{2}$$

Moreover, we have that

$$0.162015199 \le a_1 \le 0.162015200;$$

$$0.609517350 \le a_2 \le 0.609517351$$

Remark 1. *Recall the best known upper bound of ρ for the optimal $(4,\;9,\;\rho)$ code* [11]*:*

$$\rho \le 1.84639.$$

For any code X with parameters (n,ρ,M), any two points $\mathbf{x}$ and $\mathbf{y}$ from this code are called *neighbours* if $\rho(\mathbf{x},\mathbf{y}) = \rho$. For any point $\mathbf{x}$ of X the number of its neighbours is called the *valency* of $\mathbf{x}$. A point of valency 0 is called an *isolated point.* A point $\mathbf{x}$ which by a small displacement (not affecting other points of the code) can be transformed to an isolated point is said to be *loose.*

The construction of the code implies two observations.

Remark 2. *There are* 4 *points* $\mathbf{x}$ *of the code* X *of Lemma 1, such that any of them has* 6 *neighbours* $\mathbf{x}_i$ $(1 \leq i \leq 6)$*. Moreover,* 3 *points out of them belong to* X_2 *and another* 3 *belong to* X_1*.*

This remark implies the following lemma, which is a restatement of Lemma 1. The construction of the code X can be described in a different way. Enumerate the code points of the code X in the following way: let $\mathbf{x}_1, \ldots, \mathbf{x}_4$ belong to the subcode X_1, the points $\mathbf{x}_5, \ldots, \mathbf{x}_8$ belong to X_2, and the point $\mathbf{x}_9 \in X_3$. Assume that the points $\mathbf{x}_i$ and $\mathbf{x}_{i+4}$ are antipodal in the initial simplices P_3 and $\bar{P}_3$.

The following lemma provides another presentation of the code X of Lemma 1

Lemma 2. *For any* $i,\ i = 4, \ldots, 8$*, the constructed above* $(4, \rho_m, 9)$ *code* X *is equal to the union of three spherical codes:*

$$X \;=\; Z_1 \cup Z_2 \cup Z_3,$$

where

$$\begin{aligned} Z_1 &= \{\mathbf{x} \in X_1 \cup X_2 : \mathbf{x} \neq \mathbf{x}_i, \mathbf{x}_{4+i}\}, \\ Z_2 &= \{\mathbf{x}_i, \mathbf{x}_9\}, \\ Z_3 &= \{\mathbf{x}_{4+i}\}. \end{aligned}$$

Moreover, any point $\mathbf{x}_{4+i}$*,* $4 \leq i \leq 8$ *has exactly* 3 *neighbours from* Z_1 *(i.e. all points of* Z_1 *are the neighbours of* $\mathbf{x}_{4+i}$*). In particular, if* $\mathbf{x}_{4+i}$ *is of the form* $\mathbf{x}_5 = (0, \ldots, 0, 1)$ *then any point* $\mathbf{x} \in Z_1$ *is of the form*

$$\mathbf{x} = (x_1, x_2, x_3, a_1),$$

and the points $\mathbf{x}_i$ *and* $\mathbf{x}_9$ *of* Z_2 *are of the form*

$$\mathbf{x}_i = (x_1', x_2', x_3', -(b_1 b_2 + a_1 a_2)), \quad \mathbf{x}_9 = (x_1'', x_2'', x_3'', -a_2),$$

where the real numbers a_i *è* b_i *satisfy the condition (1) and Lemma 1. Moreover*

$$\rho(\mathbf{x}_i, \mathbf{x}_9) = \rho_m \;\; \textit{and} \;\; \rho(\mathbf{x}_i, \mathbf{x}_{i+4}) = 2 - 2(b_1 b_2 + a_1 a_2).$$

Observe that in the code X, the distance between any pair of points can be given by one of the four possible expressions: $\rho_m = 2 - 2a_1$,

$$\rho(X_1) = \frac{8}{3} \cdot b_1^2, \;\; \rho(X_2, X_3) = (2 + 2a_2), \;\; \rho(\mathbf{x}_i, \mathbf{x}_{i+4}) = 2 - 2(b_1 b_2 + a_1 a_2).$$

3 Uniqueness and optimality of $(4, \rho_m, 9)$ code

We start with some properties of an optimal spherical $(4, \rho, 9)$ code X. We consider this code up to any rotation on the Euclidean shpere, which is a multiplication all code points by an orthogonal matrix from the group $\mathrm{SO}(4, R)$. In particular, it implies that without loss of generality one can assume that the code X contains any point on the sphere S^3.

Since the code X constructed in Lemma 1 is a rigid set, i.e. there are no points in the code whose coordinats can be changed without decreasing the minimal distance, we have the following properties of this code:

Lemma 3. *Let X be an optimal spherical code with parameters $(4, \rho, 9)$. Then any point $\mathbf{x} \in X$ has valency at least 4.*

On the other hand, the valency of any point is bounded from above.

Lemma 4. *Let X be an optimal spherical code with parameters $(4, \rho, 9)$. Then for any $\mathbf{x} \in X$ the valency of $\mathbf{x}$ is not greater than 6.*

Since the code X has an odd number of points, we have that

Lemma 5. *Let X be an optimal spherical code with parameters $(4, \rho, 9)$. Then there exist a point whose valency is not equal to 5.*

Moreover, we can show that the valency of the points of an optimal code X cannot be equal to 5 or 6 only. So, consequently we have that

Lemma 6. *Let X be an optimal spherical code with parameters $(4, \rho, 9)$. Then, there exists a point of X whose valency is equal to 4.*

Recall that the *deep holes* for the simplex code P_n (i.e. the point on the sphere whose distance is maximum from all code points) are well known. In particular, we recall the following facts.

Lemma 7. *Suppose $P_n = \{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$ is a simplex code on the sphere S^{n-1} in R^n.*
1) For an arbitrary point $\mathbf{y} \notin P_n$ on the sphere S^{n-1} the following inequalities are true

$$\rho(\mathbf{y}, P_n) \leq 2 - \frac{2}{n}.$$

2) There exist $n+1$ points $\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$ on the sphere, so that the above inequalities become equalities.
3) The $n+1$ points $\mathbf{y}_1, \ldots, \mathbf{y}_{n+1}$, from above form a simplex $\bar{P}_n = -P_n$, which is anipodal to the given simplex P_n, i.e. $\mathbf{y}_i = \bar{\mathbf{x}}_i$, $\mathbf{x}_i \in P_n$ for $i = 1, \ldots, n+1$.

Suppose X is an optimal code of type $(4, \rho, 9)$. According to Lemma 6, there exists a point z_0 of that code with valency 4. Applying some rotation of R^4 we can assume that x_0 is of the form $(0, 0, 0, 1)$. Suppose that the neighbours of x_0 are x_1, x_2, x_3, x_4 denoted by set X_1. The remaining 4 points x_5, x_6, x_7, x_8 are denoted by X_2.

Lemma 8. *Let X be an optimal spherical code with parameters $(4, \rho, 9)$. Suppose $x_0 = (0, 0, 0, 1) \in X$ has valency 4 and its neighbours is a set $X_1 = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$. Then, the numeration of the remaining points $\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, \mathbf{x}_8$ can be chosen in such a way that the neighbours of $\mathbf{x}_i$ (for any $1 \leq i \leq 4$) are $\mathbf{x}_{4+j}$, where $1 \leq i \leq 4$ and $j \neq i$.*

The points $\mathbf{x}_i$, $1 \leq i \leq 4$, of Lemma 8 are of the form (x_i, y_i, z_i, s), where $\rho_m = 2 - 2s$. So, once orthogonally projected onto a 3 dimensinal space (by erazing the last coordinate) these points $\mathbf{x}'_i$ lie on a sphere of radius r, $r = \sqrt{1 - s^2}$ centered at zero. Let

$$\rho^* = \rho(\{\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4\}) = \rho(\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}) = 4r^2 \sin(\phi),$$

where ϕ is the half angle between the points at the minimal distance. Since $\rho \leq \rho^* \leq 8r^2/3$, then the following estimate holds

$$2(1 - s) \leq 4(1 - s^2) \sin^2(\phi) \leq \frac{8}{3}(1 - s^2),$$

once simplified we obtain that

$$\sin(\phi_0) = \frac{1}{\sqrt{2(1 + s)}} \leq \sin(\phi) \leq \sin(\phi_1) = \frac{\sqrt{2}}{\sqrt{3}},$$

so that $\phi_0 \leq \phi \leq \phi_1$.

Using Lemma 7, the configuration of points $\{\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, \mathbf{x}_8\}$ with the maximal distance from the points $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ is determined uniquely. Then, it turns out that $\rho(\{\mathbf{x}_5, \mathbf{x}_6, \mathbf{x}_7, \mathbf{x}_8\})$ can be expressed as $f(\phi)$, where function f can be written explicitly.

Studying the behaviour of the function $f(\phi)$ we show that the maximum values of this function over the interval $[\phi_0, \phi_1]$ are at the endspoints and moreover $f(\phi_0) = f(\phi_1)$, where the two endpoint correspond to two different presentation of our construction (Lemmas 1 and 2).

Thus, we arrive at the main Theorem.

Theorem 1. *Let X be an optimal spherical code with parameters $(4, \rho, 9)$. Then its minimal distance graph coincides to that of the code constructed in Lemma 1.*

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