On $m$-specially resolvable BIB designs and $q$-ary constant weight codes

Leonid Bassalygo, Vladimir Lebedev, Victor Zinoviev

Harkevich Institute for Problems of Information Transmission,
Moscow, Russia

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Outline

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2. Introduction
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We introduce $m$-specially resolvable BIB designs ($SRB_m$) which generalize known resolvable BIB designs. The coexistence theorem between $SRB_m$ designs and some class of $q$-ary constant weight codes, satisfying the Johnson upper bound, is established. Several constructions of such designs and codes are developed based on Steiner systems and super-simple $t$-designs.
Let $Q = \{0, 1, \ldots, q - 1\}$. Any subset $C \subseteq Q^n$ is a code denoted by $(n, N, d)_q$ of length $n$, cardinality $N = |C|$ and minimum (Hamming) distance $d$. A code $C$ is constant weight and denoted $(n, N, w, d)_q$ if every its codeword is of weight $w$. 
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We recall the following two classical bounds for the size $N_q(n, d)$ of a $q$-ary $(n, N, d)_q$-code and for the size $N_q(n, d, w)$ of a $q$-ary constant weight $(n, N, w, d)_q$-code:
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**Plotkin bound:**

$$N_q(n, d) \leq \frac{qd}{qd - (q - 1)n}, \text{ if } qd > (q - 1)n, \quad (1)$$
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$$N_q(n, d) \leq \frac{qd}{qd - (q - 1)n}, \quad \text{if } qd > (q - 1)n, \quad (1)$$

**Johnson bound:**

$$N_q(n, d, w) \leq \frac{(q - 1)dn}{qw^2 - (q - 1)(2w - d)n}, \quad \text{if } qw^2 > (q-1)(2w-d)n, \quad (2)$$
On \(m\)-specially resolvable BIB designs and \(q\)-ary constant weight codes

Introduction

Definition 1.

A \(T(v, k, t, \lambda)\)-design is an incidence structure \((X, B)\), where \(X = \{x_1, \ldots, x_v\}\) is a set of elements and \(B\) is a collection of blocks of size \(k\), such that every \(t\) distinct elements of \(X\) are contained in exactly \(\lambda > 0\) blocks of \(B\) (here \(1 \leq t \leq k \leq v - 1\)). If \(\lambda = 1\) then \(T(v, k, t, \lambda)\)-design is called a Steiner system and denoted by \(S(v, k, t)\).
Introduction

If $t = 2$ a design $T(v, k, 2, \lambda)$ is called also a balanced incomplete block-design $(v, b, r, k, \lambda)$ and denoted $B(v, k, \lambda)$. Respectively, a Steiner system $S(v, k, 2)$ is a design $B(v, k, 1)$. The other two parameters of a $B(v, k, \lambda)$ design are: $b = |B|$ (the number of blocks) and $r$ (the number of blocks containing one fixed element):

$$b = \lambda \frac{v(v-1)}{k(k-1)}, \quad r = \lambda \frac{v-1}{k-1}.$$ (3)
It is well known and quite evidently that a design $T(v, k, t, \lambda)$ with $t \geq 3$ is a design $B(v, k, \lambda_2)$ with parameters

$$\lambda_2 = \lambda \frac{\binom{v-2}{t-2}}{\binom{k-2}{t-2}}, \quad b = \lambda \frac{\binom{v}{t}}{\binom{k}{t}}, \quad r = \lambda \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}.$$  \hspace{1cm} (4)
Introduction

A block-design $B(v, k, \lambda)$ is completely described by its *incidence matrix* $A = [a_{i,j}]$, where $a_{i,j} = 1$ if $a_i \in B_j$ and $a_{i,j} = 0$, otherwise. So $A$ is a binary $(v \times b)$-matrix with columns of weight $k$, such that any two distinct rows contain exactly $\lambda$ common nonzero positions.
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$$A = [A_1 | \cdots | A_r],$$

where for any $i \in \{1, \ldots, r\}$ the every row of $A_i$ has the weight 1.
Preliminary results

**Definition 2.**

(Bassalygo-Lebedev-Zinoviev, 2017) A $B(v, k, \lambda)$ design is $m$-nearly resolvable ($NRB_m(v, k, \lambda)$ design) if its incidence matrix $A$ can be presented as follows:

$$A = [A_1 | \cdots | A_n], \quad n = \frac{bk}{v - m},$$

such that the following properties are valid:
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$$A = [A_1 | \cdots | A_n], \quad n = \frac{bk}{v - m}, \quad (6)$$

such that the following properties are valid:

1. the every submatrix $A_j$, of size $v \times \frac{v-m}{k}$ consists of rows of weight 1 with exception of $m$ zero rows whose indices belong to the set $V_j$, $|V_j| = m$, $V_j \subset \{1, 2, \ldots, v\}$;
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(2) the sets $V_1, \ldots, V_n$ (as a collection of $n$ blocks of size $m$) induces a block design $B(v, m, \xi)$ (which we call accompanying design) for some $\xi$.  

Preliminary results

It can be seen that in such $m$-nearly resolvable $NRB_m(v, k, \lambda)$ design, parameters $\lambda$ and $\xi$ look as follows:

$$\lambda = n \frac{(k - 1)(v - m)}{v(v - 1)}, \quad \xi = n \frac{m(m - 1)}{v(v - 1)}.$$  \hfill (7)
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$$

(7)

The case $m = 0$ corresponds to the mentioned above resolvable designs, and the case $m = 1$ gives near-resolvable designs (both widely considered in the literatures (see [Abel-Ge-Yin, 2007], [Beth-Jungnickel-Lenz, 1986], [Furino-Miao-Yin, 1996] and references there for resolvable and near-resolvable designs).
As well known [Beth-Jungnickel-Lenz, 1986], [Furino-Miao-Yin, 1996] block designs with blocks of two different sizes are also considered. Denote by $B(v, b, r, \{k_1, k_2\}, \lambda)$ such a design with blocks of two sizes $k_1$ and $k_2$. Denote for shortness $k_1 = m$ and $k_2 = k$. 
A $B(v, b, r, \{m, k\}, \lambda)$ design we call $m$-specially resolvable if the incidence matrix $A$ can be presented as follows:

$$A = [A_1 \mid \cdots \mid A_r],$$

such that the following properties are valid:

1. Each submatrix $A_i$ of size $v \times v - mk$ consists of rows of weight 1 and of columns of weight $k$ with exception that one column is of weight $m$;
2. The sets $V_1, \ldots, V_r$ formed by the elements of blocks of the size $m$ (one block from every submatrix $A_i$) induces a constant weight code.
A $B(v, b, r, \{m, k\}, \lambda)$ design we call $m$-specially resolvable if the incidence matrix $A$ can be presented as follows:

$$A = [A_1 | \cdots | A_r],$$  \hspace{1cm} (8)

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Definition 3.

A $B(v, b, r, \{m, k\}, \lambda)$ design we call $m$-specially resolvable if the incidence matrix $A$ can be presented as follows:

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Preliminary results

It is straightforward to check (by the standard counting of ones and of pairs of ones in the matrix $A$) that in a such $m$-specially design $B(v, b, r, \{m, k\}, \lambda)$ the parameters $r$ and $b$ are defined by $v, k, m, \lambda$ as follows:

$$r = \frac{\lambda v(v - 1)}{m(m - 1) + (v - m)(k - 1)}, \quad b = \frac{\lambda v(v - 1)(v - m + k)}{k(m(m - 1) + (v - m)(k - 1))}.$$
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So, denote an $m$-specially resolvable design $B(v, b, r, \{m, k\}, \lambda)$ by $SRB_m(v, k, \lambda)$. 
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So, denote an $m$-specially resolvable design $B(v, b, r, \{m, k\}, \lambda)$ by $SRB_m(v, k, \lambda)$.

The concept of $m$-specially resolvable designs is also a variant of generalization of frames [Furino-Miao-Yin, 1996], which are used for construction of resolvable and near-resolvable designs. Now we have the following simple relations between concepts introduced above.
Preliminary results

Lemma 1.

(1) Let $D$ be a resolvable RB$(v,k,\lambda)$ design. Then this design implies the existence of a $(k-1)$-specially resolvable design $SRB_{k-1}(v-1,k,\lambda)$.

(2) Let $D^{(1)}$ be a $m$-nearly resolvable NRB$_m(v,k,\lambda)$-design and let $D^{(0)}$ be its accompanying $B(v,m,\xi)$ design, formed by the sets $V_1, ..., V_n$. Then the design $D^{(1)}$ implies the existence of $m$-specialy resolvable $SRB_m(v,k,\lambda + \xi)$ design.
Preliminary results

We recall the following known results, connecting $q$-ary optimal equidistant codes meeting the upper bounds (1) and (2) and resolvable and $m$-nearly resolvable designs $B(v, k, \lambda)$, respectively.

**Theorem 1.**

([8]) An optimal equidistant $(n, d, N)_q$ code meeting the Plotkin bound (1) exists if and only if there exists a resolvable $RB(v, k, \lambda)$ design, where

$$q = v/k, \quad n = \lambda(v - 1)/(k - 1), \quad N = v, \quad d = n - \lambda.$$  \hspace{1cm} (10)
Theorem 2. ([B − Z, 2017], [B − L − Z, 2017]) Any \( m \)-nearly resolvable \( NRB_m(v, k, \lambda) \)-design with accompanying \( B(v, m, \xi) \) design induces a \( q \)-ary equidistant constant weight \( (n, N, w, d)_q \) code \( C \) with parameters

\[
q = \frac{v - m}{k} + 1, \quad n = \lambda \frac{v(v - 1)}{(k - 1)(v - m)}, \quad N = v, \quad w = \lambda \frac{k - 1}{v - 1}, \quad d = \lambda \frac{v - m}{k - 1}
\]

meeting the Johnson bound (2) with additional property that its \( n \) blocks of size \( m = (n - w)N/n \), formed by indices of zero positions, defines a \( B(v, m, \xi) \) design.
Conversely, any $q$-ary equidistant constant weight $(n, N, w, d)_q$ code $C$, satisfying Johnson bound (2), whose $n$ blocks of size $m = (n - w)N/n$, formed by indices of zero positions, defines a $B(v, m, \xi)$-design, induces an $m$-nearly resolvable $N RB_m(v, k, \lambda)$-design with parameters

\[ v = N, \quad k = \frac{wN}{(q - 1)n}, \quad \lambda = \frac{w(wN - (q - 1)n)}{n(q - 1)(N - 1)}, \quad m = \frac{(n - w)N}{n}. \]
Main results

First we formulate the main equivalence theorem, which is a natural extension of Theorem 2, involving a wider class of \( q \)-ary constant weight codes satisfying the Johnson upper bound (2).
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Main results

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**Theorem 3.**

Any $m$-specially resolvable $SRB_m(v, k, \lambda)$ design $D$ induces a $q$-ary equidistant constant weight $(n, N, w, d)_q$ code $C$ with parameters

\[
q = \frac{v - m + k}{k}, \quad n = \frac{\lambda v(v - 1)}{m(m - 1) + (v - m)(k - 1)}, \quad N = v, \quad w = n\frac{v - m}{v}
\]

meeting the Johnson bound (2).
Conversely, any $q$-ary equidistant constant weight $(n, N, w, d)_q$ code $C$, satisfying Johnson bound (2), induces an $m$-specially resolvable design $SRB_m(v, k, \lambda)$, where

\[ v = N, \quad m = \frac{(n - w)N}{n}, \quad k = \frac{wN}{(q - 1)n}, \quad \lambda = n - d. \]
Main results

In [B-L-Z, 2017] we derived several constructions of $m$-nearly resolvable designs (based on Steiner systems and super-simple designs) and corresponding $q$-ary equidistant constant weight codes meeting Johnson bound (2). In all that cases we obtain, according to Theorem 3, the corresponding $m$-specially resolvable designs $SRB_m(v, k, \lambda)$. In particularly, we have the following
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**Theorem 4.**

*Any Steiner system $S(v,k,t)$ induces a $(t - 1)$-specially resolvable design $SRB_{t-1}(v,k - t + 1,\lambda)$, where*

$$\lambda = \frac{m(m-1) + (v-m)(k-1)}{v(v-1)} \binom{v}{t-1}. \tag{11}$$
Main results

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The interesting question is, of course, the existence of $m$-specially resolvable $SRB_m(v, k, \lambda)$.
For the simplest case $k = 2$ and $\lambda = 1$ we can write out the parameters of the putative infinite family of such designs:

$$v = (m - 1)^3 + 1, \quad k = 2, \quad \lambda = 1, \quad m = 3, 4, 5, \ldots$$  \hspace{1cm} (12)

and corresponding $q$-ary codes with $q = m(m - 1)(m - 2)/2 + 1$:

$$n = (m^2 - 3m + 3)(m - 1), \quad N = (m - 1)^3 + 1, \quad w = (m - 1)^2(m - 2), \quad d = n-$$  \hspace{1cm} (13)
Main results

We give the two first \((6, 9, 4, 5)_4\) codes \(C_1\) and \(C_2\) from this family for the cases \(m = 3\) and \(m = 4\):
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\[
C_1 = \begin{bmatrix}
(0 & 1 & 1 & 0 & 2 & 1) \\
(0 & 2 & 2 & 1 & 0 & 2) \\
(0 & 3 & 3 & 2 & 1 & 0) \\
(1 & 0 & 1 & 1 & 3 & 0) \\
(2 & 0 & 2 & 0 & 1 & 3) \\
(3 & 0 & 3 & 3 & 0 & 1) \\
(1 & 1 & 0 & 2 & 0 & 3) \\
(2 & 2 & 0 & 3 & 2 & 0) \\
(3 & 3 & 0 & 0 & 3 & 2)
\end{bmatrix}
\]
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The second \((21, 28, 18, 20)_{13}\) code \(C_2\) of this family (for the case \((m = 4)\) also exists.
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$$
C_2 = \begin{bmatrix}
P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\
P_7 & P_8 & P_9 & P_{10} & P_{11} & P_{12}
\end{bmatrix}
$$

consisting of the 12 circulant matrices $P_1, ..., P_{12}$ of order 7.
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consisting of the 12 circulant matrices $P_1, \ldots, P_{12}$ of order 7.
Main results

We give the first rows $p_i$ of these matrices (the other rows are cyclic shifts of the first one):

\[
\begin{align*}
p_1 & = (0, 1, 2, 3, 3, 2, 1), \\
p_2 & = (0, 6, 5, 4, 3, 2, 1), \\
p_3 & = (0, 1, 2, 3, 4, 5, 6), \\
p_4 & = (0, 4, 5, 6, 6, 5, 4), \\
p_5 & = (1, 2, 10, 9, 8, 0, 7), \\
p_6 & = (1, 7, 0, 8, 9, 10, 2), \\
p_7 & = (0, 7, 8, 9, 9, 8, 7), \\
p_8 & = (6, 12, 11, 0, 5, 8, 9), \\
p_9 & = (6, 9, 8, 5, 0, 11, 12), \\
p_{10} & = (0, 10, 11, 12, 12, 11, 10), \\
p_{11} & = (10, 0, 3, 11, 12, 7, 4), \\
p_{12} & = (10, 4, 7, 12, 11, 3, 0).
\end{align*}
\]
Main results

We conjecture that for the case $m$, when $m^2 - 3m + 3$ is a prime power number, the corresponding code (13) and design (12) exist.


References


