A family of binary completely transitive codes and distance-transitive graphs

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Outline

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In particular, for any integer $\rho \geq 2$, there exist two codes in the constructed class of codes with $d = 3$, covering radius $\rho$ and length $\binom{4\rho}{2}$ and $\binom{4\rho+2}{2}$, respectively.

These new completely transitive codes induce as coset graphs a family of distance-transitive graphs of growing diameter.
Introduction

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Given any vector \(v \in \mathbb{F}_2^n\) its distance to the code \(C\) is

\[
d(v, C) = \min_{x \in C} \{d(v, x)\}
\]

and the covering radius of the code \(C\) is

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\rho = \max_{v \in \mathbb{F}_2^n} \{d(v, C)\}.
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$$\rho = \max_{v \in \mathbb{F}_2^n} \{d(v, C)\}.$$ 

For a given code $C$ with covering radius $\rho = \rho(C)$ define

$$C(i) = \{x \in \mathbb{F}_2^n : d(x, C) = i\}, \quad i = 1, 2, \ldots, \rho.$$
Definition 1.

(Neumaier, [1992]) A code $C$ with covering radius $\rho = \rho(C)$ is completely regular, if for all $l \geq 0$ and for every vector $x \in C(l)$ there are precisely:

- the same number $c_l$ of neighbors in $C(l - 1)$
- and
- the same number $b_l$ of neighbors in $C(l + 1)$. 

Define $a_l = n - b_l - c_l$ and note that $c_0 = b_\rho = 0$. 

Define the intersection array of $C$ as $(b_0, ..., b_{\rho - 1}; c_1, ..., c_\rho)$. 
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Define the intersection array of $C$ as $(b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho)$. 
For a given code $C$ with automorphism group $\text{Aut}(C)$ and any $\mathbf{x} \in \mathbb{F}_2^n$ and $\varphi \in \text{Aut}(C)$ the group acts on a coset $\mathbf{x} + C$ as

$$\varphi(\mathbf{x} + C) = \varphi(\mathbf{x}) + C.$$
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**Definition 2.**

(Solé, [1990]) A linear code $C$ with covering radius $\rho = \rho(C)$ and automorphism group $\text{Aut}(C)$ is completely transitive, if the set of all cosets of $C$ is partitioned into $\rho + 1$ orbits under action of $\text{Aut}(C)$. 

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Let \( \Gamma \) be a finite connected simple (i.e. undirected, without loops and multiple edges) graph. Let \( d(\gamma, \delta) \) be the distance between two vertices \( \gamma \) and \( \delta \), i.e. a numbers of edges in the minimal path between \( \gamma \) and \( \delta \).
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An automorphism of a graph $\Gamma$ is a permutation $\pi$ of the vertex set of $\Gamma$ such that, for all $\gamma, \delta \in \Gamma$ we have $d(\gamma, \delta) = 1$, if and only if $d(\pi \gamma, \pi \delta) = 1$. 
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Definition 3.

(Brouwer-Cohen-Neumaier [1989]) A simple connected graph $\Gamma$ is called *distance-regular*, if it is regular of valency $k$, and if for any two vertices $\gamma, \delta \in \Gamma$ at distance $i$ apart, there are precisely:

- $c_i$ neighbors of $\delta$ in $\Gamma_{i-1}(\gamma)$
- $b_i$ neighbors of $\delta$ in $\Gamma_{i+1}(\gamma)$. 

Furthermore, this graph is called *distance transitive*, if for any pair of vertices $\gamma, \delta \in \Gamma$ at distance $d(\gamma, \delta)$ there is an automorphism $\pi \in \text{Aut}(\Gamma)$ which moves this pair to any other given pair $\gamma', \delta' \in \Gamma$ at the same distance $d(\gamma', \delta') = d(\gamma, \delta)$. 

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(Brouwer-Cohen-Neumaier [1989]) A simple connected graph $\Gamma$ is called \textit{distance-regular}, if it is regular of valency $k$, and if for any two vertices $\gamma, \delta \in \Gamma$ at distance $i$ apart, there are precisely: $c_i$ neighbors of $\delta$ in $\Gamma_{i-1}(\gamma)$ and $b_i$ neighbors of $\delta$ in $\Gamma_{i+1}(\gamma)$. Furthermore, this graph is called \textit{distance transitive}, if for any pair of vertices $\gamma, \delta$ at distance $d(\gamma, \delta)$ there is an automorphism $\pi \in \text{Aut}(\Gamma)$ which moves this pair to any other given pair $\gamma', \delta'$ of vertices at the same distance $d(\gamma, \delta) = d(\gamma', \delta')$. 
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- length $n = \binom{m}{2}$,
- number of information symbols $k = n - m + 1$,
- minimum distance $d = 3$
- and covering radius $\rho = \lfloor m/2 \rfloor$.  

A half of these codes are non-antipodal and this implies (Borges-Rifa-Zinoviev, [2008]), that the covering set $C(\rho)$ of $C$ is a coset of $C$. In this case the union $C \cup C(\rho)$ gives also a completely regular and completely transitive code.
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Our purpose here is to describe the resulting linear completely transitive codes with growing covering radius and distance-transitive coset graphs with growing diameter.
Let $C$ be a linear completely regular code with covering radius $\rho$ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_{\rho})$. Let $\{D\}$ be the set of cosets of $C$. 
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Define the graph $\Gamma_C$ (which is called the coset graph of $C$), taking all cosets $D = C + x$ as vertices, with two vertices $\gamma = \gamma(D)$ and $\gamma' = \gamma(D')$ adjacent, if and only if the cosets $D$ and $D'$ contains neighbor vertices, i.e. $v \in D$ and $v' \in D'$ with distance $d(v, v') = 1$. 

Lemma 4.

(Brouwer-Cohen-Neumaier [1989], Rifà-Pujol, [1991]) Let $C$ be a linear completely regular code with covering radius $\rho$ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho)$ and let $\Gamma_C$ be the coset graph of $C$. 

Lemma 4.

(Brouwer-Cohen-Neumaier [1989], Rifà-Pujol, [1991]) Let $C$ be a linear completely regular code with covering radius $\rho$ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_{\rho})$ and let $\Gamma_C$ be the coset graph of $C$.

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Then $\Gamma_C$ is distance-regular of diameter $\rho$ with the same intersection array.

If $C$ is completely transitive, then $\Gamma_C$ is distance-transitive.
Lemma 5.

(Neumaier [1992]) Let $C$ be a completely regular code with covering radius $\rho$ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$. Then $C(\rho)$ is a completely regular code too, with intersection array $(c_\rho, \ldots, c_1; b_{\rho-1}, \ldots b_0)$. 
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Main results

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**Definition 6.**

Let $H_m$ be the binary matrix of size $m \times m(m - 1)/2$, whose columns are exactly all different vectors of length $m$ and weight 2. Now define the binary linear code $C^{(m)}$ whose parity check matrix is the matrix $H_m$. 
Theorem 7. (Rifa-Zinoviev, [2009]) Let $m$ be a natural number, $m \geq 3$. 

$C(m)$ is a binary linear code with parameters:

- $n = (m^2 - 2)$,
- $k = n - m + 1$,
- $d = 3$,
- $\rho = \lfloor m^2 \rfloor$.

The intersection numbers of $C(m)$ are:

- $b_i = (m - 2i^2)$,
- $c_i = (2i^2)$.

Code $C(m)$ is antipodal if $m$ is odd and non-antipodal if $m$ is even.
Theorem 7.

(Rifa-Zinoviev, [2009]) Let $m$ be a natural number, $m \geq 3$. The binary linear $[n, k, d]$ code $C^{(m)}$ has parameters:

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Code $C^{(m)}$ is completely transitive and, therefore, completely regular. The intersection numbers of $C^{(m)}$ for $i = 0, \ldots, \rho$ are:

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Code $C^{(m)}$ is antipodal if $m$ is odd and non-antipodal if $m$ is even.
Since for even $m$ the code $C^{(m)}$ is non-antipodal, its covering set $C^{(m)}(\rho)$ is a translate of $C^{(m)}$ (Borges-Rifà-Zinoviev, [2008]).
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$$C^{[m]} = C^{(m)} \cup C^{(m)}(\rho).$$
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C^{[m]} = C^{(m)} \cup C^{(m)}(\rho).
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The generating matrix \( G^{[m]} \) of this code has a very symmetric structure:

\[
G^{[m]} = \begin{bmatrix}
I_{k-1} & H_{m-1}^t \\
0 \ldots 0 & 1 \ldots 1
\end{bmatrix}.
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Using Lemma 5 and the fact that

$$C^{(m)}(\rho) = C^{(m)} + (1,1,\ldots,1),$$

we obtain the following result.
Theorem 8.

Let \( m \geq 6 \) be even. The code \( C[m] \) is completely transitive \([n, k, d]\) code with parameters
\[
    n = \frac{m(m-1)}{2}, \quad k = n - m + 2, \quad d = 3, \quad \rho = \lfloor m/4 \rfloor.
\]
Main results

Theorem 8.

Let \( m \geq 6 \) be even. The code \( C^m \) is completely transitive \([n, k, d]\) code with parameters
\[
n = m(m - 1)/2, \quad k = n - m + 2, \quad d = 3, \quad \rho = \lfloor m/4 \rfloor.
\]
The intersection numbers of \( C^m \) for \( m \equiv 0 \pmod{4} \) and \( \rho = m/4 \) are
\[
b_i = \binom{m-2i}{2}, \quad c_i = \binom{2i}{2}, \quad i = 0, 1, \ldots, \rho - 1,
\]
\[
c_\rho = 2 \binom{2\rho}{2}
\]
and, for \( m \equiv 2 \pmod{4} \) and \( \rho = (m - 2)/4 \), are
\[
b_i = \binom{m-2i}{2}, \quad c_i = \binom{2i}{2}, \quad i = 0, 1, \ldots, \rho.
\]
We note that the extension of the code $C^{[m]}$ (i.e. adding one more overall parity checking position) is not uniformly packed in the wide sense, and therefore, it is not completely regular (Brouwer et alt. [1989]).
Denote by $\Gamma^{(m)}$ (respectively, $\Gamma^{[m]}$) the coset graph, obtained from the codes $C^{(m)}$ (respectively, $C^{[m]}$) by Lemma 4. From Theorems 7 and 8 we obtain the following results, which leads to new coset graphs.
Main results

Theorem 9. For any even \( m \geq 6 \) there exist two embedded double covers \( \Gamma^{(m)} \) and \( \Gamma^{[m]} \) of complete graph \( K_n, n = \binom{m}{2} \), on \( 2^{m-1} \) and \( 2^{m-2} \) vertices, respectively, and with covering radius \( m/2 \) and \( \lfloor m/4 \rfloor \), respectively.
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The intersection arrays of graphs \( \Gamma^{(m)} \) and \( \Gamma^{[m]} \) are the same as the intersection arrays of codes, given by Theorems 7 and 8.
Theorem 9.

For any even $m \geq 6$ there exist two embedded double covers $\Gamma^{(m)}$ and $\Gamma^{[m]}$ of complete graph $K_n$, $n = \binom{m}{2}$, on $2^{m-1}$ and $2^{m-2}$ vertices, respectively, and with covering radius $m/2$ and $\lfloor m/4 \rfloor$, respectively. The intersection arrays of graphs $\Gamma^{(m)}$ and $\Gamma^{[m]}$ are the same as the intersection arrays of codes, given by Theorems 7 and 8. Both graphs $\Gamma^{(m)}$ and $\Gamma^{[m]}$ are distance transitive.
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The graphs \( \Gamma^{(m)} \) are imprimitive and the graphs \( \Gamma^{[m]} \) are primitive.

The graph \( \Gamma^{[m]} \) has eigenvalues \( \left\{ \frac{(m-4i)^2 - m}{2} : i = 0, 1, \ldots, \rho \right\} \).
The graph $\Gamma^{(m)}$ is well known. It can be obtained from the even weight binary vectors of length $m$, adjacent when their distance is 2. It is the halved $m$-cube and is a distance-transitive graph, uniquely defined from its intersection array (Brouwer et al. [1989]).
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Bibliography


