On maximal antipodal spherical codes with few distances

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Antipodal spherical codes

- $C \subset S^{n-1}$, $|C| < \infty$, spherical code
- If $C = -C$, then $C$ is called antipodal

**Problem:** Given dimension $n$, find the maximum possible cardinality of an antipodal code which under certain restrictions (for example, all inner products in $\{-1\} \cup [-s,s]$; or in $\{-1, \pm s\}$, etc.)

- Distance distribution with respect to $x \in C$ – the system $(A_t(x) : t \in [-1,1), \exists y \in C, \langle x, y \rangle = t)$, where

  $$A_t(x) = |\{y \in C : \langle x, y \rangle = t\}|.$$

  Obvious properties: $A_{-1}(x) = 1$ for every $x \in C$, $A_t(x) = A_{-t}(x)$ for every $t \in (-1,1)$ and every $x \in C$
For fixed dimension $n$, the Gegenbauer polynomials are defined by $P_0^{(n)} = 1$, $P_1^{(n)} = t$ and the three-term recurrence relation

$$(i + n - 2)P_{i+1}^{(n)}(t) = (2i + n - 2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t)$$

for $i \geq 1$

If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree $k$, then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as

$$f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$$
We use the identity

\[ |C| f(1) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^{k} f_i M_i \quad (1) \]

as a source of estimations by polynomial techniques. Here

\[ M_i := \frac{1}{r_i} \sum_{j=1}^{r_i} (\sum_{x \in C} Y_{ij}(x))^2 \]

is the \(i\)-th moment of \(C\), the functions \(\{Y_{i,j}, j = 1, 2, \ldots, r_i\}\), are the so-called spherical harmonics of degree \(i\), and \(r_i = \binom{n+i-3}{n-2}\frac{2i+n-2}{i}\).

\(C\) is antipodal iff \(M_i = 0\) for every odd \(i\). Further, a code \(C\) is a spherical \(\tau\)-design if and only if its moments satisfy \(M_i = 0\) for every positive integer \(i \leq \tau\).
Antipodal codes with inner products $-1$ and $\pm s$ (1)

- $C \subset S^{n-1}$ – antipodal, $M = |C|$, $C$ has inner products $-1$ and $\pm s$ (i.e. $C$ defines a system equiangular lines). Well known – if $M > 2n$ then $s = \frac{1}{2\ell+1}$, where $\ell$ is a positive integer. Denote by $M_{2\ell+1}(n)$ the maximum possible size of such $C$.


**Theorem**

(Barg, Yu) If $P^{(n)}_{2k}(\frac{1}{2\ell+1}) < 0$, then $M_{2\ell+1}(n) \leq 2 - \frac{2}{P^{(n)}_{2k}(\frac{1}{2\ell+1})}$.

**Proof.** Set $f(t) = P^{(n)}_{2k}(t)$ in (1).

□
Antipodal codes with inner products $-1$ and $\pm s$ (2)

- For $k = 1$ we have $P_2^{(n)}(t) = \frac{nt^2 - 1}{n - 1}$ and therefore
  \[
  M_{2\ell+1}(n) \leq \frac{8n\ell(\ell + 1)}{(2\ell + 1)^2 - n}
  \]
  (this is usually called relative bound) provided $n < (2\ell + 1)^2$.

- For $k = 2$ we have $P_4^{(n)}(t) = \frac{(n+2)(n+4)t^4 - 6(n+2)t^2 + 3}{n^2 - 1}$ and therefore
  \[
  M_{2\ell+1}(n) \leq \frac{2(n - 2)((2\ell + 1)^4(n + 2) + 6(2\ell + 1)^2 - n - 4)}{6(2\ell + 1)^2(n + 2) - 3(2\ell + 1)^4 - (n + 2)(n + 4)}
  \]
  provided $6(2\ell + 1)^2(n + 2) - 3(2\ell + 1)^4 - (n + 2)(n + 4) > 0$. The bound (2) is better than the relative bound for $n \geq 96$ and for every $\ell$. 

Generalizations?

- Free the inner products – consider codes with two possible inner products $a$ and $b$ (two-distance sets on $S^{n-1}$).

- Allow more inner products – this talk
Antipodal codes with inner products $-1, 0$ and $\pm s$ (1)

- $C \subset S^{n-1}$ – antipodal, $M = |C|$, inner products $-1, 0$ and $\pm s$, where $0 < s < 1$.

**Theorem**

If $s^2 < \frac{3}{n+2}$, then

$$M \leq \frac{2n(n + 2)(1 - s^2)}{3 - s^2(n + 2)}.$$  

(3)

**Proof.** Set $f(t) = t^2(t^2 - s^2)$ in (1).

If (3) is attained, then $M_2 = M_4 = 0$, i.e. $C$ is a spherical 5-design.

Then we compute the distance distribution $A_s(x) = A_s = \frac{M - 2n}{2ns^2}$, $A_0(x) = A_0 = M - 2 - 2A_s = \frac{M(ns^2 - 1) + n(1 - 2s^2)}{ns^2}$.
Antipodal codes with inner products $-1, 0$ and $\pm s$ (2)

We consider a derived code of $C$ to obtain a Lloyd-type theorem.

**Theorem**

If $C$ attains the bound (3) then $s$ is rational.

**Proof.** Some algebraic manipulations. □

**Theorem**

If $C$ is a spherical 3-design, $k \geq 2$ and $P_{2k}^{(n)}(s) + (ns^2 - 1)P_{2k}^{(n)}(0) < 0$, then

$$M \leq \frac{n \left( 2ns + (1 - 2s^2)P_{2k}^{(n)}(0) - P_{2k}^{(n)}(s) \right)}{|P_{2k}^{(n)}(s) + (ns^2 - 1)P_{2k}^{(n)}(0)|}.$$  \hspace{1cm} (4)

**Proof.** We set $f(t) = P_{2k}^{(n)}(t)$ in (1). □
Antipodal codes with inner products −1, ±s₁ and ±s₂ (1)

- \( C \subset S^{n-1} \) – antipodal, \( M = |C| \), inner products −1, ±s₁ and ±s₂, where \( 0 < s_1 < s_2 < 1 \). Again, we first derive the analog of the relative bound.

**Theorem**

If \( s_1^2 s_2^2 + \frac{3-(n+2)(s_1^2+s_2^2)}{n(n+2)} > 0 \) and \( 6 - (n + 4)(s_1^2 + s_2^2) > 0 \), then

\[
M \leq \frac{n(n + 2)(1 - s_1^2)(1 - s_2^2)}{n(n + 2)s_1^2 s_2^2 - (n + 2)(s_1^2 + s_2^2) + 3}.
\]

(5)

**Proof.** Set \( f(t) = (t^2 - s_1^2)(t^2 - s_2^2) \) in (1).

If (5) is attained, then \( C \) must be a spherical 5-design. Therefore

\[
A_{s_1} = \frac{M - 2n - ns_1^2(M - 2)}{2n(s_1^2 - s_2^2)}, \quad A_{s_2} = \frac{M - 2n - ns_2^2(M - 2)}{2n(s_1^2 - s_2^2)}.
\]
The investigation of the derived codes imply, similarly to the previous case, the following assertion.

**Theorem**

*If* $C$ *attains the bound* (5) *then* $s_1$ *are simultaneously rational or simultaneously irrational.*

*Proof.* By calculation of the distance distribution of the derived codes $C_{s_1}(x)$ and $C_{s_2}(x)$.

□
Analog of Theorems 1 and 4 follows from $M_{2k} \geq 0$.

**Theorem**

*If $C$ is a spherical 5-design, $k \geq 2$ and \((1 - ns_1^2)P_{2k}^{(n)}(s_1) + (1 - ns_2^2)P_{2k}^{(n)}(s_2) < 0\), then*

\[
M \leq \frac{2n \left( (1 - s_1^2)P_{2k}^{(n)}(s_1) + (1 - s_2^2)P_{2k}^{(n)}(s_2) + s_2^2 - s_1^2 \right)}{\left| (1 - ns_1^2)P_{2k}^{(n)}(s_1) + (1 - ns_2^2)P_{2k}^{(n)}(s_2) \right|}. \tag{6}
\]

*Proof. Set $f(t) = P_{2k}^{(n)}(t)$ in (1).*
Thank you for your attention!