Completely regular codes with different parameters and the same intersection arrays

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We obtain several classes of completely regular codes with different parameters, but identical intersection array.
We obtain several classes of completely regular codes with different parameters, but identical intersection array. Given a prime power $q$ and any two natural numbers $a, b$, we construct completely transitive codes over different fields with covering radius $\rho = \min\{a, b\}$ and identical intersection array, specifically, one code over $\mathbb{F}_{qr}$ for each divisor $r$ of $a$ or $b$. 
Let $\mathbb{F}_q$ be a finite field of the order $q$ and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A $q$-ary linear code $C$ of length $n$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. 
Introduction

Let $\mathbb{F}_q$ be a finite field of the order $q$ and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A $q$-ary linear code $C$ of length $n$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its distance to the code $C$ is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C} \{d(\mathbf{v}, \mathbf{x})\}$, and the covering radius of the code $C$ is $\rho = \max_{\mathbf{v} \in \mathbb{F}_q^n} \{d(\mathbf{v}, C)\}$. 
Let $\mathbb{F}_q$ be a finite field of the order $q$ and $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. A $q$-ary linear code $C$ of length $n$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. Given any vector $\mathbf{v} \in \mathbb{F}_q^n$, its distance to the code $C$ is $d(\mathbf{v}, C) = \min_{\mathbf{x} \in C} \{d(\mathbf{v}, \mathbf{x})\}$, and the covering radius of the code $C$ is $\rho = \max_{\mathbf{v} \in \mathbb{F}_q^n} \{d(\mathbf{v}, C)\}$. We say that $C$ is a $[n, k, d; \rho]_q$-code. Let $D = C + \mathbf{x}$ be a coset of $C$, where $+$ means the component-wise addition in $\mathbb{F}_q$. 
For a given $q$-ary code $C$ with covering radius $\rho = \rho(C)$ define

$$C(i) = \{ \mathbf{x} \in \mathbb{F}_q^n : d(\mathbf{x}, C) = i \}, \quad i = 1, 2, \ldots, \rho.$$
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Say that two vectors $x$ and $y$ are *neighbors* if $d(x, y) = 1$. 


Introduction

Definition 1.

(Neumaier, 1992) A $q$-ary code $C$ is completely regular, if for all $l \geq 0$ every vector $x \in C(l)$ has the same number $c_l$ of neighbors in $C(l-1)$ and the same number $b_l$ of neighbors in $C(l+1)$. 
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The group $\text{Aut}(C)$ acts on the set of cosets of $C$ in the following way: for all $\sigma \in \text{Aut}(C)$ and for every vector $v \in \mathbb{F}_q^n$ we have $(v + C)\sigma = v^\sigma + C$.

**Definition 2.**

(Sole, 1990; Giudici-Praeger, 1999) Let $C$ be a linear code over $\mathbb{F}_q$ with covering radius $\rho$. Then $C$ is completely transitive if $\text{Aut}(C)$ has $\rho + 1$ orbits when acts on the cosets of $C$.

Since two cosets in the same orbit should have the same weight distribution, it is clear, that any completely transitive code is completely regular.
In (Rifà-Zinoviev, 2010) we described an explicit construction, based on the Kronecker product of parity check matrices, which provides, for any natural number $\rho$ and for any prime power $q$, an infinite family of $q$-ary linear completely regular codes with covering radius $\rho$. 

In (Rifà-Zinoviev, 2011) we presented another class of $q$-ary linear completely regular codes with the same property, based on lifting of perfect codes. Here we extend the Kronecker product construction to the case when component codes have different alphabets and connect the resulting completely regular codes with codes obtained by lifting $q$-ary perfect codes. This gives several different infinite classes of completely regular codes with different parameters and with identical intersection arrays.
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Preliminary results

Definition 3.

For two matrices $A = [a_{r,s}]$ and $B = [b_{i,j}]$ over $\mathbb{F}_q$ define a new matrix $H$ which is the Kronecker product $H = A \otimes B$, where $H$ is obtained by changing any element $a_{r,s}$ in $A$ by the matrix $a_{r,s}B$. 

Definition 4.

Let $C$ be the $[n,k,d]_q$ code with parity check matrix $H$ where $1 \leq k \leq n - 1$ and $d \geq 3$. Denote by $C_r$ the $[n,k,d]_{q^r}$ code over $\mathbb{F}_{q^r}$ with the same parity check matrix $H$. Say that code $C_r$ is obtained by lifting $C$ to $\mathbb{F}_{q^r}$. 

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Let $C$ be the $[n, k, d]_q$ code with parity check matrix $H$ where $1 \leq k \leq n - 1$ and $d \geq 3$. Denote by $C_r$ the $[n, k, d]_{qr}$ code over $\mathbb{F}_{qr}$ with the same parity check matrix $H$. Say that code $C_r$ is obtained by lifting $C$ to $\mathbb{F}_{qr}$.
Denote by $C(H)$ the code defined by the parity check matrix $H$. 
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Denote by $C(H)$ the code defined by the parity check matrix $H$. By $H_m^q$ we denote the parity check matrix of the $q$-ary Hamming $[n, n - m, 3]_q$ code $C = C(H_m^q)$ of length $n = (q^m - 1)/(q - 1)$. By $C_r(H_m^q)$ we denote the code (of the same length $n = (q^m - 1)/(q - 1)$) obtained by lifting $C(H_m^q)$ to the field $\mathbb{F}_{q^r}$. 
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Let $C(H^q_{m_a})$ and $C(H^q_{m_b})$ be two Hamming codes with parameters $[n_a, n_a - m_a, 3]_{q^u}$ and $[n_b, n_b - m_b, 3]_q$, respectively, where $n_a = (q^{u m_a} - 1)/(q^u - 1)$, $n_b = (q^{m_b} - 1)/(q - 1)$, $q$ is a prime power, $m_a, m_b \geq 2$, and $u \geq 1$. 
Theorem 5.

(i) The code $C$ with parity check matrix $H = H_{m_a}^{q u} \otimes H_{m_b}^{q}$, the Kronecker product of $H_{m_a}^{q u}$ and $H_{m_b}^{q}$, is a completely transitive, and so completely regular, $[n, k, d; \rho]_{q^u}$ code with parameters

$$n = n_a n_b, \quad k = n - m_a m_b, \quad d = 3, \quad \rho = \min\{u m_a, m_b\}. \quad (1)$$
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(ii) The code $C$ has the intersection numbers:

\[ b_\ell = \frac{(q^u m_a - q^\ell)(q^{m_b} - q^\ell)}{(q - 1)}, \quad c_\ell = q^{\ell - 1} \frac{q^\ell - 1}{q - 1}. \]
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(iii) The lifted code $C_{m_b}(H_{u m_a}^q)$ is a completely regular code with the same intersection array as $C$. 
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(i) There exist the following completely regular codes with different parameters \([n, k, d; \rho]_q\), where \( d = 3 \) and \( \rho = \min\{ua, b\} \):

- \( C_{ua}(H_q^b) \) over \( \mathbb{F}_{qa} \) with \( n = \frac{q^b - 1}{q - 1} \), \( k = n - b \);
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(i) There exist the following completely regular codes with different parameters $[n, k, d; \rho]_{qr}$, where $d = 3$ and $\rho = \min\{ua, b\}$:

- $C_{ua}(H^q_b)$ over $\mathbb{F}^{ua}_q$ with $n = \frac{q^b - 1}{q-1}$, $k = n - b$;
- $C_b(H^q_{ua})$ over $\mathbb{F}^{b}_q$ with $n = \frac{q^{ua} - 1}{q-1}$, $k = n - ua$;
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$C_b(H^q_{ua})$ over $\mathbb{F}^b_q$ with $n = \frac{q^{ua}-1}{q-1}$, $k = n - ua$;

$C(H^q_b \otimes H^q_{ua})$ over $\mathbb{F}_q$ with $n = \frac{q^{ua}-1}{q-1} \times \frac{q^b-1}{q-1}$, $k = n - bua$.
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Let $q$ be a prime number and $a, b, u$ natural numbers. Then:
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$C_{ua}(H_b^q)$ over $\mathbb{F}_q^{ua}$ with $n = \frac{q^b-1}{q-1}$, $k = n - b$;

$C_b(H_{ua}^q)$ over $\mathbb{F}_q^b$ with $n = \frac{q^{ua}-1}{q-1}$, $k = n - ua$;

$C(H_b^q \otimes H_{ua}^q)$ over $\mathbb{F}_q$ with $n = \frac{q^{ua}-1}{q-1} \times \frac{q^b-1}{q-1}$, $k = n - bua$;

$C(H_b^q \otimes H_u^q)$ over $\mathbb{F}_q^a$ with $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$, $k = n - bu$;
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$C_b(H_{ua}^q)$ over $\mathbb{F}_q^{b}$ with $n = \frac{q^{ua-1}}{q-1}$, $k = n - ua$;

$C(H_q^b \otimes H_{ua}^q)$ over $\mathbb{F}_q$ with $n = \frac{q^{ua-1}}{q-1} \times \frac{q^{b-1}}{q-1}$, $k = n - bua$;

$C(H_q^b \otimes H_{ua}^q)$ over $\mathbb{F}_q^a$ with $n = \frac{q^{b-1}}{q-1} \times \frac{q^{ua-1}}{q^{u-1}}$, $k = n - bu$;

$C(H_q^b \otimes H_{ua}^q)$ over $\mathbb{F}_q^u$ with $n = \frac{q^{b-1}}{q-1} \times \frac{q^{ua-1}}{q^{u-1}}$, $k = n - ba$;
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Theorem 6.

Let $q$ be a prime number and $a, b, u$ natural numbers. Then:
(i) There exist the following completely regular codes with different parameters $[n, k, d; \rho]_{qr}$, where $d = 3$ and $\rho = \min\{ua, b\}$:

- $C_{ua}(H_b^q)$ over $\mathbb{F}_{aq}$ with $n = \frac{q^b-1}{q-1}$, $k = n - b$;
- $C_b(H_{ua}^q)$ over $\mathbb{F}_b^q$ with $n = \frac{q^{ua}-1}{q-1}$, $k = n - ua$;
- $C(H_b^q \otimes H_{ua}^q)$ over $\mathbb{F}_q$ with $n = \frac{q^{ua}-1}{q-1} \times \frac{q^b-1}{q-1}$, $k = n - bua$;
- $C(H_b^q \otimes H_{u}^{a})$ over $\mathbb{F}_a^q$ with $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$, $k = n - bu$;
- $C(H_b^q \otimes H_{u}^{a})$ over $\mathbb{F}_u^q$ with $n = \frac{q^b-1}{q-1} \times \frac{q^{ua}-1}{q^u-1}$, $k = n - ba$;
Main results

Theorem 6 (continuation)

(ii) All the above codes have the same intersection numbers

\[ b_\ell = \frac{(q^b - q^\ell)(q^{ua} - q^\ell)}{(q - 1)}, \quad \ell = 0, \ldots, \rho - 1, \quad c_\ell = \frac{q^{\ell - 1}q^\ell - 1}{q - 1}, \quad \ell = 1, \ldots, \rho \]
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(iii) All codes above coming from Kronecker constructions are completely transitive.
Distance-regular graphs

Let $\Gamma$ be a finite connected simple graph (i.e., undirected, without loops and multiple edges). Let $d(\gamma, \delta)$ be the distance between two vertices $\gamma$ and $\delta$ (i.e., the number of edges in the minimal path between $\gamma$ and $\delta$). The diameter $D$ of $\Gamma$ is its largest distance. Two vertices $\gamma$ and $\delta$ from $\Gamma$ are neighbors if $d(\gamma, \delta) = 1$. Define

$$\Gamma_i(\gamma) = \{\delta \in \Gamma : d(\gamma, \delta) = i\}.$$

An automorphism of a graph $\Gamma$ is a permutation $\pi$ of the vertex set of $\Gamma$ such that, for all $\gamma, \delta \in \Gamma$ we have $d(\gamma, \delta) = 1$ if and only if $d(\pi \gamma, \pi \delta) = 1$. 
Distance-regular graphs

Definition 7.

Brouwer-Cohen-Neumaier, 1989) A simple connected graph $\Gamma$ is called distance-regular if it is regular of valency $k$, and if for any two vertices $\gamma, \delta \in \Gamma$ at distance $i$ apart, there are precisely $c_i$ neighbors of $\delta$ in $\Gamma_{i-1}(\gamma)$ and $b_i$ neighbors of $\delta$ in $\Gamma_{i+1}(\gamma)$. Furthermore, this graph is called distance-transitive, if for any pair of vertices $\gamma, \delta$ at distance $d(\gamma, \delta)$ there is an automorphism $\pi$ from $\text{Aut}(\Gamma)$ which moves this pair $(\gamma, \delta)$ to any other given pair $\gamma', \delta'$ of vertices at the same distance $d(\gamma, \delta) = d(\gamma', \delta')$. 
Let $C$ be a linear completely regular code with covering radius $\rho$ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots c_\rho)$. Let $\{B\}$ be the set of cosets of $C$. Define the graph $\Gamma_C$, which is called the coset graph of $C$, taking all different cosets $B = C + x$ as vertices, with two vertices $\gamma = \gamma(B)$ and $\gamma' = \gamma(B')$ adjacent if and only if the cosets $B$ and $B'$ contain neighbor vectors, i.e., there are $v \in B$ and $v' \in B'$ such that $d(v, v') = 1$. 
Lemma 8.

(Brouwer-Cohen-Neumaier, 1989; Rifà-Pujol, 1991) Let $C$ be a linear completely regular code with covering radius $\rho$ and intersection array $(b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho)$ and let $\Gamma_C$ be the coset graph of $C$. Then $\Gamma_C$ is distance-regular of diameter $D = \rho$ with the same intersection array. If $C$ is completely transitive, then $\Gamma_C$ is distance-transitive.
Theorem 9.

Let $C_1, C_2, \ldots, C_k$ be a family of linear completely transitive codes constructed by Theorem 5 and let $\Gamma_{C_1}, \Gamma_{C_2}, \ldots, \Gamma_{C_k}$ be their corresponding coset graphs. Then:
(i) Any graph $\Gamma_{C_i}$ is a distance-transitive graph, induced by bilinear forms.
(ii) If any two codes $C_i$ and $C_j$ have the same intersection array, then the graphs $\Gamma_{C_i}$ and $\Gamma_{C_j}$ are isomorphic.
(iii) If the graph $\Gamma_{C_i}$ has $q^m$ vertices, where $m$ is not a prime, then it can be presented as a coset graph by several different ways, depending on the number of factors of $m$. 
References