A binary block concatenated code based on two convolutional codes

Igor Zhilin  Victor Zyablov  Dmitry Zigangirov

Institute for Information Transmission Problems
Russian Academy of Sciences

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Outline

- Construction Description and Encoding
- Derivation of Code Distance
  - upper bounding
  - lower bounding
- Conclusion
We consider a \textit{block} code that uses \textit{terminated} convolutional codes as component codes. Let us start with information matrix:

\[
I = \begin{bmatrix}
    k_B \\
    k_A
\end{bmatrix}
\]
Construction Description // Encoding Outer

- At first it is read in row-wise order and encoded by the outer convolutional coder, \( I = \)

\[
\begin{array}{c}
\left[ \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{array} \right]
\end{array}
\]

\[
\{ k_A \}
\]

\[
\{ k_B \}
\]

- The resulting matrix is written in row-wise order too, \( I_A = Enc_B(I) = \)

\[
\begin{array}{c}
\left[ \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\cdots \\
\end{array} \right]
\end{array}
\]

\[
\{ k_A \}
\]

\[
\{ n_B \}
\]
Then $I_A$ is read in column-wise order by inner convolutional code encoder, $I_A = $ 

And written in the same column-wise order to a matrix that is a codeword, $C = Enc_A(I_A) = Enc_A(Enc_B(I)) =$
The result is a codeword:

\[ \mathbf{C} = \text{Enc}_A(\text{Enc}_B(\mathbf{I})) = \]

Shaded cells correspond to parity-check symbols. Red cells schematically depict minimal-weight codeword.

Note: a single encoder is used for encoding all rows. Then a single encoder is used for encoding all columns.
**Notations**

\[ R_A = \frac{b_A}{c_A} \] — rate of inner code,
\[ d_A \] — free distance of inner code (in binary symbols),
\[ f_A \] — maximum length of word (packet) of inner code that has weight \( d_A \), measured in \( c_A \)-tuples,

\[ R_B, b_B, c_B, d_B, f_B \] — the same for outer code.

Let us consider code construction where \( n_A \geq f_A c_A \),
\( n_B \geq f_B c_B \). That means that the longest word of minimal weight of outer/inner code fits in a single row/column (probably with wrapping).
Theorem

There exist such sizes $n'_A$ and $n'_B$ that binary block concatenated code based on two convolutional codes with $n_A \geq n'_A$ and $n_B \geq n'_B$ has minimum Hamming distance $d = d_A d_B$, where $d_A$ and $d_B$ are free distances of inner and outer codes respectively.
Upper Bound

- To prove $d \leq d_Ad_B$ we can just provide an example of codeword of weight $w = d_Ad_B$
- Since we’ve chosen $n_B \geq f_Bc_B$, we can place a sequence of weight $w' = d_B$ in any rows of $I_A$. These rows would be independent since such sequence has length $f_Bc_B$ that is less than row width $n_B$.
- We should place that sequences in rows of $I_A$ in a such way that nonzero symbols would form information sequences of smallest weight $d_A$ in columns of $C$.
- This encoding procedure yields a codeword that has $d_A$ rows of weight $d_B$, or, alternatively, $d_B$ columns of weight $d_A$, thus its weight $w = d_Ad_B$. 
Lower Bound

Let us prove it from the encoding standpoint.

Encoding of the outer code is just a plain encoding of the convolutional code with arbitrary input. Its output sequence has at least \( d_B \) nonzero symbols. Since we’ve chosen \( n_B \geq f_B c_B \), all these bits would be in different columns of \( I_A \) yielding at least \( d_B \) nonzero columns.

Now we should consider two options:

1. In case the columns would be encoded by the inner code independently from column to column, the result is straightforward: it yields a codeword similar to the one considered for upper bound (probably with wrapped rows or columns). Encoding of each column by the inner encoder gives a word of weight at least \( d_A \), so in this case \( d \geq d_A d_B \).

2. Counting for dependencies in columns-to-column encoding requires use of active distances of inner code.
Active Distances

- A concept of active distance was introduced in 1999 by Host et. al.\textsuperscript{1}.
- Active distances lower bound weight of a code sequence generated by a coder that does not pass through two consequent zero states.
- Authors\textsuperscript{1} proved that convolutional codes with active distances that grow with sequence length and are lower-bounded by a linearly increasing function exist . . .
- and also showed a couple of examples of known codes where increasing active distances are seen.

Active Distances

▶ Example of active column distance curve from $^1$:

![Graph showing active column distances](image)

▶ Let us write a bound on the active column distance $a_j^r$ in simplified form:

$$a_j^r \geq uj + v$$

(1)

where $u > 0$ is a constant that depends on code properties, $j$ is a sequence length in $c_A$-tuples.
Lower Bound (continues)

Since we need two consequent columns to have weight of at least $2d_A$, three columns to have weight $3d_A$ and so on, we need to choose such $n_A$ that active column distance

$$a'_j \geq sd_A, s \in \overline{1, n_B},$$

(2)

where $j = sn_A/c_A$.

and (after a couple of transformations)

$$n_A \geq d_A c_A/u = \text{const}$$

(3)

This ends the proof of $d \geq d_A d_B$ and thus $d = d_A d_B$. 

Conclusion

We proved that code distance of binary block concatenated code based on two convolutional codes equals $d = d_A d_B$ — the product of free distances of component codes for large enough $n_A$ and $n_B$.

This construction differs from other constructions of concatenated codes based on convolutional codes:

- It is not a convolutional code like the one proposed in ²
- It doesn’t use separate codes for each row and each column like in, i.e., ³

Thank you for your attention.