Some parallelisms of PG(3,5) involving a definite type of spread

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Parallelisms – relations and applications

Johnson, Combinatorics of Spreads and Parallelisms, CRC Press (2010)

- translation planes
- network coding
- error-correcting codes
- cryptography

Definitions and notations

- *Spread* in *PG(n,q)* - a partition of the point set by lines
- *Parallelism* in *PG(n,q)* – a partition of the set of lines by spreads
- necessary condition for the existence of spreads: *n* is odd

Definitions and notations

- ***Isomorphic spreads*** - if there is an automorphism of PG(n,q) which takes one to the other
- ***Isomorphic parallelisms*** - if there is an automorphism of PG(n,q) which maps each spread of one parallelism to a spread of the other
- ***Automorphism*** of a parallelism – an automorphism of PG(n,q) which preserves the parallelism

Definitions and notations

- ***Automorphism group*** of the parallelism – subgroup of the automorphism group of PG(n,q)
- ***Deficiency one*** parallelism – a partial parallelism with one spread less than the parallelism
- ***Cyclic*** parallelism – it has an automorphism moving its spreads in one cycle

Definitions and notations

- PG(3,q) : q^2+1 lines in a spread; q^2+q+1 spreads in a parallelism
- ***Regulus*** of PG(3,q) – a set $R=\{l_1, \ldots, l_{q+1}\}$ of mutually skew lines

$$\left.\begin{aligned} l \cap l_i &= p_i, \\ l \cap l_j &= p_j, \\ l \cap l_k &= p_k, \end{aligned}\right\} \Rightarrow l \cap l_s \neq \emptyset, \ \forall l_s \in R$$

Definitions and notations

- ***Regular spread***

$$S=\{l_1, \ldots, l_{q^2+1}\} \text{ of } PG(3,q)\text{: } R(l_i, l_j, l_k) \subset S$$

- ***Regular parallelism*** – all its spreads are regular.
- ***Uniform parallelism*** – all its spreads are isomorphic.

History

General constructions of parallelisms:

- in ***PG(n,2)***, Zaicev, Zinoviev, Semakov, 1973; Baker, 1976.
- in ***PG(2^n-1,q)***, Beutelspacher, 1974.
- a pair of orthogonal parallelisms in ***PG(3,q)*** – Fuji-Hara,1986.
- two infinite families of regular cyclic parallelisms, ***PG(3,q), q ≡ 2 (mod 3)***, Penttila and Williams, 1998.

History

Parallelisms in *PG(3,q)*:

- ***PG(3,2)*** – all (2) are classified, regular;
- ***PG(3,3)***
  - aut(5) – Prince, 1997;
  - all (73 343) – Betten, 2016;
- ***PG(3,4)***
  - odd prime order – Topalova, Zhelezova, 2013, 2015, 2017;
  - Baer involution – Betten, Topalova, Zhelezova, 2018;
  - cyclic groups of order 4 – Betten, Topalova, Zhelezova, 2019

The number of known parallelisms of PG(3,4) with nontrivial automorphisms

Aut	2	3	4	5	6	7	8	10	12	15	16	17	20	24	30	32	48	60	64	96	960
#	$\geq$ 286 340	8 115 559	$\geq$ 25 1836	31 830	4 488	482	$\geq$ 596	76	52	40	$\geq$ 170	0	52	14	38	$\geq$ 14	12	8	$\geq$ 4	2	4

G $\cong$ PΓL(4,4) **|G|** $= 2^{13} \, . \, 3^4 \, . \, 5^2 \, . \, 7 \, . \, 17$

History

Parallelisms in ***PG(3,5)***:

- cyclic – Prince, 1998;
- regular noncyclic – Topalova, Zhelezova, 2016;
- automorphism of order 13 – Topalova, Zhelezova, 2019.

Regularity of the spreads of PG(3,5)

#	N_6	N_4
1	130	0
2	31	105
3	16	246
4	10	192
5	7	120
6	7	150
7	5	200
8	4	78
9	4	102
10	3	237

#	N_6	N_4
11	1	82
12	1	138
13	1	210
14	0	72
*15	0	104
16	0	114
17	0	180
18	0	190
19	0	225
20	0	310

N_i – the number of reguli in PG(3,5) which have i common lines with a spread, $i \in \{4,6\}$

Construction method

PG(3,5)

lexicographic order on the points $\rightarrow$ lines $\rightarrow$ parallelisms

v = $(q^{n+1} - 1)/(q-1)$ = ***156 points***

$(q^2+1)(q^2+q+1)$ = ***806 lines***

q^2+1 = ***26 lines*** in a spread

q^2+q+1 = ***31 spreads*** in a parallelism

(1,0,0,0) $\rightarrow$ 1

...

(4,4,4,1) $\rightarrow$ 156

Construction method

PΓL(4,5) and its Sylow ***2***-subgroup

$G \cong P\Gamma L(4,5)$, $|G| = 2^9.3^2.5^6.13.31$

G_{2^9} – 12 conjugacy classes, elements of orders 2, 4 and 8

class	group G_8	fixed points	fixed lines	$N(G_8)$
1	$\boxed{G_{8_2}}$	2	2	384
2	G_{8_6}	6	2	5760

$N(G_8) = \{ g \in G \mid gG_8g^{-1} = G_8\}$ – the normalizer of G_8 in G

https://www.gap-system.org/

Construction method

Types of line orbits under G_{82}

group	fixed lines	length 2		length 4		length 8	
		SL	NSL	SL	NSL	SL	NSL
G_2	38	240	144	–	–	–	–
G_4	38	–	–	120	72	–	–
G_{82}	2	12	6	–	–	36	60

SL (spread-like) line orbit – each point appears at most once;

NSL (non-spread-like) line orbit – only in a nonfixed spread;

O_l - a line orbit of length l under G_{82}

Construction method

Types of spread orbits under G_{82}

- 16 fixed spreads

a line orbit of lenght 8 under G_{82}

F_1 :

O_1^1	O_1^2	O_8^1	O_8^2	O_8^3

- 2832 spread orbits of length 2

a line orbit of length 4 under G_4

S_2 :

o_1^1	o_1^2	o_1^3	o_1^4	o_1^5	o_1^6	o_4^1	o_4^2	o_4^3	o_4^4	o_4^5
$o_1^{1'}$	$o_1^{2'}$	$o_1^{3'}$	$o_1^{4'}$	$o_1^{5'}$	$o_1^{6'}$	$o_4^{1'}$	$o_4^{2'}$	$o_4^{3'}$	$o_4^{4'}$	$o_4^{5'}$

a line orbit of length 2 under G_{82}

a line orbit of length 8 under G_{82}

- L_8: 14227090 spread orbits of lenght 8 – 26 lines from 26 different line orbits under G_{82}

Construction method

Parallelism construction

P :	F_1	S_2	S_2	S_2	L_8	L_8	L_8

$\underbrace{F_1 \; S_2 \; S_2 \; S_2}_{\text{partial parallelism}}$

partial parallelism

Normalizer automorphisms	8	16	24	32	48	96
Partial parallelisms	10279	683	11	9	7	1

automorphisms of order 3

Construction method

Computer search

- Backtrack search, construction of necessary spread orbits:
  - in advance;
  - on the fly – a line is added if it meets the requirements of the spread type, and has not been used yet.
- Isomorph rejection
  - a normalizer-based minimality test;
  - invariant calculation.

Results

Parallelisms with G_{82}

Automorphisms	**8**	**16**	**24**	**32**	**48**	**All**
Parallelisms	630	154	85	16	14	899
Selfdual	24	0	3	0	0	27

- each spread is in one of the 20 classes;
- 227 invariants:
    - the order of the full automorphism group;
    - selfduality;
    - the number of class of each spread

Results

Parallelisms with the previously missing spread

F_1	$3xS_2$	L_8	L_8	L_8	automorphisms	parallelisms
4	2	5	13	**20**	8	4
4	2	14	17	**20**	8	2
4	10	4	14	**20**	8	8
4	10	5	10	**20**	8	4
4	10	11	11	**20**	16	2
4	10	11	14	**20**	8	8
4	**20**	5	15	16	8	2
4	**20**	5	15	17	8	2
4	**20**	8	8	10	8	2
4	**20**	11	11	11	8	2
4	**20**	11	13	15	8	2
4	**20**	11	15	19	8	2

Results

Parallelisms with regular spreads

F_1	$3xS_2$	L_8	L_8	L_8	automorphisms	parallelisms
4	10	**1**	**1**	**1**	24	2
4	10	**1**	15	15	8	2

Results

Invariants of spreads which yield uniform deficiency one parallelisms

F_1	$3xS_2$	L_8	L_8	L_8	automorphisms	parallelisms
4	10	10	10	10	16	40
4	10	10	10	10	24	2
4	10	10	10	10	48	8

Thank you for the attention