

Some parallelisms of $PG(3,5)$ involving a definite type of spread

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Parallelisms – relations and applications

Johnson, Combinatorics of Spreads and Parallelisms, CRC Press (2010)

- translation planes
- network coding
- error-correcting codes
- cryptography

Definitions and notations

- **Spread** in $PG(n,q)$ - a partition of the point set by lines
- **Parallelism** in $PG(n,q)$ – a partition of the set of lines by spreads
- necessary condition for the existence of spreads: **n** is odd

Definitions and notations

- **Isomorphic spreads** - if there is an automorphism of $PG(n,q)$ which takes one to the other
- **Isomorphic parallelisms** - if there is an automorphism of $PG(n,q)$ which maps each spread of one parallelism to a spread of the other
- **Automorphism** of a parallelism – an automorphism of $PG(n,q)$ which preserves the parallelism

Definitions and notations

- **Automorphism group** of the parallelism – subgroup of the automorphism group of $PG(n,q)$
- **Deficiency one** parallelism – a partial parallelism with one spread less than the parallelism
- **Cyclic** parallelism – it has an automorphism moving its spreads in one cycle

Definitions and notations

- PG(3,q) : q^2+1 lines in a spread; q^2+q+1 spreads in a parallelism
- Regulus** of PG(3,q) – a set $R=\{l_1, \dots, l_{q+1}\}$ of mutually skew lines

$$\left. \begin{array}{l} l \cap l_i = p_i, \\ l \cap l_j = p_j, \\ l \cap l_k = p_k, \end{array} \right\} \Rightarrow l \cap l_s \neq \emptyset, \forall l_s \in R$$

Definitions and notations

- **Regular spread**

$$S = \{l_1, \dots, l_{q^2+1}\} \text{ of } PG(3, q): R(l_i, l_j, l_k) \subset S$$

- **Regular parallelism** – all its spreads are regular.

- **Uniform parallelism** – all its spreads are isomorphic.

History

General constructions of parallelisms:

- in $PG(n, 2)$, Zaicev, Zinoviev, Semakov, 1973; Baker, 1976.
- in $PG(2^n - 1, q)$, Beutelspacher, 1974.
- a pair of orthogonal parallelisms in $PG(3, q)$ – Fuji-Hara, 1986.
- two infinite families of regular cyclic parallelisms, $PG(3, q)$, $q \equiv 2 \pmod{3}$, Penttila and Williams, 1998.

History

Parallelisms in $PG(3,q)$:

- **$PG(3,2)$** – all (2) are classified, regular;
- **$PG(3,3)$**
 - $\text{aut}(5)$ – Prince, 1997;
 - all (73 343) – Betten, 2016;
- **$PG(3,4)$**
 - odd prime order – Topalova, Zhelezova, 2013, 2015, 2017;
 - Baer involution – Betten, Topalova, Zhelezova, 2018;
 - cyclic groups of order 4 – Betten, Topalova, Zhelezova, 2019

The number of known parallelisms of PG(3,4) with nontrivial automorphisms

Aut	2	3	4	5	6	7	8	10	12	15	16	17	20	24	30	32	48	60	64	96	960
#	$\geq 286\,340$	$8\,115\,559$	$\geq 25\,1836$	$31\,830$	$4\,488$	482	≥ 596	76	52	40	≥ 170	0	52	14	38	≥ 14	12	8	≥ 4	2	4

$$G \cong \text{P}\Gamma\text{L}(4,4) \quad |G| = 2^{13} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$$

History

Parallelisms in $PG(3,5)$:

- cyclic – Prince, 1998;
- regular noncyclic – Topalova, Zhelezova, 2016;
- automorphism of order 13 – Topalova, Zhelezova, 2019.

Regularity of the spreads of PG(3,5)

#	N_6	N_4
1	130	0
2	31	105
3	16	246
4	10	192
5	7	120
6	7	150
7	5	200
8	4	78
9	4	102
10	3	237

#	N_6	N_4
11	1	82
12	1	138
13	1	210
14	0	72
*15	0	104
16	0	114
17	0	180
18	0	190
19	0	225
20	0	310

N_i – the number of reguli in PG(3,5) which have i common lines with a spread, $i \in \{4,6\}$

Construction method

PG(3,5)

lexicographic order on the points \rightarrow lines \rightarrow parallelisms

$$v = (q^{n+1} - 1)/(q-1) = \mathbf{156 \text{ points}}$$

$$(q^2+1)(q^2+q+1) = \mathbf{806 \text{ lines}}$$

$$q^2+1 = \mathbf{26 \text{ lines}}$$
 in a spread

$$q^2+q+1 = \mathbf{31 \text{ spreads}}$$
 in a parallelism

$$(1,0,0,0) \rightarrow 1$$

...

$$(4,4,4,1) \rightarrow 156$$

Construction method

$P\Gamma L(4,5)$ and its Sylow **2**-subgroup

$$G \cong P\Gamma L(4,5), |G| = 2^9 \cdot 3^2 \cdot 5^6 \cdot 13 \cdot 31$$

G_{29} – 12 conjugacy classes, elements of orders 2, 4 and 8

class	group G_8	fixed points	fixed lines	$N(G_8)$
1	G_{82}	2	2	384
2	G_{86}	6	2	5760

$N(G_8) = \{ g \in G \mid gG_8g^{-1} = G_8 \}$ – the normalizer of G_8 in G

<https://www.gap-system.org/>

Construction method

Types of line orbits under G_{82}

group	fixed lines	length 2		length 4		length 8	
		SL	NSL	SL	NSL	SL	NSL
G_2	38	240	144	–	–	–	–
G_4	38	–	–	120	72	–	–
G_{82}	2	12	6	–	–	36	60

SL (spread-like) line orbit – each point appears at most once;

NSL (non-spread-like) line orbit – only in a nonfixed spread;

O_l - a line orbit of length l under G_{82}

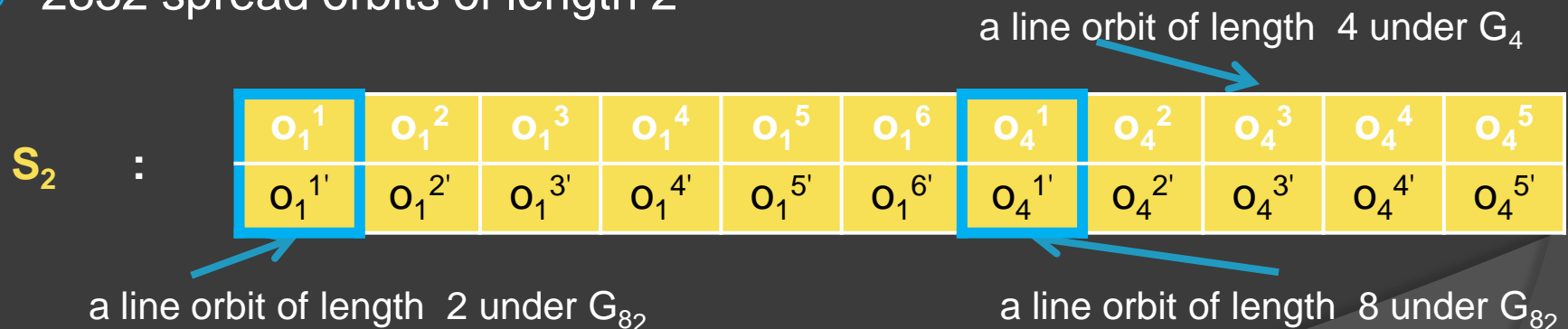
Construction method

Types of spread orbits under G_{82}

- 16 fixed spreads



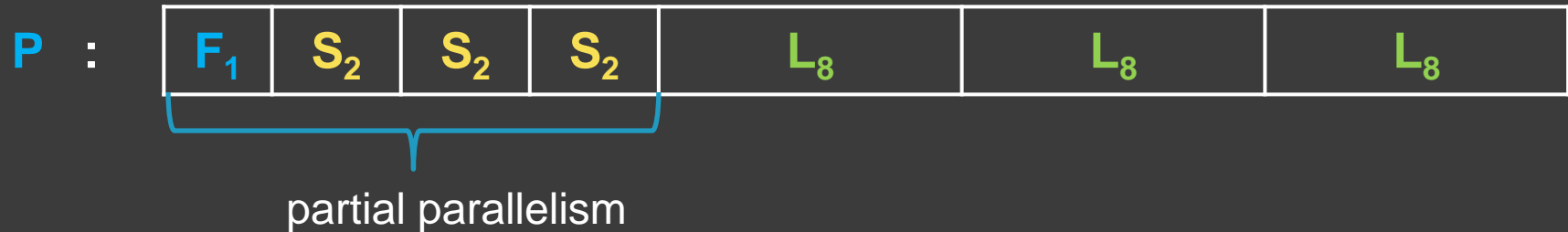
- 2832 spread orbits of length 2



- L_8 : 14227090 spread orbits of length 8 – 26 lines from 26 different line orbits under G_{82}

Construction method

Parallelism construction



Normalizer automorphisms	8	16	24	32	48	96
Partial parallelisms	10279	683	11	9	7	1

automorphisms of order 3

Construction method

Computer search

- Backtrack search, construction of necessary spread orbits:
 - in advance;
 - on the fly – a line is added if it meets the requirements of the spread type, and has not been used yet.
- Isomorph rejection
 - a normalizer-based minimality test;
 - invariant calculation.

Results

Parallelisms with G_{82}

Automorphisms	8	16	24	32	48	All
Parallelisms	630	154	85	16	14	899
Selfdual	24	0	3	0	0	27

- each spread is in one of the 20 classes;
- 227 invariants:
 - the order of the full automorphism group;
 - selfduality;
 - the number of class of each spread

Results

Parallelisms with the previously missing spread

F_1	$3xS_2$	L_8	L_8	L_8	automorphisms	parallelisms
4	2	5	13	20	8	4
4	2	14	17	20	8	2
4	10	4	14	20	8	8
4	10	5	10	20	8	4
4	10	11	11	20	16	2
4	10	11	14	20	8	8
4	20	5	15	16	8	2
4	20	5	15	17	8	2
4	20	8	8	10	8	2
4	20	11	11	11	8	2
4	20	11	13	15	8	2
4	20	11	15	19	8	2

Results

Parallelisms with regular spreads

F_1	$3xS_2$	L_8	L_8	L_8	automorphisms	parallelisms
4	10	1	1	1	24	2
4	10	1	15	15	8	2

Results

Invariants of spreads which yield
uniform deficiency one parallelisms

F_1	$3xS_2$	L_8	L_8	L_8	automorphisms	parallelisms
4	10	10	10	10	16	40
4	10	10	10	10	24	2
4	10	10	10	10	48	8

Thank you for the attention