Affine Data Modeling by Low-Rank Approximation

I. Markovsky

The low-rank approximation problem

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{D} \quad \|D - \widehat{D}\|_{\mathrm{F}} \\ \text{subject to} & \operatorname{rank}(\widehat{D}) \leq r, \end{array} \tag{1}$$

where $\|\cdot\|_{\mathrm{F}}$ is the Frobenius norm, is a ubiquitous *linear* data modeling tool. Indeed, assuming without loss of generality that D has more columns than rows, the aim of (1) is to fit optimally the columns of D (the data) by a subspace of dimension at most r (the linear model).

This paper studies the application of low-rank approximation for *affine data modeling*, i.e., optimal data fitting by an affine set. The relevant optimization problem in this case is

$$\begin{array}{ll} \text{minimize} & \text{over } \widehat{D} \text{ and } c \quad \|D - c\mathbf{1}^\top - \widehat{D}\|_{\mathrm{F}} \\ \text{subject to} & \operatorname{rank}(\widehat{D}) \leq r, \end{array} \tag{2}$$

where $\mathbf{1}$ is the vector of appropriate dimension with all elements equal to one.

We prove that a solution for the parameter c in (2) is the mean of the columns of D. Therefore, affine data modeling reduces to linear data modeling. From an optimization point of view, (2) decouples into two independent optimization problems:

$$\text{minimize} \quad \text{over } c \quad \|D - c\mathbf{1}^\top\|_{\mathrm{F}} \tag{3}$$

and (1), where D is replaced by $D - c\mathbf{1}^\top$, with c computed from (3).

A solution of (2), however, is *not unique* even when a solution of (1) is unique. This is due to nonuniqueness of the shift parameter c. The general solution for c is given by a particular solution plus a vector in the column span of $\widehat{D}$.