Revisiting Preconditioning: An Interesting Result and the Lessons Learned from It

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In 1988 L. Hemmingson considered a semi-circulant preconditioner for a finite-difference discretization of the reaction diffusion equation in the unit square Ω. That is,

$$Au = \epsilon \Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + du = f, \quad (x, y) \in \Omega$$

$$u = g(x, y), \quad (x, y) \in \partial\Omega.$$

The coefficients a, b are constant and $d \leq 0$.
Her results seemed to be in conflict with earlier results of Manteuffel and Parter (1990). In 2003 Kim and Parter [KP] returned to this problem and clarified the situation. In addition, they discussed the limiting behavior of the finite-difference equations.
In this work, we explain these matters. We then use other results ($\epsilon = 1$) of [KP] to discuss the computational results of a 2001 paper by Hemmingson and Wathen on preconditioning finite-difference equations for the Navier-Stokes equations.
Then we use the techniques developed in [KP] to study the limiting behavior ($\epsilon \downarrow 0$) of the solutions of the boundary-value problem

$$\epsilon \Delta u + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0, \quad (x, y) \in R$$

$$u = g(x, y), \quad (x, y) \in \partial R$$

where

$$R = \{(x, y); -1 < x < 1, -1 < y < 1\}.$$

We observe that the point $(0, 0)$ is a "stagnation" point at which the reduced equation becomes singular. Finally, we discuss the behavior of the solutions of the finite-difference equations for this problem.