On a New Function in the Dynamic Software

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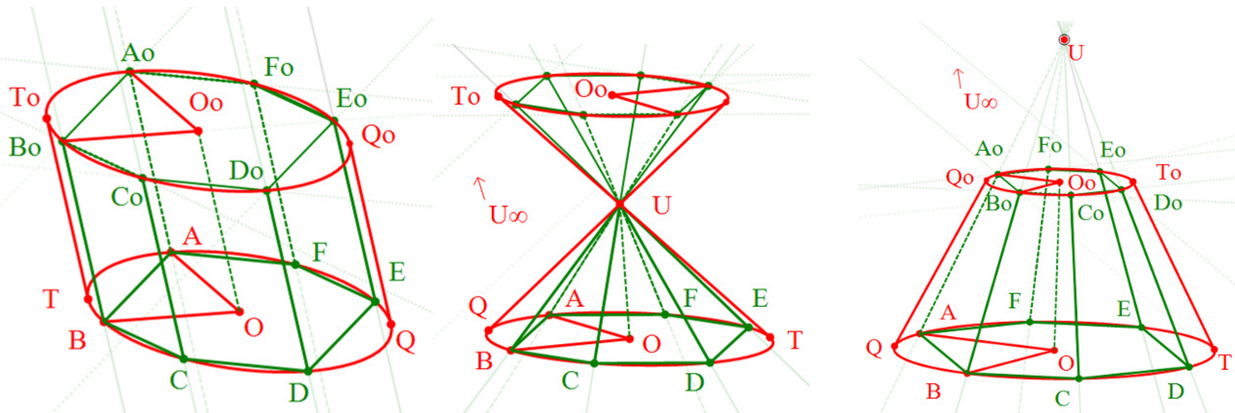
Abstract

The Dynamic Geometric System Sam (which is written in C # in the NET Framework 4 environment) [1] was created as an educational software for the needs of the subject Synthetic Geometry. The unveiling of the common projective root for large groups of problems is in the basis of our innovative approach, which is eased by the Dynamic Geometric Software Sam. This allows a generalization of propositions that are related to the school mathematics. The new propositions are easy for the university students and are understandable for the studious pupils.

The new function “Swap finite & infinite points” in the menu of the Dynamic Geometric Software Sam optimizes the drawing of the sketches; it also develops a research style of thinking in the students, which is an important element in the modernization of the teaching of mathematics.

For instance, the edges of a parallelepiped are connected with three infinite points U_∞, V_∞, W_∞. The swapping of U_∞, V_∞, W_∞ with the free points U, V, W allows the user to transform the parallelepiped in no time into prism, truncated pyramid, pyramid - with bases parallelogram, trapezium, quadrilateral or triangle, with a special situation of a chosen surrounding edge. The constructions made for one of the solids are carried in the new one.



The Sam *software proposes a fast preparation of the auxiliary sketches, which are connected with inscribed or circumscribed polyhedra.*

By the help of the Fibonacci *project we would like to share and spread our experience in teaching the students to be creative by means of the new function.*

Key words

Inquiry based approach, dynamic software, swap of finite and infinite points.

1. Introduction

The Dynamic Geometric Software *Sam* (which is written in C # in the NET Framework 4 environment) [1] was created as an educational Software for the needs of the subject Synthetic Geometry. The innovative approach in the teaching of Synthetic Geometry [2] has converted the software in a natural bridge for the knowledge obtained at university and at school. The basis of our approach is the discovery of the common projective roots for large classes of problems and all this was eased by the dynamic environment *Sam*. This allows the generalization of propositions, which are studied at school. The new propositions are easy for the university students and understandable for the studious pupils. The knowledge of the way and the instruments by which the generalizations are made teaches the future teachers how to discover new problems from old ones.

2. Swap of finite & infinite points

The presence of the infinite point in the instruments of the Dynamic Geometric Software *Sam* and the function "Swap of finite & infinite points", which is a new function for the Dynamic Geometric Software, has turned out to be an important tool for the formation of a creative style of thinking, which is a particularly important point for the modernization of the teaching in mathematics. This function, selected from the "select" menu, optimizes the drawing of the sketches. For example we consider a parallelepiped with its axis sections.

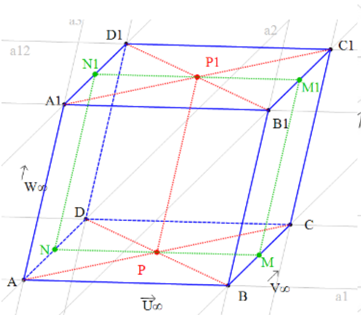


figure 1.1

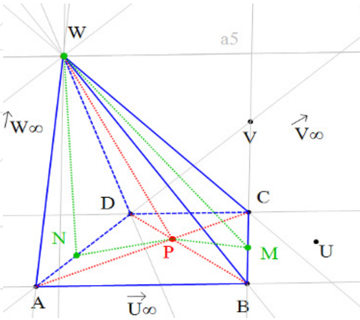


figure 1.2

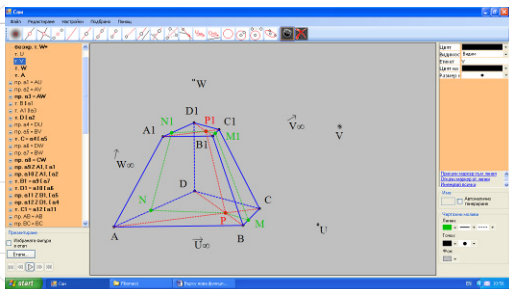


figure 1.3

The edges of a parallelepiped are connected with three infinite points U_∞, V_∞, W_∞. Swapping consecutively U_∞, V_∞, W_∞ with the free points *U*, *V*, *W* allows the user to transform the parallelepiped in no time into prism, truncated pyramid, pyramid with its base being a parallelogram, a trapezium, a quadrilateral or a triangle, with a special position of a chosen surrounding edge. Some of these possibilities are presented in figures 1.1, 1.2 and 1.3. All the constructions made for one of the figures are preserved for the new one, but which of the relations preserve their properties and which do not is part of the investigations.

We would like to share and spread via the Fibonacci project our experience with two practical examples demonstrating how to teach the students to be creative by the new function.

The first example is connected with the application of the Desargues' Theorem for perspective triangles.

Desargues' Theorem for perspective triangles: *The connecting lines of the couples of corresponding vertexes of two triangles ABC and A'B'C' are intersecting at a point S if and only if the intersection points of the couples of corresponding sides* $P=BC\cap B'C'$, $Q=AC\cap A'C'$, $R=AB\cap A'B$ *lie at a line s.*

The two triangles that satisfied the conditions of the above theorem are called ***perspective***. The point *S* is called ***perspective center*** and the line *s* is called a ***perspective axis***. They can be either finite or infinite objects.

The Desargues' Theorem allows us to situate the vertices of the two perspective triangles on two lines. In this way we get the next basic problem:

Problem 1: *Let g and g´ be two lines with $g \cap g'=T$ and let $A, B, C \in g$, and $A', B', C' \in g'$. Prove that if the lines AA´, BB´, CC´ intersect at a point S, then the points T, $P=AB' \cap A'B$, $Q=BC' \cap B'C$, lie at a line s.*

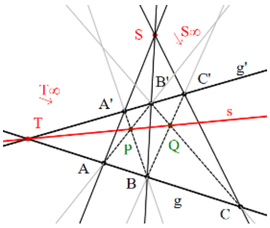


figure 2.1

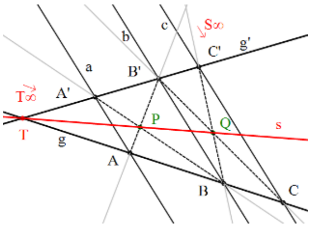


figure 2.2

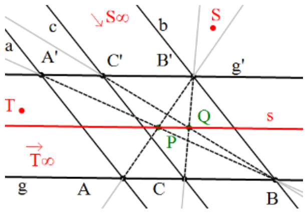


figure 2.3

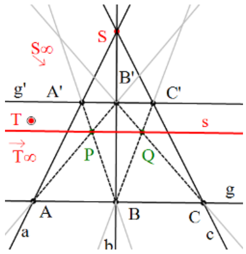


figure 2.4

Solution: (figure 2) The point *S* is a perspective center for the couple of triangles *AB´C* and *A´BC´*. The triangles satisfy the conditions of Desargues' Theorem. Then the intersection points of the couples of the corresponding sides $Q=BC' \cap B'C$, $P=AB' \cap A'B$, $T=AC \cap A'C'$ are collinear.

The statement of Problem 1 holds true in the four different cases shown in figure 2. We sketch only one of the cases and with the help of the function "Swap finite & infinite points" we obtain the other three automatically. When *S* and *T* are finite and infinite points (figure 2.2) respectively, we obtain Problem 89 of [3], which is solved by homothety and the complementary requirement for the lines *A'B* and *B'C* to be parallel. The usage of the dynamic geometric systems reveals that this complementary condition is not necessary. The projective method gives the shortest solution of this problem for all of the four cases.

Problem 1 appears as a repeating element in the solution of the next three problems. We have stated it just to help the reader.

Problem 2: *Let ABCD be a parallelogram and the point K be in its interior. Let the lines p and q be passing through K and $p \parallel AD$ u $q \parallel AB$. Let denote $R=p \cap AB$, $S=p \cap CD$, $M=q \cap AD$, $N=q \cap BC$, $O=AC \cap BD$. Prove that the points K, O, L, where L is the intersection point of the bimedians of the quadrilateral MRNS, are collinear* ([3], Problem. 90).

Solution: (figure 3) Let us denote the common infinite points of the lines *AB* and *CD* by U_∞, the common infinite point of the lines *BC* and *AD* by V_∞ and also $1=AK \cap MR$, $2=BK \cap NR$, $3=CK \cap SN$, $4=DK \cap MS$.

By the conditions $p \parallel AD \parallel BC$ and $q \parallel AB \parallel CD$ it follows that we can apply Problem 1 (figure 2.3) for the lines *AB* and *q*, *CD* and *q*, *AD* and *p*, *BC* and *p*. We consider the triads of points (*A, R, B*) и (*M, K, N*); (*D, S, C*) and (*M, K, N*); and by Problem 1 we get that the triads of points (*1, 2,* U_∞) and (*3, 4,* U_∞) are collinear. Consequently the lines *12* and *34* are parallel. Similarly we consider the triads of points (*A, M, D*) and (*R, K, S*); (*B, N, C*) and (*R, K, S*) and by Problem 1 (figure 2.3) we get that the triads of points (*1, 4,* V_∞) and (*2, 3,* V_∞) are collinear. Thus the lines *14* and *23* are parallel, too. So we conclude that the quadrilateral *1234* is a parallelogram with sides parallel to the sides of the parallelogram *ABCD*. By the condition for a collinearity of the triads of points (*A,1,K*); (*B,2,K*); (*C,3,K*); (*D,4,K*), we can conclude that the parallelograms *ABCD* and *1234* are corresponding for a homothety *h* with a center *K*. The points *1*, *2*, *3*, *4* are the intersection points of the diagonals of the four parallelograms *ARKM*, *BRKN*, *CNKS*, *DSKM*. That is why it follows that they are the middles of the sides of the quadrilateral *RNSM* and the coefficient of *h* is *2*. By *h(A,B,C,D)=1,2,3,4* we obtain that the intersection points of the diagonals $O=AC \cap BD$ and $L=13 \cap 24$ are corresponding for *h*, i.e. the points *K*, *L*, *O* are collinear and *KO=2KL*.

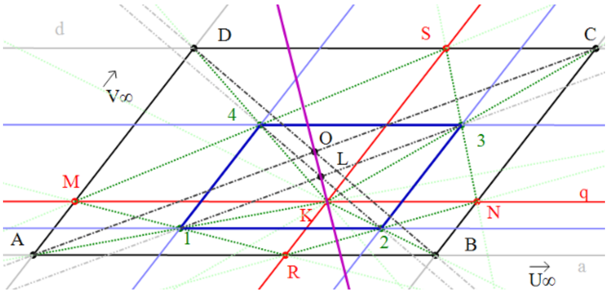


figure 3

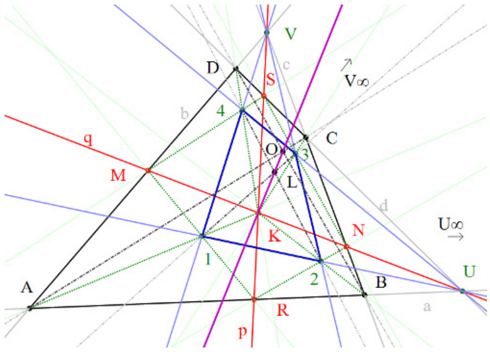


figure 4

Now we can generalize the problem for an arbitrary quadrilateral *ABCD,* because the parallelogram can be considered as a complete quadrilateral with two infinite diagonal points, and the homothety is a homology with a finite center and an axis the infinite line. The exchange of the arbitrary quadrilateral with a complete quadrilateral will remove the restriction on the point *K* to belong in the interior of the quadrilateral.

Problem 3: Let *ABCD be a complete quadrilateral with diagonal points U=AB∩CD, V=AD∩BC and let the point K be arbitrary and not lying on its sides. Construct the lines p=KV and q=KU, and denote R=p∩AB, S=p∩CD, M=q∩AD, N=q∩BC, O=AC∩BD, 1=AK∩MR, 2=BK∩NR, 3= CK∩SN, 4=DK∩MS, L=13∩24. Prove that the points K, O, L are collinear*.

Solution: (figure 4) By the condition that the lines *q*, *AB, CD* intersect at the point *U* and the lines *p, AD, BC* intersect at the point *V* it follows that we can apply Problem 1 (figure 2.1) for the lines *AB* and *q, CD* and *q, AD* and *p, BC* and *p*. Considering the triads of points (*A, R, B)* and (*M, K, N)*; (*D, S, C)* and (*M, K, N)* by Problem 1 (figure .2.1) we get that the triads of points (*1, 2, U)* and (*3, 4, U)* are collinear. Consequently *12∩34=U*. Similarly let consider the triads of points (*A, M, D)* and (*R, K, S)*; (*B, N, C)* and (*R, K, S). B*y Problem 1 (figure 2.1) we get that the triads of points (*1, 4, V) and* (*2, 3, V)* are collinear. Consequently *14∩23=V*. So we have established that the complete quadrilaterals *ABCD* and *1234* have common diagonal points *U* and *V*.

Let us consider a homology *Φ* with a center *K,* axis the line *UV* and a couple of corresponding points *A, 1*. By the properties of the homology we have that *Φ(ABCD)=1234*. Therefore *Φ(O=AC∩BD)=Φ(AC)∩Φ(BD)=13∩24=L*, which means that the points *O, K, L* are collinear.

Problem 3 is interesting for university students as well as for excellent high-school students.

The vertical integration of the teaching started upwards. Now we will change its direction downwards. It is enough to make one of the diagonal points finite by using the function "Swap finite & infinite points" – for instance let us swap *U* with U_∞. Now *ABCD* transforms into a trapezium and we get the next problem.

Problem 4: *Let ABCD be a trapezium with AB || CD and AD ∩ BC=V and let K be an arbitrary point in its interior. Construct two lines through the point K: p=KV and q‖AB‖CD. Denote R=p∩AB, S=p∩CD, M=q∩AD, N=q∩BC, 1=AK∩MR, 2=BK∩NR, 3=CK∩SN, 4=DK∩MS, O=AC∩BD, L=13∩ 24. Prove that the quadrilateral 1234 is a trapezium; with legs which intersect at the point V and prove that the points K, O, L are collinear.*

Hint: (figure 5) For the proof that the triads of points (*1, 4, V)* and (*2, 3, V)* are collinear apply Problem 1 (case 2.1.) and for the proof that the triads of points (*1, 2,* U_∞) and (*3, 4,* U_∞) are collinear apply Problem 1 (case 2.4). The homology will be with a center *K* and axis $o=VU_\infty$, i.e. a line trough the point *V*, which is parallel to the bases of the trapezium.

It is enough to sketch one of the figures 3, 4, 5 and the other two will be made by the new function "Swap finite & infinite points" of the Dynamic Geometric Software Sam.

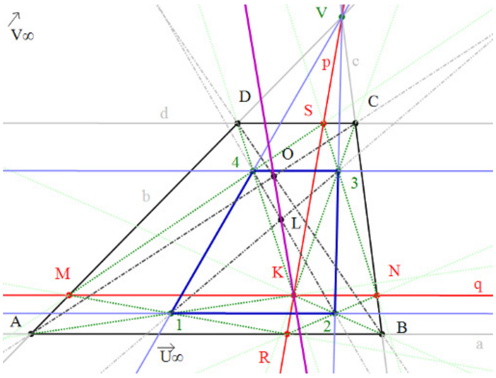


figure 5

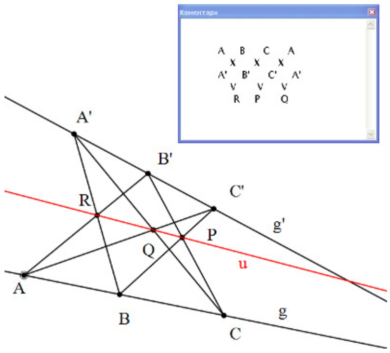


figure 6

The second example is connected with Pappus' Theorem:

Pappus' Theorem: *Let A, B, C be three different points lying on the line g and A´, B´, C´ be three different points lying on the line g´. Then the points* $P=BC'\cap B'C$, $Q=CA'\cap C'A$, $R=AB'\cap A'B$ *lie at a* line *u*. (figure 6)

A symbolic notation of obtaining the points *P*, *Q*, and *R* is given above figure 6.

Let's consider the next problem:

Problem 5: *Let ABCD be a parallelogram and K be a point in its interior. Two lines p and q, passing through the point K are constructed so that* $p\parallel AD$ *and* $q\parallel AB$*. Denote by* $R=p\cap AB$, $S=p\cap CD$, $M=q\cap AD$, $N=q\cap BC$, $T=AN\cap CR$, $Q=BS\cap DN$, $G=CM\cap AS$, $P=DR\cap BM$*. Prove that the* triads *of points (P, K, C); (T, K, D); (Q, K, A); (G, K, B) are collinear.*

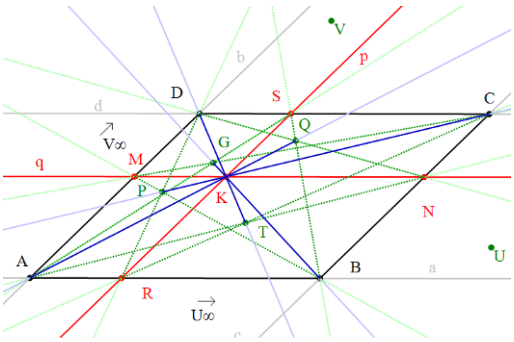


figure 7

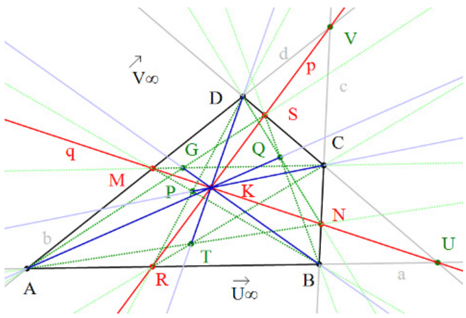


figure 8

Solution: (figure 7) Let us denote the common infinite point of the lines *AB* and *CD* by U_∞ and the common infinite point of the lines *BC* and *AD* by V_∞. By applying Pappus' Theorem four times in succession for the couples of triads of points: *(B, R,* U_∞) and (*D, M,* V_∞); (*A, R,* U_∞) and (*C, N,* V_∞); (*D, S,* U_∞) and (*B, N,* V_∞); (*C, S,* U_∞) and (*A, M,* V_∞) we obtain that the triads of points (*P, K, C); (T, K, D)*; ([3], problem 83), (*Q, K, A); (G, K, B)* are collinear.

Remark 1: The solution in ([3] problem 83) uses homothety and Thales' Theorem.

Remark 2: *We would like to point out that the projective approach for solving the problem is giving not only a short prove, but mainly it allows us to construct very easy sequences of new problems from the old one. This feature of the projective method and the function "Swap finite & infinite points" realize the vertical integration of the teaching at school and at university. First we have generalized* ([3], *Problem 83). By establishing that the point K lies in the segments DT, AQ, CP, BG; as well as in the segments RS and MN, we can use the set of the couples of perspective triangles with a perspective center K and to find new* triads *of collinear points. For example the triangles MRA and NSQ are perspective and therefore it follows that the points*

E=MR∩NS, D=MA∩NQ, B=RA∩SQ are collinear ([4] problem 28, p. 71). The describing of all the possible cases of couples of perspective triangles is a nice combinatorial problem. By this open problem we present a horizontal integration in the teaching. The investigation of the mutual position of the perspective axis is eased by the dynamic environment of the software.

Let us generalize problem 5 for an arbitrary quadrilateral *ABCD.*

Problem 6: Let *ABCD be a complete quadrilateral with diagonal points U=AB∩CD*, *V=AD∩BC* and let the point *K be arbitrary and not lying on its sides.* Let us construct the lines *p=KV* and *q=KU*. Denote *R=p∩AB, S=p∩CD, M=q∩AD, N=q∩BC, T=AN∩CR, Q=BS∩DN, G=CM∩AS, P=DR∩BM*. Prove that the following triads of points: (*P, K, C); (T, K, D); (Q, K, A); (G, K,B) are collinear.*

Solution: The usage of the function "Swap finite & infinite points" for the points U_∞, *U and* V_∞, *V* presents us in no time the sketch of the problem (figure 8). The solution is literally the same as that of Problem 5 just with replacing the points U_∞, V_∞ by *U, V,* respectively.

The last problem and the problems which it generates (Remark 2 holds true also for Problem 6), enters in the course of Synthetic Geometry for university students. They are interesting for mathematically gifted high-school students, too.

It is not difficult to realize that we can state a new problem that is suitable for all students. It is enough to choose one of the diagonal points of the complete quadrilateral *ABCD* to be an infinite point U_∞. Thus we get the next problem

Problem 7: *Let ABCD be a trapezium with legs AD and BC that intersect at a point V and let K be an arbitrary point in its interior and not lying on its sides. Construct two lines through the point K: p=KV and q*‖*AB*‖*CD. Denote R=p∩AB, S=p∩CD, M=q∩AD, N=q∩BC, T=AN∩CR, Q=BS∩DN, G=CM∩AS, P=DR∩BM. Prove that the* triads *of points (P, K, C); (T, K, D); (Q, K, A); (G, K, B) are collinear.*

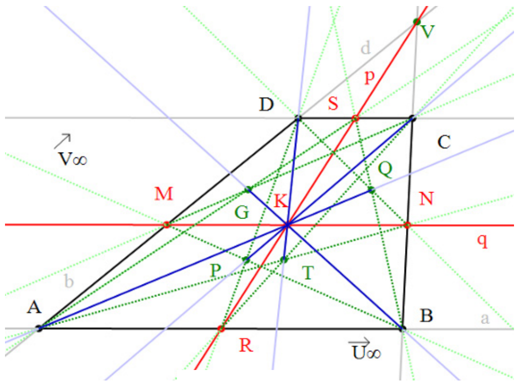


figure 9

Let us point out that it is enough to sketch just one of the figures 7, 8, 9 and the other two are generated in no time with the help of the special function "Swap finite & infinite points" of the Dynamic Geometric Software Sam.

Dynamic drawings of the paper can be found at *http://fmi-plovdiv.org/GetResource?id=1178*

3. Conclusion

The new function "Swap finite & infinite points" of the Dynamic Geometric Software Sam induces and facilitates the students' research.

Reference

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