Periodic Process Modeling via sin *x*

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Abstract

Trigonometric functions and, in particular, the $\sin x$ *function are largely used for studying, analyzing and visualizing different facts and effects. They could be of both, natural origin or as a result of human activities. Some examples of natural effects that include the usage of the* $\sin x$ *function include (and are not limited to): light spreading and electromagnetic waves, working with spring mechanisms, pendulum-based mechanisms etc. Using GeoGebra gives us the possibility to thoroughly study and analyze the characteristics and functions related to these processes, thus the possibility to deeply dive into the subject of understanding and applying trigonometric functions [1].*

The presented article can be used for out-of-class activities in mathematics, after the respective course related to trigonometric functions has already been run and the concept of angles with measure ranging from + to – infinity has been introduced.

Key words

Inquiry-based learning, Project Fibonacci, Dynamic Software, $\sin x$

1. Period and Frequency

Looking at a moving pendulum, we can state that its movement is performed with a variable velocity. The motion is shown in Fig. 1. If we stop the pendulum in its rightmost position, its velocity equals zero. Leaving the pendulum running from there, due to Earth gravitational forces, its velocity increases until the point of equilibrium is reached. Once there, the pendulum velocity starts decreasing, again due to the Earth gravitational forces. Its velocity equals again zero in its leftmost position. Given that there is no outside interaction with the object, the pendulum will perform the same movement from the left to the right limit, however, in the opposite direction.

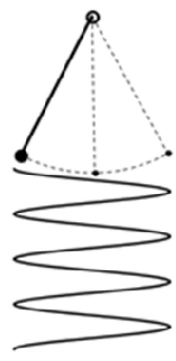


Fig. 1.

Let us remind that the time needed for the pendulum to perform one full oscillation, i.e. to move through the way described, only depends on its length. This time is called *pendulum period*. This is the period of its velocity variation.

Let us compare the velocity variation for the pendulum movement with the behaviour of the function $f(x)=\sin x$ for $x\in[0,2\pi]$. It starts with $\sin 0=0$. The values increase to $\sin\frac{\pi}{2}=1$ and decrease to $\sin\frac{3\pi}{2}=-1$ going through $\sin\pi=0$; then, again increase to $\sin 2\pi=0$.

The similarity between the two processes suggests using the sinusoid in order to describe the pendulum velocity variation. We cannot expect that the period of our pendulum is exactly 2π. Since this period is exactly defined and unvariable in each specific case, we need to find a way to change the period of the function $f(x)=\sin x$.

What do we need to change in the function $f(x)=\sin x$ in order to obtain a function for which the period is greater or lower than 2π? Let us multiply the independent *x* variable with the positive number *k* and observe the function: $g(x)=\sin kx$.

- ❋ What is the period of the function $g(x)=\sin kx$ compared to the period of the function $f(x)=\sin x$?
- ❋ Draw the graph of the function $\sin x$. Create a slider $k\in[0,5]$ with a step (granularity) of 1. Draw the graph of the function $\sin kx$.

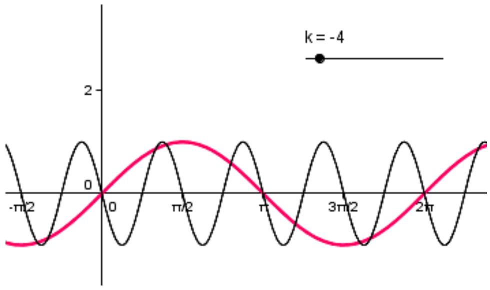


Fig. 2

- ❋ Examine the graphs for the intervals (0, 1) and (1, 5). What did you observe?
- ❋ Derive the formula for the period of the function $\sin kx$ for $k>0$.
- ❋ Perform the same analysis and draw conclusions for the negative values of *k.*
- ❋ How should we change the formula so that it is still valid for the negative parameter?
- ❋ Write down the respective conclusions.

In the real-world scenarios, in physics, techniques and electrical engineering the *k* coefficient is called *oscillation frequency* and is denoted by ω. It is closely related to the characteristics of many periodic phenomena and processes. Its unit of measure is called Hertz and is denoted by Hz; it has the format of 1/sec and it denotes the number of process cycles per second.

Some examples:

- **angle speed** or **angle frequency** – for rotating mechanisms – It is the unit of measure for the velocity obtained during rotation
- **sound frequency** – the number of vibrations per second (e.g. between 16 and 25,000 Hz), particularly for ultrasound (> 25,000 Hz) or infrasound (<16 Hz)
- **electromagnetic wave frequency** – part of them are visible: infrared, ultraviolet, roentgen and gamma waves; electric and magnetic field vibrations spreading through the space.

2. Amplitude

Let us take a look at the pendulum movement itself. It is performed over an arc that is part of a circumference. The projection of this is a segment whose center is corresponding to the point of equilibrium. Let us denote the segment's length by 2*a*. If we look upon the segment's center as on the starting point, i.e. point 0, the projection varies from $-a$ to $+a$. This process is performed periodically without any variations, unless external forces are applied. Let us make a parallel to the second characteristics described for the $f(x)=\sin x$ function. Again, we bump on certain similarities that lead us to using this function. Since we cannot expect $a = 1$ every time, we shall change its initial presentation.

Let us create a new function $g(x) = a\sin x,\ a \neq 0$.

The new coefficient that is frequently a parameter in a modeling is called *amplitude* and is usually denoted by *A*.

In the practice it happens frequently to observe and describe periodic processes, for which the amplitude is not a constant but is a function of the variable *x* instead.

- ❋ Draw sample graphs for the function $f(x) = a\sin x$ by replacing the constant *a* with a linear function (Fig. 3), a quadratic function (Fig. 4) or a trigonometric function (Fig. 5)

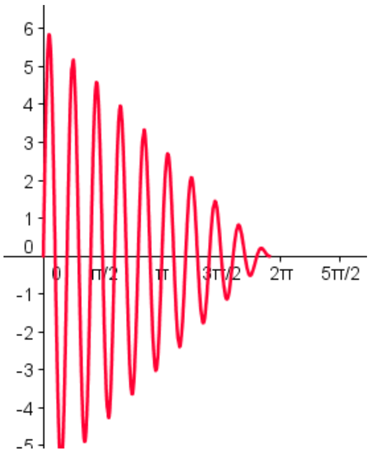


Fig. 3

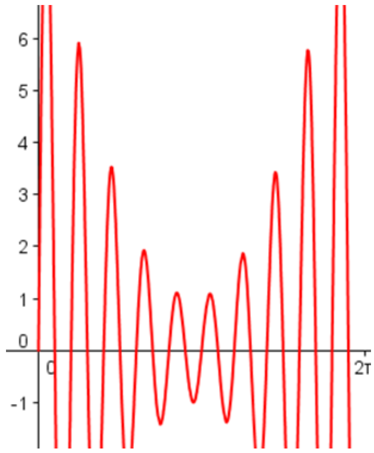


Fig. 4

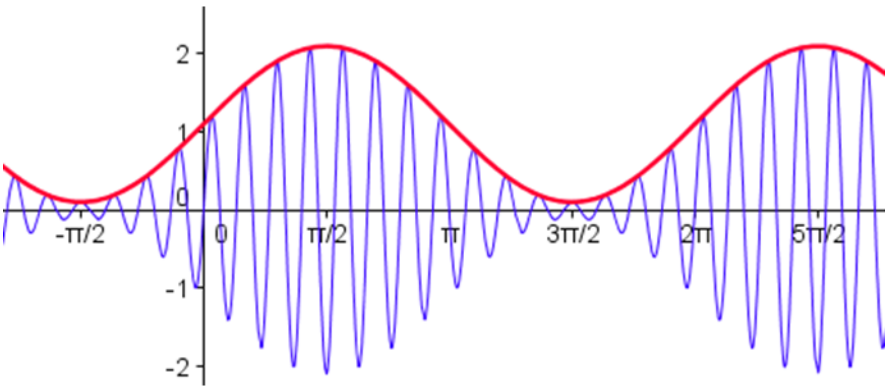
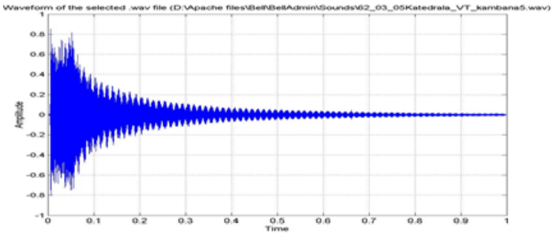


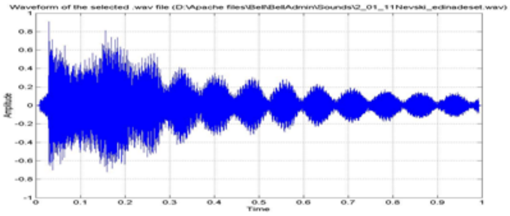
Fig. 5

In the last example the way of transmitting information through radio broadcasting is shown, specifically in the 10^6-10^7 Hz range. The sound pitch, its frequency generated by the loudspeaker, corresponds to the sinusoid's period, created by the amplitude's peaks of the radio carrier wave. This approach is called *amplitude modulation*.

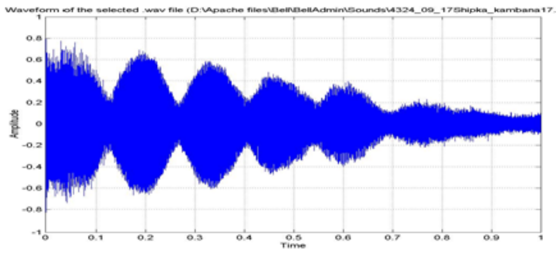
Examples of amplitude modulation that can be observed under other circumstances are shown in the graphs below (Fig. 6). They describe the amplitude variation – the sound intensity – for church bells. The graphs were obtained by means of specialized acoustic devices, used during the development of the "Bells" project [2].



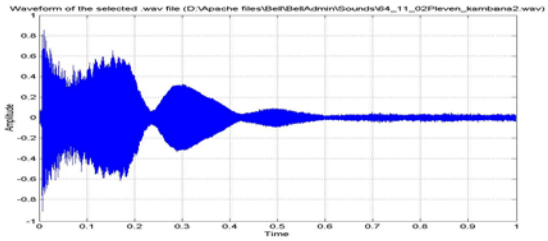
Sveto Rozhdestvo Bogorodichno Church in Veliko Tarnovo



St. Alexander Nevski Church in Sofia



Sveto Rozhdestvo Christovo Church Shipka



St. Nicholas Church in Pleven

Fig. 6. The amplitude variation for bells of different churches

3. Phase Angles – Phase of Delay and Phase of Advance

Consider an electric power generator connected to the electric power transmission network. We apply alternative current. It switches 50/60 times per second. Similar to the pendulum, the variation is achieved following the sinusoid rules – from zero to the maximum positive value, then back, again reaching zero until the minimum negative value is reached and again up until zero is reached. Now, in the same circuit a second electric power generator producing the same frequency is connected,. The current generated follows the same rules as the ones described above. However, the following question arises: do both zero points in time overlap? If so, that would be a happy coincidence. In most cases, however, both zero points in time do not match. If we describe that with two sinusoids (Fig. 6), one of them starts from point (0, 0) and the second one: from a different point from the segment [-1; 1] on the Y-axis. The functions that describe these events and their graphs are the following (Fig. 7):

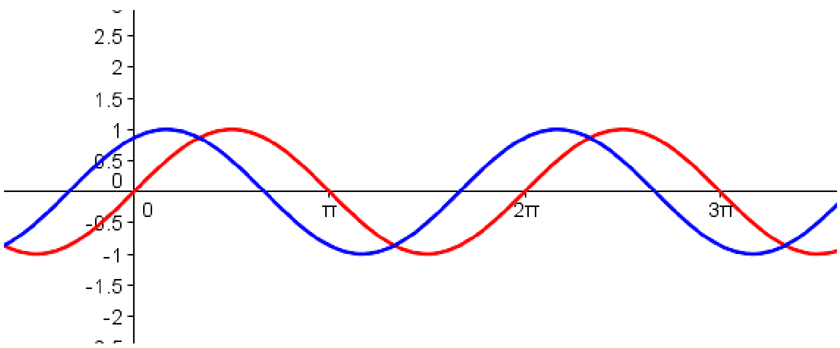


Fig. 7 The graphs of $f_1(x) = \sin x$, (the red one) and $f_2(x) = \sin(x + a)$ (the blue one)

The quantity *a* is called *phase difference* or *phase offset*, or sometimes just *phase*. It could vary in the interval $[0, 2\pi]$. In electricity supply it is limited to very small tolerances. Otherwise, serious issues in the electric power network may occur.

- Draw the graphs of the following functions: $f_1(x) = \sin x$, $f_2(x) = \sin(x + a)\ a \in [0, 2\pi]$.
- Draw the graph of the function $f_1(x) + f_2(x)$ and perform an analysis for different values of *a*.

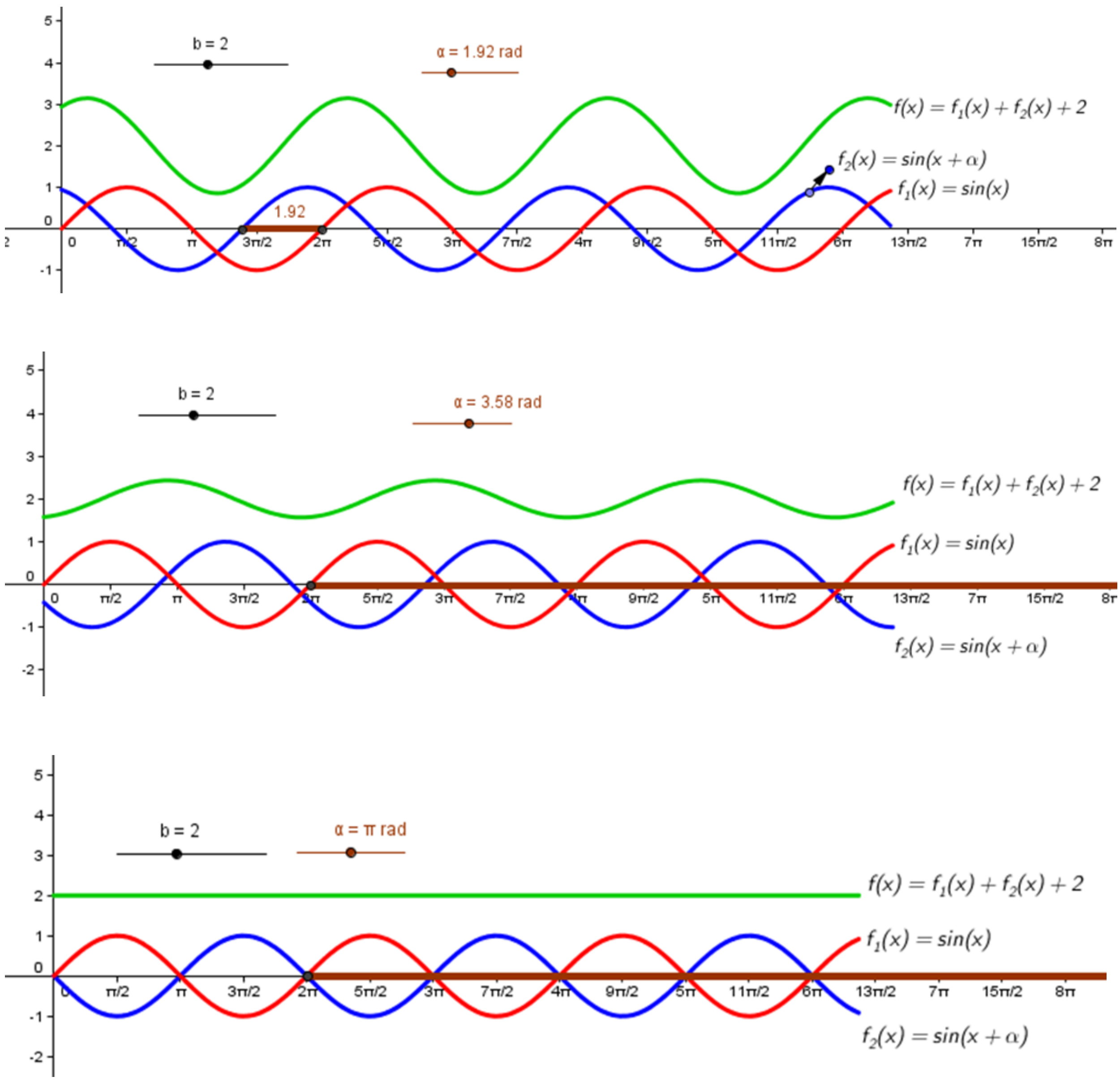


Fig. 8. Overlapping the electric power from sources with different phase difference

The three graphs above (Fig. 8), created with GeoGebra, are describing the result of overlapping (super positioning) of the electrical power from power sources with different phase difference.

4. General conclusion

We made some additions to the function $f(x)=\sin x$. Once we join them, we will obtain the following function: $f(x)=a\sin\left(kx+\alpha\right)$. However, the parameters we introduced may be called differently in real-world-scenarios. In most cases, the function is given the following form:

$$f(x)=A\sin(\omega x+\alpha),$$

where each of the parameters has a value determined according to the respective case.

References

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