When Imagination Needs Some Help

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Abstract

The well-known problem about the snail heading for the top of a well by creeping 2 meters up in daytime and slipping 1 meter down at night caused a lot of troubles to a group of future primary school teachers I worked with. The same happened to a group of second-grade schoolchildren. Both groups, however, demonstrated genuine interest to the problem and high determination to crack it. To support the students whose imagination needed some help, as well as to create a tool for exploration, I made dynamic geometry worksheets for the Snail problem. They not only allow the students to easily change the numerical data, but to reverse the direction of motion as well. For example, this is the case when the snail is up on a tree and intends to reach the ground. Experience gained from the dynamic worksheets let the students continue with a seemingly different problem about two pirates dividing a treasure found on the bottom of the sea. Discovering the analogy between the Snail problem and the Pirates' one brings excitement in the classroom. The existence of multiple solutions puts the students on the path of inquiry-based learning thoroughly discussed in [1].

Key words

Snail problem, Pirates' problem, Open-ended problems, Dynamic geometry, Inquiry-based learning

1. Introduction

Building mathematical literacy starts in childhood. It is a process that takes years and is not performed singly in a classroom. Solving number puzzles like sudoko and kakuro ([2]), playing chess or cards, practicing sports and even shopping are among the many activities which develop combinatorial thinking, counting skills, and number sense. Family and society have their fair share in raising a mathematically literate generation as well. It just happens that elderly people give children witty mathematical problems they have learned in their childhood. To solve such problems, complicated arithmetic operations are not needed, but attention, imagination, logical thinking, and common sense. Folklore mathematical problems like these in [3] train students to have a quick mind, focus on details, and think outside the box. As a result, young people become creative, imaginative, and resourceful – qualities all mathematics teachers desire to be present in their classes.

2. The Snail problem

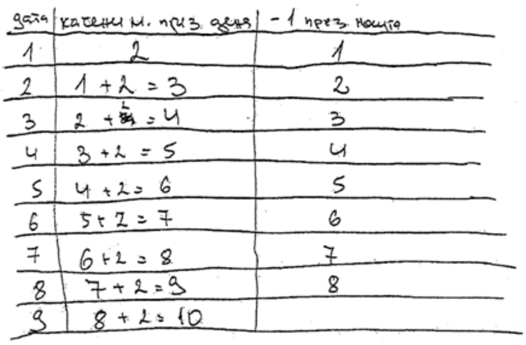
In the frame of a research I performed on how pre-service primary school teachers interpret and solve story-problems ([4], [5]) I found that applying mathematical knowledge in context was hard for many of the participants. However, if a problem was intriguing, the students did not hesitate to delve into it and come up with a solution. Here is an example of a simple, yet difficult story-problem that grasped their attention. Although well-known to the public and in the Internet, it was not familiar to the students. They worked hard to make its formulation mathematically precise:

Problem 1. *On a hot summer night in late June, the Tiny Snail found itself at the bottom of a 10-meter well. At dawn of July 1st it began its journey up. Every day it crawled* ***2 meters up*** *and every night it slid* ***1 meter down****. On which date did the Tiny Snail* ***reach the top*** *of the well?*

Lately I had the opportunity to share the Snail problem with second-grade schoolchildren. They happily took the adventure to work on a problem not included in their textbooks. Surprisingly, one of the students raised his hand asking to speak and said that "*story-problems develop brain and it's good to have such ones*". Even if these words quoted a parent's or a grandparent's opinion, they brought additional enthusiasm in the classroom. Schoolchildren's problem solving was not smooth either. But both young and adult students showed genuine interest in the problem and high determination to crack it, perceiving their errors with the stance to think about the problem again until they truly understand it.

3. Students' results and what they have led to

For their twelve years at school, undergraduate students have already been trained to write their mathematical reasoning. They managed to report their solutions on paper together with graphic or table representations and intermediate stages as well. Their results can be characterized as successful (***Fig. 1***), erroneous but corrected (***Fig. 2***), and showing total helplessness (***Fig. 3***):



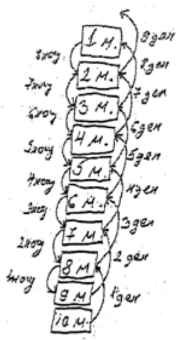
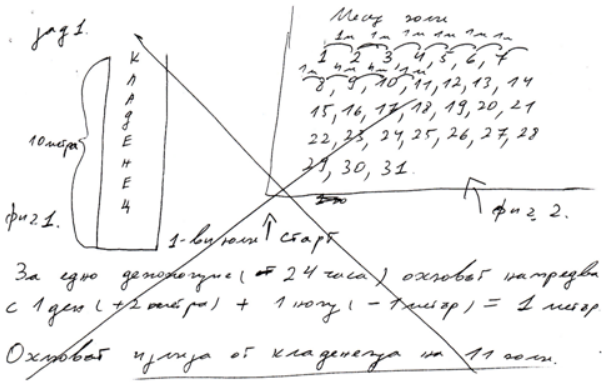


Fig. 1. *Successful solutions to the Snail problem by two undergraduate students*



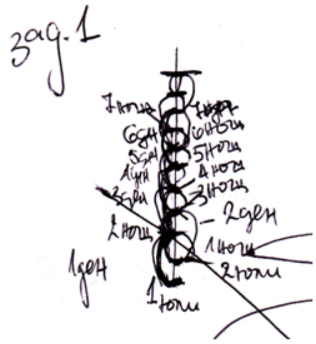


Fig. 2. *Undergraduate students' solutions, crossed out by their authors to correct them*

Analysis of the unsuccessful solutions revealed the prevalent way of reasoning. Since in the daytime the snail crawled 2 meters up and at night slipped 1 meter down, it progressed 1 meter upwards per day. Therefore, to crawl a height of 10 meters it would need 10 days. Counting 10 days from July 1st on led these students to the date of July 11th. Later they realized that on July 1st the snail had reached a height of 2 meters above the bottom, on July 2nd – 3, on July 3rd – 4 meters, etc. Exploring the pattern helped them come up with the right conclusion that the snail had fully climbed the 10-meter well on July (10 – 1)th.

From the papers presented I inferred that finding the right pattern was not the only obstacle the students faced. Many of them were lost in keeping track of their calculations. Therefore, both

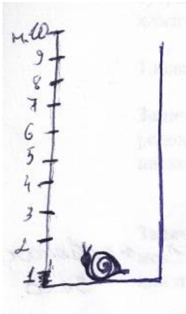


Fig. 3. No solution, only a picture presented

students' imagination and calculations needed some help. This motivated me to create dynamic geometry worksheets complementary to the Snail problem. For the purpose I used GeoGebra[1] dynamic geometry sofware (DGS) as it possesses the desired features to combine algebra and geometry and is free for non-commercial users.

Before investing time and efforts in such an activity it is worth asking if a popular amusement problem like this deserves it. According to Pollak ([6]), primary school is the right place to introduce children to the ideas of mathematical modelling. My direct experience showed that due to innate children's curiosity, providing dynamic geometry worksheets in class was like putting pen to paper: they immediately came into use, both for serious work and play.

4. Mathematical modeling of the Snail problem through DGS

To enable the students to simulate the snail motion by altering numerical data, from GeoGebra user's tools I have picked up a set of sliders. Notations are, as follows: *a* – the distance the snail climbed up in daytime, *b* – the distance the snail slipped down during the night, *t* – the depth of the well (all these in meters), *d* – the day of climbing. The algorithm I have implemented in the worksheet allows to calculate the height reached by the snail at the end of each day. This height and the corresponding day (which coincides with the current date of July) are shown by black dots which rise vertically when *d* increases (***Fig. 4, a-e***). When the snail reaches the top of the well, the two dots change their color to red. The heights the snail has reached at the end of each day during its journey up are shown in the leftmost column of dots. They mark the basic stages of its motion. The teacher can decide to hide them temporarily and ask the students to draw them on their own.

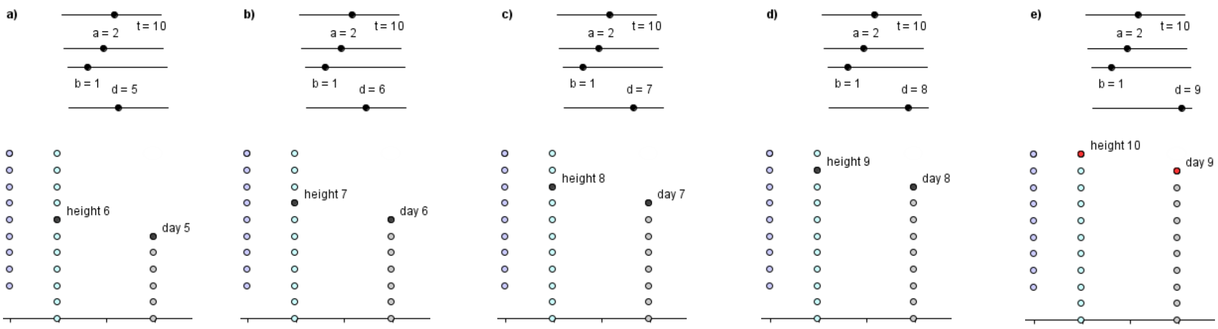


Fig. 4. *A snail climbing in a 10-meter well ($t = 10$) 2 meters up daily ($a = 2$) and slipping 1 meter down at night ($b = 1$) several successive days before reaching the edge:* ***a)*** *Day 5: height 6 (meters);* ***b)*** *Day 6: height 7;* ***c)*** *Day 7: height 8;* ***d)*** *Day 8: height 9;* ***e) Day 9: height 10****; since this is the day when the snail reaches the top of the well, the two dots become red.*

For the students who were unsure how to approach the problem (c.f. ***Fig. 3***), it makes sense to apply Polya's tip given in [7]: "*If there is a problem you can't solve, then there is an easier problem you can solve: find it*." The series of particular cases shown in ***Fig. 5*** proved useful.

Experimenting with the worksheet and guided by me, the schoolchildren soon started looking at this situation from a new perspective: "*Why does the snail need to climb up? During a summer walk in the park one can see so many snails hiding from the heat in the tree crowns. If it starts raining, they will probably need to reach the ground.*" Thus the idea to put the new situation into a new problems was born and even two problems came to life:

Problem 2. *On a hot summer night in late June, the Tiny Snail found itself in the crown of a 10-meter tree. At dawn of July 1st it began its journey down and every day crawled* ***2 meters down****.*

*However, scared of the dark, every night it climbed **1 meter up**. On which date did the Tiny Snail reach the ground?*

Problem 3. *On a hot summer night in late June, the Tiny Snail found itself in the crown of a 10-meter tree. At dawn of July 1st it began its journey down. Every day the snail crawled **2 meters down** and every night it slipped **50 centimeters down**. On which date did it reach the ground?*

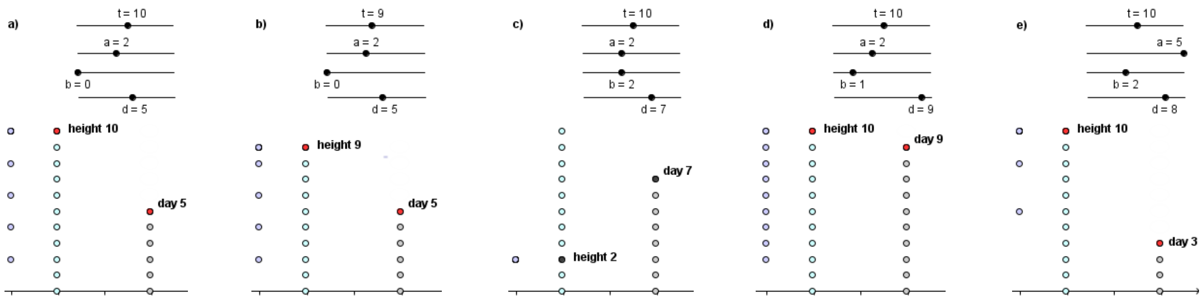


Fig. 5. a) *A snail climbing in a 10-meter well 2 meters up daily and not slipping down at night; **b)** The same in a 9-meter well; **c)** A snail climbing 2 meters up daily and slipping 2 meters down at night cannot exceed a 2-meter height; **d)** Solution to **Problem 1**; **e)** A snail climbing 5 meters up daily and slipping 2 meters down at night, reaches the top of the well on day 3, July 3.*

5. Transforming a traditional story problem through open-ended questions

Although numerous versions of the Snail problem exist in the books and the Internet, some intriguing features of the snail motion still remain unaccounted for. To reveal them to the students, I have used some open-ended questions:

Problem 4. *At dawn of a summer day the Tiny Snail found itself 10 meters above the ground, attached to the stem of a tree. It began crawling in a strange manner: **1 meter up** in day-time and **2 meters down** at night, until it reached the ground.*

- *What was the distance between the highest and the lowest point on the tree the snail had reached?*
- *How many meters did the snail travel from the very beginning of its journey to its end?*
- *During that journey, how many times was the snail 5 meters above the ground?*

The first question not only introduces the concept of maximum and minimum, but also requires students to perform operations with specific extremum values. The second question can be used by the teachers to discuss continuity of motion, as well as the units of measurement used to quantitatively characterize length. The third one summarizes students' observations from the dynamic geometry worksheets and gears up for the idea of multiple solutions (***Fig. 6***):

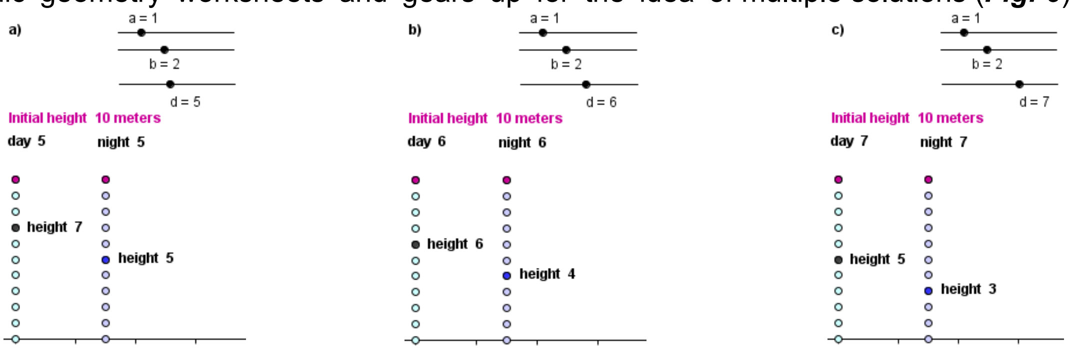


Fig. 6. *The snail that crawled 1 meter up daily and 2 meters down at night, had been 5-meter heigh above the ground **3 times**: **a)** at the the end of the 5th night; **b)** at some moment during the 6th night; and **c)** at the end of the 7th day.*

6. The Pirates' problem

Among the many fascinating pieces of advice by Polya is the following one, given in [7]: "*Look around when you have got your first mushroom or made your first discovery: they grow in clusters*." For the students I had worked with I composed a problem where the idea of multiple solutions is not so evident. However, attentive readers would be able to discern the hint given through the problem question, especially after the experience gained from the *Snail problem 4*:

Problem 5. *Two pirates named Broken Tooth and Mean Look had found a gold treasure in the sea. While Broken Tooth was diving to pull out the chests from the bottom, one at a time, Mean Look stayed on board to keep an eye on them. When the number of chests on the deck reached 10 and Broken Tooth dived for the next one, Mean Look stole two. This was repeated several times: as soon as Broken Tooth hauled one new chest on board, Mean Look stole 2. However, at one point the hard-working diver noticed that just 5 chests were present on the deck. Then not only did he accuse Mean Look of theft, but also demanded for himself all chests his deceitful partner might have stolen. How many chests of gold could Broken Tooth claim?*

In its essence *Problem 5* is very similar to *Problem 4*, but with a new twist given by the pirates' plot and characters. Its formulation excited the schoolchildren, especially after my humorous note how two pirates evenly share a prey: one divides it into halves, the other chooses his half. Such speculations helped children put themselves in *Broken Tooth's* shoes: the two solutions of *Problem 5* (***Fig. 7***) would let *Broken Tooth* to dispense justice, demanding *12 chests* for himself.

While in *Snail problem 4* lengths of continuous objects like segments are to be found, all quantities in *Pirates' problem* are discrete and need to be counted. Discussing such details is an opportunity to ask students to give other examples of continuous and discrete objects, perform measurements, or suggest their own open-ended problems. Since children love novelties, a topic like that enriches their fundamental and practical knowledge and attracts them to mathematics and science. As they love stories as well, involving story-problems can dispel their mathematical anxiety and turn the learning of mathematical concepts into a great inquiry.

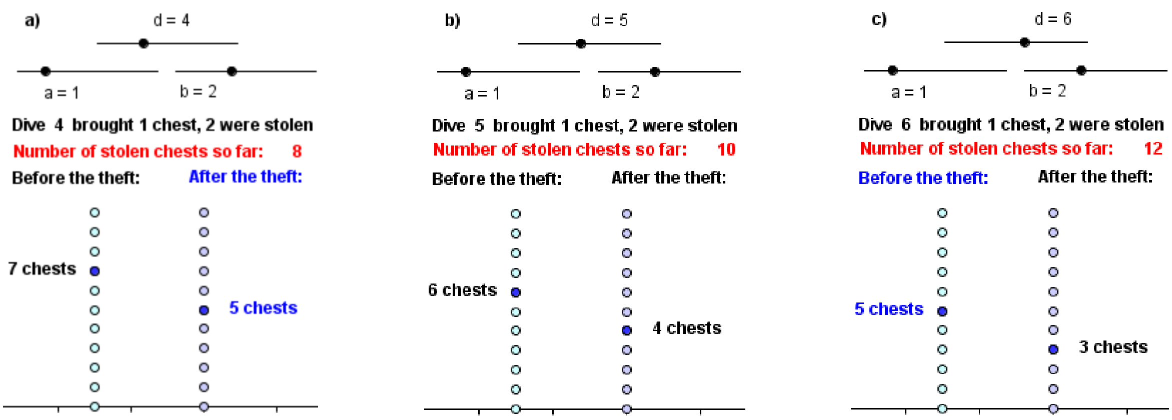


Fig. 7. *There were **two** moments when 5 chests of gold had been present on the deck: **a)** after the theft followed the fourth dive and **c)** after the sixth dive and before the theft. Since either 8 or 12 chests of gold might have been stolen, it is in Broken Tooth's interest to claim **12** chests.*

7. Concluding remarks

All reflections on the problems discussed, both from the undergraduate and primary school students, were full of admiration. The second graders' perceptions (***Fig. 8***) provide evidence that modern young people still need amusing problems as a bridge for transferring knowledge, experience, and values from preceding generations to theirs. Such feedback is a desirable corrective to inquiry-based learning. It should be studied regularly so that the teachers can be fully aware if all their students actively participate in the process.

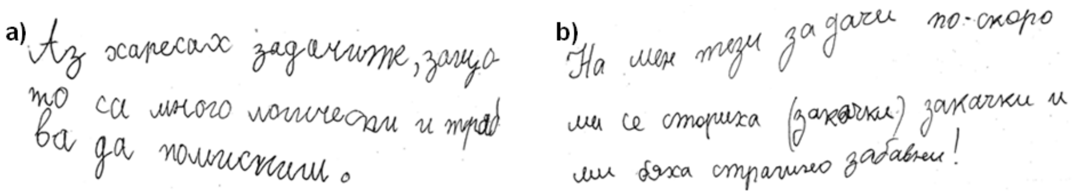


Fig. 8. *Reflections of two primary schoolchildren's on the story problems discussed:* ***a)*** *"I liked the problems because they were very logical and make you think.";* ***b)*** *"These were not the usual math problems; rather, they were brain teasers which were a lot of fun."*

The open-ended questions naturally put the students on the path of inquiry-based learning, where the small pebbles of logical or mathematical obstacles lead them to new findings. Encouraging students to invent their own problems can be both an individual and a group assignment; it can be also a DGS-task: activities of the sort are highly proactive to young digital generation[2] and match its real educational needs.

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Notes

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