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The A_f condition and relative conormal spaces for functions with nonvanishing derivative

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We introduce a join construction, as a way of completing the description of the relative conormal space of an analytic function on a complex analytic space that has a nonvanishing derivative at the origin. Then we show how to obtain a numerical criterion for Thom's A_f condition.

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1. Introduction

Throughout the paper, when dealing with complex analytic germs, we will work with suitably small representatives of these germs. Let $(X, 0)$ be a reduced complex analytic germ in $(\mathbb{C}^m, 0)$. Define the *conormal space* $C(X)$ of X as the closure in $X \times \mathbb{P}^{m-1}$ of the set of pairs (x, H) such that x is a smooth point of X and H is a tangent hyperplane to x. Suppose, f is a nonconstant analytic function on X. Then the *relative conormal space* $C(X, f)$ is defined as the closure in $X \times \mathbb{P}^{m-1}$ of the set of pairs (x, H) such that x is a smooth point of the level set $f^{-1}(f(x))$ and H is a tangent hyperplane to $f^{-1}(f(x))$ at x.

In [8, Thm. 4.2], Massey describes the fiber of the relative conormal space $C(X, f)$ over 0 assuming $f \in m_{X,0}^2$, where $m_{X,0}$ is the maximal ideal of $\mathcal{O}_{X,0}$.

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Our Thm. 2.2 gives description of this fiber in the case $f \notin m_{X,0}^2$. More precisely, let $\tilde{f}$ be an extension of f to $\mathbb{C}^m$. In addition to the irreducible components of the fiber $C(X, f)$ over 0 described by Massey, we show that whenever $f \notin m_{X,0}^2$ there are components which are the join in $\mathbb{P}^{m-1}$ of the point $d\tilde{f}(0)$ and the irreducible components of the fiber of $C(X)$ over 0.

We apply Thm. 2.2 to the numerical characterization of Thom's A_f condition (see [11]), which is a relative stratification condition for the study of functions and mappings on stratified sets. It plays an important role in Thom's second isotopy theorem, and provides a transversality condition in the development of the Milnor fibration.

Assume $(Y, 0)$ is a smooth subgerm of $(X, 0)$ such that $f(Y) = 0$. We say that the A_f condition holds for the pair $(X - Y, Y)$ at 0 if the fiber of $C(X, f)$ over 0 lies in conormal space $C(Y)$ of Y. We say that A_f holds along Y if it holds at every point of Y, or if $C(X, f)|Y \subset C(Y)$. The A_f condition is known to hold generically along Y by a result of Hironaka [5]. So it is important to understand the fiber of $C(X, f)$ over the origin and its relation to $C(X, f)|(Y - 0)$.

Set $n := m - k$. Choose an embedding of $(X, 0)$ in $\mathbb{C}^m = \mathbb{C}^n \times \mathbb{C}^k$, so that $(Y, 0)$ is represented by $0 \times V$, where V is an open neighborhood of 0 in $\mathbb{C}^k$. Let $\mathrm{pr} : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^k$ be the projection. View X as the total space of the family $\mathrm{pr}_{|X} : (X, 0) \to (Y, 0)$. For each closed point $y \in Y$ set $X_y := X \cap \mathrm{pr}^{-1}(y)$.

Assume that X and the fibers X_y are equidimensional and Y is the singular locus of X. Further, assume that $f(Y) = 0$ and $f \notin m_Y^2$, where m_Y^2 is the ideal of Y in $\mathcal{O}_{X,0}$. Then Thm. 2.2 and the lemma preceding it tell us that the irreducible components of $C(X, f)|Y$ are of two types: "big components" and "small components" which are the join of $d\tilde{f}(0)$ with the irreducible components of the fiber of $C(X)$ over the origin. By numerical control on the fibers X_y, f_y, we can ensure that the fiber of $C(X, f)$ over 0 is of minimal dimension. Then it is easy to see that all "big components" are contained in $C(Y)$. The "small components" are hard to control, because their dimension may be small.

Denote by $C(X_0)'$ the closure of the set of pairs (x, H), where x is a smooth point in X_0 and H is a tangent hyperplane of X at x. To understand the nature of the "small components", one needs to understand the relation between $C(X)|X_0$ and $C(X_0)'$. In Thm. 3.1, we give a dimensional condition which ensures that $C(X)|X_0$ and $C(X_0)'$ are the same up to embedded components. Finally, assuming that X satisfies the infinitesimal Whitney A fiber condition along Y, which is a much weaker version of Whitney condition A, we show that the components of the fiber $C(X_0)'$ over the origin are contained in $C(Y)$. All this is the content of Thm. 4.4, which is the main result of the paper, and Thm. 4.5.

2. Relative Conormal Spaces

We begin with some constructions and notation. Let $(X, 0)$ be a reduced complex analytic germ in $(\mathbb{C}^m, 0)$ and let U be an open set of $\mathbb{C}^m$ containing a representative of $(X, 0)$. Denote by $T^*_X U$ the space obtained by taking the closure of the conormal vectors to the smooth part of X in $\mathbb{C}^m \times \mathbb{C}^{m*}$. As the fibers of $T^*_X U$ over points in X are invariant under multiplication by elements from $\mathbb{C}^*$, we may projectivize $T^*_X U$ with respect to vertical homotheties of $T^*_X U$ and work with $\mathbb{P}(T^*_X U)$. This is precisely the conormal space $C(X)$ described in the introduction.

Suppose f is a function on X and $\tilde{f}$ is an extension of f to $\mathbb{C}^m$. The relative conormal space $C(X, f)$ of X with respect to f as defined in the introduction can be obtained as follows. Let $T^*_f U$ be the closure of all (x, η) in $T^*_X U$, where x is a smooth point in X and $\eta(T_x X \cap \ker(d\tilde{f})) = 0$. Then $C(X, f)$ is the projectivization of $T^*_f U$. Note that $C(X, f)$ does not depend on the choice of extension $\tilde{f}$ of f (cf. [3, Sec. 5]). Denote by $c : C(X, f) \to X$ the structure morphism. For a point $x \in X$ denote by $C(X)_x$ and $C(X, f)_x$ the fibers of $C(X)$ and $C(X, f)$ over x, respectively.

The differential $d\tilde{f}$ of $\tilde{f}$ defines an embedding of X in $\mathbb{C}^m \times \mathbb{C}^{m*}$ by the graph map. Let $z_1, \ldots, z_m$ be coordinates on U and $w_1, \ldots, w_n$ be the cotangent coordinates. Then the blowup of $T^*_X U$ along the image of the graph map is the blowup of $T^*_X U$ by the ideal $(w_1 - \frac{\partial \tilde{f}}{\partial z_1}, \ldots, w_n - \frac{\partial \tilde{f}}{\partial z_m})$ in $T^*_X U$. We denote this blowup by $\mathrm{Bl}_{\mathrm{im}\ d\tilde{f}} T^*_X U$. Thus, the blowup is contained in $X \times \mathbb{C}^{m*} \times \mathbb{P}^{m-1}$. Denote the exceptional divisor of this blowup by $E_{\tilde{f}}$. The projection of this exceptional divisor to X is the *singular locus* of f on X denoted by $S(f)$.

Let $\pi : X \times \mathbb{C}^{m*} \times \mathbb{P}^{m-1} \to X \times \mathbb{P}^{m-1}$ denote the projection. Then $\pi(E_{\tilde{f}})$ is independent of the extension of $\tilde{f}$ of f by [8, Cor. 2.12]. The following result describes the relation between $C(X, f)$ and $\mathrm{Bl}_{\mathrm{im}\ d\tilde{f}} T^*_X U$.

Lemma 2.1. *The following holds.*

(i) $E_{\tilde{f}} \cong \pi(E_{\tilde{f}})$.
(ii) *Suppose* $S(f) \subset X_{\mathrm{sing}}$. *Then* $\pi(E_{\tilde{f}}) \subset c^{-1}(X_{\mathrm{sing}})$.

Proof. Part (i) is due to Massey (see the paragraph preceding [8, Lemma 2.6]). If (x, w, η) is a point in $E_{\tilde{f}}$, then $w = d\tilde{f}(x)$, so π induces and isomorphism between $E_{\tilde{f}}$ and $\pi(E_{\tilde{f}})$.

Consider (ii). By [8, Lemma 2.6] (see also the proof of Thm. 2.2) it follows that $\pi(E_{\tilde{f}}) \subset C(X, f)$. But $E_{\tilde{f}}$ is supported over points (x, w_x) for which $w_x = d\tilde{f}(x)$, where w_x is a conormal at x. But by hypothesis $w_x = d\tilde{f}(x)$ can happen only over singular points of X, which proves the claim. $\square$

We will also use the join operation, which we now describe. Given a point a of projective space $\mathbb{P}^{m-1}$ and a subset V of $\mathbb{P}^{m-1}$ distinct from the point, the *join* of a and V consists of the set of points on all lines joining a and the points of V. If a is a point of V and $V \neq a$, then the operation is still well defined; one merely takes the join of a and $V - a$, and then takes the closure of this set. It is easy to see that if V is an analytic set and a lies in V, then the join contains the tangent cone to V at a. If V is analytic, then so is the join, for we can view the join as the inverse image of the projection of $V - a$ to $\mathbb{P}^{m-2}$ from the point a. We denote the join of a and V by $a * V$.

Let C, x be a curve on X, x. Let D_i, $i = 1, 2$ be two lifts of C to $X \times \mathbb{P}^{m-1}$, Denote by $D_{i,p}$ the fiber of D_i over $p \in C$. Suppose $D_{1,p} \neq D_{2,p}$ for p near x. Denote by $(D_1 * D_2)_C$ the family of lines parameterized by C whose fiber over p is $D_{1,p} * D_{2,p}$.

Let x be a point in X. Suppose $f \notin m^2{}_{X,x}$. Denote by $\langle d\tilde{f}(x)\rangle$ the point of $\mathbb{P}^{m-1}$ determined by $d\tilde{f}(x)$. Denote the join of $\langle d\tilde{f}(x)\rangle$ and a subset V of $\mathbb{P}^{m-1}$ by $d\tilde{f}(x) * V$ as well. It is easy to check that $d\tilde{f}(x) * C(X)_x$ is independent of the choice of extension of f to the ambient space. If $f \in m^2{}_{X,x}$, then by convention $d\tilde{f}(x) * C(X)_x$ is empty.

In the next theorem, we will be working with limits along curves, so we discuss this a little. Given $G \in \mathcal{O}^p_{\mathbb{C}}$, $G(t) \neq 0$, for $t \neq 0$, the *limit direction of* G *at* $t = 0$ is $\lim_{t\to 0}\langle G(t)\rangle$, which is a point of $\mathbb{P}^{m-1}$.

We can find the projective limit of G by working directly with G as follows. For any $g \in \mathcal{O}_{\mathbb{C}}$ denote by $o(g(t))$ the order of t in $g(t)$. If $G(t) \in \mathcal{O}^p_{\mathbb{C}}$, then $o(G(t))$ is the minimum of the orders of the component functions g_i of $G(t)$. If $o(G(t)) = k$, then the p-tuple whose entries are the coefficients of the degree k terms of the g_i is the *leading term of* G. If we denote the leading term of G by $L(G)$, then $\langle L(G)\rangle$ is the limit direction of G at $t = 0$. We can compute $L(G)$ as

$$\lim_{t\to 0}\frac{1}{t^k}(G(t)),$$

where k is the order of G.

The next proposition is the key to the description here of the relative conormal space. It grew out of an attempt to improve on some work of Massey (cf. [7, Thm. 3.11]). In particular, the idea of using the blowup of the graph of the differential of $\tilde{f}$ to study the relative conormal space is an idea we learned from him.

Theorem 2.2. *Suppose* $(X, 0)$ *is the germ of a reduced complex analytic set and* $f : X \to \mathbb{C}$ *is a submersion on a Zariski open and dense subset of* X. *Then for each point* $x \in X$, *we have the set-theoretic equality*

$$C(X, f)_x = (\pi(E_f))_x \cup d\tilde{f}(x) * C(X)_x.$$

Proof. The case $f \in m^2_{X,x}$ is part of [8, Thm. 4.2].

We first show that $C(X, f)_x$ contains $d\tilde{f}(x) * C(X)_x$. Suppose $d\tilde{f}(x) \neq 0$. Suppose $H \in C(X)_x$ and $H \neq \langle d\tilde{f}(x)\rangle$. There exists a curve

$$\phi = (\phi_1, \phi_2) : (\mathbb{C}, 0) \to (X \times \mathbb{P}^{m-1}, x \times H),$$

such that the hyperplane $\phi_2(t)$ is tangent to X at $\phi_1(t)$, and $\phi_1(t) \in X - X_{\text{sing}} - S(f)$ for $t \neq 0$, where $S(f)$ is the critical locus of f. Denote the image of ϕ_1 by C. For t sufficiently small, $t \neq 0$, we can assume that the augmented Jacobian module, which is the module generated over $\mathcal{O}_{X,x}$ by the columns of the Jacobian matrix of $(G, \tilde{f})$, has maximal rank because $\phi_1(t) \in X - X_{\text{sing}} - S(f)$ for $t \neq 0$. Then $\langle d\tilde{f}(\phi_1(t))\rangle$ and $\phi_2(t)$ give two lifts of C to $\mathbb{P}^{m-1}$ and since the rank of the augmented Jacobian module is maximal along C, then $(\langle d\tilde{f}\rangle * \phi_2)_C$ is well-defined. Since both $C(X, f)$ and $(\langle df\rangle * \phi_2)_C$ are Zariski closed and a Zariski open subset of the second lies in the first, then the second lies in the first as well. This implies that $d\tilde{f}(x) * H$ is in $C(X, f)_x$.

The rest of the proof is related to the fibers of $C(X, f)$ or $\mathrm{Bl}_{\mathrm{im}\ d\tilde{f}} T^*_X U$, so it is convenient to work along curves and take limits. Giving a curve on $C(X, f)$ at smooth points of f on X amounts to giving smoothly varying linear combinations of the rows of the Jacobian matrix of $F = (G, \tilde{f})$, where $G : (\mathbb{C}^m, 0) \to (\mathbb{C}^p, 0)$ and $X = G^{-1}(0)$ and projectivizing. We can make this precise as follows: if $(x, H) \in C(X, f)_x$, then there exist curves ϕ, ψ such that $\phi : (\mathbb{C}, 0) \to (X, x)$, and

$$\psi = (\psi_1, \psi_2) : (\mathbb{C}, 0) \to \mathbb{C}^p \times \mathbb{C}.$$

Then H is the projective limit of the curve

$$\psi_1(t) \cdot DG(\phi(t)) - \psi_2(t) d\tilde{f}(\phi(t)), \tag{1}$$

where $\psi_2(t)$ is taken with a minus sign for convenience of comparison with the blowup construction.

We can do something similar for $l \in \pi(E_{\tilde{f}})_x$. Namely, we can use ϕ and ψ as before with $\psi_2 = 1$. Then (x, l) is a point of $\pi(E_{\tilde{f}})_x$ if and only we can find curves ϕ, ψ such that $\phi(0) = x$ and l is the projective limit of the curve

$$\psi_1(t) \cdot DG(\phi(t)) - d\tilde{f}(\phi(t)).$$

This description shows that $\pi(E_{\tilde{f}})_x$ is also contained in $C(X, f)_x$.

Now suppose $(x, H) \in C(X, f)_x$. Then there exist curves ϕ, ψ such that H is the projective limit of a curve of type (1).

We deal separately with the cases, where $f \in m^2{}_{X,x}$ and $f \notin m^2{}_{X,x}$.

Assume $f \in m^2{}_{X,x}$ and that $\tilde{f}$ is chosen in such a way so that $d\tilde{f}(x) = 0$.

If

$$o(\psi_2(t)) < o(\psi_1(t) \cdot DG(\phi(t))),$$

then $\frac{\psi_1(t)}{\psi_2(t)} \cdot DG(\phi(t))$ gives a lift of ϕ to $T^*_X U$ and the projective limit of $\frac{\psi_1(t)}{\psi_2(t)} \cdot DG(\phi(t)) - d\tilde{f}(\phi(t))$ is H, showing that H lies in $\pi(E_{\tilde{f}})_x$.

We have $o(d(\tilde{f}(\phi(t)) \geq 1$, because $f \in m^2_{X,x}$. Suppose

$$o(\psi_2(t)) \geq o(\psi_1(t) \cdot DG(\phi(t))).$$

Then

$$o(\psi_2(t) d\tilde{f}(\phi(t))) > o(\psi_1(t) \cdot DG(\phi(t))),$$

so H is the projective limit of $\psi_1(t) \cdot DG(\phi(t))$. Hence, H is a limiting tangent hyperplane to X.

There are two subcases, depending on the order $d\tilde{f}(\phi)$.

If the order of the components of $d\tilde{f}(\phi)$ is greater than 1, then we use the same ϕ but replace $\psi_1(t)$ by $\psi_1(t)/t^k$, where k is chosen so that the order of $(\psi_1(t)/t^k)\cdot DG(\phi(t))$ is greater than 0, but less than the order of $d\tilde{f}(\phi(t))$. We may take $k = \max\{o(\psi_1(t)\cdot DG(\phi(t)) - o(d\tilde{f}(\phi(t))) - 1, 0\}$. Then again $(\psi_1(t)/t^k)\cdot DG(\phi(t))$ provides a lift of ϕ to T^*_XU, and the projective limit of $(\psi_1(t)/t^k)\cdot DG(\phi(t)) - d\tilde{f}(\phi(t))$ is again H. If the order of the components of $df(\phi(t))$ is 1, then we reparameterize $\phi(t)$ so that the order of $d\tilde{f}(\phi(t))$ is again greater than 1 and repeat the argument. This finishes the first case.

Now suppose $f \notin {m^2}_{X,x}$. If $o(\psi_2(t)) < o(\psi_1(t)\cdot DG(\phi(t)))$, then $H = \langle d\tilde{f}(x)\rangle$. If the order relation is reversed, then H is in $C(X)_x$. In either case $H \in d\tilde{f}(x) * C(X)_x$, unless $C(X)_x = \langle d\tilde{f}(x)\rangle$.

If $o(\psi_2(t)) = o(\psi_1(t)\cdot DG(\phi(t)))$, then $H \in d\tilde{f}(x) * C(X)_x$ unless

$$\lim_{t\to 0}\frac{\psi_1(t)}{\psi_2(t)}\cdot DG(\phi(t)) = d\tilde{f}(x).$$

In this case, we can again get a lift of ϕ to T^*_XU, using $\frac{\psi_1(t)}{\psi_2(t)}\cdot DG(\phi(t))$, so again H is in $\pi(E_{\tilde{f}})_x$.

It remains to deal with the case where $C(X)_x = \langle d\tilde{f}(x)\rangle$. We need to show that $\langle d\tilde{f}(x)\rangle$ lies in $\pi(E_{\tilde{f}})_x$. Since the dimension of $C(X)_x$ is zero, X must be a hypersurface, and by a change of coordinates, we may assume f is a linear form. There exists ϕ such that the projective limits of $DG(\phi(t))$ and of $tDG(\phi(t))$ are both $\langle d\tilde{f}(x)\rangle$. Let $k = o(DG(\phi(t)))$. Then

$$\lim_{t\to 0}\left\langle \frac{(1+t)}{t^k}DG(\phi(t)) - d\tilde{f}(\phi(t))\right\rangle = \langle d\tilde{f}(x)\rangle$$

which completes the proof. $\square$

In checking the A_f condition at the origin in a family, we need to show that $C(X,f)_0$ consists of hyperplanes which contain the tangent plane to Y at the origin. The previous theorem shows that the components of $C(X,f)_0$ are of two types — blowup components and join components. Blowup components have large dimension, and can be detected and controlled numerically. However, $C(X)_0$ may contribute small components of join type when $f \notin m^2_{X,0}$. In the next section, we prove a theorem which shows that these join components can be controlled using the fiber X_0.

3. Fibers of Generalized Conormal Spaces

Let $h : (X,0) \to (Y,0)$ be a complex analytic family such that X is equidimensional and for each closed point $y \in Y$ the fibers X_y are equidimensional of positive dimension d. Suppose Y is irreducible and Cohen–Macaulay.

The purpose of this section is to understand the relation between the closed subscheme of the conormal $C(X)$ which set-theoretically consists of limits of hyperplanes through points of X_0, and the fiber $C(X)_0$ over $0 \in Y$ of the conormal $C(X)$.

Our treatment is more general. The conormal $C(X)$ is the Projan of the Rees algebra of the Jacobian module of X (cf. [6, Sec. 1.5]). Instead of working with conormal spaces, we work below with Projans of Rees algebras of modules.

Let $\mathcal{F} := \mathcal{O}_X^p$ be a free module of rank $p \geq 1$. Let $\mathcal{M}$ be a coherent submodule which is free of rank e off a closed subset S of X. Further, assume S is finite over Y. Set $r := d + e - 1$.

Form the symmetric algebra $\mathrm{Sym}(\mathcal{F})$ of $\mathcal{F}$ and the Rees algebra $\mathcal{R}(\mathcal{M})$ of $\mathcal{M}$ which is the subalgebra of $\mathrm{Sym}(\mathcal{F})$ generated by $\mathcal{M}$ placed in degree 1. Denote the kth graded components of these algebras by $\mathcal{F}^k$ and $\mathcal{M}^k$ respectively. Given a closed point $y \in Y$ denote by $\mathcal{M}^k(y)$ the image of $\mathcal{M}^k$ in the free $\mathcal{O}_{X_y}$-module $\mathcal{F}^k(y)$.

Set $C := \mathrm{Projan}(\mathcal{R}(\mathcal{M}))$. Denote by $c\colon C \to X$ be the structure morphism. Let y be a closed point in Y. Set $C(y) := \mathrm{Projan}(\mathcal{R}(\mathcal{M}(y)))$ and denote by C_y the fiber of $h \circ c$ over $y \in Y$. For an irreducible component V of C_y, we say it is *horizontal* if it surjects onto an irreducible component of X_y or we say it is *vertical* otherwise.

Theorem 3.1. *Suppose* $\dim c^{-1}0 < r$. *Then there exists a Zariski open neighborhood U of* 0 *in Y such that for each $y \in U$ the irreducible components of C_y are horizontal. Furthermore, if $\mathcal{M}$ is a direct summand of $\mathcal{F}$ locally off S, then we have an equality of fundamental cycles*

$$[C_y] = [C(y)]. \tag{2}$$

Proof. Because h has equidimensional fibers of dimension d and X is equidimensional, then $\dim X = d + \dim Y$. Also, by assumption $d > 0$ and S is finite over Y. Thus, S is nowhere dense in X. Let x be a point in X with $x \notin S$. Because the formation of Rees algebra commutes with flat base change, we have $\mathcal{R}(\mathcal{M})_x = \mathcal{R}(\mathcal{M}_x)$. But $\mathcal{M}_x$ is free of rank e because $x \notin S$. Thus $\mathcal{R}(\mathcal{M}_x) = \mathrm{Sym}(\mathcal{O}^e_{X,x})$ whence $\dim c^{-1}x = e - 1$. The dimension formula applied for each irreducible component of C yields that C is equidimensional and $\dim C = r + \dim Y$.

Let C'_y be an irreducible component of C_y. Because Y is Cohen–Macaulay locally at each closed point $y \in Y$, then by Krull's height theorem C'_y is of codimension at most $\dim Y$. Because C is of finite type over the complex numbers, we get

$$\dim C'_y \geq r. \tag{3}$$

Replace X with Zariski neighborhood of 0 so that $\dim c^{-1}x < r$ for each point $x \in X$. Let U be a Zariski open subset of the image of h that contains $0 \in Y$. Let y be a point in U. Suppose that c maps C'_y to a point $\zeta \in X_y$. Then $C'_y \subset c^{-1}\zeta$. But $\dim c^{-1}\zeta < r$ which contradicts with (3). But S_y is zero-dimensional. Thus, there exists a Zariski open dense subset Z_y of C'_y whose image under c misses S_y.

Let $\zeta \in c(Z_y)$. Then $\mathcal{M}_\zeta$ is free of rank e. Thus $\dim c^{-1}\zeta = e - 1$. By the dimension formula

$$\dim C'_y = \dim c(C'_y) + e - 1.$$

But $\dim C'_y \geq r$. So $\dim c(C'_y) \geq d$. Because $c(C'_y) \subset X_y$ and $\dim X_y = d$, then $c(C'_y)$ is an irreducible component of X_y which proves the first claim of the theorem.

Next, assume $\mathcal{M}$ is locally a direct summand of $\mathcal{F}$ off S. Set $X = \mathrm{Specan}(R)$ and $Y = \mathrm{Specan}(Q)$. Then morphism h induces a ring homomorphism $h^{\#}: Q \to R$. Denote by $\mathfrak{n}_y$ the image under $h^{\#}$ of the ideal of y in Q. Consider the homomorphism

$$\phi_y : \mathcal{R}(\mathcal{M})/\mathfrak{n}_y\mathcal{R}(\mathcal{M}) \to \mathrm{Sym}(\mathcal{F}(y)).$$

Denote its kernel by I_{ϕ_y}. Observe that

$$(\mathcal{R}(\mathcal{M})/\mathfrak{n}_y\mathcal{R}(\mathcal{M}))/I_{\phi_y} = \mathcal{R}(\mathcal{M}(y)).$$

Let's identify I_{ϕ_y}. Consider the homomorphism

$$\tilde{\phi}_y : \mathcal{R}(\mathcal{M}) \to \mathrm{Sym}(\mathcal{F}(y)).$$

We have $\mathrm{Ker}(\tilde{\phi}_y) = \mathfrak{n}_y\mathrm{Sym}(\mathcal{F}) \cap \mathcal{R}(\mathcal{M})$. By definition I_{ϕ_y} is the kernel of ϕ_y. As $\tilde{\phi}_y$ factors through ϕ_y, we get

$$I_{\phi_y} = (\mathfrak{n}_y\mathrm{Sym}(\mathcal{F}) \cap \mathcal{R}(\mathcal{M}))/\mathfrak{n}_y\mathcal{R}(\mathcal{M}). \tag{4}$$

Because the source of ϕ_y is supported on X_y, then $\mathrm{Supp}(I_{\phi_y}) \subset X_y$. Let $x \in X_y$ with $x \notin S_y$. Then $\mathcal{M}$ is locally a direct summand of $\mathcal{F}$ at x. Write $\mathcal{F}_x = \mathcal{M}_x \oplus L(x)$. The formation of symmetric and Rees algebras commutes with localization, hence

$$(\mathfrak{n}_y\mathrm{Sym}(\mathcal{F}))_x = \mathfrak{n}_y\mathrm{Sym}(\mathcal{F}_x) \quad \text{and} \quad \mathcal{R}(\mathcal{M})_x = \mathcal{R}(\mathcal{M})_x.$$

On the other hand,

$$\mathfrak{n}_y\mathrm{Sym}(\mathcal{F}_x) = \mathfrak{n}_y\mathrm{Sym}(\mathcal{M}_x) \otimes \mathfrak{n}_y\mathrm{Sym}(L(x)).$$

Because $\mathcal{M}_x$ is free we have $\mathrm{Sym}(\mathcal{M}_x) = \mathcal{R}(\mathcal{M}_x)$. Hence, locally at x the ideals $\mathfrak{n}_y\mathcal{R}(\mathcal{M})$ and $\mathfrak{n}_y\mathrm{Sym}(\mathcal{F}) \cap \mathcal{R}(\mathcal{M})$ agree. Finally, we obtain that if I_{ϕ_y} is nonzero, then it is supported at points from S_y only. In particular, I_{ϕ_y} vanishes locally at the minimal primes of $\mathcal{R}(\mathcal{M})/\mathfrak{n}_y\mathcal{R}(\mathcal{M})$ because as we showed above each of these minimal primes contracts to a minimal prime of X_y. Therefore, $C(y)$ and C_y differ by vertical embedded components supported over S_y. This proves (2). □

Remark 3.2. Note that in general without assuming the bound on the dimension of $c^{-1}(0)$, the proof above shows that $C(y) = (C_y - W)^-$ for any $y \in Y$, where W is the union of irreducible components of C_y surjecting on S_y.

A more general version of the theorem above without assuming that S is finite over Y can be derived using Bertini's theorem for extreme morphisms from [9]. The direct summand assumption can be relaxed at the expense of mild hypothesis on X as remarked at the end of [9, Sec. 2].

4. The A_f Condition and the Main Result

Let $(X, 0)$ be a reduced complex analytic set germ with $X = G^{-1}(0)$, where $G : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^p, 0)$. Assume $(Y, 0) \subset (X, 0)$ is smooth subgerm of dimension k.

Choose an embedding of $(X,0)$ in $\mathbb{C}^{n+k} = \mathbb{C}^n \times \mathbb{C}^k$, so that $(Y,0)$ is represented by $0 \times V$, where V is an open neighborhood of 0 in $\mathbb{C}^k$. Let $\mathrm{pr} : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^k$ be the projection, $i : \mathbb{C}^k \to \mathbb{C}^n \times \mathbb{C}^k$ be the inclusion $i(y) = (0, y)$ and π_Y be the retraction $i \circ \mathrm{pr}_{|X}$. View X as the total space of the family $\pi_Y : (X,0) \to (Y,0)$. For each closed point $y \in Y$ set $X_y := X \cap \pi_Y^{-1}(y)$.

The *Jacobian module* $\mathrm{JM}(X)$ of X is the submodule of $\mathcal{O}_X^p$ generated by the partial derivatives of G. It is a direct summand of $\mathcal{O}_X^p$ locally off the singular locus of X. Denote the smooth part of X by X_{sm}. Suppose f is an analytic function on X that is a submersion on X_{sm}. Then the singular locus $S(f)$ of f is contained in X_{sing}. Denote by $\tilde{f}$ an extension of f to the ambient space. Define $H = (G, \tilde{f})$, and let $\mathrm{JM}(H)$ denote the $\mathcal{O}_{X,0}$-module defined by the partial derivatives of H. Note that $\mathrm{JM}(H)$ is independent of the choice of extension of f by the discussion in the beginning of [3, Sec. 5]. Finally, denote by c the structure morphism $c : C(X, f) \to X$ and by $C(Y)$ the conormal space of Y in $\mathbb{C}^{n+k}$.

We say that A_f condition holds for the pair X_{sm}, Y at 0 if $f(Y) = 0$ and Y lies in every hyperplane obtained as a limit of tangent hyperplanes to a level hypersurface at a point $x \in X_{\mathrm{sm}}$ as x approaches 0.

We review briefly the connection between the theory of integral closure of modules and Thom's A_f condition. Recall that given a submodule $\mathcal{M}$ of a free $\mathcal{O}_{X,0}$ module $\mathcal{F}$, we say that $u \in \mathcal{F}$ is *strictly dependent* on $\mathcal{M}$ and we write $u \in \mathcal{M}^\dagger$, if for all analytic path germs $\phi : (\mathbb{C}, 0) \to (X, 0)$, $\phi^* u$ is contained in $\phi^*(\mathcal{M}) m_1$, where m_1 is the maximal ideal of $\mathcal{O}_{\mathbb{C},0}$.

Proposition 4.1. *Assume $f(Y) = 0$. Then the following are equivalent*:

(i) *The A_f condition holds for the pair X_{sm}, Y at* 0.
(ii) $c^{-1}(Y) \subset C(Y)$.
(iii) $\frac{\partial H}{\partial y_j} \in \mathrm{JM}(H)^\dagger$ *for all* $j = 1, \ldots, k$.

Proof. The equivalence of (i) and (ii) is obvious; the equivalence of (i) and (iii) is [3, Lemma 5.1]. □

A similar result holds for the Whitney A condition. The condition we need for our main result is a much weaker version of Whitney A.

Definition 4.2. We say that $(X,0) \to (Y,0)$ satisfies the infinitesimal Whitney A fiber condition at 0 if $\frac{\partial G}{\partial y_j} \in \mathrm{JM}(X_0)^\dagger$ for all $j = 1, \ldots, k$.

This condition is equivalent to asking that limiting tangent hyperplanes to X along curves on X_0 contain the tangent space to Y (cf. [3, Lemma 4.1]). So it is much weaker than asking that Whitney A hold for the pair (X_{sm}, Y) at 0, which would require looking at all curves on X passing through the origin.

We show how weak the infinitesimal Whitney A condition is by considering a family of examples due to Trotman (see [12, Prop. 5.1, p. 147]). In these examples, the members of the families are the same, but the total space is different. The examples were used to show that a necessary and sufficient fiberwise condition for Whitney A was impossible.

Example 4.3. Consider the family of plane curves with parameter y given by $w^a - y^b v^c - v^d = 0$, so X_0 is the curve defined by $w^a - v^d = 0$ and $y = 0$. Then the infinitesimal Whitney A fiber condition holds at 0 if $b > 1$, for all a, c, d, because on X_0 we have $\frac{\partial G}{\partial y} = 0$. If $b = 1$, the condition holds if $c > \min\{d - d/a, d - 1\}$.

Indeed, let $\phi : (\mathbb{C}, 0) \to (X_0, 0)$ be a curve, and let t be the generator for the maximal ideal of $\mathcal{O}_{\mathbb{C},0}$. Write $\phi^*(w) = t^{\alpha_1} w_1(t)$ and $\phi^*(v) = t^{\beta_1} v_1(t)$, where $w_1(t)$ and $v_1(t)$ are units in $\mathcal{O}_{\mathbb{C},0}$. Because X_0 is cut out by $w^a - v^d = 0$, then $a\alpha_1 = d\beta_1$. The infinitesimal Whitney A fiber condition holds if $\phi^*(v^c) \in t(\phi^*(w^{a-1}), \phi^*(v^{d-1}))$, or equivalently if $c\beta_1 > \min\{(a-1)\alpha_1, (d-1)\beta_1\}$, which is the same as $c > \min\{d - d/a, d - 1\}$ because $\alpha_1 = \frac{d\beta_1}{a}$.

Preserve the setup from the beginning of the section. The following theorem is the main result of our paper.

Theorem 4.4. *Suppose X and X_y are equidimensional. Assume the singular locus of X is Y. Suppose f is a function on X such that f is a submersion on $X - Y$ and $f(Y) = 0$. Suppose $\dim C(X, f)_0 < n$, and the infinitesimal Whitney A fiber condition holds at 0. Then A_f holds for the pair $(X - Y, Y)$ at 0.*

Proof. We need to show that $C(X, f)_0 \subset C(Y)$. By Theorem 2.2, we know the components of $C(X, f)_0$ are of two types: the blowup components $\pi(E_{\tilde{f}})_0$ and the join components $d\tilde{f}(0) * C(X)_0$ if $d\tilde{f}(0) \neq 0$. We will show that the irreducible components of $\pi(E_{\tilde{f}})_0$ are contained in $C(Y)$, while the join components are controlled by the infinitesimal Whitney A fiber condition.

We claim that $\pi(E_{\tilde{f}})$ is of pure dimension equal to $\dim \mathrm{Bl}_{\mathrm{im}\ d\tilde{f}} T^*_X U - 1 = n + k - 1$, where U is a neighborhood of 0 in $\mathbb{C}^{n+k}$ that contains X. Indeed, by Lemma 2.1(i) $\pi(E_{\tilde{f}})$ is isomorphic to $E_{\tilde{f}}$, and $E_{\tilde{f}}$ is of pure dimension $n + k - 1$.

Because the singular locus of X is Y and because f is a submersion on $X - Y$, then $S(f) \subset Y$. Then by Lemma 2.1(ii) $\pi(E_{\tilde{f}})$ is supported over Y. By assumption, $\dim \pi(E_{\tilde{f}})_0 \leq n - 1$. Hence, by upper semi-continuity $\dim \pi(E_{\tilde{f}})_y \leq n - 1$ for each y in a neighborhood of 0. But $\dim Y = k$. Thus, the dimension formula implies that the irreducible components of $\pi(E_{\tilde{f}})$ surject onto Y. Since the A_f condition holds generically on Y, each irreducible component of $\pi(E_{\tilde{f}})$ is generically contained in $C(Y)$. Hence, each such component lies in $C(Y)$. In particular, $\pi(E_{\tilde{f}})_0$ is contained in $C(Y)$.

Now we turn to the join components. Consider a component Z of the fiber of $C(X)_0$. Since $\dim d\tilde{f}(0) * Z < n$, then $\dim Z < n - 1$. Since $Y \subset f^{-1}(0)$, then $d\tilde{f}(0) \in C(Y)$. So it suffices to show that $Z \subset C(Y)$. Apply Theorem 3.1 with $\mathcal{M} := \mathrm{JM}(X)$. Then C is the conormal space $C(X)$. Also, C_0 is $C(X)|X_0$ and $C(0)$ is the closed subscheme of $C(X)$ that consists of the closure of the pairs (x, H), where x is a smooth point of X_0 and H is a tangent hyperplane to X at x, and

$r = n - 1$. Since the dimension of Z is less than $n - 1$ then by Thm. 3.1, it follows that Z consists of limits of tangent hyperplanes at 0 along curves on X_0. Thus, the infinitesimal Whitney A fiber condition implies that Z is in $C(Y)$. □

The usefulness of the last theorem rests on our ability to control the dimension of $\dim C(X, f)_0$ by numerical means. We give an example improving [4, Thm. 5.8] that shows how this works. For another example see [10, Thm. 1.8.2]. Recall that $X \subset \mathbb{C}^{n+k}$ is a determinantal singularity if the ideal of X is generated by the minors of fixed size of an $(l + q) \times l$ matrix with entries in $\mathcal{O}_{n+k}$, and X has the expected codimension. The matrix is called the presentation matrix of X and is denoted M_X.

Denote by $\mathrm{JM}(G_y; f_y)$ the restriction to the fiber X_y of the augmented Jacobian module H as defined in the beginning of the section. Denote by $N_D(y)$ the module of first-order infinitesimal deformations of X_y coming from the deformations of the presentation matrix M_{X_y}. Let Σ^l be the $(l+q)\times l$ matrices of kernel rank l. Finally, define $e_\Gamma(\mathrm{JM}(G_y; f_y), \mathcal{O}_{n+k} \oplus N_D(y))$ to be the sum of the multiplicity of the pair of modules $(\mathrm{JM}(G_y; f_y), \mathcal{O}_{l+q} \oplus M^*_{X_y}(\mathrm{JM}(\Sigma^l)))$ and the intersection number of the image of M_{X_y} with a polar of Σ^l of complementary dimension to n.

Theorem 4.5. *Suppose $(X, 0) \to (Y, 0)$ is a family of determinantal singularities with presentation matrix $M_X : \mathbb{C}^{n+k} \to \mathrm{Hom}(\mathbb{C}^l, \mathbb{C}^{l+q})$, defined by the maximal minors of M_X. Suppose $X = G^{-1}(0)$, where $G : (\mathbb{C}^{n+k}, 0) \to (\mathbb{C}^p, 0)$ with Y a smooth subset of X, coordinates chosen so that $0 \times \mathbb{C}^k = Y$. Assume X is equidimensional with equidimensional fibers of the expected dimension and X is reduced.*

Suppose $f : (X, 0) \to (\mathbb{C}, 0)$ and set $Z = f^{-1}(0)$. Suppose the infinitesimal Whitney A fiber condition holds at 0 if $f \notin m_Y^2$.

(A) Suppose X_y and Z_y are isolated singularities, suppose the critical locus of f is Y. Suppose $e_\Gamma(\mathrm{JM}(G_y; f_y), \mathcal{O}_{n+k} \oplus N_D(y))$ is independent of y. Then the union of the singular points of f_y is Y, and the pair of strata $(X - Y, Y)$ satisfies Thom's A_f condition.

(B) Suppose the critical locus of f is Y or is empty, and the pair $(X - Y, Y)$ satisfies Thom's A_f condition. Then $e_\Gamma(\mathrm{JM}(G_y; f_y), \mathcal{O}_{n+k} \oplus N_D(y))$ is independent of y.

Proof. The proof follows the lines of the proof of [4, Thm. 5.8]. It uses Thm. 4.4 to cover the case when $f \notin m_Y^2$.

In [4], it is shown that constancy of $e_\Gamma(\mathrm{JM}(G_y; f_y), \mathcal{O}_{n+k} \oplus N_D(y))$ implies that H has no polar variety of dimension k. In turn this implies $C(X, f)_0$ has no component of dimension n or more. If $f \in m_Y^2$, this implies the result of [4]. If $f \notin m_Y^2$, then it allows us to use Thm. 4.4 to prove the above strengthening of the result of [4]. □

In a similar way, [1, Theorems 5.3 and 5.4], and [2, Theorem 5.6] can be strengthened, dropping the hypothesis of $f \in m_Y^2$.

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