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D. D. Cherkashin

ON THE ERDŐS–HAJNAL PROBLEM IN THE CASE OF 3-GRAPHS

Abstract. Let $m(n,r)$ denote the minimal number of edges in an n-uniform hypergraph which is not r-colorable. For the broad history of the problem see [10]. It is known [4] that for a fixed n the sequence

$$\frac{m(n,r)}{r^n}$$

has a limit.

The only trivial case is $n = 2$ in which $m(2,r) = \binom{r+1}{2}$. In this note we focus on the case $n = 3$. First, we compare the existing methods in this case and then improve the lower bound.

§1. Introduction

A hypergraph $H = (V,E)$ consists of a finite set of *vertices* V and a family E of the subsets of V, which are called *edges*. A hypergraph is called *n-uniform* if every edge has size n. A *vertex r-coloring* of a hypergraph $H = (V,E)$ is a map from V to $\{1,\dots,r\}$. A coloring is *proper* if there are no monochromatic edges, i.e., any edge $e \in E$ contains two vertices of different color. The *chromatic number* of a hypergraph H is the smallest number $\chi(H)$ such that there exists a proper $\chi(H)$-coloring of H. Let $m(n,r)$ be the minimal number of edges in an n-uniform hypergraph with chromatic number more than r.

Erdős and Hajnal [7] introduced problems on determining $m(n,r)$ and related quantities. We are interested in the case when n is much smaller than r (see [10] for general case and related problems).

1.1. Upper bounds. Erdős conjectured [6] that

$$m(n,r) = \binom{(n-1)r+1}{n},$$

for $r > r_0(n)$, that is achieved on the complete hypergraph.

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However Alon [3] disproved the conjecture by using the estimate

$$m(n,r) < \min_{a \geqslant 0} T(r(n+a-1)+1, n+a, n),$$

where the Turán number $T(v,k,n)$ is the smallest number of edges in an n-uniform hypergraph on v vertices such that every induced subgraph on k vertices contains an edge. Different bounds on Turán numbers refine the complete hypergraph construction when $n > 3$ (see [11] for a survey). So the case $n = 3$ is in some sense the most interesting.

Also note, that using the same inequality with better bounds on Turán numbers Akolzin and Shabanov [2] showed that

$$m(n,r) < Cn^3 \ln n \cdot r^n.$$

Alon conjectured that the sequence $m(n,r)/r^n$ has a limit which was proved by Cherkashin and Petrov [4]. Denote the corresponding limit by L_n. In this paper we are interested in estimates on L_3. The best known upper bound follows from the complete hypergraph:

$$L_3 \leqslant \frac{4}{3}.$$

1.2. Lower bounds. There are several ways to show an inequality of type $m(n,r) \geqslant c(n)r^n$ (i.e. $L_n \geqslant c(n)$). Note that Erdős conjecture implies in particular that

$$L_n = \frac{(n-1)^n}{n!}.$$

Alon [3] suggested to color vertices of an n-uniform hypergraph in $a < r$ colors uniformly and independently, and then recolor a vertex in every monochromatic edge in unused color. The expected number of monochromatic edges is

$$|E| \cdot a^{1-n}.$$

Note that we have $r - a$ remaining colors, and we can color $n-1$ vertices in each unused color such that no new monochromatic edge appears. Summing up, if

$$|E| < a^{n-1}(r-a)(n-1)$$

then a hypergraph $H = (V,E)$ has a proper r-coloring. Substituting $a = \left\lfloor \frac{n-1}{n} r \right\rfloor$, we get

$$m(n,r) \geqslant (n-1) \left\lceil \frac{r}{n} \right\rceil \left\lfloor \frac{n-1}{n} r \right\rfloor^{n-1}.$$

This method gives $L_3 \geqslant 8/27 > 0.296$.

Another way is due to Pluhár [9]. He introduced the following useful notion. A sequence of edges $a_1, \ldots, a_r$ is an *r-chain* if $|a_i \cap a_j| = 1$ if $|i-j| = 1$ and $a_i \cap a_j = \varnothing$ otherwise; it is an *ordered r-chain* if $i < j$ implies that every vertex of a_i is not bigger than any vertex of a_j (with respect to a certain fixed linear ordering on V).

Pluhár's theorem states that existence of an order on V without ordered r-chains is equivalent to r-colorability of $H = (V, E)$. Let us prove a lower bound on $m(n, r)$ via this theorem. Consider a random order on the vertex set. Note that the probability of an r-chain to be ordered is

$$\frac{[(n-1)!]^2[(n-2)!]^{r-2}}{((n-1)r+1)!}.$$

From the other hand, the number of r-chains is at most $2|E|^r/r!$ since every set of r edges generates at most 2 chains. So if

$$2\frac{|E|^r}{r!}\frac{[(n-1)!]^2[(n-2)!]^{r-2}}{((n-1)r+1)!} < 1,$$

then we have a proper r-coloring of H. After taking r-root and some calculations we have

$$m(n, r) > c\sqrt{n}r^n,$$

and in particular $L_3 \geqslant 4/e^3 > 0.199$.

Combining two previous arguments with Cherkashin–Kozik approach [5] Akolzin and Shabanov [2] proved that

$$m(n, r) \geqslant c\frac{n}{\ln n}r^n,$$

without explicit bounds on c. We show that this method gives the bound $L_3 > 0.205$ in Section 3.

Cherkashin and Petrov [4] suggested an approach, based on the evaluation of the inverse function, to show that the sequence $m(n, r)/r^n$ has a limit. Denote by $f(N)$ the maximal possible chromatic number of an n-uniform hypergraph with N edges. Also $f(0) = 1$ by agreement. The function f non-strictly increases and satisfies

$$m(n, r) = \min\{N : f(N) > r\}.$$

Therefore $m(n, r) \sim Cr^n$ if and only if $f(N) \sim (N/C)^{1/n}$. The following lemmas were proved in [4].

Lemma 1. *For any $N > 0$ and any positive integer p we have*
$$f(N) \leqslant \max_{a_1+a_2+\dots+a_p \leqslant N/p^{n-1}} f(a_1) + f(a_2) + \dots + f(a_p).$$

Lemma 2. *Denote $c_n = \lceil (1 - 2^{1/n-1})^{-n} \rceil$. For any $M > 0$ the inequality*
$$f(N) \leqslant N^{1/n} \cdot \max_{M \leqslant a < c_n M} f(a) \cdot a^{-1/n}$$
holds for all $N \geqslant M$.

It is known that $f(0) = 1$, $f(1) = \ldots = f(6) = 2$, $f(7) = \ldots = f(26) = 3$ (see [1]). Lemmas 1, 2 and computer calculations were used to get
$$L_3 > 0.324.$$

The contribution of the paper is the following theorem, which is proved by refining Pluhár approach via inducibility arguments.

Theorem 1.
$$L_3 \geqslant \frac{4}{e^2} > 0.54.$$

Structure of the paper. In Section 2 we show how to apply inducibility to the chain argument and prove Theorem 1. In Section 3 we find the constant in Akolzin–Shabanov theorem for $n = 3$ and show that even if we apply Theorem 2 to the corresponding part of the proof, the constant will be still worse than in Theorem 1.

§2. Inducibility tool

Theorem 2. *Suppose $H = (V, E)$ is a hypergraph. Then it has at most*
$$|E| \left(\frac{|E| - 1}{r - 1} \right)^{r-1}$$
r-chains.

We need a notion of inducibility. Denote by $I(G, H)$ the number of induced subgraphs of G, isomorphic to H. Let P_r be a graph with r vertices and $r - 1$ edges which form a simple path. The following basic bound was proved by Pippenger and Golumbic.

Lemma 3 (Pippenger–Golumbic [8])**.** *Let G be a graph on N vertices. Then*
$$I(G, P_r) \leqslant \frac{N}{2} \left(\frac{N-1}{r-1} \right)^{r-1}.$$

It turns out that the bound is close to optimal. The following example is about $e^2/2$ times worse than the bound in Lemma 3 (we assume that r is fixed and n tends to infinity).

Example 1. *We construct the sequence of graphs G_k inductively. Let G_1 be a copy of C_{r+1}. Define an auxiliary graph $F_k = (V_k, E_k)$ (which is the $(r+1)^{k-1}$-blow-up of C_{r+1}):*

$$V_k := W_k^1 \sqcup W_k^2 \sqcup \cdots \sqcup W_k^{r+1}$$

with $|W_k^i| = (r+1)^{k-1}$ for all i; edges connect all the pairs of vertices from parts with adjacent indices (i is adjacent to $i+1$ modulo $r+1$, in particular $r+1$ is adjacent to 1). Then, G_k is obtained from F_k by drawing the graph G_{k-1} on each vertex set W_k^i.

Now consider the graph G_k on $N = (r+1)^k$ vertices. Note that

$$\begin{aligned} I(G_k, P_r) &= I(F_k, P_r) + (r+1)I(G_{k-1}, P_r) \\ &= (r+1)\left(\frac{N}{r+1}\right)^r + (r+1)I(G_{k-1}, P_r). \end{aligned}$$

Proof of Lemma 3. Let $X(q,l)$ denote the largest possible number of ways of sequentially choosing q objects $w_0, w_1, \ldots, w_{q-1}$ from among l objects, subject to rules whereby the set of objects that are eligible to be chosen as w_i depends only on the previous choices $w_0, w_1, \ldots, w_{i-1}$, and whereby no object that is eligible to be chosen as w_i will be eligible to be chosen as w_j for any $i+1 \leqslant j \leqslant q-1$. Also, define $X(0,l) = 1$. If $q > 0$, let m denote the number of objects eligible to be chosen as w_0. For any choice of w_0, the remaining $q-1$ objects can be chosen in at most $X(q-1, l-m)$ ways. Thus

$$X(q,l) \leqslant \max_{1 \leqslant m \leqslant l} mX(q-1, l-m).$$

From these relations, we obtain

$$X(q,l) \leqslant \left(\frac{l}{q}\right)^q \tag{1}$$

by induction on q: the base $q = 1$ is obvious. To prove the step it is enough to maximize the right-hand side of

$$X(q,l) \leqslant \max_{1 \leqslant m \leqslant l} m \left(\frac{l-m}{q-1}\right)^{q-1}.$$

Taking the derivative with respect to m, we get the maximum at $m = l/q$, and we are done.

Now we are ready to prove the initial statement. Fix the first vertex v_0. The number of ways to continue an induced r-path is at most $X(r-1, N-1)$. There are N ways to choose the first vertex and every copy of induced P_r is counted twice. Substitution of (1) finishes the proof. □

Proof of Theorem 2. Consider an auxiliary graph $G = (E, F)$ with vertex set being equal to the edge set of H and edges connecting pairs of vertices which intersect (as hyperedges) on exactly one vertex.

Note that every r-chain forms induced P_r in G (note that the reverse consequence is wrong, because a non-edge in G can correspond to the pair of hyperedges with large intersection, which is impossible in r-chain). Every copy of P_r is formed by at most two different r-chains, so the number of r-chains is at most $2I(G, P_r)$. Hence, Lemma 3 finishes the proof. □

Proof of Theorem 1. Let us try to color H via Pluhár's greedy algorithm. Recall that the probability of an r-chain to be ordered is

$$\frac{[(n-1)!]^2[(n-2)!]^{r-2}}{((n-1)r+1)!} = \frac{4}{(2r+1)!}.$$

Using Theorem 2 we get that if

$$\frac{|E|(|E|-1)^{r-1}}{(r-1)^{r-1}}\frac{4}{(2r+1)!} < 1,$$

then hypergraph is r-colorable. Summing up,

$$L_3 \geqslant \lim_{r\to\infty} \sqrt[r]{\frac{(2r+1)!(r-1)^{r-1}}{4}}\frac{1}{r^3} = \frac{4}{e^2}. \qquad \square$$

§3. Analysis of the Akolzin–Shabanov proof

We rewrite the proof from [2] with optimization in the case $n = 3$.

First, for every vertex v introduce the weight $w(v)$ as randomly (accordingly to the uniform distribution and independently) chosen number from $[0, 1]$. Fix parameters $p \in [0, 1]$, $a < r$. An edge e is called *bad* if

$$\max_{v\in e} w(v) - \min_{v\in e} w(v) \leqslant \frac{1-p}{a};$$

otherwise it is called *good*.

The coloring algorithm is the following. First we color a (random) subhypergraph, consisting of all good edges, in a colors via Pluhár approach; then we color (or recolor) some vertices from bad edges in unused $r - a$ colors. If Pluhár approach succeeds (i.e. there are no ordered

a-chains) and we have at most $(n-1)(r-a)$ bad edges, then the algorithm return a proper r-coloring. Let us evaluate the probability of success.

Lemma 4 (Akolzin–Shabanov [2]). *Let e be an edge, then*

$$P\Big[e \text{ is bad}\Big] = \Big(\frac{1-p}{a}\Big)^{n-1}\Big(\frac{1-p}{a} + n\Big(1 - \frac{1-p}{a}\Big)\Big) \leqslant n\Big(\frac{1-p}{a}\Big)^{n-1} = 3\Big(\frac{1-p}{a}\Big)^{2}.$$

Let $C(A_1, \dots, A_a)$ denote the event that all the edges A_j are good and $(A_1, \dots, A_a)$ is an ordered a-chain.

Lemma 5 (Akolzin–Shabanov [2]).

$$P\left[C(A_1, \dots, A_a)\right] \leqslant a^{-a(n-2)} \frac{p^{a-1}}{(a-1)!} = a^{-a} \frac{p^{a-1}}{(a-1)!}.$$

By Theorem 2 we have at most $(|E|/(a-1))^{a-1}$ a-chains. Define $c = |E|/r^3$; we need

$$\Big(\frac{|E|}{a-1}\Big)^{a-1} a^{-a} \frac{p^{a-1}}{(a-1)!} = \Big((1+o(1))\frac{|E|pe}{a^3}\Big)^{a} = \Big((c+o(1))\frac{r^3pe}{a^3}\Big)^{a} < 1.$$

Also we need at most $(n-1)(r-a) = 2(r-a)$ bad edges:

$$P[X > 2(r-a)] \leqslant \frac{1}{2(r-a)} \frac{3(1-p)^2|E|}{a^2} < 1.$$

Define $x = r/a$. Then we need $cx^3pe < 1$ and

$$\frac{3c(1-p)x^3}{2(x-1)} < 1.$$

Computer simulations give that for $p = 0.741$ and $x = 1.05$ the algorithm with $c = 0.42$ returns a proper coloring with positive probability, which implies $L_3 > 0.42$.

If we simply follow the initial proof, the required inequalities are

$$cx^3pe^2 < 1 \quad \text{and} \quad \frac{3c(1-p)x^3}{2(x-1)} < 1.$$

So pure Akolzin–Shabanov approach gives $L_3 > 0.205$. Both constants are worse than in Theorem 1.

§4. Open problems

• First, recall that the Erdős conjecture is still open in the case $n = 3$.
• Also it is natural to ask if $m(n, r)$ is regular on the first variable, i.e.

$$\lim_{n\to\infty} \frac{m(n+1, r)}{m(n, r)} = r?$$

• In the proof of Theorem 2 we consider an auxiliary graph G. The problem is to describe the set of graphs, which may be achieved from an r-chromatic n-uniform hypergraph. Also it may be reasonable to evaluate the minimal number of vertices $N(r)$ in a graph G, which has an ordered induced r-path in every linear order of $V(G)$.

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Chebyshev Laboratory,
St.Petersburg State University;
Moscow Institute of Physics and Technology,
Moscow region, 141700, Russia;
National Research University Higher School of Economics
St. Petersburg, Russia
E-mail: matelk@mail.ru

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