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On the chromatic numbers of small-dimensional Euclidean spaces

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ABSTRACT

This paper is devoted to the study of the graph sequence $G_n = (V_n, E_n)$, where V_n is the set of all vectors $v \in \mathbb{R}^n$ with coordinates in $\{-1, 0, 1\}$ such that $|v| = \sqrt{3}$ and E_n consists of all pairs of vertices with scalar product 1. We find the exact value of the independence number of G_n. As a corollary we get new lower bounds on $\chi(\mathbb{R}^n)$ and $\chi(\mathbb{Q}^n)$ for small values of n.

1. Introduction

Let $\mathbb{R}^n$ be the standard Euclidean space, where the distance between any two points x, y is denoted by $|x-y|$. Let V be an arbitrary point set in $\mathbb{R}^n$. Let $a > 0$ be a real number. By a *distance graph* with set of vertices V, we mean the graph $G = (V, E)$ whose set of edges E contains *all* pairs of points from V that are at the distance a apart:

$$E = \{\{x, y\} : \ |x-y| = a\}.$$

Distance graphs are among the most studied objects of combinatorial geometry. First of all, they are at the ground of the classical Hadwiger–Nelson problem, which was proposed around 1950 (see [12,27]) and consists in determining the *chromatic number of the space*:

$$\chi(\mathbb{R}^n) = \min\left\{\chi : \ \mathbb{R}^n = V_1 \sqcup \cdots \sqcup V_\chi, \ \forall\, i \ \forall\, \mathbf{x}, \mathbf{y} \in V_i \ \ |\mathbf{x}-\mathbf{y}| \neq 1\right\},$$

i.e., the minimum number of colors needed to color all the points in $\mathbb{R}^n$ so that any two points at the distance 1 receive different colors. In other words, it is the chromatic number of the unit distance graph whose vertex set coincides with $\mathbb{R}^n$.

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Due to the extreme popularity of the subject, colorings of unit distance graphs are very deeply explored. Let us just refer the reader to several books and survey articles [21,2,5,14,23,25,24,26,28]. In particular, the best known lower bounds for the chromatic numbers in dimensions $\leqslant 12$ are given below [23,20,8,4,6,18,16,17,15]:

$$\chi(\mathbb{R}^2) \geqslant 4\ [23],\ \ \chi(\mathbb{R}^3) \geqslant 6\ [20],\ \ \chi(\mathbb{R}^4) \geqslant 9\ [8],\ \ \chi(\mathbb{R}^5) \geqslant 9\ [4],\ \ \chi(\mathbb{R}^6) \geqslant 11\ [6],\ \ \chi(\mathbb{R}^7) \geqslant 15\ [23],$$

$$\chi(\mathbb{R}^8) \geqslant 16\ [18],\ \ \chi(\mathbb{R}^9) \geqslant 21\ [16],\ \ \chi(\mathbb{R}^{10}) \geqslant 23\ [16],\ \ \chi(\mathbb{R}^{11}) \geqslant 25\ [17],\ \ \chi(\mathbb{R}^{12}) \geqslant 27\ [15].$$

Recently further improvements were announced [7,13]:

$$\chi(\mathbb{R}^6) \geqslant 12\ [7],\ \ \chi(\mathbb{R}^7) \geqslant 16\ [7],\ \ \chi(\mathbb{R}^8) \geqslant 19\ [13],\ \ \chi(\mathbb{R}^{10}) \geqslant 26\ [7], [13],\ \ \chi(\mathbb{R}^{11}) \geqslant 32\ [13],\ \ \chi(\mathbb{R}^{12}) \geqslant 36\ [7].$$

These improvements are essentially based on computer calculations.

In growing dimensions, the following bounds are the best known [22,18]:

$$[22]\ \ (1.239\ldots + o(1))^n \leqslant \chi(\mathbb{R}^n) \leqslant (3 + o(1))^n\ \ [18].$$

In this paper, we consider a special sequence of graphs defined in the following way.

Let V_n be the set of all vectors v from $\mathbb{R}^n$ with coordinates in $\{-1, 0, 1\}$ and $|v| = \sqrt{3}$. The set V_n can be considered as the set of vertices of a graph $G_n = (V_n, E_n)$, where an edge connects two vertices if and only if the corresponding vectors have scalar product 1. Note that G_1 and G_2 are empty and G_3 is just a cube.

Recall that an *independent set* in a graph is any set of its vertices which are pairwise non-adjacent and the *independence number* of G denoted by $\alpha(G)$ is the size of a maximum independent set in the graph G.

Theorem 1. *For $n \geqslant 1$, let $c(n)$ denote the following constant:*

$$c(n) = \begin{cases} 0 & \textit{if}\ \ n \equiv 0 \\ 1 & \textit{if}\ \ n \equiv 1 \\ 2 & \textit{if}\ \ n \equiv 2 \textit{ or } 3 \end{cases} \pmod 4.$$

Then, the independence number of G_n is given by the formula

$$\alpha(G_n) = \max\{6n - 28, 4n - 4c(n)\}.$$

Actually, the result of Theorem 1 is a far-reaching generalization of a much simpler lemma proved by Zs. Nagy (see [19]) in 1972 and used not only in combinatorial geometry, but also in Ramsey theory. In this lemma, $G'_n = (V'_n, E'_n)$, where V'_n is the set of all vectors v, $|v| = \sqrt{3}$, with coordinates in $\{0, 1\}$ and again an edge connects two vertices if and only if the corresponding vectors have scalar product 1. Lemma states that in this case $\alpha(G'_n) = n - c(n)$.

Larman and Rogers used the mentioned lemma to prove $\chi(\mathbb{R}^n) \geqslant (1 + o(1))n^2/6$ (in fact, it was suggested by Erdős and Sós), which was the first nontrivial lower bound on $\chi(\mathbb{R}^n)$. It is worth noting that the chromatic number of G'_n almost coincides with the bound $n/\alpha(G'_n)$, as was shown in [1].

On the other hand there is a natural bijection between $\{0, 1\}^n$ and the subsets of n-element set, which gives deep combinatorial sense to graphs of the mentioned types. In several recent papers [9,11,10] Frankl and Kupavskii consider analogues of some classical combinatorial problems in $\{0, \pm 1\}$ setup.

The proof of Theorem 1 is given in the following parts: some examples showing the lower bound in Theorem 1 and some preliminaries are given in Section 2; the upper bound is proved in Section 3 (for the case $n \leqslant 13$ we use computer simulations). Note that, roughly speaking, the quantity 13 is a threshold where the bound $6n - 28$ starts dominating the bound $4n$.

As a corollary of Theorem 1 we get the following bounds for the chromatic numbers of Euclidean spaces.

Theorem 2. *Let $c(n)$ be the constant defined in Theorem 1. Then, for all $n \geqslant 3$, we have*

$$\chi(\mathbb{R}^n) \geqslant \chi(\mathbb{Q}^n) \geqslant \chi(G_n) \geqslant \frac{|V_n|}{\alpha(G_n)} = \frac{8\binom{n}{3}}{\max\{6n - 28, 4n - c(n)\}}.$$

Asymptotically, the bound in this theorem is $\frac{2}{9}n^2(1 + o(1))$, which is a weak result. On the other hand, for small values of n, the theorem gives the best known bounds, namely:

$$\chi(\mathbb{R}^9) \geqslant \chi(\mathbb{Q}^9) \geqslant 21,$$

$$\chi(\mathbb{R}^{10}) \geqslant \chi(\mathbb{Q}^{10}) \geqslant 30,$$

$$\chi(\mathbb{R}^{11}) \geqslant \chi(\mathbb{Q}^{11}) \geqslant 35,$$

$$\chi(\mathbb{R}^{12}) \geqslant \chi(\mathbb{Q}^{12}) \geqslant 37.$$

Actually, we will show in Section 4 the following stronger result for $n = 9$.

Proposition 1. *The inequalities hold*

$$\chi(\mathbb{R}^9) \geqslant \chi(\mathbb{Q}^9) \geqslant 22.$$

2. Lower bounds in Theorem 1 and some preliminaries

2.1. Auxiliary definitions

Consider the graph G_n. Any of its vertices has three non-zero coordinates and $n-3$ coordinates equal to 0. We call *base* the set of non-zero coordinates of a vertex. To make our exposition more concise, we will use the word "place" instead of the word "coordinate" or instead the expression "coordinate position". For example, it will be convenient to say (a bit informally) "vertex v intersects place x", if the vector v from $\mathbb{R}^n$ corresponding to this vertex has nonzero value of the coordinate v_x. For the same reasons, we introduce the notion of a *signplace*: it is a coordinate with a fixed sign (plus or minus). In particular, from now on, we can say (again, a bit informally) "vertex v intersects signplace x^+ (x^-)", if it has the value of the coordinate v_x equal to $+1$ (-1). Finally, we define the *degree* of a place (signplace) in a set W of vertices of G_n as the number of vertices from W intersecting this place (signplace).

2.2. Constructions of independent sets in G_n

It suffices to show that $\alpha(G_n) \geqslant 4n - 4c(n)$ and that $\alpha(G_n) \geqslant 6n - 28$.

The first construction is as follows. Consider the first 4 places. Take all the 4 bases that can be taken on these places. For each base, consider 4 variants:

$$1, 1, 1; \quad 1, -1, -1; \quad -1, 1, -1; \quad -1, -1, 1.$$

Clearly any two vectors with these bases have scalar product different from 1. We call this construction (and its natural analogues) *quad*.

Take $[n/4]$ consecutive quads. If the remainder still consists of 3 places, then add 4 more bases. Eventually, we get exactly $4n - 4c(n)$ vectors that form an independent set in G_n.

Now, let us make the second construction. Take the following vectors:

$$1, -1, 0, 1, 0, \dots, 0, 0, 0, 0; \;\; 1, -1, 0, 0, 1, 0, \dots, 0, 0, 0, 0; \;\; \dots; \;\; 1, -1, 0, 0, 0, \dots, 0, 1, 0, 0, 0;$$

$$0, 1, -1, 1, 0, \dots, 0, 0, 0, 0; \;\; 0, 1, -1, 0, 1, 0, \dots, 0, 0, 0, 0; \;\; \dots; \;\; 0, 1, -1, 0, 0, \dots, 0, 1, 0, 0, 0;$$

$$-1, 0, 1, 1, 0, \dots, 0, 0, 0, 0; \;\; -1, 0, 1, 0, 1, 0, \dots, 0, 0, 0, 0; \;\; \dots; \;\; -1, 0, 1, 0, 0, \dots, 0, 1, 0, 0, 0.$$

In each line, we have a set of vectors, which is a particular case of what we will call *snake* in Section 3 and later. In every snake, we have $n-6$ vertices. Thus, the total amount of vertices here is $3n-18$. Obviously, the union of these snakes is an independent set in G_n. Moreover, we can add to it 4 more vectors, which have a common base —the three first places: say,

$$1, 1, 1, 0, \dots, 0; \quad 1, -1, -1, 0, \dots, 0; \quad -1, 1, -1, 0, \dots, 0; \quad -1, -1, 1, 0, \dots, 0.$$

The whole construction is a particular case of a *cobra* discussed later in more details. Here the cobra contains $3n-14$ vertices.

Of course, we can take one more cobra, whose "head" is on the three last places and whose "tail" consists of minus ones instead of ones. Eventually, we get exactly $6n-28$ vertices forming an independent set in G_n.

The lower bound is proven.

It is worth noting that in Section 3 we will make a rather subtle analysis of possible independent sets in G_n. One would be able to derive from this analysis a complete description of examples giving the lower bound in Theorem 1. However, we will not present such description explicitly in this paper.

It is also worth noting that in the above example having $6n-28$ vertices and avoiding the scalar product 1, the scalar product -3 is also absent. Moreover, one can exclude 6 vertices from that example so that the scalar product -2 disappears as well.

2.3. Basic lemma

Let A be an arbitrary independent set of the maximum size in G_n. We already know that $|A| \geqslant \max\{6n-28, 4n-4c(n)\}$. Assume that we exclude some signplaces and all the vertices from the graph G_n intersecting them. Then we get a new graph G' with a possibly smaller independent set A'. Denote by $a(A')$ the maximum degree of a signplace in the set A'. Denote by $m(A')$ the number of signplaces in A'.

The following lemma is an important ingredient in the proof of the upper bound.

Lemma 1. *Assume that we exclude k signplaces. Assume that the number of vertices excluded from A does not exceed $2k$. Then we have either $a(A') \geqslant 5$ or $m(A') < 14$.*

Proof of the lemma. By pigeon-hole principle $a(A') \geqslant 3(|A| - 2k)/(2n - k)$. If $|A| \geqslant 4n$, then $a(A') \geqslant 6$ and we are done. The inequality $|A| \geqslant 4n$ is true for $n = 8, 12$ and $n \geqslant 14$. Thus, it remains to consider only $n = 7, 9, 10, 11, 13$. If $n = 7$, then $|A| \geqslant \max\{14, 20\} = 20$. If $k = 1$, then $3(|A| - 2k)/(2n - k) \geqslant 54/13$, i.e., $a(A') \geqslant 5$. If $k \geqslant 2$, then it may happen that $3(|A| - 2k)/(2n - k) \leqslant 4$. But in this case, $m(A') = 2n - k \leqslant 14 - 2 < 14$. The same argument works for the 4 other values of n. The proof is complete.

3. Proof of Theorem 1

If $n \leqslant 13$, one can prove the theorem via computer simulations using the standard Bron–Kerbosch algorithm (see [3]). It is worth noting that the case $n = 7$ was considered by Cibulka in [6].

3.1. Starting the proof

Let A be an arbitrary independent set of the maximum size in G_n. Assume that we have already excluded several signplaces with the corresponding vertices (see Section 2.3). By Lemma 1 either $a(A') \geqslant 5$ or $m(A') \leqslant 13$. The second case will be considered in Section 3.4. So we assume that $a(A') \geqslant 5$.

Consider a signplace with the maximum degree. Call it x_1^+ (each time when we choose a sign we can choose plus without loss of generality) and consider the set of vertices intersecting it (denote it by $N_{x_1^+}$). Note that no base can contain more than two vertices from $N_{x_1^+}$. Thus, we have at least three different bases. Also it is clear that any two bases containing vertices from $N_{x_1^+}$ intersect in exactly two signplaces. There are two different possibilities.

(1) Among the bases, we have $\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}$. This case will be referred to as "quad" (cf. Section 2.2).
(2) All the bases contain both x_1 and x_2. This case will be referred to as "snake" (cf. Section 2.2).

The formal definition of a quad will be given in the next section, where we will analyze Case (1). The same is for a snake in Section 3.3. In Section 3.4 we will complete the proof.

3.2. The first case — "quad"

We know that $a(A') \geqslant 5$. At the same time, $a(A') \leqslant 6$, since otherwise the vertices from $N_{x_1^+}$ use at least 4 bases and therefore there is a base among $\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}$ such that it intersects the fourth base only on x_1^+, which is impossible. Put $a = a(A')$.

Thus, we have exactly three bases containing the vertices from $N_{x_1^+}$. Two of them (without loss of generality $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$) contain exactly 2 vertices from $N_{x_1^+}$ each, and the third one contains at least 1 vertex. Since $\{x_1^+, x_2, x_3\}$ contains two vertices, it intersects all the four signplaces in x_2, x_3; the same holds for $\{x_1^+, x_2, x_4\}$, which means that all the six signplaces of x_2, x_3, x_4 are necessarily intersected.

Consider the set U of all vertices intersecting $\{x_1, x_2, x_3, x_4\}$. There could be the following possibilities.

- Some vertices from U intersect x_1. There are at most $2a$ such vertices.
- Some vertices from U lie on the base $\{x_2, x_3, x_4\}$. There are at most 4 such vertices.
- Some vertices from U intersect $\{x_2, x_3, x_4\}$ in one place and are not counted above. Actually, there are no such vertices because for every signplace in $\{x_2, x_3, x_4\}$ a vertex with a base in $\{x_1, x_2, x_3, x_4\}$ exists (do not forget that $\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}$ contain exactly 2 vertices each, and the third base contains at least 1 vertex).
- Some vertices from U intersect $\{x_2, x_3, x_4\}$ in two places and are not counted above. Again, there are no such vertices. Indeed, assume that some vertex (call it v) intersects $\{x_2, x_3, x_4\}$ in $\{x_i, x_j\}$. Then $\{x_1, x_2, x_3\}$ or $\{x_1, x_2, x_4\}$ intersects $\{x_i, x_j\}$ in exactly one place. This is impossible, since we know that two vertices from $N_{x_1^+}$ lie on $\{x_1, x_2, x_3\}$ and two vertices from $N_{x_1^+}$ lie on $\{x_1, x_2, x_4\}$.

Summarizing, we have at most $2a + 4 \leqslant 16$ vertices intersecting 8 signplaces. We call any of the corresponding constructions *quad*.

Now we may assume that A was transformed into A' in the following way (more details will be given in Section 3.4).

- First, all the signplaces of degree less than 3 have been deleted one by one. Note that by Lemma 1 during this process either $a \geqslant 5$ or $m \leqslant 13$.
- Second, all the quads have been deleted one by one. Note that again by Lemma 1 during this process either $a \geqslant 5$ or $m \leqslant 13$ (at every step the number of excluded signplaces is 8 and the number of excluded vertices is at most 16).
- Third, once again, all the signplaces of degree less than 3 have been deleted one by one. Obviously, there are no new quads and still by Lemma 1 $a \geqslant 5$ or $m \leqslant 13$.

As before, we assume that $a \geqslant 5$ (since the case $m \leqslant 13$ is considered in Section 3.4), and so we are prepared to the next case, in which we have $a(A') \geqslant 5$, there are no quads, and every signplace has degree at least 3.

3.3. The second case — "snake"

We start with a formal definition of a snake.

Definition 1. **Snake** is a set of vertices intersecting a signplace and a place and containing at least 5 vertices. **Head** of a snake is a couple of places, which intersect every vertex, and **tail** of a snake is the set of the remaining signplaces in each vertex. **Size** of a snake is the number of its vertices.

Clearly in the current case we have a snake of size $a \geqslant 5$ in A'. Let it be based on $\{x_1^+, x_2\}$ (with the head being $\{x_1, x_2\}$). Note that the size of its tail is equal to a, since vertices cannot intersect on tail.

Our aim is to prove that we can exclude some t signplaces with at most $3t - 14$ vertices. Moreover, we will show that there is a special construction ("cobra", cf. Section 2.2), which has exactly $3t - 14$ vertices on t signplaces and which is the only such construction up to the graph symmetries.

We have an alternative.

(1) We can exclude $4 + a$ signplaces ($\{x_1, x_2\}$ with all possible signs and a signplaces of the tail) and $3a - 2$ vertices.
(2) We have at least $3a - 1$ vertices intersecting the signplaces mentioned in the previous point.

In the first case, our aim is realized, since we can put $t = 4 + a$ and get $3t - 14 = 3a - 2$. In the second case, the analysis will be much longer.

Let us consider the second case of the alternative. Each vertex intersecting the tail of the snake that we analyze should intersect the head as well, and each of the a initial vertices intersects the head on two signplaces. Hence the sum of the degrees of the head signplaces is at least $4a - 1$. But there is no signplace with degree exceeding a, so the degrees of the signplaces in the head are either

$$a, a, a, a \quad \text{or} \quad a, a, a, a - 1.$$

Anyway we have a place with two signplaces of degree exactly a. Without loss of generality, this place is x_1. Since all quads are already excluded, we have two snakes with signplaces on x_1: one signplace is x_1^+ and the second one is x_1^-. Consider their heads. They could both lie on $\{x_1, x_2\}$, or they could lie on $\{x_1, x_2\}$ and $\{x_1, x_3\}$, respectively.

In the first case, all the four signplaces of the head have degree a solely due to $2a$ vertices from the snakes. In addition, there are vertices intersecting the tail (since the degree of each signplace is at least 3 and two snakes could provide only two vertices on a signplace). Each vertex intersecting the tail should intersect the head as well, so the degree of some signplace in the head exceeds a, which contradicts the assumption that a is the maximum value of the degree.

We are left with the second case: there are two snakes of size a with heads on $\{x_1, x_2\}$ and $\{x_1, x_3\}$.

Let Q be the set of vertices lying fully on base $\{x_1, x_2, x_3\}$. Denote by B the set of signplaces in the intersection of the tails. Let C_1 and C_2 be the sets of the remaining signplaces in the corresponding tails. Let q, b, c_1, c_2 be the sizes of the corresponding sets. We have already described all the vertices intersecting x_1, since the maximum degree is equal to a. Consider the sum of the degrees of the signplaces on x_2 and x_3. Since the degree of each signplace is at least 3, we have a new vertex for each signplace from the intersection of the tails. Each vertex of this type should intersect both heads, and it cannot contain x_1. Therefore, it contains both x_2 and x_3 and adds 2 to our sum. We have at least two vertices intersecting each signplace of the symmetric difference of the tails. Each vertex of this type should intersect the head of a corresponding snake and could intersect two signplaces of its tail. In total, these vertices add at least

$$2(c_1 + c_2)/2 = c_1 + c_2$$

to the sum. Each of the $2a$ initial vertices intersects $\{x_2, x_3\}$. Each vertex from Q adds yet another 1 to the sum, since it intersects $\{x_2, x_3\}$ on two places. Again in total, the sum of the degrees of the four signplaces on $\{x_2, x_3\}$ is at least

$$2b + c_1 + c_2 + 2a + q.$$

On the other hand, since the degree of each signplace is at most a, this sum does not exceed $4a$. So we have

$$2b + c_1 + c_2 + q \leqslant 2a.$$

Each vertex from Q is in one snake. Consequently, $q = q_1 + q_2$ (q_i is the number of vertices lying in a corresponding snake),

$$b + q_1 + c_1 = a, \quad b + q_2 + c_2 = a,$$

and the inequality always turns to equality!

Thus, there is a set of

$$t := 6 + b + c_1 + c_2$$

signplaces intersected by

$$q + 3b + 3c_1 + 3c_2 \leqslant 3(b + c_1 + c_2) + 4 = 3t - 14$$

vertices.

The second case of the alternative is complete, and our aim is attained. However, we will also prove below an upper bound on t.

Suppose that the number of vertices is exactly $3t - 14$. It means that all the intermediate inequalities turned to equalities. The last inequality turns to equality only when $q = 4$. One can see that any vertex intersecting a signplace from C_1 or C_2 should intersect 2 vertices of the tail, so it intersects $\{x_1, x_2, x_3\}$ only on 1 vertex, which contradicts $q = 4$. Hence $C_1 = C_2 = \emptyset$ and $b = a - 2$. For every signplace from $x^+ \in B$ there is a vertex intersecting x^+ and lying on base $\{x, x_2, x_3\}$. It turns out that there is a third snake on $\{x_2, x_3\}$. We call *cobra* the union of such 3 snakes. Finally, one can see that there is no place x such that x^+, x^- lie in B, otherwise there is an edge between two vertices on the base $\{x_1, x_2, x\}$.

As a result, t does not exceed $n + 3$, and the tail of a corresponding snake cannot contain two signplaces on the same place.

Summing up the above, if we have no quad, then there is a cobra, which consists of three snakes with a common tail and pairwise intersecting heads. It has $3t - 14$ vertices on t signplaces, $8 \leqslant t \leqslant n + 3$.

3.4. Finishing the proof

In the previous sections we have shown that there are the following options.

- To exclude a signplace and at most 2 vertices intersecting it.
- To exclude 8 signplaces and at most 16 vertices intersecting it.
- To exclude t signplaces with at most $3t - 14$ vertices ($8 \leqslant t \leqslant n + 3$).
- To get $m(A') \leqslant 13$.

Clearly the two first options yield at most $2m$ vertices on m signplaces. Computer simulations give that the same holds for the fourth option. According to this, only the following cases could occur.

(1) There is no cobra. Then the number of vertices does not exceed $4n \leqslant 6n - 28$.

(2) There is one cobra and $t \leqslant n$. Then the number of vertices does not exceed

$$3t - 14 + 2(2n - t) \leqslant 5n - 14 \leqslant 6n - 28.$$

(3) There is one cobra and $t = n + 1$. We are left to prove that $n - 1$ signplaces on $n - 3$ places can contain at most $2n - 3$ vertices. Suppose the contrary, then, by pigeon-hole principle, there is a signplace of degree at least $3(2n-2)/(n-1) = 6$. Using the same arguments as in Sections 3.1–3.3 we get a quad or a cobra, but both constructions contain 3 places with 2 signplaces, which contradicts our assumptions.

(4) There is one cobra and $t = n+2$. We are left to prove that $n-2$ signplaces on $n-3$ places can contain at most $2n-6$ vertices. Then the number of vertices does not exceed

$$3(n + 2) - 14 + 2n - 6 \leqslant 5n - 14 \leqslant 6n - 28.$$

Again, suppose the contrary, so there is a singplace of degree at least $3(2n-5)/(n-2)$. For $n < 4$ the claim is obvious, and for $n \geqslant 4$ we have $3(2n - 5)/(n - 2) > 4$. Using the same arguments as in Sections 3.1–3.3 we get a quad or a cobra, but both constructions contain 3 places with 2 signplaces. Thus, we get a contradiction.

(5) There is one cobra and $t = n + 3$. All other signplaces lie on distinct places, and we can apply Nagy's lemma (see Section 1 and [19]) to get an upper bound $n - 3$ for the number of vertices. Then the number of vertices does not exceed

$$3(n + 3) - 14 + n - 3 = 4n - 8 < 5n - 14 \leqslant 6n - 28.$$

(6) There are two or more cobras. Then the bound $6n - 28$ for the number of vertices is straightforward.

The proof of Theorem 1 for $n \geqslant 14$ is complete.

4. Proof of Proposition 1

Suppose that $\chi(G_9) = 21$. Clearly $\frac{|V(G_9)|}{\alpha(G_9)} = 21$, and therefore every color has size 32. By computer simulations we see that the only way to reach 32 vertices in an independent set is by taking a couple of full quads. Thus, we have a collection of 21 pairs of full quads (denote it by A); this collection covers each base exactly two times, since every full quad has exactly 4 vertices on every covered base. Note that every pair of quads does not cover exactly one place, so one can split A into nine disjoint parts:

$$A = A_1 \sqcup \cdots \sqcup A_9.$$

Let S_1 be the set of all bases such that each of them does not contain the first place. Obviously $|S_1| = \binom{8}{3} = 56$. Consider a pair of quads $p \in A$. Note that p covers 8 bases from S_1, if $p \in A_1$, and 5 bases from S_1 otherwise. Denote the cardinalities of A_1 and $A \setminus A_1$ by a and b, respectively. Every set in S_1 is covered twice, and therefore we have $2|S_1| = 112 = 8a + 5b$. Hence there are the following possibilities: $(a = 14, b = 0)$, $(a = 9, b = 8)$ and $(a = 4, b = 16)$. But $a + b = |A| = 21$, so we get a contradiction.

Proposition 1 is proved.

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