ISSN 1064-5624, Doklady Mathematics, 2017, Vol. 95, No. 1, pp. 5–6. © Pleiades Publishing, Ltd., 2017.
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MATHEMATICS

On the Chromatic Numbers of Low-Dimensional Spaces

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Presented by Academician of the RAS V.V. Kozlov August 9, 2016

Received September 20, 2016

Abstract—New lower bounds are found for the minimum number of colors needed to color all points of a Euclidean space in such a way that any two points at a distance of 1 have different colors.

DOI: 10.1134/S106456241701001X

One of the best-known problems in combinatorial geometry is the Hadwiger–Nelson one, which was formulated in the mid-20th century (see [1]). The task is to find the chromatic number $\chi(\mathbb{R}^n)$ of a Euclidean space, i.e., the minimum number of colors required for coloring all points of $\mathbb{R}^n$ so that no two points at a distance of 1 from each other have the same color.

Now it is only known that $\chi(\mathbb{R}^1) = 2$. For the other dimensions, there are only upper and lower bounds. Specifically, for $n \le 12$, the following lower bounds were the best until recently:

$$\chi(\mathbb{R}^2) \ge 4 \ [1], \quad \chi(\mathbb{R}^3) \ge 6 \ [2], \quad \chi(\mathbb{R}^4) \ge 7 \ [3-5],$$
$$\chi(\mathbb{R}^5) \ge 9 \ [3], \quad \chi(\mathbb{R}^6) \ge 11 \ [6], \quad \chi(\mathbb{R}^7) \ge 15 \ [1],$$
$$\chi(\mathbb{R}^8) \ge 16 \ [7], \quad \chi(\mathbb{R}^9) \ge 21 \ [8], \quad \chi(\mathbb{R}^{10}) \ge 23 \ [8],$$
$$\chi(\mathbb{R}^{11}) \ge 25 \ [9], \quad \chi(\mathbb{R}^{12}) \ge 27 \ [5].$$

Better bounds have been recently announced in dimensions 8, 10, 11, and 12:

$$\chi(\mathbb{R}^8) \ge 19 \ [10], \quad \chi(\mathbb{R}^{10}) \ge 26,$$
$$\chi(\mathbb{R}^{11}) \ge 32 \ [10], \quad \chi(\mathbb{R}^{12}) \ge 36 \ [11].$$

These bounds were mainly obtained with the help of computer calculation, which somewhat devalues them. We have managed to show that

$$\chi(\mathbb{R}^9) \ge 22, \quad \chi(\mathbb{R}^{10}) \ge 30, \quad \chi(\mathbb{R}^{11}) \ge 35,$$
$$\chi(\mathbb{R}^{12}) \ge 37.$$

Moreover, these bounds are special cases of a rather general result to be discussed in the following section.

1. IDEA OF THE PROOF AND THE GENERAL THEOREM

Recall that a graph $G = (V, E)$ is called a distance graph in $\mathbb{R}^n$ if $V \subset \mathbb{R}^n$ and $E = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = a\}$, where the modulus denotes the Euclidean distance and a is a positive real number. Recall also that the chromatic number of G is the minimum number $\chi(G)$ of colors required for coloring all the vertices of G so that any two vertices joined by an edge have different colors. Clearly, if G is a distance graph in $\mathbb{R}^n$, then $\chi(\mathbb{R}^n) \ge \chi(G)$.

The independence number $\alpha(G)$ of G is defined as the maximum number k for which there exists an independent set of vertices of G of cardinality k, i.e., a set of vertices such that no two of them are connected by an edge. It is easy to see that $\chi(G) \ge \dfrac{|V|}{\alpha(G)}$.

Consider the following sequence of distance graphs:

$$G_n = (V_n, E_n),$$
$$V_n = \{\mathbf{x} = (x_1, \ldots, x_n) : \ x_i \in \{-1, 0, 1\},$$
$$|\{i : \ x_i = \pm 1\}| = 3\}, \quad E_n = \{\{\mathbf{x}, \mathbf{y}\} : \ (\mathbf{x}, \mathbf{y}) = 1\},$$

where the round brackets denote the scalar product of vectors.

We managed to find $\alpha(G_n)$.

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Theorem 1. *Assume that* $c(0) = 0$, $c(1) = 1$, $c(2) = c(3) = 2$. *Let* $\{x\}$ *denote the fractional part of the number* x. *Then*

$$\alpha(G_n) = \max\left\{6n - 28,\ 4n - 4c\left(4\left\{\frac{n}{4}\right\}\right)\right\}.$$

The bounds

$$\chi(\mathbb{R}^{10}) \geq 30, \quad \chi(\mathbb{R}^{11}) \geq 35, \quad \chi(\mathbb{R}^{12}) \geq 37$$

follow directly from this theorem and the inequality

$$\chi(\mathbb{R}^n) \geq \chi(G_n) \geq \frac{|V_n|}{\alpha(G_n)} = \frac{8C_n^3}{\max\left\{6n - 28,\ 4n - 4c\left(4\left\{\frac{n}{4}\right\}\right)\right\}}.$$

The bound $\chi(\mathbb{R}^9) \geq 22$ requires a slightly higher degree of thoroughness when we use the inequality $\chi(G_n) \geq \frac{|V_n|}{\alpha(G_n)}$, which, at first glance, gives only a previously known bound of 21.

Note also that the graphs G_n are natural and important generalizations of the graphs

$$G(n,3,1) = (V(n,3), E(n,3,1)),$$

$$V(n,3) = \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \in \{0,1\},$$
$$|\{i : x_i = 1\}| = 3\}, \quad E(n,3,1) = \{\{\mathbf{x},\mathbf{y}\} : (\mathbf{x},\mathbf{y}) = 1\}.$$

They were profoundly and comprehensively studied, starting in the classical work [12]. Specifically, they played an important role in the Hadwiger–Nelson problem; namely, they were use to derive the bounds in [7], which were long regarded as the best.

Some more new results concerning the chromatic and independence numbers of G_n are given in the next section.

2. RANDOM SUBGRAPHS OF THE GRAPHS G_n

The authors of [13] investigated the random subgraphs $G_p(n,3,1)$ of the graphs $G(n,3,1)$, which are understood as random elements with values in the set of spanning subgraphs of the original graphs and with a binomial distribution ($p = p(n)$ is the probability of preserving each individual edge).

The random subgraphs $G_{n,p}$ of the graphs G_n are defined in a similar manner. The results obtained in [13–15] are stated below as Theorem 2.

Theorem 2. *With asymptotic probability* 1,

$$\alpha(G_{1/2}(n,3,1)) \sim 2n\log_2 n,$$

$$(1 + o(1)) \cdot \frac{1}{24} \cdot \frac{n^2}{\log_2 n} \leq \chi(G_{1/2}(n,3,1)) \leq (1 + o(1)) \cdot \frac{1}{8} \cdot \frac{n^2}{\log_2 n}.$$

We have proved the following result.

Theorem 3. *With asymptotic probability* 1,

$$(1 + o(1)) \cdot 4 \cdot n\log_2 n \leq \alpha(G_{n,1/2}) \leq (1 + o(1)) \cdot 24 \cdot n\log_2 n,$$

$$(1 + o(1)) \cdot \frac{1}{18} \cdot \frac{n^2}{\log_2 n} \leq \chi(G_{n,1/2}) \leq (1 + o(1)) \cdot 3 \cdot \frac{n^2}{\log_2 n}.$$

ACKNOWLEDGMENTS

Section 2 of this work was supported by the Russian Science Foundation, project no. 16-11-10014.

REFERENCES

1. A. M. Raigorodskii, Russ. Math. Surv. **56** (1), 103–139 (2001).
2. O. Nechushtan, Discrete Math. **256**, 499–507 (2002).
3. K. Cantwell, J. Combin. Theory A **73** (2), 273–285 (1996).
4. L. L. Ivanov, Russ. Math. Surv. **61** (5), 984–986 (2006).
5. A. B. Kupavskii, Sb. Math. **202** (6), 859–886 (2011).
6. J. Cibulka, Geombinatorics **18** (2), 53–66 (2008).
7. D. G. Larman and C. A. Rogers, Mathematika **19**, 1–24 (1972).
8. A. B. Kupavskii and A. M. Raigorodskii, J. Math. Sci. (New York) **163** (6), 720–731 (2009).
9. A. Kupavskiy, Discrete Math. **311** (6), 437–440 (2011).
10. M. Kahle and B. Taha, http://arxiv.org/pdf/1409.1278v1.pdf.
11. G. Exoo and D. Ismailescu, http://arxiv.org/pdf/1408.2002v1.pdf.
12. Z. Nagy, Mat. Lapok **23** (301–302), 26 (1972).
13. L. I. Bogolyubsky, A. S. Gusev, M. M. Pyaderkin, and A. M. Raigorodskii, Sb. Math. **206** (10), 1340–1374 (2015).
14. A. S. Gusev, Math. Notes **97** (3), 326–332 (2015).
15. M. M. Pyaderkin, Math. Notes **99** (2), 312–319 (2016).

Translated by I. Ruzanova