Author's note on the scientific contributions of the thesis:

"Extremal problems in Euclidean combinatorial geometry"

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1 Introduction

The dissertation is devoted to extremal problems in the intersection of Euclidean geometry and combinatorics. Consider a *distance graph* $G(\mathbb{R}^d)$ of a Euclidean space, which is a complete weighted graph with vertex set $\mathbb{R}^d$ and the weights from Euclidean metrics. A typical framework is G or its "subgraph" $G(V, \rho) = (V, E_\rho)$, where V is a subset of $\mathbb{R}^d$ and E_ρ consists of pairs of vertices at a distance of ρ. We consider both finite and infinite V. We focus on several classical combinatorial problems: Steiner tree problem, finding a maximal independent set and finding the chromatic number. Note that these three problems belong to the initial Karp's list of 21 NP-complete problems.

2 Contributions

The first chapter of the dissertation is devoted to the Euclidean Steiner tree problem. It is well-known that the Steiner problem may have several solutions; the simplest example is the vertices of a square. We show that this situation is rare for planar configurations. Namely, the set of planar n-point configurations for which the solution of the Steiner problem us unique has Hausdorff dimension $2n-1$ (as a subset of $\mathbb{R}^{2n}$).

Also we show that the set of n-point configurations in $\mathbb{R}^d$ for which there is a unique Steiner tree is path-connected (in the standard topology of $\mathbb{R}^{dn}$.

Finally, we provide the first example of an indecomposable Steiner tree for the input of Hausdorff dimension $-\frac{\ln 2}{\ln \lambda}$, for $\lambda < 1/300$. Let $A_\infty(\lambda)$ be the (uncountable, compact) set consisting of the root and the leaves of a fractal binary tree $\Sigma(\lambda)$ with the ratio of length of edges on consecutive levels equal to λ.

The Gilbert–Steiner problem is a generalization of the Steiner tree problem on a specific optimal mass transportation. We show that a solution of the planar Gilbert–Steiner problem has no branching point of degree at least 4.

In Chapter 4 we consider the problem of minimizing the maximal distance to a given compact set M among the sets of a given length ℓ. First, we give a survey of the results on the maximal distance minimization problem. Then we find maximal distance minimizers for a closed planar curve of a small enough curvature. Such an answer was conjectured by Miranda, Paolini and Stepanov for a circle of radius $R > r$. The chapter finished with a pack of open questions.

A subset I of vertices of G is *independent* if no edge connects vertices of I. The *independence number* of a graph G is the maximal size of an independent set in G; we denote it by $\alpha(G)$. The Johnson-type graph $J_\pm(d, k, t)$ in defined the following way: the vertex set consists of all vectors from $\{-1, 0, 1\}^d$ with exactly k nonzero coordinates; edges connect the pairs of vertices with scalar product t.

We found the exact values of the independent sets in several corner cases, namely for a negative odd t and $n > n_0(k, t)$ and for $t = 0$ and $n > n_0(k, t)$.

Also for $k = 3$ and $t = -2$ we found the asymptotic of the chromatic number of Johnson-type graph; surprisingly it is double-logarithmic in n so it does not coincide with general upper and lower bounds for such classes of graphs.

In 1976 Simmons conjectured that every coloring of a 2-dimensional sphere of radius strictly greater than $1/2$ in three colors has a pair of monochromatic points at distance 1 apart. Chapter 6 contains the proof of the conjecture. This refines the result of Lovász, who show that there is a sequence of radii r_k with the limit $1/2$, such that a 2-dimensional sphere with radius r_k has the chromatic number at least 4.

The celebrated de Bruijn–Erdős theorem implies that the chromatic number of a sphere is achieved on a finite subgraph. However our proof do not provide an explicit example of a spherical subgraph with the chromatic number 4.

Finally we have a deal with the chromatic numbers of 3-dimensional slices. The main result is the inequality

$$10 \leq \chi(\mathbb{R}^3 \times [0, \varepsilon]^6)$$

which holds for any positive ε.

3 Dissemination

The results of the dissertation were reported in the following talks:

1. 2nd Russian-Hungarian combinatorial workshop, Budapest, 27–29 June 2018
2. 3rd Hungarian-Russian Combinatorics workshop, Moscow–Petrozavodsk, 20–25 May 2019.
3. Combinatorics and geometry days II at Moscow Institute of Physics and Technology, 13–16 April 2020 Online (due to COVID)
4. FMI spring science session, Sofia University “St. Kliment Ohridski”, 25.03.2023
5. Mathematics Days in Sofia, IMI BAS, 10–14 July 2023
6. Analysis Seminar in Università di Pisa, Pisa, Italy, 13 July 2023
7. Mathematical Foundations of Informatics Seminar, IMI BAS, 21 July 2023
8. Séminaire Analyse Harmonique, Université Paris-Saclay, CNRS, Orsay, France, 10 October 2023