

On minimizers of the maximal distance functional for a planar convex closed smooth curve

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Abstract

Fix a compact $M \subset \mathbb{R}^2$ and $r > 0$. A *minimizer* of the maximal distance functional is a connected set Σ of the minimal length, such that

$$\max_{y \in M} \text{dist}(y, \Sigma) \leq r.$$

The problem of finding maximal distance minimizers is connected to the Steiner tree problem.

In this paper we consider the case of a convex closed curve M , with the minimal radius of curvature greater than r (it implies that M is smooth). The first part is devoted to statements on structure of Σ : we show that the closure of an arbitrary connected component of $B_r(M) \cap \Sigma$ is a local Steiner tree which connects no more than five vertices.

In the second part we “derive in the picture”. Assume that the left and right neighborhoods of $y \in M$ are contained in r -neighborhoods of different points $x_1, x_2 \in \Sigma$. We write conditions on the behavior of Σ in the neighborhoods of x_1 and x_2 under the assumption by moving y along M .

1 Introduction

For a given compact set $M \subset \mathbb{R}^2$ consider the maximal distance functional

$$F_M(\Sigma) := \max_{y \in M} \text{dist}(y, \Sigma),$$

where Σ is a compact planar set, and $\text{dist}(y, \Sigma)$ stands for the Euclidean distance between y and Σ . Also $F_M(\emptyset) := \infty$.

Consider the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ such that $F_M(\Sigma) \leq r$ for a given $r > 0$. We are interested in the properties of sets of minimal length (one-dimensional Hausdorff measure) $\mathcal{H}^1(\Sigma)$ over the mentioned class. Further we call such sets *minimizers*.

It is known that the set of minimizers is non-empty. It is also known that every minimizer Σ of positive length satisfies $F_M(\Sigma) = r$. Also in this case the set of minimizers coincides with the set of solutions of the corresponding dual problem: to minimize F_M among the class of closed connected sets $\Sigma \subset \mathbb{R}^2$ with the prescribed bound on the length $\mathcal{H}^1(\Sigma) \leq l$ (that is the reason of calling the desired set minimizers of maximal distance functional). General statements and details of the mentioned results can be found in [4].

Let $B_r(x)$ be the open ball of radius r centered at a point x . Let $B_r(M)$ be the open r -neighborhood of M :

$$B_r(M) := \bigcup_{x \in M} B_r(x).$$

1.1 Properties of Σ for general M

In this subsection M is an arbitrary planar compact.

Note that Σ is bounded (and hence compact), since $\Sigma \subset \overline{B_r(\text{conv } M)}$, where $\text{conv } M$ stands for the convex hull of M .

Definition 1.1. A point $x \in \Sigma$ is called *energetic*, if for all $\rho > 0$ the set $\Sigma \setminus B_\rho(x)$ does not cover M i.e.

$$\text{dist}(M, \Sigma \setminus B_\rho(x)) > r.$$

Denote the set of energetic points by G_Σ .

Every minimizer Σ can be split into three disjoint subsets:

$$\Sigma = E_\Sigma \sqcup X_\Sigma \sqcup S_\Sigma,$$

where $X_\Sigma \subset G_\Sigma$ is the set of *isolated energetic* points (i.e. every $x \in X_\Sigma$ is energetic and there is a $\rho > 0$ such that $B_\rho(x) \cap G_\Sigma = \{x\}$), $E_\Sigma := G_\Sigma \setminus X_\Sigma$ is the set of *non-isolated energetic* points and $S_\Sigma := \Sigma \setminus G_\Sigma$ is the set of non-energetic points also called the *Steiner part* of Σ .

The following basic properties of minimizers has been proved in [4] (for planar M) and in [6] (for $M \subset \mathbb{R}^n$):

- (a) minimizers contain no cycles (homeomorphic images of circumference).
- (b) For every energetic $x \in G_\Sigma$ there is a point $y \in M$, such that $|x - y| = r$ and $B_r(y) \cap \Sigma = \emptyset$. Further we call y *corresponding* to x and denote by $y(x)$. Note that a corresponding point may be not unique.
- (c) For every non-energetic $x \in S_\Sigma$ there is an $\varepsilon > 0$, such that $\Sigma \cap B_\varepsilon(x)$ is either a segment or a *regular tripod*, i.e. the union of three line segments with an endpoint in x and relative angles of $2\pi/3$.

Theorem 1.2 (Teplitskaya, [7, 8]). *Let Σ be a maximal distance minimizer for a compact set $M \subset \mathbb{R}^2$, $r > 0$. We say that the ray $(ax]$ is a tangent ray of the set Σ at the point $x \in \Sigma$ if there exists non stabilized sequence of points $x_k \in \Sigma$ such that $x_k \rightarrow x$ and $\angle x_k x a \rightarrow 0$. Then*

- (i) Σ is a union of a finite number of injective images of the segment $[0, 1]$;
- (ii) the angle between each pair of tangent rays at every point of Σ is greater or equal to $2\pi/3$;
- (iii) the number of tangent rays at every point of Σ is not greater than 3. If it is equal to 3, then there exists such a neighbourhood of x that the arcs in it coincide with line segments and the pairwise angles between them are equal to $2\pi/3$.

1.2 The class of M , considered in the paper

Fix a positive real r and a closed convex curve M with the minimal radius of curvature $R > r$ (this implies $C^{1,1}$ -smoothness of M). Introduce the notation: $N := \text{conv}(M)$; let M_r be the inner part of the boundary of $B_r(M)$, and finally put $N_r = \text{conv}(M_r)$. Note that M_r also is a closed convex curve M with the minimal radius of curvature $R - r$.

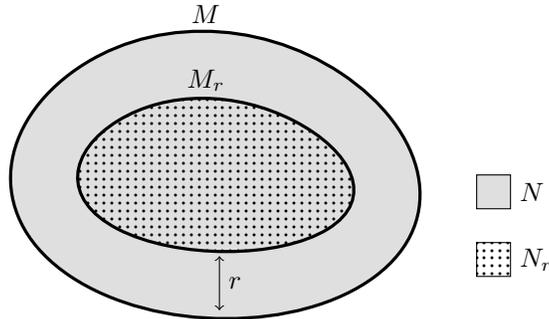


Figure 1: Definition of N , M_r and N_r

Further Σ denotes an arbitrary minimizer for M .

1.3 The problem for particular M

Finding the set of minimizers for almost every particular M is quite difficult. There are the following results.

Theorem 1.3 (Cherkashin – Teplitskaya, 2018 [1]). *Let r be a positive real, M be a convex closed curve with the radius of curvature at least $5r$ at every point, Σ be an arbitrary minimizer for M . Then Σ is a union of an arc of M_r and two segments, that are tangent to M_r at the ends of the arc (so-called horseshoe, see Fig. 2). In the case when M is a circumference with radius R , the claim is true for $R > 4.98r$.*

We prepare the paper with the following theorem.

Theorem 1.4. *Let $M = A_1A_2A_3A_4$ be a rectangle, $0 < r < r_0(M)$. Then a maximal distance minimizer has the following topology, depicted in the left part of Fig. 3. The middle part of the picture contains enlarged fragment of the minimizer near A_1 ; the labeled angles are equal to $\frac{2\pi}{3}$. The rightmost part contains much more enlarged fragment of minimizer near A_1 .*

A minimizer consists of 21 segments; an approximation of the length of a minimizer is $\text{Per} - 8.473981r$, where Per is the perimeter of the rectangle.

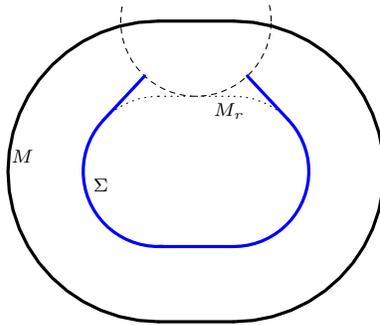


Figure 2: A horseshow

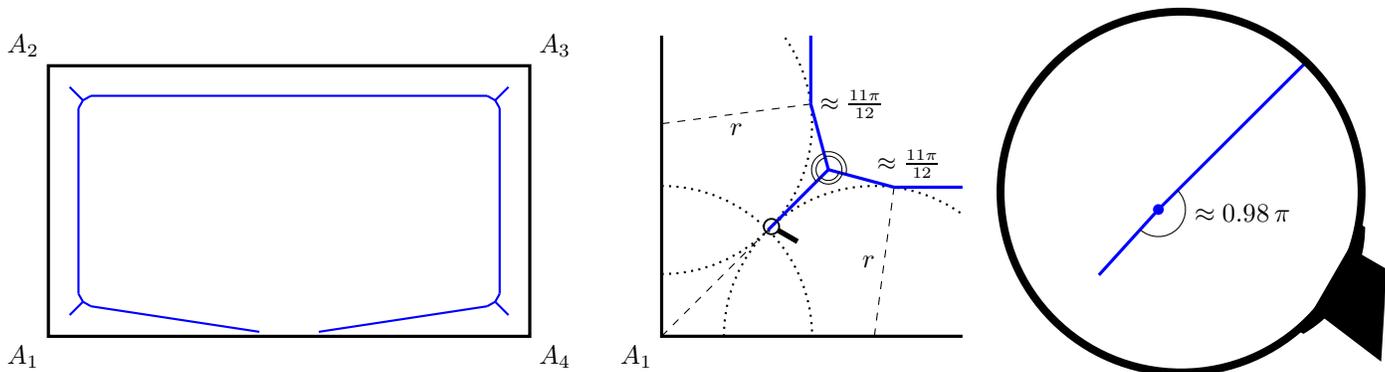


Figure 3: The minimizer for rectangle M and $r < r_0(M)$.

Structure of the paper. Section 2 contains an introduction to the Steiner problem. Section 3 is devoted to structural properties of Σ . In Section 4 we “derive in the picture”. Finally, Section 5 contains applications of our methods and open questions.

2 Steiner tree problem

Consider a finite set of points $C := \{A_1, \dots, A_n\} \subset \mathbb{R}^2$. A *Steiner tree* is a connected set $S \subset \mathbb{R}^2$, which contains C and has minimal possible length. It is known that such S always exists (but is not necessarily unique) and that it is the union of a finite set of segments. Thus, S can be represented as a plane graph, such that its set of vertices contains C , and all its edges are straight line segments. This graph is connected and does not contain cycles, i.e. is a tree, which explains the naming of S . It is known that the maximum degree of this graph is no greater than 3. Moreover, only vertices A_i can have degree 1 or 2, all the other vertices have degree 3 and are called *Steiner points*. There are no more than $n - 2$ Steiner points. The angle between any two adjacent edges is at least $2\pi/3$. That means that for a Steiner point the angle between any two edges incident to it is exactly $2\pi/3$. S is called a *full Steiner tree*, if the degree of each A_i is 1, or, equivalently, if the number of Steiner points is $n - 2$. A *(full) Steiner forest* is a set, each connected component of which is a (full) Steiner tree. Proof of the listed properties of Steiner trees and additional information on them can be found in the book [3] and in the article [2].

We define a *local Steiner tree* as a connected compact acyclic set S , which contains C , and such that for any $x \in S \setminus C$ there is a neighborhood $U \ni x$ such that $S \cap U$ coincides with the Steiner tree on the set of points $S \cap \partial U$. A local Steiner tree retains the following properties of a Steiner tree: it is the union of a finite set of segments; the angle between any two adjacent segments is at least $2\pi/3$. We are going to use the following fact: a connected closed subset of a local Steiner tree is itself a local Steiner tree.

For a given tree T we denote the set of its vertices of degree 1 or 2 as ∂T .

Definition 2.1. Define a “wind rose” as a set of six rays starting at the origin point with angle $\pi/3$ between any two adjacent rays; each ray is given a weight (a real number), which satisfy the following property: the weight of a ray is the sum of weights of two rays adjacent to it. (It follows, in particular, that the sum of the weights of two opposite rays (the ones forming a line) is zero.)

By *full Steiner pseudo-network* let us call a connected set S which contains C , if for any wind rose \mathcal{R} such that

- (i) S consists of finite number of segments which are parallel to \mathcal{R}

the following holds:

- (ii) for any $x \in S \setminus C$ and small enough $\varepsilon > 0$, sum of weights of rays of \mathcal{R} which are parallel to rays of the form $[xy)$, $y \in \partial B_\varepsilon(x) \cap S$, is zero.

It is clear that full (local) Steiner tree is a full Steiner pseudo-network.

For a given pseudo-network T let us denote by ∂T set of vertices of degree 1.

Remark 2.2. Suppose that T is a full Steiner pseudo-network, and \mathcal{R} is an arbitrary wind rose satisfying (i). Let us assign to an each vertex $x \in \partial T$ a weight of a ray of \mathcal{R} , which is parallel to a directed segment of T entering x (such segment is unique by definition of ∂T). Then sum of assigned numbers over all $x \in \partial T$ is zero.

Lemma 2.3. *Let T be a full Steiner pseudo-network, l be a line such that $T \not\subset l$. Then*

$$\#(\partial T \cap l) \leq 2\#(\partial T \setminus l).$$

Proof of Lemma 2.3: Let l^+ , l^- be the two open half-planes bounded by l . Note that it is sufficient to prove the inequality for a closure of an arbitrary component of $\overline{T \cap l^+}$ and $\overline{T \cap l^-}$, denote such closure as S .

Consider a wind rose with the origin in the same open half-plane as S , such that the rays with positive weights are exactly the ones intersecting l : such wind rose exists, because l intersects either 2 or 3 consecutive (in the counter-clockwise order) rays. In the former case we give these rays weights 1, 1, in the latter case $-1, 2, 1$. The remaining rays will have weights 0, $-1, -1, 0$ or $-1, -2, -1$ (the weights are listed in counter-clockwise order in each case). We assign weights to all leaf vertices in the way described in Remark 2.2. Then the sum of weights over the leaf vertices lying on l is at least $\#(S \cap l)$. Since, according to Remark 2.2, the sum over all leaf vertices should be zero, there are at least $\frac{\#(S \cap l)}{2}$ leaf vertices not lying on l . \square

Remark 2.4. Let T be a full Steiner pseudo-network fully lying on one side of line l , such that equality in Lemma 2.3 is achieved. Then all leaf vertices in $\partial T \setminus l$ have weight -2 , therefore all segments of T incident to vertices from $\partial T \setminus l$ are pairwise collinear.

3 Structural properties of minimizers

Recall that we work in the setting from Subsection 1.2.

Note that $\Sigma \subset N$ (N is convex, so one can project the part of Σ belonging to $\mathbb{R}^2 \setminus N$ on N and length of Σ will strictly decrease).

Consider the closure of an arbitrary connected component of $\Sigma \setminus N_r$; denote it by S . Points from $S \cap M_r$ are called *entering points*. Connectedness of S implies that $\overline{B_r(S)} \cap M$ is a closed arc; denote it by $q(S)$.

The following lemma is proved in [1] (the proof of these statements does not use the additional requirement $R > 5r$, which is inherited from the main theorem of the paper [1]).

Lemma 3.1. *Let S be the closure of a connected component of $\Sigma \setminus N_r$. Then*

- (i) S is a local Steiner tree connecting the set of entering points of S and energetic points of S ;
- (ii) S contains one or two energetic points.
- (iii) Suppose that S contains 2 energetic points x_1 and x_2 . Then
 - (i) there are unique points $y(x_1)$ and $y(x_2)$;
 - (ii) if x_i has degree 1 (i.e. x_i is the end of a line segment $[z_i x_i] \subset \Sigma$), then z_i , x_i and $y(x_i)$ are collinear;
 - (iii) if x_i has degree 2 (i.e. x_i is the end of a line segments $[z_i^1 x_i], [x_i z_i^2] \subset \Sigma$), then ray $[y(x_i)x_i)$ contains the bisector of $z_i^1 x_i z_i^2$.

In this section we prove the following statement.

Proposition 3.2. *Let S be the closure of a connected component of $\Sigma \setminus N_r$. Then*

- (i) the convex hull of S is a line segment, a triangle or a quadrangle; the vertices of convex hull are only energetic or entering points of S , the latter no more than 2;
- (ii) S has at most 3 entering points.

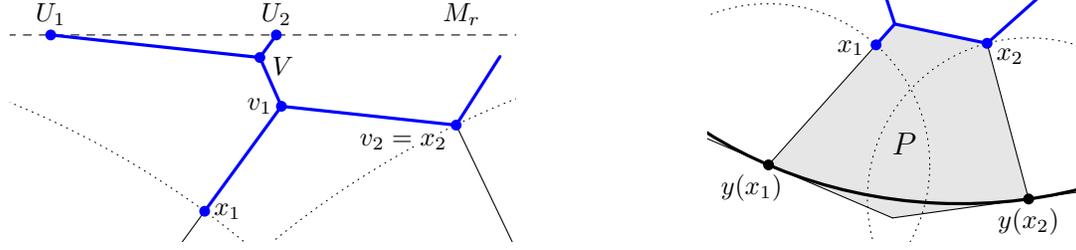


Figure 4: Illustration to the proof of Lemma 3.1 (ii)

Proof of (i). Since S is a local Steiner tree for its entering and energetic points, every other point x is a convex combination of points from a neighborhood of x . So it is enough to show that all entering points except at most two lie in the interior of $\text{conv}(S)$.

Suppose the contrary and consider maximal (by inclusion) arc $A \subset M_r$ ending by entering points of S (further we call them *extreme*), that $A \subset \overline{B_r(q(S))}$. Consider an arbitrary entering point x lying in the interior of the arc and set the tangent line to M_r at x . Since N_r is convex, connected component S contains points in the both sides of the tangent line, say t_1 and t_2 . Then x is a convex combination of t_1 and t_2 ; which is a contradiction. \square

Proof of (ii). Denote extreme (defined analogously to the previous proof) entering points by Y_1 and Y_2 . For every other entering point $Y \in S$ denote by $R = R(Y)$ a continuation of a segment of S which contains Y beyond the point Y . Let us show that R intersects with a line Y_1Y_2 . Note that Y_1, Y, Y_2 are contained in arc $M_r \cap \overline{B_r(q(S))} =: Q(S)$.

Suppose the contrary, that is R either meets again with M_r at $U \in Q(S)$ or is tangent to the M_r . Let us show that on the arc $YU \subset Q(S)$ there will be an entering point Y' that belongs to the closure of another connected component S' . If R is tangent to M_r in Y , then Y lies in a closure of another component, therefore we can put $Y' = Y$. Otherwise Y lies in a closure of a component of $\Sigma \cap \text{Int}(N_r)$, denote this component as T . Since $\Sigma \cap \text{Int}(N_r)$ lies in the Steiner part of Σ , a closure of any component of $\Sigma \cap \text{Int}(N_r)$ is a full local Steiner tree. Then $\partial T \setminus \{Y\}$ contains a point in each of closed half-planes divided by YU ; since $\partial T \subset M_r$, there is a vertex $Y' \in \partial T$ on the arc YU , since Σ is acyclic, Y' is not contained in S .

But then $q(S') \subset Q(S)$ or $S \cap S' \neq \emptyset$. It follows from the first option that S' contains no energetic points; the second option is impossible by definition.

For each entering point Y denote by $I(Y)$ an intersection of $R(Y)$ with Y_1Y_2 . By $St(S)$ we denote the union of S and all segments $[YI(Y)]$.

Let us consider two cases.

(a) Let S have one energetic point x .

- If x has degree 1, then $St(S)$ is a full Steiner pseudo-network, then by application of Lemma 2.3 to $St(S)$ and Y_1Y_2 , we have that $St(S)$ intersects with Y_1Y_2 at most two times, thus S has at most two entering points.
- If x has degree 2 then $St(S)$, being cut in the point x , falls apart into two full networks St_1 and St_2 . Let one of them have at least two entering points (say, St_1). By Lemma 2.3 for St_1 and Y_1Y_2 , it can be only a tripod; denote by V_1 the branching point of the tripod and by U_1^1 and U_2^1 the points of intersection with Y_1Y_2 . Let St_2 also be a tripod with branching point V_2 and with points U_1^2 and U_2^2 on the line Y_1Y_2 ; without loss of generality we can assume that points U_1^2 and U_2^2 are lying between points U_1^1 and U_2^1 . Then the sum of angles of pentagon $U_1^1V_1xV_2U_2^2$ is at least $10\pi/3$, because $\angle U_1^1V_1x = \angle xV_2U_2^2 = 2\pi/3$ (these angles are external for the pentagon and corresponding inside angles are equal to $4\pi/3$), $\angle V_1xV_2 \geq 2\pi/3$. That is a contradiction.

Summing up, no subtree can contain three entering points, and subtrees can not both contain two entering points simultaneously, which finishes this case.

(b) Let S have two energetic points x_1 and x_2 . Consider a polygon P (see the right-hand side of Figure 4), that bounded by S , by segments $[x_1y(x_1)]$, $[x_2y(x_2)]$ and by tangents to M in points $y(x_1)$ and $y(x_2)$ (by Lemma 3.1 points $y(x_1)$ and $y(x_2)$ are unique). Note that P is convex and its angles at vertices from S are at most $2\pi/3$. Since $B_r(y(x_1)) \cap \Sigma = B_r(y(x_2)) \cap \Sigma = \emptyset$, angles at $y(x_1)$ and $y(x_2)$ are at most $\pi/2$; by Lemma 3.1(iii) the line $y(x_1)x_1$ contains a side of P , thus angle at $y(x_1)$ is less than $\pi/2$. If P has at least 3 vertices from S , then the sum of external angles of P is strictly greater than $3\pi/3 + 2\pi/2 = 2\pi$, what is impossible. Therefore, P contains no more than two vertices from S .

Let x_i have degree 1. Then, if it is connected with x_{3-i} by a segment of Σ , segment $[x_1x_2]$ can be removed from $St(S)$ and by Lemma 2.3 remaining full Steiner network has no more than two points of intersection with Y_1Y_2 , so S has no more than two entering points. In the other case x_i is connected by a segment of Σ with a branching point V_i . Let $v_i = V_i$, if x_i has degree 1, and $v_i = x_i$, if x_i has degree 2 (see the left-hand side of Figure. 4). Then v_1 and v_2 are vertices of P . Since P contains no more than 2 vertices from S , it turns out that either $v_1 = v_2$, or $[v_1v_2] \subset \Sigma$.

If $v_1 = v_2$ then after deleting line segments $[v_1x_1]$ and $[v_2x_2]$ from S , application of Lemma 2.3 to line Y_1Y_2 gives that S has at most two entering points.

If $[v_1v_2] \subset \Sigma$, then we consider two cases.

- The case where both points x_1 and x_2 have degree 1; in this case S is a full Steiner pseudo-network. The application 2.3 to the line Y_1Y_2 gives that S contains at most 4 entering points, moreover if S has 4 entering points then equality in lemma is achieved and by Remark 2.4 rays $[x_1y(x_1))$ and $[x_2y(x_2))$ have similar direction. Then the pass between x_1 and x_2 in P has 3 branching points but P has at most 2 vertices from S ; which is a contradiction.
- The case where at least one of points x_1 and x_2 has degree 2. Removing line segment $[v_1v_2]$ splits $\mathcal{St}(S)$ into two subnetworks \mathcal{St}_1 and \mathcal{St}_2 . Suppose that one of them has at least two entering points (say, \mathcal{St}_1). Note that $\mathcal{St}_1 \setminus [x_1v_1]$ is a full pseudo-network, so by Lemma 2.3 it is a tripod; denote by V the branching point of the tripod, and by U_1, U_2 the entering points in such a way that M_r contains points Y_1, U_1, U_2, Y_2 in the mentioned order (some points may coincide). Then vector $\overrightarrow{U_1V}$ is directed away from line Y_1Y_2 , hence vector $\overrightarrow{v_1v_2}$ also is directed away from line Y_1Y_2 .

Summing up, no subtree can contain three entering points, and subtrees can not both contain two entering points simultaneously, which finishes the proof. □

4 Derivation in the picture

Consider a point $y \in M$ such that $B_r(y) \cap \Sigma = \emptyset$. Suppose there exists an energetic point $x \in \partial B_r(y) \setminus M_r$. Our goal is to determine how the length of Σ in the vicinity of point x changes with the infinitesimal movement of y along M (and the corresponding movement of x). We are going to consider all possible options for the local structure of Σ in the vicinity of x . Since radius of curvature of M is greater than r , each x corresponds to no more than two distinct y . Additionally, the degree of x is either 1 or 2. Therefore, there are 4 cases to consider.

In all cases below we are going to move point y along M a distance ε in such direction that the length of the arc covered by point x increases (it changes a minimizer in a neighborhood of x). The substitution of negative ε corresponds to moving y along M in the opposite direction.

Case 1. The degree of point x is 1 (so x is the end of some segment $[zx] \subset \Sigma$) and $y(x)$ is unique (look at the left half of Fig. 5). Points z, x , and $y(x)$ lie on one line by Lemma 3.1. Let $|zx| = l$, let α be the angle between $(zy(x))$ and M . We obtain the point $y(x_\varepsilon)$ by moving $y(x)$ a sufficiently small distance ε along M ; let $x_\varepsilon := [zy(x_\varepsilon)] \cap \partial B_r(y(x_\varepsilon))$. M is smooth, so the distance between point $y(x_\varepsilon)$ and the tangent to M at point $y(x)$ is $o(\varepsilon)$. By cosine rule for triangle $zy(x)y(x_\varepsilon)$,

$$|zy(x_\varepsilon)| = \sqrt{|zy(x)|^2 + 2|zy(x)|\varepsilon \cos \alpha + \varepsilon^2} + o(\varepsilon) = |zy(x)| + \varepsilon \cos \alpha + o(\varepsilon).$$

Therefore the derivative of the length of Σ in the vicinity of x with respect to the movement of $y(x)$ along M in this case is $\cos \alpha$.

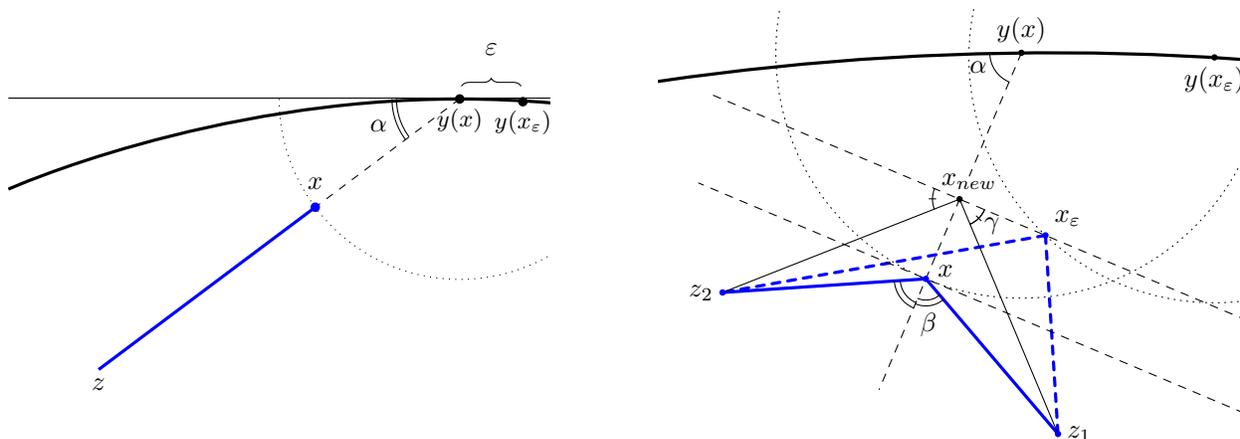


Figure 5: The first and second cases

Case 2. The degree of point x is 2 (so x is the end of some segments $[z_1x], [xz_2] \subset \Sigma$) and $y(x)$ is unique (look at the right half of Fig. 5). The ray $[y(x)x)$ contains the bisector of the angle z_1xz_2 by Lemma 3.1. Let $|z_1x| = |z_2x| = l$ (we can shorten one of the segments if needed), $\beta = \frac{1}{2}\angle z_1xz_2$; let the angle between the bisector of the angle z_1xz_2 and M be α ; $y(x_\varepsilon)$ is obtained by moving $y(x)$ along M a distance ε (which is chosen to be sufficiently small after fixing l). Let x_ε be such point on $\partial B_r(y(x_\varepsilon))$, that $[y(x_\varepsilon)x_\varepsilon)$ contains the bisector of the angle $z_1x_\varepsilon z_2$.

Consider the tangent to $B_r(y(x))$ at point x , and the parallel line going through point x_ε . Let point x_{new} be the intersection between the last line and $[y(x)x)$. Note that $|z_1x_{new}| = |z_2x_{new}|$; denote this length as l_{new} . This equality also implies that $\angle z_1x_{new}x_\varepsilon + \angle z_2x_{new}x_\varepsilon = \pi$. We denote the angle $z_1x_{new}x_\varepsilon$ as γ and write the cosine rule for triangles $z_1x_{new}x_\varepsilon$ and $z_2x_{new}x_\varepsilon$ using the fact that $|xx_{new}| = O(\varepsilon)$:

$$\begin{aligned} |z_1x_\varepsilon| &= \sqrt{l_{new}^2 + |xx_{new}|^2 - 2l_{new}|xx_{new}|\cos\gamma} = l_{new} - |xx_{new}|\cos\gamma + o(\varepsilon), \\ |z_2x_\varepsilon| &= \sqrt{l_{new}^2 + |xx_{new}|^2 + 2l_{new}|xx_{new}|\cos\gamma} = l_{new} + |xx_{new}|\cos\gamma + o(\varepsilon). \end{aligned}$$

Thus,

$$|z_1x_\varepsilon| + |z_2x_\varepsilon| - 2l_{new} = o(\varepsilon). \quad (1)$$

Note that since M is smooth,

$$|x_{new}x| = \varepsilon \cos \alpha + o(\varepsilon).$$

Finally, we write the cosine rule for the triangle z_1xx_{new} :

$$l_{new} = \sqrt{l^2 + (\varepsilon \cos \alpha + o(\varepsilon))^2 + 2l(\varepsilon \cos \alpha + o(\varepsilon))\cos\beta} = l + \varepsilon \cos \alpha \cos \beta + o(\varepsilon). \quad (2)$$

Combining (1) and (2), we conclude that the derivative is

$$2 \cos \alpha \cos \beta.$$

In cases 3 and 4 x corresponds to two points: $y_1(x)$ and $y_2(x)$. Let $y_1(x) = y_1(x_\varepsilon)$, and let the point $y_2(x_\varepsilon)$ be obtained from $y_2(x)$ by moving it along M a (possibly negative) distance ε . This uniquely determines the point $x_\varepsilon := \partial B_r(y_1(x)) \cap \partial B_r(y_2(x_\varepsilon)) \cap N$. Let us find this point explicitly (look at the left half of Fig. 6).

The triangle $xy_1(x)y_2(x)$ is isosceles with two sides of length r ; let $\angle xy_1(x)y_2(x) = \angle xy_2(x)y_1(x) =: \alpha$, $\angle x_\varepsilon y_1(x)y_2(x_\varepsilon) = \angle x_\varepsilon y_2(x_\varepsilon)y_1(x) =: \alpha_\varepsilon$.

We define the following coordinate system: the middle point of the segment $y_1(x)y_2(x)$ is the origin O ; the x axis is collinear to the ray $[y_1(x)y_2(x))$; the y axis is collinear to the ray $[Ox)$. Then

$$O = (0, 0), \quad x = (0, r \sin \alpha), \quad y_1(x) = (-r \cos \alpha, 0), \quad y_2(x) = (r \cos \alpha, 0).$$

Denote the angle between $y_1(x)y_2(x)$ and M as δ . Then

$$y_2(x_\varepsilon) = (r \cos \alpha + \varepsilon \cos \delta + o(\varepsilon), \varepsilon \sin \delta + o(\varepsilon)).$$

Therefore, by the cosine rule for triangle $y_1(x)y_2(x)y_2(x_\varepsilon)$,

$$|y_1(x)y_2(x_\varepsilon)| = \sqrt{(2r \cos \alpha + \varepsilon \cos \delta + o(\varepsilon))^2 + (\varepsilon \sin \delta + o(\varepsilon))^2} = 2r \cos \alpha + \varepsilon \cos \delta + o(\varepsilon).$$

Let O_ε be the middle point of segment $[y_1(x_\varepsilon)y_2(x_\varepsilon)]$. Then

$$O_\varepsilon = \left(\frac{\varepsilon \cos \delta}{2} + o(\varepsilon), \frac{\varepsilon \sin \delta}{2} + o(\varepsilon) \right).$$

By the definition of the cosine function,

$$\alpha_\varepsilon = \arccos \left(\frac{|y_1(x)O_\varepsilon|}{r} \right) = \arccos \left(\cos \alpha + \frac{\varepsilon \cos \delta}{2r} + o(\varepsilon) \right) = \alpha - \frac{\cos \delta}{2r \sin \alpha} \varepsilon + o(\varepsilon).$$

Let Δ be the directed angle $\angle y_2(x)y_1(x)y_2(x_\varepsilon)$ (so $\Delta < 0$ when ε is negative). By the sine rule for the triangle $y_2(x)y_1(x)y_2(x_\varepsilon)$,

$$\frac{\varepsilon}{\sin \Delta} = \frac{|y_1(x)y_2(x)|}{\sin(\delta - \Delta + o(\varepsilon))} \geq |y_1(x)y_2(x)|, \quad \text{so} \quad \Delta = O(\varepsilon).$$

Therefore,

$$\Delta = \sin \Delta + o(\varepsilon) = \frac{\varepsilon \sin(\delta + O(\varepsilon))}{|y_1(x)y_2(x)|} = \frac{\varepsilon \sin \delta}{2r \cos \alpha} + o(\varepsilon).$$

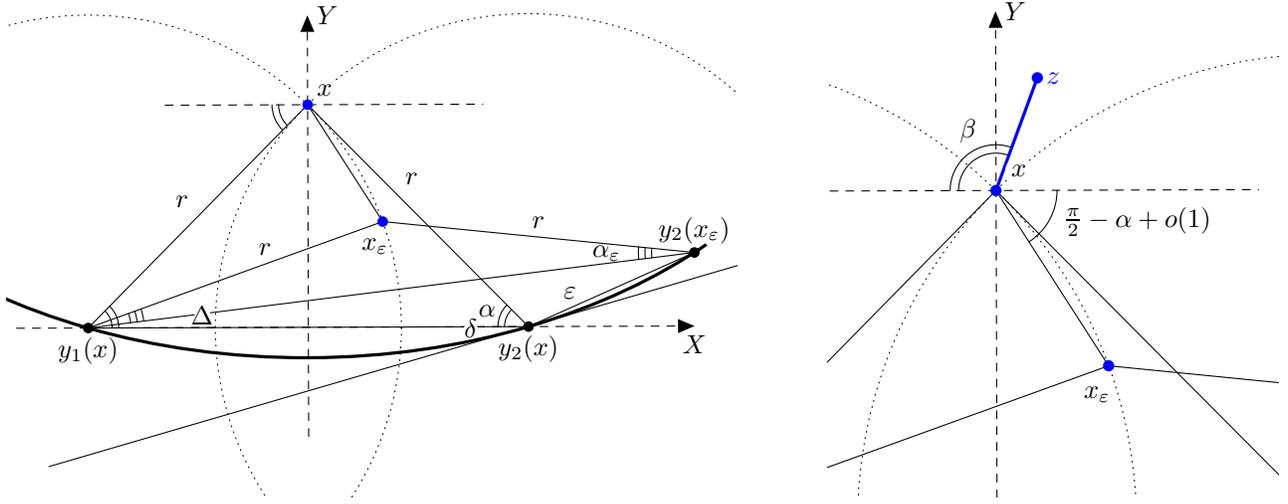


Figure 6: Finding coordinates of x_ε in the cases 3 and 4

Writing out the sum of angles in the isosceles triangle $xy_1(x)x_\varepsilon$, we get

$$\angle xy_1(x)x_\varepsilon = \alpha - \alpha_\varepsilon - \Delta = \left(\frac{\cos \delta}{2r \sin \alpha} - \frac{\sin \delta}{2r \cos \alpha} \right) \varepsilon + o(\varepsilon) = \frac{\cos(\alpha + \delta)}{r \sin(2\alpha)} \varepsilon + o(\varepsilon).$$

It follows that

$$|xx_\varepsilon| = 2r \sin \frac{\angle xy_1(x)x_\varepsilon}{2} = \frac{\cos(\alpha + \delta)}{\sin(2\alpha)} \varepsilon + o(\varepsilon),$$

and the angle between the segment xx_ε and the x axis (look at the right half of Fig. 6) is

$$\pi - \alpha - \frac{\pi - \angle xy_1(x)x_\varepsilon}{2} = \frac{\pi}{2} - \alpha + \frac{\cos(\alpha + \delta)}{2r \sin(2\alpha)} \varepsilon + o(\varepsilon) = \frac{\pi}{2} - \alpha + o(1).$$

Case 3. The degree of point x is 1 (so x is the end of some segment $[zx] \subset \Sigma$) and there are two distinct points $y_1(x)$ and $y_2(x)$.

Let β be the angle between $[zx]$ and the x axis (look at the right half of Fig. 6). Then

$$\angle zxx_\varepsilon = \frac{3\pi}{2} - \alpha - \beta + o(1).$$

By the cosine rule for the triangle zxx_ε ,

$$|zx_\varepsilon| = \sqrt{|zx|^2 - 2|zx||xx_\varepsilon| \cos \angle zxx_\varepsilon + |xx_\varepsilon|^2} = |zx| - |xx_\varepsilon| \cos \angle zxx_\varepsilon + o(\varepsilon) = |zx| + \frac{\cos(\alpha + \delta) \sin(\alpha + \beta)}{\sin(2\alpha)} \varepsilon + o(\varepsilon).$$

So the derivative is equal to

$$\frac{\cos(\alpha + \delta) \sin(\alpha + \beta)}{\sin(2\alpha)}.$$

Case 4. The degree of point x is 2 (so it is the end of some segments $[z_1x], [xz_2] \subset \Sigma$) and there are two distinct points $y_1(x)$ and $y_2(x)$. Similar to the previous case, the derivative is equal to

$$\frac{\cos(\alpha + \delta)}{\sin(2\alpha)} (\sin(\alpha + \beta) + \sin(\alpha + \gamma)),$$

where β and γ are the angles between the x axis and the segments $[z_1x]$ and $[z_2x]$, respectively.

Transitions between the cases. Note that the second case can transform into the first case, and the other way around; similarly, the third case can turn into the fourth and vice versa. The value of the derivative does not change in such transitions because

$$2 \cos \beta \cos \alpha = \cos \alpha \text{ when } \beta = \pi/3;$$

$$(\sin(\alpha + \beta) + \sin(\alpha + \gamma)) \frac{\cos(\alpha + \delta)}{\sin(2\alpha)} = 2 \sin\left(\frac{2\alpha + \beta + \gamma}{2}\right) \cos\left(\frac{\beta - \gamma}{2}\right) \frac{\cos(\alpha + \delta)}{\sin(2\alpha)} = \sin\left(\alpha + \beta + \frac{\pi}{3}\right) \frac{\cos(\alpha + \delta)}{\sin(2\alpha)}$$

when $\gamma - \beta = 2\pi/3$. That means that even if the combinatorial structure of Σ changes after moving y along M at the point y_0 , the left and right derivatives at y_0 coincide.

Proposition 4.1. *Let $x \in \Sigma$ be an energetic point, $y(x) \in M$ be an arbitrary corresponding point. Then the derivative of length of Σ in a neighborhood of x in the moving y along M is nonnegative.*

Proof. Suppose the contrary. Then one may shift y along M and the length of Σ will strictly decrease. Note that Σ is still connected and M is still covered by Σ ; which is a contradiction. \square

Proposition 4.2. *Let $y \in M$ be a point such that $B_r(y) \cap \Sigma = \emptyset$ and $\partial B_r(y)$ contains energetic points x_1 and x_2 . Define $Y = \partial B_r(y) \cap M_r$. Then*

- (i) *points x_1 and x_2 lie on the opposite sides of the line (yY) ;*
- (ii) *derivative of length of Σ in neighborhoods of x_1 and x_2 in the moving y along M are equal.*

Proof. Suppose the contrary to item (i); without loss of generality, $\angle Y y x_1 > \angle Y y x_2$. Then

$$B_r(B_\rho(x_1) \cap \Sigma) \cap M \subset B_r(x_2),$$

where $\rho > 0$ is small enough. Thus x_1 is not energetic, which is a contradiction.

Now suppose the contrary to item (ii). Without loss of generality, the derivative of the length of Σ in a neighborhood of x_1 is bigger than the derivative in a neighborhood of x_2 . Then after a shifting of y along M from x_2 to x_1 the length of Σ strictly decreases. Note that Σ is still connected and M is still covered by Σ ; this gives a contradiction. \square

5 Applications and open problems

- Sometimes it is possible to “derive in the picture” in the case of a partially smooth M . For this purpose one has to clarify the behavior of a considered competitor in a neighborhood of $B_r(y)$, with y lying in the smooth part of M .

For instance we use an analog of Statement 4.2 during the pruning of cases in the proof of Theorem 1.4.

- Miranda, Paolini and Stepanov [5] conjectured that all the minimizers for a circumference of radius $R > r$ are horseshoes. Theorem 1.3 proves this conjecture with assumption $R > 4.98r$; for $4.98r \geq R > r$ the conjecture remains open.
- At the same time, the statement of Theorem 1.3 for general M needs an assumption on the minimal radius of curvature as we show below.

Define a *stadium* as the boundary of the R -neighborhood of a segment. By the definition, stadium has the minimal radius of curvature R . If $R < 1.75r$ and a stadium is long enough, then there is a connected set Σ' that has smaller length than an arbitrary horseshoe and covers M .

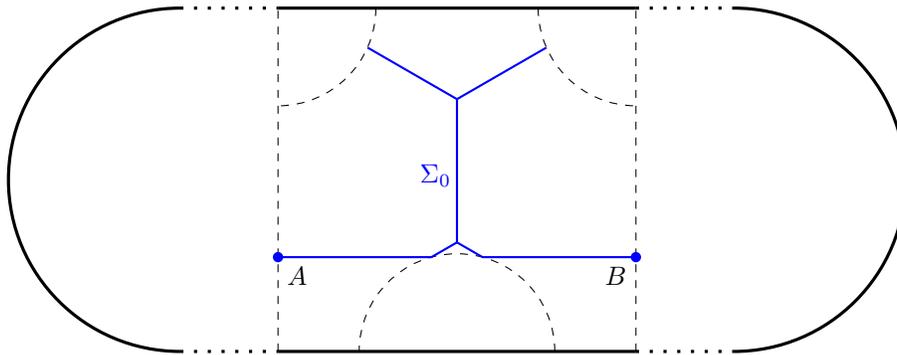


Figure 7: Horseshoe is not a minimizer for long enough stadium with $R < 1.75r$.

Define Σ_0 as a locally Steiner tree depicted in Fig. 7. Let Σ' consist of copies of Σ_0 , glued at points A and B along the length of the stadium. In the case $R < 1.75r$ the length of Σ_0 is strictly smaller than $2|AB|$. Thus for long enough stadium Σ' has length $cL + O(1)$, where L is the length of the stadium and $c < 2$ is a constant depend on Σ_0 and R . Obviously, any horseshoe has length $2L + O(1)$.

This example leads to the following problems.

Problem 5.1. *Find the minimal c such that Theorem 1.3 holds with the replacement of $5r$ with cr .*

Problem 5.2. *Find the set of minimizers for a given stadium.*

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