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Abstract	We consider a family of distance graphs in R^n and find its independence numbers in some cases. Define the graph $J_{\pm}(n,k,t)$ in the following way: the vertex set consists of all vectors from $\{-1,0,1\}^n$ with exactly k nonzero coordinates; edges connect the pairs of vertices with scalar product t. We find the independence number of $J_{\pm}(n,k,t)$ for an odd negative t and n_0 (k,t)$$]]>$n > n_0(k,t)$.
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Independence Numbers of Johnson-Type Graphs

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Abstract

We consider a family of distance graphs in $\mathbb{R}^n$ and find its independence numbers in 1
some cases. Define the graph $J_\pm(n, k, t)$ in the following way: the vertex set consists 2
of all vectors from $\{-1, 0, 1\}^n$ with exactly k nonzero coordinates; edges connect the
pairs of vertices with scalar product t. We find the independence number of $J_\pm(n, k, t)$
for an odd negative t and $n > n_0(k, t)$. 3

1 Introduction

We start with common definitions. Let $G = (V, E)$ be a graph. A subset I of vertices of G is *independent* if no edge connects vertices of I. The *independence number* of a graph G is the maximal size of an independent set in G; we denote it by $\alpha(G)$.

Generalized Johnson graphs are the graphs $J(n, k, t)$ defined as follows: the vertex set consists of vectors from the hypercube $\{0, 1\}^n$ with exactly k nonzero coordinates, edges connect vertices with scalar product t (so $J(n, k, t)$ is nonempty if $k < n$ and $2k - n \le t < k$). Generalized Kneser graphs $K(n, k, t)$ have the same vertex set but the edges connect vertices with scalar product at most t.

Now we introduce the main hero of the paper. Define graphs $J_\pm(n, k, t)$ as follows: the vertex set consists of vectors from $\{-1, 0, 1\}^n$ with exactly k nonzero coordinates, edges connect vertices with scalar product t. The graph $J_\pm(n, k, t)$ is nonempty if $k < n$ and $-k \le t < k$, and also if $k = n$ and $n - t$ is even. If $t = -k$, then the graph $J_\pm(n, k, t)$ is a matching. Note that the edges connect vertices of the Euclidean distance $\sqrt{2(k - t)}$, which means that $J_\pm(n, k, t)$ is a distance graph.

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Finally, define $K_{\pm}(n,k,t)$ as the graph which shares the vertex set with $J_{\pm}(n,k,t)$ but the edges connect vertices with scalar product at most t.

1.1 Independence and Chromatic Numbers of $J(n,k,t)$ and $K(n,k,t)$

Independent sets in these families of graphs are classical combinatorial objects. Indeed, we have a natural bijection between the set of k-subsets of $[n]$ and $V[J(n,k,t)] = V[K(n,k,t)]$. The celebrated Erdős–Ko–Rado theorem (Erdős et al. 1961) determines all maximal independent sets in $J(n,k,0) = K(n,k,0)$. A natural generalization was done by Erdős and Sós, who introduce "forbidden intersection problem", which involves finding the independence numbers of graphs $J(n,k,t)$. Then the Frankl–Wilson theorem (Frankl and Wilson 1981), the Frankl–Füredi theorem (Frankl and Füredi 1985) and the Ahlswede–Khachatryan Complete Intersection Theorem (Ahlswede and Khachatrian 1997) answered a lot of questions about the size and the structure of maximal independent sets in the graphs $J(n,k,t)$ and $K(n,k,t)$.

On the other hand a lot of questions in combinatorial geometry are related to embeddings of these graphs into $\mathbb{R}^n$. Frankl and Wilson (1981) used the graphs $J(n,k,t)$ to get an exponential lower bound on the chromatic number of the Euclidean space (Nelson–Hadwiger problem); Kahn and Kalai (1993) used them to disprove Borsuk's conjecture.

Let us describe the picture for some small k and t. Erdős et al. (1961) proved that $n \geq 2k$ implies

$$\alpha[J(n,k,0)] = \binom{n-1}{k-1}.$$

Then Lovász (1978) proved Kneser's conjecture, namely that $\chi[J(n,k,0)] = n - 2k + 2$ for $n \geq 2k$. The following result was introduced to get a constructive bound on the Ramsey number.

Proposition 1 (Nagy 1972) *Let $n = 4s + t$, where $0 \leq t \leq 3$. Then*

$$\alpha[J(n,3,1)] = \begin{cases} n & \text{if } t = 0, \\ n-1 & \text{if } t = 1, \\ n-2 & \text{if } t = 2 \text{ or } 3. \end{cases}$$

Then Larman and Rogers (1972) used the bound $\chi[J(n,3,1)] \geq \frac{|V[J(n,3,1)]|}{\alpha[J(n,3,1)]}$ to show that the chromatic number of the Euclidean space is at least quadratic in the dimension (initially it was proposed by Erdős and Sós). It turns out that the chromatic number of $J(n,3,1)$ is very close to $\frac{|V[J(n,3,1)]|}{\alpha[J(n,3,1)]}$ (and sometimes is equal to this ratio).

Theorem 1 (Balogh–Kostochka–Raigorodskii 2013) *Consider $l \geq 2$. If $n = 2^l$, then*

$$\chi[J(n,3,1)] \leq \frac{(n-1)(n-2)}{6}.$$

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If $n = 2^l - 1$, *then*

$$\chi[J(n,3,1)] \leq \frac{n(n-1)}{6}.$$

Finally, for an arbitrary n

$$\chi[J(n,3,1)] \leq \frac{(n-1)(n-2)}{6} + \frac{11}{2}n.$$

Tort (1983) proved that for $n \geq 6$,

$$\chi[K(n,3,1)] = \left[\frac{(n-1)^2}{4}\right].$$

Zakharov (2020a) showed that the existence of Steiner systems (see Sect. 2.5) implies that

$$\chi[J(n,k,t)] \leq (1+o(1))\frac{(k-t-1)!}{(2k-2t-1)!}n^{k-t}$$

for fixed $k > t$. In general $\chi[J(n,k,t)] = \Theta(n^{t+1})$ for $k > 2t+1$ and $\chi[J(n,k,t)] = \Theta(n^{k-t})$ for $k \leq 2t+1$.

1.2 Known Facts About the Graphs $J_\pm(n,k,t)$ and $K_\pm(n,k,t)$

From a geometrical point of view $J_\pm(n,k,t)$ is a natural generalization of $J(n,k,t)$. Raigorodskii (2000, 2001) used the graphs $J_\pm(n,k,t)$ to significantly refine the asymptotic lower bounds in the Borsuk's problem and the Nelson–Hadwiger problem.

Unfortunately, there is no general method to find the independence number of $J_\pm(n,k,t)$ even asymptotically. One of the reasons is that the known answers have varied and sometimes rather complicated structures. For instance the proof of the following result analogous to Proposition 1 is relatively long and the answer is quite surprising.

Theorem 2 (Cherkashin–Kulikov–Raigorodskii 2018) *For* $n \geq 1$ *define* $c(n)$ *as follows:*

$$c(n) = \begin{cases} 0 & \textit{if } n \equiv 0 \\ 1 & \textit{if } n \equiv 1 \\ 2 & \textit{if } n \equiv 2 \textit{ or } 3 \end{cases} \pmod 4.$$

Then

$$\alpha[J_\pm(n,3,1)] = \max\{6n-28, 4n-4c(n)\}.$$

In recent papers (Frankl and Kupavskii 2018a, b, 2020) Frankl and Kupavskii generalized the Erdős–Ko–Rado theorem for some subgraphs of $J_{\pm}(n,k,t)$. We need additional definitions.

$$V_{k,l} := \{v \in \{-1,0,1\}^n \mid v \text{ has exactly } k \ '1' \text{ and exactly } l \ '-1'\}.$$
$$J(n,k,l,t) := (V_{k,l}, \{(v_1,v_2) \mid \langle v_1, v_2\rangle = t\}).$$

Theorem 3 (Frankl–Kupavskii 2018a) *For $2k \le n \le k^2$ the equality*

$$\alpha[J(n,k,1,-2)] = k\binom{n-1}{k}$$

holds. In the case $n > k^2$ the following equality holds

$$\alpha[J(n,k,1,-2)] = k\binom{k^2-1}{k} + \sum_{i=k^2}^{n-1}\binom{i}{k}.$$

Paper Frankl and Kupavskii (2018b) deals with a more generic problem.

Theorem 4 (Frankl–Kupavskii 2018b) *For $2k \le n$ the following bounds hold*

$$\binom{n}{k+l}\binom{k+l-1}{l-1} \le \alpha[J(n,k,l,-2l)] \le \binom{n}{k+l}\binom{k+l-1}{l-1} + \binom{n}{2l}\binom{2l}{l}\binom{n-2l-1}{k-l-1}.$$

In the case $2k \le n \le 3k-l$ the following equality holds

$$\alpha[J(n,k,l,-2l)] = \frac{k}{n}|V_{k,l}|.$$

To introduce the next result, we will need the following definition.

Definition 1

$$S(n,D) := \begin{cases} \sum_{j=0}^{d}\binom{n}{j} & \text{if } D = 2d, \\ \binom{n-1}{d} + \sum_{j=0}^{d}\binom{n}{j} & \text{if } D = 2d+1. \end{cases}$$

In Frankl and Kupavskii (2017) (see Frankl and Kupavskii 2019 for a version with a fixed mistake) Frankl and Kupavskii determined the independence number of $K_{\pm}(n,k,t)$ for $n > n_0(k,t)$ and found the asymptotics of the independence number of $J_{\pm}(n,k,t)$ if $t < 0$ and $n > n_0(k,t)$.

Theorem 5 (Frankl–Kupavskii 2019) *For any $k \in \mathbb{N}$ and $n \ge n(k_0)$ we have:*

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1. $\alpha[K_{\pm}(n,k,t)] = \binom{n-t-1}{k-t-1}$ *for* $-1 \le t \le k-1$,
2. $\alpha[K_{\pm}(n,k,t)] = S(k,|t|-1)\binom{n}{k}$ *for odd* t *such that* $-k-1 \le t < 0$,
3. $\alpha[K_{\pm}(n,k,t)] = \alpha[J(n,k-\frac{|t|}{2},\frac{|t|}{2},t)] + S(k,|t|-2)\binom{n}{k}$ *for even* t *such that* $-k-1 \le t < 0$.

Theorem 6 (Frankl–Kupavskii 2017) *For any* $k \in \mathbb{N}$, $t < 0$ *and* $n > n_0(k,t)$ *we have*

$$\alpha[J_{\pm}(n,k,t)] \le S(k,|t|-1)\binom{n}{k} + O\left(n^{k-1}\right).$$

The main technique in the Frankl–Kupavskii theorems is shifting. It turns out that shifting can not increase a scalar product, so it preserves the independence property of a set in a Kneser-type graph. Unfortunately, the latter does not hold for Johnson-type graphs. Using additional arguments one can derive weaker results which are tight only in asymptotics. But it looks impossible to find the independence number of $J_{\pm}(n,k,t)$ for $t > -k$ using shifting.

1.3 Results

Let $J(n,k,even)$ be a graph with the vertex set $\{0,1\}^n$, where edges connect vertices with even scalar product (note that each vertex has a loop if k is even). Define $J(n,k,odd)$ in a similar way. Let $J_{\pm}(n,k,even)$ and $J_{\pm}(n,k,odd)$ be defined analogously to $J(n,k,even)$ and $J(n,k,odd)$.

Observation 1 *If* $n > n_0(k)$, *then*

$$\alpha[J_{\pm}(n,k,even)] = 2^k\alpha[J(n,k,even)],$$
$$\alpha[J_{\pm}(n,k,odd)] = 2^k\alpha[J(n,k,odd)].$$

For $n > n_0(k)$ the exact values of $\alpha[J_{\pm}(n,k,even)]$ and $\alpha[J_{\pm}(n,k,odd)]$ are determined in Theorem 15.

Proof of Observation 1 Let $parity$ stand for odd or $even$.

To prove the lower bounds consider an arbitrary maximal independent set I in the graph $J(n,k,parity)$. Then all the vertices on the supports from I form an independent set $I_{\pm}$ in $J_{\pm}(n,k,parity)$. So

$$\alpha[J_{\pm}(n,k,parity)] = 2^k\alpha[J(n,k,parity)].$$

The upper bounds simply follow from Lemma 1, since $J(n,k,parity)$ is a subgraph of $J_{\pm}(n,k,parity)$. □

Observation 2 *For every* $n \ge k$ *we have*

$$\alpha[J_{\pm}(n,k,k-1)] = 2^k\alpha[J(n,k,k-1)].$$

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Note that $\alpha[J(n,k,k-1)]$ is the size of a largest partial Steiner $(n,k,k-1)$-system. In particular, if the divisibility conditions hold, then $\alpha[J(n,k,k-1)] = \binom{n}{k-1}/k$ (see Sect. 2.5).

Proof of Observation 2 Since $J(n,k,k-1)$ is a subset of $J_\pm(n,k,k-1)$, by Lemma 1 we have

$$\alpha[J_\pm(n,k,k-1)] \le 2^k \alpha[J(n,k,k-1)].$$

To prove the lower bound consider an arbitrary maximal independent set I in the graph $J(n,k,k-1)$. Then all the vertices on the supports from I form an independent set $I_\pm$ in $J_\pm(n,k,k-1)$. $\square$

We use the Katona averaging method and Reed–Solomon codes to prove the following theorem.

Theorem 7 *Suppose that $n > k2^{k+1}$. Then*

$$\alpha[J_\pm(n,k,-1)] = \binom{n}{k}.$$

Theorem 7 can be generalized as follows.

Theorem 8 *Suppose that t is a negative odd number, $n > n_0(k)$. Then*

$$\alpha[J_\pm(n,k,t)] = S(k,|t|-1)\binom{n}{k},$$

where S is defined in Definition 1.

The next theorem is a consequence of Theorems 10 and 7.

Theorem 9 *Let $n > \frac{9}{2}k^3 2^k$. Then*

$$\alpha[J_\pm(n,k,0)] = 2\binom{n-1}{k-1}.$$

One can extract a stability version of the previous theorem from its proof.

The *support* of a vertex v is the set of nonzero coordinates of v; we denote it by $\operatorname{supp} v$. Let $\mathcal{H}_k = (V_k, E_k)$ be a k-graph such that

$$V_k := \bigcup_{u\in[n]} \{u^+, u^-\}, \qquad E_k := \left\{ A \in \binom{V(\mathcal{H})}{k} \,\middle|\, \{u^+, u^-\} \not\subset A \text{ for every } u \right\}.$$

There is a natural bijection between E_k and $V(J_\pm(n,k,t))$. Introduce notion *signplace* for a vertex of $\mathcal{H}_k$ and *place* for a pair of vertices $\{u^+, u^-\}$, $u \in [n]$; note that the latter definition does not depend on k.

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Corollary 1 *Suppose that I is an independent set in $J_{\pm}(n,k,0)$ and no place intersects all the vertices of I. Then*

$$|I| \leq C(k)\binom{n}{k-2}.$$

Structure of the paper. In Sect. 2 we describe several classical definitions and theorems, that are used in examples and proofs: Katona averaging methods, non-trivial intersecting families, isodiametric inequality for the Hamming cube, simple hypergraphs and Reed–Solomon codes, Steiner systems and finally families with intersections of prescribed parity.

Section 3 contains examples, Sect. 4 provides proofs, Sect. 5 specifies the results in the case $k \leq 3$. We finish with open questions in Sect. 6.

2 Tools

2.1 Katona Averaging Method

Properties of a graph with a rich group of automorphisms sometimes can be established via consideration of a proper subgraph. We say that a graph G is *vertex-transitive* if for every vertices v_1, v_2, G has an automorphism f such that $f(v_1) = v_2$. The following lemma is a special case of Lemma 1 from Katona (1975).

Lemma 1 (Katona 1975) *Let $G = (V, E)$ be a vertex-transitive graph. Let H be a subgraph of G. Then*

$$\frac{\alpha(G)}{|V(G)|} \leq \frac{\alpha(H)}{|V(H)|}.$$

For example Lemma 1 immediately implies that for every fixed k, t the following decreasing sequences converge

$$a_n := \frac{\alpha[J_{\pm}(n,k,t)]}{|V[J_{\pm}(n,k,t)]|} \quad \text{and} \quad b_n := \frac{\alpha[K_{\pm}(n,k,t)]}{|V[K_{\pm}(n,k,t)]|},$$

as $J_{\pm}(n-1,k,t)$ and $K_{\pm}(n-1,k,t)$ are isomorphic to subgraphs of $J_{\pm}(n,k,t)$ and $K_{\pm}(n,k,t)$, respectively, and both $J_{\pm}(n,k,t)$ and $K_{\pm}(n,k,t)$ graphs are clearly vertex-transitive.

Also since $J(n,k,t)$ is a subgraph of $J_{\pm}(n,k,t)$, Lemma 1 implies

$$\frac{\alpha[J_{\pm}(n,k,t)]}{|V[J_{\pm}(n,k,t)]|} \leq \frac{\alpha[J(n,k,t)]}{|V[J(n,k,t)]|},$$

which gives by $|V[J_{\pm}(n,k,t)]| = 2^k\binom{n}{k} = 2^k|V[J(n,k,t)]|$ the following bound

$$\alpha[J_{\pm}(n,k,t)] \leq 2^k \alpha[J(n,k,t)]. \tag{1}$$

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It turns out that bound (1) is rarely close to the optimal. On the other hand sometimes it is tight, for instance in Propositions 1 and 2.

2.2 Nontrivial Intersecting Families

A family of sets $\mathcal{A}$ is *intersecting* if every $a, b \in A$ have nonempty intersection. A *transversal* is a set that intersects each member of $\mathcal{A}$.

Theorem 10 (Erdős–Lovász 1975) *Let $\mathcal{A}$ be an intersecting family consisting of k-element sets. Then at least one of the following statements is true:*

(i) *$\mathcal{A}$ has a transversal of size at most $k-1$;*
(ii) $|\mathcal{A}| \le k^k$.

One can find better bounds in the case (ii) (Arman and Retter 2017; Cherkashin 2011; Frankl 2019; Zakharov 2020b). In particular, for $k = 3$ it is known that $3^3 = 27$ in (ii) can be replaced with 10 and this result is sharp (Frankl et al. 1996).

Theorem 11 (Deza 1974) *Let $\mathcal{A}$ be a family of k-element sets such that $|A \cap A'|$ is the same for all distinct $A, A' \in \mathcal{A}$. Then at least one of the following statements is true:*

(i) *$A \cap A'$ is the same for all distinct $A, A' \in \mathcal{A}$;*
(ii) $|\mathcal{A}| \le k^2 - k + 1$.

2.3 An Isodiametric Inequality

Define *the Hamming distance* between two subsets of $[n]$ as the size of their symmetric difference. The *Hamming distance* between two vectors $v_1, v_2 \in \{-1, 0, 1\}^n$ is the number of coordinates that differ between v_1 and v_2. The *diameter* of a family $\mathcal{A} \subset 2^{[n]}$ or $\mathcal{A} \subset \{-1, 0, 1\}^n$ is the maximal distance between its members.

Theorem 12 (Kleitman 1966) *Let $\mathcal{A} \subset 2^{[n]}$ be a family with diameter at most D for $n > D$. Then*

$$|\mathcal{A}| \le S(n, D),$$

where S is defined in Definition 1.

Theorem 12 is sharp: in the case of even D the equality holds for the family $\mathcal{K}(n, D) := \{A \subset [n] \colon |A| \le \frac{D}{2}\}$ and in the case of odd D the equality holds for the family $\mathcal{K}_x(n, D) := \{A \subset [n] \colon |A \backslash \{x\}| \le \frac{D}{2}\}$ for some fixed $x \in [n]$.

Moreover, in Frankl (2017) Frankl proved the following stability result. Let $A \Delta B$ stand for the symmetric difference of the sets A and B. We say that a family $\mathcal{A}'$ is a *translate* of a family $\mathcal{A}$ if $\mathcal{A}' = \{A \Delta T \colon A \in \mathcal{K}(n, D)\}$ for some $T \subset [n]$.

Theorem 13 (Frankl 2017) *Let $\mathcal{A} \subset 2^{[n]}$ be a family with the diameter at most D and $|\mathcal{A}| = S(n, D)$ for $n \ge D + 2$. Then in the case of even D family $\mathcal{A}$ is a translate of $\mathcal{K}(n, D)$ and in the case of odd D the family $\mathcal{A}$ is a translate of $\mathcal{K}_y(n, D)$.*

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2.4 Simple Hypergraphs and Reed–Solomon Codes

A hypergraph $H = (V, E)$ is a collection of *(hyper)edges* E on a finite set of *vertices* V. A hypergraph is called k-*uniform* if every edge has size k. A hypergraph is *simple* if every two edges share at most one vertex. The following construction is a special case of Reed–Solomon codes (MacWilliams and Sloane 1977, Chapter 10); it is also known as Kuzjurin's construction (Kuzjurin 1995).

Fix a prime $p > k$ and let the vertex set V be the union of k disjoint copies of a field with p elements $\mathbb{F} = GF(p)$; call them $\mathbb{F}_1, \ldots, \mathbb{F}_k$. Consider the following system of linear equations

$$\sum_{i=1}^{k} i^j x_i = 0, \quad j = 0, 1, \ldots, k-3$$

over $\mathbb{F}_p$. The solutions $\{x_1, \ldots x_k\} \in \mathbb{F}_1 \sqcup \cdots \sqcup \mathbb{F}_k$, where $x_i \in \mathbb{F}_i$, form the edge set E. Fixing two arbitrary variables there is a unique solution over $\mathbb{F}_p$, because the corresponding square matrix is a Vandermonde matrix with nonzero determinant. It means that there are p^2 different solutions and $|e_1 \cap e_2| \leq 1$ for every distinct $e_1, e_2 \in E$. Summing up, $H_p(k) := (V, E)$ is a p-regular k-uniform simple hypergraph with $|V| = pk$ and $|E| = p^2$.

A k-uniform hypergraph is b-*simple* if every two edges share at most b vertices. The same construction with $k - b - 1$ equations gives an example of a k-uniform b-simple hypergraph $H(p, k, b)$.

Further we use *regularity* of $H = H(p, k, b)$ in the following sense. Consider an arbitrary vertex subset A of size b. If A contains at most 1 vertex from every copy of $\mathbb{F}_p$, then H has exactly p hyperedges containing A; otherwise H contains no such edges. Slightly abusing the notation we say that b-degree of H is p.

2.5 Steiner Systems

A *Steiner system* with parameters n, k and l is a collection of k-subsets of $[n]$ such that every l-subset of $[n]$ is contained in exactly one set of the collection. There are some obvious necessary 'divisibility conditions' for the existence of Steiner (n, k, l)-system:

$$\binom{k-i}{l-i} \text{ divides } \binom{n-i}{k-i} \text{ for every } 0 \leq i \leq k-1.$$

In a breakthrough paper Keevash (2014) proved the existence of Steiner (n, k, l)-systems for fixed k and l under the divisibility conditions and for $n > n_0(k, l)$ (different proofs can be found in Glock et al. 2023 and Keevash 2018).

Partial Steiner system. When the divisibility conditions do not hold we are still able to construct a large *partial Steiner system*, that is, a collection of k-subsets of $[n]$ such that every l-subset of $[n]$ is contained in *at most* one set of the collection. Rödl confirmed a conjecture of Erdős and Hanani and proved the following theorem.

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Theorem 14 (Rödl 1985) *For every fixed k and $l < k$, and for every n there exists a partial (n, k, l)-system with*

$$(1 - o(1))\binom{n}{l} / \binom{k}{l}$$

k-subsets.

Later the result was refined in Grable (1999), Kim (2001), Kostochka and Rödl (1998). Also it follows from the mentioned results on Steiner systems.

2.6 Families with Even or Odd Intersections

Recall that $J(n, k, even)$ and $J(n, k, odd)$ were defined in Sect. 1.3. Frankl and Tokushige determined the independence numbers of these graphs.

Theorem 15 (Frankl–Tokushige 2016) *Let $n \geq n_0(k)$. Then*

$$\alpha[J(n, k, odd)] = \binom{\lfloor n/2 \rfloor}{k/2} \qquad \textit{for even } k,$$

$$\alpha[J(n, k, even)] = \binom{\lfloor (n-1)/2 \rfloor}{(k-1)/2} \qquad \textit{for odd } k.$$

In the case when k is even, the equality is achieved for the following family: we split $[n]$ into pairs and take all sets consisting of $k/2$ pairs. In the case when k is odd we also add a fixed point $x \in [n]$ to each constructed set.

3 Examples

Let us start with a simple example which is rarely close to the independence number.

Example 1 Let $t < 0$, $k > |t|$. Then $\alpha[J_{\pm}(n, k, t)] \geq 2^{|t|-1}\binom{n}{k}$.

Proof Fix an ordering of the coordinates. Take all vertices of $J_{\pm}(n, k, t)$ with the first $k - |t| + 1$ nonzero coordinates equal to 1. Any two such vertices can have different signs on at most $|t| - 1$ positions, therefore their scalar product is at least $-|t| + 1 = t + 1$. □

The following example is a part of Theorem 5.

Example 2 For any $t < 0$ and $k > |t|$ we have

$$\alpha[J_{\pm}(n, k, t)] \geq S(k, |t| - 1)\binom{n}{k},$$

and for even t we also have

$$\alpha[J_{\pm}(n, k, t)] \geq S(k, |t| - 1)\binom{n}{k} + \binom{k-1}{|t|/2}.$$

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Proof We start with the first bound for the case of odd t. Let I_{odd} be the set of all vertices of $J_{\pm}(n,k,t)$ with at most $(|t|-1)/2$ negative entries. Each k-set is the support of exactly

$$\sum_{j=0}^{(|t|-1)/2} \binom{k}{j} = S(k, |t|-1)$$

vertices in I_{odd}. Any two vectors in I_{odd} may differ in at most $2(|t|-1)/2 = |t|-1$ coordinates, so their scalar product is at least $t+1$, and I_{odd} is an independent set of the desired size.

Now we deal with the case of even t. Fix an ordering of the coordinates. For every k-set f add to I_{even} all the vertices with support f and with at most $|t|/2-1$ negative entries on f and all the vertices with -1 on the last coordinate of f and exactly $|t|/2-1$ other negative coordinates. Then each k-set is the support of exactly

$$\sum_{j=0}^{|t|/2-1} \binom{k}{j} + \binom{k-1}{|t|/2-1} = S(k, |t|-1)$$

vertices in I_{even}. Assume that I_{even} is not independent, i.e. the scalar product of some $v_1, v_2 \in I_{even}$ is equal to t. Then v_1 and v_2 together have at least $|t|$ negative entries. Hence both v_1 and v_2 have exactly $|t|/2$ negative entries, so both v_1 and v_2 have -1 at the last coordinates x_1 and x_2 of $\operatorname{supp} v_1$ and $\operatorname{supp} v_2$, respectively. But then both v_1 and v_2 can not have $+1$ at coordinates x_2 and x_1 respectively, so the scalar product is at least $t+1$. This contradiction shows that I_{even} is an independent set of the desired size.

Now we proceed to the second bound. Let us add to I_{even} all the vertices on the lexicographically first support $\{1,\ldots,k\}$ with exactly $|t|/2$ negative entries and having $+1$ at the k-th coordinate. Obviously the resulting set I has the claimed size. By definition, no edge connects two vertices from I on the support $\{1,\ldots,k\}$.

Consider a vertex v from I_{even} and a vertex $u \in I \setminus I_{even}$. Note that u and v together have at most $|t|$ negative entries. Since the largest coordinate of $\operatorname{supp} v$ is greater than k and v has -1 in this coordinate, the scalar product of u and v is at least $t+1$. Thus I is independent. □

Example 3 For $t \geq 0$ we have

$$\alpha[J_{\pm}(n,k,t)] \geq 2\alpha[J(n,k,t)].$$

Proof Let $I \subset V[J(n,k,t)]$ be an independent set of size $\alpha[J(n,k,t)]$. Define $I_{\pm}$ as a subset of $V[J_{\pm}(n,k,t)]$ consisting of vertices with all positive or all negative entries on every support $f = \operatorname{supp} v$, $v \in I$. It is easy to see that the subset $I_{\pm}$ is independent in $J_{\pm}(n,k,t)$. □

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4 Proofs

4.1 Proof of Theorem 7

We start with the lower bound. One can take the vertices only with non-negative coordinates (so exactly one vertex on each support is taken); obviously the scalar product of such vertices is always non-negative, so

$$\alpha[J_{\pm}(n,k,-1)] \geq \binom{n}{k}.$$

Now we will show the upper bound. Denote $G := J_{\pm}(n,k,-1)$. Fix a prime p, $n/(2k) \leq p \leq n/k$ (so by the statement of the theorem $p > 2^k$), and let $H := H_p(k)$ (see Sect. 2.4) be a p-regular k-uniform simple hypergraph with $V(H) \subset [n]$. Define graph $G[H]$ as a subgraph of G, consisting of the vertices with support on edges of H. So we have

$$|V(G[H])| = 2^k|E(H)|.$$

Fix an independent set I in $G[H]$; consider the set $X \subset [n]$ of coordinates on which the vertices from I have both signs. Denote by supp I the set of all supports of vertices from I (supp $I \subset E(H)$) and for a given $e \in E(H)$ put $e_X := e \cap X$.

Note that I has at most $2^{|e_X|}$ vertices on the support e ($|e_X|$ might be zero). Hence

$$\begin{aligned} |I| &\leq \sum_{e \in \operatorname{supp} I} 2^{|e_X|} \leq \sum_{e\in E(H):|e_X|=0} 2^{|e_X|} + \sum_{e\in E(H):|e_X|>0} 2^{|e_X|} \\ &\leq |\{e \in E(H) : |e_X| = 0\}| + \sum_{e\in E(H):|e_X|>0} 2^{|e_X|}. \end{aligned} \tag{2}$$

Let us show that e_X form a disjoint cover of X. Suppose the contrary, i.e. there are $e, f \in \operatorname{supp} I$ such that $e_X \cap f_X \neq \emptyset$. Since the hypergraph H is simple, and e, f correspond to its hyperedges, we have $|e_X \cap f_X| = |e \cap f| = 1$. Put $\{u\} := e \cap f$. By the definition of X there are vertices $v_1, v_2 \in I$ having different signs on u. Since I is independent and any two different supports intersect in at most 1 coordinate, v_1 and v_2 have the same support (say, not f). So every vertex of $G[H]$ with support f forms an edge in $G[H]$ with one of v_1 or v_2, thus I is not independent; contradiction.

So $\sum |e_X| = |X|$. Since the sequence $2^k/k$, $k \geq 1$, is non-decreasing and

$$\frac{a_1 + a_2 + \ldots + a_t}{b_1 + b_2 + \ldots + b_t} \leq \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_t}{b_t}\right),$$

we have

$$\sum_{e\in E(H):|e_X|>0} 2^{|e_X|} \leq \frac{|X|}{k}2^k. \tag{3}$$

By definition, every $e \in \{e \in E(H) : |e_X| = 0\}$ has empty intersection with X. Since H is p-regular, X intersects at least $\frac{p|X|}{k}$ edges of H (since every k-edge is counted at most k times), so

$$|\{e \in E(H) : |e_X| = 0\}| \le |E(H)| - \frac{p|X|}{k}. \tag{4}$$

Summing up, by (2), (3), (4) and the choice of p, we have

$$|I| \le |E(H)| - \frac{|X|}{k}p + \frac{|X|}{k}2^k \le |E(H)|,$$

which implies $\alpha(G[H]) \le |E(H)|$, hence

$$\frac{V(G[H])}{\alpha(G[H])} \ge 2^k.$$

By the definition $G[H]$ is a subgraph of the graph G, so Lemma 1 finishes the proof.

For some k one can choose a smaller H and require a weaker inequality for n, for instance in the case $k = 3$ (see Sect. 5.2).

4.2 Proof of Theorem 8

This is a generalization of the proof of Theorem 7. The lower bound is provided in the first part of Example 2.

Denote $G = J_{\pm}(n, k, t)$ during the proof. The case $t = -k$ is obvious, because $J_{\pm}(n, k, -k)$ is a matching. From now $|t| \le k - 1$. Fix n and a prime $p \le n/k$ to be large enough. Let $H = H(p, k, |t|)$ (see Sect. 2.4) be a k-uniform $|t|$-simple hypergraph with $|t|$-codegree p. Fix an embedding of $V(H)$ into $[n]$.

Define $G[H]$ as a subgraph of G, consisting of all the vertices with support on edges of H. Fix an independent set I in $G[H]$.

Let an *object* O be a pair of opposite vectors $\{o, -o\}$ with support of size $|t|$ with $\{0, \pm 1\}$ entries. Let $\mathcal{X}$ be the set of objects $O = \{o, -o\}$ such that $(v_1, o) = (v_2, -o) = |t|$ for some vertices $v_1, v_2 \in I$ (this means that v_1 and v_2 coincide on $\operatorname{supp} o$ with o and $-o$, respectively).

Let E_{tight} be the set consisting of such edges $e \in E(H)$ that $\operatorname{diam} I[e] < |t|$, where $I[e]$ stands for the set of vertices of I with the support e. Put $E_{wide} := E(H) \setminus E_{tight}$. Let I_{tight} and I_{wide} stand for the sets of vertices of I with the support from E_{tight} and E_{wide} respectively. Then $|I| = |I_{tight}| + |I_{wide}|$.

Consider an arbitrary support $e \in E_{wide}$; by the definition of E_{wide} there are an object $X = \{x, -x\} \in \mathcal{X}$ and vertices $v_1, v_2 \in I[e]$, such that $(v_1, x) = (v_2, -x) = |t|$. Since I is independent and H is $|t|$-simple, distinct e_1 and $e_2 \in E_{wide}$ cannot lead to the same $X \in \mathcal{X}$, so

$$|\mathcal{X}| \ge |E_{wide}|. \tag{5}$$

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For every $e \in E_{tight}$ we have diam $I[e] < |t|$, thus Theorem 12 implies

$$|I[e]| \leq S(k, |t| - 1). \tag{6}$$

Let us study the case of equality in (6). Fix a support $f \subset [n]$, $|f| = k$, and consider a family $\mathcal{A} \subset \{-1, 1\}^f$ with diameter at most $|t| - 1$ and size $S(k, |t| - 1)$. Also consider an object $O = \{o, -o\}$ such that supp $O \subset f$ (recall that $|\operatorname{supp} O| = |t|$). By the pigeon-hole principle and the oddity of t, one of $o, -o$ has at most $(|t| - 1)/2$ negative entries. Thus there is a vector v from $\mathcal{K}(k, |t| - 1)$ such that $(v, o) = |t|$ or $(v, -o) = |t|$. By Theorem 13 $\mathcal{A}$ is a translate of $\mathcal{K}(k, |t| - 1)$, so the previous conclusion also holds for $\mathcal{A}$.

Fix an object $X \in \mathcal{X}$ and consider an arbitrary support $e \in E_{tight}$ containing supp X. Assume that $(v, x) = \pm t$ for some $v \in I[e]$. Consider a support $g \in E_{wide}$ such that there are $u_1, u_2 \in I[g]$, satisfying $(u_1, x) = (u_2, -x) = t$ (g exists because $X \in \mathcal{X}$). Since H is $|t|$-simple, $(u_1, v) = t$ or $(u_2, v) = t$; a contradiction. By Theorem 13 we can refine the bound (6) in this case:

$$|I[e]| \leq S(k, |t| - 1) - 1. \tag{7}$$

By the construction of H for every $X \in \mathcal{X}$, supp X is contained in exactly p edges of H (because it is contained in at least one edge). Every edge of H is the support of $\binom{k}{|t|} 2^{|t|-1}$ objects, so is counted above at most $\binom{k}{|t|} 2^{|t|-1}$ times. By (5) at most $|\mathcal{X}|$ of the edges are wide. So the refined bound (7) is applicable to at least

$$\frac{p|\mathcal{X}|}{\binom{k}{|t|} 2^{|t|-1}} - |\mathcal{X}|$$

tight edges. Then

$$|I_{tight}| \leq S(k, |t| - 1) |E_{tight}| - \frac{p|\mathcal{X}|}{\binom{k}{|t|} 2^{|t|-1}} + |\mathcal{X}|.$$

On the other hand there is a straightforward bound

$$|I_{wide}| \leq 2^k |E_{wide}| \leq 2^k |\mathcal{X}|.$$

Putting it all together

$$|I| = |I_{tight}| + |I_{wide}| \leq S(k, |t| - 1) |E_{tight}| - \frac{p|\mathcal{X}|}{\binom{k}{|t|} 2^{|t|-1}} + (2^k + 1)|\mathcal{X}|. \tag{8}$$

For a large n (then p is also large enough) the inequality (8) implies $\alpha(G[H]) \leq S(k, |t| - 1) |E(H)|$. By the definition $G[H]$ is a subgraph G and

$$\frac{\alpha(G[H])}{V(G[H])} \leq \frac{S(k, |t| - 1)}{2^k},$$

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so Lemma 1 finishes the proof.

4.3 Proof of Theorem 9

Consider an arbitrary independent set I in the graph $J_{\pm}(n,k,0)$. Note that supports of the vertices of I form an intersecting family; denote it by F. Let U be a minimal (by inclusion) transversal of F. As U is minimal, for every coordinate $a \in U$ there is a vertex $x_a \in I$, such that $\operatorname{supp} x_a \cap U = \{a\}$.

In the case $|U| > 1$ we can consider the set

$$C := U \cup \operatorname{supp} x_a \cup \operatorname{supp} x_b$$

for two different $a, b \in U$. Note that $|C| \leq 3k$ and every $f \in F$ intersects C in at least two places (suppose that $|f \cap U| = 1$, then it should intersect either $(\operatorname{supp} x_a) \setminus U$ or $(\operatorname{supp} x_b) \setminus U$). Hence

$$|I| \leq 2^k \binom{|C|}{2} \binom{n}{k-2} < 2^k \frac{9k^2}{2} \binom{n}{k-2}.$$

Recall that $n > \frac{9}{2}k^3 2^k$, so

$$2^k \frac{9k^2}{2} \binom{n}{k-2} < 2^k \frac{9k^2}{2} \frac{n^{k-2}}{(k-2)!} < \frac{n}{k-1} \frac{n^{k-2}}{(k-2)!} < 2\binom{n-1}{k-1}.$$

The remaining case is $|U| = 1$, say $U = \{u\}$. Consider only vertices containing u^+, by Theorem 7 we have at most $\binom{n-1}{k-1}$ such vertices. The same bound for u^- gives the desired bound.

Example 3 and Erdős–Ko–Rado theorem give a lower bound.

4.4 Proof of Corollary 1

Let us repeat the proof of Theorem 9. Let I be an arbitrary independent set in $J_{\pm}(n,k,0)$. Then

$$|I| < 2^k \frac{9k^2}{2} \binom{n}{k-2}$$

or the family of all supports of vertices from I has a transversal of size 1. The first possibility implies

$$|I| \leq C(k) \binom{n}{k-2};$$

the latter one contradicts the condition of the corollary.

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5 The Case $k \leq 3$

We have implemented Östergård algorithm (Östergård 2002) to find independence numbers of several small graphs. All the calculations were done on a standard laptop in a few hours. The source can be found in Kiselev (2020).

5.1 The Case $k = 2$

The case $t = -1$. By simple calculations we have

$$\alpha[J_{\pm}(2,2,-1)] = \alpha[J_{\pm}(3,2,-1)] = 4, \quad \alpha[J_{\pm}(4,2,-1)] = 8, \quad \alpha[J_{\pm}(5,2,-1)] = 10.$$

In Sect. 2.1 we show that the sequence

$$\frac{\alpha[J_{\pm}(n,2,-1)]}{|V[J_{\pm}(n,2,-1)]|}$$

is non-increasing, so

$$\alpha[J_{\pm}(n,2,-1)] = \binom{n}{2}$$

for $n \geq 5$.

The case $t = 0$. It is straightforward to check that

$$\alpha[J_{\pm}(2,2,0)] = 2, \quad \alpha[J_{\pm}(3,2,0)] = \alpha[J_{\pm}(4,2,0)] = 6.$$

For the case $n > 4$ we can repeat the proof of the Theorem 9 and show that $\alpha[J_{\pm}(n,2,0)] = 2(n-1)$.

The case $t = 1$. From Proposition 2 we have

$$\alpha[J_{\pm}(n,2,1)] = 2n \qquad \text{for even } n,$$
$$\alpha[J_{\pm}(n,2,1)] = 2(n-1) \qquad \text{for odd } n.$$

5.2 The Case $k = 3, t = -1$

Proposition 2 *Let $n \geq 7$. Then*

$$\alpha[J_{\pm}(n,3,-1)] = \binom{n}{3}.$$

Proof Fano plane is the projective plane over $GF(2)$ i.e. the following simple 3-graph on 7 vertices

$$\{1,2,3\}, \{1,4,7\}, \{1,5,6\}, \{2,4,6\}, \{2,5,7\}, \{3,4,5\}, \{3,6,7\}.$$

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Consider an arbitrary embedding F of the Fano plane into $V[J_{\pm}(n, 3, -1)]$. As usual consider the subgraph $G[F]$; it has $7 \cdot 2^3 = 56$ vertices. One may check by hands or via computer that $\alpha(G[F]) = 7$. By Lemma 1

$$\alpha[J_{\pm}(n, 3, -1)] \leq \binom{n}{3}.$$

On the other hand, Example 1 implies $\alpha[J_{\pm}(n, 3, -1)] = \binom{n}{3}$. □

By the computer calculations we have

$$\alpha[J_{\pm}(6, 3, -1)] = 21 > \binom{6}{3} = 20,$$

so Proposition 2 is sharp. Also

$$\alpha[J_{\pm}(5, 3, -1)] = 14, \quad \alpha[J_{\pm}(4, 3, -1)] = 8, \quad \alpha[J_{\pm}(3, 3, -1)] = 2.$$

5.3 The Case $k = 3, t = 0$

By the computer calculations we have

$$\alpha[J_{\pm}(3, 3, 0)] = \alpha[J_{\pm}(4, 3, 0)] = 8, \quad \alpha[J_{\pm}(5, 3, 0)] = 20, \quad \alpha[J_{\pm}(6, 3, 0)] = 32,$$
$$\alpha[J_{\pm}(7, 3, 0)] = \alpha[J_{\pm}(8, 3, 0)] = \alpha[J_{\pm}(9, 3, 0)] = 56.$$

Proposition 3 *Let $n \geq 9$. Then*

$$\alpha[J_{\pm}(n, 3, 0)] = 2\binom{n-1}{2}.$$

Proof The example is inherited from Theorem 9.

Let us proceed with the upper bound. For the case $n = 9$ the computer calculations give us the desired result. Let us repeat the proof of Theorem 9, updating it for small values of n. Let I be a maximal independent set in $G := J_{\pm}(n, 3, 0)$.

Clearly supports of vertices of I form a 3-uniform intersecting family. Theorem 10 states that an intersecting family either contains at most 27 sets or has a 2-transversal. It is known (Frankl et al. 1996) that the constant 27 can be refined to 10.

In the first case the family of supports has no 2-transversal. Then $|I| \leq 8 \cdot 10$, which is enough for $n > 10$. Assume the contrary to the statement in the case $n = 10$, id est $|I| > 72$. It implies that vertices in I have exactly 10 different supports. Suppose that every pair of supports shares exactly one vertex. Then by Theorem 11 all the supports have one common vertex, so at least $1 + 2 \cdot 10 > 10$ coordinates are required. Thus there are supports f_1, f_2 such that $|f_1 \cap f_2| = 2$. The initial graph G has 16 vertices with supports f_1 and f_2; by the equality $\alpha[J_{\pm}(4, 3, 0)] = 8$, I has at least

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8 missing vertices on these supports. This refines the bound $|I| \leq 80$ to the desired $|I| \leq 72 = 2\binom{9}{2}$.

In the second case we have a one-point transversal set, say $U = \{u\}$. Let I_{sign} be a set of vertices from I containing u^{sign}, where $sign \in \{+, -\}$. Clearly $|I| = |I_+| + |I_-|$. After removing coordinate u from every vertex, I_+ becomes an independent set in $J_\pm(n-1, 2, -1)$. By the Sect. 5.1 $|I_+| \leq \binom{n-1}{2}$. The same bound for I_- finishes the proof in this case.

In the last case we have a transversal set of size 2, say $\{a, b\}$. Let I_a be the set of vertices of I containing a and not containing b, I_b is defined analogously. Both I_a and I_b are nonempty, otherwise there is a one-point transversal set which is the previous case. Define $I_{ab} = I \backslash I_a \backslash I_b$. Computer calculations show that for $n = 10$ we have at most 48 vertices in an independent set with such conditions.

Let n be greater than 10; for every set $A \subset [n]$, such that $|A| = 10$ and $a, b \in A$, we have $\alpha(G[A]) \leq 48$ (here $G[A]$ stands for the subgraph of G containing all the vertices v such that $\operatorname{supp} v \subset A$). Define $I[A]$ as the set of vertices i from I such that $\operatorname{supp} i \subset A$; note that $I[A]$ is an independent set. Every vertex from I_{ab} belongs to $\binom{n-3}{7}$ different A, every vertex from $I_a \cup I_b$ belongs to $\binom{n-4}{6}$ different A. Summing up inequalities $|I[A]| \leq \alpha(G[A]) \leq 48$ over all choices of A we got

$$\binom{n-3}{7}|I_{ab}| + \binom{n-4}{6}(|I_a| + |I_b|) \leq 48\binom{n-2}{8}$$

which is equivalent to

$$\frac{n-3}{7}|I_{ab}| + (|I_a| + |I_b|) \leq \frac{48}{56}(n-2)(n-3).$$

Finally,

$$\begin{aligned}|I| = |I_{ab}| + |I_a| + |I_b| &\leq \frac{n-3}{7}|I_{ab}| + (|I_a| + |I_b|) \\ &\leq \frac{48}{56}(n-2)(n-3) < 2\binom{n-1}{2}.\end{aligned}$$

□

5.4 The Case $k = 3, t = -2$

Example 2 gives us a lower bound $\alpha[J_\pm(n, 3, -2)] \geq 2\binom{n}{2} + 2$. Note that the Katona averaging method does not give an exact result because of the additional term of a smaller order of growth.

First, note that Theorem 6 in this case gives the bound

$$\alpha[J_\pm(n, 3, -2)] \leq 2\binom{n}{3} + 8\binom{n}{2}.$$

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Indeed, let I be an independent set in $J_{\pm}(n,3,-2)$. We call a vertex $v \in I$ *bad* if there is another vertex with the same support which differs in exactly two places. Otherwise we call a vertex *good*. From Theorem 12 there are at most $2\binom{n}{3}$ good vertices.

Let us show that the number of bad vertices is at most $8\binom{n}{2}$. Indeed, each bad vertex has a pair of signplaces such that antipodal pair of signplaces contained in another vertex. But then all vertices containing one of these two pairs of signplaces must have the same third place therefore there are at most $8\binom{n}{2}$ bad vertices. □

Using more accurate double counting we can prove the following upper bound.

Proposition 4 *For $n \geq 6$ we have*

$$\alpha[J_{\pm}(n,3,-2)] \leq 2\binom{n}{3} + \frac{8}{3}\binom{n}{2}.$$

Proof A pair of vertices $v, w \in I$ is called *tangled* if these vertices have the same support and differ exactly at two places. Define the *weight* $c_I(v,i,j)$, where $v \in I$ and $i, j \in v$, in the following way:

$$c_I(v,i,j) = \begin{cases} 1, & \text{if } v \text{ does not have tangled vertices in } G, \\ 2, & \text{if } v \text{ has a tangled vertex in } G \text{ which differs at places i, j}, \\ 0.5, & \text{otherwise.} \end{cases}$$

Note that for a vertex v sum of corresponding weights is at least 3. Let $d_{i,j}$ be the sum of weight of vertices containing places i and j and let us estimate an upper bound for $d_{i,j}$. Then there are three cases which depend on whether there are tangled vertices containing places i, j and whether these vertices have antipodal signs on places i, j.

In the first case there are no tangled vertices in I which differ in places i, j. Then for any place l the total weight of vertices with support $\{i, j, l\}$ is at most 2. Then $d_{i,j} \leq 2(n-2)$. In the second case there are tangled vertices in I which contain all four pairs of signplaces on places i, j. Then there are at most 8 vertices containing these places and $d_{i,j} \leq 16$.

In the last case there are two vertices in I which are antipodal on places i, j and there are no vertices in I which contain one of the pairs of signplaces on places i, j. Then there are at most 4 vertices which differ in places i, j and their total weight is at most 8. The rest of vertices containing places i, j have the same signs on these places therefore their total weight is at most $2(n-2)$.

Therefore, $d_{i,j} \leq 2n+4$ and

$$3|I| \leq \sum_{1 \leq i < j \leq n} d_{i,j} \leq \binom{n}{2}(2n+4) = 6\binom{n}{3} + 8\binom{n}{2}.$$

□

6 Open Questions

It seems very challenging to find a general method providing the independence number of $J_\pm(n,k,t)$. Here we discuss questions that seem for us both interesting and relatively easy.

Small values of the parameters. The smallest interesting case is $J_\pm(n,3,-2)$. We hope that for $n > n_0$ Example 2 is the best possible, i.e.

$$\alpha[J_\pm(n,3,-2)] = \alpha[K_\pm(n,3,-2)] = 2\binom{n}{3} + 2.$$

Recall that the last equality is established by Theorem 5.

Another small case leads to the following conjecture.

Conjecture 1 *Let $n > n_0$ be an even number. Then*

$$\alpha[J_\pm(n,4,1)] = 2n(n-2).$$

Obviously $\alpha[J_\pm(n,4,1)] \geq \alpha[J_\pm(n,4,odd)] = 2n(n-2)$ (see Proposition 2).

Chromatic numbers. Usually finding or evaluating the chromatic number is a more complicated problem than finding or evaluating the independence number. In particular Lovász (1978) proved Kneser's conjecture on the chromatic number of $K(n,k,0)$ 17 year after Erdős, Ko and Rado determined the independence number of this graph.

In the setting of this paper we have

$$\begin{aligned} c(k,t)n \leq \frac{|V[J_\pm(n,k,t)]|}{\alpha[J_\pm(n,k,t)]} \leq \chi[J_\pm(n,k,t)] \leq \frac{|V[J_\pm(n,k,t)]|}{\alpha[J_\pm(n,k,t)]} \log|V[J_\pm(n,k,t)]| \\ \leq C(k,t)n\log n \end{aligned}$$

for some positive constants $c(k,t)$, $C(k,t)$. The second inequality holds since $J_\pm(n,k,t)$ is a vertex-transitive graph (see Lovász 1975).

Recently Cherkashin (2022) proved that $\log\log n \leq \chi[J_\pm(n,3,-2)] \leq 4\log\log n + 6$, which means that the chromatic number of a Johnson-type graph may not coincide with simple general bounds.

Difference between $J_\pm(n,k,t)$ and $K_\pm(n,k,t)$. It turns out that for a negative odd t Theorems 5 and 8 give

$$\alpha[J_\pm(n,k,t)] = \alpha[K_\pm(n,k,t)].$$

Does it hold for all negative t? Do we have

$$\chi[J_\pm(n,k,t)] = \chi[K_\pm(n,k,t)]$$

in this case?

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4 The general comparison of the behavior of independence numbers and chromatic numbers of these graphs is also of interest.

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