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On the chromatic number of 2-dimensional spheres

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Abstract:	In 1976 Simmons conjectured that every coloring of a 2-dimensional sphere of radius strictly greater than $1/2$ in three colors has a couple of monochromatic points at distance 1 apart. We prove this conjecture.
Response to Reviewers:	Dear Reviewer, \vskip+2cm thank you for the comments, we try to implement everything, expect the swapping x_2 and y_2 (page 6, line 22-23). It looks like that the calculation is more natural without such rearrangement. \vskip+2cm \begin{flushright} Yours, authors. \end{flushright}
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On the chromatic number of 2-dimensional spheres

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Abstract

In 1976 Simmons conjectured that every coloring of a 2-dimensional sphere of radius strictly greater than 1/2 in three colors has a pair of monochromatic points at distance 1 apart. We prove this conjecture.

1 Introduction

A *coloring* of a given set M is a map from M to the set of colors. A coloring of a subset M of a metric space is *proper* if no pair of monochromatic points lie at distance 1 apart. The minimum number of colors that admits a proper coloring of M is called *the chromatic number* of M; we denote it by $\chi(M)$. In the case of $M \subset \mathbb{R}^n$, the distance typically comes from the induced Euclidean metric on M.

A slightly different point of view is to consider a *unit distance graph* $G(M)$: the points of M are the vertices of $G(M)$ and edges connect points at unit distance apart. By definition, $\chi(M) = \chi(G(M))$. The de Bruijn–Erdős theorem states that if $\chi(M)$ is finite then there is a finite subgraph H of $G(M)$ such that $\chi(H) = \chi(G(M))$.

Denote by $S^2(r)$ the two-dimensional sphere of radius r in $\mathbb{R}^3$ centered at the origin. Let $\chi(S^2(r))$ be the chromatic number of $S^2(r)$ with respect to the Euclidean metric. Obviously, if $r < 1/2$ and $r = 1/2$ then the chromatic number is equal to 1 and 2, respectively. Note that for any $r > \frac{1}{2}$ there is $r_1 < r$ such that $S^1(r_1)$ contains an odd cycle. Since $S^1(r_1) \subset S^2(r)$, we obtain that $\chi(S^2(r)) \geqslant 3$. G. Simmons [15] proved that

$$\chi(S^2(r)) \geqslant 4 \quad \text{for} \quad r \geqslant \frac{\sqrt{3}}{3}.$$

In the proof, Simmons constructs certain subgraphs of $G(S^2(r))$ that contain triangles. Obviously, for smaller values of the radius $G(S^2(r))$ is triangle-free, and so other ideas are needed.

Then L. Lovász [10] generalized the odd cycle construction to an arbitrary dimension, showing that for every $n \geqslant 3$ there exists a family of *strongly self-dual polytopes* inscribed in $S^{n-1}(r)$ whose graphs of diameters have chromatic number $n+1$ and that r can be arbitrarily close to $\frac{1}{2}$. In our notation this result can be formulated as follows:

Theorem 1 (Lovász, [10])**.** *For every $n \geqslant 2$ there exists a monotonically decreasing sequence $r_k^{(n)}$, $k = 1, 2, \dots,$ such that*

$$\lim_{k\to\infty} r_k^{(n)} = \frac{1}{2} \quad \textit{and} \quad \chi\left(S^{n-1}\left(r_k^{(n)}\right)\right) \geqslant n+1.$$

Since $S^{n-1}(r_1) \subset S^n(r)$ for $r_1 \leqslant r$, we get the following inequality.

Corollary 1.

$$\chi(S^{n-1}(r)) \geqslant n \quad \textit{for} \quad r > \frac{1}{2}.$$

Some sources state that the chromatic number of a two-dimensional sphere $S^2(r)$ is known only for $r \leqslant \frac{1}{2}$ and for $r = \frac{\sqrt{2}}{2}$ [5, 11]. But it should be clarified that the equality $\chi(S^2(r)) = n + 1 = 4$ is true for $r \in \{r_k^{(3)}\} \cap \left(\frac{1}{2}, \frac{\sqrt{3-\sqrt{3}}}{2}\right]$. Explicit formulas for algebraic numbers $r_k^{(3)}$, if such exist, seem to be too complicated, but it is not difficult to compute $r_k^{(3)}$ for a given k with an arbitrary precision by approximately solving a certain optimization problem. For example, the first non-trivial construction in the case of a two-dimensional sphere corresponds to a unit distance embedding of the Grötzsch graph at $r = 0.54003829\ldots$

It is worth noting that chromatic numbers in high dimensions were studied using algebraic, topological and combinatorial methods. A.M. Raigorodskii [14] showed that for every fixed $r > 1/2$ the chromatic number of an n-dimensional sphere grows exponentially with n. O. Kostina [7] refined asymptotic lower bounds. R. Prosanov [12] gave a new asymptotic upper bound. The paper of A. Kupavskii [9] contains several results on the number of different colors on a sphere of given radius in every proper coloring of $\mathbb{R}^n$.

A lot of results on colorings of 2-dimensional spheres were obtained by Simmons [15]. Recent discovery of a 5-chromatic unit distance subgraph of the Euclidean plane [2] spurred interest in the topic and in particular to the chromatic number of a 2-dimensional sphere.

Among the other results, in [18] the authors constructed several 5-chromatic subgraphs of 2-dimensional spheres, which lead to the bounds

$$\chi(S^2(r_1)) \geqslant 5 \quad \text{where} \quad r_1 = \cos\frac{3\pi}{10} = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} = 0.58778\ldots;$$

$$\chi(S^2(r_2)) \geqslant 5 \quad \text{where} \quad r_2 = \cos\frac{\pi}{10} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} = 0.95105\ldots.$$

The paper [16] contains a family of proper colorings of $S^2(r)$ spheres in 7 colors, provided r is large enough.

The following statement was formulated by Simmons as a conjecture [15]. The proof of Simmons' conjecture is the main result of the present paper.

Theorem 2. *For every $r > \frac{1}{2}$ we have*

$$\chi(S^2(r)) \geqslant 4.$$

We note that for $\frac{1}{2} < r \leqslant \frac{\sqrt{3-\sqrt{3}}}{2} = 0.563\ldots$ a proper 4-coloring of $S^2(r)$ can be obtained from a partition of the sphere into four equal spherical triangles [15]. It implies the following corollary.

Corollary 2. $\chi(S^2(r)) = 4$ *for* $\frac{1}{2} < r \leqslant \frac{\sqrt{3-\sqrt{3}}}{2} = 0.563\ldots$.

Structure of the paper. Section 2 contains the proof of Theorem 2. In Section 3 we summarize the results and discuss some further questions.

2 Proof of Theorem 2

Recall that for $r \geqslant \frac{\sqrt{3}}{3}$ the statement was proved in [15].

Here is the sketch of the proof. Fix $r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$. Suppose that there is a proper 3-coloring of the sphere $S^2(r)$. Further arguments consist of two steps. In the first step we use the Borsuk–Ulam theorem to show that every color is dense in the sphere. Consider a graph G_k with vertices $x_1, \ldots x_{2k+1}$, $y_1, \ldots, y_{2k+1}$ and edges $\{(y_i, y_{i+1}), (x_i, y_i) : 1 \leqslant i \leqslant 2k+1\}$ (where indices are modulo $2k+1$), i.e. an odd cycle with attached pendant vertices. We provide an explicit representation of G_k as a unit distance subgraph of the sphere. The second step is to show that this embedding is stable under small perturbations of x_i. Then one can move every x_i at a red point, which forces the odd cycle on vertices y_i to be colored in the remaining two colors. The contradiction proves the theorem.

Note that the idea of attaching an odd cycle to a finite set A in order to exclude the possibility of A to be monochromatic was used in a series of papers devoted to the existence of planar unit distance graphs with chromatic number 4 and arbitrarily large girth [4, 17, 19]. The key twist in step 2 is to find the required embedding of G_k implicitly, i.e. the corresponding A is not a constructive set. Similar ideas were used by the authors in [6].

2.1 Step 1. Each color is a dense set

All the distances are considered in the metrics induced from Euclidean space $\mathbb{R}^3$, the distance between x and y is denoted by $\|x - y\|$.

Fix $r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$ and consider $S^2(r)$. Suppose that there is a proper coloring of $S^2(r)$ in three colors. Consider the unit distance graph $G = G(S^2(r))$. Then the neighborhood of a vertex x in graph G forms a circle of diameter $d = \frac{\sqrt{4r^2-1}}{r}$ in the sphere. It is worth noting that every circle in the sphere has two centers at a pair of antipodal points and hence it has two radii; a circle of diameter $d = \frac{\sqrt{4r^2-1}}{r}$ has radii 1 and $\rho = \sqrt{4r^2-1}$ in the induced metric. Since $r < \frac{\sqrt{3}}{3}$, the smaller radius is ρ, so we refer to ρ as the radius and $-x$ as the center of the circle. Vice versa, any circle of radius ρ is a graph-neighborhood of some vertex of G, and hence contains points of at most two colors.

We need the following technical statement.

Lemma 1. *Let $D \subset S^2(r) \times S^2(r)$ be a set of pairs (x, y) such that $0 < \|x - y\| < d$. Then*

- *for every $(x, y) \in D$ there are two circles of radius ρ containing x and y. One may denote their centers by c_r and c_l in such a way that the triple of radius-vectors (x, y, c_r) is right-handed and the triple (x, y, c_l) is left-handed.*
- *The functions $c_r(x, y)$ and $c_l(x, y)$ from D to $S^2(r)$ are continuous.*

In what follows, we will call a circle passing through the points x, y with center c *right-handed* if the triple (x, y, c) is right-handed, and *left-handed* otherwise.

Let C_{red}, C_{blue}, C_{green} be the sets of red, blue and green points, respectively. A *chromaticity* of a point x is the number of sets $\overline{C_{red}}$, $\overline{C_{blue}}$, $\overline{C_{green}}$ containing x (as usual, $\overline{T}$ stands for the closure of a set T). A set $T \subset S^2(r)$ is called *dense* if $\overline{T} = S^2(r)$. Let $B_\rho(x)$ denote the set of points $y \in S^2(r)$ such that $\|x - y\| < \rho$, i.e. an open ball of radius ρ and diameter d.

Lemma 2. *If some open ball of diameter d contains points of all three colors then each of C_{red}, C_{blue}, C_{green} is dense in the sphere.*

Proof. Consider points $x \in C_{red}$, $y \in C_{blue}$ and $z \in C_{green}$ inside a ball K_0 of diameter d. Then one can continuously move K_0 to a ball K containing two points (say, x and y) on the boundary; at the first such moment the point z lies inside K. The circle ∂K contains blue and red points, and so it is colored in blue and red only. Hence, it contains a point u lying in the closures of C_{red} and C_{blue}; without loss of generality, assume that point u is red. A red-green circle (right-handed, guaranteed to exist by Lemma 1) of diameter d containing z and u and a blue-green circle (left-handed) with the diameter d containing z and blue point u' in a small neighborhood of u intersect in a green point v. Note that if $u = u'$ then $v = u = u'$. Hence, due to the continuity of circles in Lemma 1, v may be arbitrarily close to u with a proper choice of u' (see Fig. 1). It implies that the chromaticity of u is three.

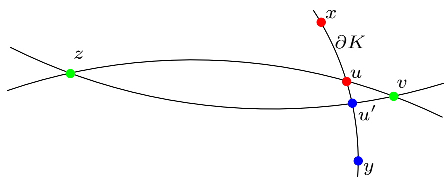

Figure 1: Finding a point with chromaticity 3 in Lemma 2

Since u has chromaticity 3, a small neighborhood of u contains a point $a \neq u$ with the chromaticity at least 2. Suppose that a has chromaticity 2 (say, a does not lie in $\overline{C_{green}}$) and $\|a - u\| < d$. Consider a green point b in a small neighborhood of u. Consider a red point e and a blue point f in a small neighborhood of a. Then the right-handed circle containing b and e is red-green and the left-handed circle containing b and f is blue-green, so they intersect in a green point g. Since the neighborhoods can be chosen arbitrarily small, g can be arbitrarily close to a. Hence, a has chromaticity 3, a contradiction.

Thus, we have shown that if a point with the chromaticity 3 and a point with the chromaticity at least 2 lie at a distance smaller than d, then they both have chromaticity 3.

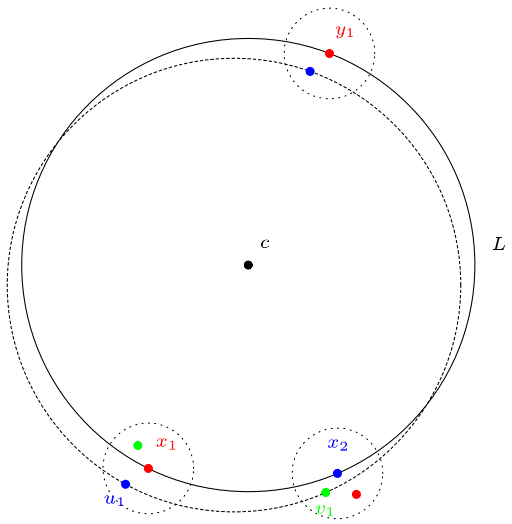

Figure 2: Propagation of 3-chromaticity along a circle in Lemma 2

Now let x_1 and x_2 be points of chromaticity 3 such that $\|x_1 - x_2\| < d$. We claim that any point on a circle L of diameter d containing x_1 and x_2 has chromaticity three. By the previous argument, it is enough to show that the chromaticity is at least 2. Without loss of generality, a triple (x_1, x_2, c) is left-handed, where c is the center of L on the sphere. Arguing indirectly, assume that a point $y_1 \in L$ has a small red neighborhood U_{y_1}. Choose a blue point u_1 in a small neighborhood of x_1 and a green point v_1 in a small neighborhood of x_2 (see Fig. 2). By Lemma 1 the left-handed circle of diameter d passing through blue point u_1, green point v_1 is close to L so it intersects red set U_{y_1}; this contradiction shows that every point on L has chromaticity 3.

Let q be an arbitrary point of $S^2(r)$. Consider a path $q_0, q_1 \dots q_t = q$ such that $q_0 \in L$ and $\|q_{i+1} - q_i\| < \rho$ for $0 \leqslant i \leqslant t-1$. A circle L_1 of diameter d that passes through q_1 and q_0 intersects L in two points, so by the previous argument every point (in particular, q_1) of L_1 has chromaticity 3. By induction, a circle L_{i+1} of diameter d that passes through q_{i+1} and q_i intersects L_i in two points, so every point in L_{i+1} (in particular q_{i+1}) has chromaticity 3. So $q = q_t$ also has chromaticity 3. Since $q \in S^2(r)$ was arbitrary, every point of $S^2(r)$ has chromaticity 3. $\square$

Suppose that the condition of Lemma 2 does not hold, i.e.

$$\textit{every open ball of diameter } d \textit{ contains points of at most two colors.} \qquad (\star)$$

Consider a continuous function

$$f : S^2(r) \to \mathbb{R}^2, \quad f(x) = (\mathrm{dist}(x, \overline{C_{red}}), \mathrm{dist}(x, \overline{C_{blue}})),$$

where $\mathrm{dist}(\cdot)$ stands for the distance between a point and a set in $\mathbb{R}^3$. By the Borsuk–Ulam theorem there exists $x^* \in S^2(r)$ such that $f(x^*) = f(-x^*)$. We have to deal with three cases.

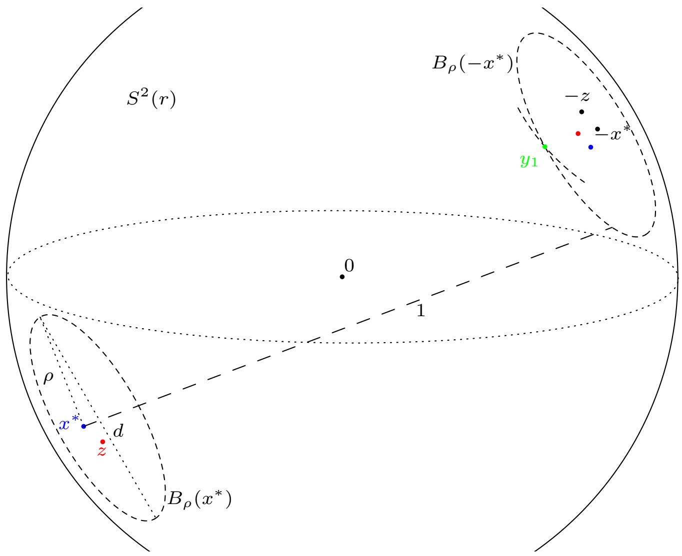

Figure 3: Case 1

Case 1: $f(x^*) = (0, 0)$. Without loss of generality, the point x^* is blue. One may pick a red point z, which is arbitrarily close to x^*. If $\|x^* - z\| < \rho$, then the intersection of circles of unit Euclidean radius with centers x^* and z consists of two green points y_1, y_2 belonging to the circle of radius ρ centered at $-x^*$. Hence, one can cover a small neighborhood of $-x^*$ and y_1 by a ball of diameter d. Every neighborhood of $-x^*$ contains red and blue points; point y_1 is green (see Fig. 3). We have a contradiction with assumption ($\star$).

Case 2: $f(x^*) = (a, b)$, $a, b > 0$. Then both points $x^*, -x^*$ are green. We may swap blue and green colors to reduce the situation to the next case with the same x^*.

Case 3: $f(x^*) = (a, 0)$, $a > 0$. We claim that $a > \rho$. Assume the contrary, i.e. $x^* \in \overline{C_{blue}}$ and for every $\eta > 0$ there is a red point $z = z_\eta$ such that $\|x^* - z\| \leqslant \rho + \eta$. Note that if x^* is green, then it contradicts ($\star$), so x^* is blue. There are distinct points $y_1, y_2 \in \overline{B_\rho(-x^*)}$ such that $\|x^* - y_1\| = \|x^* - y_2\| = \|z - y_1\| = \|z - y_2\| = 1$. Since x^* is blue and z is red $y_1, y_2 \in \overline{C_{green}}$. Recall that $f(-x^*) = f(x^*)$, so there is a point $z' \in \overline{C_{red}} \cap \overline{B_\rho(-x^*)}$. Let $y' \in \{y_1, y_2\}$ be such that z', $-x^*$ and y' do not lie on a great circle of $S^2(r)$. Then for a small enough η the neighborhoods of $-x^*$, y' and z' can be covered by a ball of diameter d. This is a contradiction with ($\star$).

So the set $\overline{B_\rho(x^*)} \cup \overline{B_\rho(-x^*)}$ is colored with blue and green.

Lemma 3. *The bipartite subgraph of $S^2(r)$ with parts $\overline{B_\rho(x^*)}$ and $\overline{B_\rho(-x^*)}$ is connected.*

Proof. Any point $x \in \overline{B_\rho(x^*)}$ has a common neighbor with x^* since the corresponding unit circles intersect. So $\overline{B_\rho(x^*)}$ belong to the same connected component; the same holds for $\overline{B_\rho(-x^*)}$. There is an edge between $\overline{B_\rho(x^*)}$ and $\overline{B_\rho(-x^*)}$, and so the subgraph is connected. □

By Lemma 3, one can color $\overline{B_\rho(x^*)} \cup \overline{B_\rho(-x^*)}$ in two colors in the unique way (up to symmetry): the first part is blue and the second one is green. Then the distance from x^* and $-x^*$ to $\overline{C_{blue}}$ is zero and nonzero simultaneously.

This contradiction implies that each color is dense in the sphere.

2.2 Step 2. Stability of embedding

In this section we will need the implicit function theorem [8] in the following weakened formulation.

Theorem 3. *Let $F : \mathbb{R}^{2s} \to \mathbb{R}^s$ be a continuously differentiable function,*

$$F = F(X, Y) = F(x_1, \ldots, x_s;\ y_1, \ldots, y_s),$$

and at some point $X = a$, $Y = b$ the following conditions are satisfied

$$F(a, b) = 0, \qquad \det\left(\frac{\partial F(X, Y)}{\partial Y}\right)_{X=a, Y=b} \neq 0.$$

Then there exists $\eta > 0$ such that the system of equations $F(X, Y) = 0$ is solvable in Y for any X satisfying the condition $\|X - a\| < \eta$.

Recall that G_k is an odd cycle of length $m = 2k + 1$ with an extra pendant (leaf) vertex attached to each vertex of the cycle. In particular, G_k has $2m$ vertices and $2m$ edges.

Denote by $y_1, \ldots, y_m$ the points of $S^2(r)$ that correspond to the cycle vertices and by $x_1, \ldots, x_m$ the points of $S^2(r)$ that correspond to the pendant vertices. For convenience, let us put $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_m)$ the vectors of dimension $s = 3m$ containing all coordinates. Then the embedding of G_k can be given by the pair (X, Y).

Lemma 4. *Fix the radius* $r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$*. Then if* k *is large enough, there exists a unit distance embedding* (X, Y) *of* G_k *into* $S^2(r)$ *and a constant* $\eta > 0$ *such that for any* $\tilde{X}$ *satisfying* $\|\tilde{X} - X\| < \eta$ *there exists* Y *such that* $(\tilde{X}, \tilde{Y})$ *is a "perturbed" unit distance embedding of* G_k*.*

In other words, for any sufficiently small perturbation of pendant vertices, it is possible to find the embedding of the cycle vertices.

Proof. We provide the desired unit distance embedding explicitly. In what follows we slightly abuse the notation and write x_i and y_i for a vertex of the graph, the corresponding point on $S^2(r)$, and its 3-dimensional vector representation. Consider the system of equations defining the embedding G_k in $S^2(r)$:

$$\begin{cases} f_i = \|y_i\|^2 - r^2 = 0, \quad 1 \leqslant i \leqslant m; \\ f_{i+m} = \|y_i - y_{i+1}\|^2 - 1 = 0, \quad 1 \leqslant i \leqslant m-1; \\ f_{2m} = \|y_m - y_1\|^2 - 1 = 0; \\ f_{i+2m} = \|x_i - y_i\|^2 - 1 = 0, \quad 1 \leqslant i \leqslant m. \end{cases} \tag{1}$$

Next, we will be interested in the family of embeddings, the $k = 2$ case of which is depicted on Fig. 4.

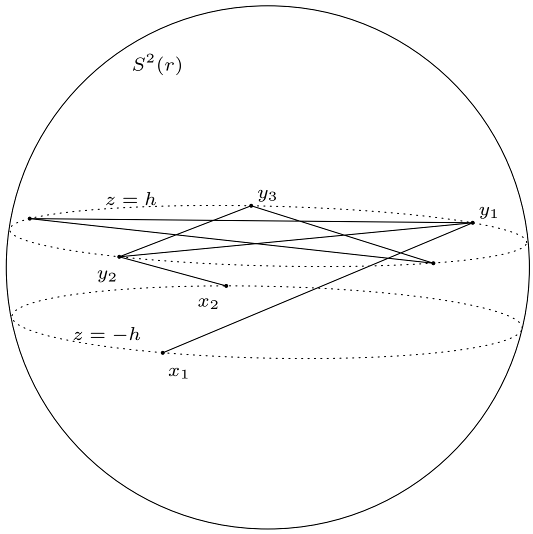

Figure 4: Unit distance embedding of G_k, the $k = 2$ case

Note that (1) allows x_i to lie in $\mathbb{R}^3$, not only $S^2(r)$, but the cycle $y_1, \ldots, y_m$ must lie on the sphere. One can consider the function corresponding to the left-hand side of the system (1).

$$F = (f_1, \ldots, f_{3m}) = F(x_{11}, x_{12}, x_{13}, \ldots, x_{m3};\ y_{11}, \ldots, y_{m3}).$$

Suppose that the Jacobian matrix $J = \left(\frac{\partial F}{\partial Y}\right)$ is nondegenerate,

$$\det J = \det\left(\frac{\partial F}{\partial Y}\right) \neq 0,$$

then the statement of the lemma follows from Theorem 3. The rest of the proof is devoted to the calculation of this determinant.

The matrix J has the following form (recall that x_i and y_i are 1×3 vectors):

$$J(X, Y) = 2\begin{pmatrix} y_1 & 0 & 0 & 0 & \ldots & & 0 \\ 0 & y_2 & 0 & 0 & \ldots & & 0 \\ 0 & 0 & y_3 & 0 & \ldots & & 0 \\ \vdots & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \ldots & & y_m \\ y_1 - y_2 & y_2 - y_1 & \ldots & 0 & \ldots & & 0 \\ 0 & y_2 - y_3 & y_3 - y_2 & 0 & \ldots & & 0 \\ \vdots & & & & & & \\ y_1 - y_m & 0 & \ldots & \ldots & & 0 & y_m - y_1 \\ y_1 - x_1 & 0 & \ldots & \ldots & & 0 & 0 \\ 0 & y_2 - x_2 & 0 & \ldots & & 0 & 0 \\ \vdots & & \vdots & & & \vdots & \\ 0 & \ldots & 0 & \ldots & & 0 & y_m - x_m \end{pmatrix}.$$

Subtracting some rows from each other, we get

$$\det J = 2^{3m} \det \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & y_3 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_m \\ y_2 & y_1 & \dots & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ y_m & \dots & 0 & \dots & 0 & y_1 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & x_m \end{pmatrix} = (-1)^s 2^{3m} \det \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_{m-1} \\ 0 & 0 & 0 & 0 & \dots & y_m \\ 0 & 0 & 0 & 0 & \dots & x_m \\ y_m & 0 & 0 & \dots & 0 & y_1 \end{pmatrix},$$

where the term $(-1)^s$, $s \in \{0,1\}$ is responsible for the parity of the permutation of the rows. Since we are not interested in the sign of the determinant, there is no point in evaluating the parity.

Then, expanding the determinant by the last row and rearranging the rows, we obtain

$$\det J = (-1)^s 2^{3m} \det \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_{m-1} \\ 0 & 0 & 0 & 0 & \dots & y_m \\ 0 & 0 & 0 & 0 & \dots & x_m \\ 0 & 0 & 0 & \dots & 0 & y_1 \end{pmatrix} + (-1)^s 2^{3m} \det \begin{pmatrix} y_m & 0 & 0 & \dots & 0 & 0 \\ y_1 & 0 & 0 & 0 & \dots & 0 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_{m-1} \\ 0 & 0 & 0 & 0 & \dots & y_m \\ 0 & 0 & 0 & 0 & \dots & x_m \end{pmatrix} =$$

$$= (-1)^s 2^{3m} \left(V_1 \dots V_m + V_1' \dots V_m' \right),$$

where

$$V_i = \det \begin{pmatrix} y_i \\ x_i \\ y_{i+1} \end{pmatrix} = -\det \begin{pmatrix} y_i \\ y_{i+1} \\ x_i \end{pmatrix}, \qquad V_i' = \det \begin{pmatrix} y_i \\ y_{i+1} \\ x_{i+1} \end{pmatrix}.$$

Here in the determinant calculations we used the fact that after decomposing the last row into a sum, each of the summands becomes block-triangular. In addition, note that the cyclic permutation of matrix rows does not change the sign of the determinant, since m is odd.

Now we fix the following embedding (Fig. 4). Let vertices y_i lie in the plane $z = h$ (and form a regular m-gon), and vertices x_i lie in the plane $z = -h$ (and also form a regular m-gon). Note that the radius of the circumcircle of the m-gon is greater than $\frac{1}{2}$, and $r^2 < \frac{1}{3}$, hence

$$h < \left(\frac{1}{3} - \frac{1}{4} \right)^{1/2} = \frac{1}{2\sqrt{3}} < \frac{1}{2}. \tag{2}$$

Denote by U_m the rotation matrix by an angle $2\pi/m$ counterclockwise around z–axis. Then $y_{i+1} = U_m y_i$, $x_{i+1} = U_m x_i$. Hence, all V_i coincide and all V_i' also coincide; put $V = V_i$ and $V' = V_i'$. Hence

$$\det J = (-1)^s 2^{3m} \left(V^m + (V')^m \right).$$

We claim that

$$V + V' = \det \begin{pmatrix} y_1 \\ y_2 \\ x_2 - x_1 \end{pmatrix} \neq 0.$$

Indeed, since $y_{13} = y_{23} = h$, $x_{13} = x_{23} = -h$, the equality

$$\alpha y_1 + \beta y_2 + \gamma (x_2 - x_1) = 0$$

implies $\alpha = -\beta$, i.e.

$$\alpha (y_1 - y_2) = \gamma (x_1 - x_2). \tag{3}$$

Recall that $\|y_1 - y_2\| = \|x_1 - x_2\| = 1$, so $\alpha = \pm\gamma$.

Since both sets of points $\mathcal{X} = \{x_1, \dots, x_m\}$, $\mathcal{Y} = \{y_1, \dots, y_m\}$ form vertices of congruent regular m-gons, in the case $\alpha = \gamma$, we have $x_1 - x_2 = y_1 - y_2$ and the projections of x_i and y_i on the plane $z = 0$ coincide, $i = 1, 2, \dots, m$, and taking into account (2), we have

$$\|x_1 - y_1\| = 2h < 1.$$

In the case $\alpha = -\gamma$, we have $x_1 - x_2 = y_2 - y_1$ and the sets $\mathcal{X}$ and $\mathcal{Y}$ are symmetric about the origin. Then $x_1x_2y_1y_2$ is a rectangle, and

$$\|x_1 - y_1\|^2 > \|x_1 - x_2\|^2 + 4h^2 > 1.$$

In both cases we got a contradiction.

Then the equation (3) does not hold and so $V + V' \neq 0$. Hence

$$\det J = (-1)^s 2^{3m} \left(V^m + (V')^m\right) \neq 0$$

as required. □

3 Open questions

Is the chromatic number of $S^2(r)$ «almost monotonically» increasing with r? Id est, is the chromatic number monotonic except for an at most countable set of values r? Recall that the known results (see Table 1) allow for such possibility.

r	Estimate for $\chi(r) = \chi(S^2(r))$	Source
$r < 1/2$	$\chi(r) = 1$	
$r = 1/2$	$\chi(r) = 2$	
$\frac{1}{2} < r \leqslant \frac{\sqrt{3-\sqrt{3}}}{2}$	$\chi(r) = 4$	Corollary 1
$r > \frac{\sqrt{3-\sqrt{3}}}{2}$	$\chi(r) \geqslant 4$	Theorem 2
$r = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$	$\chi(r) \geqslant 5$	[18]
$r = \frac{1}{\sqrt{2}}$	$\chi(r) = 4$	[15, 3]
$r = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$	$\chi(r) \geqslant 5$	[18]
$r \leqslant \frac{1}{\sqrt{3}}$	$\chi(r) \leqslant 5$	[15, 11]
$r \leqslant \sqrt{3}/2$	$\chi(r) \leqslant 6$	[11]
$r \geqslant 12.44$	$\chi(r) \leqslant 7$	[16]
$r > 1/2$	$\chi(r) \leqslant 15$	[1, 13]

Table 1: Lower and upper estimates for $\chi(S^2(r))$.

Is there a proper coloring of $S^2(r)$ in $\chi(S^2(r))$ colors such that every color is dense? It is interesting that all known upper bounds are given by explicit colorings in which every color is a finite union of regions bounded by piecewise-continuous curves.

What is the minimum number of vertices in a subgraph G of a sphere $S^2(r)$ with $\chi(G) = \chi(S^2(r))$? By the de Bruijn–Erdős theorem, this number is finite. Note that the proof of Theorem 2 does not give any finite 4-chromatic unit distance graph.

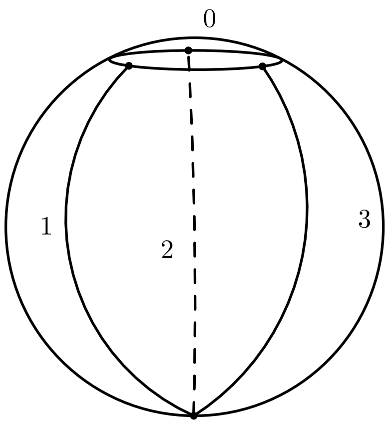

Figure 5: 4-coloring of the sphere. Here $s_0 \to 0$ as $r \to 1/2$

Let us focus on the case $r = 1/2 + \varepsilon$, $\varepsilon \to 0$. Then the sphere can be colored in 4 colors in the way shown in Figure 5. Let us denote by s_0 the area of the spherical cap of color 0. Observe that $s_0 = 4\pi\varepsilon + o(\varepsilon)$, and thus, via averaging, we have the lower bound $n_4(r) \geqslant c\varepsilon^{-1}$ for some $c > 0$, where $n_4(r)$ is the minimum number of vertices in a 4-chromatic unit distance graph. Can this obvious bound be refined?

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Dear Reviewer,

thank you for the comments, we try to implement everything, expect the swapping x_2 and y_2 (page 6, line 22-23). It looks like that the calculation is more natural without such rearrangement.

Yours, authors.