

Branching points in the planar Gilbert–Steiner problem have degree 3

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Abstract

Gilbert–Steiner problem is a generalization of the Steiner tree problem on a specific optimal mass transportation. We show that every branching point in a solution of the planar Gilbert–Steiner problem has degree 3.

1 Introduction

One of the first models for branched transport was introduced by Gilbert [3]. The difference with the optimal transportation problem is that the extra geometric points may be of use; this explains the naming in honor of Steiner. Sometimes it is also referred to as *optimal branched transport*; a large part of book [1] is devoted to this problem. Let us proceed with the formal definition.

Definition 1. Let μ^+, μ^- be two finite measures on a metric space $(X, \rho(\cdot, \cdot))$ with finite supports such that total masses $\mu^+(X) = \mu^-(X)$ are equal. Let $V \subset X$ be a finite set containing the support of the signed measure $\mu^+ - \mu^-$, the elements of V are called vertices. Further, let E be a finite collection of unordered pairs $\{x, y\} \subset V$ which we call edges. So, (V, E) is a simple undirected finite graph. Assume that for every $\{x, y\} \in E$ two non-zero real numbers $m(x, y)$ and $m(y, x)$ are defined so that $m(x, y) + m(y, x) = 0$. This data set is called a (μ^+, μ^-) -flow if

$$\mu^+ - \mu^- = \sum_{\{x, y\} \in E} m(x, y) \cdot (\delta_y - \delta_x)$$

where δ_x denotes a delta-measure at x (note that the summand $m(x, y) \cdot (\delta_y - \delta_x)$ is well-defined in the sense that it does not depend on the order of x and y).

Let $C: [0, \infty) \rightarrow [0, \infty)$ be a cost function. The expression

$$\sum_{\{x, y\} \in E} C(|m(x, y)|) \cdot \rho(x, y)$$

is called the *Gilbert functional* of the (μ^+, μ^-) -flow.

The *Gilbert–Steiner problem* is to find the flow which minimizes the Gilbert functional with cost function $C(x) = x^p$, for a fixed $p \in (0, 1)$; we call a solution *minimal flow*.

Vertices from $\text{supp}(\mu^+) \setminus \text{supp}(\mu^-)$ are called *terminals*. A vertex from $V \setminus \text{supp}(\mu^+) \setminus \text{supp}(\mu^-)$ is called a *branching point*. Formally, we allow a branching point to have degree 2, but clearly it never happens in a minimal flow.

Local structure in the Gilbert–Steiner problem was discussed in [1], and the paper [4] deals with planar case. A local picture around a branching point b of degree 3 is clear due to the initial paper of Gilbert. Similarly to the finding of the Fermat–Torricelli point in the celebrated Steiner problem one can determine the angles around b in terms of masses (see Lemma 1).

Theorem 1 (Lippmann–Sanmartín–Hamprecht [4], 2022). *A solution of the planar Gilbert–Steiner problem has no branching point of degree at least 5.*

The goal of this paper is to give some conditions on a cost function under which all branching points in a planar solution have degree 3. They are slightly stronger than the Schoenberg [6] conditions of the embedding of the metric of the form $\rho(x, y) := f(x - y)$ to a Hilbert space. In particular, this covers the case of the standard cost function x^p , $0 < p < 1$. The following main theorem is the part of a more general Theorem 3.

Theorem 2. *A solution of the planar Gilbert–Steiner problem has no branching point of degree at least 4.*

2 Preliminaries

We need the following lemmas.

Lemma 1 (Folklore). *Let PQR be a triangle and w_1, w_2, w_3 be non-negative reals. For every point $X \in \mathbb{R}^2$ consider the value*

$$L(X) := w_1 \cdot |PX| + w_2 \cdot |QX| + w_3 \cdot |RX|.$$

Then

- (i) *a minimum of $L(X)$ is achieved at a unique point X_{\min} ;*
- (ii) *if $X_{\min} = P$ then $w_1 \geq w_2 + w_3$ or there is a triangle Δ with sides w_1, w_2, w_3 and $\angle P$ is at least the outer angle between w_2 and w_3 in Δ .*

Hereafter the metric space is the Euclidean plane \mathbb{R}^2 .

The following concept only slightly changes from that of Schoenberg [6], introduced for describing which metrics of the form $\rho(x, y) = f(x - y)$ on the real line can be embedded to a Hilbert space.

Definition 2. *Let λ be a Borel measure on \mathbb{R} for which*

$$\int \min(x^2, 1) d\lambda(x) < \infty. \tag{1}$$

Assume additionally that the support of λ is uncountable. A function $f: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ of the form

$$f(t) = \sqrt{\int \sin^2(tx) d\lambda(x)} = \frac{1}{2} \|e^{2itx} - 1\|_{L^2(\lambda)} \quad (2)$$

is called admissible.

The only difference with [6] is that we require that the support of the measure λ is uncountable which guarantees that the corresponding embedding has full dimension (see below).

Remark 1. As λ is a Borel measure, a continuous function $\sin^2(tx)$ is λ -measurable. Under conditions (1), the integral in (2) is finite, so $f(t) < \infty$ for all $t \geq 0$.

Further we are going to consider only admissible cost functions. Note that admissibility implies some properties one may expect from a cost function. In particular, $f(0) = 0$ and f is subadditive: for non-negative t, s we have

$$\begin{aligned} f(t) + f(s) &= \frac{1}{2} \|e^{2itx} - 1\|_{L^2(\lambda)} + \frac{1}{2} \|e^{2isx} - 1\|_{L^2(\lambda)} \\ &= \frac{1}{2} \|e^{2itx} - 1\|_{L^2(\lambda)} + \frac{1}{2} \|e^{2i(s+t)x} - e^{2itx}\|_{L^2(\lambda)} \geq \frac{1}{2} \|e^{2i(t+s)x} - 1\|_{L^2(\lambda)} = f(t+s). \end{aligned}$$

On the other hand it does not imply monotonicity (for instance, if $\text{supp } \lambda \subset [0.9, 1.1]$ then $f(\pi) < f(\pi/2)$).

Hereafter $L^2(\lambda)$ for a measure λ on \mathbb{R} is understood as a *real* Hilbert space of complex-valued square summable w.r.t. λ functions (strictly speaking, of classes of equivalences of such functions modulo coincidence λ -almost everywhere).

Proposition 1. If λ is a Borel measure on \mathbb{R} with uncountable support such that

$$\int \min(x^2, 1) d\lambda(x) < \infty,$$

then any finite collection of functions of the form $e^{iax} - 1$, $a \in \mathbb{R}$, is affinely independent in $L^2(\lambda)$.

Proof. Assume the contrary. Then there exist distinct real numbers a_1, \dots, a_n and non-zero real coefficients t_1, \dots, t_n such that $\sum t_j = 0$ and $\sum t_j(e^{ia_j x} - 1) = 0$ λ -almost everywhere. But the analytic function $\sum t_j(e^{ia_j x} - 1)$ is either identically zero, or has at most countably many (and separated) zeroes. In the latter case, it is not zero λ -almost everywhere, since the support of λ is uncountable. The former case is not possible: indeed, if $\sum t_j e^{ia_j x} \equiv 0$, then taking the Taylor expansion at 0 we get $\sum t_j a_j^k = 0$ for all $k = 0, 1, 2, \dots$. Therefore $\sum t_j W(a_j) = 0$ for any polynomial W . Choosing $W(t) = \prod_{j=2}^n (t - a_j)$ we get $t_1 = 0$, a contradiction. \square

One can see from the proof that the condition on uncountability of the support may be weakened.

Lemma 2. Let C be an admissible cost function. Define $h(m_1, m_2)$ as the value of the outer angle between m_1 and m_2 in the triangle with sides $C(|m_1|)$, $C(|m_2|)$, $C(|m_1 + m_2|)$ (it exists by Proposition 1) for real m_1, m_2 . Suppose that OV_1, OV_2 are edges in a minimal flow with masses m_1 and m_2 . Then the angle between OV_1 and OV_2 is at least $h(m_1, m_2)$.

Proof. Assume the contrary, then by Lemma 1 with $P = O$, $Q = V_1$, $R = V_2$, $w_1 = C(|m_1|)$, $w_2 = C(|m_2|)$, $w_3 = C(|m_1 + m_2|)$ we have $X_{\min} \neq O$. Then we can replace $[OV_1] \cup [OV_2]$ with $[X_{\min}O] \cup [X_{\min}V_1] \cup [X_{\min}V_2]$ with the corresponding masses in our flow; this contradicts the minimality of the flow. \square

Lemma 3. *For $0 < p < 1$, the function $f(x) = x^p$ is admissible.*

Proof. Consider the measure $d\lambda = x^{-2p-1}dx$ on $[0, \infty)$. Then $\int_0^\infty \min(x^2, 1)d\lambda < \infty$ and for $t > 0$ we have

$$\int_0^\infty \sin^2(tx) d\lambda(x) = \int_0^\infty \sin^2(tx) x^{-2p-1}dx = t^{2p} \int_0^\infty \sin^2 y y^{-2p-1}dy,$$

thus the measure λ multiplied by an appropriate positive constant proves the result. \square

Example 1. *For another natural choice $d\lambda = 4ce^{-2cx}dx$, $c > 0$, we get an admissible function $f(t) = t/\sqrt{t^2 + c^2}$.*

The following lemma is essentially well-known, but for the sake of completeness and for covering degeneracies and the equality cases we provide a proof.

Lemma 4. *Let X be a finite-dimensional Euclidean space, let the points $A_0, A_1, A_2, \dots, A_{n-1}, A_n = A_0, A_{n+1} = A_1$ in X be chosen so that $A_i \neq A_{i+1}$ for all $i = 1, 2, \dots, n$. Denote $\varphi_i := \pi - \angle A_{i-1}A_iA_{i+1}$ for $i = 1, 2, \dots, n$. Then $\sum \varphi_i \geq 2\pi$, and if the equality holds then the points A_1, \dots, A_n belong to the same two-dimensional affine plane.*

Proof. Let u be a randomly chosen unit vector in X (with respect to a uniform distribution on the sphere). For $j = 1, 2, \dots, n$ denote by $U(j)$ the following event: $\langle u, A_j \rangle = \max_{1 \leq i \leq n} \langle u, A_i \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in X ; and by $V(j)$ the event $\langle u, A_j \rangle = \max_{j-1 \leq i \leq j+1} \langle u, A_i \rangle$. Obviously, $\text{prob } U(j) \leq \text{prob } V(j)$. Also, $\text{prob } V(j) = \frac{\varphi_j}{2\pi}$, since the set of directions of u for which $V(j)$ holds is the dihedral angle of measure φ_j . Thus, since always at least one event $U(j)$ holds, we get

$$1 \leq \sum_{j=1}^n \text{prob } U(j) \leq \sum_{j=1}^n \text{prob } V(j) = \frac{1}{2\pi} \sum_{j=1}^n \varphi_j.$$

This proves the inequality. It remains to prove that it is strict assuming that not all the points belong to a two-dimensional plane. Note that if every three consecutive points A_{j-1}, A_j, A_{j+1} are collinear, then all the points A_1, \dots, A_n are collinear that contradicts to our assumption. If A_{j-1}, A_j, A_{j+1} are not collinear, denote by α the two-dimensional plane they belong to. There exists i for which $A_i \notin \alpha$. Then $\text{prob } U(j) < \text{prob } V(j)$, since there exist planes passing through A_j which separate the triangle $A_{j-1}A_jA_{j+1}$ and the point A_i , and the measure of directions of such planes is strictly positive. Therefore, our inequality is strict. \square

3 Main result

Theorem 3. *Let μ^+, μ^- be two measures with finite support on the Euclidean plane \mathbb{R}^2 , and assume that the cost function C is admissible. Then if a (μ^+, μ^-) -flow has a branching point of degree at least 4, then there exists a (μ^+, μ^-) -flow with strictly smaller value of Gilbert functional.*

Proof. Assume the contrary. Let O be a branching point, OV_1, OV_2, \dots, OV_k , $k \geq 4$, be the edges incident to O , enumerated counterclockwise. Further the indices of V_i 's are taken modulo k , so that $V_1 = V_{k+1}$ etc. Denote $m_i = m(OV_i)$, then by the definition of flow we get $\sum m_i = 0$. By Lemma 2, $\angle V_i OV_{i+1} \geq h(m_i, m_{i+1})$.

Consider the functions $A_j(x) := e^{i(m_1 + \dots + m_j)x} - 1$ for $j = 1, 2, \dots$ (here i is the imaginary unit). Then $\sum m_j = 0$ yields that $A_{j+k} \equiv A_j$ for all $j > 0$.

Since the cost function $C(t)$ is admissible, there exists a Borel measure λ on \mathbb{R} with uncountable support such that $\int \min(x^2, 1) d\lambda(x) < \infty$ and

$$C(t) = \sqrt{\int 4 \sin^2 \frac{tx}{2} d\lambda(x)}.$$

Using the identity $|e^{ia} - e^{ib}|^2 = 4 \sin^2 \frac{a-b}{2}$ for real a, b we note that for $j, s > 0$ in the Hilbert space $L^2(\lambda)$ we have

$$\|A_{j+s} - A_j\|^2 = C(|m_{j+1} + \dots + m_{j+s}|)^2.$$

In particular, the lengths of the sides of the triangle $A_{j-1}A_jA_{j+1}$ are equal to $C(|m_j|)$, $C(|m_{j+1}|)$ and $C(|m_j + m_{j+1}|)$. Therefore $\varphi_j := \pi - \angle A_{j-1}A_jA_{j+1} = h(m_j, m_{j+1})$. By Lemma 4 we get $\sum \varphi_j \geq 2\pi$.

By Lemma 1, this yields $2\pi = \sum_{j=1}^k \angle V_j OV_{j+1} \geq \sum \varphi_j \geq 2\pi$. Therefore, the equality must take place. Again by Lemma 4 it follows that the points A_j belong to the same 2-dimensional subspace. But by Proposition 1, distinct points between A_j 's are affinely independent. Therefore, there exist at most three distinct A_j 's, and if exactly three, they are not collinear. It is easy to see that the equality $\sum \varphi_j = 2\pi$ under these conditions does not hold when $k > 3$. A contradiction. \square

4 Examples of branching points of degree 4

Let us start with an example in three dimensions. Consider four masses m_1, m_2, m_3, m_4 of zero sum, such that no two of them give zero sum. Repeat the beginning of the proof of Theorem 2 to get the simplex $A_1A_2A_3A_4$ in 3-dimensional space. Now consider unit edges OB_i in \mathbb{R}^3 with directions $A_{i-1}A_i$, $1 \leq i \leq 4$. By the construction the angles between vectors OB_i and OB_{i+1} are exactly $h(m_i, m_{i+1})$. Suppose that angles $\angle B_1OB_3$ and $\angle B_2OB_4$ are greater than $h(m_1, m_3)$ and $h(m_2, m_4)$, respectively.

Then we claim that the flow

$$\sum_{i=1}^4 m_i \cdot (\delta_O - \delta_{B_i})$$

is the unique solution of the corresponding Gilbert–Steiner problem.

First, if we fix the graph structure then the position of O is optimal by the following lemma, because the closeness of the polychain $A_1A_2A_3A_4A_1$ gives exactly (3).

Lemma 5 (Weighted geometric median, [7]). *Consider different non-collinear points $A, B, C, D \in \mathbb{R}^3$ and let w_1, w_2, w_3, w_4 be non-negative reals. Then*

$$L(X) := w_1 \cdot |AX| + w_2 \cdot |BX| + w_3 \cdot |CX| + w_4 \cdot |DX|$$

has unique local (and global) minimum satisfying

$$w_1 \bar{e}_A + w_2 \bar{e}_B + w_3 \bar{e}_C + w_4 \bar{e}_D = 0, \tag{3}$$

where $\bar{e}_A, \bar{e}_B, \bar{e}_C, \bar{e}_D$ are unit vectors codirected with XA, XB, XC, XD , respectively.

Since any two masses have nonzero sum, every flow is connected. Thus every possible competitor has 2 branching points of degree 3. Consider the case in which branching points U and V are connected with B_1, B_2 and B_3, B_4 , respectively. By the convexity of length, the Gilbert functional $L(U, V)$ considered on the set of all possible U and V ($\mathbb{R}^3 \times \mathbb{R}^3$) is a convex function. Let us show that $U = V = O$ is a local minimum. Indeed, consider $U_\varepsilon = O + \varepsilon u$ and $V_\delta = O + \delta v$ for arbitrary unit vectors u, v and small positive ε, δ . Then

$$\begin{aligned} L(U_\varepsilon, V_\delta) - L(O, O) = \\ (1 + o(1)) \cdot (w(UV) \cdot \|\varepsilon u - \delta v\| - \varepsilon \langle w_1 e_1, u \rangle - \varepsilon \langle w_2 e_2, u \rangle - \delta \langle w_3 e_3, v \rangle - \delta \langle w_4 e_4, v \rangle) = \\ (1 + o(1)) \cdot (w(UV) \cdot \|\varepsilon u - \delta v\| - \varepsilon \langle w_{12} e_{12}, u \rangle - \delta \langle w_{34} e_{34}, v \rangle), \end{aligned}$$

where $w_{12} e_{12} = w_1 e_1 + w_2 e_2$ and $w_{34} e_{34} = w_3 e_3 + w_4 e_4$ for unit e_{12} and e_{34} . By the construction one has $w_{12} = w_{34} = w(UV)$ and $e_{12} + e_{34} = 0$, so

$$L(U_\varepsilon, V_\delta) - L(O, O) = (1 + o(1)) \cdot w(UV) \cdot (\|\varepsilon u - \delta v\| - \langle e_{12}, \varepsilon u - \delta v \rangle).$$

Since e_{12} is unit, the derivative is non-negative for every u, v .

The case in which U and V are connected with B_2, B_3 and B_4, B_1 , respectively, is completely analogous. In the remaining case (U is connected with B_1, B_3 and V is connected with B_2, B_4) we have $w_{12} = w_{34} < w(UV)$ due to $\angle B_1 O B_3 > h(m_1, m_3)$ and $\angle B_2 O B_4 > h(m_2, m_4)$. Thus $U = V = O$ is also a local minimum.

It is known [3] that L has a unique local and global minimum, which finishes the example.

Now proceed with planar examples of 4-branching for some non-admissible cost-function C . Then we may repeat the 3-dimensional argument starting with planar $A_1 A_2 A_3 A_4$.

The simplest way to produce an example is to consider a trapezoid $A_1 A_2 A_3 A_4$ and apply Ptolemy's theorem. This case corresponds to $m_1 = m_3$ and $m_1 + m_2 + m_3 + m_4 = 0$. Then $|A_1 A_2| = |A_3 A_4| = C(|m_1|)$, $|A_2 A_3| = C(|m_2|)$, $|A_4 A_1| = C(|m_4|)$ and $|A_1 A_3| = |A_2 A_4| = C(|m_1 + m_2|)$. The existence of such trapezoid means

$$C(|m_1 + m_2|)^2 = C(|m_1|) \cdot (C(|m_2|) + C(|2m_1 + m_2|)). \quad (4)$$

If we assume that C is monotone and subadditive then (4) means that a trapezoid exists; note that we need values of C only at 4 points.

Now we give an example of a monotone, subadditive and concave cost function with 4-branching. For this purpose put $m_1 = m_2 = m_3 = 1$ and $m_4 = -3$, $C(1) = 1$, $C(2) = 1.9$, $C(3) = 2.61$; clearly (4) holds. Now one can easily interpolate a desired C , for instance

$$C(t) = \begin{cases} t, & t \leq 1 \\ 0.1 + 0.9t, & 1 < t \leq 2 \\ 0.48 + 0.71t, & 2 < t \leq 3 \\ 1.11 + 0.5t & 3 < t. \end{cases}$$

Finally, the inequalities $\angle B_1 O B_3 > \angle B_2 O B_3 = h(m_2, m_3) = h(m_1, m_3)$ and $\pi = \angle B_2 O B_4 > h(m_2, m_4)$ hold.

5 Open questions

It would be interesting to describe all cost functions for which the conclusion of Theorem 3 holds.

Now let us focus on the cost function $C(x) = x^p$. Having a knowledge that every branching point has degree 3 one can adapt Melzak algorithm [5] from Steiner trees to Gilbert–Steiner problem. The idea of the algorithm is that after fixing the combinatorial structure one can find two terminals t_1, t_2 connected with the same branching point b . Then one may reconstruct the solution for V from the solution for $V \setminus \{t_1, t_2\} \cup \{t'\}$ for a proper t' which depend only on t_1, t_2 (in fact one has to check 2 such t'). When the underlying graph is a matching we finish in an obvious way. Application of this procedure for all possible combinatorial structures gives a slow but mathematically exhaustive algorithm in the planar case.

However there is no known algorithm in \mathbb{R}^d for $d > 2$ (see Problem 15.12 in [1]). Recall that we have to consider a high-degree branching.

A naturally related problem is to evaluate the maximal possible degree of a branching point in the d -dimensional Euclidean space for every d . Note that the dependence on the cost function may be very complicated.

Some other questions are collected in Section 15 of [1] (some of them are solved, in particular Problem 15.1 is solved in [2]).

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