An overview of maximal distance minimizers problem

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December 13, 2022

Abstract

Consider a compact $M \subset \mathbb{R}^d$ and $l > 0$. A maximal distance minimizer problem is to find a connected compact set Σ of the length (one-dimensional Hausdorff measure $\mathcal{H}^1$) at most l that minimizes

$$\max_{y \in M} \operatorname{dist}(y, \Sigma),$$

where dist stands for the Euclidean distance.

We give a survey on the results on the maximal distance minimizers and related problems.

1 Introduction

This work is devoted to solutions of the following maximal distance minimizer problem.

Problem 1.1. *For a given compact set $M \subset \mathbb{R}^n$ and $l > 0$ to find a connected compact set Σ of length (one-dimensional Hausdorff measure $\mathcal{H}^1$) at most l that minimizes*

$$\max_{y \in M} \operatorname{dist}(y, \Sigma),$$

where dist *stands for the Euclidean distance.*

It appeared in a very general from in [3] and later has been studied in [11, 13].

A *maximal distance minimizer* is a solution of Problem 1.1. Such sets can be considered as networks of radiating Wi-Fi cables with a bounded length arriving to each customer (for the set M of customers) at the distance r, where such r is the smallest possible.

1.1 Class of problems

Maximal distance minimizers problem could be considered as a particular example of shape optimization problem. A shape optimization problem is a minimization problem where the unknown variable runs over a class of domains; then every shape optimization problem can be written in the form $\min F(\Sigma) : \Sigma \in A$ where A is the class of admissible domains and $F()$ is the cost function that one has to minimize over A.

So for a given compact set M and positive number $l \geq 0$ let the admissible set A be a set of all closed connected set Σ' with length constraint $\mathcal{H}^1(\Sigma') \leq l$; and let cost function be the *energy* $F_M(\Sigma) = \max_{y \in M} \operatorname{dist}(y, \Sigma)$.

1.2 Dual problem

Define the dual problem to Problem 1.1 as follows.

Problem 1.2. *For a given compact set $M \subset \mathbb{R}^d$ and $r > 0$ to find a connected compact set Σ of the minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) such that*

$$\max_{y \in M} \operatorname{dist}(y, \Sigma) \leq r.$$

In a nondegenerate case (i.e. for $F_M(\Sigma) > 0$) the strict and dual problems have the same sets of solutions for the corresponding r and l (see [13]) and hence an equality is reached.

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1.3 The first parallels with average distance minimizers problem

Maximal distance minimizers problem is similar to another shape optimization problem: average distance minimizers problem (see the survey of Antoine Lemenant [10]) and it seems interesting to compare the known results and open questions concerning these two problems. In the average distance minimizers problem's statement the admissible set A is the same as in Maximal distance minimizers problem, but the function $F(\Sigma_a)$ is defined as $\int_M A(\mathrm{dist}\,(y, \Sigma_a))d\phi(x)$ where $A : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function and $\phi()$ is a finite nonnegative measure with compact nonempty support in $\mathbb{R}^n$.

Minimization problems for average distance and maximum distance functionals are used in economics and urban planning with similar interpretations. If it is required to find minimizers under the cardinality constraint $\sharp\Sigma \leq k$, instead of the length and the connectedness constraints, where $k \in \mathbb{N}$ is given and $\sharp$ denotes the cardinality, then the corresponding problems are referred to as optimal facility location problems.

1.4 Notation

For a given set $X \subset \mathbb{R}^n$ we denote by $\overline{X}$ its closure, by $\mathrm{Int}\,(X)$ its interior and by ∂X its topological boundary.

Let $B_\rho(O)$ stand for the open ball of radius ρ centered at a point O, and let $B_\rho(T)$ be the open ρ-neighborhood of a set T i.e.

$$B_\rho(T) := \bigcup_{x \in T} B_\rho(x)$$

(in other words $B_\rho(T)$ is a Minkowski sum of a ball B_ρ centered in the origin and T). Note that the condition

$$\max_{y \in M} \mathrm{dist}\,(y, \Sigma) \leq r$$

is equivalent to $M \subset \overline{B_r(\Sigma)}$.

For given points B, C we use the notation $[BC]$, $[BC)$ and (BC) for the corresponding closed line segment, ray and line respectively.

1.5 Existence. Absence of loops. Ahlfors regularity and other simple properties

For the both problems existence of solutions is proved easily: according to the classical Blaschke and Golab Theorems, the class of admissible sets is compact for the Hausdorff distance and both of the functions (maximal distance and also the average distance) is continuous for this convergence because of the uniform convergence of $x \to \mathrm{dist}\,(x, \Sigma)$.

Definition 1.3. *A closed set Σ is said to be Ahlfors regular if there exists some constants C_1, $C_2 > 0$ and a radius $\varepsilon_0 > 0$ such that $C_1\varepsilon \leq \mathcal{H}^1(\Sigma \cap B_\varepsilon(x)) \leq C_2\varepsilon$ for every $x \in \Sigma$ and $\varepsilon < \varepsilon_0$.*

In the work [13] Paolini and Stepanov proved

- the absence of closed loops for maximum distance minimizers and, under general conditions on ϕ, the absence of closed loops for average distance minimizers;
- the Ahlfors regularity of maximum distance minimizers and, under the additional summability condition on ϕ, the Ahlfors regularity of average distance minimizers. Gordeev and Teplitskaya [8] refine Ahlfors constants of maximum distance minimizers to the best possible, i.e. show that $\mathcal{H}^1(\Sigma \cap B_\varepsilon(x)) = \mathrm{ord}\,_x\Sigma \cdot \varepsilon + o(\varepsilon)$, where $\mathrm{ord}\,_x\Sigma \in \{1, 2, 3\}$.
- Maximal distance minimizers problem and the dual problem have the same sets of solutions (planar case was proved before by Miranda, Paolini, Stepanov in [11]). It particularly implies that maximal distance minimizers must have maximum available length l. Paolini and Stepanov also proved that average distance minimizers, (with additional properties of ϕ) have maximum available length.

In the work [6] the following basic results were showed.

(i) Let Σ be an r-minimizer for some M. Then Σ is an r-minimizer for $\overline{B_r(\Sigma)}$.

(ii) Let Σ be an r-minimizer for $\overline{B_r(\Sigma)}$. Then Σ is an r'-minimizer for $\overline{B_{r'}(\Sigma)}$, where $0 < r' < r$.

1.6 Local maximal distance minimizers

Definition 1.4. *Let $M \subset \mathbb{R}^n$ be a compact set and let $r > 0$. A connected compact set $\Sigma \subset \mathbb{R}^n$ is called a local maximal distance minimizer if $F_M(\Sigma) \leq r$ and there exists $\varepsilon > 0$ such that for any connected set Σ' satisfying $F_M(\Sigma') \leq r$ and $\mathrm{diam}\,(\Sigma \triangle \Sigma') \leq \varepsilon$ the inequality $\mathcal{H}^1(\Sigma) \leq \mathcal{H}^1(\Sigma')$ holds, where $\triangle$ is the symmetric difference.*

Any maximal distance minimizer is also a local minimizer. Usually the properties of maximal distance minimizers are also true for local maximal distance minimizers (see [8]).

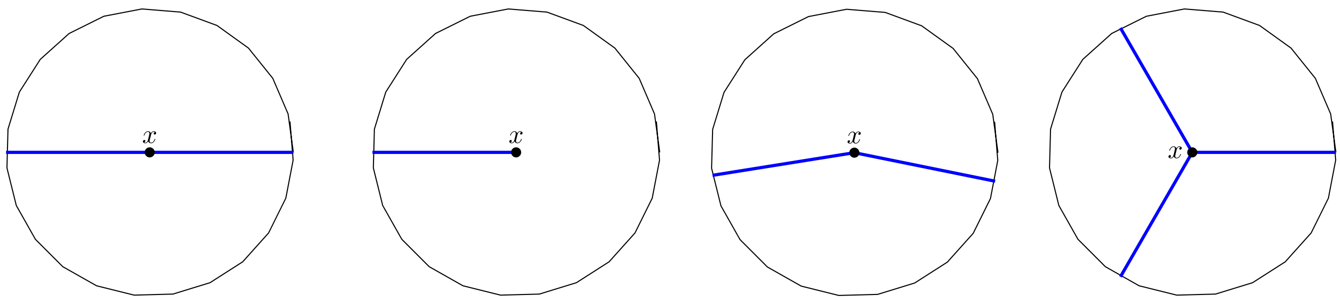
2 Regularity

2.1 Tangent rays. Blow up limits in $\mathbb{R}^n$

Definition 2.1. *We say that the ray $(ax]$ is a tangent ray of the set $\Gamma \subset \mathbb{R}^n$ at the point $x \in \Gamma$ if there exists a sequence of points $x_k \in \Gamma \setminus \{x\}$ such that $x_k \to x$ and $\angle x_k x a \to 0$.*

In the work [8] it is proved that for every maximal distance minimizer Σ at any point of Σ the pairwise angles between the tangent rays are at least $2\pi/3$. Thus every point $x \in \Sigma$ has at most 3 tangent rays of Σ.

In works concerning average distance minimizers the notion of *blow up limits* is used. Santambrogio and Tilli in [14] proved that for any average distance minimizers blow up sequence $\Sigma_\varepsilon := \varepsilon^{-1}(\Sigma_a \cap B_\varepsilon(x) - x)$ with $x \in \Sigma_a$, converges in $B_1(0)$ (for the Hausdorff distance) to some limit $\Sigma_0(x)$ when $\varepsilon \to 0$, and the limit is one of the following below (see Pic. 1 which is analogues to a picture from [10]), up to a rotation.



a general x has tangent line $\psi(x) = 0$ | x is a leaf $\psi(x) > 0$ | x is a corner point $\psi(x) > 0$ | x is a branching point $\psi(x) = 0$

Figure 1: All possible variants of tangent rays at any point of a maximal distance minimizer (or blow up limits of an average distance minimizer)

It is clear that for maximal distance minimizers blow up limits also exists and are more or less the same: Σ_0 can be a radius, a diameter, a corner points with the angle between the segments greater or equal to $2\pi/3$ or a center of a regular tripod. Herewith at the second and third case (id est when $\psi(x) > 0$) the point x has to be energetic; see the following definition.

Definition 2.2. *A point $x \in \Sigma$ is called* energetic, *if for all $\rho > 0$ one has*

$$F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma).$$

Herewith if a point x of a maximal distance minimizer Σ is energetic then there exists such a point $y \in M$ (may be not unique) such that $\operatorname{dist}(x, y) = r$ and $B_r(y) \cap \Sigma = \emptyset$; such y is called *corresponding* to x.

If a point $x \in \Sigma$ is not energetic then in a sufficiently small neighbourhood it is a center of a regular tripod or a segment (and coincides there with its one-sided tangents).

A key object in all the study of the average distance problem is the pull-back measure of μ with respect to the projection onto Σ_a, where Σ_a is a solution of the average distance minimizer problem. More precisely, if μ does not charge the Ridge set (which is defined as the set of all $x \in \mathbb{R}^n$ for which the minimum distance to Σ_a is attained at more than one point) of Σ_a (this is the case for instance when μ is absolutely continuous with respect to the Lebesgue measure), then it is possible to choose a measurable selection of the projection multimap onto Σ, i.e. a map $\pi_\Sigma : M \to \Sigma$ such that $d(x, \Sigma) = d(x, \pi_{\Sigma_a})$ (this map is uniquely defined everywhere except the Ridge set). Then one can define the measure ψ as being $\psi(A) := \mu(\pi_{\Sigma_a}^{-1}(A))$, for any Borel set $A \subset M$. In other words $\psi = \pi_{\Sigma_a} \sharp \mu$.

For the maximal distance minimizers in $\mathbb{R}^n$ we can define measure ψ the similar way, but replace M by $\partial B_r(\Sigma)$ and with $(n-1)$-dimensional Hausdorff measure as μ (or accordingly $\overline{B_r(\Sigma)}$ and n-dimensional Hausdorff measure). Thus Fig. 1 is true both for maximal and average distance minimizers.

2.2 Properties of branching points in $\mathbb{R}^2$

It is known at the plane (see [8]) that for every compact set M and a positive number r a maximal distance minimizer can have only finite number of points with 3 tangent rays.

At the plane it is also known (see [2]) that every average distance minimizer is topologically a tree composed by a finite union of simple curves joining by number of 3.

Every branching point of a planar maximal distance minimizer should be the center of a regular tripod. If $x \in \Sigma \subset \mathbb{R}^2$ has 3 tangent rays then there exists such a neighbourhood of x in which the minimizer coincides with its tangent rays. Id est, there exists such $\varepsilon > 0$ that $\Sigma \cap \overline{B_\varepsilon(x)} = [Ax] \cup [Bx] \cup [Cx]$ where $\{A, B, C\} = \Sigma \cap \partial B_\varepsilon(x)$ and $\angle AxB = \angle BxC = \angle CxA = 2\pi/3$. For planar average distance minimizers it is proved that any branching point admits such a neighbourhood in which three pieces of Σ are $C^{1,1}$.

2.3 Continuity of one-sided tangent rays in $\mathbb{R}^2$

Definition 2.3. *We will say that the ray $(ax]$ is a one-sided tangent of the set $\Gamma \subset \mathbb{R}^n$ at the point $x \in \Gamma$ if there exists a connected component Γ_1 of $\Gamma \setminus \{x\}$ with the property that any sequence of points $x_k \in \Gamma_1$ such that $x_k \to x$ satisfies $\angle x_k x a \to 0$. In this case we will also say that $(ax]$ is tangent to the connected component Γ_1.*

Lemma 2.4. *Let Σ be a (local) maximal distance minimizer and let $x \in \Sigma$. Let Σ_1 be a connected component of $\Sigma \setminus \{x\}$ with one-sided tangent $(ax]$ (it has to exist) and let $\bar{x} \in \Sigma_1$.*

1. *For any one-sided tangent $(\bar{a}\bar{x}]$ of Σ at $\bar{x}$ the equality $\angle((\bar{a}\bar{x}), (ax)) = o_{|\bar{x}x|}(1)$ holds.*
2. *Let $(\bar{a}\bar{x}]$ be a one-sided tangent at $\bar{x}$ of any connected component of $\Sigma \setminus \{\bar{x}\}$ not containing x. Then $\angle((\bar{a}\bar{x}], (ax]) = o_{|\bar{x}x|}(1)$.*

For planar average distance minimizers it is proved (see [10]) that away from branching points an average distance minimizer Σ_a is locally at least as regular as the graph of a convex function, namely that the Right and Left tangent maps admit some Right and Left limits at every point and are semicontinuous. More precisely, for a given parametrization γ of an injective Lipschitz arc $\Gamma \subset \Sigma_a$, by existence of blow up limits one can define the Left and Right tangent half-lines at every point $x \in \Gamma$ by

$$T_R(x) := x + \mathbb{R}^+ . \lim_{h \to 0} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h}$$

and

$$T_L(x) := x + \mathbb{R}^+ . \lim_{h \to 0} \frac{\gamma(t_0 - h) - \gamma(t_0)}{h}.$$

Then the following planar theorem for average distance minimizers holds.

Theorem 2.5 (Lemenant, 2011 [9])**.** *Let $\Gamma \subset \Sigma_a$ be an open injective Lipschitz arc. Then the Right and Left tangent maps $x \to T_R(x)$ and $x \to T_L(x)$ are semicontinuous, id est for every $y_0 \in \Gamma$ there holds $\lim_{y \to y_0; y <_\gamma y_0} T_L(y) = T_L(y_0)$ and $\lim_{y \to y_0; y >_\gamma y_0} T_R(y) = T_R(y_0)$. In addition the limit from the other side exists and we have $\lim_{y \to y_0; y >_\gamma y_0} T_L(y) = T_R(y_0)$ and $\lim_{y \to y_0; y <_\gamma y_0} T_R(y) = T_L(y_0)$.*

An immediate consequence of the theorem is the following corollary:

Corollary 2.6. *Assume that $\Gamma \subset \Sigma$ is a relatively open subset of Σ that contains no corner points nor branching points. Then Γ is locally a C^1-regular curve.*

2.4 Planar example of infinite number of corner points

Recall that each maximal distance minimizer at the plane is a finite union of simple curves. These curves should have continuous one-sided tangents but do not have to be C^1: there exists a minimizer with infinite number of points without tangent lines. The following example is provided in [6].

Fix positive reals r and R and let N be a large enough integer. Consider a sequence of points $\{A_i\}_{i=1}^\infty$ belonging to circumference $\partial B_R(0)$ such that $N \cdot |A_2 A_1| = r$,

$$|A_{i+1} A_{i+2}| = \frac{1}{2} |A_i A_{i+1}|$$

and $\angle A_i A_{i+1} A_{i+2} > \frac{\pi}{2}$ for every $i \in \mathbb{N}$ (see Fig. 2). Let A_∞ be the limit point of $\{A_i\}$. We claim that polyline

$$\Sigma = \bigcup_{i=1}^{\infty} A_i A_{i+1}$$

is a unique maximal distance minimizer for the following M.

Let $V_0 \in (A_1 A_2]$ be such point that $|V_0 A_1| = r$; say that $A_0 := V_0$. For $i \in \mathbb{N} \cup \{\infty\}$ define V_i as the point satisfying $|V_i A_i| = r$ and $\angle A_{i-1} A_i V_i = \angle A_{i+1} A_i V_i > \pi/2$. Finally, let $V_{\infty+1}$ be such point that $V_{\infty+1} A_\infty \perp O A_\infty$ and $|V_{\infty+1} A_\infty| = r$. Clearly $M := \{V_i\}_{i=0}^{\infty+1}$ is a compact set.

Theorem 2.7 (Cherkashin–Teplitskaya, 2022 [6])**.** *Let Σ and M be defined above. Then Σ is a unique maximal distance minimizer for M.*

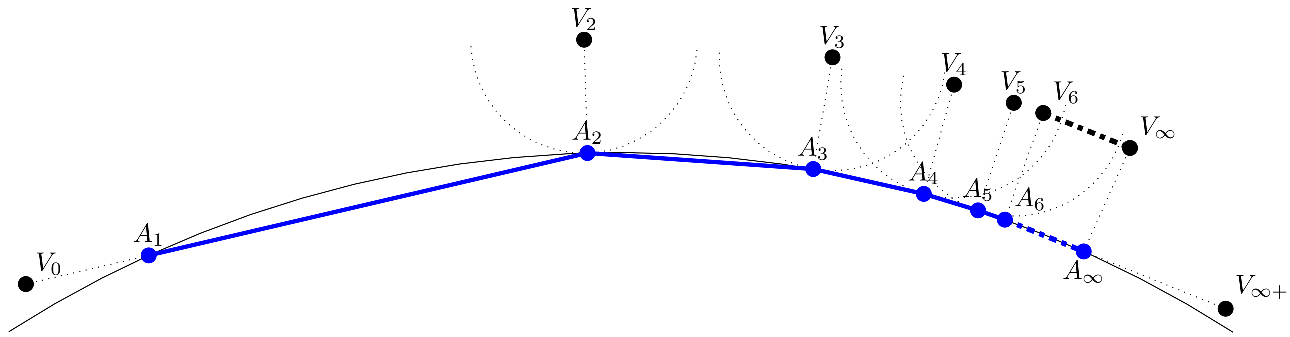


Figure 2: The example of a minimizer with infinite number of corner points

2.5 Every $C^{1,1}$-smooth simple curve is a minimizer

For planar average distance minimizers Tilli proved in [15] that any $C^{1,1}$ simple curve is a minimizer for some given data. The same thing with a similar but much simpler explanations is true for maximal distance minimizers.

Theorem 2.8. *Let γ be a $C^{1,1}$-curve. Then γ is a maximal distance minimizer for a small enough r and $M = \overline{B_r(\gamma)}$.*

3 Explicit examples for maximal distance minimizers

Recall that Theorems 2.7 and 2.8 provide explicit examples.

3.1 Simple examples. Finite number of points and r-neighbourhood. Inverse minimizers

Here we consider Problem 1.2 in a case when M is a finite set. Then it is closely related with the following Steiner problem.

Problem 3.1. *For a given finite set $P = \{x_1, \dots, x_n\} \subset \mathbb{R}^n$ to find a connected set $\mathcal{S}t(P)$ with the minimal length (one-dimensional Hausdorff measure) containing P.*

A solution of Problem 3.1 is called *Steiner tree*. Any maximal distance minimizer for any finite set in $\mathbb{R}^n$ is a finite union of at most $2\sharp M - 1$ segments. In this case maximal distance minimizers problem comes down to connecting r-neighborhoods of all the points from M. If $\overline{B_r(A)}$ are disjoint for every $A \in M$ then a maximal distance minimizer is a Steiner tree connecting points of $\partial B_r(A)$, $A \in M$.

The following observations and statements of this paragraph are from the paper [6].

Remark 3.2. (i) Let Σ be a maximal distance minimizer for some M and $r > 0$. Then Σ is a maximal distance minimizer for $\overline{B_r(\Sigma)}$, r.

(ii) Let Σ be a minimizer for $\overline{B_r(\Sigma)}$ and $r > 0$. Then Σ is a minimizer for $\overline{B_{r'}(\Sigma)}$ and r', where $0 < r' < r$.

In all known examples a $\mathcal{S}t$ with n terminals is an r-minimizer for a set M of n points and a small enough positive r if and only if $\mathcal{S}t$ in the unique Steiner tree for its terminals.

Theorem 3.3 (Cherkashin–Teplitskaya, 2022 [6])**.** *Let $\mathcal{S}t$ be a Steiner tree for terminals $A = (A_1, \dots, A_n)$, $A_i \in \mathbb{R}^d$ such that every Steiner tree for an n-tuple in the closed $2r$-neighbourhood of A (with respect to ρ) has the same topology as $\mathcal{S}t$ for some positive r. Then $\mathcal{S}t$ is an r-minimizer for an n-tuple M and such M is unique.*

Proposition 3.4. *Suppose that $\mathcal{S}t$ is a full Steiner tree for terminals $A_1, \dots, A_n \in \mathbb{R}^2$, which is not unique. Then $\mathcal{S}t$ can not be a minimizer for M being an n-tuple of points.*

Fig. 4 shows that another Steiner tree connecting the vertices of a square becomes an r-minimizer for every positive r.

3.2 Circle. Curves with big radius of curvature

Theorem 3.5 (Cherkashin–Teplitskaya, 2018 [5])**.** *Let r be a positive real, M be a convex closed curve with the radius of curvature at least $5r$ at every point, Σ be an arbitrary minimizer for M. Then Σ is a union of an arc of M_r and two segments that are tangent to M_r at the ends of the arc (so-called horseshoe, see Fig. 5). In the case when M is a circumference with the radius R, the condition $R > 4.98r$ is enough.*

Miranda, Paolini and Stepanov [11] conjectured that all the minimizers for a circumference of radius $R > r$ are horseshoes. Theorem 3.5 solves this conjecture with the assumption $R > 4.98r$; for $4.98r \geq R > r$ the conjecture remains open.

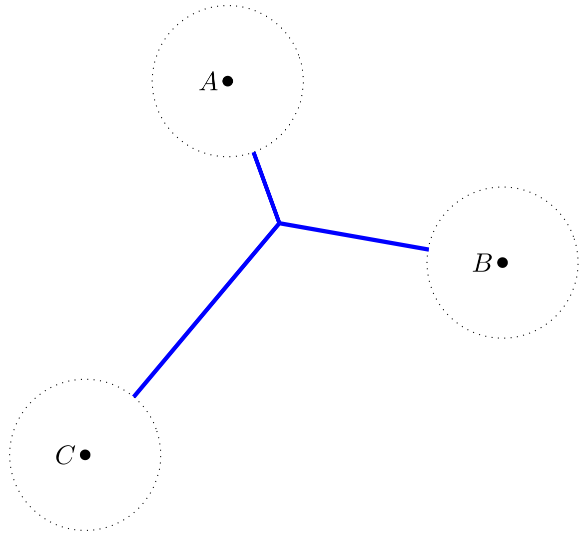


Figure 3: A maximal distance minimizer for 3-point set $M = \{A, B, C\}$

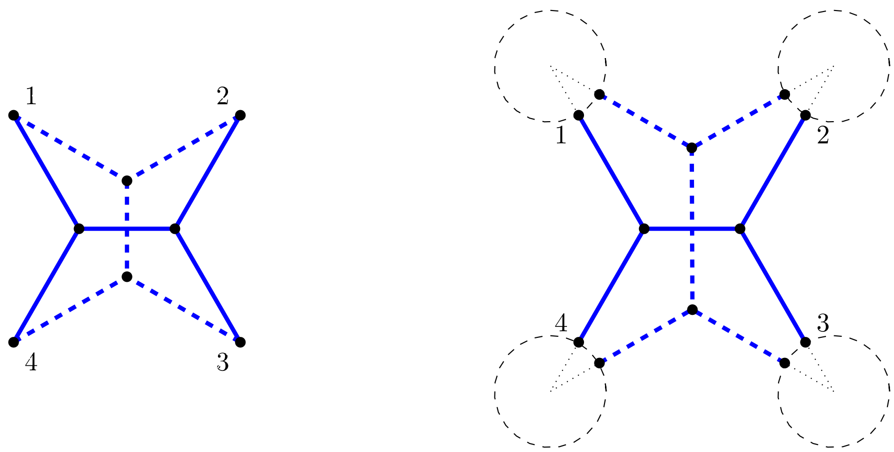


Figure 4: An example to Proposition 3.4

Problem 3.6. *Find maximal distance minimizers for a circumference of radius* $4.98r > R > r$*.*

At the same time, the statement of Theorem 3.5 does not hold for a general M if the assumption on the minimal radius of curvature is omitted as we show below.

Define a *stadium* to be the boundary of the R-neighborhood of a segment. By the definition, a stadium has the minimal radius of curvature R. Let us show that if $R < 1.75r$ and a stadium is long enough, then there is the connected set Σ' that has the length smaller than an arbitrary horseshoe and covers M.

Define Σ_0 to be the local Steiner tree depicted in Fig. 7. Let Σ' consist of copies of Σ_0, glued at points A and B along the stadium. Note that $F_M(\Sigma') \leq r$ by the construction. In the case $R < 1.75r$ the length of Σ_0 is strictly smaller than $2|AB|$. Thus for a long enough stadium Σ' has length $\alpha L + O(1)$, where L is the length of the stadium and $\alpha < 2$ is a constant depending on Σ_0 and R. On the other hand, any horseshoe has length $2L + O(1)$.

This example leads to the following problems.

Problem 3.7. *Find the minimal* α *such that Theorem 3.5 holds with the replacement of* $5r$ *with* αr*.*

Problem 3.8. *Describe minimizers for a given stadium.*

3.3 Rectangle

Theorem 3.9 (Cherkashin–Gordeev–Strukov–Teplitskaya, 2021 [4])**.** *Let* $M = A_1A_2A_3A_4$ *be a rectangle,* $0 < r < r_0(M)$*. Then a maximal distance minimizer has the following topology with 21 segments, depicted in the left part of Fig. 8. The middle part of the picture contains an enlarged fragment of the minimizer near* A_1*; the labeled angles are equal to* $\frac{2\pi}{3}$*. The rightmost part contains a much larger fragment of minimizer near* A_1*.*

All maximal distance minimizers have length approximately $Per - 8.473981r$*, where* Per *is the perimeter of the rectangle.*

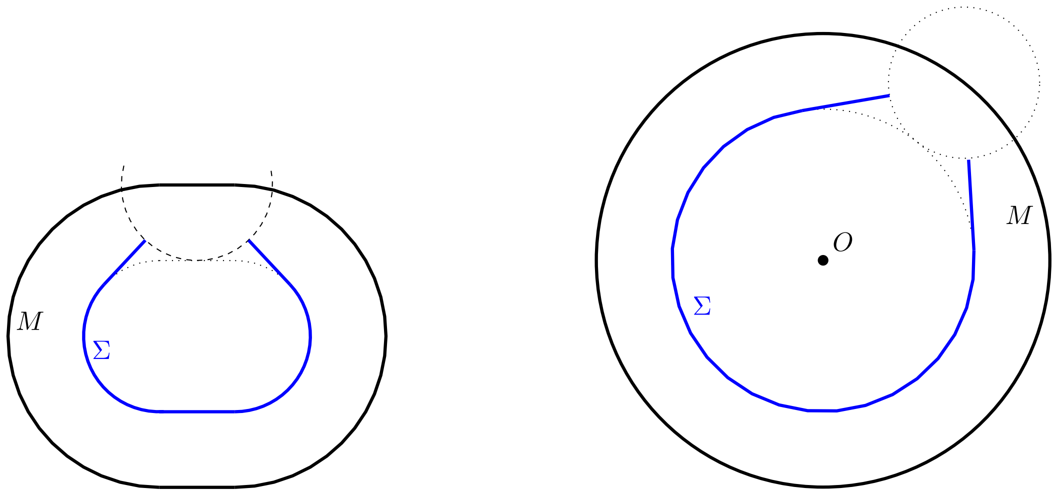


Figure 5: A minimizer for a convex closed planar curve M with the radius of curvature at least $5r$ at every point, so-called *horseshoe* (left). A minimizer for $M = \partial B_R(O)$, where $R > 4.98r$ (right)

Figure 6: M is r-neighbourhood for a sufficiently smooth curve Σ and small enough $r > 0$

In fact, every maximal distance minimizer is very close (in the sense of Hausdorff distance) to the one depicted in the picture.

Analogously to the stadium case one can easily show that for some sufficiently small $\frac{|A_1A_2|}{|A_2A_3|} < 1$ and some $r > 0$ a minimizer should have another topology than depicted at Fig. 8.

Also one may consider the following relaxation of Problem 3.8.

Problem 3.10. *Fix a real $a > 2r$. Let $M(l)$ be the union of 2 sides of length l of a rectangle $a \times l$ and $\Sigma(l)$ be a minimizer for $M(l)$. Find*

$$s(a) := \lim_{l \to \infty} \frac{\mathcal{H}^1(\Sigma(l))}{l}.$$

If $a > 10r$ one may add up $M(l)$ to a stadium and use Theorem 3.5 to get $s(a) = 2$.

4 Tools

For the planar problem the notion of energetic points (which is also correct in $\mathbb{R}^n$) is very useful.

Recall that a point $x \in \Sigma$ is called *energetic*, if for all $\rho > 0$ one has $F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma)$. The set of all energetic points of Σ is denoted by G_Σ. Each minimizer Σ can be split into three disjoint subsets:

$$\Sigma = E_\Sigma \sqcup \mathrm{X}_\Sigma \sqcup \mathcal{S}_\Sigma,$$

where $X_\Sigma \subset G_\Sigma$ is the set of isolated energetic points (i.e. every $x \in X_\Sigma$ is energetic and there is a $\rho > 0$ possibly depending on x such that $B_\rho(x) \cap G_\Sigma = \{x\}$), $E_\Sigma := G_\Sigma \setminus X_\Sigma$ is the set of non isolated energetic points and $S_\Sigma := \Sigma \setminus G_\Sigma$ is the set of non energetic points also called the *Steiner part.*

Note that it is possible for a (local) minimizer in $\mathbb{R}^n$, $n > 2$ to have no non-energetic points at all. Moreover, in some sense, any (local) minimizer does not have non-energetic points in a larger dimension:

Example 4.1. Let Σ be a (local) minimizer for a compact set $M \subset \mathbb{R}^n$ and $r > 0$. Then $\bar{\Sigma} := \Sigma \times \{0\} \subset \mathbb{R}^{n+1}$ is a (local) minimizer for $\bar{M} = (M \times \{0\}) \cup (\Sigma \times \{r\}) \subset \mathbb{R}^{n+1}$ and $\mathrm{E}_{\bar{\Sigma}} = \bar{\Sigma}$.

Recall that for every point $x \in G_\Sigma$ there exists a point $y \in M$ (may be not unique) such that $\mathrm{dist}\,(x, y) = r$ and $B_r(y) \cap \Sigma = \emptyset$. Thus all points of $\Sigma \setminus \overline{B_r(M)}$ can not be energetic and thus $\overline{\Sigma \setminus \overline{B_r(M)}}$ is so-called Steiner forest id est each connected component of it is a Steiner tree with terminal points on the $\partial B_r(M)$.

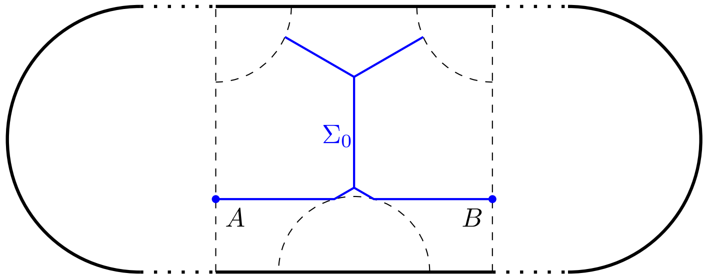


Figure 7: Horseshoe is not a minimizer for long enough stadium with $R < 1.75r$.

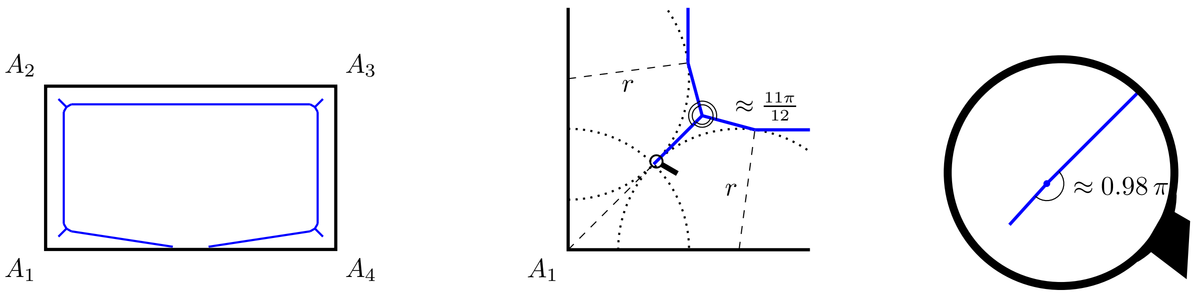


Figure 8: The minimizer for a rectangle M with $r < r_0(M)$.

At the plane it makes sense to define energetic rays.

Definition 4.2. *We say that a ray $(ax]$ is the energetic ray of the set Σ with a vertex at the point $x \in \Sigma$ if there exists non stabilized sequence of energetic points $x_k \in G_\Sigma$ such that $x_k \to x$ and $\angle x_k x a \to 0$.*

Remark 4.3. Let $\{x_k\} \subset G_\Sigma$ and let $x \in E_\Sigma$ be the limit point of $\{x_k\}$: $x_k \to x$. By basic property of energetic points for every point $x_k \in G_\Sigma$ there exists a point $y_k \in M$ (may be not unique) such that $\operatorname{dist}(x_k, y_k) = r$ and $B_r(y_k) \cap \Sigma = \emptyset$. In this case we will say that y_k *corresponds* to x_k.

Let y be an arbitrary limit point of the set $\{y_k\}$. Then set Σ does not intersect r-neighbourhood of y: $B_r(y) \cap \Sigma = \emptyset$ and the point y belongs to M and corresponds to x.

Let $[sx] \subset \Sigma$ be a simple curve. Let us define $\operatorname{turn}([\breve{sx}])$ as the upper limit (supremum) over all sequences of points of the curve:

$$\operatorname{turn}([\breve{sx}]) = \sup_{n \in \mathbb{N}, s \preceq t^1 \prec \dots \prec t^n \prec x} \sum_{i=2}^{n} \widehat{t^i, t^{i-1}},$$

where t^i denotes the ray of the one-sided tangent to the curve $[\breve{st_i}] \subset [\breve{sx}[$ at point t_i, and $t_1, \dots, t_n$ is the partition of the curve $[\breve{sx}[$ in the order corresponding to the parameterization, for which s is the beginning of the curve and x is the end. In this case, the angle $(\widehat{t^i, t^{i+1}}) \in [-\pi, \pi[$ between two rays is counted from ray t^i to ray t^{i+1}; positive direction is counterclockwise.

Let $\breve{sx}$ lay in the sufficiently small neighbourhood of x. Then if $B_r(y(x)) \cap [\breve{sx}] = \emptyset$, it is true that

$$|\operatorname{turn}([sx])| < 2\pi.$$

This property is the first one which is true for the plane and false in $\mathbb{R}^n$ with $n > 2$, so this is the main difference between planar and non-planar cases. At the plane the turn is a very useful tool, see for example the proof of Theorem 3.5 [5].

The second main differ between plane and other Euclidean spaces is also concerning angles: at the plane if you know the angles $\widehat{t^i, t^{i-1}}$ for $i = 2, \dots k$ then you know the angle $\widehat{t^1, t^k}$ which is not true for $\mathbb{R}^n$ with $n > 2$.

4.1 Derivation in the picture

During this subsection M is a planar convex closed smooth curve with the radius of curvature greater than r.

Lemma 4.4. *(i) Let x be an isolated energetic point of degree 1 (i.e. Q is the end of the segment $[QX] \subset \Sigma$) with unique $y(Q)$. Then Q, X and $y(Q)$ lie on the same line.*

(ii) Let W be an isolated energetic point of degree 2 (i.e. W is the end of the segments $[WZ_1]$ and $[WZ_2] \subset \Sigma$) with unique $y(W)$. Then $\angle Z_1 W y(W) = \angle y(W) W Z_2$.

The following proposition describes the possible situation to apply some calculus of variation.

Proposition 4.5. *Let $y \in M$ be a point such that $B_r(y) \cap \Sigma = \emptyset$ and $\partial B_r(y)$ contains energetic points x_1 and x_2. Define $Y = \partial B_r(y) \cap M_r$. Then*

(i) points x_1 and x_2 lie on opposite sides of the line (yY);

(ii) derivatives of length of Σ in neighborhoods of x_1 and x_2 when moving y along M are equal.

In [7] the derivative of length of Σ in a neighborhood of x when moving y along M is calculated. The derivative depends on the behavior of Σ in the neighborhood of x. Since M has large radius of curvature, $\partial B_r(x)$ intersects M in at most 2 points, so every energetic point has at most 2 corresponding y. Thus, we have the following 4 cases.

1. x has order 1, and there is unique corresponding $y(x)$. Then the derivative is equal to

$$\cos \alpha,$$

where $\alpha = \angle([xy(x)), l)$, l is a tangent ray to M at point $y(x)$, in the direction of increasing $\gamma(x)$.

2. x has order 2, and the unique corresponding $y(x)$. Since x has order 2, $B_\varepsilon(x) \cap \Sigma = [xz_1] \cup [xz_2]$ for small enough $\varepsilon > 0$. Then the derivative is equal to

$$2 \cos \alpha \cos \frac{\angle z_1 x z_2}{2},$$

where $\alpha = \angle([xy(x)), l)$, l is a tangent ray to M at point $y(x)$, in the direction of increasing $\gamma(x)$.

3. The degree of point x is 1, and there are two distinct points $y_1(x)$ and $y_2(x)$. Define $\alpha = \angle xy_1(x)y_2(x) = \angle xy_2(x)y_1(x)$ and let δ be the angle between $y_1(x)y_2(x)$ and M.

Let β be the angle between $[zx]$ and the x axis. Then the derivative is equal to

$$\frac{\cos(\alpha+\delta)\sin(\alpha+\beta)}{\sin(2\alpha)}.$$

4. The degree of point x is 2, and there are two distinct points $y_1(x)$ and $y_2(x)$. The derivative is equal to

$$\frac{\cos(\alpha+\delta)}{\sin(2\alpha)}(\sin(\alpha+\beta)+\sin(\alpha+\gamma)),$$

where β and γ are the angles between the x axis and the segments $[z_1x]$ and $[z_2x]$, respectively, α and δ are similar to the previous case.

If M is piece-wise smooth one can also apply such type of derivation, in particular it is heavily used in the proof of Theorem 3.9.

4.2 Convexity argument

Suppose that we fix some $M_0 \subset M$ and consider a (possible infinite) tree T which vertices are encoded by points of M_0. Let us pick an arbitrary point from $\overline{B_r(m)}$ for every $m \in M_0$ and connect such points by segments with respect to T. Consider the length L of such a representation of T; note that we allow the representation to contain cycles or edges of zero length.

Then L is a convex function from $(\mathbb{R}^d)^{M_0}$ to $\mathbb{R}$. Also if $v, u \in \overline{B_r(m)}$, then $\alpha v + (1-\alpha)u$ also lies in $\overline{B_r(m)}$. It implies that the sets of local and global minimums of L coincide and form a convex set. It usually means that L is a unique local minimum.

This approach allows to show that if one fix a topology of a solution, then Steiner problem has a unique solution.

5 The rest

5.1 Γ-convergence

Γ-convergence is an important tool in studying minimizers based on approximation of energy. For Euclidean space the following definition of Γ-convergence can be used. Let X be a first-countable space and $F_n \colon X \to \overline{\mathbb{R}}$ a sequence of functionals on X. Then F_n are said to Γ-converge to the Γ-limit $F : X \to \overline{\mathbb{R}}$ if the following two conditions hold:

- Lower bound inequality: For every sequence $x_n \in X$ such that $x_n \to x$ as $n \to +\infty$, $F(x) \le \liminf_{n\to\infty} F_n(x_n)$.
- Upper bound inequality: For every $x \in X$, there is a sequence x_n converging to x such that $F(x) \ge \limsup_{n\to\infty} F_n(x_n)$.

In the case of maximal distance minimizers for a given compact set M and number $l > 0$ we can consider a space X of connected compact sets with one-dimensional Hausdorff measure at most l; the distance in X is Hausdorff distance (the distance between $A, C \in X$ is the smallest ρ such that $A \subset \overline{B_\rho(C)}$ and $C \subset \overline{B_\rho(A)}$); for $S \in X$ let us define

$$F_n(S) := F_{M_n}(S) = \max_{y \in M_n} \operatorname{dist}(y, S)$$

and a sequence x_n in the second condition is a solution of the dual maximal distance minimizer problem for $l > 0$ and M_n (id est x_n minimizes $F_n()$ among all points of X), where a finite set $M_n \subset M$ is a finite $1/n$-network of M.

Clearly, both conditions hold as $F(x) = \lim_{n\to\infty} x_n$ for every sets $x_n \to x$.

Thus, as each maximal distance minimizer for a finite set should be a Steiner tree with a finite number of leaves, we get that every maximal distance minimizer is a limit of Steiner trees.

This result is also proved in [1]). A relations of finite Steiner trees and maximal distance minimizer are considered in Section 3.1.

5.2 Penalized form

Let M be a given compact set. Let us consider a problem of minimization $F_M(S) + \lambda\mathcal{H}^1(S)$ for some $\lambda > 0$, where $F_M(S) = \max_{y\in M} \operatorname{dist}(y, S)$ among all connected compact sets S. We will call this problem λ-*penalized*.

Clearly every set T which minimizes λ-penalized problem for some λ is a maximal distance minimizer for a given data M and the restriction of energy $r := F_M(T)$. Hence the solutions of this problem inherit all regularity properties of maximal distance minimizers.

As usual in variational calculus on a restricted class, it may happen for a small variation $\Phi_\varepsilon(\Sigma)$ of Σ, that the length constraint $\mathcal{H}^1(\Phi_\varepsilon(\Sigma)) \le l$ is violated. Hence to compute Euler–Lagrange equation associated to the maximal distance minimizers problem a possible way is to consider first the penalized functional $F_M(S) + \lambda\mathcal{H}^1(S)$ for some constant λ, for which any competitor Σ is admissible without length constraint.

Hence it is also make sense to consider *local penalisation problem*: the problem of searching such a connected compact set S that $\mathcal{H}^1(S) + \lambda F_M(S) \le \mathcal{H}^1(T) + \lambda F_M(T)$ for every connected compact T with $\operatorname{diam}(S \triangle T) < \varepsilon$ for sufficiently small $\varepsilon > 0$. The solutions of this problems also inherit properties of local maximal distance minimizers.

Proposition 5.1. *Consider*

$$\min_{\Sigma \text{ compact and connected}} F_M(\Sigma) + \lambda(\mathcal{H}^1(\Sigma) - l)^+$$

for any constant $\lambda > 1$. Then this problem is equivalent to maximal distance minimizers problem.

Proof. The same as for average distance minimizers (see Proposition 23 in [10]). We use the fact that for a connected set $S \setminus T_\varepsilon$ if S is a maximal distance minimizer and $\mathcal{H}^1(T_\varepsilon) = \varepsilon$ there holds $r - F_M(S \setminus T_\varepsilon) \le \varepsilon$. □

5.3 Lower bounds on the length of a minimizer

The proof of the following folklore inequality can be found, for instance in [12].

Lemma 5.2. *Let γ be a compact connected subset of $\mathbb{R}^d$ with $\mathcal{H}^1(\gamma) < \infty$. Then*

$$\mathcal{H}^d(\{x \in \mathbb{R}^d : \operatorname{dist}(x, \gamma) \le t\}) \le \mathcal{H}^1(\gamma)\omega_{d-1}t^{d-1} + \omega_d t^d,$$

where ω_k denotes the volume of the unit ball in $\mathbb{R}^k$.

The following corollary is very close to a theorem of Tilli on average distance minimizers [15].

Corollary 5.3. *Let V and r be positive numbers. Then for every set M with $\mathcal{H}^d(M) = V$ a maximal distance r-minimizer has the length at least*

$$\max\left(0, \frac{V - \omega_d r^d}{\omega_{d-1} r^{d-1}}\right).$$

Theorem 2.8 follows from the fact that for a $C^{1,1}$-curve and small enough r the inequality in Corollary 5.3 is sharp. Let us provide a lower bound on the length of a minimizer in planar case.

Proposition 5.4. *Let M be a planar convex set and Σ is an r-minimizer for M. Then*

$$\mathcal{H}^1(\Sigma) \ge \frac{\mathcal{H}^1(\partial M) - 2\pi r}{2}.$$

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