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# EXTREMAL PROBLEMS IN EUCLIDEAN COMBINATORIAL GEOMETRY

ABSTRACT OF DISSERTATION

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# 1 Introduction

The dissertation is devoted to extremal problems in the intersection of Euclidean geometry and combinatorics. Consider a *distance graph* $G(\mathbb{R}^d)$ of a Euclidean space, which is a complete weighted graph with vertex set $\mathbb{R}^d$ and the weights from Euclidean metrics. A typical framework is $G$ or its "subgraph" $G(V,\rho) = (V, E_\rho)$, where $V$ is a subset of $\mathbb{R}^d$ and $E_\rho$ consists of pairs of vertices at a distance of $\rho$. We consider both finite and infinite $V$. We focus on several classical combinatorial problems: Steiner tree problem, finding a maximal independent set and finding the chromatic number. Note that these three problems belong to the initial Karp's list of 21 NP-complete problems [24].

Let us start with finite and infinite versions of the Euclidean Steiner problem.

**Problem 1.1.** *For a given finite set $P = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ find a connected set $\mathcal{S}t$ with minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) containing $P$.*

For a basic results on Problem 1.1 see paper [21] and book [22]. In particular it is known that there is a finite (but possibly factorial in $n$) number of local minima. NP-hardness of the Euclidean Steiner problem was shown by Garey, Graham and Johnson [19].

**Problem 1.2.** *For a given compact set $\mathcal{A} \subset \mathbb{R}^d$ find a set $\mathcal{S}t$ with minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) such that $\mathcal{S}t \cup \mathcal{A}$ is connected.*

Existence and some local properties of a solution of Problem 1.2 are shown by Paolini and Stepanov [32]. A solution of Problem 1.1 or 1.2 is called *Steiner tree* due to the absence of cycles.

Let us go to independent and chromatic numbers of metric spaces. A *coloring* of a given set $M$ is a map from $M$ to the set of colors. A coloring of a subset $M$ of a metric space is *proper* if no pair of monochromatic points lie at distance 1 apart. The minimum number of colors that admits a proper coloring of $M$ is called *the chromatic number* of $M$; we denote it by $\chi(M)$. In the case of $M \subset \mathbb{R}^d$, the distance typically comes from the induced Euclidean metric on $M$.

A slightly different point of view is to consider a *unit distance graph* $G(M)$: the points of $M$ are the vertices of $G(M)$ and edges connect points at unit distance apart. By definition, $\chi(M) = \chi(G(M))$. The de Bruijn–Erdős theorem [13] states that if $\chi(M)$ is finite then there is a finite subgraph $H$ of $G(M)$ such that $\chi(H) = \chi(G(M))$.

To find the chromatic numbers of a metric space is a classical problem; the most famous particular question belongs to Nelson and Hadwiger and consists in the finding the chromatic number $\chi(\mathbb{R}^2)$ of the Euclidean plane. The best known bounds in this case are $5 \le \chi(\mathbb{R}^2) \le 7$, see [14].

Define graph $G(d,k,\varepsilon)$, which vertices are the point of

$$\mathrm{Slice}(d,k,\varepsilon) := \mathbb{R}^d \times [0,\varepsilon]^k$$

and edges connect points at the Euclidean distance 1 apart. Put

$$\chi[\mathrm{Slice}(d,k,\varepsilon)] := \chi[G(d,k,\varepsilon)],$$

where $\chi(H)$ is the chromatic number of a graph $H$. Obviously for every positive $\varepsilon$ one has

$$\chi(\mathbb{R}^d) \le \chi[\mathrm{Slice}(d,k,\varepsilon)] \le \chi(\mathbb{R}^{d+k}).$$

Since $\chi(\mathbb{R}^d) = (3 + o(1))^d$ (see [27]), the chromatic number of a slice is finite. Note that the best known lower bound is also exponential [36].

# 2 Results

## 2.1 Results of Chapter 2

The first chapter of the dissertation is devoted to the Euclidean Steiner tree problem. It is well-known that the Steiner problem may have several solutions; the simplest example is the vertices of a square. We show that this situation is rare for planar configurations.

Let us denote by $\mathbb{P}_d := (\mathbb{R}^d)^n \setminus diag$ the space of labelled $n$-point configurations $x_1, \dots, x_n \in \mathbb{R}^d$ of distinct points in the Euclidean space, where *diag* is the union of $(dn-d)$-dimensional subspaces $x_i = x_j$, $i \neq j$. A configuration $P \in \mathbb{P}_d$ is *ambiguous* if there are several Steiner trees for $P$. It was shown by Ivanov and Tuzhilin [23] that the set of non-ambiguous configurations contains a subset which is open and dense in $\mathbb{P}_2$. We strengthen it as follows.

**Theorem 2.1** (Basok–Cherkashin–Rastegaev–Teplitskaya [1]). *Assume that $n \geq 4, d \geq 2$. Then the set of planar ambiguous configurations in $\mathbb{P}_2$ has Hausdorff dimension $2n-1$.*

Edelsbrunner and Strelkova [16, 15] showed that is we fix a combinatorics of a solution, then for every $d \geq 2$ the subset of configuration in $\mathbb{P}_d$ for which a unique Steiner tree has a given combinatorics is path-connected. We extend this result.

**Theorem 2.2** (Basok–Cherkashin–Rastegaev–Teplitskaya [1]). *The subset of $n$-point configurations in $\mathbb{P}_d$ for which there is a unique Steiner tree is path-connected.*

In the proof of Theorem 2.2 we use the existence of a universal Steiner tree, id est a tree which contains a solution of every possible finite combinatorics. The first example of such Steiner tree was given in [34]. Also, it was the first example of an infinite indecomposable (id est cannot be represented as a union of the solutions for $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$) Steiner tree. However it was not self-similar and had zero Hausdorff dimension. We provide an example of indecomposable Steiner tree for the input of Hausdorff dimension $-\frac{\ln 2}{\ln \lambda}$, for $\lambda < 1/300$. Let $A_\infty(\lambda)$ be the (uncountable, compact) set consisting of the root and the leaves of a fractal binary tree $\Sigma(\lambda)$ with the ratio of length of edges on consecutive levels equal to $\lambda$.

**Theorem 2.3** (Cherkashin–Teplitskaya, [8]). *A binary tree $\Sigma(\lambda)$ is a Steiner tree for $A_\infty(\lambda)$ provided that $\lambda < 1/300$.*

It is worth noting that very recently Theorem 2.3 was significantly improved.

**Theorem 2.4** (Paolini–Stepanov, [33]). *A binary tree $\Sigma(\lambda)$ is a* ***unique*** *Steiner tree for $A_\infty(\lambda)$ provided that $\lambda < 1/25$.*

## 2.2 Results of Chapter 3

The Gilbert–Steiner problem [20, 2] is a generalization of the Steiner tree problem on a specific optimal mass transportation. Let us proceed with the formal definition.

**Definition 2.1.** *Let $\mu^+, \mu^-$ be two finite measures on a metric space $(X, \rho(\cdot,\cdot))$ with finite supports such that total masses $\mu^+(X) = \mu^-(X)$ are equal. Let $V \subset X$ be a finite set containing the support of the signed measure $\mu^+ - \mu^-$, the elements of $V$ are called* vertices. *Further, let $E$ be a finite collection of unordered pairs $\{x, y\} \subset V$ which we call* edges. *So, $(V, E)$ is a simple undirected finite graph.*

*Assume that for every $\{x, y\} \in E$ two non-zero real numbers $m(x, y)$ and $m(y, x)$ are defined so that $m(x, y) + m(y, x) = 0$. This data set is called a* $(\mu^+, \mu^-)$-flow *if*

$$\mu^+ - \mu^- = \sum_{\{x,y\} \in E} m(x, y) \cdot (\delta_y - \delta_x)$$

*where $\delta_x$ denotes a delta-measure at $x$ (note that the summand $m(x, y) \cdot (\delta_y - \delta_x)$ is well-defined in the sense that it does not depend on the order of $x$ and $y$).*

Let $C\colon [0, \infty) \to [0, \infty)$ be a *cost function.* The expression

$$\sum_{\{x,y\} \in E} C(|m(x, y)|) \cdot \rho(x, y)$$

is called the *Gilbert functional* of the $(\mu^+, \mu^-)$-flow.

The *Gilbert–Steiner problem* is to find the flow which minimizes the Gilbert functional with cost function $C(x) = x^p$, for a fixed $p \in (0, 1)$; we call a solution *minimal flow.*

Vertices from $\operatorname{supp}(\mu^+) \setminus \operatorname{supp}(\mu^-)$ are called *terminals.* A vertex from $V \setminus \operatorname{supp}(\mu^+) \setminus \operatorname{supp}(\mu^-)$ is called a *branching point.* Formally, we allow a branching point to have degree 2, but clearly it never happens in a minimal flow.

Local structure in the Gilbert–Steiner problem was discussed in [2], and the paper [28] deals with planar case. A local picture around a branching point $b$ of degree 3 is clear due to the initial paper of Gilbert. Similarly to the finding of the Fermat–Torricelli point in the celebrated Steiner problem one can determine the angles around $b$ in terms of masses.

**Theorem 2.5** (Lippmann–Sanmartín–Hamprecht [28], 2022)**.** *A solution of the planar Gilbert–Steiner problem has no branching point of degree at least 5.*

The goal of this section is to give some conditions on a cost function under which all branching points in a planar solution have degree 3. They are slightly stronger than the Schoenberg [37] conditions of the embedding of the metric of the form $\rho(x, y) := f(x - y)$ to a Hilbert space. In particular, this covers the case of the standard cost function $x^p$, $0 < p < 1$.

**Definition 2.2.** *Let $\lambda$ be a Borel measure on $\mathbb{R}$ for which*

$$\int \min(x^2, 1) d\lambda(x) < \infty. \tag{1}$$

*Assume additionally that the support of $\lambda$ is uncountable. A function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of the form*

$$f(t) = \sqrt{\int \sin^2(tx)\, d\lambda(x)} = \frac{1}{2}\|e^{2itx} - 1\|_{L^2(\lambda)} \tag{2}$$

*is called* admissible*.*

The main result of Chapter 3 is the following theorem.

**Theorem 2.6** (Cherkashin–Petrov [5])**.** *Let $\mu^+, \mu^-$ be two measures with finite support on the Euclidean plane $\mathbb{R}^2$, and assume that the cost function $C$ is admissible. Then if a $(\mu^+, \mu^-)$-flow has a branching point of degree at least 4, then there exists a $(\mu^+, \mu^-)$-flow with strictly smaller value of Gilbert functional.*

In particular it has the following corollary.

**Theorem 2.7** (Cherkashin–Petrov [5])**.** *A solution of the planar Gilbert–Steiner problem has no branching point of degree at least 4.*

## 2.3 Results of Chapter 4

In Chapter 4 we consider the problem of minimizing the maximal distance to a given compact set $M$ among the sets of a given length $\ell$. This problem appeared in [3] and later has been studied in [30, 31].

First, we give a survey of the results on the maximal distance minimization problem. Then we find maximal distance minimizers for a closed planar curve of a small enough curvature. Such an answer was conjectured by Miranda, Paolini and Stepanov [30] for a circle of radius $R > r$. Let $M_r$ be the curve equidistant to $M$ at a distance $r$ in the inner direction.

**Theorem 2.8** (Cherkashin–Teplitskaya, 2018 [6]). *Let $r$ be a positive real, $M$ be a convex closed curve with the radius of curvature at least $5r$ at every point, $\Sigma$ be an arbitrary minimizer for $M$. Then $\Sigma$ is a union of an arc of $M_r$ and two segments that are tangent to $M_r$ at the ends of the arc (so-called horseshoe, see Fig. 1). In the case when $M$ is a circumference with the radius $R$, the condition $R > 4.98r$ is enough.*

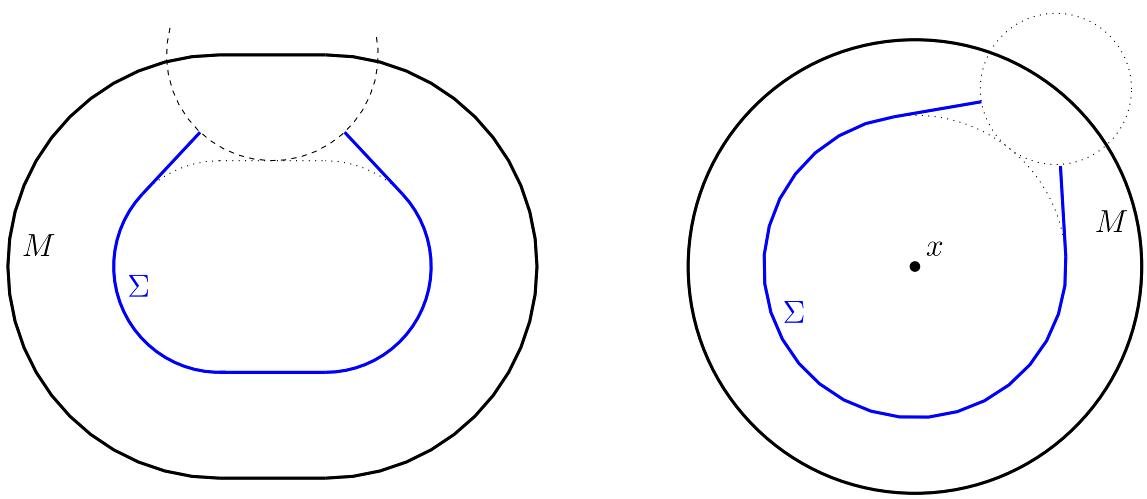


Figure 1: A minimizer for a convex closed planar curve $M$ with the radius of curvature at least $5r$ at every point, so-called *horseshoe* (left). A minimizer for $M = \partial B_R(x)$, where $R > 4.98r$ (right)

The proof is technically complicated, and the main idea is to reduce the comparison of length to the comparison of the angular measure. Some technical moments are simplified and generalized in [11].

The chapter finished with a pack of open questions from [7]; let us emphasize the following.

**Question 2.1.** *Does there exist a nonplanar maximal distance minimizer with infinite number of branching points?*

## 2.4 Results of Chapter 5

A subset $I$ of vertices of $G$ is *independent* if no edge connects vertices of $I$. The *independence number* of a graph $G$ is the maximal size of an independent set in $G$; we denote it by $\alpha(G)$.

Chapter 5 has a deal with independence and chromatic numbers of Johnson-type graphs. The Johnson-type graph $J_{\pm}(d, k, t)$ in defined the following way: the vertex set consists of all vectors from $\{-1, 0, 1\}^d$ with exactly $k$ nonzero coordinates; edges connect the pairs of vertices with scalar product $t$.

We found the exact values of the independent sets in several corner cases. Let $S(d, D)$ be the constant of the corresponding solution of the isodiametric problem on the Hamming cube defined in the paper by Kleitman [26],

$$S(d, D) := \begin{cases} \sum_{j=0}^{m} \binom{d}{j} & \text{if } D = 2m, \\ \binom{d-1}{m} + \sum_{j=0}^{m} \binom{d}{j} & \text{if } D = 2m + 1, \end{cases}$$

**Theorem 2.9** (Cherkashin–Kiselev [4]). *Suppose that $t$ is a negative odd number. Then there exists such $d_0(k)$ that for $d > d_0(k)$ one has*

$$\alpha[J_{\pm}(d, k, t)] = S\left(k, |t| - 1\right) \binom{d}{k}.$$

The case $t = -1$ and additional arguments imply

**Theorem 2.10** (Cherkashin–Kiselev [4]). *Let $d > \frac{9}{2}k^3 2^k$. Then*

$$\alpha[J_{\pm}(d, k, 0)] = 2\binom{d-1}{k-1}.$$

Note that for fixed $k$ and negative $t$ the equality

$$\alpha[J_{\pm}(d, k, t)] = (1 + o(1)) S\left(k, |t| - 1\right) \binom{d}{k}$$

was obtained by Frankl and Kupavskii [17, 18], but for even negative $t$ one can not exclude $o(1)$-term. Main technique in the Frankl–Kupavskii proofs is *shifting*, which cannot be directly applied to Johnson type graphs, which explained this $o(1)$-term. Our proofs are based on the Katona averaging method [25] which states

$$\frac{\alpha(G)}{|V(G)|} \leq \frac{\alpha(H)}{|V(H)|}$$

for every subgraph $H$ of a vertex-transitive graph $G$. To find a proper subgraph $H$ is a Johnson-type graph we use some constructions of simple hypergraphs which come from Reed–Solomon codes.

Also for $k = 3$ and $t = -2$ we found the asymptotic of the chromatic number of Johnson-type graphs.

**Theorem 2.11** (Cherkashin [10]). *For all $d \geq 3$ the inequalities are satisfied*

$$\lceil \log_2 \lceil \log_2 d \rceil \rceil \leq \chi(J_{\pm}[d, 3, -2]) \leq 4 \lceil \log_2 \lceil \log_2 d \rceil \rceil + 6.$$

The intuition of the proof of Theorem 2.11 comes from the fact that edges of a complete $n$-vertex graph cannot be covered by less than $\lceil \log_2(n) \rceil$ complete bipartite graphs.

## 2.5 Results of Chapter 6

In 1976 Simmons [38] conjectured that every coloring of a 2-dimensional sphere of radius strictly greater than 1/2 in three colors has a pair of monochromatic points at distance 1 apart. Chapter 6 contains the proof of the conjecture. This refines the result of Lovász [29], who show that there is a sequence of radii $r_k$ with the limit 1/2, such that a 2-dimensional sphere with radius $r_k$ has the chromatic number at least 4.

**Theorem 2.12** (Cherkashin–Voronov [9]). *For every $r > \frac{1}{2}$ we have*

$$\chi(S^2(r)) \geq 4,$$

*where $S^2(r) \subset \mathbb{R}^3$ is a two-dimensional sphere.*

For the case $\frac{1}{2} < r \leq \frac{\sqrt{3-\sqrt{3}}}{2}$ theorem is tight, i.e. $\chi(S^2(r)) = 4$.

Recall that the chromatic number of a sphere is finite, so the de Bruijn–Erdős theorem implies that it is achieved on a finite subgraph. However our proof do not provide an explicit example of a spherical subgraph with the chromatic number 4. We assume that there is a proper 3-coloring of a sphere; then the first step of the proof shows that every color is dense in the sphere. The second step use implicit function theorem applied to an embedding of an odd cycle with additional pendant of every vertices. This application and density condition allow us to move every pendant vertex to a point of color 1, which immediately gives a contradiction.

## 2.6 Results of Chapter 7

Chapter 7 deals with the chromatic numbers of 3-dimensional slices. The main result gives the following bounds.

**Theorem 2.13** (Cherkashin–Kanel-Belov–Strukov–Voronov [39]). *There is $\varepsilon_0 > 0$, such that for every positive $\varepsilon < \varepsilon_0$ holds*

$$10 \leq \chi[\text{Slice}(3, 6, \varepsilon)] \leq 15.$$

Note that the upper bound is a modification of a well-known bound which comes from a well-known permutohedron tiling [35, 12]. The proof combines combinatorial and topological arguments. In particular, we use the following result which is of independent interest.

**Theorem 2.14** (Cherkashin–Kanel-Belov–Strukov–Voronov [10]). *Let $T \subset \mathbb{R}^d$ be a regular simplex with the edge length $a = \sqrt{2d(d+1)}$. Then every proper coloring of $\mathbb{R}^d$ in a finite number of colors contains a point from $T$ belonging to the closures of at least $d+1$ colors.*

# 3 Publications

The thesis is based on the following papers and preprints:

1. "On the horseshoe conjecture for maximal distance minimizers", D. Cherkashin, Y. Teplitskaya, ESAIM: Control, Optimisation and Calculus of Variations 24 (3), 1015–1041, 2018 (IF2018 1.295, Q2)

2. “A self-similar infinite binary tree is a solution to the Steiner problem”, D. Cherkashin, Y. Teplitskaya, Fractal and Fractional 7 (5), 414, 2023 (IF2022 5.4, Q1)
3. “Independence numbers of Johnson-type graphs”, D. Cherkashin, S. Kiselev, Bulletin of the Brazilian Mathematical Society, New Series 54 (3), 30, 2023 (IF2022 0.7, Q3)
4. “On the chromatic number of 2-dimensional spheres”, D. Cherkashin, V. Voronov, Discrete & Computational Geometry, 71, 467–479, 2024 (IF2022 0.8, Q3)
5. “On the chromatic numbers of 3-dimensional slices”, V.A. Voronov, A.Y. Kanel-Belov, G.A. Strukov, D.D. Cherkashin, Zapiski Nauchnykh Seminarov POMI 518, 94–113, 2022 (in Russian, English translation to appear in Journal of Mathematical Sciences, SJR2022 0.314, Q3)
6. “On the chromatic numbers of Johnson-type graphs”, D.D. Cherkashin, Zapiski Nauchnykh Seminarov POMI 518, 192–200, 2022 (in Russian, English translation to appear in Journal of Mathematical Sciences, SJR2022 0.314, Q3)
7. “On uniqueness in Steiner problem”, M. Basok, D. Cherkashin, N. Rastegaev, Y. Teplitskaya, preprint arXiv:1809.01463, to appear in IMRN, 2024 (IF2022 1.0, Q2)
8. “On minimizers of the maximal distance functional for a planar convex closed smooth curve” D.D. Cherkashin, A.S. Gordeev, G.A. Strukov, Y.I. Teplitskaya, preprint arXiv:2011.10463, submitted to St. Petersburg Mathematical Journal
9. “Branching points in the planar Gilbert–Steiner problem have degree 3”, D. Cherkashin, F. Petrov, preprint arXiv:2309.04202, to appear in Pure and Applied Functional Analysis, the volume is dedicated to the memory of Anatoly Moiseevich Vershik
10. “An overview of maximal distance minimizers problem”, D. Cherkashin, Y. Teplitskaya, preprint arXiv:2212.05607, submitted to Serdica Mathematical Journal

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