Extremal problems in Euclidean combinatorial geometry

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Chapter 1

Introduction

The dissertation is devoted to extremal problems in the intersection of Euclidean geometry and combinatorics. Consider a *distance graph* $G(\mathbb{R}^d)$ of a Euclidean space, which is a complete weighted graph with vertex set $\mathbb{R}^d$ and the weights from Euclidean metrics. A typical framework is G or its "subgraph" $G(V, \rho) = (V, E_\rho)$, where V is a subset of $\mathbb{R}^d$ and E_d consists of pairs of vertices at a distance of ρ. We consider both finite and infinite V. We focus on several classical combinatorial problems: Steiner tree problem, finding a maximal independent set and finding the chromatic number. Note that these three problems belong to the initial Karp's list of 21 NP-complete problems.

Chapter 2 is devoted to the Euclidean Steiner tree problem. Theorem 2.1.1 states that for $d = 2$ a random n-point input leads to a unique solution. Then Theorem 2.1.4 shows the connectedness of the set of d-dimensional n-point configurations having a unique Steiner tree (as a subset of $(\mathbb{R}^d)^n$). Also, Theorem 2.2.1 provides an example of a Steiner tree for an input $\mathcal{A}$ of a positive Hausdorff dimension, which cannot be considered as a union of the solutions for $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$.

The Gilbert–Steiner problem is a generalization of the Steiner tree problem on a specific optimal mass transportation. The cost function for the transportation of a mass m on distance l is chosen to be $m^p \cdot l$, where $p \in (0, 1)$ is a parameter. The difference with the optimal transportation problem is that the extra geometric points may be of use; such points are called *branching*. Chapter 3 proves that every branching point in a solution of the planar Gilbert–Steiner problem has degree 3, see Theorem 3.1.2 and more general Theorem 3.3.1.

In Chapter 4 we consider the problem of minimizing the maximal distance to a given compact set M among the sets of a given length ℓ. First, we give a survey on the results in this problem. Then we find maximal distance minimizers for a closed planar curve of a small enough curvature (see Theorem 4.3.2) and finish with open questions.

Chapter 5 has a deal with independence and chromatic numbers of Johnson-type graphs. The Johnson-type graph $J_\pm(d, k, t)$ in defined the following way: the vertex set consists of all vectors from $\{-1, 0, 1\}^d$ with exactly k nonzero coordinates; edges connect the pairs of vertices with scalar product t. Theorems 5.1.7 and 5.1.8 determine the independence number of $J_\pm(d, k, t)$ for an odd negative t and $d > d_0(k, t)$. Theorem 5.1.12 shows that the asymptotic of the chromatic numbers for $k = 3$, $t = -2$ is doubly logarithmic in d.

In Chapter 6 we show that for a positive $\rho < 2$ the chromatic number of $G(\mathbb{S}^2, \rho)$ is at least four, where $\mathbb{S}^2$ is a 2-dimensional sphere with unit radius, see Theorem 6.1.2. Note that for $\rho = 2$ the corresponding graph is a matching, so its chromatic number is two.

Finally, Theorem 7.1.2 in Chapter 7 establishes that for every positive ε the chromatic number of $\mathbb{R}^3 \times [0, \varepsilon]^6$ is at least 10. For a small enough ε the upper bound is 15 and it comes from a well-known permutohedron tiling.

The thesis is based on the following papers and preprints:

1. "On the horseshoe conjecture for maximal distance minimizers", D. Cherkashin, Y. Teplitskaya, ESAIM: Control, Optimisation and Calculus of Variations 24 (3), 1015–1041, 2018 (IF2018 1.295, Q2)

2. "A self-similar infinite binary tree is a solution to the Steiner problem", D. Cherkashin, Y. Teplitskaya, Fractal and Fractional 7 (5), 414, 2023 (IF2022 5.4, Q1)

3. "Independence numbers of Johnson-type graphs", D. Cherkashin, S. Kiselev, Bulletin of the Brazilian Mathematical Society, New Series 54 (3), 30, 2023 (IF2022 0.7, Q3)

4. "On the chromatic number of 2-dimensional spheres", D. Cherkashin, V. Voronov, Discrete & Computational Geometry, 71, 467–479, 2024 (IF2022 0.8, Q3)

5. "On the chromatic numbers of 3-dimensional slices", V.A. Voronov, A.Y. Kanel-Belov, G.A. Strukov, D.D. Cherkashin, Zapiski Nauchnykh Seminarov POMI 518, 94–113, 2022 (in Russian, English translation to appear in Journal of Mathematical Sciences, SJR2022 0.314, Q3)

6. "On the chromatic numbers of Johnson-type graphs", D.D. Cherkashin, Zapiski Nauchnykh Seminarov POMI 518, 192–200, 2022 (in Russian, English translation to appear in Journal of Mathematical Sciences, SJR2022 0.314, Q3)

7. "On uniqueness in Steiner problem", M. Basok, D. Cherkashin, N. Rastegaev, Y. Teplitskaya, preprint arXiv:1809.01463, to appear in IMRN, 2024 (IF2022 1.0, Q2)

8. "On minimizers of the maximal distance functional for a planar convex closed smooth curve" D.D. Cherkashin, A.S. Gordeev, G.A. Strukov, Y.I. Teplitskaya, preprint arXiv:2011.10463, submitted to St. Petersburg Mathematical Journal

9. "Branching points in the planar Gilbert–Steiner problem have degree 3", D. Cherkashin, F. Petrov, preprint arXiv:2309.04202, to appear in Pure and Applied Functional Analysis, the volume is dedicated to the memory of Anatoly Moiseevich Vershik

10. "An overview of maximal distance minimizers problem", D. Cherkashin, Y. Teplitskaya, preprint arXiv:2212.05607, submitted to Serdica Mathematical Journal

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Chapter 2

Steiner trees

This chapter is based of papers [5, 20]. We consider both finite and infinite forms of the Euclidean Steiner tree problem:

Problem 2.0.1. *For a given finite set $P = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ find a connected set $\mathcal{S}t$ with minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) containing P.*

Problem 2.0.2. *For a given compact set $\mathcal{A} \subset \mathbb{R}^d$ find a set $\mathcal{S}t$ with minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) such that $\mathcal{S}t \cup \mathcal{A}$ is connected.*

Results on Problem 2.0.1. We prove that the set of n-point configurations for which the solution to the planar Steiner problem is not unique has the Hausdorff dimension at most $2n-1$ (as a subset of $\mathbb{R}^{2n}$). Moreover, we show that the Hausdorff dimension of the set of n-point configurations for which at least two locally minimal trees have the same length is also at most $2n-1$. The methods we use essentially rely upon the theory of subanalytic sets developed in [9]. Motivated by this approach we develop a general setup for the similar problem of uniqueness of the Steiner tree where the Euclidean plane is replaced by an arbitrary analytic Riemannian manifold M. In this setup we argue that the set of configurations possessing two locally-minimal trees of the same length either has dimension equal to $n \dim M - 1$ or has a non-empty interior. We provide an example of a two-dimensional surface for which the last alternative holds.

We study the set of n-point configurations for which there is a unique solution to the Steiner problem in $\mathbb{R}^d$. We show that this set is path-connected.

Results on Problem 2.0.2. A solution to Problem 2.0.2 for $\mathcal{A}$ is called *indecomposable* if it cannot be represented as a union of solutions for $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$. We construct several self-similar indecomposable solutions, in particular for $\mathcal{A}$ having a positive Hausdorff dimension.

2.1 Basics

Throughout this chapter $n \geq 4, d \geq 2$ are natural numbers. All the solutions of Problem 2.0.1 for $n \leq 3$ are known in the explicit form since 17-th century. The finite Euclidean Steiner problem has an intricate history, which is studied in the paper [10]. Brazil, Graham, Thomas and Zachariasen done a detailed research and discovered that a statement and basic results were rediscovered (at least) three times. Up to a modern knowledge, it was first stated by Gergonne in 1811, then by Gauss in 1836. The first known publication is dated from 1934 and belongs to Jarník and Kössler [66]. The problem has become well-known as "Steiner problem" after the great success of the book "What is Mathematics?" by Courant and Robbins [29].

A solution to Problem 2.0.1 is called *Steiner tree*. It is known that such a set $\mathcal{S}t$ always exists (but is not necessarily unique, see Fig. 2.1) and that every such $\mathcal{S}t$ is a finite union of segments. Thus, $\mathcal{S}t$ can be represented as a graph, embedded into the Euclidean space, such that its set of vertices contains P and all its edges are straight line segments. This graph is connected and does not contain cycles, i.e. is a tree, which explains the naming of $\mathcal{S}t$. It is known that the maximal degree of the vertices of $\mathcal{S}t$ is at most 3. Moreover, only vertices x_i can have degree 1 or 2, all the other vertices have degree 3 and are called *Steiner points* while the vertices x_i are called *terminals*. Vertices of degree 3 are called *branching points*. The angle between any two adjacent edges of $\mathcal{S}t$ is at least $2\pi/3$. That means that for a branching point the angle between any two segments incident to it is exactly $2\pi/3$, and these three segments belong to the same 2-dimensional plane.

The number of Steiner points in $\mathcal{S}t$ does not exceed $n-2$. A Steiner tree with exactly $2n-2$ vertices is called *full*. Every terminal point of a full Steiner tree has degree one.

For a given finite set $P \subset \mathbb{R}^d$ consider a connected acyclic set S containing P. Then S is called a *locally minimal tree* if $\overline{S \cap B_\varepsilon(x)}$ is a Steiner tree for $(\{x\} \cap P) \cup (S \cap \partial B_\varepsilon(x))$ for every point $x \in S$ and small enough $\varepsilon > 0$. Clearly every Steiner tree is locally minimal and not vice versa. Locally minimal trees have all the mentioned properties of Steiner trees except the minimal length condition. So locally minimal trees inherit the definitions of terminals, Steiner points, branching points and fullness. Proof of the listed properties of Steiner and locally minimal trees together with an additional information on them can be found in the book [63] and in article [55].

Garey, Graham and Johnson [53] proved that the Steiner problem is NP-hard, then Rubinstein, Thomas and Wormald [112] proved that the hardness property remains even in the case of terminals, belonging to a pair of parallel lines as well as in the case of terminals on the sides of the angle which smaller than $2\pi/3$.

Similar problems could also be considered in abstract metric spaces. In the most general form the problem would be to connect a set (not necessarily finite or countable) of subsets of an arbitrary metric space in a minimal way with respect to the metric [99], see Section 2.5.1.

Problem 2.0.1 may have several solutions starting with $n = 4$ (see Fig. 2.1). Theorem 2.1.1 implies the uniqueness of a solution for a general input.

Let us proceed to basic properties of Problem 2.0.2. A general setting for the problem was given in [99]: the ambient space X can be any connected complete metric space with the Heine–Borel property (closed bounded sets are compact) and the given set of points can be any compact subset of the ambient space. In this setting there always exists a set $\mathcal{S}t$, with minimal 1-dimensional Hausdorff measure $\mathcal{H}^1$, such that $\mathcal{S}t \cup \mathcal{A}$ is connected.

As shown in [99] every solution $\mathcal{S}t$ having a finite length has the following properties:

- $\mathcal{S}t \cup \mathcal{A}$ is compact,
- $\mathcal{S}t \setminus \mathcal{A}$ has at most countably many connected components, each of which has positive length,
- $\overline{\mathcal{S}t}$ contains no loops (homeomorphic images of the circle $\mathbb{S}^1$),
- the closure of every connected component of $\mathcal{S}t$ is a topological tree (a connected, locally connected compact set without loops) with endpoints on $\mathcal{A}$ (so that in particular it has at most a countable number of branching points), with at most one endpoint on each connected component of $\mathcal{A}$ and all the branching points having finite order (i.e. finite number of branches leaving them),

- if $\mathcal{A}$ has a finite number of connected components, then $\mathcal{S}t \setminus \mathcal{A}$ also has finitely many connected components, the closure of each of which is a finite geodesic embedded graph with endpoints on $\mathcal{A}$, and with at most one endpoint on each connected component of $\mathcal{A}$,
- the set $\mathcal{S}t \setminus \mathcal{A}$ is a locally finite geodesic embedded minimal graph.

We also call a solution to Problem 2.0.2 *Steiner tree*; the above properties explain such naming in the case of $\mathcal{A}$ being a totally disconnected set. It has similar properties with a solution of Problem 2.0.1 and we specify it below. Denote by $\mathbb{M}(\mathcal{A})$ the set of Steiner trees for $\mathcal{A}$. The points of $\mathcal{A}$ touched by the Steiner tree $\mathcal{S}t$ will be called *terminals*. All points of $\mathcal{A}$ are terminal points if $\mathcal{A}$ is totally disconnected. From now we focus on $X = \mathbb{R}^2$ (but the claims of this paragraph also hold for $\mathbb{R}^d$). A combination of the last enlisted property from [99] with well-known facts on Euclidean Steiner trees (see [55, 63]) gives the following properties. The edges of the locally finite graph $\mathcal{S}t \setminus \mathcal{A}$ are straight line segments. The maximal degree (in graph-theoretic sense) of a vertex is at most 3. Moreover, only terminals can have degree 1 or 2, all the other vertices have degree 3 and are called *Steiner points*. Vertices of the degree 3 are called *branching points*. The angle between any two adjacent edges of a Steiner tree is at least $2\pi/3$, in particular a Steiner tree in a neighbourhood of a branching point is a regular tripod: all three angles are equal to $2\pi/3$.

In 1980-s and 1990-s explicit solutions of the Steiner problem attracted the attention of several notable mathematicians, in particular Graham. It is worth noting that Du, Hwang and Weng [33] completely solved the Steiner problem when $\mathcal{A}$ is the set of vertices of a regular polygon. Rubinstein and Thomas [111] generalizes the result for when the points of $\mathcal{A}$ are uniformly enough distributed on a circle.

Let us also mention that Chung and Graham [25] and Burkard, Dudás and Maier [11] determined the set of Steiner trees for ladders. A *ladder* is a collection of $2n$ lattice points of $\mathbb{Z}^2$ which forms a rectangle $1 \times (n-1)$; the structure of a solution is absolutely different for odd and even n.

More recently, following the paper [99], the question of finding examples of non trivial infinite Steiner trees was raised. Of course it is easy to find infinite trees by merging together an infinite number of finite trees. Much more difficult is to find an infinite Steiner tree which is indecomposable.

The first example of an infinite, indecomposable Steiner tree was given in [101].

Let us denote by $\mathbb{P}_d := (\mathbb{R}^d)^n \setminus diag$ the space of labelled n-point configurations $x_1, \dots, x_n \in \mathbb{R}^d$ of distinct points in the Euclidean space, where $diag$ is the union of $(dn-d)$-dimensional subspaces $x_i = x_j$, $i \neq j$. Note that every point of $\mathbb{P}_d$ corresponds to some labelled non-degenerate configuration; so let us consider $\mathbb{P}_d$ as a configuration space.

A configuration $P \in \mathbb{P}_d$ is *ambiguous* if there are several Steiner trees for P. Ivanov and Tuzhilin proved [65] that the complement to the set of ambiguous configurations contains an open dense subset of $\mathbb{P}_2$. Edelsbrunner and Strelkova [36] asked whether the measure of ambiguous configurations is zero or not. We provide a positive answer by proving the following stronger statement.

Theorem 2.1.1 (Basok–Cherkashin–Rastegaev–Teplitskaya [5]). *Assume that $n \geq 4$. Then the set of planar ambiguous configurations in $\mathbb{P}_2$ has Hausdorff dimension $2n-1$.*

2.1.1 Topology and embedding class of a tree

For the sake of convenience and completeness, we would like to begin our discussion with a careful introduction of the concept of "topology" of a tree often used in the context of the Steiner problem. As it is usually done in the literature (see, for instance [55, 63]), we define the *topology* of a Steiner tree

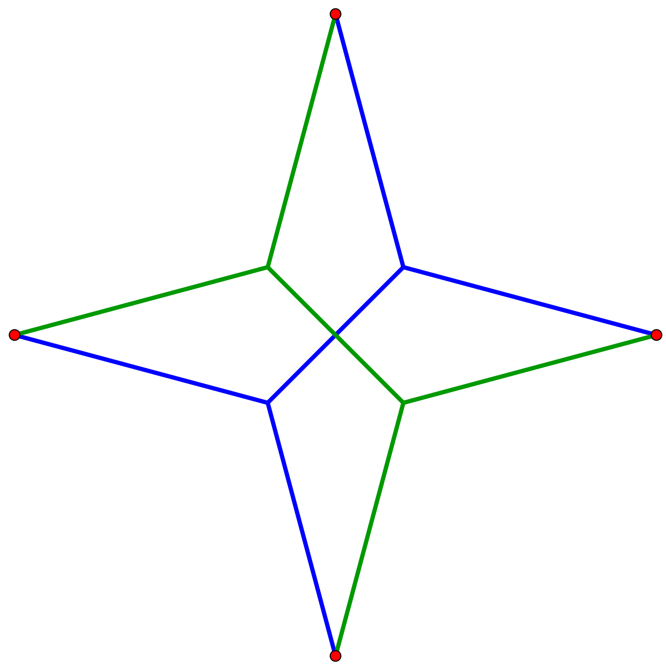


Figure 2.1: An example of non-unique solution. Terminals form a square

to be the corresponding abstract topological graph with labelled terminals and unlabelled Steiner points. Thus, a topology T is a topological space with a tree structure, and some vertices of T, including all its leaves and vertices of degree 2, are labelled. Moreover, we assume that all vertices of T have degrees at most 3, as it naturally holds for any Steiner tree.

Note that two trees embedded in a different (non-homotopic) way into the plane may have the same topology (on the other hand, for $d \geq 3$ homotopic equivalence means exactly that the topologies coincide). To distinguish non-homotopic embeddings Edelsbrunner and Strelkova [35] introduced another invariant way to describe the topological type of the tree which we call the "embedding class". Below we introduce several ways to define the embedding class of a tree commonly used. We include the proof of their equivalence for the sake of completeness.

Let T be a combinatorial tree, id est $T = (V, E)$ as a graph. Let $\vec{E}(T)$ denote the set of oriented edges of T (in particular, $|\vec{E}(T)| = 2|E(T)|$). Given an edge $\vec{e} \in \vec{E}$ denote by $o(\vec{e})$ the origin and by $t(\vec{e})$ the tail. Let us say that a bijection $\sigma : \vec{E}(T) \to \vec{E}(T)$ determines *a cyclic order around each vertex of T* if $o(\sigma(\vec{e})) = o(\vec{e})$ for any $\vec{e}$ and for any pair $(v, \vec{e})$ such that $o(\vec{e}) = v$ the set $\vec{e}, \sigma(\vec{e}), \sigma^2(\vec{e}), \dots$ is exactly the set of oriented edges emanating from v.

The following classical lemma defines the embedding class:

Lemma 2.1.1. *Let a positive integer n be fixed. The following three sets are in natural bijection:*

1. *The set PM_1 of pairs (T, σ), where T is a combinatorial tree with all vertices of degree at most 3, with n labelled vertices, including all leaves of T and vertices of degree 2, and $\sigma : \vec{E}(T) \to \vec{E}(T)$ is a bijection determining a cyclic order around each vertex.*

2. *The set PM_2 of pairs $(T, [f])$, where T is a combinatorial tree with all vertices of degree at most 3, with n labelled vertices, including all leaves of T and vertices of degree 2, f is a bijection between $\vec{E}(T)$ and the set ∂D of edges of the regular $|\vec{E}(T)|$-gon D (0-gon is assumed to be empty) oriented clockwise such that if $o(f(\vec{e}_1)) = t(f(\vec{e}_2))$, then $o(\vec{e}_1) = t(\vec{e}_2)$, and $[f]$ is the equivalence class of f with respect to the cyclic shift on ∂D.*

3. *The set PM_3 of pairs $(T,[\iota])$, where T is a topology with n labelled vertices, ι is some embedding of T into the plane and $[\iota]$ is the homotopy class of ι in the space of embeddings.*

Remark 2.1.1. *Each of these three sets can be considered as the set of plane maps or ribbon graphs with tree-like skeletons and some labelled vertices (see [81]).*

Proof. We construct a map $F_i : PM_i \to PM_{i+1}$ for $i = 1,2,3$ (where we compute indices $\mod 3$), and then show that the composition of these maps is identity.

Let $(T,\sigma) \in PM_1$ be given, let $N = |\vec{E}(T)|$. If $N = 0$, then we just take an empty D. Assume that $N > 0$. Define $\alpha : \vec{E}(T) \to \vec{E}(T)$ to be the involution reversing the orientation and set $\varphi = \alpha \circ \sigma$. It is easy to verify that φ is a cyclic permutation of $\vec{E}(T)$. Set $f : \vec{E}(T) \to \partial D$ to be any bijection which respects the cyclic order imposed by φ. It is easy to check that $(T,[f])$ belongs to PM_2, thus we get the map F_1.

Let $(T,[f]) \in PM_2$ be given. If T is one point, then we define ι arbitrary. Assume that T has at least two vertices, let $N = |\vec{E}(T)|$ and D be the regular N-gon. As above, let $\alpha : \vec{E}(T) \to \vec{E}(T)$ to be the involution reversing the orientation. Now, glue each edge $\vec{e} \in \partial D$ with $f \circ \alpha \circ f^{-1}(\vec{e})$ in opposite direction. It is straightforward to see that in this way we get an oriented surface S out of D, and a natural embedding ι of T into S. Computing the Euler characteristic we find out that S is a sphere and hence ι corresponds to a planar embedding of T. Set $F_2(T,[f]) = (T,[\iota])$ (where the topology on T comes from ι naturally).

Let $(T,[\iota]) \in PM_3$ be given. If T is one point, then we take σ to be the only map between empty sets. Assume that T has at least two vertices. Let v be a vertex and $\vec{E}_v(T)$ be the set of oriented edges emanating from v. Then given $\vec{e} \in \vec{E}_v(T)$ define $\sigma(\vec{e})$ to be first edge in $\vec{E}_v(T)$ coming after $\vec{e}$ when going around $\iota(v)$ in counterclockwise direction. Set $F_3(T,[\iota]) = (T,\sigma)$.

The fact that $F_3 \circ F_2 \circ F_1 = \mathrm{id}$ is a simple exercise which we leave to the reader. Note that given a labelled tree T the amount of all possible σ such that $(T,\sigma) \in PM_1$, or f such that $(T,[f]) \in PM_2$ is finite; this shows that F_1 is a bijection. On the other hand, the fact that the number of homotopy classes of embeddings ι for a given topology is finite is not obvious. Hence, at the moment we have only a right inverse for F_3. Let us sketch the construction the inverse map for F_2 to overcome this difficulty. Choose an embedding $\iota : T \to \mathbb{C}$ and consider the simply-connected surface $\widehat{\mathbb{C}} \setminus \iota(T)$, where $\widehat{\mathbb{C}}$ is the Riemann sphere. Let $\mathbb{D}$ be the unit disc and $\psi : \mathbb{D} \to \widehat{\mathbb{C}} \setminus \iota(T)$ be the uniformization map. One can show that ψ extends to the boundary of $\mathbb{D}$ in a unique way such that $\psi : \overline{\mathbb{D}} \to \widehat{\mathbb{C}}$ is continuous. Moreover, each point of $\iota(T)$ corresponds to several prime ends of the domain $\widehat{\mathbb{C}} \setminus \iota(T)$ (see [104, Chapter 2]); there are two prime ends for each inner point of an edge, and $\deg v$ prime ends for each vertex v. Let $v_1, \ldots, v_N$ be all the preimages of vertices of T on $\partial\mathbb{D}$, the count of the prime ends implies that $N = |\vec{E}(T)|$. Then $\overline{\mathbb{D}}$ together with these points has the combinatorics of the regular N-gon, whence we get the morphism f such that $(T,f) \in PM_2$. The fact that this construction inverses F_2 is straightforward. □

Note that the regular polygon D from the set PM_2 naturally corresponds to the outer face of the planar graph $\iota(T)$ for ι coming from PM_3. Using Lemma 2.1.1, we identify PM_1, PM_2 and PM_3, so that given, say $(T,\sigma) \in PM_1$ we will always assume that we are also given the corresponding $(T,[f]) \in PM_2$ and $(T,[\iota]) \in PM_3$ and will use the corresponding notation if it does not lead to a confusion.

Now, we introduce another (fourth) way to encode embeddings of a topological tree, which was originally used by Edelsbrunner and Strelkova.

Let $(T,\sigma) \in PM_1$ and $\varphi = \alpha \circ \sigma$, where $\alpha : \vec{E}(T) \to \vec{E}(T)$ is the involution reversing the orientation. Assume that T has n labels. Let $A = \{1,2,3,\ldots,n,b\}$ be the alphabet on $n+1$ letters,

n of them are numbers from 1 to n, and $(n+1)$-th is the special letter b. Let

$$\mathbf{C} = \left(\bigcup_{k\geq 0} A^k\right) /_{\text{cyclic shift}}$$

be the set of all words build from this alphabet considered up to the cyclic shift. Let $\vec{e} \in \vec{E}(T)$ be arbitrary and $N = |\vec{E}(T)|$. Then, given (T,σ), define the word $C(T,\sigma) \in \mathbf{C}$ by the following rule: fix a vector $\vec{e}_0 \in \vec{E}(T)$ and set $\vec{e}_i = \varphi(\vec{e}_{i-1})$, then define $C(T,\sigma) = a_0 a_1 \ldots a_{N-1}$, where a_i is the label of $o(\varphi(\vec{e}_i))$ if $o(\varphi(\vec{e}_i))$ is labelled, and $a_i = b$ else; if T consists of one vertex, then the word $C(T,\sigma)$ is the empty word.

Let $(T,[f]) \in PM_2$ correspond to (T,σ) and D be the regular N-gon. As D can be seen as the outer face of the planar graph $\iota(T)$, there is a many-to-one correspondence between the vertices of D and the vertices of T. Then the word $C(T,\sigma)$ is nothing but the list of vertices obtained by going along the boundary of D; each time we met a vertex walking along ∂D, we add its label to $C(T,\sigma)$, or the letter b if the vertex does not have a label. For example, we have $C = 1bb2b3bb4b$ and $C = 1b2bb3b4bb$ (and we could also write $C = b2bb3b4bb1$ in the latter case as we factorized by a cyclic shift) for the left and the right trees on the Fig. 2.1 respectively.

Lemma 2.1.2. *The morphism $(T,\sigma) \mapsto C(T,\sigma)$ is injective from the set of pairs (T,σ) to $\mathbf{C}$.*

Proof of Lemma 2.1.2. Let (T,σ) be given and $N = |\vec{E}(T)|$. Then the length of the word $C(T,\sigma)$ is N, hence C distinguishes pairs (T,σ) with different cardinality N of the set of edges of the tree. We will show that C distinguishes different pairs with the same N by induction. If $N = 0, 2, 4$, then there is nothing to prove, assume that $N > 4$ and $W = C(T,\sigma)$. We need to show that if $W = C(T_1,\sigma_1)$, then $(T,\sigma) = (T_1,\sigma_1)$. Define

$$I_1 = \{i \mid a_i \text{ occurs 1 time in } W\}$$
$$I_2 = \{i \mid a_i \text{ occurs 2 times in } W\}.$$

We clearly have a bijection between the labels $\{a_i \mid i \in I_1\}$ and $\{a_i \mid i \in I_2\}$ and the vertices of degree 1 and 2 in T respectively, and the same for T_1. Assume that we can find $i \in I_1$ such that $i+1 \in I_2$ (here $N+1 = 1$). Then consider the word W' obtained from W by removing a_i and a_{i+1}. Then $W' = C(T',\sigma')$, where T' is obtained from T by removing the edge $a_i a_{i+1}$ and keeping all labels, and σ' is computed from σ in the natural way (note that T has at least one edge since we assume that $N > 4$). In the same time, $W' = C(T_1',\sigma_1')$, where (T_1',σ_1') is obtained from (T_1,σ_1) in the same procedure. By the induction hypothesis $(T',\sigma') = (T_1',\sigma_1')$. From here, it is easy to see that $(T,\sigma) = (T_1,\sigma_1)$.

Assume now that for any $i \in I_1$ we have $i+1 \notin I_2$. It follows that one can find $i \in I_1$ such that $i+2 \in I_1$ also. Consider the word W' obtained from W by removing a_{i-1}, a_i, a_{i+1} and a_{i+3}. This word corresponds to $C(T',\sigma')$, where T' is obtained from T by removing two vertices a_i and a_{i+2} and labelling their common parent by a_{i+2} (note that their parent must have degree 3). Note that T' has at least one edge since $N > 4$. Doing the same with T_1 we again get two pairs (T',σ') and (T_1',σ_1') such that $W' = C(T',\sigma') = C(T_1',\sigma_1')$, which implies that $(T',\sigma') = (T_1',\sigma_1')$ by the induction and, eventually, $(T,\sigma) = (T',\sigma')$. $\square$

Starting from now we will call the word $C(T,\sigma)$ an *embedding class*. Using Lemma 2.1.1 and Lemma 2.1.2 we will feel free to identify the embedding class with the homotopy class of embeddings defined in several ways presented in aforementioned lemmas.

2.1.2 Connectedness in $\mathbb{P}_d$

Let us return to our analysis of Steiner trees. We say that a topology T of a tree S is *full* if the corresponding tree is full. Further, let us call a topology T *realizable* for a configuration $P \in \mathbb{P}_d$ if there exists such a locally minimal tree $S(P)$ with topology T; we will denote this tree by $S_T(P)$.

Proposition 2.1.1 (Melzak, [91]). *If a topology T is realizable for $P \in \mathbb{P}_2$ then the realization $S_T(P)$ is unique.*

Proposition 2.1.1 shows that $S_T(P)$ is uniquely defined. Moreover one can construct (or show that it is impossible) $S_T(P)$ in a linear time $O(n)$, see [62]. However, we rarely know a priori, which topology gives a Steiner tree. Although the number of possible topologies for an n points configuration is finite, checking all of them may consume a lot of time, since this number of topologies grows very fast with n, see [55, 63]. Indeed, the Steiner tree problem is NP-complete [53].

We need the following generalization of Proposition 2.1.1. For a full topology T define $D(T)$ as the set of topologies that can be obtained from T by shrinking some edges connecting a terminal with a Steiner point (these edges should have pairwise different ends).

Proposition 2.1.2 (Gilbert–Pollak [55], Hwang–Weng [64]). *Let T be a full topology and $P \in \mathbb{P}_2$. Consider the function $L(y_1, \dots, y_{n-2}) : (\mathbb{R}^2)^{n-2} \to \mathbb{R}$ which is the length of a tree on the vertex set $P \cup \{y_1, \dots, y_{n-2}\}$ with straight edges and topology T (we allow y_i coincide with terminals). Then L has a unique local minimum and so there is at most one realization with a topology from $D(T)$.*

A *generic topology* is a topology without terminals of degree 3.

Observation 2.1.1. *(i) Every generic topology R belongs to exactly one set $D(T)$, because the reverse procedure (replacing every vertex A of degree 2 in R on a Steiner point b and add edge bA) leads to a full topology T.*

(ii) Suppose that $\mathcal{S}t$ is the unique Steiner tree for some $P \in \mathbb{P}_2$ and has a generic topology $R \in D(T)$ for some full topology T. Then for some positive $\eta > 0$ and any other full topology T' the length of the realization from $D(T')$ exceeds $\mathcal{H}^1(\mathcal{S}t)$ by at least η. If one perturbs every terminal by at most $\eta/(2n)$, then by triangle inequality a perturbed configuration P' has a unique Steiner tree $\mathcal{S}t(P')$ and the topology of $\mathcal{S}t(P')$ belongs to $D(T)$.

(iii) Configurations $P \in \mathbb{P}_2$ for which there exists a locally minimal tree with non-generic topology have the Hausdorff dimension $2n-2$.

In this section we study the way realizations and minimal realizations of different embedding classes divide the configuration space.

A similar research topic appears in [36, 35], where the connectedness of some sets related to an embedding class EC, is studied. Let $\Omega(EC)$ be the subset of $\mathbb{P}_d$ consisting of all $P \in \mathbb{P}_d$ for which EC is realizable. Note that for every embedding class EC the set $\Omega(EC)$ is path-connected.

Theorem 2.1.2 (Edelsbrunner–Strelkova, [36, 35]). *Let EC be an embedding class. Then the subset of $\mathbb{P}_d$ for which the Steiner tree is unique and has the embedding class EC is path-connected.*

In the planar case they also obtained the following result.

Theorem 2.1.3 (Edelsbrunner–Strelkova, [36, 35]). *Let EC be a full embedding class. Then the subset of $\mathbb{P}_2$ for which the Steiner tree has the embedding class EC is path-connected.*

The second result of this chapter is the following.

Theorem 2.1.4 (Basok–Cherkashin–Rastegaev–Teplitskaya [5])**.** *The subset of $\mathbb{P}_d$ for which there is a unique Steiner tree is path-connected.*

The proof of Theorem 2.1.4 is constructive (modulo Theorem 2.1.2) and planar (again modulo Theorem 2.1.2). Moreover the embedding class of the Steiner tree is known at every point of a constructed path.

2.2 A universal Steiner tree

In this subsection we provide a construction of a unique Steiner tree with an infinite number of Steiner points. It appeared in [101] and then was simplified and improved in [20, 100].

Let S_∞ be an infinite tree with vertices y_0, y_1, $y_2, \dots$ and edges given by y_0y_1 and y_ky_{2k}, y_ky_{2k+1}, $k \geq 1$. Thus, S_∞ is an infinite binary tree with an additional vertex y_0 attached to the common parent y_1 of all other vertices $y_k, k \geq 2$. The goal of the mentioned papers is to embed S_∞ in the plane in such a way that the image of each finite subtree of S_∞ will be the unique Steiner tree for the set of its vertices having degree 1 or 2. We define the embedding below by specifying the positions of $y_0, y_1, y_2, \dots$ on the plane.

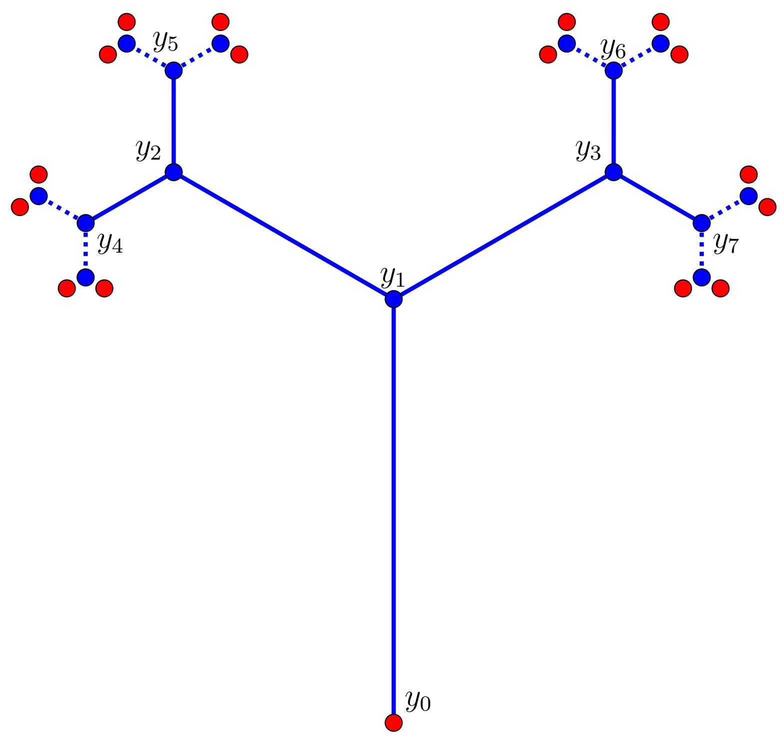


Figure 2.2: The first three levels in the construction of Σ_∞. The set Σ_3 is thick blue.

Let $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of positive real numbers. Define an embedding $\Sigma(\Lambda)$ of S_∞ as a rooted binary tree with the root $y_0 = (0, 0)$ the first descendant $y_1 = (1, 0)$ and the ratio between edges of $(i+1)$-th and i-th levels being λ_i. For small enough $\{\lambda_i\}$ the set $\Sigma(\Lambda)$ is a tree, see Fig. 2.2.

Let $A_\infty(\Lambda)$ be the union of the set of all leaves (limit points) of $\Sigma(\Lambda)$ and $\{y_0\}$.

Theorem 2.2.1 (Cherkashin–Teplitskaya, [20]). *A binary tree $\Sigma(\Lambda)$ is a Steiner tree for $A_\infty(\Lambda)$ provided that $\lambda_i = \lambda < 1/300$.*

Very recently Theorem 2.2.1 was significantly improved.

Theorem 2.2.2 (Paolini–Stepanov, [100]). *A binary tree $\Sigma(\Lambda)$ is a* ***unique*** *Steiner tree for $A_\infty(\Lambda)$ provided that $\lambda_i = \lambda < 1/25$.*

Proof of Theorem 2.2.1. Let the values of $\lambda < \frac{1}{300}$, $\varepsilon = \frac{\lambda^2}{1-\lambda}$ be fixed during the proof. The following auxiliary constructions are drawn in Figure 2.3. Let $Y_1B_1C_1$ be an isosceles triangle with the Fermat–Torricelli point T_1, such that $|Y_1T_1| = 1$, $|T_1B_1| = |T_1C_1| = \lambda$; then, by the cosine rule $|Y_1B_1| = |Y_1C_1| = \sqrt{1+\lambda+\lambda^2}$ and $|B_1C_1| = \sqrt{3}\lambda$. Analogously, let $Y_2B_2C_2$ be an isosceles triangle with the Fermat–Torricelli point T_2, such that $|Y_2T_2| = 1/4$, $|T_2B_2| = |T_2C_2| = \lambda$; then $|Y_2B_2| = |Y_2C_2| = \sqrt{1/16+\lambda/4+\lambda^2}$ and $|B_2C_2| = \sqrt{3}\lambda$.

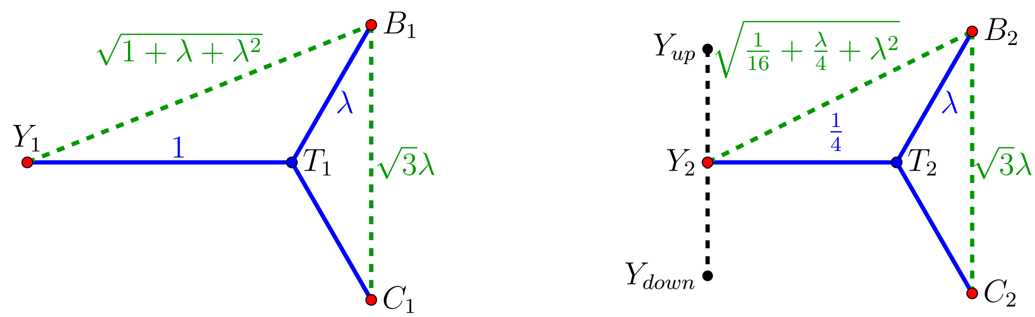


Figure 2.3: The construction of triangles in lemmas.

Let $b_i \subset B_\varepsilon(B_i)$ and $c_i \subset B_\varepsilon(C_i)$ be symmetric sets with respect to the axis of symmetry l_i of $Y_iB_iC_i$, where $i = 1, 2$. Finally, let Y_{up}, Y_{down} be such points that $Y_{up}Y_{down} \parallel B_2C_2$, $Y_2 \in [Y_{up}Y_{down}]$ and $|Y_2Y_{up}| = |Y_2Y_{down}| = 1/2$.

The following proposition is more-or-less known (see, for instance, Lemma A.6 in [101]), but we prove is for the sake of completeness. Recall that a *regular tripod* is a union of three segments with a common end and pairwise angles equal to $2\pi/3$.

Proposition 2.2.1. *(i) For every $\mathcal{S} \in \mathbb{M}(\{Y_1\} \cup b_1 \cup c_1)$, the set $\mathcal{S} \setminus B_{10\varepsilon}(B_1) \setminus B_{10\varepsilon}(C_1)$ is a regular tripod.*

(ii) Every $\mathcal{S} \in \mathbb{M}([Y_{up}Y_{down}] \cup b_2 \cup c_2)$ is a regular tripod outside of $B_{10\varepsilon}(B_2) \cup B_{10\varepsilon}(C_2)$.

Proof. In this proof, $i \in \{1, 2\}$. Suppose that $\mathcal{S}$ intersects with every circle $\partial B_\rho(B_i)$ in at least 2 points for $\varepsilon \le \rho \le 10\varepsilon$ (see Fig. 2.4). Then, we may replace $\mathcal{S}$ with a shorter competitor, as follows. Put $\mathcal{S}_b = \mathcal{S} \cap B_\varepsilon(B_i)$. By the definition and the co-area inequality,

$$\mathcal{H}^1(\mathcal{S}) \ge \mathcal{H}^1(\mathcal{S}_b) + 2 \cdot 9\varepsilon + \mathcal{H}^1(\mathcal{S}_i),$$

where $\mathcal{S}_1 \in \mathbb{M}(\{Y_1\} \cup \partial B_{10\varepsilon}(B_1) \cup c_1)$, $\mathcal{S}_2 \in \mathbb{M}([Y_{up}Y_{down}] \cup \partial B_{10\varepsilon}(B_2) \cup c_2)$. Now, take $\mathcal{S}_i \cup \mathcal{S}_b \cup \partial B_\varepsilon(B_i) \cup \mathcal{R}_B$, where $\mathcal{R}_B$ is the radius connecting $\mathcal{S}_i$ with $\partial B_\varepsilon(B_i)$. The length of this competitor is

$$\mathcal{H}^1(\mathcal{S}_b) + 2\pi\varepsilon + 9\varepsilon + \mathcal{H}^1(\mathcal{S}_i),$$

which gives a contradiction since $2\pi < 9$. The symmetric construction shows that the situation where $\mathcal{S}$ intersects with every circle $\partial B_\rho(C_i)$ in at least 2 points for $\varepsilon \le \rho \le 10\varepsilon$ is also impossible.

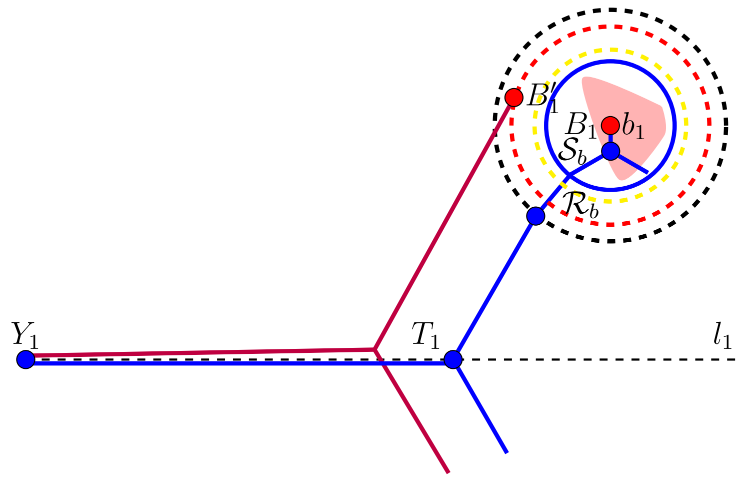


Figure 2.4: Picture of the proof of Proposition 1.

Thus, there are ρ_b, $\rho_c \in [\varepsilon, 10\varepsilon]$, such that $\mathcal{S} \cap \partial B_{\rho_b}(B_i)$ is a point B_i' and $\mathcal{S} \cap \partial B_{\rho_c}(C_i)$ is a point C_i'. Clearly, $\mathcal{S} = \mathcal{S}_i \cup \mathcal{S}_b \cup \mathcal{S}_c$, where $\mathcal{S}_b = \mathcal{S} \cap B_{\rho_b}(B_i)$, $\mathcal{S}_c = \mathcal{S} \cap B_{\rho_c}(C_i)$ and $\mathcal{S}_1 \in \mathbb{M}(\{Y_1\} \cup \{B_i'\} \cup \{C_i'\})$, $\mathcal{S}_2 \in \mathbb{M}([Y_{up}Y_{down}] \cup \{B_i'\} \cup \{C_i'\})$. Clearly $\mathcal{S}_i$ is a tripod or the union of two segments. We claim that $\mathcal{S}_i$ is a tripod. By the triangle inequality:

$$|\mathcal{H}^1([T_iB_i] \cup [T_iC_i]) - \mathcal{H}^1([T_iB_i'] \cup [T_iC_i'])| < 20\varepsilon. \tag{2.1}$$

Now, let us prove item (i). By (2.1), the length of the (non-regular) tripod $[T_1Y_1] \cup [T_1C_1'] \cup [T_1B_1']$ connecting Y_1, B_1' and C_1' is, at most, $1 + 2\lambda + 20\varepsilon$. For the same reason, the length of two segments is at least

$$\sqrt{1 + \lambda + \lambda^2} + \sqrt{3}\lambda - 30\varepsilon > 1 + \left(\frac{1}{2} + \sqrt{3}\right)\lambda - 30\varepsilon.$$

Recall that $\varepsilon = \frac{\lambda^2}{1-\lambda}$; it is straightforward to check that

$$1 + \left(\frac{1}{2} + \sqrt{3}\right)\lambda - 30\varepsilon > 1 + 2\lambda + 20\varepsilon$$

for $\lambda < 1/300$. Thus, we show that $\mathcal{S}_1$ contains a tripod connecting Y_1, B_1' and C_1'; by the minimality argument, it is regular.

Let us deal with item (ii). By (2.1), the length of the (non-regular) tripod $[T_2Y_2] \cup [T_2C_2'] \cup [T_2B_2']$ connecting Y_2, B_2' and C_2' is, at most, $1/4 + 2\lambda + 20\varepsilon$. Again, the two-segment construction has a length of at least

$$1/4 + \lambda/2 + \sqrt{3}\lambda - 30\varepsilon.$$

The rest of the calculations coincide with the first item. □

Lemma 2.2.1. *There exists $\mathcal{S} \in \mathbb{M}(\{Y_1\} \cup b_1 \cup c_1)$, which is symmetric with respect to l_1.*

Proof. Let F be a point at the ray $[Y_1T_1)$, such that $|Y_1F| = 1 + \frac{3}{2}\lambda$ (see Fig. 2.5), and denote by DEF the equilateral triangle, such that Y_1 is the middle of the segment $[DE]$ and B_1C_1 is parallel to DE. Consider segments $[Z_lZ_r] \subset [DF]$ and $[V_lV_r] \subset [EF]$, such that $|Z_lZ_r| = [V_lV_r] = \lambda$ and $Z := DF \cap T_1B_1, V := EF \cap T_1C_1$ are centers of the segments. Note that l_1 is a symmetry axis of DEF, and $[Z_lZ_r]$ and $[V_lV_r]$ are also symmetric with respect to l_1.

By Proposition 2.2.1(i), every minimal set $\mathcal{S}$ is a regular tripod $Y_1B_1'C_1'$ out of $B_{10\varepsilon}(B_1)\cup B_{10\varepsilon}(C_1)$. We claim that the tripod $Y_1B_1'C_1'$ intersects segments $[Z_lZ_r]$ and $[V_lV_r]$. Indeed, consider Cartesian coordinates in which $Y_1=(0,0)$, $B_1=(1+\lambda/2,\sqrt{3}\lambda/2)$ and $C_1=(1+\lambda/2,-\sqrt{3}\lambda/2)$. Then $Z=(1+3\lambda/8,3\sqrt{3}\lambda/8)$, $Z_l=(1+3\lambda/8-\sqrt{3}\lambda/4,3\sqrt{3}\lambda/8+\lambda/4)$, and $Z_r=(1+3\lambda/8+\sqrt{3}\lambda/4,3\sqrt{3}\lambda/8-\lambda/4)$. Since the center T_1' of $Y_1B_1'C_1'$ lies inside triangle $Y_1B_1'C_1'$, it has an x-coordinate smaller than the x-coordinate of B_1' and a y-coordinate smaller than the y-coordinate of B_1'.

We consider the following auxiliary data for the Steiner problem: $A_{mid}=[Z_lZ_r]\cup[V_lV_r]\cup\{Y_1\}$, $A_{up}=[Z_lZ_r]\cup b_1$, $A_{down}=[V_lV_r]\cup c_1$. By the results from [99], as mentioned in the introduction, every $\mathbb{M}(A_i)$ is not empty. Segments $[Z_lZ_r]$ and $[V_lV_r]$ split every $\mathcal{S}\in\mathbb{M}(\{Y_1\}\cup b_1\cup c_1)$ into three parts, connecting A_{mid}, A_{up}, and A_{down}, so

$$\mathcal{H}^1(\mathcal{S})\geq\mathcal{H}^1(\mathcal{S}_{mid})+\mathcal{H}^1(\mathcal{S}_{up})+\mathcal{H}^1(\mathcal{S}_{down}), \tag{2.2}$$

where $\mathcal{S}_i\in\mathbb{M}(A_i)$. We claim that the equality in (2.2) holds.

It is known (see the barycentric coordinate system) that the sum of distances from a point inside a closed equilateral triangle to the sides does not depend on a point. Thus, $\mathbb{M}(A_{mid})$ is a set of regular tripods, and each tripod is symmetric with respect to l_1. Moreover, for every point $x\in[Z_lZ_r]$, there is a unique regular tripod $\mathcal{S}_x\in\mathbb{M}(A_{mid})$, and $\mathcal{S}_x$ is orthogonal to $[Z_lZ_r]$ at x.

Now consider any $\mathcal{S}_{down}\in\mathbb{M}(A_{down})$. Let $\mathcal{S}_{up}$ be a set that is symmetric to $\mathcal{S}_{down}$ with respect to l_1; clearly, $\mathcal{S}_{up}\in\mathbb{M}(A_{up})$. For $x\in\mathcal{S}_{down}\cap[V_lV_r]$, the set $\mathcal{S}_x\cup\mathcal{S}_{up}\cup\mathcal{S}_{down}$ connects $\{Y_1\}\cup b_1\cup c_1$, and reaches the equality in (2.2), so $\mathcal{S}_x\cup\mathcal{S}_{up}\cup\mathcal{S}_{down}$ is a Steiner tree for $\{Y_1\}\cup b_1\cup c_1$. By the construction, it is symmetric with respect to l_1. $\square$

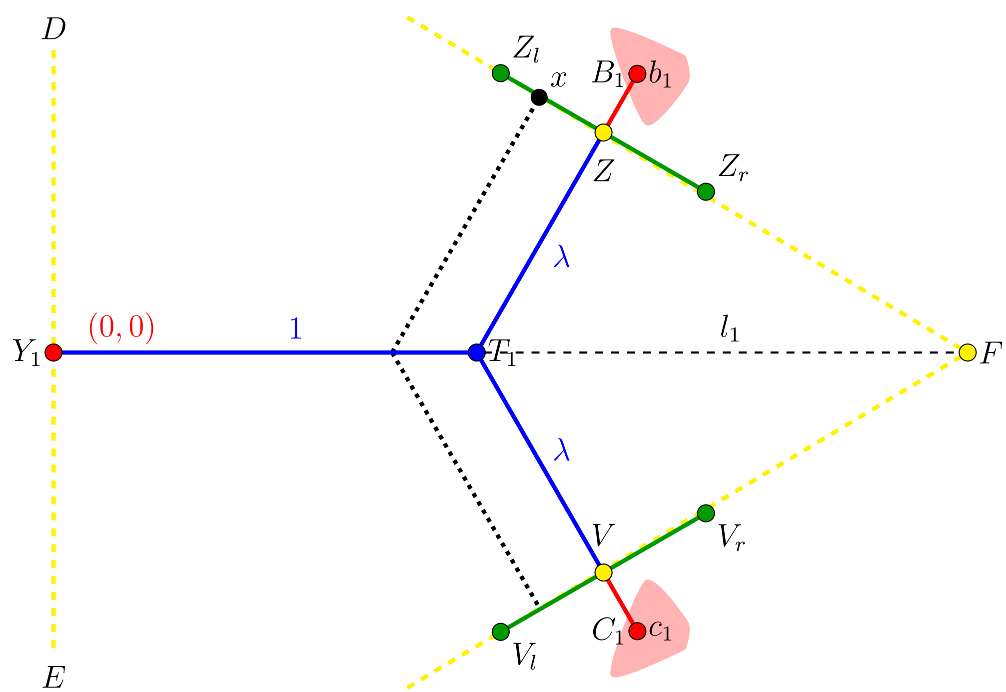


Figure 2.5: Picture of the proof of Lemma 1.

Lemma 2.2.2. *There exists $\mathcal{S}\in\mathbb{M}([Y_{up}Y_{down}]\cup b_2\cup c_2)$, which is symmetric with respect to l_2.*

Proof. By Proposition 2.2.1(ii), every Steiner tree $\mathcal{S}$ coincides with a regular tripod outside of $B_{10\varepsilon}(B_2)\cup B_{10\varepsilon}(C_2)$. Clearly, its longest segment is perpendicular to $Y_{up}Y_{down}$ (see Fig. 2.6). We want to show that it touches $Y_{up}Y_{down}$ in Y_2, i.e., one of the three segments is a subset of l_2. Assuming the contrary, suppose that $l_2 \cap \mathcal{S}$ is a point, denote it by L, and let $n \parallel B_2C_2$ be the line containing L. Then n divides $\mathcal{S}$ into three connected components; denote them by $\mathcal{S}_Y$, $\mathcal{S}_b$, and $\mathcal{S}_c$, respectively.

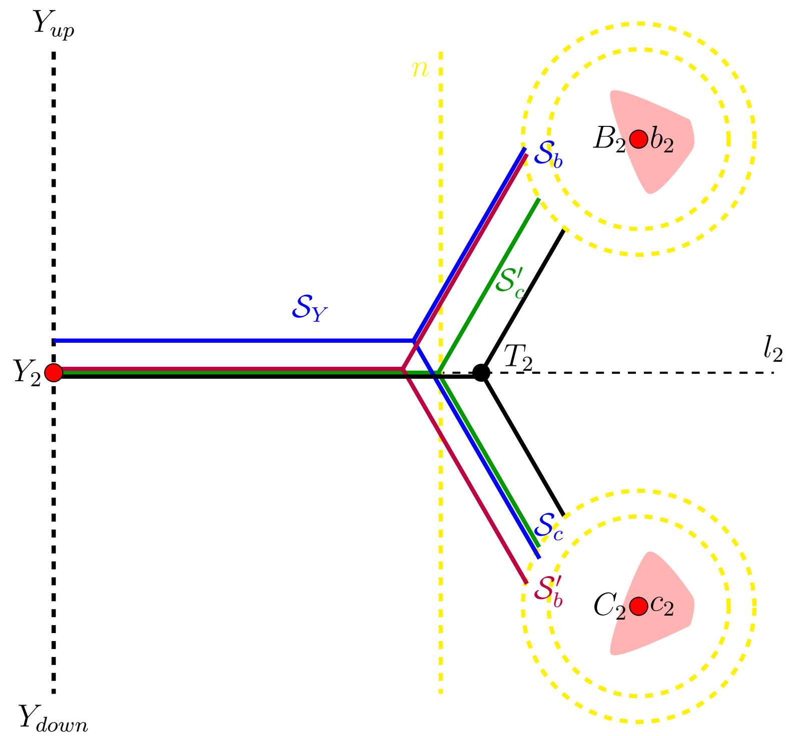


Figure 2.6: Picture of the proof of Lemma 2.

Without the loss of generality, L belongs to $\mathcal{S}_c$.

Let us construct competitors $\mathcal{S}_1$ and $\mathcal{S}_2$, connecting $[Y_{up}Y_{down}]$, b_2, and c_2. Let $\mathcal{S}_1 = [Y_2L] \cup \mathcal{S}_c \cup \mathcal{S}'_c$, where $\mathcal{S}'_c$ is a reflection of $\mathcal{S}_c$ with respect to l_2. Put $h := \operatorname{dist}(Y_2, \mathcal{S}_Y \cap [Y_{up}Y_{down}])$. Thus

$$\mathcal{H}^1(\mathcal{S}_1) = \mathcal{H}^1(\mathcal{S}_Y) - \sqrt{3}h + 2\mathcal{H}^1(\mathcal{S}_c).$$

Let $\mathcal{S}_2 := \mathcal{T} \cup \mathcal{S}_b \cup \mathcal{S}'_b$, where $\mathcal{S}'_b$ is a reflection of $\mathcal{S}_b$, with respect to l_2, and $\mathcal{T}$ is a regular tripod connecting Y_2 with $n \cap \mathcal{S}_b$ and $n \cap \mathcal{S}_{b'}$. Thus,

$$\mathcal{H}^1(\mathcal{S}_2) = \mathcal{H}^1(\mathcal{S}_Y) + \sqrt{3}h + 2\mathcal{H}^1(\mathcal{S}_b).$$

Since $\mathcal{S}$ is a Steiner tree, one has $\mathcal{H}^1(\mathcal{S}) \le \mathcal{H}^1(\mathcal{S}_1)$, $\mathcal{H}^1(\mathcal{S}) \le \mathcal{H}^1(\mathcal{S}_2)$ and clearly $\mathcal{H}^1(\mathcal{S}) = \frac{\mathcal{H}^1(\mathcal{S}_1)+\mathcal{H}^1(\mathcal{S}_2)}{2}$. Then $\mathcal{H}^1(\mathcal{S}) = \mathcal{H}^1(\mathcal{S}_1) = \mathcal{H}^1(\mathcal{S}_2)$, and so $\mathcal{S}_1, \mathcal{S}_2$ belong to $\mathbb{M}([Y_{up}Y_{down}] \cup b_2 \cup c_2)$. As $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric with respect to l_2, the statement is proven. □

Now we are ready to prove Theorem 2.2.1, i.e., to show that for $\lambda_j = \lambda < \frac{1}{300}$, the set $\Sigma(\Lambda)$ is a Steiner tree for the set of terminals A_∞.

Let b_1 and c_1 be the subsets of terminals that are descendants of y_2 and y_3, respectively. Since $\varepsilon = \lambda^2 + \lambda^3 + \dots + \lambda^k + \dots$, we have $b_i \subset B_\varepsilon(B_i)$, $c_i \subset B_\varepsilon(C_i)$. Applying Lemma 2.2.1 to $Y_1 = y_0$, $B_1 = y_2$, $C_1 = y_3$, b_1, and c_1, we show that there is a Steiner tree for A_∞, which is symmetric with respect to the line (y_0y_1).

Let $[Z_l Z_r]$ and $[V_l V_r]$ be the segments from the previous application of Lemma 2.2.1. Now define b_2 and c_2 as descendants of y_4 and y_5, respectively. Then, applying Lemma 2.2.2 to $[Y_{up}Y_{down}] = [Z_l Z_r]$, $B_2 = y_4$, $C_2 = y_5$, b_2, and c_2 (these data are similar to those required with the scale factor λ), we show that there is a Steiner tree containing $[y_0 y_1]$ and branching at y_1 (because y_1 belongs to the axis of the symmetries of b and c).

Since λ_i is constant, the upper and lower components of $\Sigma(\Lambda) \setminus [y_0 y_1]$ are similar (with the scale factor λ) to $\Sigma(\Lambda)$. Thus, the second application of Lemmas 2.2.1 and 2.2.2 shows that there is a Steiner tree containing $[y_0 y_1] \cup [y_1 y_2] \cup [y_1 y_3]$. This procedure recovers $\Sigma(\Lambda)$ step by step; so after the k-th step, we know that the length of every Steiner tree for A_∞ is at least

$$\sum_{i=0}^{k-1} (2\lambda)^i.$$

Thus, the length of every Steiner tree for A_∞ is at least the length of $\Sigma(\Lambda)$, which implies $\Sigma(\Lambda) \in \mathbb{M}(A_\infty)$. □

Now fix some $\lambda < \frac{1}{25}$, put $\Sigma_\infty = \Sigma(\Lambda)$ and define Σ_k as the union of the edges from the first k levels of Σ_∞. We will use the following corollary of Theorem 2.2.2, which explains why a full binary Steiner tree is universal, i.e. it contains a subtree with a given combinatorial structure.

Corollary 2.2.1. *In the conditions of Theorem 2.2.2 each connected closed subset S of Σ_∞ contained in Σ_k for some k has a natural tree structure. Moreover, every such an S is the unique Steiner tree for any set P containing the set of the vertices with the degree* 1 *and* 2 *of S.*

Proof. Let $S \subset \Sigma_\infty$ and $P \subset S$ satisfy the conditions of the corollary. The fact that S is a tree is straightforward. Let $S' \neq S$ be any Steiner tree for S and assume that $\mathcal{H}^1(S') \leq \mathcal{H}^1(S)$. Then it is clear that $\mathcal{H}^1((\Sigma_k \setminus S) \cup S') \leq \mathcal{H}^1(\Sigma_k)$, but on the other hand $\{y_0\} \cup A_k \subset (\Sigma_k \setminus S) \cup S'$, which contradicts to Theorem 2.2.2. □

2.3 Connectivity of the subset of $\mathbb{P}_d$ with a unique Steiner tree

2.3.1 Canonical realization of an embedding class

Using the construction of the tree Σ_∞ from the previous section we define the *canonical realization* tree $\mathcal{S}t_{EC}$ for any embedding class EC. Note that a canonical realization is planar.

Fix a topological tree S with the embedding class EC and pick some vertex v of S of degree one. Then identify S with the subtree of Σ_∞ by mapping v to the root y_0 of Σ_∞ and mapping all the other vertices following the steps of the breadth-first search algorithm started from v, where at every vertex of degree 2 of S we choose the left direction in Σ_∞ (i.e. map the only child to y_{2k} if the parent was mapped to y_k).

2.3.2 Proof of Theorem 2.1.4

In this section we prove Theorem 2.1.4. Note that there are no ambiguous configurations on at most 3 points, so Theorem 2.1.4 clearly holds for $n \leq 3$. Thus we have to deal with $n \geq 4$ to prove the theorem. First we deal with the planar case.

Let us denote by $\mathbb{P}^u_2 \subset \mathbb{P}_2$ the subset of configurations having a unique Steiner tree. Observe that, due to Theorem 2.1.2, Theorem 2.1.4 will follow from the following

Theorem 2.3.1. *Let T_1, T_2 be two embedding classes and $P_1, P_2 \in \mathbb{P}_2^u$ be the two configurations of terminal points of the corresponding canonical realizations $\mathcal{S}t_{EC_1}, \mathcal{S}t_{EC_2}$. Then there is a path in $\mathbb{P}_2^u$ connecting P_1 and P_2.*

Define the special (non-labelled) *all-left linear tree ALT* to be the path on n vertices starting at y_0 and turning left at every branching point of Σ_∞, i.e. ALT is the subgraph of Σ_∞ with the vertices $y_0, y_1, \dots, y_{2^k}, \dots, y_{2^{n-2}}$. To establish Theorem 2.3.1 we will show that any $\mathcal{S}t_{EC}$ corresponding to an embedding class with n terminal vertices can be continuously deformed to ALT inside the space of unique Steiner trees with some deformation preserving the labeling of terminal vertices. We construct such a deformation in several steps described below. In each step we continuously deform the set P of the terminal points of $\mathcal{S}t_{EC}$ to the set P' of the terminals of $\mathcal{S}t_{EC'}$ inside $\mathbb{P}_2^u$ by moving several points from P one by one.

We need a preliminary lemma.

Lemma 2.3.1. *Let EC be an embedding class with generic topology R. Suppose that $\mathcal{S}t_{EC}$ contains a leaf $B = y_{2k}$ adjacent to the terminal $A = y_k$ of degree 2. Then one can continuously move B to y_{2k+1} along a path γ in such a way that the whole configuration will remain in $\mathbb{P}_2^u$ at any point of γ and has the embedding class EC.*

Proof. We construct a desired part of γ explicitly (see Fig. 2.7). First move B into μ-neighborhood of A inside the segment $[y_{2k}y_k]$, where a small enough μ will be defined in the next paragraph; by Corollary 2.2.1 the Steiner tree is unique and has the embedding class EC at any configuration from this part of γ.

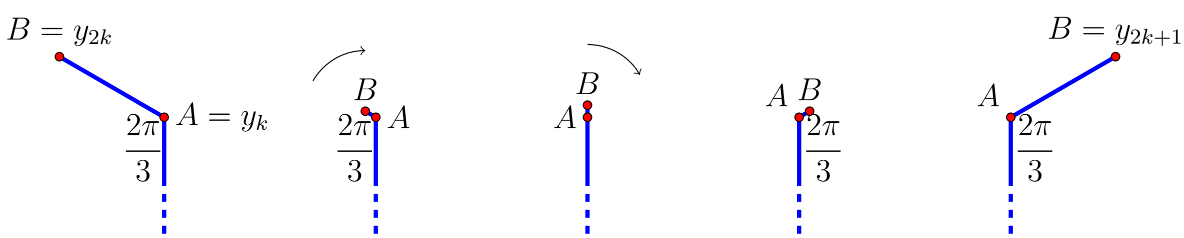


Figure 2.7: The construction of γ in Lemma 2.3.1.

Let $\bar{R}$ be the topology of $\mathcal{S}t_{EC} \setminus [AB]$; obviously $\bar{R}$ is also generic. Observation 2.1.1 (i) states that $\bar{R}$ lies in exactly one set $D(\bar{O})$, where $\bar{O}$ is a full topology. By Corollary 2.2.1 $\mathcal{S}t_{EC} \setminus [AB]$ is a unique Steiner tree with $n-1$ terminals, so by Observation 2.1.1 (ii) there is $\eta > 0$ such that for any other full topology R' the length of the realization from $D(R')$ exceeds $\mathcal{H}^1(\mathcal{S}t \setminus [AB])$ by at least η. Put $\mu = \eta/2$.

Now rotate B around A: let $B(\alpha)$, $\alpha \in [0, 2\pi/3]$ be a such point that $|BA| = |B(\alpha)A|$ and the clockwise-oriented angle $y_{[k/2]}AB(\alpha)$ is equal to $2\pi/3 + \alpha$. In particular $B(0) \in [Ay_{2k})$, $B(2\pi/3) \in [Ay_{2k+1})$.

Let $\mathcal{S}t(\alpha)$ be a Steiner tree for the terminals of $\mathcal{S}t \setminus [AB]$ and $B(\alpha)$. Then

$$\mathcal{H}^1(\mathcal{S}t(\alpha)) \le \mathcal{H}^1(\mathcal{S}t \setminus [AB] \cup [AB(\alpha)]) = \mathcal{H}^1(\mathcal{S}t \setminus [AB]) + \mu < \mathcal{H}^1(\mathcal{S}t \setminus [AB]) + \eta.$$

Let O be the full topology such that $R \in D(O)$. Then the topology of $\mathcal{S}t(\alpha)$ belongs to $D(O)$. By Proposition 2.1.2 $\mathcal{S}t(\alpha)$ is uniquely defined. The set $\mathcal{S}t \setminus [AB] \cup [AB(\alpha)]$ is locally minimal, and has the topology from $D(R)$, so it coincides with $\mathcal{S}t(\alpha)$. Thus not only the topology but the embedding class is preserved during this part of the path.

Finally, move B from $B(2\pi/3)$ to y_{2k} inside the segment $[y_{2k}y_k]$. □

Let us now fix an embedding class EC and construct the desired deformation of $\mathcal{S}t_{EC}$ to ALT inside the space of unique Steiner trees.

Step 1. Transform $\mathcal{S}t_{EC}$ into a full Steiner tree $\mathcal{S}t_{EC'}$ inside $\mathbb{P}_2^u$. To make such a transformation we need to move all terminal vertices of $\mathcal{S}t_{EC}$ of degree 2 or 3 to make them leaves.

Suppose first that $\mathcal{S}t_{EC}$ contains a terminal $A = y_j$ of degree 2. By the construction of $\mathcal{S}t_{EC}$, the vertex A is adjacent to vertices $B = y_{2j}$ and $C = y_{\lfloor j/2 \rfloor}$, which may be terminals or Steiner points. Move A towards y_{2j+1} along the edge $y_j y_{2j+1}$ of Σ_∞ until it hits y_{2j+1} (see. Fig. 2.8). Corollary 2.2.1

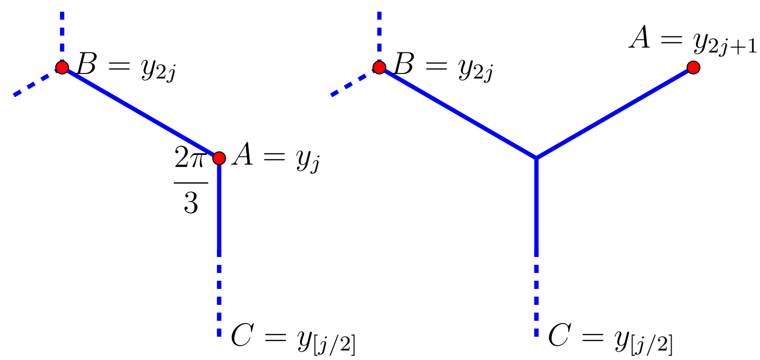


Figure 2.8: Elimination of points with degree 2 in $\mathcal{S}t_{EC}$

ensures that the obtained deformation of the set of terminal points lies inside $\mathbb{P}_2^u$. Applying this deformation to each terminal vertex of degree 2 one by one we eventually get rid of those.

Assume now that $\mathcal{S}t_{EC}$ has a terminal point of degree 3. Since $\mathcal{S}t_{EC}$ has no terminal of degree two and the number of Steiner points is at most the number of leafs minus two, one may move terminals of degree three one by one in the neighborhood of different leaves by a path in $\mathbb{P}_2^u$. From now on, the topology of a tree is generic.

Now consider any point A in an ε-neighborhood of a leaf B for some small ε. Then continue moving A while moving B simultaneously in the same direction until A reaches y_k and B reaches $B(\pi/3)$ (see Fig. 2.9). Now stop moving A, but rotate B around A until it hits the ray Ay_{2k+1}, then extend B to y_{2k+1} and A to y_{2k}. Now all our terminal points again belong to the set $\{y_0, y_1, \dots\}$ and the unique Steiner tree is given by the canonical realization $\mathcal{S}t_{EC'}$ for some new embedding class EC'. The fact that the set of terminal points was staying inside $\mathbb{P}_2^u$ while we were moving them follows from Corollary 2.2.1 and the proof of Lemma 2.3.1.

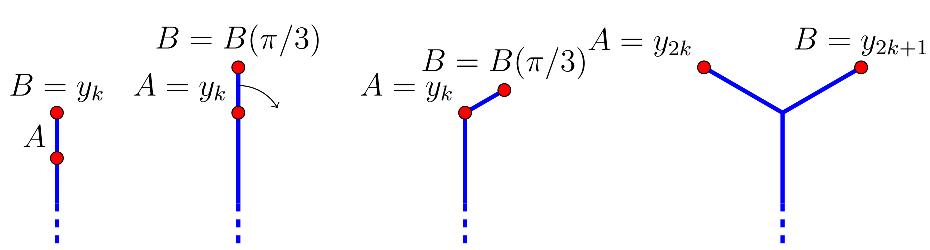


Figure 2.9: Elimination of points with degree 3 in $\mathcal{S}t_{EC}$

Note that $\mathcal{S}t_{EC'}$ still has no terminal points of degree 2 and has one less terminal point of degree 3 than $\mathcal{S}t_{EC}$. Hence we can do this procedure until we obtain a canonically realized full tree.

Step 2. Permute the labels of terminal points of $\mathcal{S}t_{EC}$ if necessary. Now we can assume that $\mathcal{S}t_{EC}$ is a full tree. By Theorem 2.1.2 one may put label 1 into the root by a path in $\mathbb{P}_2^u$ and do not touch the root of the tree later on. Let A and B be the two terminal points which we want to swap.

Since $\mathcal{S}t_{EC}$ is a full tree, it has exactly $n-2$ Steiner points. In particular, we can choose two Steiner points of $\mathcal{S}t_{EC}$ that are connected with two terminal points of $\mathcal{S}t_{EC}$; let y_k be the one of them which is not adjacent to the root of Σ_∞. Denote the terminals adjacent to y_k by $B = y_{2k}$ and $A = y_{2k+1}$.

We may swap A with any label. First swap A and B as shown at Fig. 2.10: move B into y_k and A in a small neighborhood of y_k, then turn and finally make a reverse procedure. By Lemma 2.3.1 the Steiner tree is unique during the middle part of this procedure; by Corollary 2.2.1 the Steiner tree is unique during other parts.

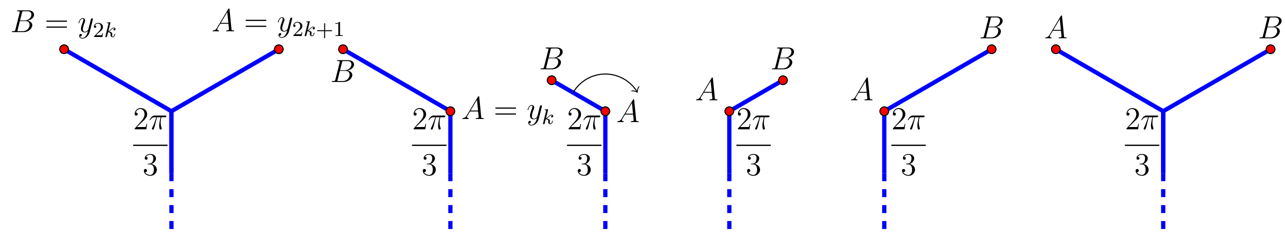


Figure 2.10: Swapping the labels of terminals connecting with a common branching point

Then swap A with any terminal $C \neq B$ of $\mathcal{S}t_i'$ (see Fig. 2.11). Start with the previous procedure and stop it at the point $B = B(\pi/3)$ (in the notation of Lemma 2.3.1). Then A moves inside the tree into a neighborhood of $C = y_l$ and B comes to y_k. We are going to apply Lemma 2.3.1 to A and C: move C to $C(\pi/3)$ and A to y_l. Then C rotates to y_{2l+1}, after that A moves to y_{2l}. Now the positions of A and C are symmetric so we may do the reverse procedure after swapping A and C.

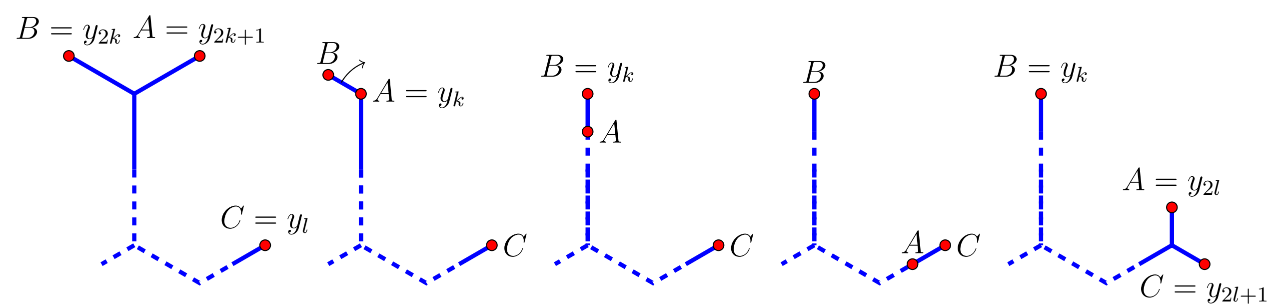


Figure 2.11: Swapping the labels of arbitrarily terminals

Finally to swap labels of arbitrary terminals C and D we swap $A = y_{2k+1}$ with C, $C = y_{2k+1}$ with D and $D = y_{2k+1}$ with A. Since the set of all transpositions spans the symmetric group we may construct a path in $\mathbb{P}_2^u$ connecting $\mathcal{S}t_{EC}$ and the same tree with an arbitrary permutation of its labels. Till the end of the section all trees are not labelled.

Step 3. Connect $\mathcal{S}t_i'$ with the all-left linear tree ALT by a path in $\mathbb{P}_2^u$. While there is a terminal point $A = y_j$ of $\mathcal{S}t_i'$ not belonging to ALT, consider such a vertex with the largest j. It implies that the degree of A is 1. Our aim is to move A inside Σ_∞ to the first vertex y_w of ALT which does not belong to $\mathcal{S}t_i'$.

Consider the case when A is adjacent to a branching point y_l then $j = 2l + 1$ and $B = y_{2l}$ is also a terminal of $\mathcal{S}t_i'$ because of the maximality of j. Move A into y_l and rotate B into $B(\pi/3)$ (in the notation of Lemma 2.3.1). Then A moves into the tree and B moves into y_l.

Now A is either inside the tree or A is a terminal connected with a vertex of degree 2. Move A into y_w, the only problem is that A cannot coincide with the terminal of degree 2. Movement through a terminal of degree 2 is depicted in Fig. 2.12.

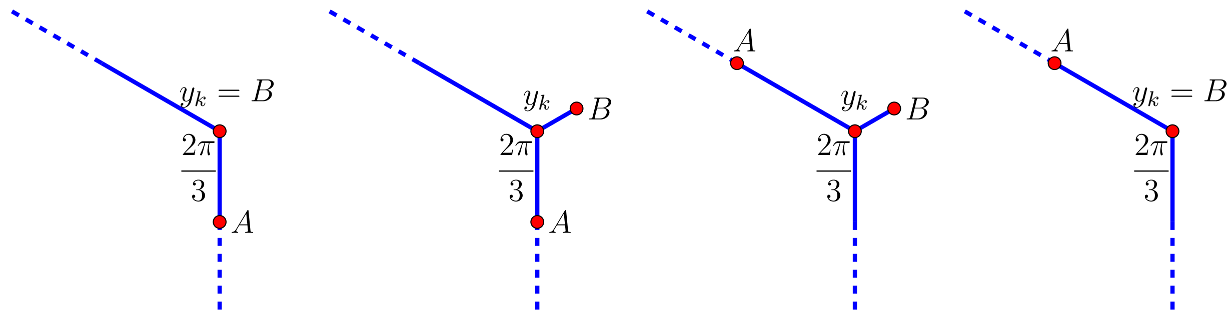


Figure 2.12: Movement through a terminal of degree 2

Finally all the vertices of $\mathcal{S}t_i'$ belong to ALT, so we are done. Since we connect $\mathcal{S}t_{EC_1}$ with ALT and $\mathcal{S}t_{EC_2}$ with ALT, the desired γ is constructed.

Proof of Theorem 2.1.4. Let $P_1, P_2 \in \mathbb{P}_d$ be configurations with unique Steiner trees $\mathcal{S}t(P_1)$ and $\mathcal{S}t(P_2)$ having embedding classes T_1 and T_2, respectively. By Theorem 2.1.2 there is a path γ_i between $\mathcal{S}t(P_i)$ and $\mathcal{S}t_{T_i}$ in $\mathbb{P}_d$ such that the Steiner tree is unique during γ_i. We have constructed the path γ between $\mathcal{S}t_{T_1}$ and $\mathcal{S}t_{T_2}$ in $\mathbb{P}_2 \subset \mathbb{P}_d$; the Steiner tree is also unique during γ. The gluing of γ_1, γ and γ_2^{-1} finishes the proof. $\square$

2.4 Proof of Theorem 2.1.1

This section is devoted to the proof of Theorem 2.1.1. The proof is using the theory of subanalytic sets, and for the sake of completeness we begin our exposition with a brief reminder of some definitions and facts from this theory.

2.4.1 Subanalytic subsets of a real analytic manifold

All the facts expounded in this section are well-known and may be skipped by an advanced reader. During our exposition we mostly follow Sections 2 and 3 from the paper [9].

Let M be a real analytic manifold and $\mathcal{O}_M$ denote the *sheaf* of real analytic functions on M, that is, for any open $U \subset M$ the set $\mathcal{O}_M(U)$ is the space of real analytic functions defined on U. We introduce the following definitions:

1. A subset $A \subset M$ is called an *analytic submanifold* if for each $p \in A$ there exists a neighborhood $U \subset M$ of it such that either $A \cap U = U$, or there exist a finite collection $f_1, f_2, \ldots, f_k \in \mathcal{O}_M(U)$

such that $A \cap U$ is the set of common zeros of $f_1, \dots, f_k$ and for any $x \in A \cap U$ the gradients $\nabla f_1(x), \dots, \nabla f_k(x)$ are linearly independent.

2. A subset $A \subset M$ is called *analytic* if for each $p \in M$ there exists a neighborhood U of p such that either $A \cap U = U$, or there exists a finite set of functions $f_1, \dots, f_k \in \mathcal{O}_M(U)$ such that $A \cap U$ is the set of common zeros of $f_1, \dots, f_k$. Note that we require this property for all $p \in M$, not only for $p \in A$.

3. A subset $A \subset M$ is called *semianalytic* if for each point $p \in M$ there exists a neighborhood $U \subset M$ and a finite number of subsets $A_{i,j} \subset U$ such that $A \cap U = \cup_i \cap_j A_{i,j}$ and each $A_{i,j}$ is of the form $\{f > 0\}$ or $\{f = 0\}$ for some $f \in \mathcal{O}_M(U)$. A semianalytic subset A is called *smooth* if it is an analytic submanifold.

The following lemma follows from [9, Proposition 2.10]:

Lemma 2.4.1. *Let M be a real analytic manifold, $A \subset M$ be a semianalytic subset and $p \in M$ be an arbitrary point. Then there exists a neighborhood $U \subset M$ of p and a finite collection of disjoint subsets $A_1, A_2, \dots, A_k \subset U$ such that*

1. *each of $A_1, \dots, A_k$ is a semianalytic subset of U and an analytic submanifold of M, and*
2. *$A \cap U$ is a disjoint union of $A_1, \dots, A_k$.*

Semianalytic sets admit many properties similar to those of semialgebraic sets (i.e. those given by polynomial inequalities), but the theories are not identical. An important difference is that projections of semianalytic sets are not necessarily semianalytic (see [9, Example 2.14]), while projections of semialgebraic sets are always semialgebraic. This motivates the following definition:

Definition 2.4.1. *A subset $X \subset M$ is called subanalytic if for any $p \in M$ there is a neighborhood U of p, an analytic manifold N and a relatively compact semianalytic subset $A \subset M \times N$ such that $X \cap U = \pi(A)$, where π is the projection on M.*

The following lemma follows immediately from this definition:

Lemma 2.4.2. *Let N, M be two analytic manifolds and $f : N \to M$ be an analytic map. Assume that $A \subset N$ is semianalytic and for each $p \in M$ there is a neighborhood U of it such that $f^{-1}(U) \cap A$ is relatively compact in N. Then $f(A)$ is a subanalytic subset of M.*

Proof. Let $\Gamma_f \subset M \times N$ be the graph of the mapping f. The graph is an analytic subset of $M \times N$ since f is analytic. Let $p \in M$ and $U \subset M$ be a semianalytic relatively compact neighborhood such that $f^{-1}(U) \cap A$ is relatively compact in N. Define $B = (U \times A) \cap \Gamma_f \subset M \times N$, then B is semianalytic and relatively compact. We have $f(A) \cap U = \pi(B)$, where π is the projection on M. Since p were arbitrary, we conclude that $f(A)$ is subanalytic. □

Subanalytic sets, although not being semianalytic in general, still have a lot of nice properties. Direct products, finite intersections and unions, closures, complements and, thus, interiors of subanalytic sets are still subanalytic (see [9, Chapter 3]). The following lemma describes a local structure of subanalytic sets, see [9, Lemma 3.4]):

Lemma 2.4.3. *Let N, M be analytic manifolds and $A \subset N \times M$ be a relatively compact semianalytic subset. Then there exists a finite collection of smooth connected semianalytic subsets $A_1, \dots, A_k \subset N \times M$ such that*

1. $A = \sqcup_{j=1}^{k} A_j$,

2. for any j the rank of $d\pi$ on $T_x A_j$ does not depend on $x \in A_j$.

From this lemma we get an immediate corollary:

Corollary 2.4.1. *Let M be an analytic manifold and $X \subset M$ be a subanalytic subset. Then there exists a countable collection of connected analytic submanifolds $X_1, X_2, X_3, \ldots$ of M such that $X = X_1 \cup X_2 \cup X_3 \cup \ldots$.*

Proof. Since the topology of M has a countable base, it is enough to prove the statement of the corollary locally. Passing to a neighborhood of some point if necessary we can assume that there is a relatively compact semianalytic subset $A \subset M \times N$ for some real analytic manifold N such that $X = \pi(A)$. Let $A_1, \ldots, A_k \subset M \times N$ be such as in Lemma 2.4.3. For each $j = 1, \ldots, k$ there is a countable collection of open subsets $U_{j1}, U_{j2}, \ldots$ of $M \times N$ covering A_j and such that $\pi(U_{ji} \cap A_j)$ is a connected analytic submanifold of M. Then we have

$$X = \pi(A) = \bigcup_{j=1}^{k} \bigcup_{i \geq 1} \pi(U_{ji} \cap A_j).$$

□

For future needs we now recall the notion of a fiber product. Let X, Y, U be some sets and $f : X \to U$, $g : Y \to U$ be some maps between these sets. The *fiber product* of X and Y over the base U is defined as

$$X \times_{f=g} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}. \tag{2.3}$$

Note that we have a natural projection $X \times_{f=g} Y \to U$ which sends (x, y) to $f(x)$.

Lemma 2.4.4. *Assume that M, N, U are real analytic manifolds and $f : M \to U$ and $g : N \to U$ are real analytic maps. Let $X \subset M$ and $Y \subset N$ be sub- or semianalytic subsets. Then $X \times_{f=g} Y$ is a sub- or semianalytic subset of $M \times_{f=g} N$ respectively.*

Proof. Follows immediately from definitions. Indeed, $X \times_{f=g} Y$ is the intersection of the sub- or semianalytic set $X \times Y$ and the subset $\{(x, y) \in M \times N \mid f(x) = g(y)\}$ inside $M \times N$, hence is sub- or semianalytic respectively. □

2.4.2 Proof of Theorem 2.1.1

In this section we prove Theorem 2.1.1. Recall that a number $n \geq 4$ of terminals is fixed and $\mathbb{P}_2$ is equal to $(\mathbb{R}^2)^n$ with diagonals removed. Denote the subset of ambiguous configurations by $\mathcal{A}$. Let also $\mathcal{A}_{\text{non-generic}}$ denote the set of all configurations admitting non-generic Steiner tree. Recall that $\dim \mathcal{A}_{\text{non-generic}} = 2n - 2$ by Observation 2.1.1.

We begin with the following

Lemma 2.4.5. *The Hausdorff dimension of $\mathcal{A}$ is at least $2n - 1$.*

Proof. Let $T_1, \ldots, T_N$ be all possible full topologies on n points and $V(T_j)$ denotes the set of vertices of T_j. Given a map $f : V(T_j) \to \mathbb{R}^2$, let $L(f)$ the total length of the segments connecting $f(V(T_j))$

accordingly to topology T_j (note that we do not claim any restrictions, in particular absence of cycles or local minimality), i.e.

$$L(f) = \sum_{vw \text{ is an edge of } T_j} |f(v) - f(w)|.$$

As it follows from Proposition 2.1.2, for any $P \in \mathbb{P}_2$ and $j = 1, \ldots, N$ there exists precisely one map $f : V(T_j) \to \mathbb{R}^2$ which maps the terminals of T_j to the points from P keeping the enumeration and which minimizes $L(f)$ among all such maps. Set $L_j(P) = L(f)$ in this case. Note that L_j is a continuous function on $\mathbb{P}_2$. Define

$$B_j = \{P \in \mathbb{P}_2 \ | \ L_j(P) < L_i(P) \ \forall \ i \neq j\}.$$

It follows that B_j's are open and disjoint sets. We also have $B_j \neq \varnothing$; indeed, by Corollary 2.2.1 each T_j is the topology of some Steiner tree which is unique. Note that $A = \mathbb{P}_2 \setminus \left(\cup_{j=1}^N B_j\right) \subset \mathcal{A} \cup \mathcal{A}_{\text{non-generic}}$ by Observation 2.1.1. The lemma now follows from $\dim \mathcal{A}_{\text{non-generic}} = \dim \mathbb{R}^{2n} \setminus \mathbb{P}_2 = 2n - 2$ and Lemma 2.4.6. $\square$

Lemma 2.4.6. *Assume that* $N \geq 2$, $m \geq 1$ *and* $B_1, \ldots, B_N \subset \mathbb{R}^m$ *are non-empty disjoint open sets. Put* $A = \mathbb{R}^m \setminus \left(\cup_{j=1}^N B_j\right)$. *Then* $\dim A \geq m - 1$, *where* $\dim$ *is the Hausdorff dimension.*

Proof. Let $P_1, P_2 \in \mathbb{R}^m$ be such that $P_1 \in B_1$ and $P_2 \in B_2$. Let l be the line passing through these points and $V \subset \mathbb{R}^m$ be the subspace of codimension 1 orthogonal to l. Let $\pi : \mathbb{R}^m \to V$ be the orthogonal projection. Note that $v \in \pi(A)$ if and only if the line $\pi^{-1}(v)$ intersects A. In particular, $v_0 = \pi(l) \in \pi(A)$ and moreover there exists $\varepsilon > 0$ such that $v \in \pi(A)$ if $|v - v_0| \leq \varepsilon$ since B_1, B_2 are open. It follows that $\pi(A)$ has a non-empty interior as a subset of V and $\dim \pi(A) = m - 1$. Since π is 1-Lipschitz, it implies that $\dim A \geq m - 1$. $\square$

The converse estimate $\dim \mathcal{A} \leq 2n-1$ is more involved and requires some additional constructions. Let T be some (not necessarily full or generic) topology with n terminals. Enumerate the Steiner points of T arbitrarily, let k be the total amount of them. Given $(P, q) = (p_1, \ldots, p_n, q_1, \ldots, q_k) \in \mathbb{P}_2 \times (\mathbb{R}^2)^k$, we can identify the corresponding points on the plane with the vertices of T following the enumeration and connect a pair of corresponding points by a straight segment for each edge of T. Thus, any such (P, q) defines a map from T to the plane. Let $\mathcal{R}_{\text{geoemb}}(T) \subset \mathbb{P}_2 \times (\mathbb{R}^2)^k$ be the following:

$$\mathcal{R}_{\text{geoemb}}(T) = \{(P, q) \in \mathbb{P}_2 \times (\mathbb{R}^2)^k \ | \ (P, q) \text{ defines an embedding of } T\}.$$

Obviously, $\mathcal{R}_{\text{geoemb}}(T)$ is an open subset of $\mathbb{P}_2 \times (\mathbb{R}^2)^k$. Let $L_T : \mathbb{P}_2 \times (\mathbb{R}^2)^k \to \mathbb{R}$ be the function that computes the length of the image of T; note that L_T is real analytic on $\mathcal{R}_{\text{geoemb}}(T)$ and continuous everywhere. Define

$$\mathcal{R}_{\text{lencrit}}(T) = \{(P, q) \in \mathcal{R}_{\text{geoemb}}(T) \ | \ \nabla_q L_T(P, q) = 0\},$$

where ∇_q is the gradient with respect to the variable q. Since L_T is real-analytic, $\mathcal{R}_{\text{lencrit}}(T)$ is an analytic subset of $\mathcal{R}_{\text{geoemb}}(T)$. We have the following

Lemma 2.4.7. *The following statements hold:*

1. *Let* $(P, q) \in \mathcal{R}_{\text{lencrit}}(T)$. *Consider the function* $L_T(P, \cdot)$ *as a continuous function from* $(\mathbb{R}^2)^k$ *to* $\mathbb{R}$. *Then* q *is the unique point of the global minimum of* $L_T(P, \cdot)$.

2. *Let $P_T : \mathcal{R}_{\mathrm{geoemb}}(T) \to \mathbb{P}_2$ be the projection. Then P_T restricted to $\mathcal{R}_{\mathrm{lencrit}}(T)$ is injective and the set $P_T(\mathcal{R}_{\mathrm{lencrit}}(T))$ is open in $\mathbb{P}_2$.*

Proof. Note that for any fixed P the function $L_T(P,q)$ tends to infinity as the distance between q and the origin tends to infinity. It follows that for any P there is a point $q(P) \in (\mathbb{R}^2)^k$ where $L_T(P,\cdot)$ attains its global minimum. By Proposition 2.1.2 such a point is always unique and there are no other local minima of $L_T(P,\cdot)$. In particular, the mapping $P \mapsto q(P)$ is a well-defined mapping $\mathbb{P}_2 \to (\mathbb{R}^2)^k$. From the continuity of L_T it follows that this mapping is continuous.

To prove the first item, it is enough to show that whenever $(P,q) \in \mathcal{R}_{\mathrm{lencrit}}(T)$, the value $L_T(P,q)$ is a local minimum of $L_T(P,\cdot)$. Let $v_1,\dots,v_n$ be the terminals of T, let $\tilde{T}_1,\dots,\tilde{T}_l$ be the connected components of $T \setminus \{v_1,\dots,v_n\}$ containing Steiner points and let T_i be the closure of $\tilde{T}_i$ in T. Then each T_i is a full topology (with terminals being a subset of terminals of T). Let $f : T \to \mathbb{R}^2$ be the embedding corresponding to (P,q), then it is easy to see that the differential condition $\nabla_q L_T(p,q) = 0$ is equivalent to the fact that $f(T_i)$ is a locally minimal tree, which implies the statement.

For the second item, consider the mapping $M : \mathbb{P}_2 \to \mathbb{P}_2 \times (\mathbb{R}^2)^k$ which sends P to $(P, q(P))$. Then M is continuous. But since $P_T(\mathcal{R}_{\mathrm{lencrit}}(T)) = M^{-1}(\mathcal{R}_{\mathrm{geoemb}}(T))$ and $\mathcal{R}_{\mathrm{geoemb}}(T)$ is open in $\mathbb{P}_2 \times (\mathbb{R}^2)^k$, we conclude that $\mathcal{R}_{\mathrm{lencrit}}(T)$ is open. □

Recall that $P_T : \mathcal{R}_{\mathrm{geoemb}}(T) \to \mathbb{P}_2$ is the projection. Given two topologies T_1, T_2 with n labelled vertices, define

$$\begin{aligned}\mathcal{A}_{T_1,T_2} = \{P \in P_{T_1}(\mathcal{R}_{\mathrm{lencrit}}(T_1)) \cap P_{T_2}(\mathcal{R}_{\mathrm{lencrit}}(T_2)) \ \mid \\ L_{T_1}(P,q_1) = L_{T_2}(P,q_2), \text{ where } (P,q_i) \in \mathcal{R}_{\mathrm{lencrit}}(T_i)\}.\end{aligned}$$

In the next lemma we will use the notion of a subanalytic subset introduced in Section 2.4.1.

Lemma 2.4.8. *Let $T_1 \neq T_2$ be two generic topologies with n terminals. Then there exists an open set $U \subset \mathbb{P}_2$ such that $\mathcal{A}_{T_1,T_2}$ is a subanalytic subset of U. In particular, $\mathcal{A}_{T_1,T_2}$ is a union of a countable collection of connected analytic submanifolds of U.*

Proof. Recall that for any T the map $P_T : \mathcal{R}_{\mathrm{geoemb}}(T) \to \mathbb{P}_2$ is the projection. Define $U = P_{T_1}(\mathcal{R}_{\mathrm{lencrit}}(T_1)) \cap P_{T_2}(\mathcal{R}_{\mathrm{lencrit}}(T_2))$. By Lemma 2.4.7 we have that U is an open subset of $\mathbb{P}_2$, and we have $\mathcal{A}_{T_1,T_2} \subset U$ by the definition of $\mathcal{A}_{T_1,T_2}$.

Introduce the temporary notation $\mathcal{R}_{\mathrm{lencrit}}(T_i)_U = \mathcal{R}_{\mathrm{lencrit}}(T_i) \cap P_{T_i}^{-1}(U)$ and

$$\mathcal{R}_{\mathrm{geoemb}}(T_i)_U = \mathcal{R}_{\mathrm{geoemb}}(T_i) \cap P_{T_i}^{-1}(U)$$

for simplicity. Recall the definition of a fiber product was introduced in (2.3). As we can see from the definition,

$$\mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{geoemb}}(T_2)_U \subset U \times (\mathbb{R}^2)^{k_1} \times (\mathbb{R}^2)^{k_2}$$

is an open subset, where k_i is the number of Steiner points of T_i and we identify $U \times (\mathbb{R}^2)^{k_1} \times (\mathbb{R}^2)^{k_2}$ with $(U \times (\mathbb{R}^2)^{k_1}) \times_{P_{T_1}=P_{T_2}} (U \times (\mathbb{R}^2)^{k_2})$. Therefore $\mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{geoemb}}(T_2)_U$ is a real analytic submanifold of $\mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times \mathcal{R}_{\mathrm{geoemb}}(T_2)_U$. The set $\mathcal{R}_{\mathrm{lencrit}}(T_i)_U$ is an analytic subset of $\mathcal{R}_{\mathrm{geoemb}}(T_i)_U$, hence by Lemma 2.4.4 $\mathcal{R}_{\mathrm{lencrit}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{lencrit}}(T_2)_U$ is an analytic subset of $\mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{geoemb}}(T_2)_U$.

Define now

$$\mathcal{R} = \{(P,q_1,q_2) \in \mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{geoemb}}(T_2)_U \ \mid \ L_{T_1}(P,q_1) = L_{T_2}(P,q_2)\}.$$

Then $\mathcal{R}$ is an analytic subset of $\mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{geoemb}}(T_2)_U$. Denote by π the natural projection $\pi : \mathcal{R}_{\mathrm{geoemb}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{geoemb}}(T_2)_U \to U$. Then we have

$$\mathcal{A}_{T_1,T_2} = \pi\Big(\mathcal{R} \cap \Big(\mathcal{R}_{\mathrm{lencrit}}(T_1)_U \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\mathrm{lencrit}}(T_2)_U\Big)\Big).$$

It follows from Lemma 2.4.2 that $\mathcal{A}_{T_1,T_2}$ is a subanalytic subset of U.

The last assertion of the lemma follows from Corollary 2.4.1. $\square$

Let us say that trees S_1 and S_2 are *codirected* at a terminal v is for a small enough neighborhood $U \ni v$ one has $S_1 \cap U = S_2 \cap U$.

Lemma 2.4.9. *Let $T_1 \neq T_2$ be two generic topologies with n terminals, and assume that $P \in \mathrm{Int}(\mathcal{A}_{T_1,T_2})$ (here* Int *stands for the interior in $\mathbb{P}_2$) and $S_1(P), S_2(P)$ are the images of T_1, T_2 on the plane. Then for each terminal v we have $\deg_{T_1} v = \deg_{T_2} v$ and the trees $S_1(P), S_2(P)$ are codirected at v.*

Proof. Given $P \in \mathrm{Int}\,\mathcal{A}_{T_1,T_2}$, denote by $q_1(P)$ and $q_2(P)$ the configurations of Steiner points such that we have $(P, q_i(P)) \in \mathcal{R}_{\mathrm{lencrit}}(T_i)$, $i = 1, 2$. Note that it is enough to prove the lemma for the points P from a dense subset of the interior, since $q_i(P)$ depends continuously on P (cf. the proof of Lemma 2.4.7). Recall that we denote by $P_{T_i} : \mathcal{R}_{\mathrm{geoemb}}(T_i) \to \mathbb{P}_2$ the projection and $P_{T_i}(\mathcal{R}_{\mathrm{lencrit}}(T_i))$ is open in $\mathbb{P}_2$; recall also that P_{T_i} restricted to $\mathcal{R}_{\mathrm{lencrit}}(T_i)$ is one-to-one by Lemma 2.4.7 and q_i is the inverse mapping. From Lemma 2.4.3 and the fact that P_{T_i} is one-to-one restricted to $\mathcal{R}_{\mathrm{lencrit}}(T_i)$ we deduce that q_i is differentiable on $P_{T_i}(\mathcal{R}_{\mathrm{lencrit}}(T_i))$ outside a subset which is nowhere dense in $P_{T_i}(\mathcal{R}_{\mathrm{lencrit}}(T_i))$. Thus, deforming P inside $\mathrm{Int}(\mathcal{A}_{T_1,T_2})$ a little bit we can achieve that that both q_1 and q_2 are differentiable in a neighborhood of P.

Further, we claim that deforming P a little bit more we can assume that for any terminal vertex of degree 2 in $S_i(P)$ the angle between the corresponding edges in $S_i(P)$ is not equal to π or $2\pi/3$. Indeed, let v be such a vertex in, say, $S_1(P)$. Then v divides $S_1(P)$ into two subtrees $S_1^+(P)$ and $S_1^-(P)$. Rotating $S_1^-(P)$ around v a little bit we can assure that the angle at v in $S_1(P)$ is not equal to π or $2\pi/3$, and q_1, q_2 are still differentiable in a neighborhood of P. Repeating this for all terminal vertices of degree 2 in $S_1(P)$ and $S_2(P)$ we get the result; note that directions of edges in $S_1(P), S_2(P)$ depend on P continuously, because $q_1(P), q_2(P)$ are continuous functions of P.

Given a point v from P and an oriented edge $\vec{e}$ of $S_i(P)$ emanating from v in $S_i(P)$ denote by $\eta_i(v, \vec{e}) \in \mathbb{R}^2$ the unit vector in $\mathbb{R}^2$ codirected with $\vec{e}$. Set

$$\eta_i(v) = \sum_{\vec{e} \in \vec{E}(S_i(P)):\ \ o(\vec{e})=v} \eta_i(v, \vec{e});$$

note that the sum consists of one or two elements for each v since all terminal vertices have degrees 1 or 2. A direct computation using angle condition at Steiner points of $S_i(P)$ shows that for any $\mu \in (\mathbb{R}^2)^n$ the derivative in the direction μ of $L_{T_i}(P, q_i(P))$ is given by

$$\frac{\partial}{\partial \mu} L_{T_i}(P, q_i(P)) = -\sum_{j=1}^{n} \eta_i(v_j) \cdot \mu_j,$$

where v_j is the j-th terminal. Since $L_{T_1}(\cdot, q_1(\cdot))$ and $L_{T_2}(\cdot, q_2(\cdot))$ are equal in a neighborhood of P, we conclude that $\eta_1(v_j) = \eta_2(v_j)$ for any $j = 1, \dots, n$. If $\deg_{S_1(P)} v_j = \deg_{S_2(P)} v_j = 1$, then this means that $S_1(P)$ and $S_2(P)$ are codirected at v_j. Assume that $\deg_{S_1(P)} v_j = 2$; then $|\eta_1(v_j)| \neq 1$

since the angle between the two edges emanating from v_j is not equal to $2\pi/3$ by our assumption. From this inequality and the equality $\eta_1(v_j) = \eta_2(v_j)$ we find out that $\deg_{S_2(P)} v_j = 2$ also. Further, we have $\eta_1(v_j) \neq 0$ since the angle between the two edges emanating from v_j is not equal to π by our assumption. As a consequence, there is only one unordered pair of unit vectors (μ_+, μ_-) such that $\eta_1(v_j) = \mu_+ + \mu_-$, hence the pair of edges emanating from v_j in both $S_1(P)$ and $S_2(P)$ must have these directions, so that $S_1(P)$ and $S_2(P)$ are codirected at v_j. We conclude that $S_1(P)$ and $S_2(P)$ are codirected at all terminals. □

Let us now formulate the following theorem by Oblakov:

Theorem 2.4.1 (Oblakov [96])**.** *Assume that S_1 and S_2 are two locally minimal trees connecting the same set of terminals $P \in \mathbb{P}_2$ and codirected at this set. Then S_1 and S_2 coincide.*

Given two generic topologies T_1 and T_2 with n terminals and $P \in \mathcal{A}_{T_1,T_2}$, denote by $S_1(P)$ and $S_2(P)$ the embeddings of T_1 and T_2 as in the lemma above. Define

$$\mathcal{A}^{\min}_{T_1,T_2} = \{P \in \mathcal{A}_{T_1,T_2} \mid S_1(P) \text{ and } S_2(P) \text{ are both locally minimal trees}\}.$$

We have the following

Corollary 2.4.2. *Let $T_1 \neq T_2$ be two generic topologies with n terminals. Then we have*

$$\mathrm{Int}(\mathcal{A}_{T_1,T_2}) \cap \mathcal{A}^{\min}_{T_1,T_2} = \varnothing,$$

where Int *stands for the interior in $\mathbb{P}_2$.*

Proof. Follows from Lemma 2.4.9 and Theorem 2.4.1. □

Lemma 2.4.10. *We have*

$$\mathcal{A} \subset \bigcup_{\substack{T_1 \neq T_2 \\ T_1,T_2 \text{ are generic}}} \mathcal{A}^{\min}_{T_1,T_2} \cup \{P \in \mathbb{P}_2 \mid \textit{there is a Steiner tree for } P \textit{ with a non-generic topology}\}.$$

Proof. Let $P \in \mathcal{A}$, then P is connected by two Steiner trees S_1 and S_2. If both S_1 and S_2 have the same topology T, then we get the contradiction with Lemma 2.4.7. Thus the topologies of S_1 and S_2 are different and the lemma follows. □

We can now prove Theorem 2.1.1:

Proof of Theorem 2.1.1. By Lemma 2.4.5 the dimension of the set $\mathcal{A}$ of ambiguous configurations is at least $2n-1$. Therefore, by Lemma 2.4.10 and Observation 2.1.1, (iv), we have

$$\dim \mathcal{A} \leq \max\{\dim \mathcal{A}^{\min}_{T_1,T_2} \mid T_1 \neq T_2,\ T_1, T_2 \text{ are generic}\}.$$

Let two generic topologies $T_1 \neq T_2$ be fixed. Let $P \in \mathcal{A}^{\min}_{T_1,T_2}$. By Lemma 2.4.8 we have $\mathcal{A}_{T_1,T_2} = X_1 \cup X_2 \cup \ldots$, where $X_1, X_2, \ldots$ are connected analytic submanifolds of an open subset $U \subset \mathbb{P}_2$. Therefore, by Corollary 2.4.2

$$\mathcal{A}^{\min}_{T_1,T_2} \subset \bigcup_{i\ :\ \dim X_i \leq 2n-1} X_i.$$

It follows that $\dim \mathcal{A}^{\min}_{T_1,T_2} \leq 2n-1$. Since T_1, T_2 were arbitrary, we conclude that $\dim \mathcal{A} \leq 2n-1$. □

Remark 2.4.1. *A reasonable question would be if the dimension of the whole $\mathcal{A}_{T_1,T_2}$ (not only $\mathcal{A}^{\min}_{T_1,T_2}$) is less or equal to $2n-1$. We claim that it can be proven in a similar way we prove it for $\mathcal{A}^{\min}_{T_1,T_2}$. Indeed, due to Lemma 2.4.8 it is enough to prove that the interior of $\mathcal{A}_{T_1,T_2}$ is empty. If $P \in \operatorname{Int} \mathcal{A}_{T_1,T_2}$ and $S_1(P)$ and $S_2(P)$ are the corresponding embeddings of T_1, T_2, then by Lemma 2.4.9 trees $S_1(P)$ and $S_2(P)$ are codirected. Our claim is now that Theorem 2.4.1 can still be applied in this case to say that $S_1(P)$ and $S_2(P)$ coincide. Note that $S_1(P)$ and $S_2(P)$ are not necessarily locally minimal networks as we allow them to have angles less than $\frac{2\pi}{3}$ at terminals of degree 2. Nevertheless, the proof of Theorem 2.4.1 given by Oblakov [96] still applies to this case.*

2.5 Steiner trees in real analytic Riemannian manifolds

A question on the uniqueness of Steiner trees in a Riemannian manifold was raised in [34].

We should not expect that Theorem 2.1.1 can be directly generalized to the case when $\mathbb{R}^2$ is replaced with an arbitrary manifold M (cf. Section 2.5.4). Nevertheless, if M is a real analytic manifold, then we still can expect that the set of ambiguous configurations of terminals either has a non-empty interior, or dimension strictly less than the set of all configurations of terminals.

The aim of this section is to build a similar framework to the one used in the proof of Theorem 2.1.1 in Section 2.4.2 in the case of arbitrary real analytic manifold M. Using this framework we reduce the alternative stated above to Conjecture 2.5.1 about analytic sets.

2.5.1 Realization space $\mathcal{R}_{\text{geoemb}}$ for an arbitrary metric space

Let us begin with rephrasing Problem 2.0.1 with an arbitrary proper metric space M instead of $\mathbb{R}^d$.

Problem 2.5.1. *Let M be a metric space. For a given finite set $Q = \{p_1, \dots, p_n\} \subset M$ find a connected set $\mathcal{S}t$ with minimal length (one-dimensional Hausdorff measure) containing Q.*

Due to the following theorem, solutions to Problem 2.5.1 still lie among geodesically embedded trees:

Theorem 2.5.1 (Paolini–Stepanov, [99])**.** *Assume that M is proper (i.e. all closed balls in M are compact) and pathwise connected. Then a solution to Problem 2.5.1 exists. Moreover, for any solution $\mathcal{S}t(Q)$ the following statements hold:*

(i) $\mathcal{S}t$ is compact;

(ii) $\mathcal{S}t$ contains no embedded loops (homeomorphic images of $\mathbb{S}^1$);

(iii) $\mathcal{S}t \setminus Q$ has a finite number of connected components, each component has strictly positive length, and the closure of each component is a finite geodesic embedded graph with endpoints in Q;

(iv) the closure of every connected component of $\mathcal{S}t \setminus Q$ is a topological tree with endpoints in Q, and all the branching points having finite degree.

This theorem motivates the following definition. Given a positive integer n and a topology T, define

$$\mathcal{R}_{\text{geoemb}}(T) = \{f : T \to M \mid f \text{ is an embedding which maps all edges to (shortest) geodesics}\}/_{\sim}, \tag{2.4}$$

where $f_1 \sim f_2$ if and only if $f_1(v) = f_2(v)$ for any labelled vertex v and $f_1(T) = f_2(T)$ as subsets of M. Note that this $\mathcal{R}_{\text{geoemb}}(T)$ is a straightforward generalization of $\mathcal{R}_{\text{geoemb}}(T)$ introduces in Section 2.4.2. Let $\mathcal{K}(M)$ be the set of all compact subsets of M endowed with the Hausdorff metric. Introduce two maps: first, $\mathcal{R}_{\text{geoemb}}(T) \to \mathcal{K}(M)$ which maps f to $f(T)$, second, $P_T : \mathcal{R}_{\text{geoemb}}(T) \to M^n$ which sends f to the collection $(f(v_1), \ldots, f(v_n))$ of images of n terminals. The topology on $\mathcal{R}_{\text{geoemb}}(T)$ is defined to be the pullback of the product topology under the map $\mathcal{R}_{\text{geoemb}}(T) \to \mathcal{K}(M) \times M^n$ given by the two maps above (the map P_T is added to keep track of the enumeration of terminals).

2.5.2 Manifold structure on $\mathcal{R}_{\text{geoemb}}$

Let us now assume that M is a connected real analytic manifold with a Riemannian metric d which depends analytically on the point of M. We define the intrinsic metric d_{in} on M as usual; note that (M, d_{in}) is a proper metric space. Given a point p, denote by $\exp_p$ the exponential map defined with respect to d; since we have not required (M, d_{in}) to be complete, $\exp_p$ is defined only for an open subset of the tangent space T_pM. Set

$$\widetilde{T_pM} = \{w \in T_pM \mid w \neq 0 \text{ and } \exp_p(w) \text{ is defined}\}.$$

Denote by TM the tangent bundle of the manifold M. Elements of TM are parameterized by pairs (p, w) where $p \in M$ and $w \in T_pM$. Let $\widetilde{TM} \subset TM$ be the union of $\widetilde{T_pM}$ over all $p \in M$. Then $\widetilde{TM}$ is an open subset of TM and $\exp : \widetilde{TM} \to TM$, given by $(p, w) \mapsto (\exp_p(w), \mathrm{d}\exp_p(w))$ on each fiber, is a real analytic map mapping $\widetilde{TM}$ onto its image in TM diffeomorphically (see [26, Section 8]). Moreover, this diffeomorphism is analytic at any point.

Let us show that for any topology T the set $\mathcal{R}_{\text{geoemb}}(T)$ has a natural structure of an analytic manifold. Let $e_1, \ldots, e_m$ be the edges of T and ϑ be an arbitrary orientation on edges of T. Define the map $\varphi_\vartheta : \mathcal{R}_{\text{geoemb}}(T) \to (\widetilde{TM})^m$ by

$$\varphi_\vartheta(f) = ((f(o(e_1)), w_1), \ldots, (f(o(e_m)), w_m)) \in (\widetilde{TM})^m,$$

where $o(e_k)$ is the origin of e_k oriented according to ϑ and w_k is the tangent vector at $o(e_k)$ such that the geodesic $f(e_k)$ is given by $\{\exp_{o(e_k)}(tw_k)\}_{0 \le t \le 1}$.

Lemma 2.5.1. *The following statements hold true:*

(i) For any orientation ϑ, the map φ_ϑ is a homeomorphism between $\mathcal{R}_{\text{geoemb}}(T)$ and a smooth real analytic submanifold $\mathcal{U}_\vartheta \subset (\widetilde{TM})^m$ of dimension $(m+1) \cdot \dim M$.

(ii) The morphism $P_T \circ \varphi_\vartheta^{-1} : \mathcal{U}_\vartheta \to M^n$ is analytic.

(iii) Given two orientations ϑ_1, ϑ_2, the map $\varphi_{\vartheta_2} \circ \varphi_{\vartheta_1}^{-1} : \mathcal{U}_{\vartheta_1} \to \mathcal{U}_{\vartheta_2}$ is analytic.

Proof. We first prove (i). Note that φ_ϑ is injective. Let us show now that $\mathcal{U}_\vartheta = f(\mathcal{R}_{\text{geoemb}}(T))$ is a real analytic submanifold. In fact we have

$$\mathcal{U}_\vartheta = \{((x_1, w_1), \ldots, (x_m, w_m)) \in (\widetilde{TM})^m \mid \exp_{x_i}(w_i) = x_j \text{ if } x_j = \mathrm{tail}(e_i),\ x_i = x_j \text{ if } o(e_i) = o(e_j)\}, \tag{2.5}$$

where $\mathrm{tail}(e_i)$ is the tail of e_i oriented according to ϑ. It follows that $\mathcal{U}_\vartheta$ is an analytic subset of $(\widetilde{TM})^m$. The smoothness easily follows from the fact that exp is a diffeomorphism.

Finally, we have that the inverse mapping $\varphi_\vartheta^{-1} : \mathcal{U}_\vartheta \to \mathcal{R}_{\text{geoemb}}(T)$ is continuous because exp is continuous. We conclude that φ_ϑ is continuous because $\mathcal{U}_\vartheta$ is locally compact.

The items (ii) and (iii) follow easily from the fact that exp is analytic. $\square$

From Lemma 2.5.1 we see that $\mathcal{R}_{\text{geoemb}}(T)$ has a natural structure of a real analytic manifold.

Lemma 2.5.2. *The map $P_T : \mathcal{R}_{\text{geoemb}}(T) \to M^n$ is analytic and $P_T(\mathcal{R}_{\text{geoemb}}(T))$ is open in M^n. The differential $\mathrm{d}P$ has the maximal rank at any point. In particular, the fiber of P is smooth and has the dimension $\dim \mathcal{R}_{\text{geoemb}}(T) - n \dim M$.*

Proof. Choose an arbitrary orientation ϑ on the edges of T having the property that any terminal is an origin of some edge. The lemma now follows from Lemma 2.5.1 and the description (2.5) of $\mathcal{U}_\vartheta$. □

2.5.3 The subvariety $\mathcal{R}_{\text{locmin}}$ of $\mathcal{R}_{\text{geoemb}}$

Define the length function $L_T : \mathcal{R}_{\text{geoemb}}(T) \to \mathbb{R}$ by

$$L_T(f) = \mathcal{H}^1(f(T)).$$

Let $\mathcal{R}_{\text{locmin}}(T) \subset \mathcal{R}_{\text{geoemb}}(T)$ be the subset given by

$$\mathcal{R}_{\text{locmin}}(T) = \{f \in \mathcal{R}_{\text{geoemb}}(T) \ \mid \ f \text{ is a local minimum of } L_T \text{ on the set } P_T^{-1}(P_T(f))\}.$$

Note that, given the orientation ϑ on the edges of T we have

$$L \circ \varphi_\vartheta^{-1}((x_1, w_1), \dots, (x_m, w_m)) = |w_1| + \cdots + |w_m|$$

(cf. Lemma 2.5.1), hence L_T is an analytic function on $\mathcal{R}_{\text{geoemb}}(T)$. Recall that due to Lemma 2.5.2, the fibers of P_T are smooth. Define the vertical gradient of the function L_T at a point $f \in \mathcal{R}_{\text{geoemb}}$ to be the restriction of ∇L_T to the tangent space to the fiber $P_T^{-1}(P_T(f))$ of P_T over f. Define

$$\mathcal{R}_{\text{lencrit}}(T) = \{f \in \mathcal{R}_{\text{geoemb}} \ \mid \ \text{vertical gradient of } L_T \text{ at } f \text{ is zero}\}.$$

Clearly, $\mathcal{R}_{\text{locmin}}(T) \subset \mathcal{R}_{\text{lencrit}}(T)$; note that if $M = \mathbb{R}^2$ with the Euclidean metric, then the converse inclusion is also true, but in general we do not have equality between these two sets. We expect nevertheless that $\mathcal{R}_{\text{locmin}}(T)$ is a subanalytic subset of $\mathcal{R}_{\text{geoemb}}(T)$. By Lemma 2.5.2, this statement would immediately follow from the following assertion:

Conjecture 2.5.1. *Assume that $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^k$ are open subsets, $f : U \times V \to \mathbb{R}$ is real analytic. Put*

$$A = \{(x, y) \in U \times V \ \mid \ x \text{ is a local minimum for } f(\cdot, y) \text{ on } U, \text{ if } y \text{ is fixed}\}.$$

Then A is a semianalytic subset of $U \times V$.

Assume that for any T the set $\mathcal{R}_{\text{locmin}}(T)$ is subanalytic for a moment. In this case we immediately have the following

Proposition 2.5.1. *Let T_1, T_2 be two topologies with n terminals and assume that*

$$\mathcal{A}_{T_1,T_2} = \{P \in M^n \ \mid \ \exists f_1 \in \mathcal{R}_{\text{locmin}}(T_1),\ f_2 \in \mathcal{R}_{\text{locmin}}(T_2) \ :$$

$$P_{T_1}(f_1) = P_{T_2}(f_2) = P \text{ and } L_{T_1}(f_1) = L_{T_2}(f_2)\},$$

Then either $\mathcal{A}_{T_1,T_2}$ has a non-empty interior, or the Hausdorff dimension of $\mathcal{A}_{T_1,T_2}$ is strictly less than $n \dim M$.

Note that both alternatives in Proposition 2.5.1 can occur, see Section 2.5.4.

Proof. The proof essentially follows proof of Lemma 2.4.8. We consider the fiber product

$$\mathcal{R}_{\text{locmin}}(T_1) \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\text{locmin}}(T_2)$$

which is subanalytic due to Lemma 2.4.4. Inside $\mathcal{R}_{\text{locmin}}(T_1) \times_P \mathcal{R}_{\text{locmin}}(T_2)$ we consider the set $\tilde{\mathcal{A}}_{T_1,T_2}$ cut out by the equation $L_{T_1}(f_1) = L_{T_2}(f_2)$. Then $\tilde{\mathcal{A}}_{T_1,T_2}$ is subanalytic and $\mathcal{A}_{T_1,T_2} = P(\tilde{\mathcal{A}}_{T_1,T_2})$, where $P = P_{T_1} : \mathcal{R}_{\text{locmin}}(T_1) \times_{P_{T_1}=P_{T_2}} \mathcal{R}_{\text{locmin}}(T_2) \to M^n$ is the projection. The proposition now follows from Corollary 2.4.1. □

Let us finalize this section with a short discussion on Conjecture 2.5.1. Note that if $k = 0$ so that V is just a point, then the statement in the conjecture is straightforward (see also [41] for a much more general statement). Indeed, define first

$$Z = \{x \in U \mid \nabla f(x) = 0\},$$

Clearly, $A \subset Z$. Using [9, Proposition 2.10] we find that each $(x, y) \in U$ has a neighborhood U' such that $Z \cap U' = Z_1 \cup \cdots \cup Z_k$ where each Z_j is a connected semianalytic subset of U' which is also an analytic submanifold of U. In particular, f restricted to each Z_j is a constant, say, C_j. Define

$$B_j = \overline{\{x \in U \mid f(x) < C_j\}}.$$

Then B_j is a semianalytic subset (as the closure of a semianalytic subset is semianalytic [9, Proposition 2.10]). We have

$$A \cap U' = \bigcup_{j=1}^{k} (Z_j \setminus B_j)\,.$$

Since x was chosen arbitrarily, this proves that A is semianalytic.

With some more involved arguments using Weierstrass preparation theorem we can prove the conjecture when $\dim U = 1$ and V is arbitrary, but the general case remains unclear to us.

2.5.4 Example of $\mathcal{A}_{T_1,T_2}$ with non-empty interior

In this subsection we construct an example of a Riemann surface M with a locally flat metric such that there exist two topologies T_1, T_2 on 8 terminals for which $\mathcal{A}_{T_1,T_2}$ has non-empty interior (see Proposition 2.5.1 for the definition of $\mathcal{A}_{T_1,T_2}$).

Let T_1 and T_2 be the two topologies introduced by Ivanov and Tuzhilin in [65, Fig. 1], see Figure 2.13. Let T_1 be the topology of the tree drawn by solid lines for certainty. We fix a set of terminals $x = (x_1, \ldots, x_8) \in \mathbb{P}_2$ and fix *immersions* $f_i : T_i \to \mathbb{R}^2$ into plane as on Figure 2.13. Following [65] we have the following

Lemma 2.5.3. *For any configuration $(\tilde{x}_1, \ldots, \tilde{x}_8)$ sufficiently close to $(x_1, \ldots, x_8)$ the corresponding immersions $\tilde{f}_1, \tilde{f}_2$ of the topologies T_1, T_2 realizing them as locally minimal trees are codirected and have the same length.*

Proof. Follows immediately from the Melzak algorithm. Note that being codirected at some point implies being codirected in a neighborhood, which in turn is equivalent to the equality of length (cf. Lemma 2.4.9). □

Let

$$T = T_1 \sqcup T_2 /_{\text{glued along terminal half-edges}},$$

i.e. we glue T_1 and T_2 along the portion of edges emanating from $x_1, \dots, x_8$ which coincide on the picture (other intersection points on the picture are not glued). Let $f : T \to \mathbb{R}^2$ be the corresponding immersion.

Note that T inherits a metric from $f(T)$: use Euclidean metric on $f(T)$ to measure distances along an edge of T, and use the inner metric to measure the distance between two arbitrary points.

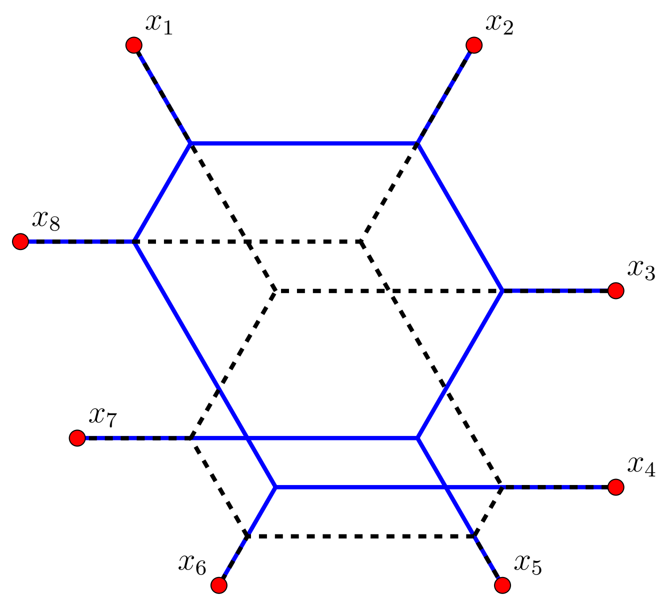


Figure 2.13: Two locally minimal trees with self-intersections

Fix an $\varepsilon > 0$ and for each $t \in T$ define the surface M_t to be a copy the ε-neighborhood of $f(t)$ in $\mathbb{R}^2$. The map f extends to the embedding $F_t : M_t \to \mathbb{R}^2$. Given $t_1, t_2 \in T$ on the distance at most 10ε from each other, glue M_{t_1} and M_{t_2} such that $F_{t_1} = F_{t_2}$. As a result we obtain Riemann surface M containing T, and an immersion $F : M \to \mathbb{R}^2$ such that the image $F(M)$ is the ε-neighborhood of $f(T)$. Note that M is endowed with a locally flat metric for which F is a local isometry.

Let $g_i : T_i \to M, i = 1, 2$, be the natural maps. Note that $f_i = F \circ g_i$ and g_i are injective. Denote by $X_1, \dots, X_8 \in M$ the points such that $F(X_i) = x_i$. The following proposition is straightforward.

Proposition 2.5.2. *Trees $g_1(T_1), g_2(T_2)$ are locally minimal on M and $(g_1, g_2) \in \operatorname{Int} \mathcal{A}_{T_1,T_2}$.*

We are not able to extend this example on minimal trees.

Chapter 3

Gilbert–Steiner problem

Here we follow paper [17].

3.1 Basics

One of the first models for branched transport was introduced by Gilbert [54]. The difference with the optimal transportation problem is that the extra geometric points may be of use; this explains the naming in honor of Steiner. Sometimes it is also referred to as *optimal branched transport*; a large part of book [8] is devoted to this problem. Let us proceed with the formal definition.

Definition 3.1.1. *Let μ^+, μ^- be two finite measures on a metric space $(X, \rho(\cdot,\cdot))$ with finite supports such that total masses $\mu^+(X) = \mu^-(X)$ are equal. Let $V \subset X$ be a finite set containing the support of the signed measure $\mu^+ - \mu^-$, the elements of V are called* vertices. *Further, let E be a finite collection of unordered pairs $\{x, y\} \subset V$ which we call* edges. *So, (V, E) is a simple undirected finite graph. Assume that for every $\{x, y\} \in E$ two non-zero real numbers $m(x, y)$ and $m(y, x)$ are defined so that $m(x, y) + m(y, x) = 0$. This data set is called a (μ^+, μ^-)-*flow *if*

$$\mu^+ - \mu^- = \sum_{\{x,y\}\in E} m(x, y) \cdot (\delta_y - \delta_x)$$

where δ_x denotes a delta-measure at x (note that the summand $m(x, y) \cdot (\delta_y - \delta_x)$ is well-defined in the sense that it does not depend on the order of x and y).

Let $C\colon [0, \infty) \to [0, \infty)$ be a *cost function*. The expression

$$\sum_{\{x,y\}\in E} C(|m(x, y)|) \cdot \rho(x, y)$$

is called the *Gilbert functional* of the (μ^+, μ^-)-flow.

The *Gilbert–Steiner problem* is to find the flow which minimizes the Gilbert functional with cost function $C(x) = x^p$, for a fixed $p \in (0, 1)$; we call a solution *minimal flow*.

Vertices from $\mathrm{supp}\,(\mu^+) \setminus \mathrm{supp}\,(\mu^-)$ are called *terminals*. A vertex from $V \setminus \mathrm{supp}\,(\mu^+) \setminus \mathrm{supp}\,(\mu^-)$ is called a *branching point*. Formally, we allow a branching point to have degree 2, but such points may be easily eliminated.

Local structure in the Gilbert–Steiner problem was discussed in [8], and the paper [85] deals with planar case. A local picture around a branching point b of degree 3 is clear due to the initial paper of Gilbert. Similarly to the finding of the Fermat–Torricelli point in the celebrated Steiner problem one can determine the angles around b in terms of masses (see Lemma 3.2.1).

Theorem 3.1.1 (Lippmann–Sanmartín–Hamprecht [85], 2022)**.** *A solution of the planar Gilbert–Steiner problem has no branching point of degree at least 5.*

A similar setup was independently considered in Minkowski spaces by Volz, Brazil, Ras, Swanepoel and Thomas [121]. In a Euclidean space (of an arbitrary dimension) they obtained that a degree of a branching is at most 3 provided that c is a concave monotone function, the function $(c^2)'$ is convex and $c(0) > 0$. Note that for $p \in (1/2, 1)$ the convexity condition fails.

The goal of this chapter is to give some conditions on a cost function under which all branching points in a planar solution have degree 3. They are slightly stronger than the Schoenberg [115] conditions of the embedding of the metric of the form $\rho(x, y) := f(x - y)$ to a Hilbert space. In particular, this covers the case of the standard cost function x^p, $0 < p < 1$. The following main theorem is the part of a more general Theorem 3.3.1.

Theorem 3.1.2. *A solution of the planar Gilbert–Steiner problem has no branching point of degree at least 4.*

3.2 Preliminaries

We need the following lemmas.

Lemma 3.2.1 (Folklore)**.** *Let PQR be a triangle and w_1, w_2, w_3 be non-negative reals. For every point $X \in \mathbb{R}^2$ consider the value*

$$L(X) := w_1 \cdot |PX| + w_2 \cdot |QX| + w_3 \cdot |RX|.$$

Then

(i) a minimum of $L(X)$ is achieved at a unique point X_{min};

(ii) if $X_{min} = P$ then $w_1 \geq w_2 + w_3$ or there is a triangle Δ with sides w_1, w_2, w_3 and $\angle P$ is at least the outer angle between w_2 and w_3 in Δ.

Hereafter the metric space is the Euclidean plane $\mathbb{R}^2$.

The following concept only slightly changes from that of Schoenberg [115], introduced for describing which metrics of the form $\rho(x, y) = f(x - y)$ on the real line can be embedded to a Hilbert space.

Definition 3.2.1. *Let λ be a Borel measure on $\mathbb{R}$ for which*

$$\int \min(x^2, 1) d\lambda(x) < \infty. \tag{3.1}$$

Assume additionally that the support of λ is uncountable. A function $f \colon \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ of the form

$$f(t) = \sqrt{\int \sin^2(tx)\, d\lambda(x)} = \frac{1}{2} \|e^{2itx} - 1\|_{L^2(\lambda)} \tag{3.2}$$

is called admissible.

The only difference with [115] is that we require that the support of the measure λ is uncountable which guarantees that the corresponding embedding has full dimension (see below).

Remark 3.2.1. *As λ is a Borel measure, a continuous function $\sin^2(tx)$ is λ-measurable. Under conditions (3.1), the integral in (3.2) is finite, so $f(t) < \infty$ for all $t \geqslant 0$.*

Further we are going to consider only admissible cost functions. Note that admissibility implies some properties one may expect from a cost function. In particular, $f(0) = 0$ and f is subadditive: for non-negative t, s we have

$$\begin{aligned} f(t) + f(s) &= \frac{1}{2}\|e^{2itx} - 1\|_{L^2(\lambda)} + \frac{1}{2}\|e^{2isx} - 1\|_{L^2(\lambda)} \\ &= \frac{1}{2}\|e^{2itx} - 1\|_{L^2(\lambda)} + \frac{1}{2}\|e^{2i(s+t)x} - e^{2itx}\|_{L^2(\lambda)} \geq \frac{1}{2}\|e^{2i(t+s)x} - 1\|_{L^2(\lambda)} = f(t+s). \end{aligned}$$

On the other hand it does not imply monotonicity (for instance, if $\operatorname{supp}\lambda \subset [0.9, 1.1]$ then $f(\pi) < f(\pi/2)$).

Hereafter $L^2(\lambda)$ for a measure λ on $\mathbb{R}$ is understood as a *real* Hilbert space of complex-valued square summable w.r.t. λ functions (strictly speaking, of classes of equivalences of such functions modulo coincidence λ-almost everywhere).

Proposition 3.2.1. *If λ is a Borel measure on $\mathbb{R}$ with uncountable support such that*

$$\int \min(x^2, 1) d\lambda(x) < \infty,$$

then any finite collection of functions of the form $e^{iax} - 1$, $a \in \mathbb{R}$, is affinely independent in $L^2(\lambda)$.

Proof. Assume the contrary. Then there exist distinct real numbers $a_1, \ldots, a_n$ and non-zero real coefficients $t_1, \ldots, t_n$ such that $\sum t_j = 0$ and $\sum t_j(e^{ia_jx} - 1) = 0$ λ-almost everywhere. But the analytic function $\sum t_j(e^{ia_jx} - 1)$ is either identically zero, or has at most countably many (and isolated) zeroes. In the latter case, it is not zero λ-almost everywhere, since the support of λ is uncountable. The former case is not possible: indeed, if $\sum t_j e^{ia_jx} \equiv 0$, then taking the Taylor expansion at 0 we get $\sum t_j a_j^k = 0$ for all $k = 0, 1, 2, \ldots$. Therefore $\sum t_j W(a_j) = 0$ for any polynomial W. Choosing $W(t) = \prod_{j=2}^n (t - a_j)$ we get $t_1 = 0$, a contradiction. □

One can see from the proof that the condition on uncountability of the support may be weakened.

Lemma 3.2.2. *Let C be an admissible cost function. For real numbers m_1 and m_2 we define $h(m_1, m_2)$ as the value of the outer angle between $C(|m_1|)$ and $C(|m_2|)$ in the triangle with sides $C(|m_1|)$, $C(|m_2|)$, $C(|m_1 + m_2|)$ (it exists by the discussion before Proposition 3.2.1). Suppose that OV_1, OV_2 are edges in a minimal flow with masses m_1 and m_2. Then the angle between OV_1 and OV_2 is at least $h(m_1, m_2)$.*

Proof. Assume the contrary, then by Lemma 3.2.1 with $P = O$, $Q = V_1$, $R = V_2$, $w_1 = C(|m_1|)$, $w_2 = C(|m_2|)$, $w_3 = C(|m_1 + m_2|)$ we have $X_{min} \neq O$. Then we can replace $[OV_1] \cup [OV_2]$ with $[X_{min}O] \cup [X_{min}V_1] \cup [X_{min}V_2]$ with the corresponding masses in our flow; this contradicts the minimality of the flow. □

Lemma 3.2.3. *For $0 < p < 1$, the function $f(x) = x^p$ is admissible.*

Proof. Consider the measure $d\lambda = x^{-2p-1}dx$ on $[0,\infty)$. Then $\int_0^\infty \min(x^2,1)d\lambda < \infty$ and for $t>0$ we have

$$\int_0^\infty \sin^2(tx)\,d\lambda(x) = \int_0^\infty \sin^2(tx)\,x^{-2p-1}dx = t^{2p}\int_0^\infty \sin^2 y\,y^{-2p-1}dy,$$

thus the measure λ multiplied by an appropriate positive constant proves the result. □

Example 3.2.1. *For another natural choice $d\lambda = 4ce^{-2cx}dx$, $c>0$, we get an admissible function $f(t) = t/\sqrt{t^2+c^2}$.*

The following lemma is essentially well-known, but for the sake of completeness and for covering degeneracies and the equality cases we provide a proof.

Lemma 3.2.4. *Let X be a finite-dimensional Euclidean space, let the points A_0, A_1, $A_2,\dots,A_{n-1}$, $A_n = A_0$, $A_{n+1} = A_1$ in X be chosen so that $A_i \neq A_{i+1}$ for all $i = 1,2,\dots,n$. Denote $\varphi_i := \pi - \angle A_{i-1}A_iA_{i+1}$ for $i = 1,2,\dots,n$. Then $\sum\varphi_i \geqslant 2\pi$, and if the equality holds then the points $A_1,\dots,A_n$ belong to the same two-dimensional affine plane.*

Proof. Let u be a randomly chosen unit vector in X (with respect to a uniform distribution on the sphere). For $j=1,2,\dots,n$ denote by $U(j)$ the following event: $\langle u,A_j\rangle = \max_{1\leqslant i\leqslant n}\langle u,A_i\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in X; and by $V(j)$ the event $\langle u,A_j\rangle = \max_{j-1\leqslant i\leqslant j+1}\langle u,A_i\rangle$. Obviously, $\operatorname{prob}U(j)\leqslant \operatorname{prob}V(j)$. Also, $\operatorname{prob}V(j) = \frac{\varphi_j}{2\pi}$, since the set of directions of u for which $V(j)$ holds is the dihedral angle of measure φ_j. Thus, since always at least one event $U(j)$ holds, we get

$$1 \leqslant \sum_{j=1}^n \operatorname{prob}U(j) \leqslant \sum_{j=1}^n \operatorname{prob}V(j) = \frac{1}{2\pi}\sum_{j=1}^n \varphi_j.$$

This proves the inequality. It remains to prove that it is strict assuming that not all the points belong to a two-dimensional plane. Note that if every three consecutive points A_{j-1},A_j,A_{j+1} are collinear, then all the points $A_1,\dots,A_n$ are collinear that contradicts to our assumption. If A_{j-1},A_j,A_{j+1} are not collinear, denote by α the two-dimensional plane they belong to. There exists i for which $A_i\notin\alpha$. Then $\operatorname{prob}U(j) < \operatorname{prob}V(j)$, since there exist planes passing through A_j which separate the triangle $A_{j-1}A_jA_{j+1}$ and the point A_i, and the measure of directions of such planes is strictly positive. Therefore, our inequality is strict. □

3.3 Main result

Theorem 3.3.1. *Let μ^+,μ^- be two measures with finite support on the Euclidean plane $\mathbb{R}^2$, and assume that the cost function C is admissible. Then if a (μ^+,μ^-)-flow has a branching point of degree at least 4, then there exists a (μ^+,μ^-)-flow with strictly smaller value of Gilbert functional.*

Proof. Assume the contrary. Let O be a branching point, $OV_1,OV_2,\dots,OV_k$, $k\geqslant 4$, be the edges incident to O, enumerated counterclockwise. Further the indices of V_i's are taken modulo k, so that $V_1 = V_{k+1}$ etc. Denote $m_i = m(OV_i)$, then by the definition of flow we get $\sum m_i = 0$. By Lemma 3.2.2, $\angle V_iOV_{i+1}\geqslant h(m_i,m_{i+1})$.

Consider the functions $A_j(x) := e^{i(m_1+\ldots+m_j)x}-1$ for $j=1,2,\dots$ (here i is the imaginary unit). Then $\sum m_j = 0$ yields that $A_{j+k}\equiv A_j$ for all $j>0$.

Since the cost function $C(t)$ is admissible, there exists a Borel measure λ on $\mathbb{R}$ with uncountable support such that $\int \min(x^2, 1) d\lambda(x) < \infty$ and

$$C(t) = \sqrt{\int 4 \sin^2 \frac{tx}{2} d\lambda(x)}.$$

Using the identity $|e^{ia} - e^{ib}|^2 = 4 \sin^2 \frac{a-b}{2}$ for real a, b we note that for $j, s > 0$ in the Hilbert space $L^2(\lambda)$ we have

$$\|A_{j+s} - A_j\|^2 = C(|m_{j+1} + \ldots + m_{j+s}|)^2.$$

In particular, the lengths of the sides of the triangle $A_{j-1}A_jA_{j+1}$ are equal to $C(|m_j|)$, $C(|m_{j+1}|)$ and $C(|m_j + m_{j+1}|)$. Therefore $\varphi_j := \pi - \angle A_{j-1}A_jA_{j+1} = h(m_j, m_{j+1})$. By Lemma 3.2.4 we get $\sum \varphi_j \geqslant 2\pi$.

By Lemma 3.2.1, this yields $2\pi = \sum_{j=1}^{k} \angle V_jOV_{j+1} \geqslant \sum \varphi_j \geqslant 2\pi$. Therefore, the equality must take place. Again by Lemma 3.2.4 it follows that the points A_j belong to the same 2-dimensional subspace. But by Proposition 3.2.1, distinct points between A_j's are affinely independent. Therefore, there exist at most three distinct A_j's, and if exactly three, they are not collinear. It is easy to see that the equality $\sum \varphi_j = 2\pi$ under these conditions does not hold when $k > 3$. A contradiction. □

3.4 Examples of branching points of degree 4

Let us start with an example in three dimensions. Consider four masses m_1, m_2, m_3, m_4 of zero sum, such that no two of them give zero sum. Repeat the beginning of the proof of Theorem 3.3.1 to get the simplex $A_1A_2A_3A_4$ in 3-dimensional space. Now consider unit edges OB_i in $\mathbb{R}^3$ with directions $A_{i-1}A_i$, $1 \leq i \leq 4$. By the construction the angles between vectors OB_i and OB_{i+1} are exactly $h(m_i, m_{i+1})$. Suppose that angles $\angle B_1OB_3$ and $\angle B_2OB_4$ are at least $h(m_1, m_3)$ and $h(m_2, m_4)$, respectively (for instance, it happens for $C(x) = x^p$, $1/2 \leq p < 1$, $m_1 = m_2 = m_3 = 1$, $m_4 = -3$).

Then we claim that for a concave monotone admissible cost function C the flow

$$\sum_{i=1}^{4} m_i \cdot (\delta_O - \delta_{B_i})$$

is a solution of the corresponding Gilbert–Steiner problem.

First, if we fix the graph structure then the position of O is optimal by the following lemma, because the closeness of the polychain $A_1A_2A_3A_4A_1$ gives exactly (3.3).

Lemma 3.4.1 (Weighted geometric median, [125]). *Consider different non-collinear points A, B, C, $D \in \mathbb{R}^3$ and let w_1, w_2, w_3, w_4 be non-negative reals. Then*

$$L(X) := w_1 \cdot |AX| + w_2 \cdot |BX| + w_3 \cdot |CX| + w_4 \cdot |DX|$$

has unique local (and global) minimum satisfying

$$w_1\bar{e_A} + w_2\bar{e_B} + w_3\bar{e_C} + w_4\bar{e_D} = 0, \qquad (3.3)$$

where $\bar{e_A}, \bar{e_B}, \bar{e_C}, \bar{e_D}$ are unit vectors codirected with XA, XB, XC, XD, respectively.

For a concave monotone cost function c there is an optimal flow without cycles (see Proposition 7.8 in [8]).

Since any two masses have nonzero sum, every flow is connected. Thus every possible competitor has 2 branching points of degree 3. Consider the case in which branching points U and V are connected with B_1, B_2 and B_3, B_4, respectively. By the convexity of length, the Gilbert functional $L(U, V)$ considered on the set of all possible U and V ($\mathbb{R}^3 \times \mathbb{R}^3$) is a convex function. Let us show that $U = V = O$ is a local minimum. Indeed, consider $U_\varepsilon = O + \varepsilon u$ and $V_\delta = O + \delta v$ for arbitrary unit vectors u, v and small positive ε, δ. By the convexity of length

$$L(U_\varepsilon, V_\delta) - L(O, O) \geq$$

$$w(UV) \cdot \|\varepsilon u - \delta v\| - \varepsilon\langle w_1 e_1, u\rangle - \varepsilon\langle w_2 e_2, u\rangle - \delta\langle w_3 e_3, v\rangle - \delta\langle w_4 e_4, v\rangle =$$
$$w(UV) \cdot \|\varepsilon u - \delta v\| - \varepsilon\langle w_{12} e_{12}, u\rangle - \delta\langle w_{34} e_{34}, v\rangle,$$

where $w_{12}e_{12} = w_1 e_1 + w_2 e_2$ and $w_{34}e_{34} = w_3 e_3 + w_4 e_4$ for unit e_{12} and e_{34}. By the construction one has $w_{12} = w_{34} = w(UV)$ and $e_{12} + e_{34} = 0$, so

$$L(U_\varepsilon, V_\delta) - L(O, O) \geq w(UV) \cdot (\|\varepsilon u - \delta v\| - \langle e_{12}, \varepsilon u - \delta v\rangle).$$

Since e_{12} is unit, the derivative is non-negative for every u, v.

The case in which U and V are connected with B_2, B_3 and B_4, B_1, respectively, is completely analogous. In the remaining case (U is connected with B_1, B_3 and V is connected with B_2, B_4) we have $w_{12} = w_{34} \leq w(UV)$ due to $\angle B_1 O B_3 \geq h(m_1, m_3)$ and $\angle B_2 O B_4 \geq h(m_2, m_4)$. Thus $U = V = O$ is also a local minimum.

It is known [54] that L has a unique local and global minimum, which finishes the example.

Now proceed with planar examples of 4-branching for some non-admissible cost-function C. Then we may repeat the 3-dimensional argument starting with planar $A_1A_2A_3A_4$.

The simplest way to produce an example is to consider an isosceles trapezoid $A_1A_2A_3A_4$ and apply Ptolemy's theorem. This case corresponds to $m_1 = m_3$ and $m_1 + m_2 + m_3 + m_4 = 0$. Then $|A_1A_2| = |A_3A_4| = C(|m_1|)$, $|A_2A_3| = C(|m_2|)$, $|A_4A_1| = C(|m_4|)$ and $|A_1A_3| = |A_2A_4| = C(|m_1 + m_2|)$. The existence of such trapezoid means

$$C(|m_1 + m_2|)^2 = C(|m_1|)^2 + C(|m_2|) \cdot C(|2m_1 + m_2|). \tag{3.4}$$

If we assume that C is monotone and subadditive then (3.4) means that the isosceles trapezoid exists; note that we need values of C only at 4 points.

Now we give an example of a monotone, subadditive and concave cost function with 4-branching. For this purpose put $m_1 = m_2 = m_3 = 1$ and $m_4 = -3$, $C(1) = 1$, $C(2) = 1.9$, $C(3) = 2.61$; clearly (3.4) holds. Now one can easily interpolate a desired C, for instance

$$C(t) = \begin{cases} t, & t \leq 1 \\ 0.1 + 0.9t, & 1 < t \leq 2 \\ 0.48 + 0.71t, & 2 < t \leq 3 \\ 1.11 + 0.5t & 3 < t. \end{cases}$$

Finally, the inequalities $\pi = \angle B_1 O B_3 > h(m_1, m_3)$ and $\angle B_2 O B_4 > \angle B_2 O B_3 = h(m_2, m_3) = h(m_2, m_4)$ hold.

3.5 Open questions

It would be interesting to describe all cost functions for which the conclusion of Theorem 3.3.1 holds.

Now let us focus on the cost function $C(x) = x^p$. Having a knowledge that every branching point has degree 3 one can adapt Melzak algorithm [91] from Steiner trees to Gilbert–Steiner problem. The idea of the algorithm is that after fixing the combinatorial structure one can find two terminals t_1, t_2 connected with the same branching point b. Then one may reconstruct the solution for V from the solution for $V \setminus \{t_1, t_2\} \cup \{t'\}$ for a proper t' which depend only on t_1, t_2 (in fact one has to check 2 such t'). When the underlying graph is a matching we finish in an obvious way. Application of this procedure for all possible combinatorial structures gives a slow but mathematically exhaustive algorithm in the planar case.

However there is no known algorithm in $\mathbb{R}^d$ for $d > 2$ (see Problem 15.12 in [8]). Recall that we have to consider a high-degree branching.

A naturally related problem is to evaluate the maximal possible degree of a branching point in the d-dimensional Euclidean space for every d. Note that the dependence on the cost function may be very complicated. In particular, an upper bound on the degree which does not depend on p is of interest. It is worth noting that $p < 1/2$ implies that $h(m_1, m_2) > \pi/2$ for every $m_1, m_2 \neq 0$ and thus the degree is at most $d + 1$.

Some other questions are collected in Section 15 of [8] (some of them are solved, in particular Problem 15.1 is solved in [27]).

Chapter 4

Maximal distance minimizers

4.1 Introduction

This chapter is based on papers [19, 18, 24]. We are interested in the solutions of the following maximal distance minimizer problem.

Problem 4.1.1. *For a given compact set $M \subset \mathbb{R}^d$ and $l > 0$ to find a connected compact set Σ of length (one-dimensional Hausdorff measure $\mathcal{H}^1$) at most l that minimizes*

$$\max_{y \in M} \operatorname{dist}(y, \Sigma),$$

where dist *stands for the Euclidean distance.*

It appeared in a very general form by Buttazzo, Oudet and Stepanov in [13] and then in was specified by Miranda, Paolini and Stepanov in [92, 98].

A *maximal distance minimizer* is a solution of Problem 4.1.1. Such sets can be considered as networks of radiating Wi-Fi cables with a bounded length arriving to each customer (for the set M of customers) at the distance r, where such r is the smallest possible.

4.1.1 Class of problems

Maximal distance minimization problem could be considered as a particular example of shape optimization problem. A shape optimization problem is a minimization problem where the unknown variable runs over a class of domains; then every shape optimization problem can be written in the form $\min F(\Sigma) : \Sigma \in A$ where A is the class of admissible domains and $F()$ is the cost function that one has to minimize over A.

So for a given compact set M and positive number $l \geq 0$ let the admissible set A be a set of all closed connected set Σ' with length constraint $\mathcal{H}^1(\Sigma') \leq l$; and let cost function be the *energy* $F_M(\Sigma) = \max_{y \in M} \operatorname{dist}(y, \Sigma)$. Also $F_M(\emptyset) := \infty$.

4.1.2 Dual problem

Define the dual problem to Problem 4.1.1 as follows.

Problem 4.1.2. *For a given compact set $M \subset \mathbb{R}^d$ and $r > 0$ to find a connected compact set Σ of the minimal length (one-dimensional Hausdorff measure $\mathcal{H}^1$) such that*

$$\max_{y \in M} \operatorname{dist}(y, \Sigma) \leq r.$$

In a nondegenerate case (i.e. for $F_M(\Sigma) > 0$) the primal and dual problems have the same sets of solutions for the corresponding r and l (see [98]) and hence an equality $F_M(\Sigma) = r$ is reached for a minimizer Σ.

4.1.3 The first parallels with average distance minimization problem

Maximal distance minimization problem is somehow similar to another shape optimization problem: average distance minimization problem (see the survey of Lemenant [84]) and it seems interesting to compare the known results and open questions concerning these two problems. In the average distance minimization problem's statement the admissible set A is the same as in Maximal distance minimization problem, but the function $F(\Sigma_a)$ is defined as $\int_M A(\operatorname{dist}(y, \Sigma_a))d\phi(x)$ where $A : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function and $\phi()$ is a finite nonnegative measure with compact nonempty support in $\mathbb{R}^d$.

Minimization problems for average distance and maximum distance functionals are used in economics and urban planning with similar interpretations. If it is required to find minimizers under the cardinality constraint $\sharp\Sigma \le k$, instead of the length and the connectedness constraints, where $k \in \mathbb{N}$ is given and $\sharp$ denotes the cardinality, then the corresponding problems are referred to as optimal facility location problems.

4.1.4 Notation

For a given set $X \subset \mathbb{R}^d$ we denote by $\overline{X}$ its closure, by $\operatorname{Int}(X)$ its interior and by ∂X its topological boundary.

Let $B_\rho(x)$ stand for the open ball of radius ρ centered at a point x, and let $B_\rho(T)$ be the open ρ-neighborhood of a set T i.e.

$$B_\rho(T) := \bigcup_{x\in T} B_\rho(x)$$

(in other words $B_\rho(T)$ is Minkowski sum of a ball B_ρ centered in the origin and T). Note that the condition

$$\max_{y\in M} \operatorname{dist}(y, \Sigma) \le r$$

is equivalent to $M \subset \overline{B_r(\Sigma)}$.

For given points b, c we use the notation $[bc]$, $[bc)$ and (bc) for the corresponding closed line segment, ray and line respectively.

4.1.5 Existence. Absence of loops. Ahlfors regularity and other simple properties

For the both problems existence of solutions is proved easily: according to the classical Blaschke and Gołąb Theorems, the class of admissible sets is compact for the Hausdorff distance and both of the functions (maximal distance and also the average distance) is continuous for this convergence because of the uniform convergence of $x \to \operatorname{dist}(x, \Sigma)$.

Definition 4.1.1. *A closed set Σ is said to be Ahlfors regular if there exists some constants C_1, $C_2 > 0$ and a radius $\varepsilon_0 > 0$ such that $C_1\varepsilon \le \mathcal{H}^1(\Sigma \cap B_\varepsilon(x)) \le C_2\varepsilon$ for every $x \in \Sigma$ and $\varepsilon < \varepsilon_0$.*

In the work [98] Paolini and Stepanov proved

- the absence of closed loops for maximum distance minimizers and, under general conditions on ϕ, the absence of closed loops for average distance minimizers;
- the Ahlfors regularity of maximum distance minimizers and, under the additional summability condition on ϕ, the Ahlfors regularity of average distance minimizers. Gordeev and Teplitskaya [58] refine Ahlfors constants of maximum distance minimizers to the best possible, i.e. show that $\mathcal{H}^1(\Sigma \cap B_\varepsilon(x)) = \mathrm{ord}_x\Sigma \cdot \varepsilon + o(\varepsilon)$, where $\mathrm{ord}_x\Sigma \in \{1, 2, 3\}$.
- Recall that maximal distance minimization problem and the dual problem have the same sets of solutions (the planar case was proved before by Miranda, Paolini, Stepanov in [92]). It particularly implies that maximal distance minimizers must have maximum available length l. Paolini and Stepanov also proved that average distance minimizers (with additional assumptions on ϕ) have maximum available length.

In the work [6] the following basic results were shown.

(i) Let Σ be an r-minimizer for some M. Then Σ is an r-minimizer for $\overline{B_r(\Sigma)}$.

(ii) Let Σ be an r-minimizer for $\overline{B_r(\Sigma)}$. Then Σ is an r'-minimizer for $\overline{B_{r'}(\Sigma)}$, where $0 < r' < r$.

4.1.6 Local maximal distance minimizers

Definition 4.1.2. *Let $M \subset \mathbb{R}^d$ be a compact set and let $r > 0$. A closed connected set $\Sigma \subset \mathbb{R}^d$ with $\mathcal{H}^1(\Sigma) < \infty$ is called a local minimizer if $\mathcal{F}_M(\Sigma) \leq r$ and there exists $\varepsilon > 0$ such that for any connected set Σ' satisfying $\mathcal{F}_M(\Sigma') \leq r$ and $\mathrm{diam}\,(\Sigma \triangle \Sigma') \leq \varepsilon$ the inequality $\mathcal{H}^1(\Sigma) \leq \mathcal{H}^1(\Sigma')$ holds, where $\triangle$ is the symmetric difference.*

Any maximal distance minimizer is also a local minimizer. Usually the properties of maximal distance minimizers are also true for local maximal distance minimizers (see [58]).

4.2 Regularity

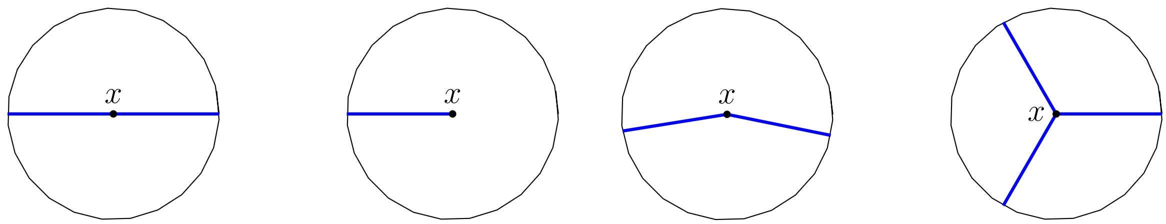
4.2.1 Tangent rays. Blow up limits in $\mathbb{R}^d$

Definition 4.2.1. *We say that a ray $(ax]$ is a* tangent ray *of a set $\Gamma \subset \mathbb{R}^d$ at the point $x \in \Gamma$ if there exists a sequence of points $x_k \in \Gamma \setminus \{x\}$ such that $x_k \to x$ and $\angle x_k x a \to 0$.*

Then we have the following regularity theorem.

Theorem 4.2.1 (Gordeev–Teplitskaya [58])**.** *Let Σ be a minimizer for a compact set $M \subset \mathbb{R}^d$ and $r > 0$. Then there are at most three tangent rays at any point of Σ, and the pairwise angles between the tangent rays are at least $2\pi/3$. Furthermore, tangent rays coincide with one-sided tangents, particularly the angles between one-sided tangents cannot be equal to 0, i.e. there is one to one correspondence between tangent rays at an arbitrary point $x \in \Sigma$ and connected components of $\Sigma \setminus \{x\}$. Moreover, if $d = 2$, then Σ is a finite union of simple curves with one-sided tangents continuous from the corresponding side.*

In works concerning average distance minimizers the notion of *blow up limits* is used. Santambrogio and Tilli in [113] proved that for any average distance minimizer blow up sequence $\Sigma_\varepsilon := \varepsilon^{-1}(\Sigma_a \cap B_\varepsilon(x) - x)$ with $x \in \Sigma_a$, converges in $B_1(0)$ (for the Hausdorff distance) to some limit $\Sigma_0(x)$ when $\varepsilon \to 0$, and the limit is one of the following below (see Fig. 4.1 which is analogues to a picture from [84]), up to a rotation.



a general x has tangent line $\psi(x) = 0$ | x is a leaf $\psi(x) > 0$ | x is a corner point $\psi(x) > 0$ | x is a branching point $\psi(x) = 0$

Figure 4.1: All possible variants of tangent rays at any point of a maximal distance minimizer (or blow up limits of an average distance minimizer)

It is clear that for maximal distance minimizers blow up limits also exists and are more or less the same: Σ_0 can be a radius, a diameter, a corner points with the angle between the segments greater or equal to $2\pi/3$ or a center of a regular tripod. Herewith at the second and third case (id est when $\psi(x) > 0$) the point x has to be energetic; see the following definition.

Definition 4.2.2. *A point $x \in \Sigma$ is called* energetic*, if for all $\rho > 0$ one has*

$$F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma).$$

Herewith if a point x of a maximal distance minimizer Σ is energetic then there exists such a point $y \in M$ (may be not unique) such that $\mathrm{dist}\,(x, y) = r$ and $B_r(y) \cap \Sigma = \emptyset$; such y is called *corresponding* to x.

If a point $x \in \Sigma$ is not energetic then in a sufficiently small neighbourhood it is a center of a regular tripod or a segment (and coincides there with its one-sided tangents).

A key object in all the study of the average distance problem is the pull-back measure of μ with respect to the projection onto Σ_a, where Σ_a is a solution of the average distance minimizer problem. More precisely, if μ does not charge the Ridge set (which is defined as the set of all $x \in \mathbb{R}^d$ for which the minimum distance to Σ_a is attained at more than one point) of Σ_a (this is the case for instance when μ is absolutely continuous with respect to the Lebesgue measure), then it is possible to choose a measurable selection of the projection multimap onto Σ, i.e. a map $\pi_\Sigma : M \to \Sigma$ such that $d(x, \Sigma) = d(x, \pi_{\Sigma_a})$ (this map is uniquely defined everywhere except the Ridge set). Then one can define the measure ψ as being $\psi(A) := \mu(\pi_{\Sigma_a}^{-1}(A))$, for any Borel set $A \subset M$. In other words $\psi = \pi_{\Sigma_a} \sharp \mu$.

For the maximal distance minimizers in $\mathbb{R}^d$ we can define measure ψ the similar way, but replace M by $\partial B_r(\Sigma)$ and with $(n-1)$-dimensional Hausdorff measure as μ (or accordingly $\overline{B_r(\Sigma)}$ and n-dimensional Hausdorff measure). Thus Fig. 4.1 is true both for maximal and average distance minimizers.

4.2.2 Properties of branching points in $\mathbb{R}^2$

Recall that by Theorem 4.2.1 that for every planar compact set M and a positive number r a maximal distance minimizer can have only a finite number of points with 3 tangent rays.

In the plane it is also known (see [12]) that every average distance minimizer is topologically a tree composed of a finite union of simple curves joining with a number of 3.

Every branching point of a planar maximal distance minimizer should be the center of a regular tripod. If $x \in \Sigma \subset \mathbb{R}^2$ has 3 tangent rays then there exists such a neighbourhood of x in which the minimizer coincides with its tangent rays. Id est, there exists such $\varepsilon > 0$ that $\Sigma \cap \overline{B_\varepsilon(x)} = [ax] \cup [bx] \cup [cx]$ where $\{a, b, c\} = \Sigma \cap \partial B_\varepsilon(x)$ and $\angle axb = \angle bxc = \angle cxa = 2\pi/3$. For planar average distance minimizers it is proved that any branching point admits such a neighbourhood in which three pieces of Σ are $C^{1,1}$.

4.2.3 Continuity of one-sided tangents in $\mathbb{R}^2$

Definition 4.2.3. *We will say that the ray $(ax]$ is a* one-sided tangent *of a set $\Gamma \subset \mathbb{R}^d$ at a point $x \in \Gamma$ if there exists a connected component Γ_1 of $\Gamma \setminus \{x\}$ such that $x \in \overline{\Gamma_1}$ and that any sequence of points $x_k \in \Gamma_1$ with the property $x_k \to x$ satisfies $\angle x_k x a \to 0$. In this case we will also say that $(ax]$ is tangent to the connected component Γ_1.*

In the plane the continuity of one-sided tangents from the corresponding side holds (see [58]):

Lemma 4.2.1. *Let $\Sigma \subset \mathbb{R}^2$ be a (local) maximal distance minimizer and let $x \in \Sigma$. Let Σ_1 be a connected component of $\Sigma \setminus \{x\}$ with one-sided tangent $(ax]$ (it has to exist) and let $\bar{x} \in \Sigma_1$.*

1. *For any one-sided tangent $(\bar{a}\bar{x}]$ of Σ at $\bar{x}$ the equality $\angle((\bar{a}\bar{x}), (ax)) = o_{|\bar{x}x|}(1)$ holds.*
2. *Let $(\bar{a}\bar{x}]$ be a one-sided tangent at $\bar{x}$ of any connected component of $\Sigma \setminus \{\bar{x}\}$ not containing x. Then $\angle((\bar{a}\bar{x}], (ax]) = o_{|\bar{x}x|}(1)$.*

For planar average distance minimizers it is proved (see [84]) that away from branching points an average distance minimizer Σ_a is locally at least as regular as the graph of a convex function, namely that the Right and Left tangent maps admit some Right and Left limits at every point and are semicontinuous. More precisely, for a given parametrization γ of an injective Lipschitz arc $\Gamma \subset \Sigma_a$, by existence of blow up limits one can define the Left and Right tangent half-lines at every point $x \in \Gamma$ by

$$T_R(x) := x + \mathbb{R}^+ . \lim_{h \to 0} \frac{\gamma(t_0 + h) - \gamma(t_0)}{h}$$

and

$$T_L(x) := x + \mathbb{R}^+ . \lim_{h \to 0} \frac{\gamma(t_0 - h) - \gamma(t_0)}{h}.$$

Then the following planar theorem for average distance minimizers holds.

Theorem 4.2.2 (Lemenant, 2011 [83])**.** *Let $\Gamma \subset \Sigma_a$ be an open injective Lipschitz arc. Then the Right and Left tangent maps $x \to T_R(x)$ and $x \to T_L(x)$ are semicontinuous, id est for every $y_0 \in \Gamma$ there holds $\lim_{y \to y_0; y <_\gamma y_0} T_L(y) = T_L(y_0)$ and $\lim_{y \to y_0; y >_\gamma y_0} T_R(y) = T_R(y_0)$. In addition the limit from the other side exists and we have $\lim_{y \to y_0; y >_\gamma y_0} T_L(y) = T_R(y_0)$ and $\lim_{y \to y_0; y <_\gamma y_0} T_R(y) = T_L(y_0)$.*

An immediate consequence of the theorem is the following corollary:

Corollary 4.2.1. *Assume that $\Gamma \subset \Sigma$ is a relatively open subset of Σ that contains no corner points nor branching points. Then Γ is locally a C^1-regular curve.*

4.2.4 Planar example of infinite number of corner points

Recall that each maximal distance minimizer in the plane is a finite union of simple curves. These curves should have continuous one-sided tangents but do not have to be C^1: there exists a minimizer with an infinite number of points without tangent lines. The following example is provided in [6].

Fix positive reals r, R and let N be a large enough integer. Consider a sequence of points $\{a_i\}_{i=1}^{\infty}$ chosen from the circumference $\partial B_R(o)$ such that $N \cdot |a_2a_1| = r$,

$$|a_{i+1}a_{i+2}| = \frac{1}{2}|a_ia_{i+1}|$$

and $\angle a_ia_{i+1}a_{i+2} > \frac{\pi}{2}$ for every $i \in \mathbb{N}$ (see Fig. 4.2). Let a_∞ be the limit point of $\{a_i\}$. Finally, let $a_{\infty+1}$ be the point in the tangent line to $B_r(o)$ at a_∞, such that

$$|a_\infty a_{\infty+1}| = r/N.$$

We claim that polyline

$$\Sigma = \bigcup_{i=1}^{\infty}[a_ia_{i+1}]$$

is a unique maximal distance minimizer for the following M.

Let $v_1 \in (a_1a_2]$ be such point that $|v_1a_1| = r$. For $i \in \mathbb{N} \cup \{\infty\} \setminus \{1\}$ define v_i as the point satisfying $|v_ia_i| = r$ and $\angle a_{i-1}a_iv_i = \angle a_{i+1}a_iv_i > \pi/2$. Define v_∞ as the limit point of $\{v_i\}$. Finally, let $v_{\infty+1}$ be such point that $v_{\infty+1}a_\infty \perp v_\infty a_\infty$ and $|v_{\infty+1}a_\infty| = r$. Clearly $M := \{v_i\}_{i=1}^{\infty+1}$ is a compact set.

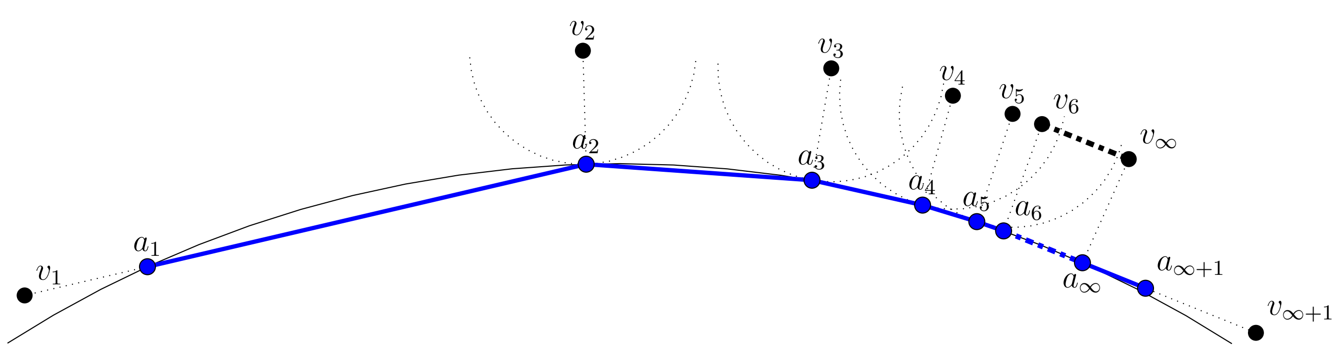


Figure 4.2: The example of a minimizer with infinite number of corner points

Theorem 4.2.3 (Basok–Cherkashin–Teplitskaya, 2022 [6])**.** *Let Σ and M be defined above. Then Σ is a unique maximal distance minimizer for M.*

4.2.5 Every $C^{1,1}$-smooth simple curve is a minimizer

For planar average distance minimizers Tilli proved in [119] that any simple $C^{1,1}$-curve is a minimizer for some given data. Paper [6] generalizes Tilli's result on d-dimensional space. The same statement with a similar but much simpler explanation is true for maximal distance minimizers.

Theorem 4.2.4 (Basok–Cherkashin–Teplitskaya, 2022 [6])**.** *Let $\gamma \subset \mathbb{R}^d$ be a simple $C^{1,1}$-curve. Then γ is a maximal distance minimizer for a small enough r and $M = \overline{B_r(\gamma)}$.*

4.3 Explicit examples for maximal distance minimizers

Recall that Theorems 4.2.3 and 4.2.4 provide explicit examples; however they are obtained by "reverse engineering": the input M is constructed in a way to give the minimizer property to a desired Σ. This section is devoted to known explicit results.

4.3.1 Simple examples. Finite number of points and r-neighbourhood. Inverse minimizers

Here we consider Problem 4.1.2 in a case when M is a finite set. Then it is closely related with the Steiner problem (Problem 2.0.1).

Any maximal distance minimizer for any finite set $M \subset \mathbb{R}^d$ is a finite union of at most $2\sharp M - 1$ segments. In this case maximal distance minimization problem comes down to connecting r-neighborhoods of all the points from M. If $\overline{B_r(a)}$ are disjoint for every $a \in M$ then a maximal distance minimizer is a Steiner tree connecting some points from $\partial B_r(a)$, $a \in M$.

The following observations and statements of this paragraph are from the paper [6].

Remark 4.3.1. *(i) Let Σ be a maximal distance minimizer for some M and $r > 0$. Then Σ is a maximal distance minimizer for $\overline{B_r(\Sigma)}$ and r.*

(ii) Let Σ be a minimizer for $\overline{B_r(\Sigma)}$ and $r > 0$. Then Σ is a minimizer for $\overline{B_{r'}(\Sigma)}$ and r', where $0 < r' < r$.

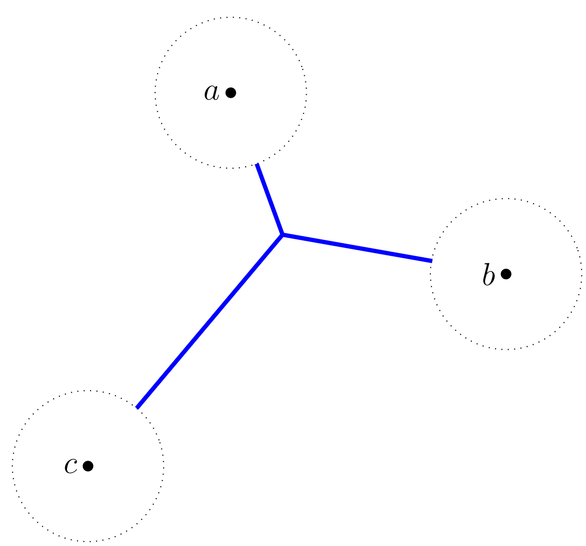


Figure 4.3: A maximal distance minimizer for a certain 3-point set $M = \{a, b, c\}$

A *topology* T of a labelled Steiner tree (or a labelled locally minimal tree) $\mathcal{S}t$ is the corresponding abstract graph with labelled terminals and unlabelled Steiner points.

Theorem 4.3.1 (Basok–Cherkashin–Teplitskaya, 2022 [6])**.** *Let $\mathcal{S}t$ be a Steiner tree for a labelled set of terminals $A = (a_1, \dots, a_n)$, $a_i \in \mathbb{R}^d$ such that every Steiner tree for an n-tuple in the closed $2r$-neighbourhood of A (with respect to ρ) has the same topology as $\mathcal{S}t$ for some positive r. Then $\mathcal{S}t$ is an r-minimizer for an n-tuple M.*

In the plane a Steiner tree for a random input is unique with unit probability, see [5]. Also in the plane we have a general inverse statement to Theorem 4.3.1.

Proposition 4.3.1 (Basok–Cherkashin–Teplitskaya, 2022 [6])**.** *Suppose that $\mathcal{S}t$ is a full Steiner tree for terminals $a_1, \dots, a_n \in \mathbb{R}^2$, which is not unique. Then $\mathcal{S}t$ can not be a minimizer for M being an n-tuple of points.*

To illustrate Proposition 4.3.1 consider a square $a_1a_2a_3a_4$. There are two Steiner trees for a_1, a_2, a_3, a_4 (see the left-hand side of Fig. 4.4), let us pick the solid one. The right-hand side of Fig. 4.4 shows that an r-minimizer for every positive r has the topology of the dotted Steiner tree.

In all known examples a $\mathcal{S}t$ with n terminals is an r-minimizer for a set M of n points and a small enough positive r if and only if $\mathcal{S}t$ in the unique Steiner tree for its terminals. So the planar case of several non-full solutions is open, and also it is interesting to derive any analogue of Proposition 4.3.1 for $d > 2$.

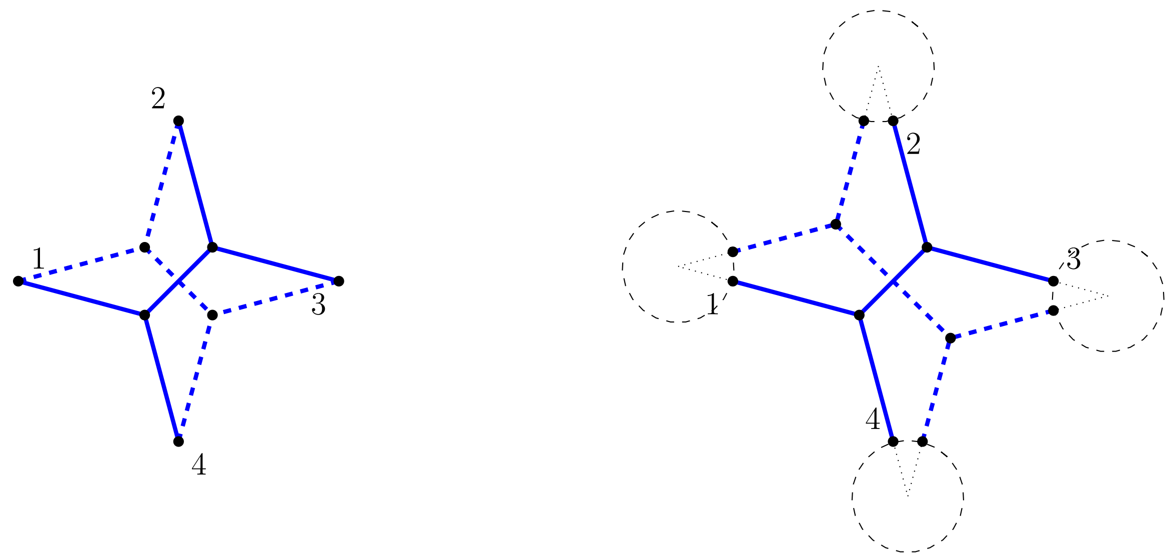


Figure 4.4: An example to Proposition 4.3.1

4.3.2 Circle. Curves with big radius of curvature

Theorem 4.3.2 (Cherkashin–Teplitskaya, 2018 [18])**.** *Let r be a positive real, M be a convex closed curve with the radius of curvature at least $5r$ at every point, Σ be an arbitrary minimizer for M. Then Σ is a union of an arc of M_r and two segments that are tangent to M_r at the ends of the arc (so-called horseshoe, see Fig. 4.5). In the case when M is a circumference with the radius R, the condition $R > 4.98r$ is enough.*

Also Theorem 4.3.2 admits a corollary on local minimizers in the sense of Definition 4.1.2.

Corollary 4.3.1 (Cherkashin–Teplitskaya, 2018 [18])**.** *Let $\hat{\Sigma}$ be a local minimizer for some closed convex curve M with minimal radius of curvature $R > 5r$. Then if $\hat{\Sigma}$ is not a horseshoe, one has $\mathcal{H}^1(\hat{\Sigma}) - \mathcal{H}^1(\Sigma) \geq (R - 5r)/2$, where Σ is an arbitrary (global) minimizer.*

Miranda, Paolini and Stepanov [92] conjectured that all the minimizers for a circumference of radius $R > r$ are horseshoes. Theorem 4.3.2 solves this conjecture with the assumption $R > 4.98r$; for $4.98r \geq R > r$ the conjecture remains open.

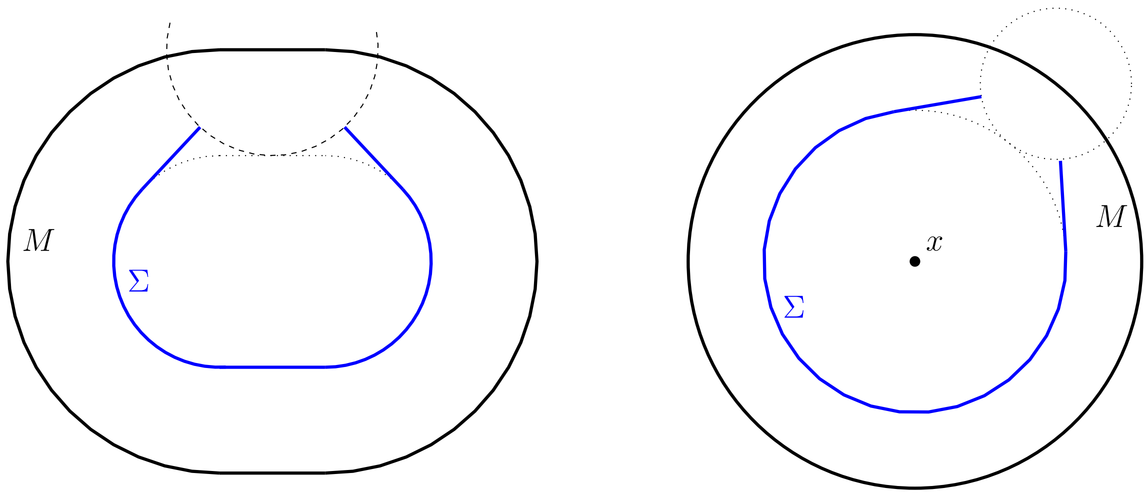


Figure 4.5: A minimizer for a convex closed planar curve M with the radius of curvature at least $5r$ at every point, so-called *horseshoe* (left). A minimizer for $M = \partial B_R(x)$, where $R > 4.98r$ (right)

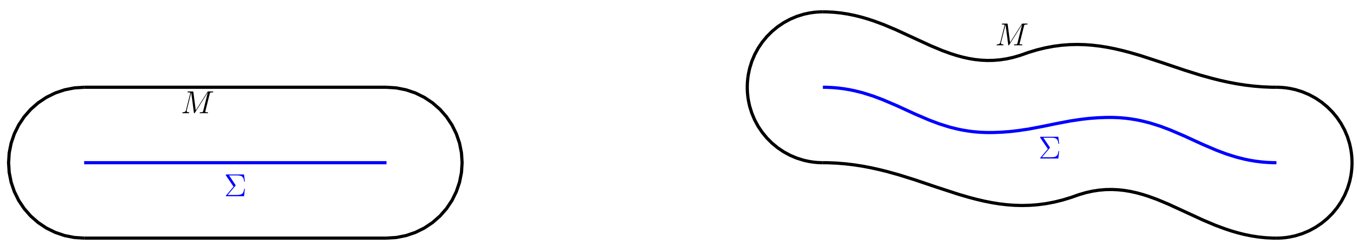


Figure 4.6: M is r-neighbourhood for a sufficiently smooth curve Σ and small enough $r > 0$

4.3.3 Rectangle

Theorem 4.3.3 (Cherkashin–Gordeev–Strukov–Teplitskaya, 2021 [14]). *Let $M = a_1a_2a_3a_4$ be a rectangle. Then there is a positive number $r_0(M)$ such that for any positive $r < r_0(M)$ every minimizer of the maximum distance functional has a topology of 21 segments, shown on the leftmost side of Fig. 4.7. The middle part of the figure shows an enlarged fragment of the minimizer in the vicinity of point a_1; the marked angles are equal to $\frac{2\pi}{3}$. The rightmost side of the figure shows an even more enlarged fragment of the minimizer in the vicinity of a_1.*

Any minimizer of the maximum distance functional has length $\mathrm{Per}(M) - cr + o(r)$*, where* $\mathrm{Per}(M)$ *is the perimeter of the rectangle M, and c is a constant approximately equal to 8.473981.*

In fact, every maximal distance minimizer is very close (in the sense of Hausdorff distance) to the one depicted in the picture.

4.4 Tools

4.4.1 Energetic points

For the planar problem the notion of energetic points (which is also correct in $\mathbb{R}^d$) is very useful.

Recall that a point $x \in \Sigma$ is called *energetic*, if for all $\rho > 0$ one has $F_M(\Sigma \setminus B_\rho(x)) > F_M(\Sigma)$. The set of all energetic points of Σ is denoted by G_Σ. Each minimizer Σ can be split into three disjoint

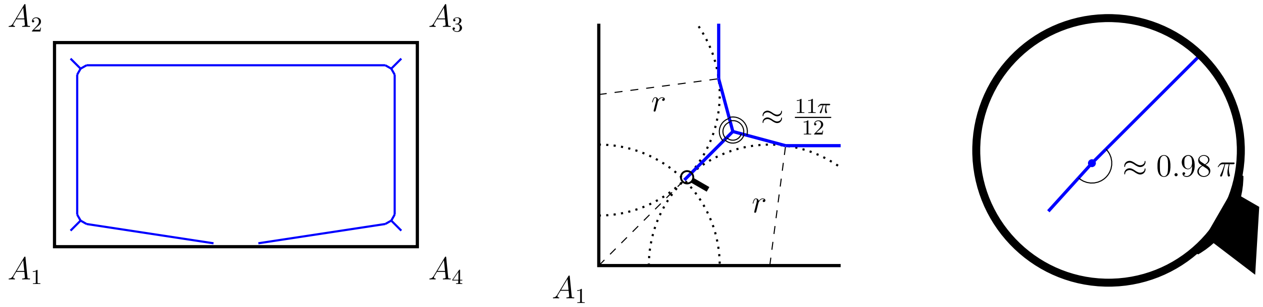


Figure 4.7: The minimizer for a rectangle M with $r < r_0(M)$.

subsets:

$$\Sigma = E_\Sigma \sqcup \mathcal{X}_\Sigma \sqcup \mathcal{S}_\Sigma,$$

where $X_\Sigma \subset G_\Sigma$ is the set of isolated energetic points (i.e. every $x \in X_\Sigma$ is energetic and there is a $\rho > 0$ possibly depending on x such that $B_\rho(x) \cap G_\Sigma = \{x\}$), $E_\Sigma := G_\Sigma \setminus X_\Sigma$ is the set of non isolated energetic points and $S_\Sigma := \Sigma \setminus G_\Sigma$ is the set of non energetic points also called the *Steiner part*.

Note that it is possible for a (local) minimizer in $\mathbb{R}^d$, $d > 2$ to have no non-energetic points at all. Moreover, in some sense, any (local) minimizer does not have non-energetic points in a larger dimension:

Example 4.4.1. *Let Σ be a (local) minimizer for a compact set $M \subset \mathbb{R}^d$ and $r > 0$. Then $\bar\Sigma := \Sigma \times \{0\} \subset \mathbb{R}^{d+1}$ is a (local) minimizer for $\bar M = (M \times \{0\}) \cup (\Sigma \times \{r\}) \subset \mathbb{R}^{d+1}$ and $\mathcal{E}_{\bar\Sigma} = \bar\Sigma$.*

Recall that for every point $x \in G_\Sigma$ there exists a point $y \in M$ (may be not unique) such that $\operatorname{dist}(x, y) = r$ and $B_r(y) \cap \Sigma = \emptyset$. Thus all points of $\Sigma \setminus \overline{B_r(M)}$ can not be energetic and thus $\Sigma \setminus \overline{B_r(M)}$ is so-called Steiner forest id est each connected component of it is a Steiner tree with terminal points on the $\partial B_r(M)$.

In the plane it makes sense to define energetic rays.

Definition 4.4.1. *We say that a ray $(ax]$ is the* energetic ray *of the set Σ with a vertex at the point $x \in \Sigma$ if there exists non stabilized sequence of energetic points $x_k \in G_\Sigma$ such that $x_k \to x$ and $\angle x_k x a \to 0$.*

Remark 4.4.1. *Let $\{x_k\} \subset G_\Sigma$ and let $x \in E_\Sigma$ be the limit point of $\{x_k\}$: $x_k \to x$. By basic property of energetic points for every point $x_k \in G_\Sigma$ there exists a point $y_k \in M$ (may be not unique) such that* $\operatorname{dist}(x_k, y_k) = r$ *and $B_r(y_k) \cap \Sigma = \emptyset$. In this case we will say that y_k* corresponds *to x_k.*

Let y be an arbitrary limit point of the set $\{y_k\}$. Then the set Σ does not intersect r-neighbourhood of y: $B_r(y) \cap \Sigma = \emptyset$ and the point y belongs to M and corresponds to x.

Let $[sx] \subset \Sigma$ be a simple curve. Let us define $\operatorname{turn}([\breve{sx}])$ as the upper limit (supremum) over all sequences of points of the curve:

$$\operatorname{turn}([\breve{sx}]) = \sup_{n \in \mathbb{N}, s \preceq t^1 \prec \cdots \prec t^n \prec x} \sum_{i=2}^{n} \widehat{t^i, t^{i-1}},$$

where t^i denotes the ray of the one-sided tangent to the curve $[\breve{st_i}] \subset [\breve{sx}[$ at point t_i, and $t_1, \ldots, t_n$ is the partition of the curve $[\breve{sx}[$ in the order corresponding to the parameterization, for which s is the beginning of the curve and x is the end. In this case, the angle $(\widehat{t^i, t^{i+1}}) \in [-\pi, \pi[$ between two rays is counted from ray t^i to ray t^{i+1}; positive direction is counterclockwise.

Let $\breve{sx}$ lay in the sufficiently small neighbourhood of x. Then if $B_r(y(x)) \cap [\breve{sx}] = \emptyset$, it is true that

$$|\operatorname{turn}([sx])| < 2\pi.$$

This property is the first one which is true for the plane and false in $\mathbb{R}^d$ with $d > 2$, so this is the main difference between planar and non-planar cases. In the plane the turn is a very useful tool, see for example the proof of Theorem 4.3.2 [18].

The second main differ between plane and other Euclidean spaces is also concerning angles: in the plane if you know the angles $\widehat{t^i, t^{i-1}}$ for $i = 2, \dots k$ then you know the angle $\widehat{t^1, t^k}$ which is not true for $\mathbb{R}^d$ with $d > 2$.

4.4.2 Convexity argument

Suppose that we fix some $M_0 \subset M$ and consider a (possibly infinite) tree T whose vertices are encoded by points of M_0. Let us pick an arbitrary point from $\overline{B_r(m)}$ for every $m \in M_0$ and connect such points by segments with respect to T. Consider the length L of such a representation of T; note that we allow the representation to contain cycles or edges of zero length.

Then L is a convex function from $(\mathbb{R}^d)^{M_0}$ to $\mathbb{R}$. Also if $v, u \in \overline{B_r(m)}$, then $\alpha v + (1-\alpha)u$ also lies in $\overline{B_r(m)}$. It implies that the sets of local and global minimums of L coincide and form a convex set. It usually means that L is a unique local minimum.

This approach allows us to show that if one fixes a topology of a solution, then the corresponding Steiner-type problem has a unique solution. The proofs of Theorems 4.2.3 and 4.3.1 heavily use it.

4.4.3 Lower bounds on the length of a minimizer

The proof of the following folklore inequality can be found, for instance in [93].

Lemma 4.4.1. *Let γ be a compact connected subset of $\mathbb{R}^d$ with $\mathcal{H}^1(\gamma) < \infty$. Then*

$$\mathcal{H}^d(\{x \in \mathbb{R}^d : \operatorname{dist}(x, \gamma) \le t\}) \le \mathcal{H}^1(\gamma)\omega_{d-1}t^{d-1} + \omega_d t^d,$$

where ω_k denotes the volume of the unit ball in $\mathbb{R}^k$.

The following corollary is very close to a theorem of Tilli on average distance minimizers [119].

Corollary 4.4.1. *Let V and r be positive numbers. Then for every set M with $\mathcal{H}^d(M) = V$ a maximal distance r-minimizer has the length at least*

$$\max\left(0, \frac{V - \omega_d r^d}{\omega_{d-1} r^{d-1}}\right).$$

Theorem 4.2.4 follows from the fact that for a $C^{1,1}$-curve and small enough r the inequality in Corollary 4.4.1 is sharp. Let us provide a lower bound from [6] on the length of a minimizer in the planar case.

Proposition 4.4.1. *Let M be a planar convex set and Σ is an r-minimizer for M. Then*

$$\mathcal{H}^1\Sigma) \ge \frac{\mathcal{H}^1\partial M) - 2\pi r}{2}.$$

4.5 More properties of minimizers

4.5.1 Γ-convergence

Γ-convergence is an important tool in studying minimizers based on approximation of energy. For Euclidean space the following definition of Γ-convergence can be used. Let X be a first-countable space and F_n: $X \to \overline{\mathbb{R}}$ a sequence of functionals on X. Then F_n are said to Γ-*converge* to a Γ-limit $F : X \to \overline{\mathbb{R}}$ if the following two conditions hold.

- Lower bound inequality. For every sequence $x_n \in X$ such that $x_n \to x$ as $n \to +\infty$,
$$F(x) \le \liminf_{n\to\infty} F_n(x_n).$$
- Upper bound inequality. For every $x \in X$, there is a sequence x_n converging to x such that
$$F(x) \ge \limsup_{n\to\infty} F_n(x_n).$$

In the case of maximal distance minimizers for a given compact set M and a number $l > 0$ we can consider the space X of connected compact sets with one-dimensional Hausdorff measure at most l, equipped with the Hausdorff distance (the distance d_H between $A, C \in X$ is the smallest ρ such that $A \subset \overline{B_\rho(C)}$ and $C \subset \overline{B_\rho(A)}$).

Proposition 4.5.1. *If a sequence of compacts M_i converges to M then F_{M_i} Γ-converges to F_M.*

Proof. By the definition of F_M and triangle inequalities we have

$$|F_{M_i}(S_i) - F_M(S)| \le |F_{M_i}(S_i) - F_M(S_i)| + |F_M(S_i) - F_M(S)| \le d_H(M_i, M) + d_H(S_i, S) \tag{4.1}$$

for every connected S_i and S. So by (4.1) every sequence of S_i with limit S we have the first condition of Γ-convergence. For the second condition consider S_i being a Steiner tree for a finite $1/i$-network $N_i \subset S$. By the definition $\mathcal{H}^1(S_i) \le \mathcal{H}^1(S) \le l$. Again, by (4.1) $F_{M_i}(S_i)$ converges to $F_M(S)$. □

4.5.2 Approximation by Steiner trees

A crucial property of Γ-convergence is that in the notation of Proposition 4.5.1 every cluster point of the sequence of minimizers of F_{M_i} is a minimizer of F_M. Now let M_n be a finite $1/n$-network for M, so that every minimizer for M_n is a finite Steiner tree.

Unfortunately, in the case of several minimizers for M we cannot be sure that every minimizer is approximated. On the other hand it can be approximated a posteriori. Let Σ be a minimizer for M and let $\mathcal{E}_k \subset \Sigma$ be a finite $1/k$-network and Σ_k be an arbitrary solution of the Steiner problem for $\mathcal{E}_k$. By the definition we have

$$\mathcal{H}^1(\Sigma_k) \le \mathcal{H}^1(\Sigma).$$

On the other hand, for any subsequential limit (with respect to the Hausdorff distance) Σ' of the sequence Σ_k we have $\Sigma \subset \Sigma'$ and so

$$\mathcal{H}^1(\Sigma) \le \mathcal{H}^1(\Sigma') \le \liminf_{k\to\infty} \mathcal{H}^1(\Sigma_k)$$

by Gołąb's theorem. It follows that Σ_k converges to Σ and $\mathcal{H}^1(\Sigma_k)$ converges to $\mathcal{H}^1(\Sigma)$.

Summing up, every maximal distance minimizer is a limit of finite Steiner trees. Similar results are also proved in [2]. More detailed and structural relations of finite Steiner trees and maximal distance minimizer are considered in Section 4.3.1.

4.5.3 NP-hardness

It is well-known that Euclidean Steiner problem is NP-hard [53] even is we restrict the terminals to two lines in the plane [112]. The first source of hardness is that if we fix a topology then one can write the length in the explicit form. However the expression may have $\Omega(n)$ square roots.

To avoid it Garey, Graham and Johnson [53] introduce a discrete version of the Steiner problem. Of course a minimizer of a new problem does not inherit any geometric properties, in particular we have no $2\pi/3$-condition at a branching point. Such a discretization appears to be NP-complete (and so the initial one is NP-hard), namely, Garey, Graham and Johnson used a reduction of the X3C problem to this version of the Steiner problem. The X3C problem is to decide whether a family of 3-sets $\mathcal{F} \subset 2^{[3n]}$ has a subfamily of n sets which cover $[3n]$. It is well-known that X3C is NP-complete.

First we need the following reduction to the classical Steiner problem.

Theorem 4.5.1 (Garey–Graham–Johnson [53])**.** *For a given $\mathcal{F} \subset 2^{[3n]}$ one can construct in a polynomial time in n an input $X(\mathcal{F}) \subset \mathbb{R}^2$ whose size is also polynomial in n such that*

(i) if $\mathcal{F}$ has an n-set covering then a solution of the Steiner problem for $X(\mathcal{F})$ has the length at most L;

(ii) if $\mathcal{F}$ does not have an n-set covering then a solution of the Steiner problem for $X(\mathcal{F})$ has the length at least $L + 12|X(\mathcal{F})|$.

Moreover $L = L(\mathcal{F})$ can be extracted from the construction of $X(\mathcal{F})$ in an explicit form.

Now let us repeat Garey–Graham–Johnson rounding in the case of maximal distance minimizers. The following problem is a discrete approximation of Problem 4.1.2 analogous to the discrete version of Steiner problem used in [53]. Following [53] we replace the length function with is ceiling because it is not known if the problem of determining whether $\sum \sqrt{n_i} < L$ is NP or not (n_i, L are integer).

Problem 4.5.1. *Let M be a finite set of points in the plane with integer coordinates and $r, \ell \in \mathbb{N}$. Decide whether exists a connected graph whose vertices have integer coordinates and edges are segments with the sum of the ceiling function of the length over edges at most ℓ such that every point of M lies at a distance at most r from some vertex of the graph.*

Now we are ready to obtain the following corollary of Garey–Graham–Johnson results and the approximation.

Proposition 4.5.2. *Problem 4.5.1 is NP-complete.*

Proof. Let $\mathcal{F} \subset 2^{[3n]}$ be an arbitrary family. Consider the set $X(\mathcal{F})$ from Theorem 4.5.1. Fix any $r \in \mathbb{N}$ and let $k > 10r|X(\mathcal{F})|$ be a large integer number. Define $kX(\mathcal{F})$ as a set homothetic to $X(\mathcal{F})$ with the scale factor k. Let M be the set of closest points of $kX(\mathcal{F})$ in the integer grid $\mathbb{Z}^2$. Put also $\ell = kL(\mathcal{F}) + k$.

Then if F has an n-set covering, then a solution $\mathcal{S}t$ of the Steiner problem for $kX(\mathcal{F})$ has the length at most $kL(\mathcal{F})$. Now we replace in $\mathcal{S}t$ every vertex with the closest point from $\mathbb{Z}^2$; denote the resulting set by $\mathcal{S}t^D$. By the definition $\mathcal{S}t^D$ is a graph whose vertices have integer coordinates and it connects M; also it has at most $2|X(\mathcal{F})| - 3$ segments. After the rounding the length of every segment of $\mathcal{S}t$ grows by at most $\sqrt{2}$. Hence the ceiling function of the length of an edge in $\mathcal{S}t^D$ is at most the length of the corresponding edge of $\mathcal{S}t$ plus 3. Thus sum of the ceiling function of the length over edges $\mathcal{S}t^D$ is at most

$$kL + 6|X(\mathcal{F})| < \ell$$

and the answer to Problem 4.5.1 is positive.

On the other hand let us show that in the case when F has no n-set covering, the answer to Problem 4.5.1 is negative. Consider a solution Σ^D of Problem 4.5.1 for $X(\mathcal{F})$. Assume the contrary, so that the sum of the ceiling function of length over the edges of Σ^D is at most ℓ. It implies $\mathcal{H}^1(\Sigma^D) \le \ell$. Consider the homothety $\Sigma^D_{1/k}$ of Σ^D with the scale factor $1/k$, one has

$$\mathcal{H}^1(\Sigma^D_{1/k}) = \frac{\mathcal{H}^1\Sigma^D)}{k} \le \frac{\ell}{k}.$$

By definition for every $x \in M)$ there is a point $\sigma \in \Sigma^D$ at a distance at most r from x. Hence for every $x \in X(\mathcal{F})$ there is a point $\sigma \in \Sigma^D_{1/k}$ at a distance at most $(r+1)/k < 1$ from x. Thus the length of a Steiner tree for $X(\mathcal{F})$ is at most

$$\frac{\mathcal{H}^1(\Sigma^D)}{k} + |X(\mathcal{F})| \cdot \frac{r+1}{k} \le \frac{\ell + (r+1)\cdot |X(\mathcal{F})|}{k} \le L + |X(\mathcal{F})|.$$

We got a contradiction with Theorem 4.5.1 and thus finished the reduction of the X3C problem to Problem 4.5.1 with the input M, r, ℓ.

Finally one can easily compute the sum of the ceiling function of length over edges of a competitor for Problem 4.5.1 in polynomial time.

□

4.5.4 Penalized form

Let M be a given compact set. Let us consider a problem of minimization $F_M(S) + \lambda\mathcal{H}^1(S)$ for some $\lambda > 0$, where $F_M(S) = \max_{y\in M} \operatorname{dist}(y, S)$ among all connected compact sets S. We will call this problem *λ-penalized*.

Clearly every set T which minimizes λ-penalized problem for some λ is a maximal distance minimizer for a given data M and the restriction of energy $r := F_M(T)$. Hence the solutions of this problem inherit all regularity properties of maximal distance minimizers.

As usual in variational calculus on a restricted class, it may happen for a small variation $\Phi_\varepsilon(\Sigma)$ of Σ, that the length constraint $\mathcal{H}^1(\Phi_\varepsilon(\Sigma)) \le l$ is violated. Hence to compute Euler–Lagrange equation associated to the maximal distance minimization problem a possible way is to consider first the penalized functional $F_M(S) + \lambda\mathcal{H}^1(S)$ for some constant λ, for which any competitor Σ is admissible without length constraint.

Hence it is also make sense to consider *local penalisation problem*: the problem of searching a connected compact set S of a finite length, such that $\mathcal{H}^1(S) + \lambda F_M(S) \le \mathcal{H}^1(T) + \lambda F_M(T)$ for every connected compact T with $\operatorname{diam}(S\triangle T) < \varepsilon$ for sufficiently small $\varepsilon > 0$. The solutions of these problems also inherit properties of local maximal distance minimizers.

Proposition 4.5.3. *Consider*

$$\min_{\Sigma \text{ compact and connected}} F_M(\Sigma) + \lambda(\mathcal{H}^1\Sigma) - l)^+$$

for any constant $\lambda > 1$. Then this problem is equivalent to the maximal distance minimization problem.

Proof. The same as for average distance minimizers (see Proposition 23 in [84]). We use the fact that for a connected set $S \setminus T_\varepsilon$ if S is a maximal distance minimizer and $\mathcal{H}^1(T_\varepsilon) = \varepsilon$ there holds $r - F_M(S \setminus T_\varepsilon) \le \varepsilon$. □

4.5.5 Uniqueness

Let us start with the following simple observation. The set of minimizers for M being a circle $B_R(o)$ is uncountable for $r < R$. Indeed, any minimizer has no loops and does not reduce to a point, so its rotations rarely coincide.

Note that for every compact $M \subset \mathbb{R}^d$ and r equal to the radius of the smallest ball containing M, there is a unique point o such that $M \subset \overline{B_r(o)}$, id est the solution of Problem 4.1.2 is unique. For a larger r one has an uncountable number of solutions. This motivate us to consider only small enough r. Let us call a finite point configuration M *ambiguous* if Problem 4.1.2 has several solutions for M and $r < r_0(M)$. The following statement is a straightforward corollary from the Theorem 2.1.1.

Proposition 4.5.4. *For $n \geq 4$ the set of planar n-point ambiguous configurations M has Hausdorff dimension $2n-1$ (as a subset of $\mathbb{R}^{2n}$).*

Proof. Fix $n \geq 4$. Theorem 2.1.1 states that the Hausdorff dimension of planar n-point configurations with multiply Steiner trees is $2n-1$.

Recall that a topology T of a labelled Steiner tree is the corresponding abstract graph; a topology is *full* if every its terminal has degree 1. We call a topology *generic* if it has no terminals of degree 3. For a generic topology R one can replace vertex A of degree 2 with a Steiner point b and add edge bA; the resulting topology $T(R)$ is full.

By Proposition 2.1.1 the set of all configurations for which there is a realization of any degenerate topology has Hausdorff dimension $2n-2$.

Let us show that if a Steiner tree for a finite M is unique then M is not ambiguous. Consider any n-point planar configuration M with unique Steiner tree $\mathcal{S}t$ whose topology is generic. Let the length of the second locally minimal tree be $\mathcal{H}^1(\mathcal{S}t) + a$ and choose $r < a/(2n)$.

Then a maximal distance minimizer for a given M and r is obtained by a convexity argument for a topology T. Thus the Hausdorff dimension of planar n-point ambiguous configuration is at most $2n-1$.

To show that the Hausdorff dimension $2n-1$ of the set of planar n-point ambiguous configurations is at least M we word-by-word repeat the argument of Lemma 2.4.5.

□

Note that we need $n \geq 4$ is the proof since there is only one full topology for each $n \leq 3$.

4.6 On minimizers for a planar convex closed smooth curve

4.6.1 The class of M considered in the section

Fix a positive real r and a closed convex curve M with the minimal radius of curvature $R > r$ (this implies $C^{1,1}$-smoothness of M). Introduce the notation: $N := \text{conv}(M)$; let M_r be the inner part of the boundary of $B_r(M)$, and finally put $N_r = \text{conv}(M_r)$. In the literature, M_r is often called parallel or equidistant curve. Note that M_r also is a closed convex curve M with the minimal radius of curvature $R - r$ and inherits $C^{1,1}$-smoothness. For each point $y \in M$, the circle $\partial B_r(y)$ touches M_r; let us call the tangent point n_y. The segment $[yn_y]$ is orthogonal to the curves M and M_r.

Further Σ denotes an arbitrary minimizer for M. We need the following simple observation. The definitions of ∂T are given in Section 4.6.2.

Lemma 4.6.1. *Under the assumptions of this section*

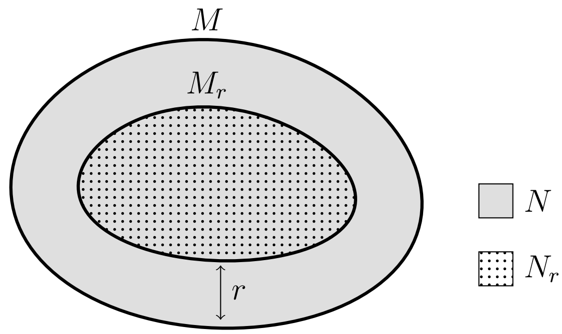


Figure 4.8: Definitions of N, M_r, N, and N_r

(i) a connected component T of the set $\overline{\Sigma \cap \operatorname{Int} N_r}$ is a Steiner tree, moreover, $\partial T \subset M_r$;

(ii) the set $\Sigma \cap \operatorname{Int} N_r$ is a subset of the Steiner part of $\mathcal{S}_\Sigma$.

Proof. Note that, by the basic property of minimizers (b), an energetic point $x \in \Sigma$ is located at a distance r apart the point $y(x) \in M$, therefore $\operatorname{dist}(x, M) \leq r$, which entails item (ii).

Now we prove item (i). Consider a connected component T of the set $\overline{\Sigma \cap \operatorname{Int} N_r}$; by already proven item (ii), the set ∂T belongs to M_r. By Theorem 4.2.1 (i) the set ∂T is finite. If T is not a Steiner tree for $T \cap M_r$, then there is a shorter tree T' for the same points. Then replacing Σ with $\Sigma \setminus T \cup T'$ preserves connectivity, does not increase energy, and reduces the length of the minimizer. Contradiction. $\square$

4.6.2 Pseudo-networks

We will also often use the surprising fact that there is a unique simple path between two points in a path-connected acyclic set, in particular in a Steiner tree or a locally minimal tree.

For a given tree T we denote the set of its vertices of degree 1 or 2 as ∂T.

Remark 4.6.1. *Let T be an arbitrary (full) Steiner tree, and L an arbitrary line. Then the closure of an arbitrary connected component $T \setminus L$ is also a (full) Steiner tree for a finite subset of $\partial T \cup (T \cap L)$.*

Definition 4.6.1. *Define a "wind rose" as a set of six rays starting at the origin point with angle $\pi/3$ between any two adjacent rays; each ray is given a weight (a real number), which satisfy the following property: the weight of a ray is the sum of weights of two rays adjacent to it.*

It follows, in particular, that the sum of the weights of two opposite rays (the ones forming a line) is zero.

By *full Steiner pseudo-network* let us call a connected set S which contains C, if for any wind rose $\mathcal{R}$ such that

(i) S consists of finite number of segments which are parallel to $\mathcal{R}$

the following holds:

(ii) no point of C is incident to exactly two segments,

(iii) for any $x \in S \setminus C$ and small enough $\varepsilon > 0$, sum of weights of rays of $\mathcal{R}$ which are parallel to rays of the form $[xy)$, $y \in \partial B_\varepsilon(x) \cap S$, is zero.

It is clear that a full Steiner tree or a full locally minimal tree is a full Steiner pseudo-network. Note that a full Steiner pseudo-networks may contain cycles, and a point may be incident to 1, 2, 3, 4 or 6 segments; in what follows this number is called the *degree* of the point. For a given full Steiner pseudo-network T we denote by ∂T the set of its points of degree 1. The diagram in Fig. 4.9 illustrates the inclusions of certain classes of sets.

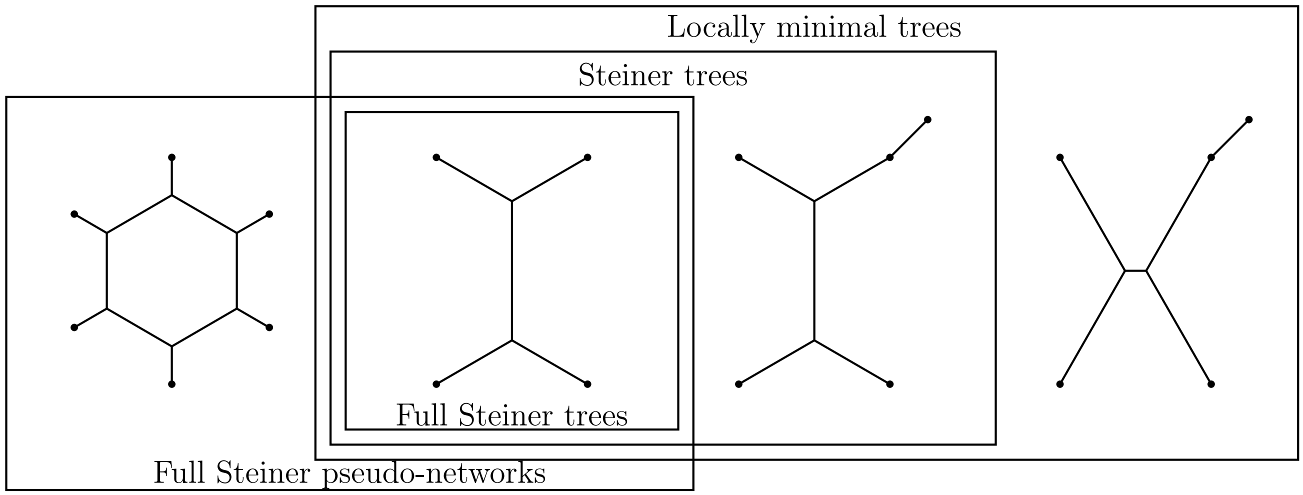


Figure 4.9: Inclusion relations between the used notions

For a given pseudo-network T let us denote by ∂T set of vertices of degree 1.

Remark 4.6.2. *Suppose that T is a full Steiner pseudo-network, and $\mathcal{R}$ is an arbitrary wind rose satisfying (i). Let us assign to an each vertex $x \in \partial T$ a weight of a ray of $\mathcal{R}$, which is parallel to a directed segment of T entering x (such segment is unique by definition of ∂T). Then sum of assigned numbers over all $x \in \partial T$ is zero.*

Lemma 4.6.2. *Let T be a full Steiner pseudo-network, L an arbitrary line intersecting T at a finite number of points. Then*

$$\sharp(\partial T \cap L) \leq 2\sharp(\partial T \setminus L).$$

Proof of Lemma 4.6.2: Let L^+, L^- be the two open half-planes bounded by L. Note that it is sufficient to prove the inequality for a closure of an arbitrary component of $\overline{T \cap L^+}$ and $\overline{T \cap L^-}$, denote such closure as S. By definition, S is also a full Steiner pseudo-network.

Without loss of generality, we can assume that the center of $\mathcal{R}$ lies in the same half-plane as S. Let us choose the weights on the rays of the rose $\mathcal{R}$ in such a way that all those and only those rays that intersect L have positive weights: obviously, such a choice exists since L intersects 3 or 2 neighboring rays $\mathcal{R}$. In the first case we provide them with weights $1, 2, 1$, in the second case $1, 1$. Then the remaining rays will have weights $-1, -2, -1$ or $0, -1, -1, 0$ (we list all weights counterclockwise). For each vertex $x \in \partial S$ there is a unique ray from $\mathcal{R}$ codirected with the segment S included in x; let us assign the weight of this ray to vertex x. Then the sum over endpoints S belonging to L is at least $\sharp(S \cap L)$. On the other hand, the sum over all other endpoints of S is at least $-2\sharp(\partial S \setminus L)$. Due to Remark 4.6.2 the sum of the weights over all endpoints of S is equal to zero, that is

$$\sharp(\partial S \cap L) - 2\sharp(\partial S \setminus L) \leq 0,$$

which completes the proof of the lemma. $\square$

Remark 4.6.3. *Let T be a full Steiner pseudo-network fully lying on one side of line L, such that equality in Lemma 4.6.2 is achieved. Then all leaf vertices in $\partial T \setminus L$ have weight -2, therefore all segments of T incident to vertices from $\partial T \setminus L$ are pairwise collinear.*

Lemma 4.6.3. *Consider a regular tripod with ends a, b and c and a branch point f. Let g be the intersection point of the lines (af) and (bc). Then*

$$\frac{\pi}{3} < \angle agb, \angle agc < \frac{2\pi}{3}.$$

Proof. The angle $\angle agb$ is an exterior angle of the triangle formed by the bisector of the angle $\angle bfc$ and the lines (bc) and (fc). Since the angle at f in this triangle is equal to $\pi/3$, the angle $\angle agb$ is strictly greater than $\pi/3$. Similar reasoning for $\angle agc$ and the identity $\angle agb + \angle agc = \pi$ imply the required inequalities. □

Observation 4.6.1. *A full Steiner pseudo-network with three endpoints is a regular tripod.*

4.6.3 Structural properties of minimizers in the annulus $N \setminus N_r$

Recall that we work in the setting from Subsection 4.6.1. The proofs of the next few lemmas are essentially contained in the paper [18], but we will rewrite them almost verbatim for our more general case.

Lemma 4.6.4. *The inclusion $\Sigma \subset N$ holds.*

Proof. Assume the contrary and consider the projection of the closure of a connected component of the set $\Sigma \setminus N$ onto the (closed convex) set N. It is well known (see Chapter 1.2 in [114]) that the projection $\mathrm{pr}(x)$ of a point x onto a convex body is defined, and also that

$$|\,\mathrm{pr}(x) - \mathrm{pr}(y)| \leq |x - y|.$$

Thus, the length does not increase after the projection of a connected closed set onto a convex set. Obviously, equality is achieved only if the set and its image are parallel segments.

If there are at least two connected components of the set $\Sigma \setminus N$, then none of them is a segment parallel to the corresponding segment N. If the only component of $\Sigma \setminus N$ is a segment parallel to the segment N, then $\Sigma \setminus N = \Sigma$. Hence M does not lie in $B_r(\Sigma)$, since $N = \mathrm{conv}(M)$ contains a ball of radius $R > r$, where R is defined in Section 4.6.1, and the width of $B_r(\Sigma)$ in the direction orthogonal to Σ is equal to $2r$.

On the other hand, for any $x \in \Sigma \setminus N$, $y \in M \setminus \{\mathrm{pr}(x)\}$ angle $\mathrm{pr}(x)$ in triangle $x\,\mathrm{pr}(x)y$ is at least $\pi/2$, therefore the distance between any pair of points from Σ and M does not increase during projection, which means that the energy does not increase. Preservation of connectivity follows from the fact that each connected component of the set $\Sigma \setminus N$ remains connected under projection, and the set Σ inside N does not change. The resulting contradiction with the optimality of Σ finishes the proof. □

Consider the closure of an arbitrary connected component $\Sigma \setminus N_r$; denote it by S and reserve this designation in the current section. Let us call points from $S \cap M_r$ *entering points*. We will call the continuous image of an (open, half-open, closed) segment (respectively *open, half-open, closed) arc*. From the connectedness and closedness of $S \subset (N \setminus \mathrm{Int}\, N_r)$ it follows that $\overline{B_r(S)} \cap M$ is a closed arc; let us denote it $Q(S)$. In what follows we will show that $Q(S) \neq M$. Since M at each point has a strictly positive radius of curvature, it is $C^{1,1}$ smooth and has a tangent at each point. Then by *angular measure of the arc* $\gamma \subset M$ we mean the directed angle between the tangent rays to M at the ends of γ. This value is also equal to the limit of the sum of external angles in polychains approximating γ, with nodes consequentially lying on M. The angular measure of an arc γ is a non-negative quantity not exceeding 2π.

We say that a subset $m \subset M$ *is covered* by a subset $\sigma \subset \Sigma$ if $m \subset \overline{B_r(\sigma)}$. Recall that point n_q is defined in Section 4.6.1.

Proposition 4.6.1. *Let $y \in M$ correspond to two energetic points $x_1, x_2 \in \Sigma \setminus N_r$. Then the points x_1 and x_2 lie on opposite sides of the line (yn_y).*

Proof. Since M is a smooth curve with a radius of curvature greater than r at each point, for each of the points x_1 and x_2 the set covered by the point is an arc; let us call these arcs $Q(x_1)$ and $Q(x_2)$ respectively. Note that the arcs $Q(x_1)$ and $Q(x_2)$ have a common endpoint y. Suppose that x_1 and x_2 lie on the same side of the line (yn_y), then one of the arcs is a subset of the other, without loss of generality, $Q(x_2) \subset Q(x_1)$. If $Q(x_2)$ is a proper subset of $Q(x_1)$, then for a sufficiently small $\rho > 0$ the arc covered by the set $B_\rho(x_2) \cap \Sigma$ is contained in $Q(x_1)$, that is, x_2 is not an energetic point. So $Q(x_1) = Q(x_2) =: Q$.

Let for any $\rho > 0$ set $B_\rho(x_1) \cap \Sigma$ covers a larger arc than Q. Then, similarly to the previous argument, x_2 is not energetic. Finally, if there is $\rho_1 > 0$ such that $B_{\rho_1}(x_1) \cap \Sigma$ covers exactly Q, then x_1 is not an energetic point, because the entire arc Q is covered by point x_2. □

Lemma 4.6.5. *Under the assumptions of this section, $Q(S) \neq M$.*

Proof. If $Q(S) = M$, then S is connected and covers M, hence $\Sigma = S$. Then $S = \overline{S \setminus N_r}$, and since S is the closure of the connected component $\Sigma \setminus N_r = S \setminus N_r$, the set $S \setminus N_r$ is connected.

Let us assume that there are two different corresponding points $y_1, y_2 \in M$ (they can correspond to either the same energetic point or different ones). Then the ring $N \setminus N_r$ is divided by the balls $B_r(y_1)$ and $B_r(y_2)$ into two connected components C_1 and C_2. Since $\Sigma \cap B_r(y_i) = \emptyset, i = 1, 2$, and $S \setminus N_r$ is connected, the set S belongs to the closure of one of the components, say $\overline{C_1}$. Then Σ does not cover one of the two arcs $[y_1 \breve{} y_2]$ in M. Therefore, there is at most one corresponding point.

So, all energetic points in S correspond to a single point y. Then, according to Proposition 4.6.1, any two energetic points in $S \setminus N_r$ lie on opposite sides of the line (yn_y), which means there are no more than two energetic points in $S \setminus N_r$. Also, the point n_y can belong to S and be energetic. Thus, there are no more than three energetic points in S, and all of them lie on the circle $\partial B_r(y)$. Then $\Sigma = S$ is a Steiner tree on the set of its energetic points, and if there are at least two of these points, Σ intersects $B_r(y)$, which contradicts to the definition of the corresponding point. This means that Σ contains unique energetic point, so Σ is itself a point. But the set M has a radius of curvature greater than r at each point, which means it cannot be covered by one point, a contradiction. □

Having proved Lemma 4.6.5, we can consider the ends of $Q(S)$; let us call them e_1 and e_2. Directly from the definitions $B_r(e_i) \cap S = \emptyset$, where $i = 1, 2$. Consider the arc $Z_{Q(S)} \subset M_r$, consisting of all possible points n_q, where $q \in Q(S)$. By definition, S belongs to the (closed) domain $D = D(S)$ bounded by $\partial B_r(e_1)$, $\partial B_r(e_2)$, the arc $Z_{Q(S)}$ and sometimes arc $Q(S)$, see Fig. 4.10. It is easy to see that the angular measures of the arcs $\partial B_r(e_1) \cap D$, $\partial B_r(e_2) \cap D$ do not exceed $\pi/2$.

The following lemma essentially appeared in [18] (the proof of these statements does not use the additional requirement $R > 5r$, which is inherited from the main theorem of the paper [18]).

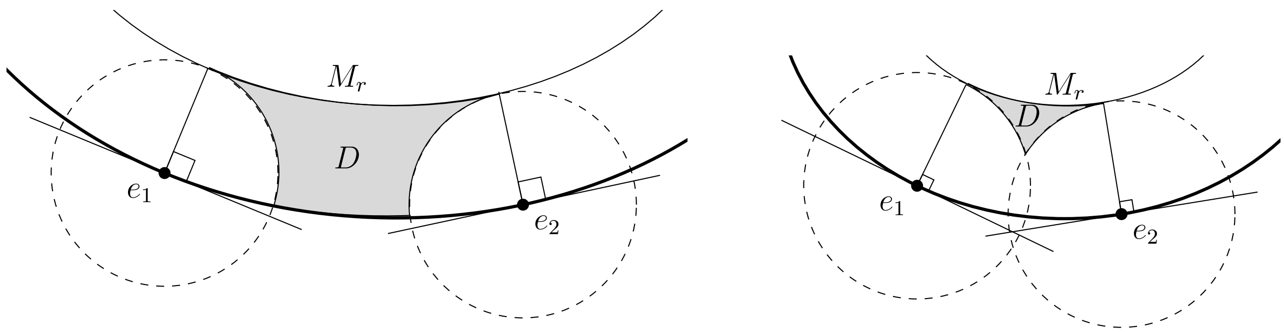


Figure 4.10: Two possibilities for a domain $D \supset S$

Lemma 4.6.6. *The following statements are true.*

(i) Let $x \in S$ be an energetic point. Then an arbitrary corresponding point $y(x)$ belongs to $\{e_1, e_2\}$.

(ii) Each of the ends e_1, e_2 corresponds to at most one energetic point from $S \setminus N_r$.

(iii) The set of non-isolated energetic points $\mathcal{E}_\Sigma$ is a subset of M_r.

(iv) The set S is a locally minimal tree for its entering points and energetic points.

(v) The set $S \setminus N_r$ contains one or two energetic points.

(vi) Let the energetic point $x \in S \setminus N_r$ have a unique corresponding point $y(x)$. Then

(a) if x has degree 1 (that is, by item (iii) it is the end of some segment $[zx] \subset \Sigma$), then z, x and $y(x)$ lie on the same line;

(b) if x has degree 2 (that is, by item (iii) it is the end of the segments $[z_1x], [xz_2] \subset \Sigma$), then the ray $[y(x)x)$ contains the bisector of angle z_1xz_2.

Proof. Assume the contrary to item (i). Then $B_r(y) \cup \{n_y\}$ divides D into two non-empty regions, with $\partial B_r(e_1) \setminus B_r(y)$ and $\partial B_r(e_2) \setminus B_r(y)$ lie in different regions. But the set S should intersect both arcs $\partial B_r(e_i)$, since $e_1, e_2 \in Q(S)$, and therefore S contains n_y. But then $S \setminus \{n_y\}$ consists of two connected components, which contradicts the definition of S.

Assume the contrary to item (ii). Then $\partial B_r(e_i) \cap D$ contains two energetic points $x_1, x_2 \in S \setminus N_r$, which contradicts the Proposition 4.6.1.

Let the point $x \in \Sigma \setminus N_r$ belong to $\mathcal{E}_\Sigma$. Since items (i) and (ii) imply that each connected component $\Sigma \setminus N_r$ has at most two energetic points, there is an infinite set of components containing a point from an arbitrarily small neighborhood x. This contradicts the finiteness of the length of Σ. Lemma 4.6.1(ii) completes the proof of item (iii).

Note that the neighborhood of each point of the Steiner part of the set $S \setminus M_r$ is a segment or a regular tripod. Theorem 4.2.1(i) and the connection of S complete the proof of item (iv).

It immediately follows from items (i) and (ii) that there are no more than two energetic points in the set $S \setminus N_r$. Let us assume that $S \setminus N_r$ does not contain energetic points. Then S is a locally minimal tree for vertices on M_r, hence $S \subset N_r$; the resulting contradiction shows the validity of item (v).

Suppose the contrary to item (vi)(a), then the point $t := \partial B_r(y(x)) \cap [y(x)z]$ is different from x. But then the set $(\Sigma \setminus [xz]) \cup [zt]$ is connected, covers M and has a length strictly less than that of Σ, a contradiction.

Suppose the contrary to item (vi)(b), that is, that $\angle z_1 x y(x) \neq \angle y(x) x z_2$. Let L be the tangent line to $B_r(y(x))$ at point x. Then

$$\mathcal{H}^1(\Sigma \cap B_\varepsilon(x)) - \mathcal{H}^1([z_1' t] \cup [t z_2']) = \Omega(\varepsilon),$$

where z_1' and z_2' are the intersections of the segments $[z_1 x]$ and $[z_2 x]$ with $\partial B_\varepsilon(x)$, and t is such a point on the line L such that $\angle z_1' t x + \angle x t z_2' = \pi$ (see the left side of Fig. 4.11). On the other hand, $\operatorname{dist}(t, \partial B_r(y(x))) = O(\varepsilon^2)$; let us define $v \in \partial B_r(y(x))$ as the point for which $\operatorname{dist}(t, \partial B_r(y(x))) = \operatorname{dist}(t, v)$ holds. Then the set $(\Sigma \setminus B_\varepsilon(x)) \cup [z_1' t] \cup [z_2' t] \cup [tv]$ is connected, covers M and has length strictly less than Σ. The resulting contradiction completes the proof of the lemma. $\square$

Remark 4.6.4. *Note that if the entering point x is energetic, then by Lemma 4.6.6(i) one has $x = n_{e_1}$ or $x = n_{e_2}$. Moreover, such a point x, in contrast to energetic points that are not also entering points, can belong to a set of non-isolated energetic points.*

Since S covers exactly $Q(S)$, the sets $S \cap B_r(e_1)$ and $S \cap B_r(e_2)$ are empty, and the sets $S \cap \partial B_r(e_1)$ and $S \cap \partial B_r(e_2)$ are non-empty. For each pair of points $x_1 \in S \cap \partial B_r(e_1)$, $x_2 \in S \cap \partial B_r(e_2)$ consider the area bounded by the arc $Q(S)$, with radii $[e_1 x_1]$ and $[e_2 x_2]$ and the path from x_1 to x_2 to S. Note that the intersection of any two regions of the described type is also a region of the described type. Consequently, there is a minimal such region, let us call it $E(S)$ (one example of the $E(S)$ region is shown on the right side of Fig. 4.11). Note that in view of items (iv) and (i) of the Lemma 4.6.6 and the minimality of $E(S)$, one has $S \cap \operatorname{Int} E(S) = \emptyset$.

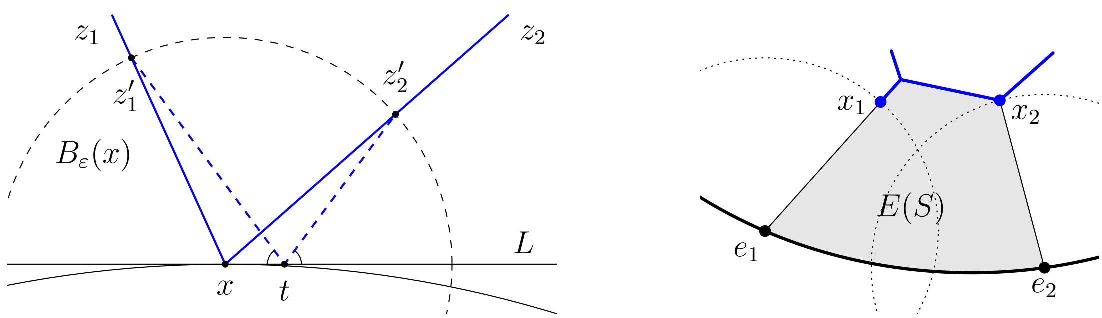


Figure 4.11: The left side shows the situation near the point x in part (vi)(b) of the Lemma 4.6.6. On the right is one of the possible options $E(S)$.

Lemma 4.6.7. *The region $E(S)$ is convex.*

Proof. The boundary $E(S)$ consists of the arc $Q(S)$ and a polychain. Recall that M is convex, so it suffices to show that all interior (with respect to $E(S)$) angles of the polychain are at most π, including the angles between the polychain and the tangents to $Q(S)$ at points e_1 and e_2.

The angle between the polychain and the tangent to $Q(S)$ at point e_i does not exceed $\pi/2$, since the corresponding radius $[e_i x_i]$ lies inside the right angle between the radius $[e_i n_{e_i}]$ and tangent to $Q(S)$ at point e_i.

If the polychain consists of one point, then $x_1 = x_2$ and the interior angle at this point is not greater than π, since otherwise $[e_1 e_2] \subset B_r(e_1) \cup B_r(e_2)$, then there is $[e_1 e_2] \cap S = \emptyset$, but $[e_1 e_2]$ separates x_1 from M_r, which contradicts the connectedness of S.

Consider the angle of the polychain at point x_i. If x_i is not energetic, then its neighborhood is a segment tangent to the circle $\partial B_r(e_i)$, and the angle of interest to us is equal to $\pi/2$. If x_i is an energetic point of degree 1, then by Lemma 4.6.6(vi)(a) the angle is equal to π. If x_i is an energetic point of degree 2, then by Lemma 4.6.6(vi)(b) and in view of the condition $S \cap \operatorname{Int} E(S) = \emptyset$ the angle is not greater than $2\pi/3$.

It remains to deal with the angles of the polychain at the internal vertices. By Lemma 4.6.6(i) the polychain does not contain energetic points, except, possibly, the ends. This means that all internal (relative to $E(S)$) angles of the polychain at internal vertices are equal to $2\pi/3$ or $4\pi/3$. But in the latter case $S \cap \operatorname{Int} E(S) \neq \emptyset$, which is impossible.

□

The following theorem is the main statement of this subsection.

Theorem 4.6.1 (Cherkashin–Gordeev–Strukov–Teplitskaya [24]). *Let S be the closure of the connected component $\Sigma \setminus N_r$. Then*

(i) convex hull of S is a segment, a triangle or a quadrilateral, the vertices of $\operatorname{conv}(S)$ *are energetic points of $S \setminus N_r$ and entering points of S, with at most two points of each type;*

(ii) S contains at most three entering points.

Proof of item (i). If S has less than three entering points, then the assertions of item (i) immediately follow from Lemma 4.6.6(iv) and (v): S is a local Steiner tree whose only entering points and energetic points are not a convex combination of points from the neighborhood, and there are at most two energetic points $S \setminus N_r$.

By Lemma 4.6.5 the set $A_0 = \overline{B_r(Q(S))} \cap M_r$ is an arc. Let us consider the largest subarc $A \subset A_0$ by inclusion, both ends of which are entering points S (we will call them *extreme*). Let us consider an arbitrary entering point x contained in the interior of this arc (there is one, because we assumed that there are at least three entering points), and draw a tangent L to M_r in it. By Remark 4.6.4 point x is not energetic, therefore its neighborhood in Σ is a segment or tripod. By definition of S, a neighborhood x in S is a segment not belonging to L. Due to the convexity of N_r, the tangent L does not intersect $\operatorname{Int} N_r$. This means that the set S intersects L on both sides of x, say at points t_1 and t_2. Then the entering point x is a convex combination of points t_1 and t_2. Thus, we have proven that all entering points, except the two extreme ones, lie inside the convex hull S. □

Proof of item (ii). Let us assume that there are at least three entering points and define the arc $A \subset M_r$ similarly to item (i); let us denote its ends w_1 and w_2. By Remark 4.6.4 if the entering point is energetic, then it is inevitably extreme.

Let w be an arbitrary entering point from the interior of A. Let us denote by $R = R(w)$ the ray starting at the point w and extending beyond the point w the segment S containing it.

Lemma 4.6.8. *The ray R intersects the segment $[w_1 w_2]$.*

Proof. If the ray R does not intersect the segment $[w_1 w_2]$, then it either touches M_r at point w or intersects M_r at some point $u \in A$. Let us recall that the point w cannot be energetic according to the Remark 4.6.4, hence the tangent case contradicts the connectedness of $S \setminus N_r$.

Thus, the ray R intersects M_r at some point $u \in A$. Since w belongs to the Steiner part of the minimizer, its neighborhood in Σ is a segment or tripod. In the first case, there is a single connected component $\Sigma \cap \operatorname{Int}(N_r)$ containing w. In view of the Lemma 4.6.1, the closure T of this component is a full Steiner tree, and therefore, in view of the Remark 4.6.1, contains an end vertex different from w in each of the closed half-planes separated by the line (wu). Moreover, all end vertices T belong

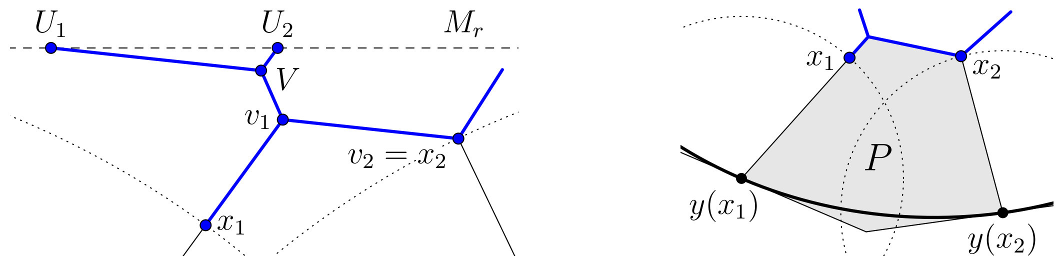


Figure 4.12: Illustration to the proof of Proposition 4.6.1 (ii)

to the set M_r, which means there is a vertex w' belonging to the half-open arc $[\breve{wu}] \setminus \{w\} \subset A \subset M_r$; Moreover, w' does not belong to the set S due to the absence of cycles in Σ. The point w' is not energetic, since in this case $B_r(y(w')) \cup \{w'\}$ divides the set D, and hence S, into two non-empty parts; on the other hand, $B_r(y(w')) \cap \Sigma = \emptyset = B_r(y(w')) \cap S$ and $w' \notin S$. Therefore w' is the entering point of the closure of some connected component of the set $\Sigma \setminus N_r$; let us call this closure S'. Since Σ does not contain cycles, $D(S') \subset D(S)$ and $Q(S') \subset Q(S)$. The last inclusion implies the absence of energetic points in S', which contradicts Lemma 4.6.6(v).

The case of a tripod is analyzed in exactly the same way. $\square$

For each entering point $w \notin \{w_1, w_2\}$ we denote by $i(w)$ the point of intersection of the ray $R(w)$ with the segment $[w_1w_2]$. We also put $i(w_1) = w_1$, $i(w_2) = w_2$. Let $\mathcal{S}t(S)$ denote the union of S and all segments $[wi(w)]$. By Lemma 4.6.6(iv) the set $\mathcal{S}t(S)$ is the union of several full Steiner pseudo-networks. Let us consider two cases.

(a) Let the set $S \setminus N_r$ have one energetic point x. By Lemma 4.6.6(iii) $x \in \mathcal{X}_\Sigma$.

- Let x have degree 1. In this case, $\mathcal{S}t(S)$ is a full Steiner pseudo-network. By Lemma 4.6.2 applied to $\mathcal{S}t(S)$ and the line (w_1w_2), the pseudo-network $\mathcal{S}t(S)$ intersects (w_1w_2) at most twice, since x is the only endpoint of $\mathcal{S}t(S)$ outside (w_1w_2). Then, by Lemma 4.6.8, the set S has at most two entering points.
- Let x have degree 2. Then the sets $S \setminus \{x\}$ and $\mathcal{S}t(S) \setminus \{x\}$ have exactly two connected components; let us denote by S_1, S_2 and $\mathcal{S}t_1$, $\mathcal{S}t_2$, respectively, their closures. Obviously, $\mathcal{S}t_1$ and $\mathcal{S}t_2$ are full Steiner pseudo-networks, with $\mathcal{S}t_1 \cup \mathcal{S}t_2 = \mathcal{S}t(S)$.

  Let us show that each of the sets S_1 and S_2 has at most two entering points. Let us assume the opposite: without loss of generality, let the set S_1 have at least three entering points. Then the full Steiner pseudo-network $\mathcal{S}t_1$ has at least three endpoints on the line (w_1w_2) and only one outside this line. Contradiction with Lemma 4.6.2.

  Thus, it is necessary to exclude only the situation in which both sets have two entering points, that is, according to Observation 4.6.1 they are tripods.

  For each tripod $\mathcal{S}t_i$ (where $i \in \{1; 2\}$) we denote its branch point by v_i. Let g_i be the intersection point of the lines (xv_i) and (w_1w_2). Then by Lemma 4.6.3

$$\frac{\pi}{3} < \angle x g_i w_j < \frac{2\pi}{3},$$

  where $i, j \in 1, 2$. But then $\angle v_1 x v_2 = \angle g_1 x g_2 = \pi - \angle x g_1 w_2 - \angle x g_2 w_1 < \pi/3 < 2\pi/3$, which contradicts the local minimality of S, namely the Lemma 4.6.6(iv).

  Thus, in this case S cannot have more than three entering points.

(b) Let the set $S \setminus N_r$ have two energetic points x_1 and x_2. Let us show that $\partial E(S) \cap S$ is the path between x_1 and x_2 in S; let us call this path P. Assuming the contrary, then, without loss of generality, there exists a point $x \in S \cap \partial B_r(e_1)$ such that $\angle x e_1 n_{e_1} > \angle x_1 e_1 n_{e_1}$. By Lemma 4.6.7 and the fact that $S \cap \operatorname{Int} E(S) = \emptyset$ the segment $[e_1 e_2]$ does not intersect S, that is, from the inequality $\angle x e_1 n_{e_1} > \angle x_1 e_1 n_{e_1}$ it follows that $Q(x_1)$ is a proper subset of $Q(x)$, where $Q(x^*)$ denotes the arc $\overline{B_r(x^*)} \cap M$. Then x_1 is not energetic, a contradiction.

By Lemma 4.6.7 the domain $E(S)$ is convex, therefore the sum of the exterior angles of the broken part $\partial E(S)$ and the angular measure of the arc $Q(S)$ is equal to 2π.

By Lemma 4.6.6(iv), the vertices of $\partial E(S)$ from the set S are x_1, x_2, and branch points. The external angle at the branch points is $\pi/3$. By Lemma 4.6.6(vi) at energetic points of degree 1 the external angle is equal to 0, and at energetic points of degree 2 it is at least $\pi/3$. Recall that the sum of the external angles $\partial E(S)$ at the vertices e_1 and e_2 is at least π in total. If the sum of these angles is π, then $x_1 = n_{e_1}$ and $x_2 = n_{e_2}$, which contradicts the fact that $x_1, x_2 \in S \setminus N_r$. Also, the angular measure $Q(S)$ is non-negative. Therefore, the exterior angle is nonzero in at most two vertices of S.

Let x_i (where $i \in \{1;2\}$) have degree 1 and be connected by the segment Σ to x_{3-i}. Then we can remove the segment $[x_1 x_2]$ from $\mathcal{S}t(S)$ and, by Lemma 4.6.2, the remaining full Steiner pseudo-network $\overline{\mathcal{S}t(S) \setminus [x_1 x_2]}$ has no more than two intersection points with $(w_1 w_2)$, which means S has at most two entering points.

In the remaining situation, if the point x_i is a point of degree 1, then the segment $[x_i x_{3-i}]$ does not belong to Σ. In this case, we denote by v_i the branch point to which point x_i is connected by the segment Σ. If x_i has degree 2, we define $v_i = x_i$ (see Fig. 4.12).

Regardless of the degrees of the points x_i, the points v_1 and v_2 are vertices of P at which $\partial E(S)$ has a nonzero exterior angle. Since $E(S)$ contains at most 2 such vertices from S, either $v_1 = v_2$, or Σ contains a non-degenerate segment $[v_1 v_2]$.

If $v_1 = v_2$, then removing the segments $[v_1 x_1]$ and $[v_2 x_2]$ from $\mathcal{S}t(S)$ allows us to apply the Lemma 4.6.2 to the line $(w_1 w_2)$ and the full pseudo-network $\overline{\mathcal{S}t(S) \setminus ([v_1 x_1] \cup [v_2 x_2])}$, and find that in this case S has at most two entering points.

If Σ contains a non-degenerate segment $[v_1 v_2]$, then three cases are possible.

- Let both points x_1 and x_2 be of degree 1, then $\mathcal{S}t(S)$ is a full Steiner pseudo-network. Application of Lemma 4.6.2 to the line $(w_1 w_2)$ shows that $\mathcal{S}t(S)$ intersects $w_1 w_2$ in no more than four points, which means S contains no more than 4 entering points, and if there are 4 entering points, then equality is achieved in the lemma, and by Remark 4.6.3 the rays $[v_1 x_1)$ and $[v_2 x_2)$ are codirected. This contradicts the fact that P contains exactly 2 branch points v_1 and v_2.
- Let both points x_1 and x_2 have degree 2 and, accordingly, coincide with points v_1 and v_2. Then $P = [x_1 x_2] = [v_1 v_2]$. Then removing $[v_1 v_2] \setminus \{v_1, v_2\}$ splits $\mathcal{S}t(S)$ into two full pseudo-networks $\mathcal{S}t_1 \ni v_1$ and $\mathcal{S}t_2 \ni v_2$, in neither of which, In view of Lemma 4.6.2, there cannot be more than two entering points. Let each of them have two entering points, then by Observation 4.6.1 $\mathcal{S}t_1$ and $\mathcal{S}t_2$ are tripods. Let us denote by u_i the branching point of the corresponding tripod, and by g_i the intersection of the lines $(v_i u_i)$ and $(w_1 w_2)$. By Lemma 4.6.3 the angles g_1 and g_2 of the quadrilateral $g_1 g_2 v_2 v_1$ are strictly greater than $\pi/3$, and by the property of the local Steiner tree the angles v_2 and v_1 are at least $2\pi/3$, which when summed gives an immediate contradiction.

– Let one of the points (without loss of generality, x_1) have degree 1, and the other have degree 2 (see Fig. 4.12). Removing the set $[v_1v_2] \cup [v_1x_1] \setminus \{v_1, v_2\}$ from $\mathcal{S}t(S)$ splits $\mathcal{S}t(S)$ into two pseudo-networks $\mathcal{S}t_1 \ni v_1$ and $St_2 \ni v_2$. Then, by Lemma 4.6.2, each of the pseudo-networks $\mathcal{S}t_1$ or $\mathcal{S}t_2$ has at most two entering points, and if they both have two entering points, by Observation 4.6.1 they are tripods. Let us denote by u_i the branching point of the corresponding tripod, and by g_i the intersection of the lines (v_iu_i) and (w_1w_2). By Lemma 4.6.3 the angles g_1 and g_2 of the quadrilateral $g_1g_2v_2v_1$ are strictly greater than $\pi/3$, by the property of the local Steiner tree the value of the angle v_2 is at least $2\pi/3$, and finally the angle v_1 is equal to $2\pi/3$. The resulting contradiction completes the proof of the theorem.

□

Now we are going to focus on the relations between the components. Denote the set of closures of a connected components of $\Sigma \setminus N_r$ by $V_C(G)$ and the set of maximal (with respect to the inclusion) arcs of $\Sigma \cap M_r$ of positive length by $V_A(G)$ (further $V_A(G) \cap V_C(G)$ will be associated with a subset of the vertex set of a graph).

Lemma 4.6.9. *If $S \in V(G) := V_C(G) \sqcup V_A(G)$ does not reduce to a point, then*

$$Q_S \not\subset \bigcup_{S' \in V(G) \setminus \{S\}} Q_{S'}.$$

Proof of Lemma 4.6.9. The fact that S has an energetic point immediately implies that Q_S does not belong to the union of $Q_{S'}$ over $S' \in V(G) \setminus \{S\}$. Suppose the contrary, i.e. that S has no energetic point.

If S is the closure of a connected component of $\Sigma \setminus N_r$, then by Lemma 4.7.3 S is a locally minimal tree for its entering points, but $m(S) \leq 2$, hence S is a segment with endpoints on M_r, which is impossible for a connected component of $\Sigma \setminus N_r$.

If S is a non degenerate arc $[\breve{bc}]$, then $[\breve{bc}] \subset S_\Sigma$, which is impossible by the definition of $V_A(G)$. □

Lemma 4.6.10. *The set $V(G) = V_C(G) \sqcup V_A(G)$ is finite.*

Proof of Lemma 4.6.10. Suppose the contrary. Consider an arbitrary $\varepsilon > 0$ (which later will be chosen sufficiently small). First, note that Lemma 4.6.9 implies that every point of M belongs to at most two different arcs q_S, where $S \in V(G)$ (otherwise, there are three arcs of M containing a point $x \in M$, so one of them is contained in the union if others, which is impossible by Lemma 4.6.9). Thus the sum of $\mathcal{H}^1(q_S)$ over $V(G)$ is at most $2\mathcal{H}^1(M)$, and therefore there is only a finite number of connected components and arcs with $\mathcal{H}^1(q_S) \geq \varepsilon$. Denote by $V_\varepsilon(G)$ the infinite set of such $S \in V(G)$ that $\mathcal{H}^1(q_S) < \varepsilon$.

Obviously, if $V(G)$ is an infinite set, then $V_C(G)$ is an infinite set. Let us show that there are infinitely many chords of M_r in Σ intersecting $\mathrm{Int}(N_r)$ (if N, and hence N_r, is strictly convex then in fact every chord of M_r intersects $\mathrm{Int}(N_r)$). Suppose the contrary. Then $\Sigma \setminus \mathrm{Int}(N_r)$ has a finite number of connected components; but $V_C(G)$ is infinite, hence there are components containing infinitely many elements of $V_C(G)$; let K be one of these components containing at least five different elements of $V_C(G)$. Obviously, $q_K := \overline{B_r(K)} \cap \Sigma$ is connected. By Lemma 4.6.9 $K \setminus M_r$ contains 5 energetic points, such that they belong to different elements of $V_C(G)$. Call them W_1, W_2, W_3, W_4, W_5 such that Q_{W_1}, Q_{W_2}, Q_{W_3}, Q_{W_4}, $Q_{W_5} \in q_K$ belong to M_r in the natural (clockwise) order. Then $B_r(Q_{W_i}) \cap \Sigma = \emptyset$, $i = 1, \ldots, 5$ and therefore K should contain the points I_2, I_3, $I_4 \in M_r$ such that

$$\mathrm{dist}\,(Q_{W_2}, I_2) = \mathrm{dist}\,(Q_{W_3}, I_3) = \mathrm{dist}\,(Q_{W_4}, I_4) = r$$

(because $K \setminus I_j$ must be disconnected, $j = 2, 3, 4$). Consider the path between I_2 and I_4 in K. It should coincide with $[I_2 \breve{I}_4] \subset M_r$, otherwise we reduce the length of Σ, projecting the path on M_r. So W_3 should belong to M_r which is impossible by the choice of W_i, $i = 1, \dots, 5$ and gives the desired contradiction. Thus the set Ch of chords of M_r in Σ intersecting $\mathrm{Int}(N_r)$ is infinite.

There is at most a finite number of chords of length at least ε because $\mathcal{H}^1(\Sigma)$ is finite. Let us exclude from the infinite set Ch a finite set of chords of length at least ε and a finite set of chords adjacent to a component not in $V_\varepsilon(G)$; denote the resulting set by Ch': chords in Ch' are adjacent only to the elements of $V_\varepsilon(G)$ and have length strictly less than ε. Let us show that any of the chords in Ch' connects components without Steiner points. Suppose the contrary. The following three cases have to be considered:

(i) A chord in Ch' is adjacent to a connected component $S \in V_\varepsilon(G)$ with $m(S) = 2$ containing a Steiner point. Then the angle between the entering segments of the component is at most $2\pi/3$ (in fact, it must be between $\pi/3$ and $2\pi/3$). Recall that $\mathcal{H}^1(q_S) < \varepsilon$, hence by the triangle inequality S is a subset of an ε-neighbourhood of M_r (otherwise $\mathrm{dist}\,(x, y) \le r - \varepsilon$ for some $x \in S$, $y \in M$, so $B_\varepsilon(y) \cap M \subset q_S$ which contradicts $\mathcal{H}^1(q_S) < \varepsilon$). So, when ε is sufficiently small, recalling smoothness of M_r one has that one of the entering segments has angle with M_r at least $\pi/12$. It implies that the entering point I of this segment is not energetic, so by Lemma 4.7.2 its neighbourhood is a segment and it is an end of a chord $[IJ] \subset \Sigma$ of M_r. So by the constraint on the radius of curvature of M chord $[IJ]$ has length more than ε, which gives a contradiction with the assumption that our chord is in Ch'.

(ii) A chord in Ch' is adjacent to a connected component $S \in V_\varepsilon(G)$ with $m(S) = 1$ containing a Steiner point. Then it has the combinatorial type (b) on Fig. 4.33. Let us consider the triangle ΔQCI, where Q is an end of q_S, C is the branching point of S, I is the entering point of S. Since $\angle QCI = 2\pi/3$, we have $\angle QIC \le \pi/3$, so the angle between the entering segment $[CI]$ and M_r is at least $\pi/6$. Then again the chord $[IJ]$ has length more than ε, that contradicts the choice of the chord.

(iii) Finally, a chord in Ch' is adjacent to an arc $S \in V_\varepsilon(G)$ containing a Steiner point x. Then $x \in M_r$, and x is an end of a chord of M_r in Σ which forms angle $\pi/3$ with M_r. Again by the condition on the radius of curvature of M_r and with the choice of ε sufficiently small, this chord has length more than ε which is impossible.

Let us consider any chord $[I_1I_2] \in Ch'$, such that it connects some components from $V_\varepsilon(G)$ (which do not have Steiner points as proven). Note that the set $]I_2I_1) \cap \Sigma$ (resp. $]I_1I_2) \cap \Sigma$) contains an energetic point (it may coincide with I_1 (I_2); if I_1 (I_2) is not energetic, an energetic point on $]I_2I_1) \cap \Sigma$ (resp. $]I_1I_2) \cap \Sigma$) exists by Lemma 4.7.2 and the absence of Steiner points in the considered connected components and arcs); denote the nearest to I_1 (resp. I_2) energetic point of $]I_2I_1) \cap \Sigma$ (resp. $]I_1I_2) \cap \Sigma$) by W_1 (resp. W_2).

Consider the region P bounded by the segments $[W_1Q_{W_1}]$, $[W_2Q_{W_2}]$, $[W_1W_2]$ and the lesser arc $[Q_{W_1}\breve{Q}_{W_2}]$ of M. Let us show that the intersection of $\mathrm{Int}(P)$ with Σ is nonempty. There are two tangent lines to M_r parallel to $[W_1W_2]$; let l be the nearest line to $[W_1W_2]$. Note that $[I_1I_2] \in Ch' \subset Ch$, so $[I_1I_2] \cap \mathrm{Int}(N_r) \neq \emptyset$ and $l \cap [W_1W_2] = \emptyset$. Consider a point $w \in l \cap M_r$ and note that Q_w is not covered by Σ, because $\mathrm{dist}\,(Q_W, \Sigma) = \mathrm{dist}\,(Q_w, [W_1, W_2]) > \mathrm{dist}\,(Q_w, l) = r$. We got a contradiction, so $\mathrm{Int}(P) \cap \Sigma \neq \emptyset$.

Let us pick a point $x \in \mathrm{Int}(P) \cap \Sigma$ and consider the path in Σ connecting x with the segment $[W_1W_2]$. The existence of this path gives that for some $i \in \{1, 2\}$ (say, without loss of generality,

$i = 1$) one has $W_i = I_i$ (in fact, $]W_1W_2[\subset S_\Sigma$, which means that this path connects x with W_1 without touching $]W_1W_2]$, but a neighbourhood in Σ of an energetic point of $\Sigma \setminus N_r$ is either a single line segment or two line segments with angle at least $2\pi/3$, see Fig. 4.33 and 4.34, and thus $W_1 \in M_r$) and $B_\delta(I_1) \cap \mathrm{Int}(P) \cap \Sigma \neq \emptyset$ for sufficiently small $\delta > 0$. Let k be the tangent line to M_r at $I_1 = W_1$. Since $|I_1I_2| \leq \varepsilon$, the angle between k and $[I_1I_2]$ is $O(\varepsilon)$. Consider an arbitrary point $y \in \partial B_\delta(I_1) \cap \mathrm{Int}(P) \cap \Sigma$. Since $B_r(Q_{I_1}) \cap \Sigma = \emptyset$ and $|yI_1| = \delta$ the angle between k and $[yI_1]$ is $O(\delta)$. Let z be a projection of y on $[I_1I_2]$. Then $\angle yI_1z = O(\varepsilon + \delta)$ is the smallest angle (for sufficiently small ε, δ) in right-angled triangle ΔyI_1z. Hence one can replace $]I_1z[$ by $[zy]$ in Σ. The new set is still connected, covers M and has strictly lower length than Σ. We got in this way a contradiction with the optimality of Σ, concluding the proof.

□

Note that a singleton of $\Sigma \cap M_r$ (a maximal arc $\xi \subset \Sigma \cap M_r$ of zero length not contained in the closure of a connected component of $\Sigma \setminus N_r$) cannot be energetic (by the previous Lemma the union of Q_S over $S \in V(G) \setminus \xi$ is closed as a finite union of closed sets, hence it coincides with M because $q_\xi = \{q_\xi\}$), so a neighbourhood of ξ is a segment or a tripod (the latter is impossible by Lemma 4.7.2). Summing up, every point of $\Sigma \cap M_r$ is contained in a maximal arc of M_r of positive length or in the closure of a connected component of $\Sigma \setminus N_r$. Also by Lemma 4.7.3 every connected component of $\Sigma \setminus N_r$ contains at most 5 segments, thus Σ consists of a finite number of segments and arcs of M_r.

Lemma 4.6.11. *Let $[bi] \subset \Sigma$ be a chord of M_r. Then $i \in S_\Sigma$ and moreover there exists such an $\varepsilon > 0$ that $\overline{B_\varepsilon(i)} \cap \Sigma = [i_1i_2]$, for some $i_1, i_2 \in \partial B_\varepsilon(i)$.*

Proof of Lemma 4.6.11. Note that in Σ there are at most two chords of M_r ending at i. It is true because of the properties of a locally minimal tree: the angle between two segments ending at the same point is greater or equal to $2\pi/3$.

Let us show that $i \in S_\Sigma$. Assume the contrary: let $i \in G_\Sigma$. Then $B_r(q_i) \cap \Sigma = \emptyset$. There are two possibilities:

(1) $i \in S$, where $S \in V_C(G)$;

(2) $i \in S$, where $S \in V_A(G)$ (as mentioned after Lemma 4.6.10 S is non degenerate i.e. does not reduce to a single point i).

Recall that Σ consists of a finite number of segments and a finite number of arcs of M_r. In the case (1) the smoothness of M_r, Lemma 4.7.3 and the fact $B_r(q_i) \cap \Sigma = \emptyset$ imply that the intersection of a small neighbourhood of i with $S \setminus N_r$ is a subset of the tangent line to M_r at i.

Thus the set $\Sigma \cap B_\varepsilon(i) \setminus \mathrm{Int}(N_r)$ is contained in the union of the tangent line τ to M_r at i and the arc $M_r \cap \partial B_\varepsilon(i)$. Both $\tau \cap B_\varepsilon(i)$ and $M_r \cap B_\varepsilon(i)$ are split by i into 2 segments $[ie'_1]$, $[ie'_2]$ and 2 arcs $[\breve{ie_1}]$, $[\breve{ie_2}]$ of M_r, respectively, where $e_1, e_2, e'_1, e'_2 \in \partial B_\varepsilon(i)$. we may assume e_1 in the same halfplane with e'_1 bounded by the normal to M_r passing throw i. At least one arc and one segment (say, $[\breve{ie_1}]$ and $[ie'_1]$) have angle at most $\pi/2$ with the chord $[ib]$. The cases (i) and (ii) below deal with the situation with nonempty set $\Sigma \cap ([\breve{ie_1}] \cup [ie'_1])$. In the remaining cases $\Sigma \cap B_\varepsilon(i) \setminus \mathrm{Int}(N_r)$ is a subset of $[ie'_2] \cup [\breve{ie_2}]$ and therefore in (iii)-(vi) we deal with all the possible cases of $B_\varepsilon(i) \cap [ie'_2]$ and $B_\varepsilon(i) \cap [\breve{ie_2}]$ empty/nonempty:

(i) There is such a segment $[ie] \subset \Sigma$, that (ie) is the tangent line to M_r, $|ie| = \varepsilon$ and $\angle bie \leq \pi/2$;

(ii) There is such an $\varepsilon > 0$ and an arc $[\breve{ie}] \subset \Sigma \cap M_r$ that $|ie| = \varepsilon$ and $\angle bie \leq \pi/2$;

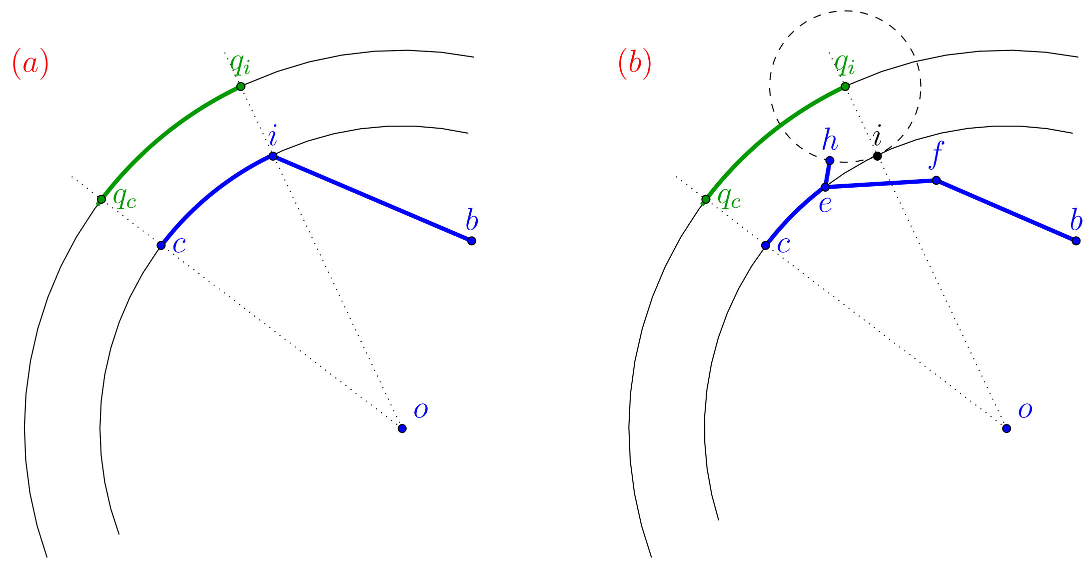


Figure 4.13: The case (iv) from Lemma 4.6.11: (a) the (impossible) part of the minimizer; (b) a better competitor.

(iii) There is such a small $\varepsilon > 0$ that $\overline{B_\varepsilon(i)} \cap \Sigma$ is equal to $[fi] \cup [ie]$ where $f, e \in \partial B_\varepsilon(i)$, $[fi] \subset [bi]$ and $[ie]$ is a subset of the tangent line to M_r at point i;

(iv) There is such a small $\varepsilon > 0$ that $\overline{B_\varepsilon(i)} \cap \Sigma$ is equal to $[fi] \cup [\breve{ie}]$, where $f, e \in \partial B_\varepsilon(i)$, $[fi] \subset [ib]$ and $[\breve{ie}] \subset M_r$;

(v) There is such a small $\varepsilon > 0$ that $\overline{B_\varepsilon(i)} \cap \Sigma$ contains $[fi] \cup [ic] \cup [\breve{id}]$ where $[ic]$ is a subset of the tangent line to M_r at point i, $[fi] \subset [bi]$, $[\breve{id}] \subset M_r$ and $\angle cid < \pi/6$;

(vi) There is such an $\varepsilon > 0$ that $\overline{B_\varepsilon(i)}$ is a subset of chord $[ib]$.

we will show that all these cases are impossible. Let ξ stand for the segment $[ie]$ in the cases (i) and (iii), and for $[\breve{ie}]$ in the cases (ii) and (iv).

CASES (i), (ii): Let $f := [bi] \cap B_\varepsilon(i)$ and l^ε be the lesser arc of $\partial B_\varepsilon(i)$ limited by intersections with $\partial B_r(q_i)$ and M_r. It is easy to see that $\mathcal{H}^1(l^\varepsilon) = O(\varepsilon^2)$ and $|fi| + \mathcal{H}^1(\xi) - \mathcal{H}^1(\mathcal{S}t(f, i, e)) = c\varepsilon + o(\varepsilon)$ with $c > 0$, where $\mathcal{S}t(f, i, e)$ is a Steiner tree connecting points f, i, e. Then the length of $\Sigma' := \Sigma \setminus ([fi] \cup \xi) \cup l^\varepsilon \cup \mathcal{S}t(f, i, e)$ is less than $\mathcal{H}^1(\Sigma)$ for sufficiently small ε. Moreover Σ' is still connected and $F_M(\Sigma') \leq F_M(\Sigma)$. This gives us a contradiction with optimality of Σ.

CASES (iii), (iv): Note that $|fi| = |ie| = \varepsilon$ (see Fig. 4.13(a)), so $\mathcal{H}^1(\xi) = \varepsilon + o(\varepsilon)$ when $\varepsilon \to 0^+$, because M_r is smooth. Let h be the point of intersection of $[eq_i]$ and $\partial B_r(q_i)$ (see Fig. 4.13(b)). Note that (iq_i) is perpendicular to the tangent line to M_r at the point i. Thus

$$\begin{aligned} |eh| &= |eq_i| - |q_i h| = \sqrt{|ei|^2 + r^2} - r = \sqrt{\varepsilon^2 + r^2} - r \\ &= r\sqrt{1 + o(\varepsilon)} - r = o(\varepsilon). \end{aligned}$$

Now, since the angle between ξ and the segment $[fi]$ is less than π, we get

$$|ef| = \sqrt{2\varepsilon^2 - 2\varepsilon^2 \cos \angle eif} = \sqrt{2}\varepsilon\sqrt{1 - \cos \angle eif} < 2\varepsilon - c\varepsilon, \quad \text{for some} \quad c > 0$$

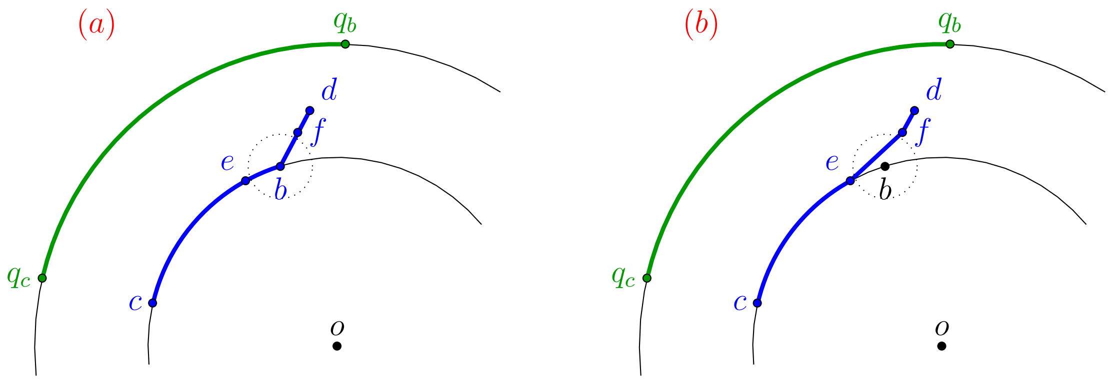


Figure 4.14: The case of an outer segment in the proof of Lemma 4.6.12: (a) the (impossible) part of the minimizer; (b) the better competitor.

and therefore

$$|eh| + |ef| < \mathcal{H}^1(\xi) + |if| = 2\varepsilon + o(\varepsilon)$$

for sufficiently small $\varepsilon > 0$. So we have a contradiction with the optimality of Σ, because we show that $(\Sigma \setminus B_\varepsilon(i)) \cup [eh] \cup [ef]$ is the better competitor.

CASE (v): Let $h \in [ic)$ be such a point that $(dh) \perp (ic)$. Then the set

$$\Sigma' = \Sigma \setminus]\breve{id}] \cup [hd]$$

is still connected, has energy F_M not greater than Σ and strictly smaller length, since $|hd| < |id|/2 \leq \mathcal{H}^1([\breve{id}])/2$. It means Σ' is the better competitor than Σ, again a contradiction.

CASE (vi): In this case $S \in V_A(G)$ and $S = \{i\}$, which is impossible.

So all cases are impossible and we have a contradiction which implies $i \in S_\Sigma$. Because of Lemma 4.7.2 i can not be a Steiner point. Then there exists an $\varepsilon > 0$ such that $S_\Sigma \cap B_\varepsilon(i)$ is a segment. $\square$

Lemma 4.6.12. *Every (maximal with respect to inclusion) arc $[\breve{bc}] \in V_A(G)$ is continued by segments lying on tangent lines to M_r in the sense that there exists such an open $U \supset [\breve{bc}]$ that $\Sigma \cap \overline{U} = [b'b] \cup [\breve{bc}] \cup [cc']$, where $[b'b]$ and $[cc']$ are subsets of tangent lines to M_r at points b, c respectively.*

Proof of Lemma 4.6.12. Let $\breve{bc}$ be as in the statement being proven.

Suppose that there is a segment $[ij] \subset \Sigma$ such that $i =]\breve{bc}[\cap [ij]$. we claim that $B_\varepsilon(i) \cap \Sigma \subset [\breve{bc}]$. In fact, by Lemma 4.6.11 $[ij]$ cannot be a part of a chord of M_r, so $]ij] \subset \Sigma \setminus \mathrm{Int}(N_r)$. Note that in this case i is energetic (because $B_\varepsilon(i)$ is not a segment or a tripod for every $\varepsilon > 0$). Hence $B_r(q_i) \cup \Sigma = \emptyset$, so $[ij]$ is a part of the tangent line to M_r at i. Let us choose an $\varepsilon > 0$ and set $\{d_1, d_2\} := [\breve{bc}] \cap \partial B_\varepsilon(i)$, $e := [ij] \cap \partial B_\varepsilon(i)$. If $\varepsilon > 0$ is sufficiently small one of the angles $\angle d_1 ie$, $\angle d_2 ie$ is less than $\pi/6$ (say $\angle d_1 ie$). Let $h \in [ij)$ be such a point that $(d_1 h) \perp (ij)$. Then the set

$$\Sigma' := \Sigma \setminus]\breve{id_1}] \cup [hd_1]$$

is still connected, has energy F_M not greater than $F_M(\Sigma)$ and strictly smaller length, since $|hd_1| < |id_1|/2 \leq \mathcal{H}^1([\breve{id_1}])/2$. It means that Σ' is better competitor than Σ. we got a contradiction, showing thus $B_\varepsilon(i) \cap \Sigma \subset [\breve{bc}]$ for $i \in]\breve{bc}[$.

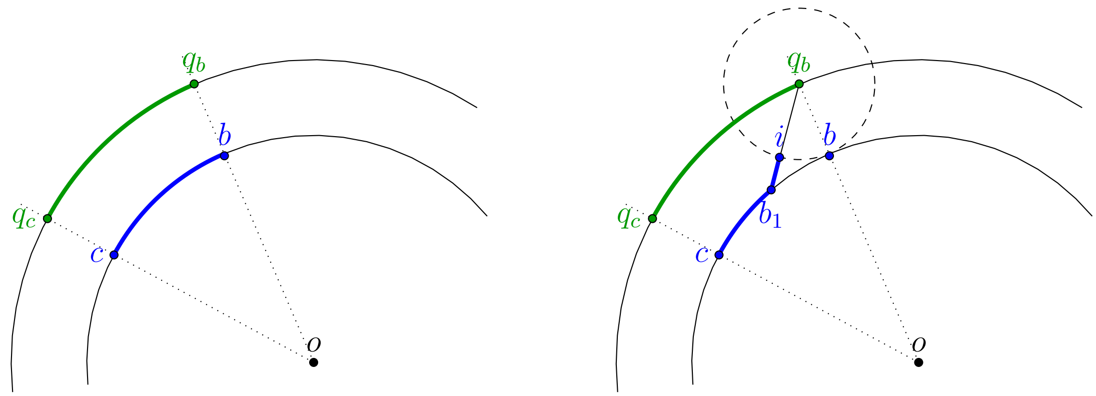


Figure 4.15: Picture to Lemma 4.6.12. An end of an arc of $M_r \cap \Sigma$ cannot be an endpoint of Σ.

Let us prove now that $B_\varepsilon(b) \setminus [\breve{bc}]$ is a subset of the tangent line to M_r at b (the analogous statement for the point c is completely symmetric). By Lemma 4.6.11 there is no chord of M_r in Σ with endpoint b. So the set $B_\varepsilon(b) \setminus [\breve{bc}]$ is a subset of $\Sigma \setminus N_r$.

we claim first that b is not an endpoint of Σ i.e. $B_\varepsilon(b) \setminus [\breve{bc}] \neq \emptyset$. Assume the contrary and recall that $q_b, q_c \in M$ are such points that $\operatorname{dist}(b, q_b) = \operatorname{dist}(c, q_c) = r$. Then one can set $b_1 := \partial B_\varepsilon(b) \cap [\breve{bc}]$ and replace $[\breve{b_1b}]$ by the segment $[b_1i] := [b_1q_b] \setminus B_r(q_b)$, producing the competitor of strictly lower length because $[\breve{bc}] \setminus [\breve{b_1b}] \cup [b_1i] = [\breve{b_1c}] \cup [b_1i]$ still covers the arc $[\breve{q_bq_c}]$ of M (when ε is sufficiently small), see Fig. 4.15.

Therefore we have proven that for sufficiently small $\varepsilon > 0$ the set $B_\varepsilon(b) \setminus [\breve{bc}]$ is a nonempty subset of $\Sigma \setminus N_r$. If b is energetic then $B_r(q_b) \cap \Sigma = \emptyset$, hence $B_\varepsilon(b) \setminus [\breve{bc}]$ is a subset of the tangent line to M_r at point b showing the claim. So $b \in S_\Sigma$, hence $B_\varepsilon(b)$ is a segment or a tripod for sufficiently small $\varepsilon > 0$. But the case of a tripod is impossible by Lemma 4.7.2, while the case of a segment is only possible recalling smoothness of M_r (and part of M_r in a neighbourhood of b is in fact flat).

Summing up, the only segments intersecting $[\breve{bc}]$ are segments tangent to M_r at points b and c. As a consequence of Lemma 4.6.10 Σ consists of a finitely many segments and maximal arcs of M_r, so when ε is small, $B_\varepsilon([\breve{bc}])$ contains only 2 segments which is proven to be tangent to M_r at points b and c, respectively. The statement is proven. $\square$

Lemma 4.6.13. *Let $c \in M_r \cap \Sigma$. Then Σ has the tangent line at c, in particular for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every couple of points $b, d \in \Sigma \cap b_\delta(c) \setminus c$, holds* $\min(|\angle bcd - \pi|, |\angle bcd|) < \varepsilon$.

Proof of Lemma 4.6.13. Consider a point $c \in M_r \cap \Sigma$. By Lemma 4.6.12 if c belongs to some non degenerate arc of $\Sigma \cap M_r$ with an energetic point in its interior (i.e. an element of $V_A(G)$) the statement is true. Note that if there is a chord $[ic] \subset \Sigma$ of M_r then Lemma 4.6.11 implies the claim. Thus $B_\varepsilon(c) \cap \operatorname{Int}(N_r) = \emptyset$. If $c \in S_\Sigma$ then by Lemma 4.7.2 its neighbourhood cannot be a tripod, so it is a segment and the statement of Lemma is obvious. It remains to consider the case when $B_\varepsilon(c) \cap \operatorname{Int}(N_r) = \emptyset$ and c is energetic, which implies $B_r(q_c) \cap \Sigma = \emptyset$ so the set $B_\varepsilon(c) \cap \Sigma$ is just a segment (because Σ consists of a finite number of arcs of M_r and segments by Lemma 4.6.10) which must be a subset of the tangent line to M_r at c, the claim follows. $\square$

4.6.4 Derivation in the picture

Motivation. Let the point $y \in M$ correspond to two energetic points $x_1, x_2 \in \Sigma$ from the ring $N \setminus N_r$. Then by Lemma 4.6.6(i) and (ii) they belong to the closures of different connected components $\Sigma \setminus N_r$; let us call them S_1 and S_2. The purpose of this section is to describe the structure of the Σ minimizer in small neighborhoods of the points x_1 and x_2.

Informally speaking, we can try to "move" y along M, that is, change the sets S_1 and S_2 in small neighborhoods of x_1 and x_2 in such a way that the resulting sets together are still covered all the same points M as before; but so that the boundary between the arcs M they cover would no longer pass at y, but at point M at a small distance from it. Since Σ is a minimizer, such an operation cannot reduce the length of the entire set, that is, the sum of changes in lengths is non-negative.

Below we formally define the described operation for any of the sets S_1 and S_2 when y is shifted by any sufficiently small distance. By directing the shift value y to zero, we obtain "the derivative of the length Σ in the vicinity of the point x_1 (or x_2)" when y moves along M, we calculate this derivative explicitly and describe the structure Σ in small neighborhoods of x_1 and x_2 in terms of conditions on "derivatives of the lengths of Σ in neighborhoods of x_1 and x_2" when y moves along M.

Definition of derivative. Let S be the closure of the connected component $\Sigma \setminus N_r$. We denote one of the ends of the arc $Q(S)$ by $y_1 \in M$. Let $x \in (\partial B_r(y_1) \setminus N_r) \cap S$ be the energetic point for which y_1 is corresponding. By Lemma 4.6.6(iv) the set S is a local Steiner tree for its entering points and energetic points; in addition, $x \in \partial S$, since the energetic point x cannot have a degree 3. By Lemma 4.6.6(i) x cannot have more than two corresponding points. If there are two corresponding points, we denote the second by y_2. Also, x can have degree 1 or 2. Let us denote the degree of x by $d \in \{1, 2\}$, and the number of points corresponding to x by $k \in \{1, 2\}$. Thus, there are 4 possible cases, each of which we will consider in detail below.

Let us fix a $l > 0$ such that $\overline{B_l(x)} \cap \Sigma$ is the union of d segments of the form $[z_i x]$, $z_i \in \partial B_l(x)$, $1\ leqi \leq d$. For a sufficiently small $0 \leq \varepsilon < \varepsilon_0(l, r, \{y_j\}_{j=1}^k, \{z_i\}_{i=1}^d, x)$ for y_1^ε denote the point obtained by shifting the point y_1 along M by ε (that is, such that the arc M with ends at y_1 and y_1^ε has length ε) in such a direction that $y_1^\varepsilon \notin Q(S)$. For a sufficiently small modulo $0 > \varepsilon > -\varepsilon_0(l, r, \{y_j\}_{j=1}^k \{z_i\}_{i=1}^d, x)$ we denote y_1^ε is the point obtained by shifting y_1 along M by $-\varepsilon$ in the opposite direction (that is, in such a way that $y_1^\varepsilon \in Q(S)$). Let us denote $y_1^0 = y_1$. In the case of $k = 2$, we denote $y_2^\varepsilon = y_2$ for any ε. Let

$$\Gamma(\varepsilon) = \min_{x'} \sum_{i=1}^d |z_i x'|,$$

where the minimum is taken over all points x' such that $|y_j^\varepsilon x'| = r$ for $1 \leq j \leq k$. Let us denote by x_ε the point at which the value $\Gamma(\varepsilon)$ is reached.

Note that $x_0 = x$, since Σ is a minimizer. The derivative $\Gamma(\varepsilon)$ at the origin $\Gamma'(0)$ will be called *the derivative of the length Σ in the neighborhood of the point x as $y_1(x)$ moves along M*. We will show that this derivative exists by calculating it explicitly in each of the four cases. From the explicit form of the derivative, in particular, it will be clear that it does not depend on the auxiliary parameter l used in the definition.

In further discussions on the angle between a curve and a ray with a vertex on the curve, we always mean the smaller of the angles between the ray and the tangent to the curve drawn at the intersection point of the curve and the ray.

Case 1. Let $d=1$, $k=1$ (see the left part of Fig. 4.16). By Lemma 4.6.6(vi)(a) the points z_1, x and y_1 lie on the same line. Since x_ε is the closest to z_1 among the points $\partial B_r(y_1^\varepsilon)$, $x_\varepsilon = [z_1 y_1^\varepsilon] \cap \partial B_r(y_1^\varepsilon)$. Since M is smooth, the distance from the point y_1^ε to the tangent to M at the point y_1 is $o(\varepsilon)$. Let the angle between $(z_1 y_1]$ and M be equal to α. Note that $\alpha < \pi/2$ since $x \notin N_r$. Moreover, $\angle y_1^\varepsilon y_1 x = \pi - \alpha - o(\varepsilon)$ for $\varepsilon > 0$, since in this case $y_1^\varepsilon \notin Q(S)$, and $\angle y_1^\varepsilon y_1 x = \alpha + o(\varepsilon)$ for $\varepsilon < 0$, since in this case $y_1^\varepsilon \in Q(S)$. By the cosine theorem for the triangle $z_1 y_1 y_1^\varepsilon$

$$|z_1 y_1^\varepsilon| = \sqrt{|z_1 y_1|^2 + 2|z_1 y_1|\varepsilon \cos\alpha + \varepsilon^2} + o(\varepsilon) = |z_1 y_1| + \varepsilon\cos\alpha + o(\varepsilon).$$

Then, since

$$\Gamma(\varepsilon) - \Gamma(0) = (|z_1 y_1^\varepsilon| - r) - (|z_1 y_1| - r) = |z_1 y_1^\varepsilon| - |z_1 y_1| = \varepsilon\cos\alpha + o(\varepsilon),$$

$$\Gamma'(0) = \cos\alpha.$$

So, in this case, the derivative of the length Σ in the neighborhood of the point x when the point y_1 moves along M is equal to $\cos\alpha$.

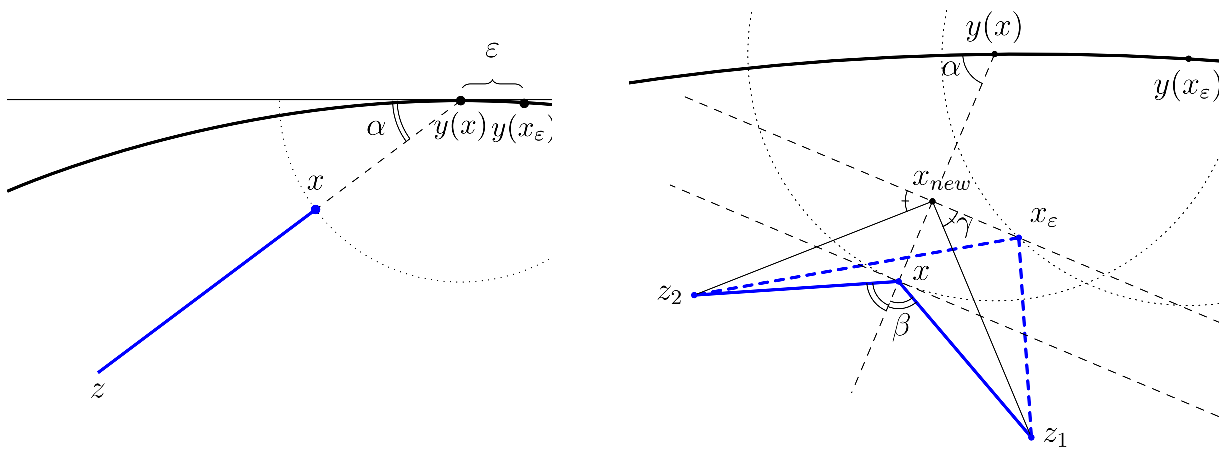


Figure 4.16: The first and second cases

Case 2. Let $d=2$, $k=1$ (see the right side of Fig. 4.16). By Lemma 4.6.6(vi)(b) the ray $[y_1 x)$ contains the bisector of the angle $z_1 x z_2$. Let $\beta = \frac{1}{2}\angle z_1 x z_2$, the angle between $[y_1 x)$ and M is equal to α. Since at point x_ε the minimum value $|z_1 x_\varepsilon| + |z_2 x_\varepsilon|$ among the points $\partial B_r(y_1^\varepsilon)$, the ray $[y_1^\varepsilon x_\varepsilon)$ contains the bisector of the angle $z_1 x_\varepsilon z_2$, which can be understood by repeating the proof verbatim Lemmas 4.6.6(vi)(b).

If we write these two statements about bisectors algebraically, then $x, x_\varepsilon \in N$ are defined as solutions to the following system

$$f_1(x^*) = (x^* - y, x^* - y) = r^2, \qquad f_2(x^*) = \frac{(x^* - z_1, x^* - y)}{\sqrt{(x^* - z_1, x^* - z_1)}} - \frac{(x^* - z_2, x^* - y)}{\sqrt{(x^* - z_2, x^* - z_2)}} = 0$$

for $y = y_1$ and $y = y_1^\varepsilon$ respectively. Outside the points z_1, z_2, the system is smooth, and the gradient f_1 is equal to $2(x^* - y)$, that is, parallel to the bisector of the angle $z_1 x^* z_2$, which is the level line f_2. Therefore the implicit function theorem implies $|x x_\varepsilon| = O(\varepsilon)$.

Let us draw a tangent to $B_r(y_1)$ through x, and parallel to it draw a straight line through x_ε. Let the last straight line intersect the ray $[y_1 x)$ at point x_{new}. Since $|z_1 x| = |z_2 x|$, and the ray $[x_{new} x)$

contains the bisector of the angle $\angle z_1xz_2$, the triangles z_1xx_{new} and z_2xx_{new} are equal. Therefore $\angle z_1x_{new}x_\varepsilon + \angle z_2x_{new}x_\varepsilon = \pi$ and $|z_1x_{new}| = |z_2x_{new}|$; let us denote the last value by l_{new}. Let us denote by γ the angle $z_1x_{new}x_\varepsilon$ and write the cosine theorems for the triangles $z_1x_\varepsilon x_{new}$ and $z_2x_\varepsilon x_{new}$ and use the fact that $|x_\varepsilon x_{new}| \le |xx_\varepsilon| = O(\varepsilon)$:

$$|z_1x_\varepsilon| = \sqrt{l_{new}^2 + |x_\varepsilon x_{new}|^2 - 2l_{new}|x_\varepsilon x_{new}|\cos\gamma} = l_{new} - |x_\varepsilon x_{new}|\cos\gamma + o(\varepsilon),$$

$$|z_2x_\varepsilon| = \sqrt{l_{new}^2 + |x_\varepsilon x_{new}|^2 + 2l_{new}|x_\varepsilon x_{new}|\cos\gamma} = l_{new} + |x_\varepsilon x_{new}|\cos\gamma + o(\varepsilon).$$

Thus,

$$|z_1x_\varepsilon| + |z_2x_\varepsilon| = 2l_{new} + o(\varepsilon). \tag{4.2}$$

Since $|xy_1| = |x_\varepsilon y_1^\varepsilon| = r$, and $|xx_\varepsilon|, |y_1y_1^\varepsilon| = O(\varepsilon)$, the angle between the lines (xy_1) and $(x_\varepsilon y_1^\varepsilon)$ is $O(\varepsilon)$, its cosine is equal to $1 - o(\varepsilon)$, therefore projection length the segment $[x_\varepsilon y_1^\varepsilon]$ to the straight line (xy_1) is equal to $r - o(\varepsilon)$. Let y' be the projection of the point y_1^ε onto the straight line (xy_1). Due to the smoothness of M, the length of the segment y_1y' is equal to $\varepsilon\cos\alpha + o(\varepsilon)$, therefore

$$|x_{new}x| = \varepsilon\cos\alpha + o(\varepsilon).$$

It remains to write the cosine theorem for the triangle z_1xx_{new}:

$$l_{new} = \sqrt{l^2 + (\varepsilon\cos\alpha + o(\varepsilon))^2 + 2l(\varepsilon\cos\alpha + o(\varepsilon))\cos\beta} = l + \varepsilon\cos\alpha\cos\beta + o(\varepsilon). \tag{4.3}$$

Combining (4.2) and (4.3), we get

$$\Gamma(\varepsilon) - \Gamma(0) = 2l_{new} + o(\varepsilon) - 2l = 2\varepsilon\cos\alpha\cos\beta + o(\varepsilon),$$

that is, the desired derivative is equal to

$$\Gamma'(0) = 2\cos\alpha\cos\beta.$$

Cases 3 and 4. In these cases $k = 2$. Point x lies at the intersection of $\partial B_r(y_1) \cap \partial B_r(y_2)$. Note that the circles $\partial B_r(y_1)$ and $\partial B_r(y_2)$ intersect at two points. Indeed, suppose that this is not the case, that is, the circles touch at point x. If $d = 2$, then x lies on the segment $[z_1z_2]$; in this case, the length of Σ can be reduced by removing the connected component $S \setminus \{x\}$ containing one of the z_i that has no entering points. So $d = 1$. In addition, the segment xz_1 lies on the common tangent to the circles. But this contradicts the fact that Σ is a minimizer: consider a point x' on the interval $[xz_1]$ such that $|xx'| = \chi$ for small $\chi > 0$. Since x' lies on the common tangent to the circles, $|x'y_1| = r + o(\chi)$, $|x'y_2| = r + o(\chi)$. Then, for a sufficiently small χ, the length of Σ can be reduced by replacing the segment $[xx']$ with segments connecting x' with the circles $B_r(y_1)$ and $B_r(y_2)$.

Since Σ is connected, lies in N and does not intersect $B_r(y_1) \cup B_r(y_2)$, then x is that point from $\partial B_r(y_1) \cap \partial B_r(y_2)$, which lies on the same side relative to the line (y_1y_2) as the points z_i, $1 \le i \le d$. Let ε be so small that the circles $\partial B_r(y_1^\varepsilon)$ and $\partial B_r(y_2^\varepsilon)$ also intersect at two points. Then x_ε is that point in $\partial B_r(y_1^\varepsilon) \cap \partial B_r(y_2^\varepsilon)$ that lies in the same half-plane relative to the straight line $(y_1^\varepsilon y_2^\varepsilon)$, as the points z_i, $1 \le i \le d$. Let us find x_ε explicitly (see the left side of Fig. 4.17).

Triangles xy_1y_2 and $x_\varepsilon y_1^\varepsilon y_2^\varepsilon$ are isosceles with side r; let $\angle xy_1y_2 = \angle xy_2y_1 =: \alpha$, and also $\angle x_\varepsilon y_1^\varepsilon y_2^\varepsilon = \angle x_\varepsilon y_2^\varepsilon y_1^\varepsilon =: \alpha_\varepsilon$.

Let us introduce the following coordinates: midpoint o of the segment $[y_1y_2]$ be the origin of coordinates; X axis is aligned with the beam $[y_2y_1)$; the Y axis is codirected with the ray $[ox)$. Then

$$o = (0,0), \quad x = (0, r\sin\alpha), \quad y_1 = (r\cos\alpha, 0), \quad y_2 = (-r\cos\alpha, 0).$$

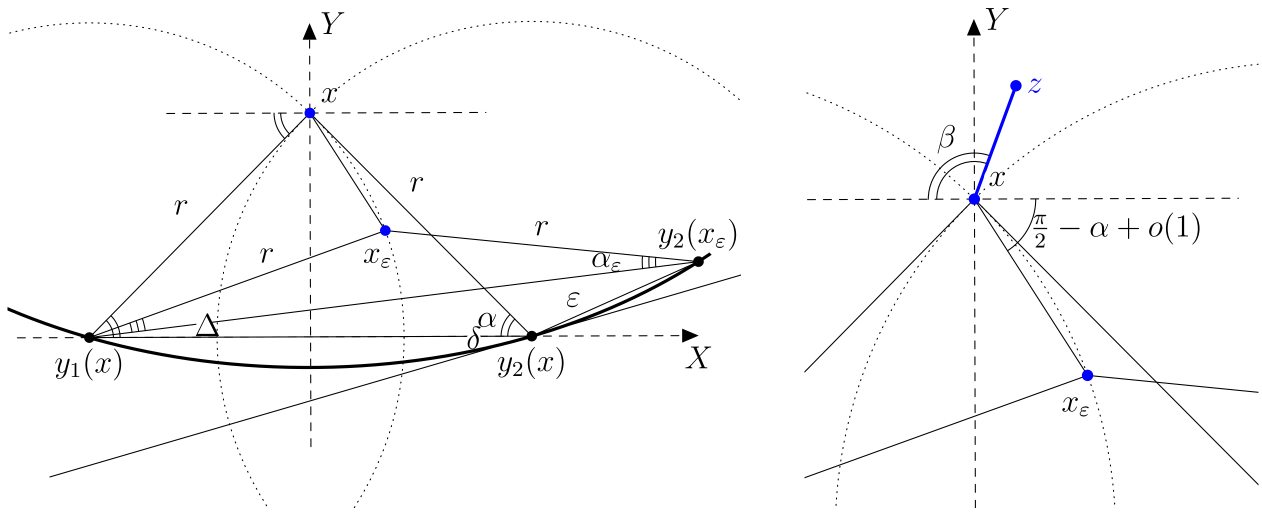


Figure 4.17: Finding coordinates of x_ε in the cases 3 and 4

Let the angle between straight line (y_1y_2) and M at point y_1 be equal to δ. Then

$$y_1^\varepsilon = (r\cos\alpha + \varepsilon\cos\delta + o(\varepsilon), \varepsilon\sin\delta + o(\varepsilon)).$$

Therefore, by the Pythagorean theorem

$$|y_2y_1^\varepsilon| = \sqrt{(2r\cos\alpha + \varepsilon\cos\delta + o(\varepsilon))^2 + (\varepsilon\sin\delta + o(\varepsilon))^2} = 2r\cos\alpha + \varepsilon\cos\delta + o(\varepsilon).$$

Let o_ε be the midpoint of the segment $[y_1^\varepsilon y_2^\varepsilon]$. Then

$$o_\varepsilon = \left(\frac{\varepsilon\cos\delta}{2} + o(\varepsilon), \frac{\varepsilon\sin\delta}{2} + o(\varepsilon)\right).$$

By definition of cosine

$$\alpha_\varepsilon = \arccos\left(\frac{|y_2o_\varepsilon|}{r}\right) = \arccos\left(\cos\alpha + \frac{\varepsilon\cos\delta}{2r} + o(\varepsilon)\right) = \alpha - \frac{\cos\delta}{2r\sin\alpha}\varepsilon + o(\varepsilon).$$

Let us denote by Δ the oriented angle $\angle y_1y_2y_1^\varepsilon$ (that is, for negative ε we have $\Delta < 0$). By the sine theorem for the triangle $y_1y_2y_1^\varepsilon$,

$$\frac{\varepsilon - o(\varepsilon)}{\sin\Delta} = \frac{|y_1y_2|}{\sin(\delta - \Delta + o(\varepsilon))} \geq |y_1y_2|, \quad \text{that is} \quad \Delta = o(\varepsilon)$$

(here we use the fact that $\angle y_2y_1^\varepsilon y_1 = \delta - \Delta + o(\varepsilon)$, since M is a smooth curve). Hence,

$$\Delta = \sin\Delta + o(\varepsilon) = \frac{\varepsilon\sin(\delta + O(\varepsilon))}{|y_1y_2|} = \frac{\varepsilon\sin\delta}{2r\cos\alpha} + o(\varepsilon).$$

Counting the angles in an isosceles triangle xy_2x_ε gives

$$\angle xy_2x_\varepsilon = \alpha - \alpha_\varepsilon - \Delta = \left(\frac{\cos\delta}{2r\sin\alpha} - \frac{\sin\delta}{2r\cos\alpha}\right)\varepsilon + o(\varepsilon) = \frac{\cos(\alpha+\delta)}{r\sin(2\alpha)}\varepsilon + o(\varepsilon).$$

Now it's clear that

$$|xx_\varepsilon| = 2r\sin\frac{\angle xy_2x_\varepsilon}{2} = \frac{\cos(\alpha+\delta)}{\sin(2\alpha)}\varepsilon + o(\varepsilon),$$

and the angle between the segment $[xx_\varepsilon]$ and the axis X (the straight line passing through the point x and parallel to (y_1y_2)) (see the right side of Fig. 4.17) is equal

$$\pi - \alpha - \frac{\pi - \angle xy_2x_\varepsilon}{2} = \frac{\pi}{2} - \alpha + \frac{\cos(\alpha+\delta)}{2r\sin(2\alpha)}\varepsilon + o(\varepsilon) = \frac{\pi}{2} - \alpha + o(1).$$

Case 3. Let $d = 1$, $k = 2$. Let us denote by β the angle between $[z_1x]$ and the X axis (see the right side of Fig. 4.17). Then

$$\angle z_1xx_\varepsilon = \frac{3\pi}{2} - \alpha - \beta + o(1).$$

By the cosine theorem for the triangle z_1xx_ε

$$|z_1x_\varepsilon| = \sqrt{|z_1x|^2 - 2|xx_\varepsilon||z_1x|\cos\angle z_1xx_\varepsilon + |xx_\varepsilon|^2} = |z_1x| - |xx_\varepsilon|\cos\angle z_1xx_\varepsilon + o(\varepsilon) =$$

$$|z_1x| + \frac{\cos(\alpha+\delta)\sin(\alpha+\beta)}{\sin(2\alpha)}\varepsilon + o(\varepsilon).$$

Then

$$\Gamma(\varepsilon) - \Gamma(0) = |z_1x_\varepsilon| - |z_1x| = \frac{\cos(\alpha+\delta)\sin(\alpha+\beta)}{\sin(2\alpha)}\varepsilon + o(\varepsilon),$$

that is, the derivative is equal

$$\Gamma'(0) = \frac{\cos(\alpha+\delta)\sin(\alpha+\beta)}{\sin(2\alpha)}.$$

Case 4. Let $d = 2$, $k = 2$. As in case 3, let β denote the angle between $[z_1x]$ and the X axis; similarly, we denote by γ the angle between $[z_2x]$ and the X axis. Repeating the reasoning from the previous case separately for the segment $[z_1x]$ and for the segment $[z_2x]$, we obtain the value of the derivative

$$\Gamma'(0) = \frac{\cos(\alpha+\delta)}{\sin(2\alpha)}(\sin(\alpha+\beta) + \sin(\alpha+\gamma)).$$

Transitions between the cases. Let in the second case the angle between the segments $[z_1x]$ and $[z_2x]$ be equal to $\frac{2\pi}{3}$. Then for $\varepsilon > 0$ the angle between the segments $[z_1x_\varepsilon]$ and $[z_2x_\varepsilon]$ is less than $\frac{2\pi}{3}$. If we replace $[z_1x_\varepsilon] \cup [x_\varepsilon z_2]$ with the Steiner tree for the triangle $z_1x_\varepsilon z_2$, the length changes to $o(\varepsilon)$. Thus, we can consider this case as a degenerate first case. In this case, the value of the derivative will not change, since

$$2\cos\beta\cos\alpha = \cos\alpha \text{ with } \beta = \pi/3.$$

Under a similar assumption, the fourth case can be considered as a degenerate third case, and the value of the derivative will again remain unchanged:

$$(\sin(\alpha+\beta) + \sin(\alpha+\gamma))\frac{\cos(\alpha+\delta)}{\sin(2\alpha)} = 2\sin\left(\frac{2\alpha+\beta+\gamma}{2}\right)\cos\left(\frac{\beta-\gamma}{2}\right)\frac{\cos(\alpha+\delta)}{\sin(2\alpha)} =$$

$$\sin\left(\alpha+\beta+\frac{\pi}{3}\right)\frac{\cos(\alpha+\delta)}{\sin(2\alpha)}$$

for $\gamma - \beta = 2\pi/3$.

Structural statements.

Proposition 4.6.2. *Let $x \in \Sigma \setminus N_r$ be an energetic point, $y(x) \in M$ be any of the points corresponding to it. Then the derivative of the length Σ in a neighborhood of the point x as y moves along M is non-negative.*

Proof. Let us assume the opposite. Let d, z_i be taken from the definition of derivative. Let us consider

$$\Sigma_\varepsilon = \Sigma \setminus \left(\bigcup_{i=1}^{d}[z_i x]\right) \cup \left(\bigcup_{i=1}^{d}[z_i x_\varepsilon]\right).$$

For a sufficiently small $\varepsilon > 0$, the length of Σ_ε is less than the length of Σ, the set Σ_ε is connected, and $F_M(\Sigma_\varepsilon) \le r$. This contradicts the fact that Σ is a minimizer. □

Proposition 4.6.3. *Let $y \in M$ correspond to two energetic points $x_1, x_2 \in \Sigma \setminus N_r$. According to the Proposition 4.6.1, points x_1 and x_2 lie on opposite sides of the straight line (yn_y). Then the derivatives of the length Σ in the vicinity of the points x_1 and x_2 as y moves along M are equal.*

Proof. Let us assume the contrary, without loss of generality, that the derivative of the length Σ in the neighborhood of the point x_1 is greater than the derivative of the length Σ in the neighborhood of the point x_2. Let d_1, z_i^1 and d_2, z_i^2 be taken from from the definition of the derivatives in the neighborhood of x_1 and x_2, respectively. Let us consider

$$\Sigma_\varepsilon = \Sigma \setminus \left(\bigcup_{j=1}^{2}\bigcup_{i=1}^{d_j}[z_i^j x_j]\right) \cup \left(\bigcup_{j=1}^{2}\bigcup_{i=1}^{d_j}[z_i^j (x_j)_\varepsilon]\right).$$

For a sufficiently small $\varepsilon > 0$, the length of Σ_ε is less than the length of Σ, Σ_ε is connected, and $F_M(\Sigma_\varepsilon) \le r$. This contradicts the fact that Σ is a minimizer. □

4.7 Horseshoe theorem

4.7.1 Sketch of the proof

The proof of Theorem 4.3.2 consists of two main steps. The first one is to show that a minimizer is the union of chords of M_r, arcs of M_r and closures of connected components of $N \setminus M_r$, which are local Steiner trees with at most 4 terminals. Moreover the graph, whose vertices are these Steiner trees and edges connect the trees which are connected by arc of chord, is a finite path.

On the second step we enclose this path into a cycle and consider the inner region T (see Fig. 4.18). Then we compare the turn of S and $\overline{B_r(S)} \cap M$ where S is an arc of $\Sigma \cap M_r$ or a connected component $\partial T \setminus M_r$. It turns out that for every S one has

$$\mathrm{turn}(S \cap \partial T) \ge \mathrm{turn}(\overline{B_r(S)} \cap M). \tag{4.4}$$

Also one can take into account the boundary terms to get the following key inequality

$$2\pi = \mathrm{turn}(\partial T) = \sum_S \mathrm{turn}(S \cap \partial T) + \text{ boundary terms } \ge \sum_S \mathrm{turn}(\overline{B_r(S)} \cap M) \ge \mathrm{turn}(M) = 2\pi.$$

Thus all the inequalities in (4.4) are equalities. The proof of (4.4) immediately imply that this is possible only for a horseshoe.

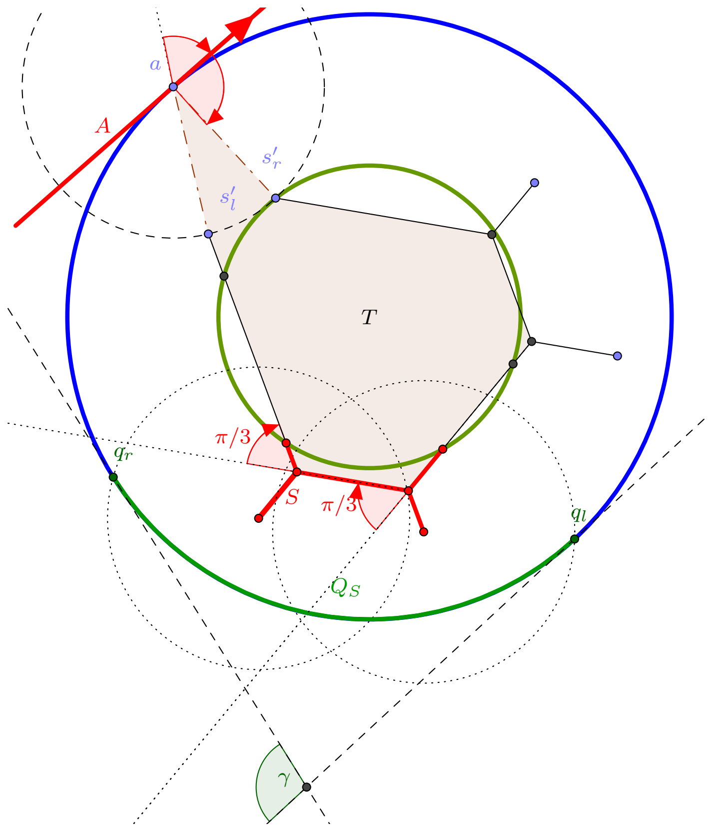


Figure 4.18: Figure to the construction of T.

4.7.2 Lemmas for the first step

In the sequel the union of the closures of all connected components of $\Sigma \cap \operatorname{Int}(N_r)$ is denoted by Σ_r. Recall that by Lemma 4.6.1 and Theorem 4.2.1 the set $\Sigma \cap \operatorname{Int}(N_r)$ is a finite union of line segments. Note that the number of line segments in Σ sometimes it can be infinite, see Theorem 4.2.3.

Lemma 4.7.1. *Let M be a convex closed curve with minimal radius of curvature R and Σ be a minimizer with the energy $r < R$. Then the length of each line segment in Σ_r does not exceed $a_M(r)$ for some $a_M(r) \leq 2r$. For the circumference $\partial B_R(o)$ one can take $a_{\partial B_R(o)}(r) = 2r\sqrt{1 - \frac{r^2}{4R^2}}$.*

Proof of Lemma 4.7.1. Proof of (i): No change in the set $\operatorname{Int}(\Sigma \cap N_r)$ influences the value of $F_M(\Sigma)$, so if we take the closure S of any connected component of $\Sigma \cap \operatorname{Int}(N_r)$ and substitute it by a Steiner tree connecting $S \cap M_r$ (which must be nonempty if $\Sigma \cap \operatorname{Int}(N_r) \neq \emptyset$ because of connectedness of Σ and the requirement $F_M(\Sigma) \leq r$ which gives $\Sigma \setminus \operatorname{Int}(N_r) \neq \emptyset$), then the length of the resulting set should remain the same by optimality of Σ, and thus S is itself a Steiner tree connecting $S \cap M_r$ as claimed.

Proof of (ii): Recall that $\Sigma = E_\Sigma \sqcup X_\Sigma \sqcup S_\Sigma$, where X_Σ is a discrete set of points, S_Σ consists of Steiner trees (hence of line segments) and $E_\Sigma \subset M_r$ by Lemma 4.6.6(iii).

Proof of (iii): Remove an arbitrary open line segment Δ from the set $\Sigma \cap \operatorname{Int}(N_r)$. The value of F_M does not change, i.e. $F_M(\Sigma \setminus \Delta) = F_M(\Sigma)$, and by the absence of cycles $\Sigma \setminus \Delta$ splits into two connected components Σ_1 and Σ_2, so that $\Sigma \setminus \Delta = \Sigma_1 \sqcup \Sigma_2$ (Σ is closed, so Σ_1, Σ_2 are closed too). Obviously $M \subset \overline{B_r(\Sigma_1)} \cup \overline{B_r(\Sigma_2)}$. Then by connectedness of M there is such a point $a \in M$ that $a \in \overline{B_r(\Sigma_1)} \cap \overline{B_r(\Sigma_2)}$, but then there are points $b \in \overline{\Sigma_1}$ and $c \in \overline{\Sigma_2}$ such that $|ab| \leq r$, $|ab| \leq r$. Hence the distance between Σ_1 and Σ_2 does not exceed $|bc| \le 2r$ but the length of the deleted segment Δ does not exceed the distance between the Σ_1 and Σ_2 in view of optimality of Σ (otherwise one could connect Σ_1 with Σ_2 with a shorter segment). we let then $a_M(r)$ be the supremum of $|bc|$ over all the possible choices of Δ, so that we have proven $a_M(r) \le 2r$.

In the case $M = \partial B_R(o)$ the length of the segment $[bc]$ reaches its maximal value when $[bc]$ is a chord and $|ab| = |ac| = r$. Then we can calculate the maximal value of length of $[bc]$ in this case:

$$\sin\frac{\angle aoc}{2} = \frac{|ac|}{2|oc|} = \frac{r}{2R},$$

so that

$$|bc| = 2|oc|\sin\angle aoc = 4|oc|\sin\frac{\angle aoc}{2}\cos\frac{\angle aoc}{2} = 2r\sqrt{1-\frac{r^2}{4R^2}}.$$

□

Lemma 4.7.2. *Let Σ be an r-minimizer for a closed convex curve M with minimal radius of curvature $R > 2a_M(r) + r$, where a_M is defined in Lemma 4.7.1). Then Σ has no Steiner point in* $\mathrm{Int}(N_r) \cup (S_\Sigma \cap N_r)$ *and moreover $\Sigma \cap N_r$ consists of chords of M_r with disjoint interiors.*

Proof. Assume the contrary i.e. that Σ has a Steiner point $x \in \mathrm{Int}(N_r) \cup (S_\Sigma \cap N_r)$. By the condition on the raduis of curvature there is a point $o \in N$ such that $X \in B_R(o)$ and $B_R(o) \subset \mathrm{Int}(N)$ (hence $B_{R-r}(o) \subset \mathrm{Int}(N_r)$, and in particular, $o \in \mathrm{Int}(N_r)$). Recall that as defined in Lemma 4.7.1 Σ_r is the union of the closures of all connected components of $\Sigma \cap \mathrm{Int}(N_r)$. Now denote by x_0 one of the Steiner points of $\Sigma_r \cup (S_\Sigma \cap M_r)$ nearest to o, and let $t := |ox_0|$. we claim that $x_0 \in \mathrm{Int}(N_r)$. In fact, otherwise $x_0 \in M_r$ and hence

$$t = \mathrm{dist}\,(o, M_r) = \mathrm{dist}\,(o, M) - r \ge R - r > 3.98r,$$

but x_0 is a Steiner point, hence, in view of the smoothness and convexity of M_r there are two line segments $[x_0 i] \subset \Sigma$, $i = 1, 2$ at angle $2\pi/3$ with respect to each other, intersecting $\mathrm{Int}(N_r)$. Suppose without loss of generality that $\angle ox_0z_1 \le \pi/3$. Then $z_1 \in B_t(o) \subset \mathrm{Int}(N_r)$, since otherwise there is an $y \in [x_0z_1] \cap \partial B_t(o) \subset \Sigma \cap \partial B_t(o)$ such that the line segment $[x_0y] \subset \Sigma$ is a chord of $\partial B_t(o)$, which provides the estimate

$$|x_0y| = 2t\cos\angle ox_0z_1 \ge t > 3.98r$$

contrary to Lemma 2.7(iii), this contradiction proving the claim.

Let Σ' stand for the closure of the connected component of $\Sigma \cap \mathrm{Int}(N_r)$ containing x_0. By the structure of a Steiner tree since x_0 belongs to $\mathrm{Int}(N_r)$ then there are three maximal line segments of Σ' starting from x_0. Consider such a pair of them $[x_0x_{-1}], [x_0x_1]$ that the point o belongs to the angle $\angle x_{-1}x_0x_1$ (not excluding the case it belongs to one of the sides of this angle). Recall that $\angle x_{-1}x_0x_1 = 2\pi/3$. Also note that points x_{-1}, x_1 lie outside of $B_t(o)$. Hence either $[x_0x_1]$ or $[x_0x_{-1}]$ intersects $B_t(o)$. We assume without loss of generality that it is $[x_0x_1]$. Denote the intersection of the segment $[x_0x_1]$ and the circumference $\partial B_t(o)$ by c.

we claim that $t \le a_M(r)$. Supposing the contrary, since $|x_0c| \le a_M(r)$ and $|ox_0| = |oc| = t > a_M(r) \ge |x_0c|$, we have $\angle ox_0c > \pi/3$, hence the segment $[x_0x_{-1}]$ also intersects $B_t(o)$. Denote the intersection of the segment $[x_0x_{-1}]$ with $\partial B_t(o)$ by d and note that also $\angle ox_0d > \pi/3$, and hence $\angle cx_0d > 2\pi/3$ which contradicts the local optimality of Σ, showing the claim.

Note that x_1, x_{-1} belong to $\mathrm{Int}(N_r)$ because $R - r > 2a_M(r) \ge t + a_M(r)$, and hence x_1, x_{-1} are Steiner points. Also by Lemma 4.7.1 the lengths $[x_0x_{-1}]$ and $[x_0x_1]$ do not exceed $a_M(r)$. Consider a regular hexagon P with sidelength $a_M(r)$ such that x_0 is a vertex of P and the segments $[x_0x_1]$, $[x_0x_{-1}]$ belong to two sides of P. The following assertions hold.

- $\operatorname{diam} P = 2a_M(r)$.
- The line segment $[ox_0]$ splits the angle $\angle x_{-1}x_0x_1 = 2\pi/3$ in two angles, at least one of them is acute. Denote the latter angle by $\angle ox_0b$, where b is the corresponding vertex of P (so that $|x_0b| = a_M(r)$). Then the angle $\angle obx_0$ is also acute because $|ox_0| = t \le a_M(r) = |x_0b|$. Therefore the perpendicular from o to the line (x_0b) intersects the latter inside $[x_0b]$, so that o is inside the square built on $[x_0b]$. But this square is a subset of P hence $o \in P$.
- The above assertions imply that $P \subset \overline{B_{2a_M(r)}(o)}$, and hence $P \subset \operatorname{Int}(N_r)$.

Now let us pick such vertices x_{-2} and x_2 that $[x_1x_2], [x_{-1}x_{-2}] \subset \Sigma_r$ and o belongs to both angles $\angle x_0x_1x_2$ and $\angle x_0x_{-1}x_{-2}$. Clearly $x_2, x_{-2} \in P \subset \operatorname{Int}(N_r)$ so they again are Steiner points. Let us define the points x_3, x_{-3} in the same way: $[x_2x_3], [x_{-2}x_{-3}] \in \Sigma_r$ and o belongs to the angles $\angle x_1x_2x_3$ and $\angle x_{-1}x_{-2}x_{-3}$. Points x_3, x_{-3} also belong to P, hence to $\operatorname{Int}(N_r)$, hence they also are Steiner points. The six constructed line segments belong to $\operatorname{Int}(N_r)$, so there is no endpoint there. continuing inductively this construction, we arrive at two paths in $P \subset \operatorname{Int}(N_r)$: one path (starting from $x_0, x_1, x_2, x_3 \dots$) turns left every time and the other one (starting from $x_0, x_{-1}, x_{-2}, x_{-3} \dots$) turns right every time. Thus $\Sigma \cap P \subset \Sigma \cap \operatorname{Int}(N_r)$ contains a cycle or an endpoint of Σ in $\operatorname{Int}(N_r)$, but both cases are impossible for a Steiner tree by the absence of cycles and Lemma 4.7.1. □

Let us recall several definitions we used in the second part of Section 4.6.3. Consider the set of the closures of connected components of $\Sigma \setminus N_r$ and denote it by $V_C(G)$ (further it will be associated with a subset of the vertex set of a graph). Note that Σ is connected (and does not reduce to a single point), so every $S \in V_C(G)$ has positive length. In our setting M is compact, thus every Σ has finite length, hence the set $V_C(G)$ is at most countable.

Consider an arbitrary $S \in V_C(G)$. Note that by connectedness of S the set $\overline{B_r(S)} \cap M$ is always a closed arc. We denote it by Q_S.

Consider the set of all maximal arcs of M_r in the set Σ, which are not contained in the closure of a connected component of $\Sigma \setminus N_r$. Let us denote by $V_A(G)$ the subset of such arcs having an energetic point in their interior. Note that if M is not strictly convex, then an arc $[\breve{bc}]$ of M_r can be a chord of M_r. In this situation if $]\breve{bc}[$ has no energetic point then we will consider it as a chord of M_r: note that if $\Sigma \setminus]\breve{bc}[$ does not cover $q_x \in M$ for some $x \in]bc[$, then x is energetic; thus if $]\breve{bc}]$ has no energetic point then $[bc] = [\breve{bc}]$ has all the properties of a standard chord of M_r.

Obviously, an arc $[\breve{bc}] \in V_A(G)$ of M_r covers an arc $q_{[\breve{bc}]} := [\breve{q_bq_c}]$ of M, where $q_b, q_c \in M$ are the unique points such that $\operatorname{dist}(b, q_b) = \operatorname{dist}(c, q_c) = r$.

Definition 4.7.1. *Denote by $n(S)$ and $m(S)$ the numbers of energetic and entering points in S, respectively.*

Theorem 4.6.1 says $n(S) \le 2$, $m(S) \le 3$ and S is a locally minimal tree for its energetic and entering points. We need to strengthen this statement using the bound curvature.

Lemma 4.7.3. *Let M be a closed convex curve with minimal radius of curvature $R > 2a_M(r) + r$, Σ. Let S be the closure of a connected component of $\Sigma \setminus N_r$. Then $m(S) \le 2$.*

By the previous Lemma, S is a locally minimal tree for at most $n(S) + m(S) \le 4$ points. All the possible combinatorial types of such networks are listed in Figures 4.33 and 4.34.

Proof of Lemma 4.7.3. Let $S \in V_C(G)$ be the closure of a connected component of $\Sigma \setminus N_r$.

Assume the contrary i.e. the existence of at least three different entering points in S. Let us denote them i_1, i_2 and i_3 such that $q_{i_2} \in [\breve{q_{i_1} q_{i_3}}] \subset Q_S$. Note that i_2 cannot be energetic, because q_{i_2} is not an end of Q_S. So i_2 has such a neighbourhood U that $U \cap \Sigma$ is a segment or a regular tripod; by Lemma 4.7.2 it is a segment.

We claim that Σ contains a chord $[i_2 j]$ of M_r. It is true if Σ is not tangent to M_r at i_2. Now, let Σ be tangent to M_r at i_2, so i_2 belongs to two closures of different connected components of $\Sigma \setminus N_r$; one of them is S; denote the second one by S'. Let P_1 be the region bounded by the arc $[\breve{i_1 i_2}]$ of M_r (choosing in such a way that P_1 does not contain N_r) and the unique path between i_1 and i_2 in S. Define P_3 analogously (with i_3 in place of i_1). Obviously, $S' \subset P_1$ or $S' \subset P_3$. Hence $q(S') \subset q(S)$ and replacing S' in Σ by a Steiner tree for $S' \cap M_r$ we get a connected competitor to Σ still covering M. Also, any Steiner tree for $S' \cap M_r$ belongs to N_r by the convexity of M_r, so this replacement decreases the length, which is impossible. Hence, we get the claim, i.e. there is a chord $[i_2 j] \subset \Sigma$ of M_r.

Then $|i_2 j| \leq |i_1 j|$ (otherwise we can replace $[i_2 j]$ by $[i_1 j]$ in Σ producing the competitor of strictly lower length), and analogously $|i_2 j| \leq |i_3 j|$. Note that $j \notin S$ because Σ has no loops. One can see that points i_1, i_2, i_3, j belong to M_r in the natural (clockwise) order otherwise the arc Q_{S_j} is a subset of Q_S, where S_j is the closure of the connected component of $\Sigma \setminus N_r$ containing j, which is impossible.

Hence $|j i_2|$ is at least the diameter d of the maximal ball inscribed in N_r and touching M_r at point i_2, i.e. the double *inradius* of M_r. Since $d \geq 2(R - r)$, we have $|j i_2| \geq 2(R - r) > 2r$ contradicting Lemma 4.7.1, showing the claim $m(S) \leq 2$.

□

Definition 4.7.2. *Under conditions of Theorem 4.3.2 consider the following abstract graph $G = (V(G), E(G))$ (recall that the set of vertices $V(G) = V_C(G) \sqcup V_A(G)$; by Lemma 4.6.10 it is finite), where the set of edges $E(G)$ is defined as follows:*

- *in the case S_1, $S_2 \in V_C(G)$ there is an edge between them if they are connected in Σ by a chord of M_r or if $S_1 \cap S_2 \neq \emptyset$;*
- *in the case $S_1 \in V_C(G)$, $[\breve{bc}] \in V_A(G)$ there is an edge between S_1 and $[\breve{bc}]$, if $S_1 \cap [\breve{bc}] \neq \emptyset$;*
- *and finally in the case $[\breve{b_1 c_1}]$, $[\breve{b_2 c_2}] \in V_A(G)$ there is no edge between them.*

Corollary 4.7.1. *Under conditions of Theorem 4.3.2 graph G has no cycles; it has exactly two vertices of degree 1 and all the other vertices have degree 2. In other words G is a path with at least one edge.*

Proof. First, by Lemma 4.6.10 the graph is finite. By Lemma 4.6.11 every chord of M_r in Σ connects exactly two vertices in $V(G)$. Thus, the inequality $m(S) \leq 2$ (Lemma 4.7.3) implies $\deg(v) \leq 2$ for $v \in V_C(G)$; for $v \in V_A(G)$ the inequality $\deg(v) \leq 2$ holds by Lemma 4.6.12.

Note that if $(S_1, S_2) \in E(G)$ then there is a path between S_1 and S_2 in Σ not intersecting other sets $S \in V(G)$, $S \notin \{S_1, S_2\}$. It means that if G has a cycle c then so has Σ, contradicting to the absence of cycles. Moreover, the path between two points in Σ belonging to two different vertices of $V(G)$ naturally induces a path in G (in fact, if a path in Σ connects two different vertices $S_1, S_2 \in V(G)$ without touching other vertices, then $(S_1, S_2) \in E(G)$; therefore for a generic path in Σ connecting two different vertices of G it is enough to split it in a finite number of paths connecting different vertices in G and not passing throw other vertices). Therefore, connectedness of Σ gives us that G is connected. we conclude that G is a path.

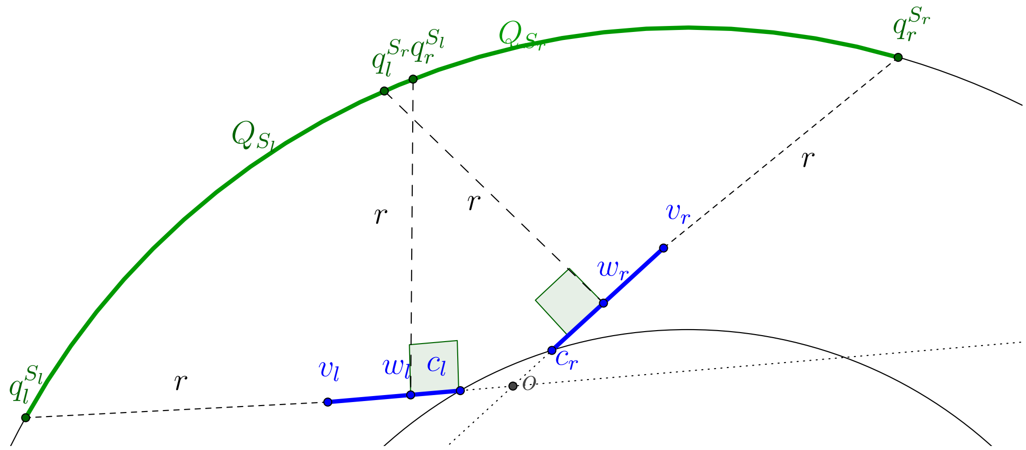


Figure 4.19: Picture to Lemma 4.7.4

Now we have to show that $\#V(G) > 1$. Suppose the contrary, i.e. $V(G) = \{v\}$. If $v \in V_C(G)$, then $m(v) = 0$, so v is a segment that is impossible. Otherwise v is an arc, but $q_v = M$, so $v = M_r$ contains a loop. We got again a contradiction with the absence of cycles. $\square$

Thus under conditions of Theorem 4.3.2 there are two connected components of $\Sigma \setminus N_r$ with one entering point; these components correspond to the leaves of our graph. we call them *ending components* and denote by S_l and S_r (calling them *left* and *right* respectively); the other components will be called *middle components*.

By Lemma 4.6.9 every point of M is covered by at most two sets from $V(G)$. By Corollary 4.7.1 graph G is a path, so if S_1, S_2 are connected by an edge in G, then $Q_{S_1} \cap Q_{S_2} \neq \emptyset$. Moreover, the same reasoning gives $Q_{S_l} \cap Q_{S_r} \neq \emptyset$, because otherwise there would be some part of M not covered by Σ.

Lemma 4.7.4. *The arcs Q_{S_l} and Q_{S_r} have disjoint interiors.*

Denote by a an arbitrary point of the intersection of Q_{S_l} and Q_{S_r} (see Fig. 4.18); by Lemma 4.7.4 there are at most 2 such points. Consider the set $\hat{\Sigma} := \Sigma \cup [as_l] \cup [as_r]$, where $[as_l]$ and $[as_r]$ are segments of length r connecting a with s_l and s_r respectively. From the absence of cycles and the fact that $B_r(a) \cap \Sigma = \emptyset$, the set $\hat{\Sigma}$ bounds the unique region which we further denote by T (see Fig.4.18).

Previous lemmas give us the following corollary.

Corollary 4.7.2. *The boundary of T is a closed curve consisting of a finite number of arcs of M_r and a finite number of line segments.*

Consider the behavior of the tangent line to the boundary of T. Corollary 4.7.2 and Lemma 4.6.13 imply that all points where tangent direction is discontinuous (i.e. points where the tangent line to ∂T does not exist) except a belong to connected components of $\Sigma \setminus N_r$.

Proof of Lemma 4.7.4. Recall that $m(S_l) = m(S_r) = 1$. Denote the ends of Q_{S_l} and Q_{S_r} in the following way: $Q_{S_l} = [q_l^{S_l} \breve{\ } q_r^{S_l}]$, $Q_{S_r} = [q_l^{S_r} \breve{\ } q_r^{S_r}]$. Suppose the contrary, i.e. that $q_r^{S_l} \in]q_l^{S_r} \breve{\ } q_r^{S_r}[$, $q_l^{S_r} \in]q_l^{S_l} \breve{\ } q_r^{S_l}[$. Suppose that $n(S_l) = 2$ or $n(S_r) = 2$ (let $n(S_l) = 2$, the case $n(S_r) = 2$ is completely analogous). Then by Lemma 4.6.6(v) there is an energetic point of S_l corresponding to the point $q_r^{S_l}$.

But $B_r(q_r^{S_l}) \cap \Sigma \neq \emptyset$, because $q_r^{S_l} \in]q_l^{S_r} \breve{q}_r^{S_r}[= Q_{S_r}$. So we have a contradiction with the assumption $n(S_l) = 2$, and hence S_l coincides with the segment $[c_l v_l]$. Clearly, v_l, c_l and $q_l^{S_l}$ lie on the same line (otherwise one can replace $[v_l V']$ by the part of the segment $[V' q_l^{S_l}]$, where $V' := \partial B_\varepsilon(v_l) \cap [v_l c_l]$ producing a competitor of strictly lower length). Hence $[c_l v_l]$ is tangent to $B_r(Q_r^{S_l})$ (see Fig. 4.19).

Let w_l be such a point of $[c_l v_l]$ that $\operatorname{dist}(w_l, q_r^{S_l}) = r$ and w_r be such a point of $[c_r v_r]$ that $\operatorname{dist}(w_r, q_l^{S_r}) = r$. Note that the points c_l, v_l, $q_l^{S_l}$ lie on the same line, so $\operatorname{dist}(w_l q_l^{S_l}) \geq r = \operatorname{dist}(w_l, q_r^{S_l})$, so $\angle q_r^{S_l} q_l^{S_l} w_l \leq \angle q_l^{S_l} q_r^{S_l} w_l$. The segment $[c_l v_l]$ is tangent to $B_r(Q_r^{S_l})$, hence $(q_r^{S_l} w_l) \perp (v_l c_l)$. Calculating angles in triangle $\Delta q_r^{S_l} q_l^{S_l} w_l$ we have $\angle q_r^{S_l} q_l^{S_l} w_l \leq \pi/4$. Obviously, $\angle q_r^{S_r} q_l^{S_l} w_l \leq \angle q_r^{S_l} q_l^{S_l} w_l$, so $\angle q_r^{S_r} q_l^{S_l} w_l \leq \pi/4$. By symmetry we have inequality $\angle q_l^{S_l} q_r^{S_r} w_r \leq \pi/4$. Denote by o the intersection point of $(v_l c_l)$ and $(v_r c_r)$. From the triangle $\Delta q_r^{S_r} q_l^{S_l} o$ we have $\angle q_r^{S_r} o q_l^{S_l} \geq \pi/2$.

Note that $2r > |w_l w_r| \geq |c_l c_r|$ and $\angle q_r^{S_r} o q_l^{S_l} = \angle c_l o c_r \geq \pi/2$. It means that $|c_l o| < 2r$ and $|c_r o| < 2r$. Hence the intersection point of the rays $[v_l c_l)$ and $[v_r c_r)$ belongs to N_r, that contradicts the optimality of Σ. □

4.7.3 Central lemma

Now we are ready to state the central Lemma. Figure 4.18 should simplify the reading of its statement.

Lemma 4.7.5. *Under conditions of Theorem 4.3.2 let Σ be a minimizer, $S \in V(G)$ be the closure of a connected component of $\Sigma \setminus N_r$ or an arc of M_r. Then the following assertions hold.*

- *If S is a middle component or an arc of M_r then* $\operatorname{turn}(Q_S) \leq \operatorname{turn}(S)$. *The equality holds if and only if S is an arc of M_r.*

- *If S is an ending component then for the left and the right components we have*

$$\operatorname{turn}(Q_{S_l}) \leq \operatorname{turn}(S_l) + \angle([c_l s_l), [s_l a)) + \angle([s_l a), a),$$

$$\operatorname{turn}(Q_{S_r}) \leq \angle(a, [a s_r)) + \angle([a s_r), [s_r c_r)) + \operatorname{turn}(S_r),$$

  where A stands for the tangent ray to M at the point a directed from the left to the right (see Fig. 4.18, angles $\angle([s_l a), A)$, $\angle(A, [a s_r))$ are marked red) and c_i is the branching point of S_i if S_i is a tripod and the entering point of S_i in other cases, where $i \in \{l, r\}$ (the definition is correct by Lemma 4.7.3). The equality holds if and only if S is a segment of the tangent line to M_r.

Remark 4.7.1. *If in Lemma 4.7.5 we assume that Σ has no Steiner points in N_r then it is enough to request the inequality $r < R/2.9$ (see proof of Lemma 4.7.5, case 1a).*

Proof of Lemma 4.7.5. Obviously, if S is an arc, then the compared values are equal.

It suffices thus to consider the case when S is the closure of a connected component of $\Sigma \setminus N_r$. Denote by q_l and q_r the ends of Q_S. Let o be an intersection point of the normals to M at points q_l and q_r. It exists unless $\operatorname{turn}(Q_S) = 0$ in which case the claim is obvious. Note that $\operatorname{turn}(Q_S) = \angle q_l o q_r$ and denote for brevity thus value by γ. Also one has $|q_l o| \geq R$, $|q_r o| \geq R$. Note that Lemmas 4.7.2 and 4.7.3 as well as Corollary 4.7.1 hold true when $R > 2a_M(r) + r$ which is guaranteed when $R > 5r$ (or $R > 4.98r$ in the case when M is a circumference of radius R), i.e. under the conditions of the statement being proven.

By Lemma 4.7.3 S is a locally minimal tree for at most $n(S) + m(S) \leq 4$ points. All the possible combinatorial types of such networks are listed in Figures 4.33 and 4.34. Note that if S is a middle component then $m(S) = 2$, otherwise $m(S) = 1$. Let us analyze all the possible types one by one, first when S is a middle component, then for S an ending component.

1. Let S be a middle component. By Lemma 4.6.6(iv) S is a locally minimal tree, moreover it has two entering points (if one, then it is an ending component) and one or two energetic points.

   (a) *The case $n = 2$, $m = 2$, the combinatorial type (a) on Fig. 4.34 (see Fig. 4.20).* Denote

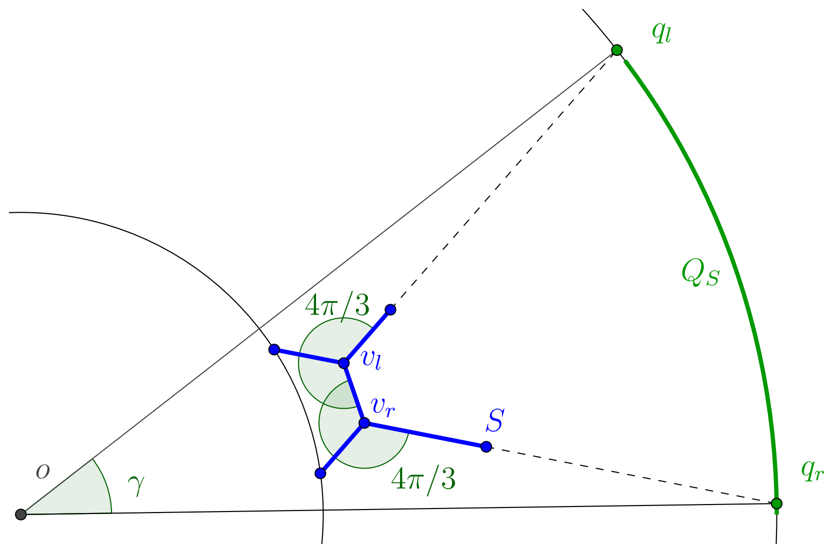


Figure 4.20: Picture to the case 1a: middle component, $n = 2, m = 2$.

the Steiner points of S by v_l and v_r. In this case $\text{turn}(S) = \pi/3 + \pi/3 = 2\pi/3$. Assuming the contrary (it means that $\gamma \geq 2\pi/3$) and connecting o with q_l and q_r, we get a (non convex) pentagon $q_l v_l v_r q_r o$ with two angles equal to $4\pi/3$ and one angle at least $2\pi/3$, which is impossible.

   (b) *The case $n = 2, m = 2$, the combinatorial type (b) on Fig. 4.34 (see Fig. 4.21).* Note that

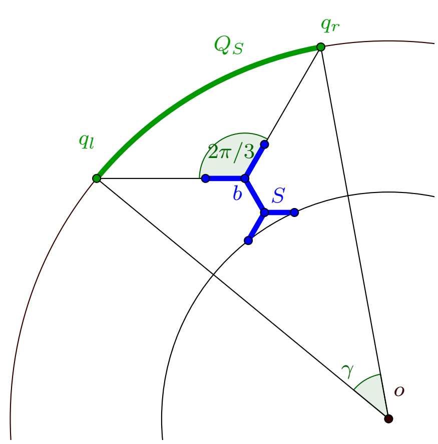


Figure 4.21: A general picture to the case 1b: middle component, $n = 2, m = 2$.

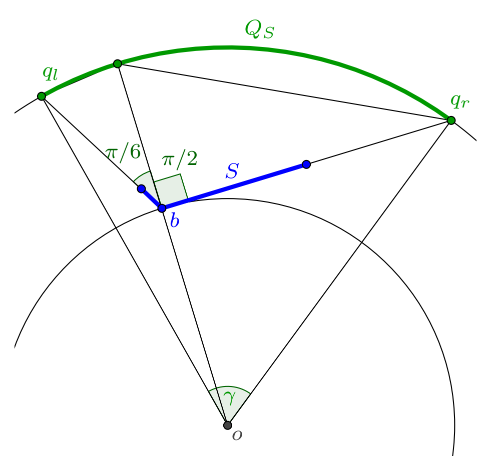


Figure 4.22: A marginal picture to the case 1b: middle component, $n = 2, m = 2$.

in this case there exists a Steiner point adjacent to both entering points, and also there exists a Steiner point (we call it b) adjacent to both energetic points. Clearly $\text{turn}(S) = \pi/3$.

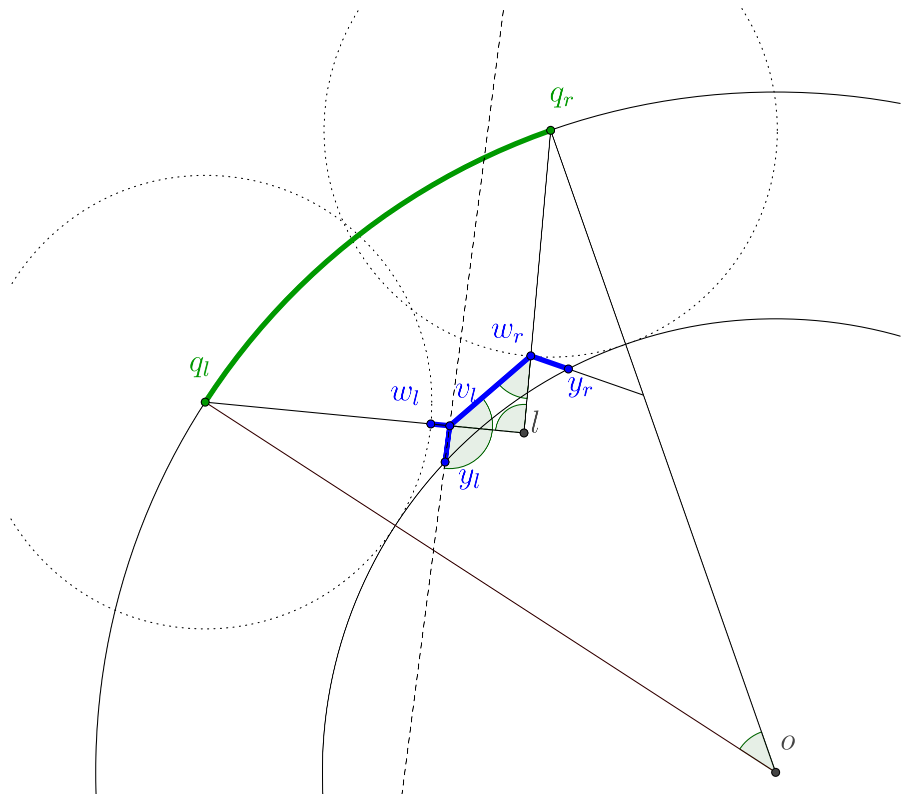


Figure 4.23: Picture to the case 1c: middle component, $n = 2$, $m = 2$.

Let us prove that $\mathrm{turn}(Q_S) < \pi/3$. we evaluate the arc of M bounded by continuations of segments starting from b. Clearly this arc is maximal when b belongs to M_r (it is the marginal case). Hence it is enough to look at the angle in $N \setminus N_r$ of size $2\pi/3$ with vertex b on M_r. It is well-known that the arc is maximal when S is tangent to M_r and when M is a circumference. In this case the normal to M_r at b splits the angle $\angle q_l b q_r = 2\pi/3$ in two angles: one of size $\pi/2$ and another of size $\pi/6$ (see Fig. 4.22), so that the size of the arc is

$$\arccos\left(1 - \frac{1}{\delta}\right) + \frac{\pi}{6} - \arcsin\left(\frac{1}{2}\left(1 - \frac{1}{\delta}\right)\right),$$

where $\delta := R/r$, hence it is strictly less than $\pi/3$ for $\delta \geq 2.9$.

(c) *The case $n = 2$, $m = 2$, the combinatorial type (c) on Fig. 4.34.*

There are two possibilities for S in this case, see Fig. 4.23 and Fig. 4.24.

The case on Fig. 4.24 can be reduced to the previous case 1b. Obviously, $\mathrm{turn}(S) = \pi/3$. Let us fix the entering points y_l, y_r and the left energetic point w_l and move the right energetic point w_r to the right (in the direction of the ray $[w_l w_r)$). Then at some time the combinatorial type changes to (b) on Fig. 4.34, during this process $\mathrm{turn}(S) = \pi/3$, and $\mathrm{turn}(Q_S)$ grows, but $\mathrm{turn}(Q_S) \leq \pi/3$. By case 1b.

The case on Fig. 4.23: denote the energetic points of S by w_l and w_r, and the entering points by y_l, y_r respectively, and the branching point by v_l (without loss of generality it is connected with w_l and y_l). Let $2\beta := \angle v_l w_r y_r$, and note that $\angle y_l v_l w_r = 2\pi/3$. Then $\mathrm{turn}(S) = (\pi - 2\pi/3) + (\pi - 2\beta) = 4\pi/3 - 2\beta$. Assume the contrary (i.e. in

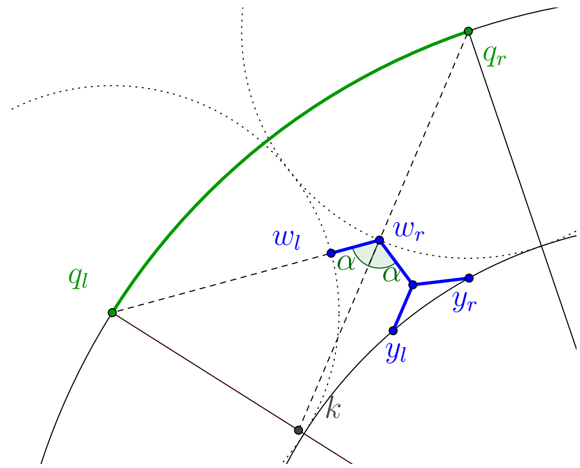


Figure 4.24: Picture to the case 1c: middle component, $n = 2$, $m = 2$.

this case $\gamma \geq 4\pi/3 - 2\beta$) and call l the point of intersection of $(q_l w_l)$ and $(q_r w_r)$. By Lemma 4.6.6(v)(b) $\angle l w_r v_l = \angle y_r w_r v_l / 2 = \beta$. Then

$$\pi - \pi/3 - \beta = \angle q_l l q_r > \angle q_l o q_r = \gamma,$$

(the first equality coming from $\Delta v_l w_r l$) which implies

$$\gamma \geq 4\pi/3 - 2\beta > 2\pi/3 - \beta = \angle q_l l q_r > \gamma,$$

a contradiction.

(d) *The case $n = 2$, $m = 2$, the combinatorial type (d) on Fig. 4.34 (see Fig. 4.25).* Denote the energetic points of S by w_l and w_r, and the entering points by y_l, y_r respectively. Let $2\alpha := \angle y_l w_l w_r$, $2\beta := \angle w_l w_r y_r$. Then $\mathrm{turn}(S) = (\pi - 2\alpha) + (\pi - 2\beta)$. Assume the contrary (it means that $\gamma \geq 2\pi - 2\alpha - 2\beta$) and denote by l the point of intersection of $(q_l w_l)$ and $(q_r w_r)$. By Lemma 4.6.6(v)(b) $\angle l w_l w_r = \angle y_l w_l w_r / 2 = \alpha$, $\angle l w_r w_l = \angle y_r w_r w_l / 2 = \beta$. Then

$$\pi - \alpha - \beta = \angle q_l l q_r > \angle q_l o q_r = \gamma,$$

(the first equality coming from $\Delta w_l w_r l$) which implies

$$\gamma \geq 2\pi - 2\alpha - 2\beta > \pi - \alpha - \beta = \angle q_l l q_r > \gamma,$$

a contradiction.

(e) *The case $n = 1$, $m = 2$, the combinatorial type (b) on Fig. 4.33 (see Fig. 4.26, Fig. 4.27, Fig. 4.28).*

Clearly, $\mathrm{turn}(S) = \pi/3$. To prove the statement, assume the contrary (i.e. $\gamma \geq \pi/3$) and as in the previous case connect o with q_l and q_r. Denote the energetic point of S by w. Let us consider three subcases:

- the point w covers both q_r and q_l (see Fig. 4.26);
- the point w covers q_l and q_r is covered by an entering point (see Fig. 4.27);

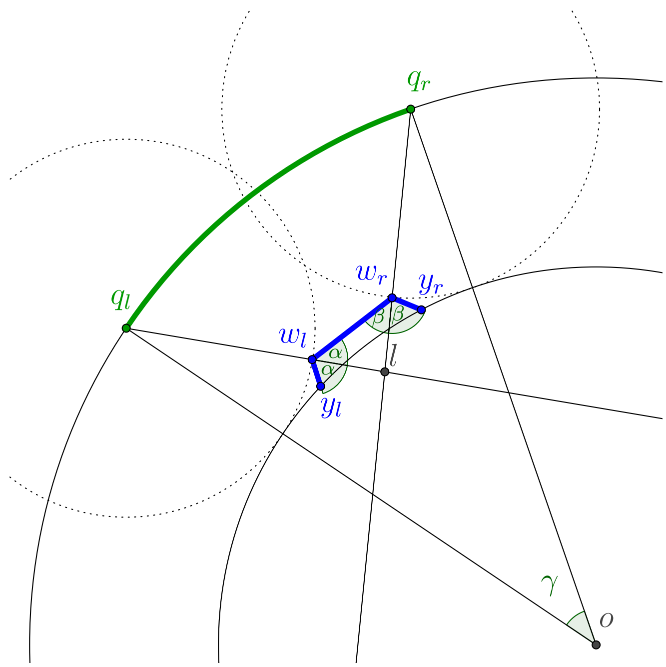


Figure 4.25: Picture to the case 1d: middle component, $n = 2$, $m = 2$.

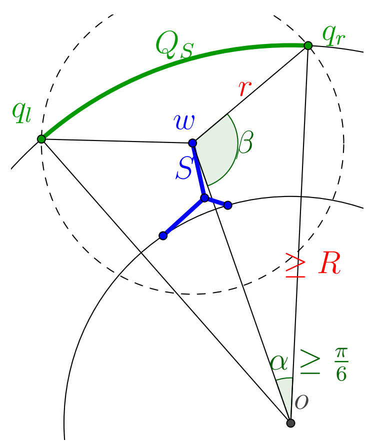


Figure 4.26: Picture to the case 1e: middle component, $m = 2, n = 1$.

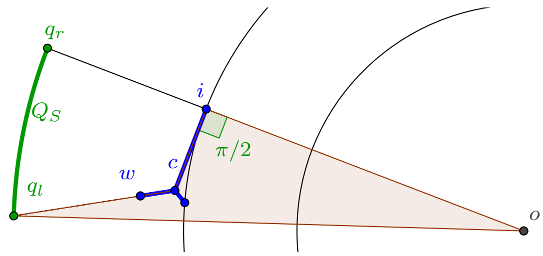


Figure 4.27: Picture to the case 1e: middle component, $m = 2, n = 1$.

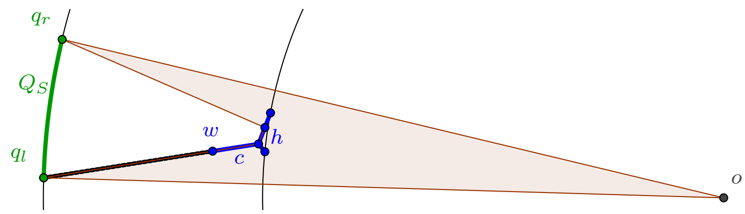


Figure 4.28: Picture to the case 1e: middle component, $m = 2, n = 1$.

- w covers q_l and q_r is covered by $h \in S \setminus (M_r \cup w)$ (see Fig. 4.28).

In the subcase (i) $|wq_r| = |wq_l| = r$. Let us connect o with w, and note that the angle $\angle q_l o q_r = \gamma$ splits into two parts; let us pick the largest one (without loss of generality it is $\angle woq_r$). Consider the triangle $\Delta oq_r w$ with side $|oq_r| \geq R$ and acute angle (α on the Fig. 4.26) at least $\pi/6$ against the side $|wq_r| = r$. Recalling that $R > 2r$ and denoting by $\beta := \angle owq_r$, by the law of sines for triangle $\Delta oq_r w$ we get

$$\sin\beta = \frac{|oq_r|}{r}\sin\alpha \geq \frac{R}{2r} > 1,$$

a contradiction.

In the subcase (ii) q_r is covered by the entering point i. Then (ci) is perpendicular to (iq_r), where c is the branching point of S, so points q_r, o, i lie on the same line. Consider the sum of the angles in the non convex quadrilateral $q_l cio$: it is $\angle q_l + \angle c + \angle i + \angle o \geq \angle q_l + 4\pi/3 + \pi/2 + \pi/3 > 2\pi$, a contradiction.

In the subcase (iii) q_r is covered by $h \in]ci[$, where c is the branching point of S, i is an entering point of S. Note that (ci) is perpendicular to (hq_r); points q_l, w, c lie on the same line. Consider the sum of the angles in the non convex pentagon $q_l chq_r o$: it is $\angle q_l + \angle c + \angle h + \angle q_r + \angle o \geq \angle q_l + 4\pi/3 + 3\pi/2 + \angle q_r + \pi/3 > 3\pi$, a contradiction.

(f) *The last case $n = 1$, $m = 2$, the combinatorial type (c) on Fig. 4.33 (see Fig. 4.29).* Then S consists of two segments, i.e. $S = [bw] \cup [wd]$, where $b, d \in M_r$ are entering points, w is energetic and $\angle bwd \geq 2\pi/3$. In this case $\mathrm{turn}(S) = \pi - \angle bwd$.

First, connect o with q_l and q_r then denote $k_l = [oq_l] \cap M_r$ and $k_r = [oq_r] \cap M_r$. Now consider the convex quadrilateral $P = k_l o k_r w$. The sum of the angles $\angle k_l + \angle k_r + \angle w$ of P is at least $\pi/2 + \angle bwd + \pi/2$, so that the remaining angle (which is equal to γ) is at most $\pi - \angle bwd = \mathrm{turn}(S)$ as claimed.

If one has the equality then both $[bw]$ and $[wd]$ are tangent to M_r, but w is not energetic point in this case, because q_l is covered by $b = k_l$, q_r is covered by $d = k_r$, so we got a contradiction.

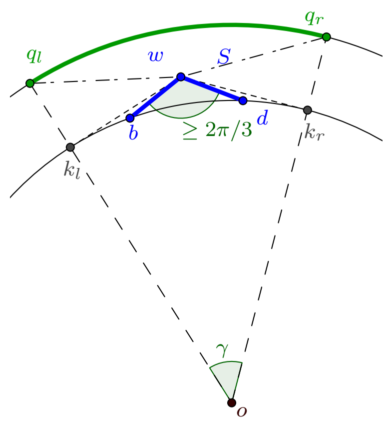
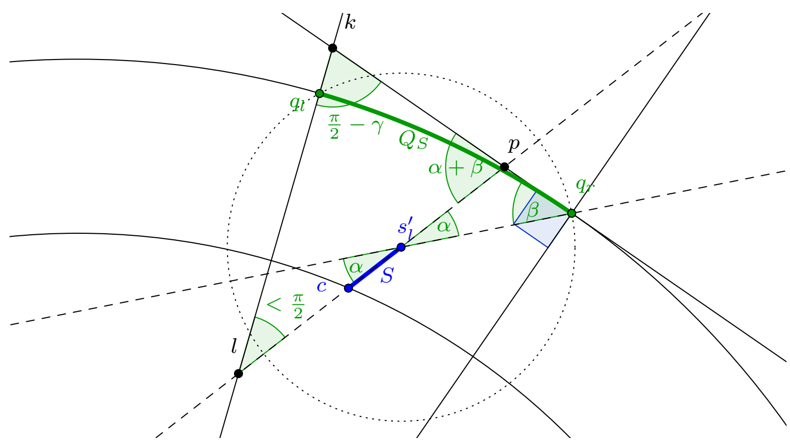


Figure 4.29: Picture to the case 1f: middle component, $m = 2$, $n = 1$.



2. *Let S be an ending component (without loss of generality let it be the left one, so $q_r = A$). Recall that c denotes the branching point if S is a tripod and the entering point if S is a segment. Then there are two options:*

   (a) *The case $n = 1$, $m = 1$, the combinatorial type (a) on Fig. 4.33 (see Fig. 4.30).* In this case $S = [cs_l]$, where $c \in M_r$, $|s_l q_r| = r$, and $\mathrm{turn}(S) = 0$. Denote by k such a point that $k \in [oq_l)$ and $\angle oq_r k = \pi/2$. Define the points $l := [s_l c) \cap (oq_l)$ and $p := [cs_l) \cap (q_r k)$, and introduce the angles $\alpha := \angle ps_l q_r$ and $\beta := \angle s_l q_r k$.

   The following two situations have to be considered. Note that $|s_l q_l| = r$, otherwise one can replace $[cs_l] \cap B_\varepsilon(s_l)$ in Σ by the part $[df]$ of the segment $[dq_r]$ where $d = [cs_l] \cap \partial B_\varepsilon(s_l)$, f is the point satisfying $\mathrm{dist}\,(f, q_r) = r$, producing the competitor of strictly lower length.

   - case $\angle cs_l q_r \leq \pi$ (see the top picture on Fig. 4.30).
     Then $\angle([as_l), A) = \beta$ and $\angle([cs_l), [s_l a)) = \alpha$, so that

     $$\mathrm{turn}(S) + \angle([cs_l), [s_l a)) + \angle([s_l a), A) = \alpha + \beta.$$

     Note that $\angle s_l pk = \alpha + \beta$ and $\angle okq_r = \pi/2 - \gamma$. If $\alpha + \beta \leq \gamma$ (contrary to the claim being proven), then $\angle okp + \angle kps_l < \pi/2$ so $\angle klp > \pi/2$, which is impossible because then $|cq_l| < |s_l q_l|$ which contradicts $|s_l q_l| = r$, $|cq_l| \geq r$.
   - case $\angle cs_l q_r > \pi$ (see the bottom picture on Fig. 4.30). In this case $\angle([s_l a), A) = \beta$

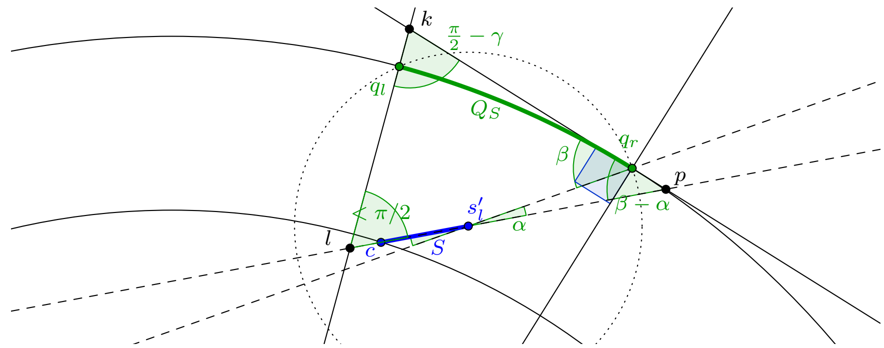


Figure 4.30: Picture to the case 2a: ending component, $n = 1$, $m = 1$.

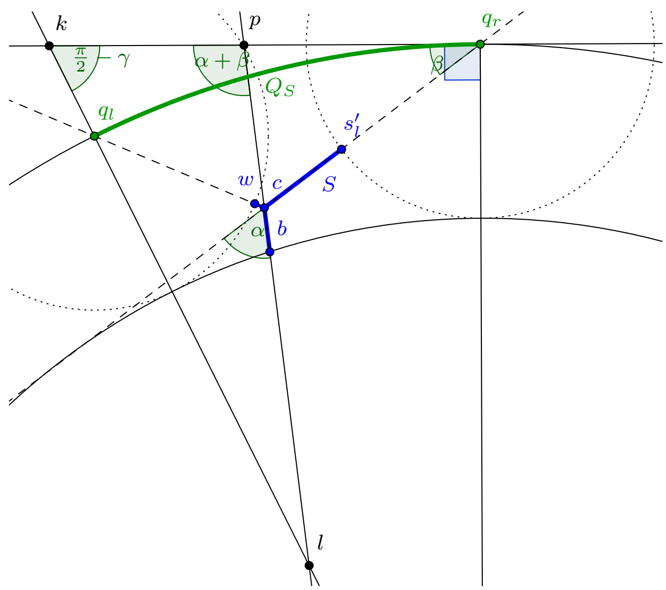


Figure 4.31: Picture to the case 2b: ending component, $n = 2$, $m = 1$.

and $\angle([cs_l), [s_la)) = -\alpha$, so that

$$\text{turn}(S) + \angle([cs_l), [s_la)) + \angle([s_la), A) = \beta - \alpha$$

and we know that $\angle kpc = \beta - \alpha$. If $\beta - \alpha \leq \gamma$ (the contrary to the claim being proven), then $\angle okp + \angle kpc < \pi/2$, which is impossible because then $|cq_l| < |s_lq_l|$ which contradicts $|s_lq_l| = r$, $|cq_l| \geq r$.

(b) *The case $n = 2$, $m = 1$, the combinatorial type (b) on Fig. 4.33 (see Fig. 4.31).*

Note that S is a tripod: $S = [bc] \cup [cw] \cup [cs_l] \subset \overline{(N \setminus N_r)}$, where $b \in M_r$. Let us prove that $q_r = [cs_l) \cap M$ and $q_l = [cw) \cap M$. Suppose the contrary i.e. without loss of generality c, s_l, and q_r do not lie on the same line. Let us pick a sufficiently small $\varepsilon > 0$ and denote by j the intersection point of $\partial B_\varepsilon(s_l)$ with $[cs_l]$. Then one may replace $[js_l]$ by $[ji]$ in Σ, where i stands for the intersection point of $\partial B_r(q_r)$ with $[jq_r]$. Clearly the resulting set covers Q_{s_l}, so it has the same energy F_M; by the triangle inequality it has strictly lower length, so we got a contradiction.

Note that $|s_lq_r| = r = |wq_l|$; $B_r(q_r) \cap \Sigma = B_r(q_l) \cap \Sigma = \emptyset$. Let $k \in [oq_l)$ be the point satisfying $(q_rk) \perp (oq_r)$. Then $\alpha := \text{turn}(S) = \angle([bc), [cq_r)) = \pi/3$, $\angle([cs_l), [s_la)) = 0$

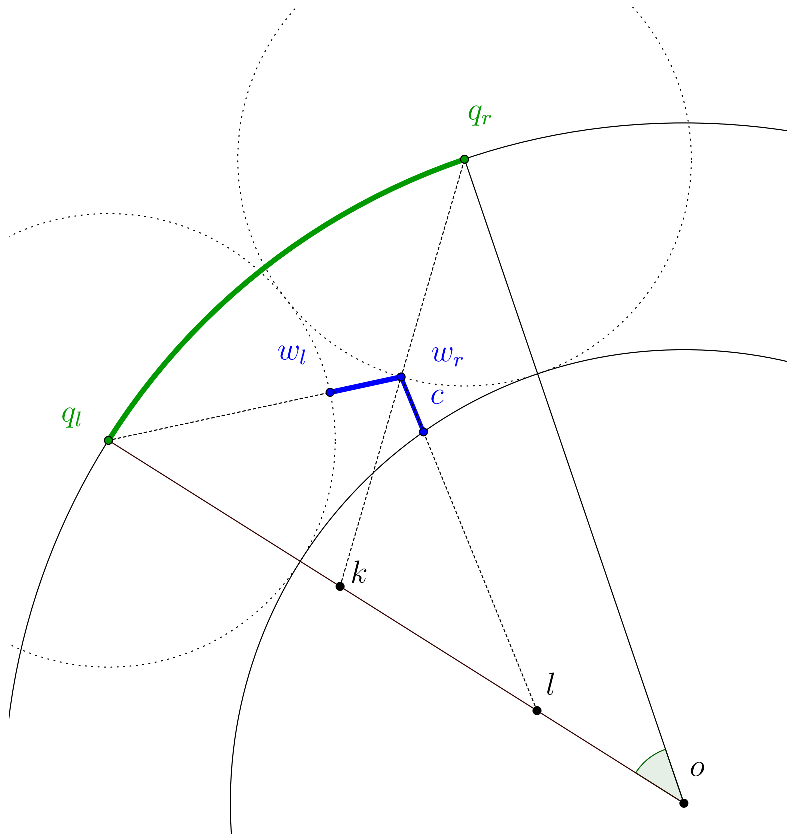


Figure 4.32: Picture to the case 2c: ending component, $n = 2$, $m = 1$.

and $\beta := ([cq_r), [q_r k)) = \angle([s_l a), A)$. we have to show $\alpha + \beta > \gamma$. Let p be the point of intersection of $(kq_r]$ and $[bc)$. Then $\angle okp = \pi/2 - \gamma$ and $\angle kpc = \alpha + \beta$. Assume the contrary, i.e. $\alpha + \beta \leq \gamma$. Then $\angle okp + \angle kpc \leq \pi/2$ hence $\angle klp \geq \pi/2$, where l is the point of intersection of (bc) and (ok), but since $\angle q_l cl = 2\pi/3$, then the sum of the angles of the triangle Δclq_l exceeds π, which is impossible.

(c) *The case $n = 2$, $m = 1$, the combinatorial type (c) on Fig. 4.33 (see Fig. 4.32).* In this case $a = q_r$, $s_l = w_r$. Denote $\angle([cw_r), [w_r q_r))$ by α, $\angle([s_l a), A) = \angle([w_r q_r), A)$ by β, clearly $\mathrm{turn}(S) = \alpha + \beta$, $\mathrm{turn}(Q_S) = \gamma$. Let l be the point of intersection of $(w_r c)$ and $(q_l o)$. Suppose the contrary, i.e. $\gamma \geq \alpha + \beta$. Then

$$\angle w_r l q_l = \pi - \angle w_r l o = \pi - (2\pi - \angle l w_r q_r - \angle w_r q_r o - \angle q_r o l) =$$

$$\pi - (2\pi - (\pi - \alpha) - (\pi/2 - \beta) - \gamma) = \pi/2 - \beta - \alpha + \gamma \geq \pi/2,$$

which is impossible because then $|cq_l| < |s_l q_l|$, which contradicts $|s_l q_l| = r$, $|cq_l| \geq r$.

□

4.7.4 Finishing the proof

Now the proof of Theorem 4.3.2 is just few lines.

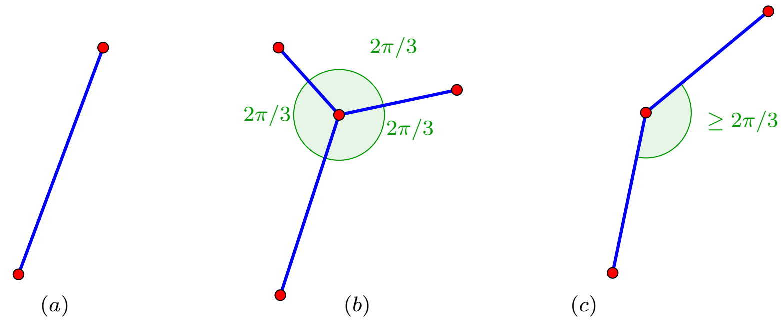


Figure 4.33: Locally miminal trees for sets of 2 and 3 points.

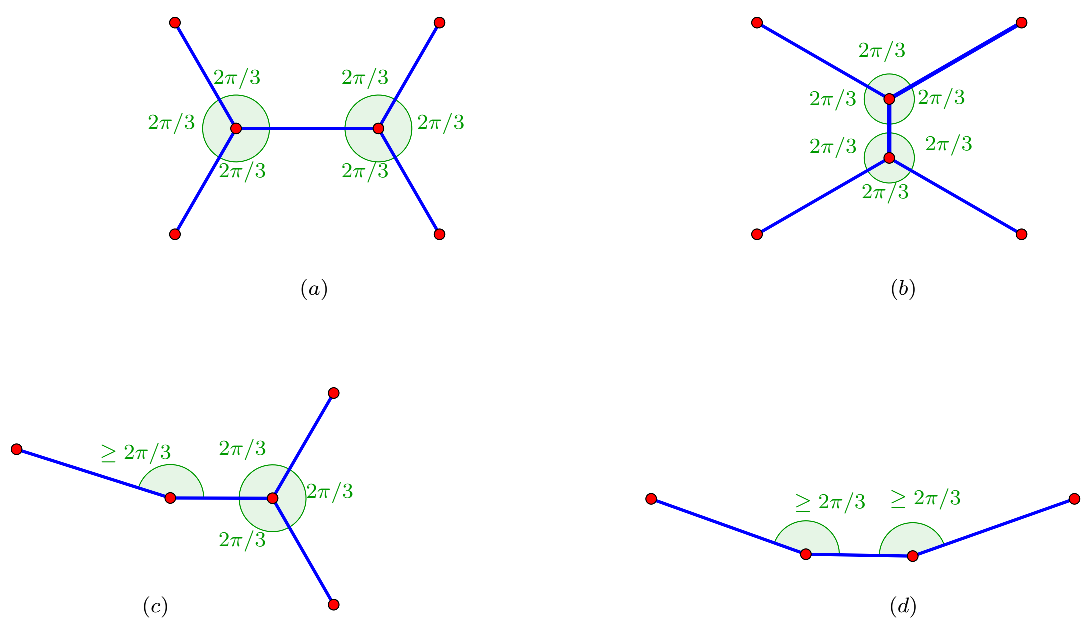


Figure 4.34: Locally miminal trees for sets of 4 points.

Proof of Theorem 4.3.2. By Lemma 4.7.1 $2a_M(r) + r < 5r$ for general M, and $2a_M(r) + r < 4.98r$ when M is the circumference. Note that

$$2\pi = \mathrm{turn}(\partial T) = \sum_{S \in V(G)} \mathrm{turn}(S) + \angle([c_l s_l), [s_l a)) + \angle([s_l a), a) + \angle([as_r), [s_r c_r)) + \angle(a, [as_r))$$

by Lemma 4.6.12 and Lemma 4.6.13, and also $\mathrm{turn}(M) = 2\pi$. Hence by Lemma 4.7.5

$$\begin{aligned} 2\pi &= \sum_{S \in V(G)} \mathrm{turn}(S) + \angle([c_l s_l), [s_l a)) + \angle([s_l a), a) + \angle([as_r), [s_r c_r)) + \angle(a, [as_r)) \\ &\geq \sum_{S \in V(G)} \mathrm{turn}(Q_S) \geq \mathrm{turn}(M) = 2\pi. \end{aligned}$$

Thus all the inequalities in Lemma 4.7.5 are equalities. Summing up, every global minimizer Σ consists of arcs of M_r and segments of tangent lines to M_r, i.e. components of the combinatorial type (a) on Fig. 4.33, tangent to M_r. Every vertex, corresponding to a component of the combinatorial type (a) on Fig. 4.33 has degree 1 in G. Thus Σ has the unique arc of M_r, and because of the absence of loops it cannot coincide with M_r. By Lemma 4.6.12 every maximal arc $[\breve{bc}] \in V_A(G)$ is connected in the graph G with two vertices, corresponding to connected components of $\Sigma \setminus N_r$. Hence any minimizer is a horseshoe. $\square$

Proof of Corollary 4.3.1. Let $\hat{\Sigma}$ be a local minimizer in the sense of Definition 4.1.2. Suppose the claim is false, i.e.

$$\mathcal{H}^1(\hat{\Sigma}) - \mathcal{H}^1(\Sigma) < (R - 5r)/2 \tag{4.5}$$

and $\hat{\Sigma}$ is not a horseshoe. Suppose first that $\hat{\Sigma}_r$ contains no line segment of length exceeding

$$a'_M(r) := 2r + \mathcal{H}^1(\hat{\Sigma}) - \mathcal{H}^1(\Sigma) < 2r + (R - 5r)/2.$$

Then Lemma 4.7.2 remains true for this situation with a'_M instead of a_M, because $2a'_M(r) + r < R$. Lemma 4.7.3 also remains true with $a'_M(r)$ instead of a_M by the same reason. we may repeat now line by line the proof of Theorem 4.3.2 without any change because all the arguments used in this proof as well as in Lemma 4.7.5 are local, except the Lemma 4.7.2 and Lemma 4.7.3 which hold true with a'_M instead of a_M. This proves that $\hat{\Sigma}$ is a horseshoe in the considered case.

On the other hand it is impossible to $\hat{\Sigma}_r$ to have a segment of length at least $a'_M(r)$, otherwise using the replacement from Lemma 4.7.1 and get a contradiction with (4.5).

$\square$

4.8 Open questions

4.8.1 Regularity

The first question, especially if the answer in negative, might be difficult.

Question 4.8.1. *Does there exist a nonplanar maximal distance minimizer with infinite number of branching points?*

An easier question should be to construct an example of a minimizer with a branching point, whose neighbourhood does not coincide with a regular tripod:

Question 4.8.2. *To construct a (nonplanar) maximal distance minimizer Σ containing a locally nonplanar branching point x, i.e. for every $\varepsilon > 0$ the set $B_\varepsilon(x) \cap \Sigma$ does not belong to a plane.*

Thus the question if there exists a nonplanar maximal distance minimizer with an infinite number of points with three tangent rays also makes sense.

The following question asks if one-sided tangents should have continuity from the corresponding side.

Question 4.8.3. *Does Lemma 4.2.1 holds for a d-dimensional maximal distance minimizer?*

All the questions in this subsection can be also asked for a local minimizers.

4.8.2 Explicit solutions

Recall that the horseshoe conjecture is still open.

Question 4.8.4. *Find maximal distance minimizers for a circumference of radius $4.98r > R > r$.*

At the same time, the statement of Theorem 4.3.2 does not hold for a general M if the assumption on the minimal radius of curvature is omitted as we show below.

Define a *stadium* to be the boundary of the R-neighborhood of a segment. By the definition, a stadium has the minimal radius of curvature R. Let us show that if $R < 1.75r$ and a stadium is long enough, then there is the connected set Σ' that has the length smaller than an arbitrary horseshoe and covers M.

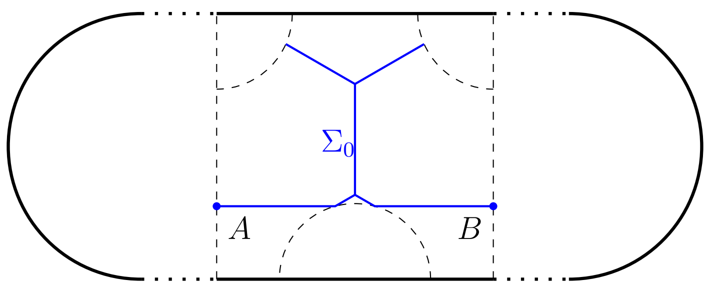


Figure 4.35: Horseshoe is not a minimizer for long enough stadium with $R < 1.75r$.

Define Σ_0 to be the locally minimal tree depicted in Fig. 4.35. Let Σ' consist of copies of Σ_0, glued at points a and b along the stadium. Note that $F_M(\Sigma') \leq r$ by the construction. In the case $R < 1.75r$ the length of Σ_0 is strictly smaller than $2|ab|$. Thus for a long enough stadium Σ' has length $\alpha L + O(1)$, where L is the length of the stadium and $\alpha < 2$ is a constant depending on Σ_0 and R. On the other hand, any horseshoe has length $2L + O(1)$.

This example leads to the following problems.

Question 4.8.5. *Find the minimal α such that Theorem 4.3.2 holds with the replacement of $5r$ with αr.*

Question 4.8.6. *Describe the set of r-minimizers for a given stadium.*

Analogously to the stadium case one can easily show that for some sufficiently small $\frac{|a_1a_2|}{|a_2a_3|} < 1$ and some $r > 0$ a minimizer should have another topology than depicted at Fig. 4.7.

Also one may consider the following relaxation of Problem 4.8.6.

Question 4.8.7. *Fix a real $a > 2r$. Let $M(l)$ be the union of two sides of length l of a rectangle $a \times l$ and $\Sigma(l)$ be a minimizer for $M(l)$. Find*

$$s(a) := \lim_{l \to \infty} \frac{\mathcal{H}^1(\Sigma(l))}{l}.$$

If $a > 10r$ one may add up $M(l)$ to a stadium and use Theorem 4.3.2 to get $s(a) = 2$.

4.8.3 Uniqueness

Recall that if Σ be an r-minimizer for some M, then it is a minimizer for $\overline{B_r(\Sigma)}$. This motivates the following question.

Question 4.8.8. *Let Σ be an r-minimizer for some M. Is Σ the unique r-minimizer for $\overline{B_r(\Sigma)}$?*

A weaker form of this question is if we replace r with some positive $r_0 < r$ in the hypothesis. Again we are interested whether Proposition 4.5.4 holds in larger dimensions.

Question 4.8.9. *Fix $d \geq 3$ and $n \geq 4$. Find the Hausdorff dimension of d-dimensional n-point ambiguous configurations M (as a subset of $\mathbb{R}^{dn}$).*

A weaker question is to determine whether the set of d-dimensional n-point ambiguous configurations has measure zero.

Chapter 5

Johnson-type graphs

This chapter is based on papers [15] and [23]. We consider a family of distance graphs in $\mathbb{R}^d$ and find its independence numbers in some cases.

Define the graph $J_{\pm}(d,k,t)$ in the following way: the vertex set consists of all vectors from $\{-1,0,1\}^d$ with exactly k nonzero coordinates; edges connect the pairs of vertices with scalar product t. We find the independence number of $J_{\pm}(d,k,t)$ for an odd negative t and $d > d_0(k,t)$.

5.1 Basics

We start with common definitions. Let $G = (V,E)$ be a graph. A subset I of vertices of G is *independent* if no edge connects vertices of I. The *independence number* of a graph G is the maximal size of an independent set in G; we denote it by $\alpha(G)$.

Generalized Johnson graphs are the graphs $J(d,k,t)$ defined as follows: the vertex set consists of vectors from the hypercube $\{0,1\}^d$ with exactly k nonzero coordinates, edges connect vertices with scalar product t (so $J(d,k,t)$ is nonempty if $k < d$ and $2k - d \le t < k$). Generalized Kneser graphs $K(d,k,t)$ have the same vertex set but the edges connect vertices with scalar product at most t.

Now we introduce the main hero of the chapter. Define graphs $J_{\pm}(d,k,t)$ as follows: the vertex set consists of vectors from $\{-1,0,1\}^d$ with exactly k nonzero coordinates, edges connect vertices with scalar product t. The graph $J_{\pm}(d,k,t)$ is nonempty if $k < d$ and $-k \le t < k$, and also if $k = d$ and $d - t$ is even. If $t = -k$, then the graph $J_{\pm}(d,k,t)$ is a matching. Note that the edges connect vertices of the Euclidean distance $\sqrt{2(k-t)}$, which means that $J_{\pm}(d,k,t)$ is a distance graph.

Finally, define $K_{\pm}(d,k,t)$ as the graph which splits the vertex set with $J_{\pm}(d,k,t)$ but the edges connect vertices with scalar product at most t.

The *support* of a vertex is the set of its non-zero coordinates. For $k = 2$ we use the notation $a^i b^j$ for a vertex with support $\{a,b\}$ and signs $i,j \in \{+,-\}$ on coordinates a and b, respectively; We use similar notation for $k = 3$.

An *automorphism* of a graph is a bijection from a set of vertices onto itself that preserves adjacency. A graph is called vertex-transitive if for any vertices u and v there is a graph automorphism that takes u to v.

Finally, let $m(a,b)$ be the number of the most significant unequal digit in the binary notation of the numbers a and b (bits are numbered starting from one).

5.1.1 Independence and chromatic numbers of $J(d, k, t)$ and $K(d, k, t)$

Independent sets in these families of graphs are classical combinatorial objects. Indeed, we have a natural bijection between the set of k-subsets of $[d]$ and $V[J(d, k, t)] = V[K(d, k, t)]$. The celebrated Erdős–Ko–Rado theorem [37] determines all maximal independent sets in $J(d, k, 0) = K(d, k, 0)$. A natural generalization was done by Erdős and Sós, who introduce "forbidden intersection problem", which involves finding the independence numbers of graphs $J(d, k, t)$. Then the Frankl–Wilson theorem [52], the Frankl–Füredi theorem [44] and the Ahlswede–Khachatryan Complete Intersection Theorem [1] answered a lot of questions about the size and the structure of maximal independent sets in the graphs $J(d, k, t)$ and $K(d, k, t)$.

On the other hand a lot of questions in combinatorial geometry are related to embeddings of these graphs into $\mathbb{R}^d$. Frankl and Wilson [52] used the graphs $J(d, k, t)$ to get an exponential lower bound on the chromatic number of the Euclidean space (Nelson–Hadwiger problem); Kahn and Kalai [68] used them to disprove Borsuk's conjecture.

Let us describe the picture for some small k and t. Erdős, Ko and Rado [37] proved that $d \geq 2k$ implies

$$\alpha[J(d, k, 0)] = \binom{d-1}{k-1}.$$

Then Lovász [87] proved Kneser's conjecture, namely that $\chi[J(d, k, 0)] = d - 2k + 2$ for $d \geq 2k$. The following result was introduced to get a constructive bound on the Ramsey number.

Proposition 5.1.1 (Nagy, [94]). *Let $d = 4s + t$, where $0 \leq t \leq 3$. Then*

$$\alpha[J(d, 3, 1)] = \begin{cases} d & \textit{if } \ t = 0, \\ d - 1 & \textit{if } \ t = 1, \\ d - 2 & \textit{if } \ t = 2 \textit{ or } 3. \end{cases}$$

Then Larman and Rogers [82] used the bound $\chi[J(d, 3, 1)] \geq \frac{|V[J(d,3,1)]|}{\alpha[J(d,3,1)]}$ to show that the chromatic number of the Euclidean space is at least quadratic in the dimension (initially it was proposed by Erdős and Sós). It turns out that the chromatic number of $J(d, 3, 1)$ is very close to $\frac{|V[J(d,3,1)]|}{\alpha[J(d,3,1)]}$ (and sometimes is equal to this ratio).

Theorem 5.1.1 (Balogh–Kostochka–Raigorodskii [4]). *Consider $l \geq 2$. If $d = 2^l$, then*

$$\chi[J(d, 3, 1)] \leq \frac{(d-1)(d-2)}{6}.$$

If $d = 2^l - 1$, then

$$\chi[J(d, 3, 1)] \leq \frac{d(d-1)}{6}.$$

Finally, for an arbitrary d

$$\chi[J(d, 3, 1)] \leq \frac{(d-1)(d-2)}{6} + \frac{11}{2}n.$$

Tort [120] proved that for $d \geq 6$,

$$\chi[K(d, 3, 1)] = \left\lceil \frac{(d-1)^2}{4} \right\rceil.$$

Zakharov [127] showed that the existence of Steiner systems (see Subsection 5.2.6) implies that

$$\chi[J(d,k,t)] \le (1+o(1))\frac{(k-t-1)!}{(2k-2t-1)!}d^{k-t}$$

for fixed $k > t$. In general $\chi[J(d,k,t)] = \Theta(d^{t+1})$ for $k > 2t+1$ and $\chi[J(d,k,t)] = \Theta(d^{k-t})$ for $k \le 2t+1$.

5.1.2 Known facts about the graphs $J_{\pm}(d,k,t)$ and $K_{\pm}(d,k,t)$

From a geometrical point of view $J_{\pm}(d,k,t)$ is a natural generalization of $J(d,k,t)$. Raigorodskii [107, 108] used the graphs $J_{\pm}(d,k,t)$ to significantly refine the asymptotic lower bounds in the Borsuk's problem and the Nelson–Hadwiger problem.

Unfortunately, there is no general method to find the independence number of $J_{\pm}(d,k,t)$ even asymptotically. One of the reasons is that the known answers have varied and sometimes rather complicated structures. For instance the proof of the following result analogous to Proposition 5.1.1 is relatively long and the answer is quite surprising.

Theorem 5.1.2 (Cherkashin–Kulikov–Raigorodskii, [16])**.** *For $d \ge 1$ define $c(d)$ as follows:*

$$c(d) = \begin{cases} 0 & \text{if } n \equiv 0 \\ 1 & \text{if } n \equiv 1 \\ 2 & \text{if } n \equiv 2 \text{ or } 3 \end{cases} \pmod 4.$$

Then

$$\alpha[J_{\pm}(d,3,1)] = \max\{6d-28, 4d-4c(d)\}.$$

In recent papers [46, 47, 49] Frankl and Kupavskii generalized the Erdős–Ko–Rado theorem for some subgraphs of $J_{\pm}(d,k,t)$. We need additional definitions.

$$V_{k,l} := \{v \in \{-1,0,1\}^d \mid v \text{ has exactly } k \ \ '1' \text{ and exactly } l \ \ '-1'\}.$$

$$J(d,k,l,t) := (V_{k,l}, \{(v_1,v_2) \mid \langle v_1, v_2\rangle = t\}).$$

Theorem 5.1.3 (Frankl–Kupavskii, [46])**.** *For $2k \le n \le k^2$ the equality*

$$\alpha[J(d,k,1,-2)] = k\binom{d-1}{k}$$

holds. In the case $d > k^2$ the following equality holds

$$\alpha[J(d,k,1,-2)] = k\binom{k^2-1}{k} + \sum_{i=k^2}^{d-1}\binom{i}{k}.$$

Paper [47] deals with a more generic problem.

Theorem 5.1.4 (Frankl–Kupavskii, [47]). *For $2k \le d$ the following bounds hold*

$$\binom{d}{k+l}\binom{k+l-1}{l-1} \le \alpha[J(d,k,l,-2l)] \le \binom{d}{k+l}\binom{k+l-1}{l-1} + \binom{d}{2l}\binom{2l}{l}\binom{d-2l-1}{k-l-1}.$$

In the case $2k \le n \le 3k-l$ the following equality holds

$$\alpha[J(d,k,l,-2l)] = \frac{k}{d}|V_{k,l}|.$$

To introduce the next result, we will need the following definition.

Definition 5.1.1.

$$S(d,D) := \begin{cases} \sum_{j=0}^{m}\binom{d}{j} & \text{if } D = 2m, \\ \binom{d-1}{m} + \sum_{j=0}^{m}\binom{d}{j} & \text{if } D = 2m+1, \end{cases}$$

In [45] (see [48] for a version with a fixed mistake) Frankl and Kupavskii determined the independence number of $K_{\pm}(d,k,t)$ for $d > d_0(k,t)$ and found the asymptotics of the independence number of $J_{\pm}(d,k,t)$ if $t < 0$ and $d > d_0(k,t)$.

Theorem 5.1.5 (Frankl–Kupavskii, [48]). *For any $k \in \mathbb{N}$ and $d \ge n(k_0)$ we have:*

1. $\alpha[K_{\pm}(d,k,t)] = \binom{d-t-1}{k-t-1}$ *for $-1 \le t \le k-1$,*
2. $\alpha[K_{\pm}(d,k,t)] = S(k,|t|-1)\binom{d}{k}$ *for odd t such that $-k-1 \le t < 0$,*
3. $\alpha[K_{\pm}(d,k,t)] = \alpha[J(d,k-\frac{|t|}{2},\frac{|t|}{2},t)] + S(k,|t|-2)\binom{d}{k}$ *for even t such that $-k-1 \le t < 0$.*

Theorem 5.1.6 (Frankl–Kupavskii, [45]). *For any $k \in \mathbb{N}$, $t < 0$ and $d > d_0(k,t)$ we have*

$$\alpha[J_{\pm}(d,k,t)] \le S(k,|t|-1)\binom{d}{k} + O\left(d^{k-1}\right).$$

The main technique in the Frankl–Kupavskii theorems is shifting. It turns out that shifting can not increase a scalar product, so it preserves the independence property of a set in a Kneser-type graph. Unfortunately, the latter does not hold for Johnson-type graphs. Using additional arguments one can derive weaker results which are tight only in asymptotics. But it looks impossible to find the independence number of $J_{\pm}(d,k,t)$ for $t > -k$ using shifting.

5.1.3 Results

Let $J(d,k,even)$ be a graph with the vertex set $\{0,1\}^d$, where edges connect vertices with even scalar product (dote that each vertex has a loop if k is even). Define $J(d,k,odd)$ in a similar way. Let $J_{\pm}(d,k,even)$ and $J_{\pm}(d,k,odd)$ be defined analogously to $J(d,k,even)$ and $J(d,k,odd)$.

Observation 5.1.1. *If $d > d_0(k)$, then*

$$\alpha[J_{\pm}(d,k,even)] = 2^k\alpha[J(d,k,even)],$$

$$\alpha[J_{\pm}(d,k,odd)] = 2^k\alpha[J(d,k,odd)].$$

For $d > d_0(k)$ the exact values of $\alpha[J_\pm(d, k, even)]$ and $\alpha[J_\pm(d, k, odd)]$ are determined in Theorem 5.2.6.

Proof of Observation 5.1.1. Let *prt* stand for *odd* or *even*.

To prove the lower bounds consider an arbitrary maximal independent set I in $J(d, k, prt)$. Then all the vertices on the supports from I form an independent set $I_\pm$ in $J_\pm(d, k, prt)$. So

$$\alpha[J_\pm(d, k, prt)] = 2^k \alpha[J(d, k, prt)].$$

The upper bounds simply follow from Lemma 5.2.1, since $J(d, k, prt)$ is a subgraph of $J_\pm(d, k, prt)$. □

Observation 5.1.2. *For every $d \geq k$ we have*

$$\alpha[J_\pm(d, k, k-1)] = 2^k \alpha[J(d, k, k-1)].$$

Note that $\alpha[J(d, k, k-1)]$ is the size of a largest partial Steiner $(d, k, k-1)$-system. In particular, if the divisibility conditions hold, then $\alpha[J(d, k, k-1)] = \binom{d}{k-1}/k$ (see Subsection 5.2.6).

Proof of Observation 5.1.2. Since $J(d, k, k-1)$ is a subset of $J_\pm(d, k, k-1)$, by Lemma 5.2.1 we have

$$\alpha[J_\pm(d, k, k-1)] \leq 2^k \alpha[J(d, k, k-1)].$$

To prove the lower bound consider an arbitrary maximal independent set I in the graph $J(d, k, k-1)$. Then all the vertices on the supports from I form an independent set $I_\pm$ in $J_\pm(d, k, k-1)$. □

We use the Katona averaging method and Reed–Solomon codes to prove the following theorem.

Theorem 5.1.7 (Cherkashin–Kiselev [15])**.** *Suppose that $d > k2^{k+1}$. Then*

$$\alpha[J_\pm(d, k, -1)] = \binom{d}{k}.$$

Theorem 5.1.7 can be generalized as follows.

Theorem 5.1.8 (Cherkashin–Kiselev [15])**.** *Suppose that t is a negative odd number, $d > d_0(k)$. Then*

$$\alpha[J_\pm(d, k, t)] = S(k, |t| - 1) \binom{d}{k},$$

where S is defined in Definition 5.1.1.

The next theorem is a consequence of Theorems 5.2.1 and 5.1.7.

Theorem 5.1.9 (Cherkashin–Kiselev [15])**.** *Let $d > \frac{9}{2}k^3 2^k$. Then*

$$\alpha[J_\pm(d, k, 0)] = 2\binom{d-1}{k-1}.$$

One can extract a stability version of the previous theorem from its proof.

The *support* of a vertex v is the set of nonzero coordinates of v; we denote it by $\operatorname{supp} v$. Let $\mathcal{H}_k = (V_k, E_k)$ be a k-graph such that

$$V_k := \bigcup_{u \in [d]} \{u^+, u^-\}, \qquad E_k := \left\{ A \in \binom{V(\mathcal{H})}{k} \,\middle|\, \{u^+, u^-\} \not\subset A \text{ for every } u \right\}.$$

There is a natural bijection between E_k and $V(J_\pm(d,k,t))$. Introduce notion *signplace* for a vertex of $\mathcal{H}_k$ and *place* for a pair of vertices $\{u^+, u^-\}$, $u \in [d]$; note that the latter definition does not depend on k.

Corollary 5.1.1. *Suppose that I is an independent set in $J_\pm(d,k,0)$ and no place intersects all the vertices of I. Then*

$$|I| \leq C(k)\binom{d}{k-2}.$$

Let us proceed with the chromatic numbers in some corner cases. To warm up, let us correctly color $J_\pm(d,2,-1)$ in $2\lceil \log_2 d \rceil + 2$ colors. Let the first and second colors get vertices with non-negative and non-positive values, respectively. Only vertices of the form a^+b^- remain. Let us color the vertex a^+b^- with the color $m(a,b)$ if in the bit $m(a,b)$ the number a has 1, and the number b, respectively, has 0; let us paint the vertex a^-b^+ in the color $-m(a,b)$. It is easy to see that all the vertices are colored, and each edge connects vertices of different colors. The total is just $2\lceil \log_2 d \rceil + 2$ colors.

The following theorem shows that the asymptotic behavior of the chromatic number is approximately two times less than in the example given.

Theorem 5.1.10 (Cherkashin [23]). *For all $d \geq 2$ the inequalities are satisfied*

$$\log_2 n \leq \chi(J_\pm[d,2,-1]) \leq \log_2 d + \left(\frac{1}{2} + o(1)\right) \log_2 \log_2 d.$$

In the case $k = 3$, $t = -1$ the picture is asymptotically the same.

Theorem 5.1.11 (Cherkashin [23]). *For some positive constants c, C and arbitrary $d > 3$ the following inequalities hold:*

$$c \log_2 d \leq \chi(J_\pm[d,3,-1]) \leq C \log_2 d.$$

And for $k = 3$, $t = -2$ we have an interesting picture.

Theorem 5.1.12 (Cherkashin [23]). *For all $d \geq 3$ the inequalities are satisfied*

$$\lceil \log_2 \lceil \log_2 d \rceil \rceil \leq \chi(J_\pm[d,3,-2]) \leq 4\lceil \log_2 \lceil \log_2 d \rceil \rceil + 6.$$

5.2 Tools

5.2.1 Trivial bounds on the chromatic numbers

Let k be fixed and t be negative. Then all vectors with non-negative coordinates form an independent set I with a fraction of vertices $\frac{1}{2^k}$. Thus, for any Johnson type graph $G = J_\pm(d,k,t)$ the classical inequality

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)} \tag{5.1}$$

gives a lower bound for the chromatic number not exceeding 2^k. We will see later that this inequality is often very inaccurate.

On the other hand, all graphs considered in the section are vertex-transitive. Consequently, for $G = J_\pm(d,k,t)$ the inequality holds, which is true for all vertex-transitive graphs [86]

$$\chi(G) \le (1 + \ln \alpha(G)) \frac{|V(G)|}{\alpha(G)} \le C(k) \log_2 d, \tag{5.2}$$

where $C(k)$ is a constant depending only on k. We will see that sometimes this estimate gives the exact order of growth in n, in particular for $J_\pm(d,3,-1)$.

5.2.2 Katona averaging method

Properties of a graph with a rich group of automorphisms sometimes can be established via consideration of a proper subgraph. We say that a graph G is *vertex-transitive* if for every vertices v_1, v_2, G has an automorphism f such that $f(v_1) = v_2$. The following lemma is a special case of Lemma 1 from [70].

Lemma 5.2.1 (Katona, [70]). *Let $G = (V, E)$ be a vertex-transitive graph. Let H be a subgraph of G. Then*

$$\frac{\alpha(G)}{|V(G)|} \le \frac{\alpha(H)}{|V(H)|}.$$

For example Lemma 5.2.1 immediately implies that for every fixed k, t the following decreasing sequences converge

$$a_n := \frac{\alpha[J_\pm(d,k,t)]}{|V[J_\pm(d,k,t)]|} \quad \text{and} \quad b_n := \frac{\alpha[K_\pm(d,k,t)]}{|V[K_\pm(d,k,t)]|},$$

as $J_\pm(d-1,k,t)$ and $K_\pm(d-1,k,t)$ are isomorphic to subgraphs of $J_\pm(d,k,t)$ and $K_\pm(d,k,t)$, respectively, and both $J_\pm(d,k,t)$ and $K_\pm(d,k,t)$ graphs are clearly vertex-transitive.

Also since $J(d,k,t)$ is a subgraph of $J_\pm(d,k,t)$, Lemma 5.2.1 implies

$$\frac{\alpha[J_\pm(d,k,t)]}{|V[J_\pm(d,k,t)]|} \le \frac{\alpha[J(d,k,t)]}{|V[J(d,k,t)]|},$$

which gives by $|V[J_\pm(d,k,t)]| = 2^k\binom{d}{k} = 2^k|V[J(d,k,t)]|$ the following bound

$$\alpha[J_\pm(d,k,t)] \le 2^k \alpha[J(d,k,t)]. \tag{5.3}$$

It turns out that bound (5.3) is rarely close to the optimal. On the other hand sometimes it is tight, for instance in Propositions 5.1.1 and 5.1.2.

5.2.3 Nontrivial intersecting families

A family of sets $\mathcal{A}$ is *intersecting* if every $a, b \in A$ have nonempty intersection. A *transversal* is a set that intersects each member of $\mathcal{A}$.

Theorem 5.2.1 (Erdős–Lovász, [38]). *Let $\mathcal{A}$ be an intersecting family consisting of k-element sets. Then at least one of the following statements is true:*

(i) $\mathcal{A}$ *has a transversal of size at most* $k-1$*;*

(ii) $|\mathcal{A}| \le k^k$*.*

One can find better bounds in the case (ii) [3, 22, 43, 128]. In particular, for $k = 3$ it is known that $3^3 = 27$ in (ii) can be replaced with 10 and this result is sharp [50].

Theorem 5.2.2 (Deza, [32])**.** *Let* $\mathcal{A}$ *be a family of* k*-element sets such that* $|A \cap A'|$ *is the same for all distinct* $A, A' \in \mathcal{A}$*. Then at least one of the following statements is true:*

(i) $A \cap A'$ *is the same for all distinct* $A, A' \in \mathcal{A}$*;*

(ii) $|\mathcal{A}| \le k^2 - k + 1$*.*

5.2.4 An isodiametric inequality

Define *the Hamming distance* between two subsets of $[d]$ as the size of their symmetric difference. The *Hamming distance* between two vectors $v_1, v_2 \in \{-1, 0, 1\}^d$ is the number of coordinates that differ between v_1 and v_2. The *diameter* of a family $\mathcal{A} \subset 2^{[d]}$ or $\mathcal{A} \subset \{-1, 0, 1\}^d$ is the maximal distance between its members.

Theorem 5.2.3 (Kleitman, [75])**.** *Let* $\mathcal{A} \subset 2^{[d]}$ *be a family with diameter at most* D *for* $d > D$*. Then*

$$|\mathcal{A}| \le S(d, D),$$

where S *is defined in Definition 5.1.1.*

Theorem 5.2.3 is sharp: in the case of even D the equality holds for the family $\mathcal{K}(d, D) := \{A \subset [d] \colon |A| \le \frac{D}{2}\}$ and in the case of odd D the equality holds for the family $\mathcal{K}_x(d, D) := \{A \subset [d] \colon |A \setminus \{x\}| \le \frac{D}{2}\}$ for some fixed $x \in [d]$.

Moreover, in [42] Frankl proved the following stability result. Let $A \Delta B$ stand for the symmetric difference of the sets A and B. We say that a family $\mathcal{A}'$ is a *translate* of a family $\mathcal{A}$ if $\mathcal{A}' = \{A \Delta T \colon A \in \mathcal{K}(d, D)\}$ for some $T \subset [d]$.

Theorem 5.2.4 (Frankl, [42])**.** *Let* $\mathcal{A} \subset 2^{[d]}$ *be a family with the diameter at most* D *and* $|\mathcal{A}| = S(d, D)$ *for* $d \ge D + 2$*. Then in the case of even* D *family* $\mathcal{A}$ *is a translate of* $\mathcal{K}(d, D)$ *and in the case of odd* D *the family* $\mathcal{A}$ *is a translate of* $\mathcal{K}_y(d, D)$*.*

5.2.5 Simple hypergraphs and Reed–Solomon codes

A hypergraph $H = (V, E)$ is a collection of *(hyper)edges* E on a finite set of *vertices* V. A hypergraph is called k*-uniform* if every edge has size k. A hypergraph is *simple* if every two edges share at most one vertex. The following construction is a special case of Reed–Solomon codes ([89], Chapter 10); it is also known as Kuzjurin's construction [80].

Fix a prime $p > k$ and let the vertex set V be the union of k disjoint copies of a field with p elements $\mathbb{F} = GF(p)$; call them $\mathbb{F}_1, \dots, \mathbb{F}_k$. Consider the following system of linear equations

$$\sum_{i=1}^{k} i^j x_i = 0, \quad j = 0, 1, \dots, k-3$$

over $\mathbb{F}_p$. The solutions $\{x_1, \dots x_k\} \in \mathbb{F}_1 \sqcup \dots \sqcup \mathbb{F}_k$, where $x_i \in \mathbb{F}_i$, form the edge set E. Fixing two arbitrary variables there is a unique solution over $\mathbb{F}_p$, because the corresponding square matrix is a Vandermonde matrix with nonzero determinant. It means that there are p^2 different solutions and $|e_1 \cap e_2| \leq 1$ for every distinct $e_1, e_2 \in E$. Summing up, $H_p(k) := (V, E)$ is a p-regular k-uniform simple hypergraph with $|V| = pk$ and $|E| = p^2$.

A k-uniform hypergraph is *b-simple* if every two edges share at most b vertices. The same construction with $k - b - 1$ equations gives an example of a k-uniform b-simple hypergraph $H(p, k, b)$.

Further we use *regularity* of $H = H(p, k, b)$ in the following sense. Consider an arbitrary vertex subset A of size b. If A contains at most 1 vertex from every copy of $\mathbb{F}_p$, then H has exactly p hyperedges containing A; otherwise H contains no such edges. Slightly abusing the notation we say that b-degree of H is p.

5.2.6 Steiner systems

A *Steiner system* with parameters d, k and l is a collection of k-subsets of $[d]$ such that every l-subset of $[d]$ is contained in exactly one set of the collection. There are some obvious necessary 'divisibility conditions' for the existence of Steiner (d, k, l)-system:

$$\binom{k-i}{l-i} \text{ divides } \binom{d-i}{k-i} \text{ for every } 0 \leq i \leq k-1.$$

In a breakthrough paper [71] Keevash proved the existence of Steiner (d, k, l)-systems for fixed k and l under the divisibility conditions and for $d > d_0(k, l)$ (different proofs can be found in [56, 72]).

Partial Steiner system. When the divisibility conditions do not hold we are still able to construct a large *partial Steiner system*, that is, a collection of k-subsets of $[d]$ such that every l-subset of $[d]$ is contained in *at most* one set of the collection. Rödl confirmed a conjecture of Erdős and Hanani and proved the following theorem.

Theorem 5.2.5 (Rödl, [110])**.** *For every fixed k and $l < k$, and for every d there exists a partial (d, k, l)-system with*

$$(1 - o(1))\binom{d}{l} / \binom{k}{l}$$

k-subsets.

Later the result was refined in [59, 73, 77]. Also it follows from the mentioned results on Steiner systems.

5.2.7 Families with even or odd intersections

Recall that $J(d, k, even)$ and $J(d, k, odd)$ were defined in Subsection 5.1.3. Frankl and Tokushige determined the independence numbers of these graphs.

Theorem 5.2.6 (Frankl–Tokushige, [51])**.** *Let $d \geq d_0(k)$. Then*

$$\alpha[J(d, k, odd)] = \binom{\lfloor d/2 \rfloor}{k/2} \qquad \textit{for even } k,$$
$$\alpha[J(d, k, even)] = \binom{\lfloor (d-1)/2 \rfloor}{(k-1)/2} \qquad \textit{for odd } k.$$

In the case when k is even, the equality is achieved for the following family: we split $[d]$ into pairs and take all sets consisting of $k/2$ pairs. In the case when k is odd we also add a fixed point $x \in [d]$ to each constructed set.

5.3 Examples

Let us start with a simple example which is rarely close to the independence number.

Example 5.3.1. *Let $t < 0$, $k > |t|$. Then $\alpha[J_\pm(d,k,t)] \geq 2^{|t|-1}\binom{d}{k}$.*

Proof. Fix an ordering of the coordinates. Take all vertices of $J_\pm(d,k,t)$ with the first $k - |t| + 1$ nonzero coordinates equal to 1. Any two such vertices can have different signs on at most $|t| - 1$ positions, therefore their scalar product is at least $-|t| + 1 = t + 1$. □

The following example is a part of Theorem 5.1.5.

Example 5.3.2. *For any $t < 0$ and $k > |t|$ we have*

$$\alpha[J_\pm(d,k,t)] \geq S(k, |t|-1)\binom{d}{k},$$

and for even t we also have

$$\alpha[J_\pm(d,k,t)] \geq S(k, |t|-1)\binom{d}{k} + \binom{k-1}{|t|/2}.$$

Proof. We start with the first bound for the case of odd t. Let I_{odd} be the set of all vertices of $J_\pm(d,k,t)$ with at most $(|t|-1)/2$ negative entries. Each k-set is the support of exactly

$$\sum_{j=0}^{(|t|-1)/2}\binom{k}{j} = S(k, |t|-1)$$

vertices in I_{odd}. Any two vectors in I_{odd} may differ in at most $2(|t|-1)/2 = |t|-1$ coordinates, so their scalar product is at least $t+1$, and I_{odd} is an independent set of the desired size.

Now we deal with the case of even t. Fix an ordering of the coordinates. For every k-set f add to I_{even} all the vertices with support f and with at most $|t|/2 - 1$ negative entries on f and all the vertices with -1 on the last coordinate of f and exactly $|t|/2 - 1$ other negative coordinates. Then each k-set is the support of exactly

$$\sum_{j=0}^{|t|/2-1}\binom{k}{j} + \binom{k-1}{|t|/2-1} = S(k, |t|-1)$$

vertices in I_{even}. Assume that I_{even} is not independent, i.e. the scalar product of some $v_1, v_2 \in I_{even}$ is equal to t. Then v_1 and v_2 together have at least $|t|$ negative entries. Hence both v_1 and v_2 have exactly $|t|/2$ negative entries, so both v_1 and v_2 have -1 at the last coordinates x_1 and x_2 of $\operatorname{supp} v_1$ and $\operatorname{supp} v_2$, respectively. But then both v_1 and v_2 can not have $+1$ at coordinates x_2 and x_1 respectively, so the scalar product is at least $t+1$. This contradiction shows that I_{even} is an independent set of the desired size.

Now we proceed to the second bound. Let us add to I_{even} all the vertices on the lexicographically first support $\{1, \dots, k\}$ with exactly $|t|/2$ negative entries and having $+1$ at the k-th coordinate. Obviously the resulting set I has the claimed size. By definition, no edge connects two vertices from I on the support $\{1, \dots, k\}$.

Consider a vertex v from I_{even} and a vertex $u \in I \setminus I_{even}$. Note that u and v together have at most $|t|$ negative entries. Since the largest coordinate of $\operatorname{supp} v$ is greater than k and v has -1 in this coordinate, the scalar product of u and v is at least $t+1$. Thus I is independent.

□

Example 5.3.3. *For $t \geq 0$ we have*

$$\alpha[J_\pm(d,k,t)] \geq 2\alpha[J(d,k,t)].$$

Proof. Let $I \subset V[J(d,k,t)]$ be an independent set of size $\alpha[J(d,k,t)]$. Define $I_\pm$ as a subset of $V[J_\pm(d,k,t)]$ consisting of vertices with all positive or all negative entries on every support $f = \operatorname{supp} v$, $v \in I$. It is easy to see that the subset $I_\pm$ is independent in $J_\pm(d,k,t)$. □

5.4 Proofs

5.4.1 Proof of Theorem 5.1.7

We start with the lower bound. One can take the vertices only with non-negative coordinates (so exactly one vertex on each support is taken); obviously the scalar product of such vertices is always non-negative, so

$$\alpha[J_\pm(d,k,-1)] \geq \binom{d}{k}.$$

Now we will show the upper bound. Denote $G := J_\pm(d,k,-1)$. Fix a prime p, $d/(2k) \leq p \leq n/k$ (so by the statement of the theorem $p > 2^k$), and let $H := H_p(k)$ (see Subsection 5.2.5) be a p-regular k-uniform simple hypergraph with $V(H) \subset [d]$. Define graph $G[H]$ as a subgraph of G, consisting of the vertices with support on edges of H. So we have

$$|V(G[H])| = 2^k |E(H)|.$$

Fix an independent set I in $G[H]$; consider the set $X \subset [d]$ of coordinates on which the vertices from I have both signs. Denote by $\operatorname{supp} I$ the set of all supports of vertices from I ($\operatorname{supp} I \subset E(H)$) and for a given $e \in E(H)$ put $e_X := e \cap X$.

Note that I has at most $2^{|e_X|}$ vertices on the support e ($|e_X|$ might be zero). Hence

$$|I| \leq \sum_{e \in \operatorname{supp} I} 2^{|e_X|} \leq \sum_{e \in E(H): |e_X|=0} 2^{|e_X|} + \sum_{e \in E(H): |e_X|>0} 2^{|e_X|} \leq |\{e \in E(H) : |e_X| = 0\}| + \sum_{e \in E(H): |e_X|>0} 2^{|e_X|}. \tag{5.4}$$

Let us show that e_X form a disjoint cover of X. Suppose the contrary, i.e. there are $e, f \in \operatorname{supp} I$ such that $e_X \cap f_X \neq \emptyset$. Since the hypergraph H is simple, and e, f correspond to its hyperedges, we have $|e_X \cap f_X| = |e \cap f| = 1$. Put $\{u\} := e \cap f$. By the definition of X there are vertices v_1, $v_2 \in I$ having different signs on u. Since I is independent and any two different supports intersect in at most 1 coordinate, v_1 and v_2 have the same support (say, not f). So every vertex of $G[H]$ with support f forms an edge in $G[H]$ with one of v_1 or v_2, thus I is not independent; contradiction.

So $\sum |e_X| = |X|$. Since the sequence $2^k/k$, $k \geq 1$, is non-decreasing and

$$\frac{a_1 + a_2 + \ldots + a_t}{b_1 + b_2 + \ldots + b_t} \leq \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_t}{b_t}\right),$$

we have

$$\sum_{e \in E(H): |e_X| > 0} 2^{|e_X|} \leq \frac{|X|}{k} 2^k. \tag{5.5}$$

By definition, every $e \in \{e \in E(H) : |e_X| = 0\}$ has empty intersection with X. Since H is p-regular, X intersects at least $\frac{p|X|}{k}$ edges of H (since every k-edge is counted at most k times), so

$$|\{e \in E(H) : |e_X| = 0\}| \leq |E(H)| - \frac{p|X|}{k}. \tag{5.6}$$

Summing up, by (5.4), (5.5), (5.6) and the choice of p, we have

$$|I| \leq |E(H)| - \frac{|X|}{k} p + \frac{|X|}{k} 2^k \leq |E(H)|,$$

which implies $\alpha(G[H]) \leq |E(H)|$, hence

$$\frac{V(G[H])}{\alpha(G[H])} \geq 2^k.$$

By the definition $G[H]$ is a subgraph of the graph G, so Lemma 5.2.1 finishes the proof.

For some k one can choose a smaller H and require a weaker inequality for d, for instance in the case $k = 3$ (see Subsection 5.5.2).

5.4.2 Proof of Theorem 5.1.8

This is a generalization of the proof of Theorem 5.1.7. The lower bound is provided in the first part of Example 5.3.2.

Denote $G = J_{\pm}(d, k, t)$ during the proof. The case $t = -k$ is obvious, because $J_{\pm}(d, k, -k)$ is a matching. From now $|t| \leq k - 1$. Fix d and a prime $p \leq n/k$ to be large enough. Let $H = H(p, k, |t|)$ (see Subsection 5.2.5) be a k-uniform $|t|$-simple hypergraph with $|t|$-codegree p. Fix an embedding of $V(H)$ into $[d]$.

Define $G[H]$ as a subgraph of G, consisting of all the vertices with support on edges of H. Fix an independent set I in $G[H]$.

Let an *object* O be a pair of opposite vectors $\{o, -o\}$ with support of size $|t|$ with $\{0, \pm 1\}$ entries. Let $\mathcal{X}$ be the set of objects $O = \{o, -o\}$ such that $(v_1, o) = (v_2, -o) = |t|$ for some vertices $v_1, v_2 \in I$ (this means that v_1 and v_2 coincide on $\operatorname{supp} o$ with o and $-o$, respectively).

Let E_{tight} be the set consisting of such edges $e \in E(H)$ that $\operatorname{diam} I[e] < |t|$, where $I[e]$ stands for the set of vertices of I with the support e. Put $E_{wide} := E(H) \backslash E_{tight}$. Let I_{tight} and I_{wide} stand for the sets of vertices of I with the support from E_{tight} and E_{wide} respectively. Then $|I| = |I_{tight}| + |I_{wide}|$.

Consider an arbitrary support $e \in E_{wide}$; by the definition of E_{wide} there are an object $X = \{x, -x\} \in \mathcal{X}$ and vertices $v_1, v_2 \in I[e]$, such that $(v_1, x) = (v_2, -x) = |t|$. Since I is independent and H is $|t|$-simple, distinct e_1 and $e_2 \in E_{wide}$ cannot lead to the same $X \in \mathcal{X}$, so

$$|\mathcal{X}| \geq |E_{wide}|. \tag{5.7}$$

For every $e \in E_{tight}$ we have $\operatorname{diam} I[e] < |t|$, thus Theorem 5.2.3 implies

$$|I[e]| \leq S\left(k, |t| - 1\right). \tag{5.8}$$

Let us study the case of equality in (5.8). Fix a support $f \subset [d]$, $|f| = k$, and consider a family $\mathcal{A} \subset \{-1, 1\}^f$ with diameter at most $|t| - 1$ and size $S(k, |t| - 1)$. Also consider an object $O = \{o, -o\}$ such that $\operatorname{supp} O \subset f$ (recall that $|\operatorname{supp} O| = |t|$). By the pigeon-hole principle and the oddity of t, one of $o, -o$ has at most $(|t| - 1)/2$ negative entries. Thus there is a vector v from $\mathcal{K}(k, |t| - 1)$ such that $(v, o) = |t|$ or $(v, -o) = |t|$. By Theorem 5.2.4 $\mathcal{A}$ is a translate of $\mathcal{K}(k, |t| - 1)$, so the previous conclusion also holds for $\mathcal{A}$.

Fix an object $X \in \mathcal{X}$ and consider an arbitrary support $e \in E_{tight}$ containing $\operatorname{supp} X$. Assume that $(v, x) = \pm t$ for some $v \in I[e]$. Consider a support $g \in E_{wide}$ such that there are $u_1, u_2 \in I[g]$, satisfying $(u_1, x) = (u_2, -x) = t$ (g exists because $X \in \mathcal{X}$). Since H is $|t|$-simple, $(u_1, v) = t$ or $(u_2, v) = t$; a contradiction. By Theorem 5.2.4 we can refine the bound (5.8) in this case:

$$|I[e]| \leq S\left(k, |t| - 1\right) - 1. \tag{5.9}$$

By the construction of H for every $X \in \mathcal{X}$, $\operatorname{supp} X$ is contained in exactly p edges of H (because it is contained in at least one edge). Every edge of H is the support of $\binom{k}{|t|}2^{|t|-1}$ objects, so is counted above at most $\binom{k}{|t|}2^{|t|-1}$ times. By (5.7) at most $|\mathcal{X}|$ of the edges are wide. So the refined bound (5.9) is applicable to at least

$$\frac{p|\mathcal{X}|}{\binom{k}{|t|}2^{|t|-1}} - |\mathcal{X}|$$

tight edges. Then

$$|I_{tight}| \leq S\left(k, |t| - 1\right)|E_{tight}| - \frac{p|\mathcal{X}|}{\binom{k}{|t|}2^{|t|-1}} + |\mathcal{X}|.$$

On the other hand there is a straightforward bound

$$|I_{wide}| \leq 2^k |E_{wide}| \leq 2^k |\mathcal{X}|.$$

Putting it all together

$$|I| = |I_{tight}| + |I_{wide}| \leq S\left(k, |t| - 1\right)|E_{tight}| - \frac{p|\mathcal{X}|}{\binom{k}{|t|}2^{|t|-1}} + (2^k + 1)|\mathcal{X}|. \tag{5.10}$$

For a large d (then p is also large enough) the inequality (5.10) implies

$$\alpha(G[H]) \leq S\left(k, |t| - 1\right)|E(H)|.$$

By the definition $G[H]$ is a subgraph G and

$$\frac{\alpha(G[H])}{V(G[H])} \leq \frac{S\left(k, |t| - 1\right)}{2^k},$$

so Lemma 5.2.1 finishes the proof.

5.4.3 Proof of Theorem 5.1.9

Consider an arbitrary independent set I in the graph $J_{\pm}(d,k,0)$. Note that supports of the vertices of I form an intersecting family; denote it by F. Let U be a minimal (by inclusion) transversal of F. As U is minimal, for every coordinate $a \in U$ there is a vertex $x_a \in I$, such that $\operatorname{supp} x_a \cap U = \{a\}$.

In the case $|U| > 1$ we can consider the set

$$C := U \cup \operatorname{supp} x_a \cup \operatorname{supp} x_b$$

for two different $a, b \in U$. Note that $|C| \leq 3k$ and every $f \in F$ intersects C in at least two places (suppose that $|f \cap U| = 1$, then it should intersect either $(\operatorname{supp} x_a) \setminus U$ or $(\operatorname{supp} x_b) \setminus U$). Hence

$$|I| \leq 2^k \binom{|C|}{2} \binom{d}{k-2} < 2^k \frac{9k^2}{2} \binom{d}{k-2}.$$

Recall that $d > \frac{9}{2}k^3 2^k$, so

$$2^k \frac{9k^2}{2} \binom{d}{k-2} < 2^k \frac{9k^2}{2} \frac{d^{k-2}}{(k-2)!} < \frac{d}{k-1} \frac{d^{k-2}}{(k-2)!} < 2\binom{d-1}{k-1}.$$

The remaining case is $|U| = 1$, say $U = \{u\}$. Consider only vertices containing u^+, by Theorem 5.1.7 we have at most $\binom{d-1}{k-1}$ such vertices. The same bound for u^- gives the desired bound.

Example 3 and Erdős–Ko–Rado theorem give a lower bound.

5.4.4 Proof of Corollary 5.1.1

Let us repeat the proof of Theorem 5.1.9. Let I be an arbitrary independent set in $J_{\pm}(d,k,0)$. Then

$$|I| < 2^k \frac{9k^2}{2} \binom{d}{k-2}$$

or the family of all supports of vertices from I has a transversal of size 1. The first possibility implies

$$|I| \leq C(k) \binom{d}{k-2};$$

the latter one contradicts the condition of the corollary.

5.4.5 Proof of Theorem 5.1.10

Proof. Let us start with the lower estimate. Consider a subset of vertices containing coordinates of different signs; let us call it $V_{\pm}$. Suppose that we covered $V_{\pm}$ with independent sets $I_1, \ldots, I_q$. Consider the set I_j. If at some coordinate a vertices from I_j take values of both signs, then all vertices from I_j whose support contains a are a^+b^- and a^-b^+ for some coordinate b. Thus, on coordinate b, the vertices from I_j also take values of different signs; let us call such coordinates *diverse*, and the remaining coordinates *positive* and *negative*, respectively.

Let us define an auxiliary graph G_j whose vertices are coordinates, and an edge connects a pair of coordinates if this pair is the support of a vertex from I_j. Note that G_j is a bipartite graph: indeed, the support either contains positive and negative coordinates (since we consider only vertices from $V_{\pm}$) or two different ones; We showed above that G_j is a matching on various coordinates.

Since $I_1, \dots, I_q$ cover $V_\pm$, the graphs G_j cover the complete graph $K_{[d]}$ on the set of coordinates. It is well known that such coverage requires at least $\log_2 d$ bipartite subgraphs. Indeed, if $F_i := G_1 \cup \dots \cup G_i$, then $\alpha(F_{i+1}) \geq \alpha(F_i)/2$; on the other hand, $\alpha(K_{[d]}) = 1$.

Let us move on to an example. We will color $J_\pm(d, 2, -1)$ with $2m+2$ colors for $n \leq \binom{2m+1}{m}$. Let us associate with each coordinate an m-element subset $[2m+1]$; let us denote the matching by f. For $1 \leq i \leq 2m+1$ the color I_i consists of vertices a^+b^+ for which $i \in f(a), f(b)$, of vertices a^+b^- for which $i \in f(a)$, $i \notin f(b)$ and vertices a^-b^- for which $i \notin f(a), f(b)$. Note that all vertices of the form a^+b^- are covered by these colors. Indeed, for any m-element subsets $f(a)$, $f(b)$ there is an element i that belongs to $f(a)$ but not to $f(b)$. Similarly, all vertices of the form a^-b^- are covered with colors, since for any two m-element subsets of a $(2m+1)$-element set there is an element from their common complement. The last color with number $2m+2$ contains all vertices of the form a^+b^+.

Since $J_\pm(d_1, 2, -1)$ is a subgraph of $J_\pm(d_2, 2, -1)$ for $d_1 < d_2$, it remains to check that the inequality $d \geq \binom{2m-1}{m-1}$ implies the inequality $2m+2 \leq \log_2 d + (1+o(1))\frac{1}{2}\log_2\log_2 d$. This is true because

$$\binom{2m-1}{m-1} = \frac{1}{2}\binom{2m}{m} = \Omega\left(\frac{4^m}{\sqrt{m}}\right).$$

□

5.4.6 Proof of Theorem 5.1.11

Let I be an independent set of the graph $J_\pm(d, 3, -1)$. We call a coordinate $i \in [d]$ *diverse* if the vertices from I take both non-zero values on i. We call a vertex $v \in I$ *special* if the support of v contains a diverse coordinate.

Lemma 5.4.1. *Let I be an independent set of the graph $J_\pm(d, 3, -1)$ for which the t coordinates are diverse. Then*

$$|I| \leq 8t(d-2) + \binom{d-t}{3}.$$

Proof. Consider a varied coordinate i. Let I_i be a subset of vertices I whose support contains i. Then I_i contains vertex v_+ with support $\{i, a, b\}$ and sign $+$ on i, as well as vertex v_- with support $\{i, c, e\}$ and the sign $-$ on i (the sets $\{a, b\}$ and $\{c, e\}$ must intersect). Then any vertex $v \in I_i$ with sign $+$ on i intersects $\{c, e\}$ (otherwise there is an edge between v_+ and v_-, which contradicts the independence of I); similarly, any v with a $-$ sign on i intersects $\{a, b\}$. Thus

$$|I_i| \leq 8(d-2) \qquad \text{and} \qquad \bigcup_{1 \leq i \leq t} I_i \leq 8t(d-2).$$

We counted all vertices for which at least one coordinate is diverse. There are vertices left whose support lies on $d-t$ uniform coordinates. There are no more than $\binom{d-t}{3}$. □

Corollary 5.4.1. *Let I be an independent set of the graph $J_\pm(d, 3, -1)$. Then the number of special vertices does not exceed cn^2.*

The upper bound follows from the fact that the graph $J_\pm(d, 3, -1)$ is vertex-transitive, as was shown in the introduction.

Let us move on to the lower estimate. Consider the partition of the set of vertices V of the graph $J_\pm(d, 3, -1)$ into independent sets $I_1, \dots, I_k$. Consider the set $V_\pm \subset V$, consisting of vertices that have coordinates of different signs; the cardinality of $V_\pm$ is $6\binom{d}{3}$. Note that for each pair of coordinates (i, j) there are $4(d-2)$ vertices from $V_\pm$ in which i and j have different signs. We say that a set of vertices *splits* a pair of coordinates (i, j) if the set contains all these $4(d-2)$ vertices. Then, by Corollary 5.4.1, the special vertices of all independent sets in the union split $O(kd)$ pairs of coordinates.

Completing the proof is similar to Theorem 5.1.10. Let independent sets $I_1, \dots, I_q$ cover the graph G. Consider a bipartite graph F_i on a set of coordinates whose parts are uniform coordinates I_i of positive and negative signs. As was shown in the proof of Theorem 5.1.10, the graph $F := F_1 \cup \dots \cup F_q$ has an independent set of size at least $d/2^q$. For $q < \frac{1}{3}\log_2 d$ we have

$$\alpha(F) \geq d^{2/3}.$$

Then all pairs of coordinates on an independent set of size $d^{2/3}$ must be separated by different coordinates; there are asymptotically more of them than kd, which is a contradiction.

5.4.7 Proof of Theorem 5.1.12

Let us start with the lower estimate. Let us assume that we have covered all the vertices of the graph with independent sets $I_1, \dots, I_q$. Let us fix the order of coordinates $[d]$ and consider only the subset of vertices V_{alt} in which the signs alternate, that is, vertices with support $\{a, b, c\}$, where $a < b < c$, have the form $a^+b^-c^+$ and $a^-b^+c^-$.

Consider an auxiliary graph H, the vertices of which are unordered pairs of coordinates, and the edges are drawn between pairs of the form $\{a, b\}$ and $\{b, c\}$, where $a < b < c$. Then, for each edge H, the union of vertices, as a support, contains two vertices of the graph G from V_{alt}; and vice versa, the support of any vertex from V_{alt} is uniquely obtained as the union of vertices corresponding to the edge H.

Lemma 5.4.2. *For each $1 \leq j \leq q$, the edges corresponding to the supports of I_j form a bipartite subgraph in H.*

Let us call this subgraph H_j.

Proof. Let us put two labels for each vertex from $I_j \cap V_{alt}$: if the vertex has the form $a^+b^-c^+$ (in accordance with the order on $[d]$), then pairs of coordinates $\{a, b\}$ and $\{b, c\}$ receive labels L and R, respectively, and if the form is $a^-b^+c^-$, then vice versa. Let some pair of coordinates $\{e, f\}$ receive both label L and label R from vertices v_1 and v_2. Then the supports of v_1 and v_2 coincide, otherwise their scalar product is equal to -2, which contradicts the independence of I_j. But then the support of any other vertex from $I_j \cap V_{alt}$ does not contain a pair of coordinates $\{e, f\}$, otherwise an edge appears with either v_1 or v_2.

Now we define vertices H only with the label L in one part, and only with the label R in the other. The subgraph on these vertices is bipartite. Vertices with both labels, as we showed above, are pendant, so adding them leaves the graph bipartite. □

Let complete bipartite subgraphs on parts of subgraphs H_i partition $V(H)$ into independent (in the graph H) sets $J_1, \dots, J_w$.

Lemma 5.4.3. *Let J be an independent set of the graph H. Then there is a partition $[d] = B \sqcup E$ such that for any vertex $\{b, e\} \in J$ we have $b \in B$, $e \in E$ and $b < e$.*

Proof. No coordinate can be the first at vertex $j_1 \in J$ and the second at $j_2 \in J$, since J is an independent set, and such j_1 and j_2 would form an edge. This allows us to define B as the set of first coordinates of the vertices $j \in J$, and E as $[d] \setminus B$. □

For each graph J_i, consider the partition $B_i \sqcup E_i$ from Lemma 5.4.2. Let some pair of coordinates not lie in different parts of any partition $B_i \sqcup E_i$. Then the corresponding vertex of the graph H does not lie in the union of independent sets $J_1, \ldots, J_w$, a contradiction. Consider the union F of bipartite graphs on the parts B_i, E_i for $1 \le i \le w$. All parts of F have size 1, hence $w \ge \lceil \log_2 d \rceil$, similar to the corresponding part of the proof of Theorem 5.1.10.

It turns out that subgraphs $H_1, \ldots, H_q$ partition $V(H)$ into at least $\lceil \log_2 d \rceil$ sets, that is, subgraphs at least $\lceil \log_2 \lceil \log_2 d \rceil \rceil$, which completes the proof of the lower bound.

Let us have a deal with the upper bound and demonstrate the coloring of the graph in

$$4 \lceil \log_2 \lceil \log_2 d \rceil \rceil + 6$$

colors.

First, let us color all vertices of the form $a^+b^-c^+$, where $a < b < c$, in $2 \lceil \log_2 \lceil \log_2 d \rceil \rceil$ colors. Let us assign the color sign $(m(a,b) - m(b,c)) \cdot m(m(a,b), m(b,c))$ to such a vertex. Note that for natural numbers $x < y < z$ the inequality $m(x,y) \ne m(y,z)$ holds; indeed: from $x < y$ it follows that in the bit $m(x,y)$ the number x takes the value 0, and the number y takes 1; similar reasoning for y and z entails a contradiction.

Let there be a pair of vertices of color j on supports $\{a,b,c\}$, where $a < b < c$, and $\{b,c,e\}$ with scalar product -2. Then, since the first vertex has the form $a^+b^-c^+$, the second vertex has the signs b^+c^-, hence the coordinates are of order $a < b < c < e$. Since the vertices are of the same color, the expressions $m(a,b) - m(b,c)$ and $m(b,c) - m(c,e)$ have the same sign. Let $m(a,b) < m(b,c) < m(c,e)$, then $m(m(a,b), m(b,c)) = m(m(b,c), m(c,e))$ means that $m(b,c)$ has a one on one side and a zero on the other side; contradiction. The case $m(a,b) > m(b,c) > m(c,e)$ is treated in the same way.

Similarly, one can color vertices of the form $a^-b^+c^-$ $(a < b < c)$ in $2 \lceil \log_2 \lceil \log_2 d \rceil \rceil$ colors.

Finally, the last six colors consist of vertices of the form $a^+b^+c^+$, $a^+b^+c^-$, $a^+b^-c^-$, $a^-b^+c^+$, $a^-b^-c^+$ and $a^-b^-c^-$, respectively, where $a < b < c$. A direct check shows that with such a coloring there are no edges of the same color.

5.5 Independent numbers in the case $k \le 3$

We have implemented Östergård algorithm [97] to find independence numbers of several small graphs. All the calculations were done on a standard laptop in a few hours. The source can be found in [74].

5.5.1 The case $k = 2$

The case $t = -1$. By simple calculations we have

$$\alpha[J_\pm(2,2,-1)] = \alpha[J_\pm(3,2,-1)] = 4, \quad \alpha[J_\pm(4,2,-1)] = 8, \quad \alpha[J_\pm(5,2,-1)] = 10.$$

In Section 5.2.2 we show that the sequence

$$\frac{\alpha[J_\pm(d,2,-1)]}{|V[J_\pm(d,2,-1)]|}$$

is non-increasing, so

$$\alpha[J_\pm(d,2,-1)] = \binom{d}{2}$$

for $d \ge 5$.

The case $t = 0$. It is straightforward to check that

$$\alpha[J_{\pm}(2,2,0)] = 2, \quad \alpha[J_{\pm}(3,2,0)] = \alpha[J_{\pm}(4,2,0)] = 6.$$

For the case $d > 4$ we can repeat the proof of the Theorem 5.1.9 and show that $\alpha[J_{\pm}(d,2,0)] = 2(d-1)$.

The case $t = 1$. From Proposition 5.1.2 we have

$$\begin{aligned} &\alpha[J_{\pm}(d,2,1)] = 2n && \text{for even } d, \\ &\alpha[J_{\pm}(d,2,1)] = 2(d-1) && \text{for odd } d. \end{aligned}$$

5.5.2 The case $k = 3$, $t = -1$

Proposition 5.5.1. *Let $d \geq 7$. Then*

$$\alpha[J_{\pm}(d,3,-1)] = \binom{d}{3}.$$

Proof. Fano plane is the projective plane over $GF(2)$ i.e. the following simple 3-graph on 7 vertices

$$\{1,2,3\}, \{1,4,7\}, \{1,5,6\}, \{2,4,6\}, \{2,5,7\}, \{3,4,5\}, \{3,6,7\}.$$

Consider an arbitrary embedding F of the Fano plane into $V[J_{\pm}(d,3,-1)]$. As usual consider the subgraph $G[F]$; it has $7 \cdot 2^3 = 56$ vertices. One may check by hands or via computer that $\alpha(G[F]) = 7$. By Lemma 5.2.1

$$\alpha[J_{\pm}(d,3,-1)] \leq \binom{d}{3}.$$

On the other hand, Example 5.3.1 implies $\alpha[J_{\pm}(d,3,-1)] = \binom{d}{3}$. □

By the computer calculations we have

$$\alpha[J_{\pm}(6,3,-1)] = 21 > \binom{6}{3} = 20,$$

so Proposition 5.5.1 is sharp. Also

$$\alpha[J_{\pm}(5,3,-1)] = 14, \quad \alpha[J_{\pm}(4,3,-1)] = 8, \quad \alpha[J_{\pm}(3,3,-1)] = 2.$$

5.5.3 The case $k = 3$, $t = 0$

By the computer calculations we have

$$\alpha[J_{\pm}(3,3,0)] = \alpha[J_{\pm}(4,3,0)] = 8, \quad \alpha[J_{\pm}(5,3,0)] = 20, \quad \alpha[J_{\pm}(6,3,0)] = 32,$$

$$\alpha[J_{\pm}(7,3,0)] = \alpha[J_{\pm}(8,3,0)] = \alpha[J_{\pm}(9,3,0)] = 56.$$

Proposition 5.5.2. *Let $d \geq 9$. Then*

$$\alpha[J_{\pm}(d,3,0)] = 2\binom{d-1}{2}.$$

Proof. The example is inherited from Theorem 5.1.9.

Let us proceed with the upper bound. For the case $d = 9$ the computer calculations give us the desired result. Let us repeat the proof of Theorem 5.1.9, updating it for small values of d. Let I be a maximal independent set in $G := J_\pm(d, 3, 0)$.

Clearly supports of vertices of I form a 3-uniform intersecting family. Theorem 5.2.1 states that an intersecting family either contains at most 27 sets or has a 2-transversal. It is known [50] that the constant 27 can be refined to 10.

In the first case the family of supports has no 2-transversal. Then $|I| \leq 8 \cdot 10$, which is enough for $d > 10$. Assume the contrary to the statement in the case $d = 10$, id est $|I| > 72$. It implies that vertices in I have exactly 10 different supports. Suppose that every pair of supports splits exactly one vertex. Then by Theorem 5.2.2 all the supports have one common vertex, so at least $1 + 2 \cdot 10 > 10$ coordinates are required. Thus there are supports f_1, f_2 such that $|f_1 \cap f_2| = 2$. The initial graph G has 16 vertices with supports f_1 and f_2; by the equality $\alpha[J_\pm(4, 3, 0)] = 8$, I has at least 8 missing vertices on these supports. This refines the bound $|I| \leq 80$ to the desired $|I| \leq 72 = 2\binom{9}{2}$.

In the second case we have a one-point transversal set, say $U = \{u\}$. Let I_{sign} be a set of vertices from I containing u^{sign}, where $sign \in \{+, -\}$. Clearly $|I| = |I_+| + |I_-|$. After removing coordinate u from every vertex, I_+ becomes an independent set in $J_\pm(d-1, 2, -1)$. By Subsection 5.5.1 $|I_+| \leq \binom{d-1}{2}$. The same bound for I_- finishes the proof in this case.

In the last case we have a transversal set of size 2, say $\{a, b\}$. Let I_a be the set of vertices of I containing a and not containing b, I_b is defined analogously. Both I_a and I_b are nonempty, otherwise there is a one-point transversal set which is the previous case. Define $I_{ab} = I \setminus I_a \setminus I_b$. Computer calculations show that for $d = 10$ we have at most 48 vertices in an independent set with such conditions.

Let d be greater than 10; for every set $A \subset [d]$, such that $|A| = 10$ and $a, b \in A$, we have $\alpha(G[A]) \leq 48$ (here $G[A]$ stands for the subgraph of G containing all the vertices v such that $\operatorname{supp} v \subset A$). Define $I[A]$ as the set of vertices i from I such that $\operatorname{supp} i \subset A$; note that $I[A]$ is an independent set. Every vertex from I_{ab} belongs to $\binom{d-3}{7}$ different A, every vertex from $I_a \cup I_b$ belongs to $\binom{d-4}{6}$ different A. Summing up inequalities $|I[A]| \leq \alpha(G[A]) \leq 48$ over all choices of A we got

$$\binom{d-3}{7}|I_{ab}| + \binom{d-4}{6}(|I_a| + |I_b|) \leq 48\binom{d-2}{8}$$

which is equivalent to

$$\frac{d-3}{7}|I_{ab}| + (|I_a| + |I_b|) \leq \frac{48}{56}(d-2)(d-3).$$

Finally,

$$|I| = |I_{ab}| + |I_a| + |I_b| \leq \frac{d-3}{7}|I_{ab}| + (|I_a| + |I_b|) \leq \frac{48}{56}(d-2)(d-3) < 2\binom{d-1}{2}.$$

□

5.5.4 The case $k = 3$, $t = -2$

Example 5.3.2 gives us a lower bound $\alpha[J_\pm(d, 3, -2)] \geq 2\binom{d}{2} + 2$. Note that the Katona averaging method does not give an exact result because of the additional term of a smaller order of growth.

First, note that Theorem 5.1.6 in this case gives the bound

$$\alpha[J_\pm(d, 3, -2)] \leq 2\binom{d}{3} + 8\binom{d}{2}.$$

Indeed, let I be an independent set in $J_{\pm}(d,3,-2)$. We call a vertex $v \in I$ *bad* if there is another vertex with the same support which differs in exactly two places. Otherwise we call a vertex *good*. From Theorem 5.2.3 there are at most $2\binom{d}{3}$ good vertices.

Let us show that the number of bad vertices is at most $8\binom{d}{2}$. Indeed, each bad vertex has a pair of signplaces such that antipodal pair of signplaces contained in another vertex. But then all vertices containing one of these two pairs of signplaces must have the same third place therefore there are at most $8\binom{d}{2}$ bad vertices. □

Using more accurate double counting we can prove the following upper bound.

Proposition 5.5.3. *For $d \geq 6$ we have*

$$\alpha[J_{\pm}(d,3,-2)] \leq 2\binom{d}{3} + \frac{8}{3}\binom{d}{2}.$$

Proof. A pair of vertices $v, w \in I$ is called *tangled* if these vertices have the same support and differ exactly at two places. Define the *weight* $c_I(v,i,j)$, where $v \in I$ and $i, j \in v$, in the following way:

$$c_I(v,i,j) = \begin{cases} 1, & \text{if } v \text{ does not have tangled vertices in } G, \\ 2, & \text{if } v \text{ has a tangled vertex in } G \text{ which differs at places } i, j, \\ 0.5, & \text{otherwise.} \end{cases}$$

Note that for a vertex v sum of corresponding weights is at least 3. Let $d_{i,j}$ be the sum of weight of vertices containing places i and j and let us estimate an upper bound for $d_{i,j}$. Then there are three cases which depend on whether there are tangled vertices containing places i, j and whether these vertices have antipodal signs on places i, j.

In the first case there are no tangled vertices in I which differ in places i, j. Then for any place l the total weight of vertices with support $\{i, j, l\}$ is at most 2. Then $d_{i,j} \leq 2(d-2)$. In the second case there are tangled vertices in I which contain all four pairs of signplaces on places i, j. Then there are at most 8 vertices containing these places and $d_{i,j} \leq 16$.

In the last case there are two vertices in I which are antipodal on places i, j and there are no vertices in I which contain one of the pairs of signplaces on places i, j. Then there are at most 4 vertices which differ in places i, j and their total weight is at most 8. The rest of vertices containing places i, j have the same signs on these places therefore their total weight is at most $2(d-2)$.

Therefore, $d_{i,j} \leq 2n + 4$ and

$$3|I| \leq \sum_{1 \leq i < j \leq n} d_{i,j} \leq \binom{d}{2}(2n+4) = 6\binom{d}{3} + 8\binom{d}{2}.$$

□

5.6 Open questions

It seems very challenging to find a general method providing the independence number of $J_{\pm}(d,k,t)$. Here we discuss questions that seem for us both interesting and relatively easy.

Small values of the parameters. The smallest interesting case is $J_{\pm}(d,3,-2)$. We hope that for $d > d_0$ Example 5.3.2 is the best possible, i.e.

$$\alpha[J_{\pm}(d,3,-2)] = \alpha[K_{\pm}(d,3,-2)] = 2\binom{d}{3} + 2.$$

Recall that the last equality is established by Theorem 5.1.5.

Another small case leads to the following conjecture.

Conjecture 5.6.1. *Let $d > d_0$ be an even number. Then*

$$\alpha[J_{\pm}(d,4,1)] = 2n(d-2).$$

Obviously $\alpha[J_{\pm}(d,4,1)] \geq \alpha[J_{\pm}(d,4,odd)] = 2n(d-2)$ (see Proposition 5.1.2).

Chromatic numbers. Usually finding or evaluating the chromatic number is a more complicated problem than finding or evaluating the independence number. In particular Lovász [87] proved Kneser's conjecture on the chromatic number of $K(d,k,0)$ 17 year after Erdős, Ko and Rado determined the independence number of this graph.

In the setting of this chapter we have

$$c(k,t)n \leq \frac{|V[J_{\pm}(d,k,t)]|}{\alpha[J_{\pm}(d,k,t)]} \leq \chi[J_{\pm}(d,k,t)] \leq \frac{|V[J_{\pm}(d,k,t)]|}{\alpha[J_{\pm}(d,k,t)]} \log |V[J_{\pm}(d,k,t)]| \leq C(k,t)n\log n$$

for some positive constants $c(k,t)$, $C(k,t)$. The second inequality holds since $J_{\pm}(d,k,t)$ is a vertex-transitive graph (see [86]).

Recall that Theorem 5.1.12 shows that for $k=3$, $t=-2$ the chromatic number of a Johnson-type graph may not coincide with simple general bounds.

Difference between $J_{\pm}(d,k,t)$ and $K_{\pm}(d,k,t)$. It turns out that for a negative odd t Theorems 5.1.5 and 5.1.8 give

$$\alpha[J_{\pm}(d,k,t)] = \alpha[K_{\pm}(d,k,t)].$$

Does it hold for all negative t? Do we have

$$\chi[J_{\pm}(d,k,t)] = \chi[K_{\pm}(d,k,t)]$$

in this case?

The general comparison of the behavior of independence numbers and chromatic numbers of these graphs is also of interest.

Chapter 6

Chromatic numbers of 2-dimensional spheres

In 1976 Simmons conjectured that every coloring of a 2-dimensional sphere of radius strictly greater than 1/2 in three colors has a pair of monochromatic points at distance 1 apart. Paper [21] proves this conjecture and we repeat the proof here.

6.1 Introduction

A *coloring* of a given set M is a map from M to the set of colors. A coloring of a subset M of a metric space is *proper* if no pair of monochromatic points lie at distance 1 apart. The minimum number of colors that admits a proper coloring of M is called *the chromatic number* of M; we denote it by $\chi(M)$. In the case of $M \subset \mathbb{R}^d$, the distance typically comes from the induced Euclidean metric on M.

A slightly different point of view is to consider a *unit distance graph* $G(M)$: the points of M are the vertices of $G(M)$ and edges connect points at unit distance apart. By definition, $\chi(M) = \chi(G(M))$. The de Bruijn–Erdős theorem states that if $\chi(M)$ is finite then there is a finite subgraph H of $G(M)$ such that $\chi(H) = \chi(G(M))$.

Denote by $S^2(r)$ the two-dimensional sphere of radius r in $\mathbb{R}^3$ centered at the origin. Let $\chi(S^2(r))$ be the chromatic number of $S^2(r)$ with respect to the Euclidean metric. Obviously, if $r < 1/2$ and $r = 1/2$ then the chromatic number is equal to 1 and 2, respectively. Note that for any $r > \frac{1}{2}$ there is $r_1 < r$ such that $S^1(r_1)$ contains an odd cycle. Since $S^1(r_1) \subset S^2(r)$, we obtain that $\chi(S^2(r)) \geq 3$. G. Simmons [116] proved that

$$\chi(S^2(r)) \geq 4 \quad \text{for} \quad r \geq \frac{\sqrt{3}}{3}.$$

In the proof, Simmons constructs certain subgraphs of $G(S^2(r))$ that contain triangles. Obviously, for smaller values of the radius $G(S^2(r))$ is triangle-free, and so other ideas are needed.

Then L. Lovász [88] generalized the odd cycle construction to an arbitrary dimension, showing that for every $d \geq 3$ there exists a family of *strongly self-dual polytopes* inscribed in $S^{d-1}(r)$ whose graphs of diameters have chromatic number $d+1$ and that r can be arbitrarily close to $\frac{1}{2}$. In our notation this result can be formulated as follows:

Theorem 6.1.1 (Lovász, [88]). *For every $d \geq 2$ there exists a monotonically decreasing sequence $r_k^{(d)}, k = 1, 2, \dots,$ such that*

$$\lim_{k\to\infty} r_k^{(d)} = \frac{1}{2} \quad \textit{and} \quad \chi\left(S^{d-1}\left(r_k^{(d)}\right)\right) \geq n + 1.$$

Since $S^{d-1}(r_1) \subset S^n(r)$ for $r_1 \le r$, we get the following inequality.

Corollary 6.1.1.
$$\chi(S^{d-1}(r)) \ge n \quad \textit{for} \quad r > \frac{1}{2}.$$

Some sources state that the chromatic number of a two-dimensional sphere $S^2(r)$ is known only for $r \le \frac{1}{2}$ and for $r = \frac{\sqrt{2}}{2}$ [67, 90]. But it should be clarified that the equality $\chi(S^2(r)) = n + 1 = 4$ is true for $r \in \{r_k^{(3)}\} \cap \left(\frac{1}{2}, \frac{\sqrt{3-\sqrt{3}}}{2}\right]$. Explicit formulas for algebraic numbers $r_k^{(3)}$, if such exist, seem to be too complicated, but it is not difficult to compute $r_k^{(3)}$ for a given k with an arbitrary precision by approximately solving a certain optimization problem. For example, the first non-trivial construction in the case of a two-dimensional sphere corresponds to a unit distance embedding of the Grötzsch graph at $r = 0.54003829...$

It is worth noting that chromatic numbers in high dimensions were studied using algebraic, topological and combinatorial methods. A.M. Raigorodskii [109] showed that for every fixed $r > 1/2$ the chromatic number of an d-dimensional sphere grows exponentially with d. O. Kostina [76] refined asymptotic lower bounds. R. Prosanov [105] gave a new asymptotic upper bound. The paper of A. Kupavskii [79] contains several results on the number of different colors on a sphere of given radius in every proper coloring of $\mathbb{R}^d$.

A lot of results on colorings of 2-dimensional spheres were obtained by Simmons [116]. Recent discovery of a 5-chromatic unit distance subgraph of the Euclidean plane [31] spurred interest in the topic and in particular to the chromatic number of a 2-dimensional sphere.

Among the other results, Voronov, Neopryatnaya, and Dergachev [122] constructed several 5-chromatic subgraphs of 2-dimensional spheres, which lead to the bounds

$$\chi(S^2(r_1)) \ge 5 \quad \text{where} \quad r_1 = \cos\frac{3\pi}{10} = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} = 0.58778\ldots;$$

$$\chi(S^2(r_2)) \ge 5 \quad \text{where} \quad r_2 = \cos\frac{\pi}{10} = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} = 0.95105\ldots.$$

The paper [117] contains a family of proper colorings of $S^2(r)$ spheres in 7 colors, provided r is large enough.

The following statement was formulated by Simmons as a conjecture [116]. The proof of Simmons' conjecture is the main result of the chapter.

Theorem 6.1.2 (Cherkashin–Voronov [21])**.** *For every $r > \frac{1}{2}$ we have*
$$\chi(S^2(r)) \ge 4.$$

We note that for $\frac{1}{2} < r \le \frac{\sqrt{3-\sqrt{3}}}{2} = 0.563\ldots$ a proper 4-coloring of $S^2(r)$ can be obtained from a partition of the sphere into four equal spherical triangles [116]. It implies the following corollary.

Corollary 6.1.2. $\chi(S^2(r)) = 4$ *for* $\frac{1}{2} < r \le \frac{\sqrt{3-\sqrt{3}}}{2} = 0.563\ldots.$

6.2 Proof of Theorem 6.1.2

Recall that for $r \geq \frac{\sqrt{3}}{3}$ the statement was proved in [116].

Here is the sketch of the proof. Fix $r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$. Suppose that there is a proper 3-coloring of the sphere $S^2(r)$. Further arguments consist of two steps. In the first step we use the Borsuk–Ulam theorem to show that every color is dense in the sphere. Consider a graph G_k with vertices $x_1, \dots x_{2k+1}$, $y_1, \dots, y_{2k+1}$ and edges $\{(y_i, y_{i+1}), (x_i, y_i) : 1 \leq i \leq 2k+1\}$ (where indices are modulo $2k+1$), i.e. an odd cycle with attached pendant vertices. We provide an explicit representation of G_k as a unit distance subgraph of the sphere. The second step is to show that this embedding is stable under small perturbations of x_i. Then one can move every x_i at a red point, which forces the odd cycle on vertices y_i to be colored in the remaining two colors. The contradiction proves the theorem.

Note that the idea of attaching an odd cycle to a finite set A in order to exclude the possibility of A to be monochromatic was used in a series of papers devoted to the existence of planar unit distance graphs with chromatic number 4 and arbitrarily large girth [61, 118, 126]. The key twist in step 2 is to find the required embedding of G_k implicitly, i.e. the corresponding A is not a constructive set. Similar ideas were used in [69].

6.2.1 Step 1. Each color is a dense set

All the distances are considered in the metrics induced from Euclidean space $\mathbb{R}^3$, the distance between x and y is denoted by $\|x - y\|$.

Fix $r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$ and consider $S^2(r)$. Suppose that there is a proper coloring of $S^2(r)$ in three colors. Consider the unit distance graph $G = G(S^2(r))$. Then the neighborhood of a vertex x in graph G forms a circle of diameter $d = \frac{\sqrt{4r^2-1}}{r}$ in the sphere. It is worth noting that every circle in the sphere has two centers at a pair of antipodal points and hence it has two radii; a circle of diameter $d = \frac{\sqrt{4r^2-1}}{r}$ has radii 1 and $\rho = \sqrt{4r^2-1}$ in the induced metric. Since $r < \frac{\sqrt{3}}{3}$, the smaller radius is ρ, so we refer to ρ as the radius and $-x$ as the center of the circle. Vice versa, any circle of radius ρ is a graph-neighborhood of some vertex of G, and hence contains points of at most two colors.

We need the following technical statement.

Lemma 6.2.1. *Let $D \subset S^2(r) \times S^2(r)$ be a set of pairs (x, y) such that $0 < \|x - y\| < d$. Then*

- *for every $(x, y) \in D$ there are two circles of radius ρ containing x and y. One may denote their centers by c_r and c_l in such a way that the triple of radius-vectors (x, y, c_r) is right-handed and the triple (x, y, c_l) is left-handed.*

- *The functions $c_r(x, y)$ and $c_l(x, y)$ from D to $S^2(r)$ are continuous.*

In what follows, we will call a circle passing through the points x, y with center c *right-handed* if the triple (x, y, c) is right-handed, and *left-handed* otherwise.

Let C_{red}, C_{blue}, C_{green} be the sets of red, blue and green points, respectively. A *chromaticity* of a point x is the number of sets $\overline{C_{red}}$, $\overline{C_{blue}}$, $\overline{C_{green}}$ containing x (as usual, $\overline{T}$ stands for the closure of a set T). A set $T \subset S^2(r)$ is called *dense* if $\overline{T} = S^2(r)$. Let $B_\rho(x)$ denote the set of points $y \in S^2(r)$ such that $\|x - y\| < \rho$, i.e. an open ball of radius ρ and diameter d.

Lemma 6.2.2. *If some open ball of diameter d contains points of all three colors then each of C_{red}, C_{blue}, C_{green} is dense in the sphere.*

Proof. Consider points $x \in C_{red}$, $y \in C_{blue}$ and $z \in C_{green}$ inside a ball K_0 of diameter d. Then one can continuously move K_0 to a ball K containing two points (say, x and y) on the boundary; at the first such moment the point z lies inside K. The circle ∂K contains blue and red points, and so it is colored in blue and red only. Hence, it contains a point u lying in the closures of C_{red} and C_{blue}; without loss of generality, assume that point u is red. A red-green circle (right-handed, guaranteed to exist by Lemma 6.2.1) of diameter d containing z and u and a blue-green circle (left-handed) with the diameter d containing z and blue point u' in a small neighborhood of u intersect in a green point v. Note that if $u = u'$ then $v = u = u'$. Hence, due to the continuity of circles in Lemma 6.2.1, v may be arbitrarily close to u with a proper choice of u' (see Fig. 6.1). It implies that the chromaticity of u is three.

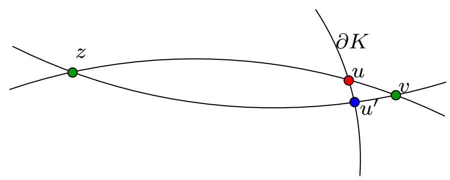


Figure 6.1: Finding a point with chromaticity 3 in Lemma 6.2.2

Since u has chromaticity 3, a small neighborhood of u contains a point $a \neq u$ with the chromaticity at least 2. Suppose that a has chromaticity 2 (say, a does not lie in $\overline{C_{green}}$) and $\|a - u\| < d$. Consider a green point b in a small neighborhood of u. Consider a red point e and a blue point f in a small neighborhood of a. Then the right-handed circle containing b and e is red-green and the left-handed circle containing b and f is blue-green, so they intersect in a green point g. Since the neighborhoods can be chosen arbitrarily small, g can be arbitrarily close to a. Hence, a has chromaticity 3, a contradiction.

Thus, we have shown that if a point with the chromaticity 3 and a point with the chromaticity at least 2 lie at a distance smaller than d, then they both have chromaticity 3.

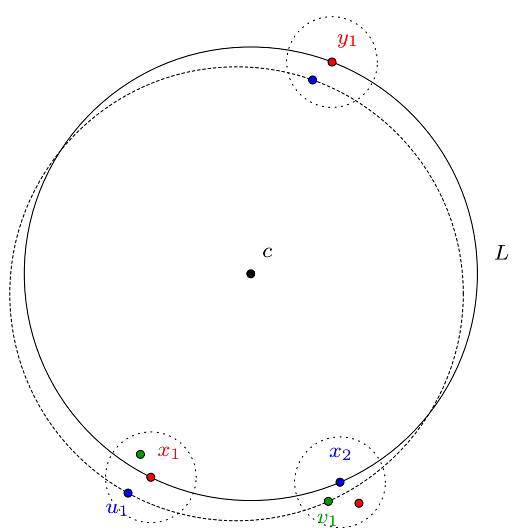


Figure 6.2: Propagation of 3-chromaticity along a circle in Lemma 6.2.2

Now let x_1 and x_2 be points of chromaticity 3 such that $\|x_1 - x_2\| < d$. We claim that any point on a circle L of diameter d containing x_1 and x_2 has chromaticity three. By the previous argument, it is enough to show that the chromaticity is at least 2. Without loss of generality, a triple (x_1, x_2, c) is left-handed, where c is the center of L on the sphere. Arguing indirectly, assume that a point $y_1 \in L$ has a small red neighborhood U_{y_1}. Choose a blue point u_1 in a small neighborhood of x_1 and a green point v_1 in a small neighborhood of x_2 (see Fig. 6.2). By Lemma 6.2.1 the left-handed circle of diameter d passing through blue point u_1, green point v_1 is close to L so it intersects red set U_{y_1}; this contradiction shows that every point on L has chromaticity 3.

Let q be an arbitrary point of $S^2(r)$. Consider a path $q_0, q_1 \dots q_t = q$ such that $q_0 \in L$ and $\|q_{i+1} - q_i\| < \rho$ for $0 \le i \le t-1$. A circle L_1 of diameter d that passes through q_1 and q_0 intersects L in two points, so by the previous argument every point (in particular, q_1) of L_1 has chromaticity 3. By induction, a circle L_{i+1} of diameter d that passes through q_{i+1} and q_i intersects L_i in two points, so every point in L_{i+1} (in particular q_{i+1}) has chromaticity 3. So $q = q_t$ also has chromaticity 3. Since $q \in S^2(r)$ was arbitrary, every point of $S^2(r)$ has chromaticity 3. $\square$

Suppose that the condition of Lemma 6.2.2 does not hold, i.e.

$$\textit{every open ball of diameter } d \textit{ contains points of at most two colors.} \qquad (\star)$$

Consider a continuous function

$$f : S^2(r) \to \mathbb{R}^2, \quad f(x) = (\operatorname{dist}(x, \overline{C_{red}}), \operatorname{dist}(x, \overline{C_{blue}})),$$

where dist $(\cdot)$ stands for the distance between a point and a set in $\mathbb{R}^3$. By the Borsuk–Ulam theorem there exists $x^* \in S^2(r)$ such that $f(x^*) = f(-x^*)$. We have to deal with three cases.

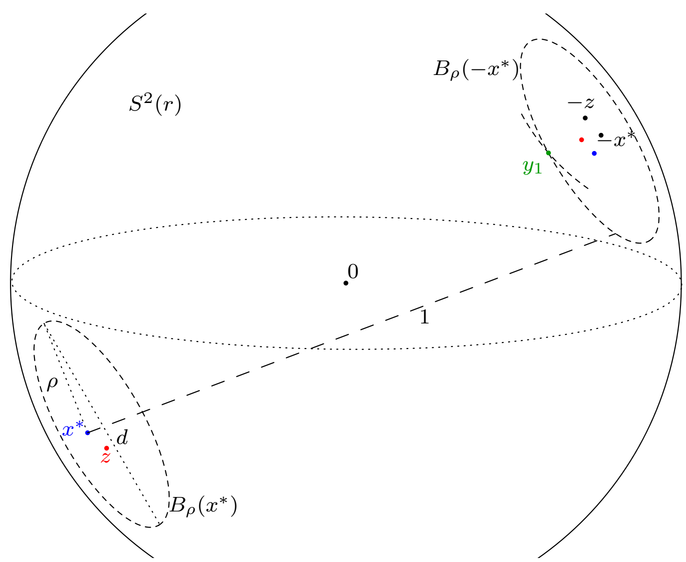


Figure 6.3: Case 1

Case 1: $f(x^*) = (0,0)$**.** Without loss of generality, the point x^* is blue. One may pick a red point z, which is arbitrarily close to x^*. If $\|x^* - z\| < \rho$, then the intersection of circles of unit Euclidean radius with centers x^* and z consists of two green points y_1, y_2 belonging to the circle of radius ρ centered at $-x^*$. Hence, one can cover a small neighborhood of $-x^*$ and y_1 by a ball of diameter d. Every neighborhood of $-x^*$ contains red and blue points; point y_1 is green (see Fig. 6.3). We have a contradiction with assumption $(\star)$.

Case 2: $f(x^*) = (a, b)$, $a, b > 0$. Then both points $x^*, -x^*$ are green. We may swap blue and green colors to reduce the situation to the next case with the same x^*.

Case 3: $f(x^*) = (a, 0)$, $a > 0$. We claim that $a > \rho$. Assume the contrary, i.e. $x^* \in \overline{C_{blue}}$ and for every $\eta > 0$ there is a red point $z = z_\eta$ such that $\|x^* - z\| \le \rho + \eta$. Note that if x^* is green, then it contradicts ($\star$), so x^* is blue. There are distinct points $y_1, y_2 \in \overline{B_\rho(-x^*)}$ such that $\|x^* - y_1\| = \|x^* - y_2\| = \|z - y_1\| = \|z - y_2\| = 1$. Since x^* is blue and z is red $y_1, y_2 \in \overline{C_{green}}$. Recall that $f(-x^*) = f(x^*)$, so there is a point $z' \in \overline{C_{red}} \cap \overline{B_\rho(-x^*)}$. Let $y' \in \{y_1, y_2\}$ be such that z', $-x^*$ and y' do not lie on a great circle of $S^2(r)$. Then for a small enough η the neighborhoods of $-x^*$, y' and z' can be covered by a ball of diameter d. This is a contradiction with ($\star$).

So the set $\overline{B_\rho(x^*)} \cup \overline{B_\rho(-x^*)}$ is colored with blue and green.

Lemma 6.2.3. *The bipartite subgraph of $S^2(r)$ with parts $\overline{B_\rho(x^*)}$ and $\overline{B_\rho(-x^*)}$ is connected.*

Proof. Any point $x \in \overline{B_\rho(x^*)}$ has a common neighbor with x^* since the corresponding unit circles intersect. So $\overline{B_\rho(x^*)}$ belong to the same connected component; the same holds for $\overline{B_\rho(-x^*)}$. There is an edge between $\overline{B_\rho(x^*)}$ and $\overline{B_\rho(-x^*)}$, and so the subgraph is connected. $\square$

By Lemma 6.2.3, one can color $\overline{B_\rho(x^*)} \cup \overline{B_\rho(-x^*)}$ in two colors in the unique way (up to symmetry): the first part is blue and the second one is green. Then the distance from x^* and $-x^*$ to $\overline{C_{blue}}$ is zero and nonzero simultaneously.

This contradiction implies that each color is dense in the sphere.

6.2.2 Step 2. Stability of embedding

In this section we will need the implicit function theorem [78] in the following weakened formulation.

Theorem 6.2.1. *Let $F: \mathbb{R}^{2s} \to \mathbb{R}^s$ be a continuously differentiable function,*

$$F = F(X, Y) = F(x_1, \dots, x_s;\ y_1, \dots, y_s),$$

and at some point $X = a$, $Y = b$ the following conditions are satisfied

$$F(a, b) = 0, \qquad \det\left(\frac{\partial F(X, Y)}{\partial Y}\right)_{X=a, Y=b} \ne 0.$$

Then there exists $\eta > 0$ such that the system of equations $F(X, Y) = 0$ is solvable in Y for any X satisfying the condition $\|X - a\| < \eta$.

Recall that G_k is an odd cycle of length $m = 2k + 1$ with an extra pendant (leaf) vertex attached to each vertex of the cycle. In particular, G_k has $2m$ vertices and $2m$ edges.

Denote by $y_1, \dots, y_m$ the points of $S^2(r)$ that correspond to the cycle vertices and by $x_1, \dots, x_m$ the points of $S^2(r)$ that correspond to the pendant vertices. For convenience, let us put $X = (x_1, \dots, x_m)$ and $Y = (y_1, \dots, y_m)$ the vectors of dimension $s = 3m$ containing all coordinates. Then the embedding of G_k can be given by the pair (X, Y).

Lemma 6.2.4. *Fix the radius $r \in \left(\frac{1}{2}, \frac{\sqrt{3}}{3}\right)$. Then if k is large enough, there exists a unit distance embedding (X, Y) of G_k into $S^2(r)$ and a constant $\eta > 0$ such that for any $\tilde{X}$ satisfying $\|\tilde{X} - X\| < \eta$ there exists Y such that $(\tilde{X}, \tilde{Y})$ is a "perturbed" unit distance embedding of G_k.*

In other words, for any sufficiently small perturbation of pendant vertices, it is possible to find the embedding of the cycle vertices.

Proof. We provide the desired unit distance embedding explicitly. In what follows we slightly abuse the notation and write x_i and y_i for a vertex of the graph, the corresponding point on $S^2(r)$, and its 3-dimensional vector representation. Consider the system of equations defining the embedding G_k in $S^2(r)$:

$$\begin{cases} f_i = \|y_i\|^2 - r^2 = 0, \quad 1 \le i \le m; \\ f_{i+m} = \|y_i - y_{i+1}\|^2 - 1 = 0, \quad 1 \le i \le m-1; \\ f_{2m} = \|y_m - y_1\|^2 - 1 = 0; \\ f_{i+2m} = \|x_i - y_i\|^2 - 1 = 0, \quad 1 \le i \le m. \end{cases} \tag{6.1}$$

Next, we will be interested in the family of embeddings, the $k = 2$ case of which is depicted on Fig. 6.4.

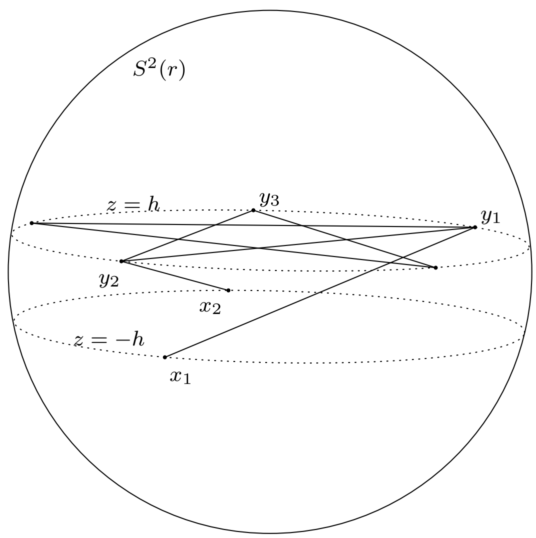


Figure 6.4: Unit distance embedding of G_k, the $k = 2$ case

Note that (6.1) allows x_i to lie in $\mathbb{R}^3$, not only $S^2(r)$, but the cycle $y_1, \ldots, y_m$ must lie on the sphere.

One can consider the function corresponding to the left-hand side of the system (6.1).

$$F = (f_1, \ldots, f_{3m}) = F(x_{11}, x_{12}, x_{13}, \ldots, x_{m3};\ y_{11}, \ldots, y_{m3}).$$

Suppose that the Jacobian matrix $J = \left(\frac{\partial F}{\partial Y}\right)$ is nondegenerate,

$$\det J = \det \left(\frac{\partial F}{\partial Y}\right) \neq 0,$$

then the statement of the lemma follows from Theorem 6.2.1. The rest of the proof is devoted to the calculation of this determinant.

The matrix J has the following form (recall that x_i and y_i are 1×3 vectors):

$$J(X,Y) = 2 \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & y_3 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_m \\ y_1 - y_2 & y_2 - y_1 & \dots & 0 & \dots & 0 \\ 0 & y_2 - y_3 & y_3 - y_2 & 0 & \dots & 0 \\ \vdots & & & & & \\ y_1 - y_m & 0 & \dots & \dots & 0 & y_m - y_1 \\ y_1 - x_1 & 0 & \dots & \dots & 0 & 0 \\ 0 & y_2 - x_2 & 0 & \dots & 0 & 0 \\ \vdots & & \vdots & & \vdots & \\ 0 & \dots & 0 & \dots & 0 & y_m - x_m \end{pmatrix}.$$

Subtracting some rows from each other, we get

$$\det J = 2^{3m} \det \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & y_3 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_m \\ y_2 & y_1 & \dots & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ y_m & \dots & 0 & \dots & 0 & y_1 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & x_m \end{pmatrix} = (-1)^s 2^{3m} \det \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_{m-1} \\ 0 & 0 & 0 & 0 & \dots & y_m \\ 0 & 0 & 0 & 0 & \dots & x_m \\ y_m & 0 & 0 & \dots & 0 & y_1 \end{pmatrix},$$

where the term $(-1)^s$, $s \in \{0,1\}$ is responsible for the parity of the permutation of the rows. Since we are not interested in the sign of the determinant, there is no point in evaluating the parity.

Then, expanding the determinant by the last row and rearranging the rows, we obtain $\det J =$

$$= (-1)^s 2^{3m} \det \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_{m-1} \\ 0 & 0 & 0 & 0 & \dots & y_m \\ 0 & 0 & 0 & 0 & \dots & x_m \\ 0 & 0 & 0 & \dots & 0 & y_1 \end{pmatrix} + (-1)^s 2^{3m} \det \begin{pmatrix} y_m & 0 & 0 & \dots & 0 & 0 \\ y_1 & 0 & 0 & 0 & \dots & 0 \\ x_1 & 0 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & \dots & 0 \\ 0 & y_2 & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & 0 & \dots & 0 \\ 0 & y_3 & y_2 & 0 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \dots & y_{m-1} \\ 0 & 0 & 0 & 0 & \dots & y_m \\ 0 & 0 & 0 & 0 & \dots & x_m \end{pmatrix} =$$

$$= (-1)^s 2^{3m} \left(V_1 \ldots V_m + V_1' \ldots V_m' \right),$$

where

$$V_i = \det \begin{pmatrix} y_i \\ x_i \\ y_{i+1} \end{pmatrix} = -\det \begin{pmatrix} y_i \\ y_{i+1} \\ x_i \end{pmatrix}, \qquad V_i' = \det \begin{pmatrix} y_i \\ y_{i+1} \\ x_{i+1} \end{pmatrix}.$$

Here in the determinant calculations we used the fact that after decomposing the last row into a sum, each of the summands becomes block-triangular. In addition, note that the cyclic permutation of matrix rows does not change the sign of the determinant, since m is odd.

Now we fix the following embedding (Fig. 6.4). Let vertices y_i lie in the plane $z = h$ (and form a regular m-gon), and vertices x_i lie in the plane $z = -h$ (and also form a regular m-gon). Note that the radius of the circumcircle of the m-gon is greater than $\frac{1}{2}$, and $r^2 < \frac{1}{3}$, hence

$$h < \left(\frac{1}{3} - \frac{1}{4} \right)^{1/2} = \frac{1}{2\sqrt{3}} < \frac{1}{2}. \tag{6.2}$$

Denote by U_m the rotation matrix by an angle $2\pi/m$ counterclockwise around z–axis. Then $y_{i+1} = U_m y_i$, $x_{i+1} = U_m x_i$. Hence, all V_i coincide and all V_i' also coincide; put $V = V_i$ and $V' = V_i'$. Hence

$$\det J = (-1)^s 2^{3m} \left(V^m + (V')^m \right).$$

We claim that

$$V + V' = \det \begin{pmatrix} y_1 \\ y_2 \\ x_2 - x_1 \end{pmatrix} \neq 0.$$

Indeed, since $y_{13} = y_{23} = h$, $x_{13} = x_{23} = -h$, the equality

$$\alpha y_1 + \beta y_2 + \gamma (x_2 - x_1) = 0$$

implies $\alpha = -\beta$, i.e.

$$\alpha (y_1 - y_2) = \gamma (x_1 - x_2). \tag{6.3}$$

Recall that $\|y_1 - y_2\| = \|x_1 - x_2\| = 1$, so $\alpha = \pm\gamma$.

Since both sets of points $\mathcal{X} = \{x_1, \ldots, x_m\}$, $\mathcal{Y} = \{y_1, \ldots, y_m\}$ form vertices of congruent regular m-gons, in the case $\alpha = \gamma$, we have $x_1 - x_2 = y_1 - y_2$ and the projections of x_i and y_i on the plane $z = 0$ coincide, $i = 1, 2, \ldots, m$, and taking into account (6.2), we have

$$\|x_1 - y_1\| = 2h < 1.$$

In the case $\alpha = -\gamma$, we have $x_1 - x_2 = y_2 - y_1$ and the sets $\mathcal{X}$ and $\mathcal{Y}$ are symmetric about the origin. Then $x_1 x_2 y_1 y_2$ is a rectangle, and

$$\|x_1 - y_1\|^2 > \|x_1 - x_2\|^2 + 4h^2 > 1.$$

In both cases we got a contradiction.

Then the equation (6.3) does not hold and so $V + V' \neq 0$. Hence

$$\det J = (-1)^s 2^{3m} \left(V^m + (V')^m \right) \neq 0$$

as required. □

6.3 Open questions

Is the chromatic number of $S^2(r)$ "almost monotonically" increasing with r? Id est, is the chromatic number monotonic except for an at most countable set of values r? Recall that the known results (see Table 1) allow for such possibility.

r	Estimate for $\chi(r) = \chi(S^2(r))$	Source
$r < 1/2$	$\chi(r) = 1$	
$r = 1/2$	$\chi(r) = 2$	
$\frac{1}{2} < r \le \frac{\sqrt{3-\sqrt{3}}}{2}$	$\chi(r) = 4$	Corollary 1
$r > \frac{\sqrt{3-\sqrt{3}}}{2}$	$\chi(r) \ge 4$	Theorem 2
$r = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$	$\chi(r) \ge 5$	[122]
$r = \frac{1}{\sqrt{2}}$	$\chi(r) = 4$	[116, 57]
$r = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$	$\chi(r) \ge 5$	[122]
$r \le \frac{1}{\sqrt{3}}$	$\chi(r) \le 5$	[116, 90]
$r \le \sqrt{3}/2$	$\chi(r) \le 6$	[90]
$r \ge 12.44$	$\chi(r) \le 7$	[117]
$r > 1/2$	$\chi(r) \le 15$	[28, 106]

Table 6.1: Lower and upper estimates for $\chi(S^2(r))$.

Is there a proper coloring of $S^2(r)$ in $\chi(S^2(r))$ colors such that every color is dense? It is interesting that all known upper bounds are given by explicit colorings in which every color is a finite union of regions bounded by piecewise-continuous curves.

What is the minimum number of vertices in a subgraph G of a sphere $S^2(r)$ with $\chi(G) = \chi(S^2(r))$? By the de Bruijn–Erdős theorem, this number is finite. Note that the proof of Theorem 2 does not give any finite 4-chromatic unit distance graph.

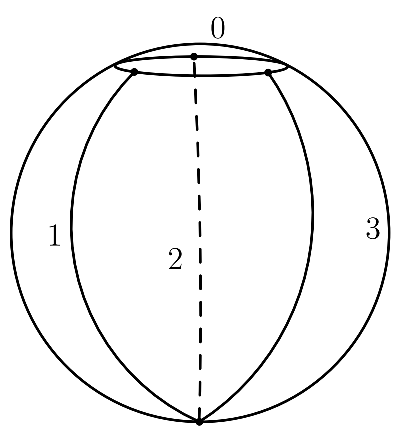


Figure 6.5: 4-coloring of the sphere. Here $s_0 \to 0$ as $r \to 1/2$

Let us focus on the case $r = 1/2 + \varepsilon$, $\varepsilon \to 0$. Then the sphere can be colored in 4 colors in the way shown in Figure 6.5. Let us denote by s_0 the area of the spherical cap of color 0. Observe that

$s_0 = 4\pi\varepsilon + o(\varepsilon)$, and thus, via averaging, we have the lower bound $n_4(r) \geq c\varepsilon^{-1}$ for some $c > 0$, where $n_4(r)$ is the minimum number of vertices in a 4-chromatic unit distance graph. Can this obvious bound be refined?

Chapter 7

Chromatic numbers of 3-dimensional slices

We follow the paper [123] and prove that for an arbitrary $\varepsilon > 0$ holds

$$\chi(\mathbb{R}^3 \times [0, \varepsilon]^6) \geq 10,$$

where $\chi(M)$ stands for the chromatic number of an (infinite) graph with the vertex set M and the edge set consists of pairs of points at the distance 1 apart.

7.1 Introduction

We study colorings of a set $\text{Slice}(d, k, \varepsilon) = \mathbb{R}^d \times [0, \varepsilon]^k$ in a finite number of colors with the forbidden distance 1 between monochromatic points; further such sets are called *slices*. Slightly abusing the notation we say that d is the *dimension* of a slice.

Define graph $G(d, k, \varepsilon)$, which vertices are the point of $\text{Slice}(d, k, \varepsilon)$ and edges connect points at the Euclidean distance 1 apart. Put

$$\chi[\text{Slice}(d, k, \varepsilon)] := \chi[G(d, k, \varepsilon)],$$

where $\chi(H)$ is the chromatic number of a graph H. Obviously for every positive ε one has

$$\chi(\mathbb{R}^d) \leq \chi[\text{Slice}(d, k, \varepsilon)] \leq \chi(\mathbb{R}^{d+k}).$$

Since $\chi(\mathbb{R}^d) = (3 + o(1))^d$ (see [82]), the chromatic number of a slice is finite. So by the de Bruijn–Erdős theorem it is achieved on a finite subgraph.

7.1.1 Nelson–Hadwiger problem and its planar generalizations

In this notation the classical Nelson–Hadwiger problem is to determine $\chi[\text{Slice}(2, 0, 0)]$, but as usual we write $\chi(\mathbb{R}^2)$ for this quantity. The best known bounds up to the date are

$$5 \leq \chi(\mathbb{R}^2) \leq 7.$$

The upper bound is a classical coloring of a regular hexagon tiling due to Isbell. The lower bound were obtained by de Grey [31] in 2018, breaking a 70 year-old record (another constructions are contained in [60, 40, 122, 102, 103]).

The study of slice colorings started at [69] with the following main result.

Theorem 7.1.1. *For every positive* ε *holds*

$$6 \leq \chi[\text{Slice}(2, 2, \varepsilon)].$$

Theorem 7.1.1 is a strengthening of the result $\chi_\varepsilon(\mathbb{R}^2) \geq 6$ (Currie–Eggleton [30]), where χ_ε stands for the minimal number of colors, for which there is a coloring of plane without a pair of monochromatic points at the distance in the range $[1, 1+\varepsilon]$. Exoo [39] conjectured that for every $\varepsilon > 0$ holds $\chi_\varepsilon(\mathbb{R}^2) \geq 7$. Recently Voronov [124] proved this conjecture.

On the other hand Isbell' coloring implies inequality

$$\chi_\varepsilon(\mathbb{R}^2) \leq 7$$

for $\varepsilon < 0.13\ldots$. As a corollary, for every k there is $\varepsilon_k > 0$ such that for every positive $\varepsilon < \varepsilon_k$ holds

$$\chi[\text{Slice}(2, k, \varepsilon)] \leq 7.$$

7.1.2 The chromatic numbers of real 3-dimensional slices

First recall the best bounds on $\chi(\mathbb{R}^3)$. The best known lower bound $\chi(\mathbb{R}^3) \geq 6$ is due to Nechushtan [95]. The upper bound $\chi(\mathbb{R}^3) \leq 15$ is obtained independently by Coulson [28] and by Radoičić and Tóth [106].

The main result of this section is the following theorem.

Theorem 7.1.2 (Cherkashin–Kanel-Belov–Strukov–Voronov [123])**.** *There is* $\varepsilon_0 > 0$*, such that for every positive* $\varepsilon < \varepsilon_0$ *holds*

$$10 \leq \chi[\text{Slice}(3, 6, \varepsilon)] \leq 15.$$

The upper bound immediately follows from the coloring of $\chi(\mathbb{R}^3)$ from [28, 106], similarly to 2-dimensional case. The lower bound requires somewhat more complicated arguments than in two dimensions.

The following theorem is a quantitative strengthening of Theorem 10 from [69] and is of an independent interest.

Theorem 7.1.3 (Cherkashin–Kanel-Belov–Strukov–Voronov [23])**.** *Let* $T \subset \mathbb{R}^d$ *be a regular simplex with the edge length* $a = \sqrt{2d(d+1)}$*. Then every proper coloring of* $\mathbb{R}^d$ *in a finite number of colors contains a point from* T *belonging to the closures of at least* $d+1$ *colors.*

Corollary 7.1.1. *For every positive* ε' *holds*

$$\chi_{\varepsilon'}(\mathbb{R}^3) \geq 10.$$

Indeed, the orthogonal projection of a unit distance graph from the proof of Theorem 7.1.2 for a fixed ε has distances between adjacent vertices in the range $[\sqrt{1-6\varepsilon^2}, 1]$.

7.1.3 The chromatic numbers of 2-dimensional rational slices

Denote by $[0, \varepsilon]_\mathbb{Q}$ the set of rational numbers from $[0, \varepsilon]$. In paper [69] it is shown that

$$\chi(\mathbb{Q} \times [0, \varepsilon]^3_\mathbb{Q}) = 3.$$

Benda and Perles [7] show that $\chi(\mathbb{Q}^4) = 4$. Thus the chromatic number of $\mathbb{Q}^2 \times [0, \varepsilon]^2_\mathbb{Q}$ is at most 4.

Proposition 7.1.1. *For every* $\varepsilon > 0$ *holds*

$$\chi(\mathbb{Q}^2 \times [0, \varepsilon]^2_\mathbb{Q}) = 4.$$

7.2 Notation and auxiliary lemmas

Here and after we focus on the following Slice$(3, 6, \varepsilon) \subset \mathbb{R}^9$. By $S^d_r(x)$ we denote a d-dimensional sphere of the radius r and centered at x.

Definition 7.2.1. *An attached sphere of a simplex with vertices $\{v_i\}_{1\le i\le k}$, $3 \le k \le 4$ is a set of points at the distance 1 from each v_i:*

$$S(v_1, \dots, v_k; 1) := \bigcap_i S(v_i; 1) \subset \mathbb{R}^9.$$

Note that if the radius r of a circumscribed $(k-2)$-dimensional sphere $v_1, \dots, v_k$ is smaller than 1, then $S(v_1, \dots, v_k; 1)$ is a $(9-k)$-dimensional sphere with the radius $\sqrt{1-r^2}$.

Definition 7.2.2. *A t-equator of a sphere S is a subsphere of the dimension t which radius is equal to the radius of S.*

As usual, $\overline{T}$ stands for the closure of a set T.

Definition 7.2.3. *Let a metric X be colored in a finite number of colors; denote these colors by $C_1, \dots, C_m$. A chromaticity of a point $x \in X$ is the number of sets $\overline{C_i}$, $1 \le i \le m$ containing x.*

Lemma 7.2.1 (Knaster–Kuratowski–Mazurkiewicz)**.** *Suppose that $(d-1)$-dimensional simplex is covered by closed sets $X_1, \dots, X_d$ in such a way that every face $I \subset [d]$ is contained in the union of X_i over $i \in I$. Then all sets X_i have a common point.*

The following lemma is a spherical analogue of the planar lemma from [69]. The proof is also analogous; we provide it in the interest of completeness.

Lemma 7.2.2. *Let S^2_r be a sphere of radius $r > \sqrt{\frac{1}{2}}$, ε be a positive number, and $Q \subset S^2_r$ be a ε-neighbourhood of a curve $\xi \subset S^2_r$, such that*

$$\operatorname{diam} \xi > \frac{\sqrt{4r^2-1}}{r}.$$

Then $\chi(Q) \ge 3$.

Proof. Without loss of generality $\varepsilon < 1$. Denote by $G(Q)$ the corresponding graph; we are going to find an odd cycle in $G(Q)$.

Consider a point $u \in \xi$. Since $\operatorname{diam} \xi > \frac{\sqrt{4r^2-1}}{r} = \operatorname{diam} S(u; 1)$, the intersection of $S(u; 1)$ and ξ is non-empty. Let $v \in S(u; 1) \cap \xi$; consider such points v_1, v_2, v_3, $v_4 \in S^2_r$ that $\|u - v_1\| = 1$; $\|v_i - v_{i+1}\| = 1$; $i = 1, 2, 3$. If the angles at the vertices of polygonal chain $vuv_1v_2v_3v_4$ are at most $\frac{\varepsilon}{2}$, then $\|v - v_1\| < \frac{\varepsilon}{2}$, $\|u - v_2\| < \frac{\varepsilon}{2}$, $\|v - v_3\| < \varepsilon$, $\|u - v_4\| < \varepsilon$, and hence $v_i \in Q$, $i = 1, 2, 3, 4$. Note that

$$l_1 = \|u - v_2\| \in \left[0; 2\sin\frac{\varepsilon}{4}\right],$$

$$l_2 = \|v_2 - v_4\| \in \left[0; 2\sin\frac{\varepsilon}{4}\right]$$

can be chosen arbitrarily, and the oriented angle between vectors $\overrightarrow{v_2u}$ and $\overrightarrow{v_2v_4}$ can be independently chosen from $[-\frac{\varepsilon}{4}; \frac{\varepsilon}{4}]$. Fix the line containing vector $\overrightarrow{v_2u}$; one may choose it orthogonal to uv. Then a set of all possible v_4 contains a rhombus centered at u with the side length $2\sin\frac{\varepsilon}{4}$ and the angle $\frac{\varepsilon}{2}$. Then $G(Q)$ contains a path of length 4 between u and an arbitrary point from a γ-neighbourhood of u, where $\gamma = \sin\frac{\varepsilon}{2}\sin\frac{\varepsilon}{4}$.

Thus one may move from u to v along ξ by steps of size at most γ. Every such step corresponds to a path of length 4 in $G(Q)$; since v is adjacent to u we find a desired odd cycle in $G(Q)$. □

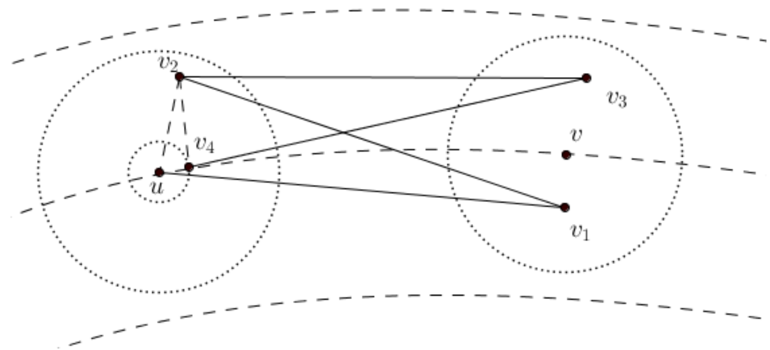


Figure 7.1: A path of length four between u and v_4.

Lemma 7.2.3. *Suppose that a sphere $S_r^2 \subset \mathbb{R}^3$, $r > \sqrt{\frac{1}{2}}$ has a proper coloring a finite number of colors. Then it has a point with the chromaticity at least 3.*

Proof. Note that for $r > \sqrt{\frac{1}{2}}$

$$\frac{\sqrt{4r^2-1}}{r} < 2r.$$

By compactness of the sphere it is sufficient to show that there is a spherical ball with an arbitrarily small radius containing points of at least 3 colors. Suppose the contrary: there is a proper coloring of the sphere and $\varepsilon > 0$ such that every spherical ball with the radius ε is colored in at most 2 colors. Consider a partition of the sphere onto cells such that every cell contains a ball with the radius $\delta = \varepsilon_0/100$ and is contained in a ball of the radius $\varepsilon_0/10$. Then every cell contains points of at most two colors, moreover all the adjacent cells are colored in the same two colors.

Consider an arbitrary cell with two colors (say, colors 1 and 2). Let A_0 be the region which is maximal by inclusion, that contains cells with one- or two-colored cells of colors 1 and 2. The diameter of A_0 is smaller than $\frac{\sqrt{4r^2-1}}{r} < 2r$ otherwise, by looking at any path between diametrally opposed points, we have a contradiction with Lemma 7.2.2. Let us consider the outer boundary p of the reigon A_0. Every cell adjacent to p is adjacent to some cell not in A_0, hence it is monochromatic; moreover, colors of all cells from A_0 along p are the same, otherwise there would be a ball that contains cells of two different colors and cell not from A_0, which contradicts our assumption. So we may assume that all cells from A_0 along p are of color 1. Same argument shows that cells not from A_0 along p cannot contain two different colors that are not 1 or 2, and cannot contain the color 2. Therefore all cells adjacent to p are colored in colors 1 or 3 (maybe both). Consider the region A_1, that contains cells along p of colors 1 and 3, and is maximal by inclusion. We can apply to A_1 the smae argument, and by induction we obtain the sequence of 2-colored regions A_i. Note that (spherical) diameter of A_i is increasing by at least δ each step, so eventually we obtain the contradiction to Lemma 7.2.2. $\square$

The proof of the main result require technical statement on stability of circumscribed circle of a triangle with vertices of a form $(0, 0, 0, b_1, \dots, b_6)$ with respect to a shifts by vectors from the main subspace $\mathbb{R}^3$, i.e. vectors of type $(p, q, r, 0, \dots, 0)$. Such shifts will be called *orthogonal*. The next lemma will be applied for the case of S^5, but we prove it in the general case.

Lemma 7.2.4. *Suppose that several points are chosen on S^k so that minimal distance between two chosen points is $\Omega(m^{-2})$. Then there is the triangle $\mathcal{T}$ which vertices are amongst the chosen points satisfying the following condition. Every orthogonal shift of it vertices by $O(m^{-3})$ causes change of radius R of circumscribed circle of $\mathcal{T}$ by $O\big(\frac{R}{m^2}\big)$ and shift of its center by $O(\frac{R}{m})$.*

Proof. Let us find a triangle $\mathcal{T}_0$ from selected points with heights $\Omega(m^{-2})$. Assume the contrary, id est that there is no such triangle. Let us consider the maximum distance between these points; say, it is achieved between points A and B. Then all other points should lie in the $o(m^{-2})$-neighborhood of the great circle AB (any great circle AB, if the points A and B were diametrically opposite): indeed, otherwise the height from point C to AB is equal to $\Omega(m^{-2})$ and it is the smallest of the heights of triangle ABC, since points A and B were chosen at the maximum distance and ABC is suitable for the role of $\mathcal{T}_0$.

Let us consider the projections of the selected points onto the great circle AB (they are uniquely determined). Since the pairwise distances between the selected points are equal to $\Omega(m^{-2})$, and the points lie in the $o(m^{-2})$-neighborhood of the great circle AB, the projections are separated from each other by at least by $\Omega(m^{-2})$. One of the arcs AB contains the projection of at least $m_1 \geq m/2$ points. Let us number the points according to the projection on this arc AB; let C be the point with the number $[m_1/2]$. Then AC and BC are equal to $\Omega(1/m)$. Let O be the circumcenter of triangle ABC. Let us denote the lengths of the sides AB, BC, AC by c, a, b, respectively; let the lengths of the heights be equal to h_a, h_b, h_c.

It is clear that $\angle ACB$ is the largest of the angles of triangle ABC and

$$\angle ACB \leq \angle OCA + \angle OCB = \arccos\frac{b}{2R} + \arccos\frac{a}{2R} \leq 2\arccos\frac{\Omega(1/m)}{2R},$$

since the triangles ACO and BCO are isosceles. Then

$$2\arccos\frac{\Omega(1/m)}{2R} = \pi - \frac{\Omega(1/m)}{R}.$$

Therefore, $\sin\angle ACB = \sin(\pi - \angle ACB) = \Omega(1/m)$. Since AB is the longest side, the height from point C is the smallest. Then, since the sine of at least one of the angles A and B is also equal to $\Omega(1/m)$, the height from point C is equal to $\Omega(1/m^2)$.

The triangle $\mathcal{T}_0 = ABC$ has been found; let us show that it is suitable as $\mathcal{T}$. Let us keep the notation for the parameters of the triangle $\mathcal{T}_0$ introduced above. Let the shifted points be A', B' and C'. Let us denote by Δq the change in the value of q during the transition from ABC to $A'B'C'$.

Let us show that an orthogonal shift of the ends of the segment y_1y_2 by $O(m^{-3})$ changes (increases) the length of the segment l by $O(m^{-6}l^{-1})$. Let us denote the shifted points z_1, z_2, respectively. Due to orthogonality, $(y_i - y_j, z_j - y_j) = 0$. The square of the new length is

$$(z_1 - z_2, z_1 - z_2) = \|(z_1 - y_1) + (y_1 - y_2) + (y_2 - z_2)\|^2 =$$

$$= (z_1 - y_1, z_1 - y_1) + (y_1 - y_2, y_1 - y_2) + (z_2 - y_2, z_2 - y_2) - 2(z_1 - y_1, z_2 - y_2);$$

That is, the difference in the squares of the lengths is estimated as

$$(z_1 - z_2, z_1 - z_2) - (y_1 - y_2, y_1 - y_2) = O(m^{-6}).$$

It remains to apply the difference of squares formula.

It turns out that $\Delta a = O(m^{-6}a^{-1})$, similarly for other sides. Let H be the base of the height CH, since AB is the greatest, H belongs to the segment AB. Note that the length of the height h_c cannot decrease, and on the other hand, the distance from the shifted vertex C to the point that is projected into H changes by no more than $O(m^{-6}h_c^{-1})$, and the length of the new height h_c' does not exceed this distance. Let S be the area of triangle ABC, then

$$\Delta S = O(c\Delta h_C + \Delta c \cdot h_C) = O(h_c m^{-6}c^{-1}) + O(cm^{-6}h_c^{-1}),$$

hence,

$$\frac{\Delta S}{S} = O(m^{-6}c^{-2}) + O(m^{-6}h_c^{-2}) = O(m^{-2}).$$

Using the well-known formula

$$R = \frac{abc}{4S},$$

we get

$$\Delta R = O\left(\max\left(\frac{\Delta a \cdot bc}{S}, \frac{\Delta b \cdot ac}{S}, \frac{\Delta c \cdot ab}{S}, \frac{abc\Delta S}{S^2}\right)\right) =$$
$$O\left(\max\left(\frac{\Delta a}{a}R, \frac{\Delta b}{b}R, \frac{\Delta c}{c}R, \frac{\Delta S}{S}R\right)\right) = O\left(\frac{R}{m^2}\right),$$

which is what was required.

Let us limit the shift of the center of the circumscribed circle when changing along one coordinate. We showed above that the heights and sides of a triangle change slightly when the vertices are orthogonally shifted by $O(m^{-3})$, which means it will be possible to repeat the following estimate several times.

Let us consider three-dimensional Cartesian coordinates in which C is the center, the plane ABC is generated by the first two coordinates, and the latter corresponds to the infinitesimal shift. Then the normal to the plane ABC has the form

$$\bar{v}_1 = \bar{AC} \times \bar{BC} = (0, 0, 2S).$$

Then the normal to the plane $A'B'C'$ is equal to

$$\bar{v}_2 = \bar{A'C'} \times \bar{B'C'} = (A_yB'_z - A'_zB_y, A_xB'_z - A'_zB_x, 2S).$$

Without loss of generality, $a \geq b$ and then $|A_x|, |B_x|, |A_y|, |B_y| \leq a$. Therefore $|A_yB'_z - A'_zB_y|, |A_xB'_z - A'_zB_x| = O(am^{-3})$. Recall that $S = 0.5ah_a = \Omega(am^{-2})$, which implies $|A_yB'_z - A'_zB_y|, |A_xB'_z - A'_zB_x| = O(Sm^{-1})$. Let us estimate the angle ϕ between the planes ACB and $A'B'C'$:

$$\cos\phi = \frac{(v_1, v_2)}{\sqrt{(v_1, v_1) \cdot (v_2, v_2)}} = \frac{4S^2}{2S \cdot \sqrt{4S^2 + O(S^2m^{-2})}} = 1 - O(m^{-2}), \qquad \phi = O(m^{-1}).$$

Consequently, the change in the center does not exceed $O(R \sin\phi) = O(R/m)$.

$\square$

7.3 Proof of Theorem 7.1.3

Suppose the contrary. Let C_i — be a set of points $\mathbb{R}^d$ of color i, $1 \leq i \leq m$. Define

$$C_i^* := \overline{\operatorname{Int} \overline{C_i}} \qquad \text{(the closure of the interior of the closure).}$$

Split every C_i^* into connected components (with respect to the standard topology):

$$C_i^* = \bigcup_{\alpha \in A_i} D_\alpha.$$

Put also $\{D_\alpha\} = \bigcup_{i=1}^{m} \bigcup_{\alpha \in A_i} D_\alpha$.

(i) *Sets C_i^* cover $\mathbb{R}^d$.* Suppose the contrary, i.e. $\exists v : \forall i \;\; v \notin C_i^*$. Then there is an open ball $B(v;\eta)$ such that

$$B(v;\eta) \cap C_i^* = \emptyset; \quad B(v;\eta) \subset \bigcup C_i.$$

Consider an arbitrary ball

$$B^1 \subset B(v;\eta) \setminus \overline{C}_1.$$

Then B^1 cannot be a subset of $\overline{C}_i$, otherwise the intersection of the interior of $\overline{C}_i$ and $B(v;\eta)$ is nonempty. Define a sequence of balls

$$B^{k+1} \subset B^k \setminus \overline{C}_k.$$

Note that points of B^{m+1} do not belong to any $\overline{C}_i$, which is a contradiction.

(ii) *Suppose that a point $v \in T$ belongs to exactly k sets C_i^*. Assume that $k \le n$ (otherwise the chromaticity of v is at least $d+1$). Then for every $\mu_0 > 0$ there is $\mu < \mu_0$ such that the sphere $S(v;1-\mu)$ does not intersect at least one of those k sets.*

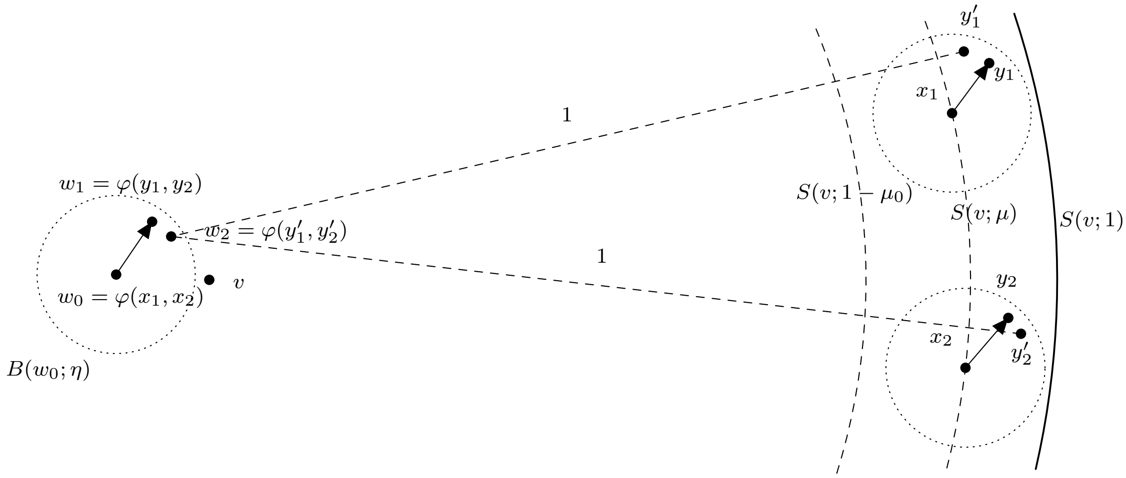


Figure 7.2: Illustration to item (ii).

We can assume without loss of generality that $v \in C_i^*$, $1 \le i \le k$. Suppose the contrary, i.e. there is such a $\mu_0 > 0$ that for every $\mu \in (0,\mu_0)$ holds

$$S(v;1-\mu) \cap C_i^* \neq \emptyset, \quad 1 \le i \le k.$$

By the definition of C_i^* any neighbourhood of an arbitrary point $x \in C_i^*$ contains a point from $\operatorname{Int}\overline{C}_i$. Hence, the set

$$\mathcal{M}_0 := \{\mu \in (0,\mu_0) \mid \exists S(v;1-\mu) \cap \operatorname{Int}\overline{C}_i = \emptyset\}$$

is closed and nowhere dense.

Fix some $\mu \in (0,\mu_0) \setminus \mathcal{M}_0$. One may choose points $x_1, ..., x_k$ in such a way that

$$x_i \in S(v;1-\mu) \cap \operatorname{Int}\overline{C}_i, \qquad 1 \le i \le k;$$

and $\{v, x_1, ..., x_k\}$ are in a general position (i.e. all the simplices are non-degenerate). Consider any $\eta > 0$ such that $B(x_i;\eta) \subset C_i^*$, $1 \le i \le k$. Put

$$z \in B(0;\eta); \quad y_i = x_i + z.$$

Define

$$w_0 = \phi(x_1, \ldots, x_k) := \mathrm{Argmin}_{u \in U} \|u - v\|, \quad U = \bigcap_{1 \le i \le k} S(y_i; 1),$$

$$w_1 = \phi(y_1, \ldots, y_k).$$

By the construction the color of w_1 differs from the colors of $y_1, \ldots, y_k$.

In a small neighbourhood of $\{y_i\}$ function $w(\cdot)$ is properly defined and continuous. Choose points

$$y_i' \in C_i, \quad 1 \le i \le k,$$

for which exists $w_2 = \phi(y_1', \ldots, y_k')$. Then

$$w_2 \in \bigcup_{j=k+1}^{m} C_j.$$

At the same time the quantity

$$\delta(y_1', ..., y_k') = \max_{1 \le i \le k} \|y_i' - y_i\|$$

can be chosen arbitrarily small and hence

$$w_1 \in \bigcup_{j=k+1}^{m} \overline{C_j}.$$

Since $z \in B(0; \eta)$ was chosen arbitrarily

$$B(w; \eta) \subset \bigcup_{j=k+1}^{m} \overline{C_j}.$$

Hence an arbitrary neighbourhood of w_0 has an inner point of at least one set $\overline{C_j}$, $k+1 \le j \le m$, so $w_0 \in C_j^*$ for some j. Note that $w_0 = \phi(y_1, \ldots, y_k) \to v$ with $\mu \to 0$, thus v belongs to at least one of sets C_j^*, $k+1 \le j \le m$, which contradicts to the initial assumption.

(iii) *There is a cover of T by sets from $\{D_\alpha\}$, such that every set from the cover is contained in a closed unit ball.* By (ii) every point is covered by at least one set that satisfies the condition. Axiom of choice finishes the proof of the item. For every color i denote by $\{D_\beta^{(i)}\}$ the chosen sets.

(iv) *There is a finite cover of T by closed sets D_{ik}', $1 \le i \le m$, $1 \le k \le K_i$, such that every set from the cover is the union of some sets from $\{D_\alpha\}$ and also is contained in a closed unit ball.*

For every i, $1 \le i \le m$ consider a sequence $v_1^i, v_2^i, \dots \in \bigcup D_\beta^{(i)}$ such that

$$\gamma(v_j^i) = i;$$

$$v_{s+1}^i \in \bigcup D_\beta^{(i)} \setminus \bigcup_{1 \le j \le s} B(v_j^i; 1).$$

Let the sequence be maximal (with respect to inclusion). The pairwise distances v_j^i, $j = 1, 2, \ldots$ are at least 1, so the sequence is finite because T is bounded. Now let us define

$$D_{ik}' = B(v_k^i; 1) \cap \left(\bigcup D_\beta^{(i)} \right) \setminus \bigcup_{1 \le j \le k-1} D_{ij}'.$$

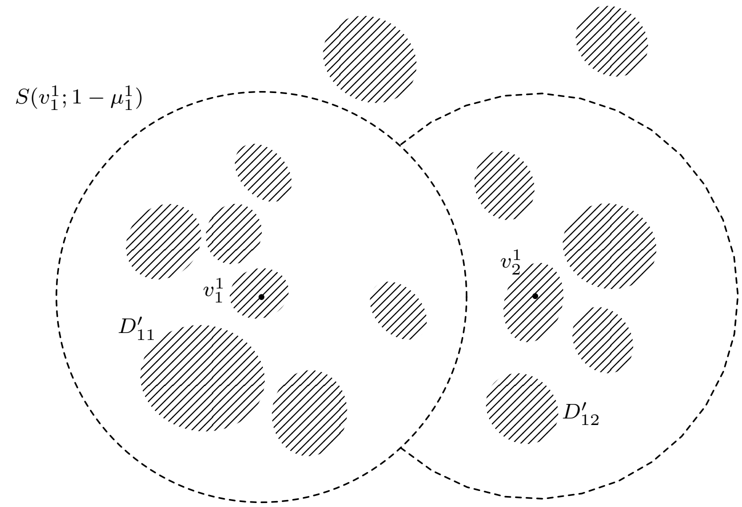


Figure 7.3: Illustration to item (iv). The construction of sets D'_{ik}

Every set D'_{ik} is separated from other connected components of C_i^* by a neighbourhood of a sphere, without points from C_i^* (see Fig. 7.3). Thus these sets are closed.

Come back to the main construction and note that every set D'_{ik} cannot intersect every face of simplex T, because it is contained in an open unit ball while the inner radius of T is equal to 1. Split the cover $\mathcal{D}' = \bigcup_i \{D'_{ik}\}$ into $d+1$ subfamilies, in the way that every set subfamily consists of sets that do not intersect a face of T. Clearly there is a bijection between subfamilies and the vertices of T. Let X_i, $i = 1, \dots, d+1$ be the unions of sets over corresponding subfamilies. By Lemma 7.2.1 sets X_i have a common point x_*, and thus an arbitrary neighbourhood of x_* intersects with at least $d+1$ sets from $\{D_\alpha\}$. They belong to at least $d+1$ different $\{C_i^*\}$, because $\{D_\alpha\}$ are connected components. Hence, x^* has the chromaticity at least $d+1$.

7.4 Proof of Theorem 7.1.2

Outline of the proof. Suppose the contrary to the statement. First, we find points v_1, v_2, v_3, v_4 of different colors, such that the intersection I of attached sphere $S(v_1, v_2, v_3, v_4; 1)$ and the slice contains 2-equator S^2 of the sphere and we also require the radius of the sphere to be close to 1.

Then I is close (in the sense of Hausdorff distance) to $S^2_{1-\eta} \times [0, \varepsilon]^3$, where η is small enough. Then one can follow the arguments from [69], that were applied in the case of 2-dimensional slices. Note that the sets of colors of I and $\{v_1, v_2, v_3, v_4\}$ are disjoint.

Let us find points $v_5, v_6, v_7 \in I$, such that an equator of the corresponding attached sphere belongs to the slice. The attached sphere of $v_1, \dots, v_7$ has an equator belonging to the slice, so the intersection of $v_1, \dots, v_7$ contains a spherical neighbourhood of a circle. It requires 3 additional colors in addition to the colors of points $v_1, \dots, v_7$.

Step 1. Finding of points v_1, v_2, v_3, v_4, which attached sphere has the radius closed to 1 and the great circle belonging to the slice.

This requires the 3-dimensional subspace U spanned by v_1, v_2, v_3, v_4, to be "almost orthogonal" to the main subspace $\mathbb{R}^3$, and the circumradius of the simplex $v_1v_2v_3v_4$ in U to be small enough.

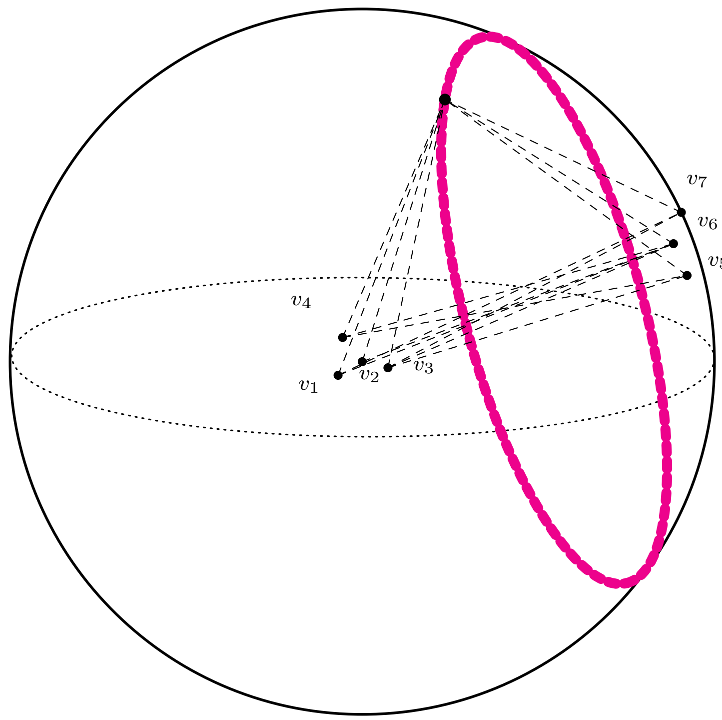


Figure 7.4: Construction of a rainbow 10-point set.

Consider the standard Cartesian coordinate system in slice $\mathbb{R}^3 \times [0, \varepsilon]^6$:

$$v = (x_1, x_2, x_3, y_1, \ldots, y_6), \quad x_i \in \mathbb{R}, \quad y_i \in [0, \varepsilon].$$

For a given point $v = (x_1, x_2, x_3, y_1, \ldots, y_6)$ define projections

$$p_R(v) = (x_1, x_2, x_3, 0, \ldots, 0) \quad \text{and} \quad p_\varepsilon(v) = (0, 0, 0, y_1, \ldots, y_6).$$

Consider sphere $S := S^5_{\varepsilon_1}$ of the radius $\varepsilon_1 < \varepsilon/2$ centered at $(0, 0, 0, \varepsilon/2, \varepsilon/2, \ldots, \varepsilon/2)$; note that $S \subset (0, 0, 0) \times [0, \varepsilon]^6$. Let $T \subset \mathbb{R}^3$ be an arbitrary regular tetrahedron with the edge length $2\sqrt{6}$ and the center at the origin and u be an arbitrary point of sphere S. By Lemma 7.1.3 every set $T \times \{u\} \subset T \times S$ has a point with chromaticity at least 4.

Fix the parameters $\delta, h > 0$, which values will be chosen later.

Consider a set of points $U = \{u_1, \ldots, u_m\} \subset S$ such that $\|u_i - u_j\| \geq \delta$, $i \neq j$ and m is maximal. Obviously $m = \Omega(\delta^{-5})$. Match every point $u_i \in U$ with an arbitrary point $u_i^* \in T \times \{u_i\}$ with chromaticity at least 4.

Consider how T is cut by a cubic mesh with edge length h:

$$T_{i,j,k} := \bigsqcup_{i,j,k} T \cap Z_{i,j,k}; \qquad Z_{i,j,k} := [ih, (i+1)h) \times [jh, (j+1)h) \times [kh, (k+1)h),$$

where i, j, k are integers. Since $T \subset \mathbb{R}^3$ is bounded, one has

$$\#\{(i, j, k) : T \cap Z_{i,j,k} \neq \emptyset\} = O(h^{-3}).$$

Consider points $w_i = p_R(u_i^*) \in T$. Put $h = \delta^{3/2}$. There is a cell $T_{a,b,c}$ such that it contains at least

$$m = \frac{\Omega(\delta^{-5})}{O(\delta^{-9/2})} = \Omega\left(\delta^{-\frac{1}{2}}\right)$$

points from $\{w_i\}$. Note that $h = O(m^{-3})$, $\delta = O(m^{-2})$ and

$$\operatorname{diam} T_{a,b,c} \leq \sqrt{3}h = \sqrt{3}\delta^{3/2} = O(m^{-3}).$$

Now apply Lemma 7.2.4 for these m points. It gives a triangle $\mathcal{T} = w_1w_2w_3$ such that its arbitrary small orthogonal shift, in particular the triangle $u_1^*u_2^*u_3^*$ has circumradius at most $(1/4 + O(m^{-2}))\varepsilon$ and its circumcircle ω belongs to the slice. Let us construct a (five-dimensional) sphere S^* on ω as the diameter and choose v_4 as the most distant point from the plane $u_1^*u_2^*u_3^*$ on the sphere S^*. Then the simplex $u_1^*u_2^*u_3^*v_4$ is a non-degenerate simplex whose circumscribed sphere belongs to the interior of the slice.

It remains to choose in arbitrarily small neighborhoods of the points u_1^*, u_2^* and u_3^* the points v_1, v_2 and v_3, respectively, in such a way that the points v_1, v_2, v_3 and v_4 have pairwise different colors.

Step 2. Finding in sphere $S(v_1, v_2, v_3, v_4; 1)$ points v_5, v_6, v_7 of different colors such that attached (2-dimensional) sphere $S(v_1, \ldots, v_7; 1)$ has a 2-equator belonging to the slice. Note that $S(v_1, \ldots, v_7, 1)$ is the intersection of $S(v_1, v_2, v_3, v_4; 1)$ and $S(v_5, v_6, v_7; 1)$. A proper choice of ε_1, h makes radii of the spheres and the distance between its centers close to 1. Then the radius of $S(v_1, \ldots, v_7, 1)$ tends to $\frac{\sqrt{3}}{2} > \frac{1}{2}$.

Suppose the intersection of an attached sphere with the slice

$$M := S(v_1, v_2, v_3, v_4; 1) \bigcap \mathbb{R}^3 \times [0, \varepsilon]^6 = S^5_{1-\eta} \bigcap \mathbb{R}^3 \times [0, \varepsilon]^6$$

is properly colored and the equator $M_E = S^2_{1-\eta}$ belongs to the slice.

Let $H \subset \mathbb{R}^9$ be the 6-dimensional subspace containing $S^5_{1-\eta}$. Consider a coordinates in H such that M_E belongs to the subspace spanned by the first 3 coordinates. For every point $u \in M_E$, $u = (u_1, u_2, u_3, 0, 0, 0)$ consider a sphere

$$S^2(u; \nu) = \{\sqrt{1 - \nu^2}u + \xi \mid \quad \xi = (0, 0, 0, \xi_4, \xi_5, \xi_6); \ \|\xi\| = \nu\}.$$

Note that $S^2(u; \nu)$ is a subset of M when ν is small enough.

For every u consider the following regular pentagon belonging to $S^2(u; \nu)$:

$$w_{u,k} = \left(u_1, u_2, u_3, \cos\frac{2\pi k}{5}\nu, \sin\frac{2\pi k}{5}\nu, 0\right), \qquad k = 1, \ldots, 5.$$

Let u be a point. If one can find among $w_{u,1}, \ldots, w_{u,5}$ points of three different colors, then they can be taken as v_5, v_6, v_7. Otherwise vertices of every pentagon are colored in at most 2 different colors, i.e. there is a color with at least three representatives. Call this color *dominating* at u.

Consider an auxiliary coloring π in which every point of M_E has its dominating color. Let us show that π is proper. Indeed if the distance between $p, q \in M_E$ is equal to 1, then $\|w_{p,k} - w_{q,k}\| = 1$ for every k, so by the pigeonhole principle dominating colors at p and q are different.

By Lemma 7.2.3 sphere M_E has a point u^* with chromaticity at least 3, i.e. an arbitrary neighbourhood of u^* has three points of different dominating colors. Then one may choose from corresponding pentagons 3 points of different colors in a way that chosen points lie in three small neighbourhoods of points $w_{u^*,1}, \ldots, w_{u^*,5}$. Every triangle with vertices in these points is non-degenerate, and has sides of length at least ν.

Step 3. Recall that every point from v_1, v_2, v_3, v_4 and every point from v_5, v_6, v_7 lie at the distance 1 apart. Moreover, v_1, v_2, v_3, v_4 have pairwise different colors; the same holds for v_5, v_6, v_7. Moreover, by Lemma 7.2.2 (applied to equator that lies in the slice) the intersection of attached sphere $S(v_1, \dots, v_7; 1)$ and the slice has the chromatic number at least 3. Hence we show that a proper coloring of the slice requires at least 4+3+3=10 colors, as desired.

7.5 Proof of Proposition 7.1.1

Consider the following 4 points in $\mathbb{Q}^2 \times [0, \varepsilon]^2_{\mathbb{Q}}$:

$$A = (0,\ 0,\ 0,\ 0), \tag{7.1}$$

$$B = (q,\ \tfrac{1}{2},\ \alpha,\ \beta), \qquad C = (q,\ -\tfrac{1}{2},\ \alpha,\ \beta), \tag{7.2}$$

$$D = (2q,\ 0,\ 0,\ 0). \tag{7.3}$$

So we have

$$|AB|^2 = |AC|^2 = |BD|^2 = |CD|^2 = q^2 + \frac{1}{4} + \alpha^2 + \beta^2. \tag{7.4}$$

Our goal is to choose numbers $q \in \mathbb{Q}$ and $\alpha, \beta \in [0, \varepsilon]_{\mathbb{Q}}$ such that expression (7.4) is equal to 1. Let $q = a/2b$, where a and b are some integers. Then we need

$$\alpha^2 + \beta^2 = \frac{3}{4} - \frac{a^2}{(2b)^2} = \frac{3b^2 - a^2}{4b^2}. \tag{7.5}$$

It is enough for (a, b) to satisfy

$$3b^2 - a^2 = 2, \tag{7.6}$$

so if b is large enough, we can put $\alpha = \beta = \frac{1}{2b}$.

Let us construct a series of solutions to (7.6) as follows. Given the solution (a_n, b_n), we build next pair as

$$(a_{n+1},\ b_{n+1}) = (7a_n + 12b_n,\ 4a_n + 7b_n). \tag{7.7}$$

One can check that (a_{n+1}, b_{n+1}) is a solution to (7.6) by straightforward computation and use of assumption that so is (a_n, b_n). Now by taking $(a_0, b_0) = (1, 1)$ we get an infinite sequence of solutions with b_n strictly increasing without limit. So for any given ε there is some n_ε such that for $n > n_\varepsilon$

$$\frac{3b_n^2 - a_n^2}{4b_n^2} = \frac{1}{2b_n^2} < 2\varepsilon^2, \tag{7.8}$$

which implies $1/2b < \varepsilon$

Now we are going to find such integers x and y that

$$x \cdot \frac{a_n}{b_n} + y \cdot \frac{a_{n+1}}{b_{n+1}} = 1 \tag{7.9}$$

or

$$x \cdot a_n b_{n+1} + y \cdot a_{n+1} b_n = b_n b_{n+1}. \tag{7.10}$$

So existence of such x and y is equivalent to

$$\gcd(a_n b_{n+1},\ a_{n+1} b_n) \mid b_n b_{n+1}. \tag{7.11}$$

It is sufficient to show that $\gcd(\ldots) = 1$:

$$\gcd(a_n b_{n+1},\ a_{n+1} b_n) \mid (a_n b_{n+1} - a_{n+1} b_n), \tag{7.12}$$

$$a_n b_{n+1} - a_{n+1} b_n = a_n(4a_n + 7b_n) - b_n(7a_n + 12b_n) = 4a_n^2 - 12b_n^2 = -4(3b_n^2 - a_n^2) = -8. \tag{7.13}$$

And from (7.7) it is clear that

$$a_{n+1} \equiv a_n \equiv \ldots \equiv a_0 = 1 \pmod 2, \tag{7.14}$$

$$b_{n+1} \equiv b_n \equiv \ldots \equiv b_0 = 1 \pmod 2. \tag{7.15}$$

So $\gcd(\ldots) = 1$ as required.

Finally, let $\chi(\mathbb{Q}^2 \times [0, \varepsilon]^2_{\mathbb{Q}}) = 3$. Then points A and D have the same color. Hence, each point at the distance $k \cdot a_n/b_n + l \cdot a_{n+1}/b_{n+1}$ (where $1/2b_n^2 < 2\varepsilon^2$ and k, l are integers) from 0 has the same color. Taking $k = x$ and $l = y$, one can obtain that $(1, 0, 0, 0)$ has the same color. A contradiction.

Remark 7.5.1. *Recursion formula* (7.7) *was obtained the following way. Consider a ring* $\mathbb{Z}[\sqrt{3}]$. *It has the norm*

$$N(a + b\sqrt{3}) = (a + b\sqrt{3})(a - b\sqrt{3}) = a^2 - 3b^2.$$

Then (7.6) *transforms to an equation* $N(\alpha) = -2$. *Norm is multiplicative:* $N(\alpha\beta) = N(\alpha)N(\beta)$ *for any* $\alpha, \beta \in \mathbb{Z}[\sqrt{3}]$. *So if* $N(\alpha) = -2$ *and* $N(\zeta) = 1$, *then* $N(\alpha\zeta) = -2$. *For* (7.7) *one can take* $\zeta = 7 + 4\sqrt{3}$.

7.6 Further questions

Question 7.6.1. *Let* $\mathcal{M}_d$ *be a family of compact convex set* $\mathbb{R}^d$ *such that a proper coloring of any* $\mathbb{R}^d$ *have a point of chromaticity at least* $d + 1$ *in every* $M \in \mathcal{M}_d$. *Evaluate* $V_d^* = \inf_{M \in \mathcal{M}_d} \operatorname{Vol} M$ *from above.*

Theorem 7.1.3 gives the bound $V_d^* \le \frac{\sqrt{d+1}}{d!\sqrt{2^d}} \cdot \left(\sqrt{2d(d+1)}\right)^d = \frac{\sqrt{d^d(d+1)^{d+1}}}{d!}$.

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