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# Finite Complete Rewriting Systems for Semigroups and Groups

*Deko V. Dekov*

*Presented by P. Kenderov*

This paper deals with unified representations of complete rewriting systems for classes of semigroups and groups that have been established earlier by other methods. The results allow a representation of the elements of the semigroup or group in a canonical form, related to some problems of the combinatorial and computer algebras.

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## 1. Introduction

A finite complete rewriting system $R$ for a semigroup $S$ (or for a group $G$) gives a simple solution to the word problem for the semigroup $S$ (or for the group $G$) as follows: Two words are equivalent if and only if their $R$-irreducible forms (often called normal forms, or canonical forms) are the same.

In 1942 M.H.A. Newman [24] introduced the basic concepts and gave the basic results concerning complete rewriting systems. In 1951 T. Evans [13] introduced a completion procedure in order to solve the word problem for loops and other classes of algebras. In 1970 D.E. Knuth and P.B. Bendix [17] extended Evans' completion procedure to an arbitrary finitely presented variety of universal algebras. For an overview on the critical-pair completion approaches see e.g. B. Buchberger [1]. In the recent years, the Knuth-Bendix completion procedure has been used for creating complete rewriting systems for many classes of semigroups and groups.

In the present paper we give finite complete rewriting systems for a number of semigroups and groups whose word problem have been solved earlier by different methods.

All calculations in the present paper are by hand. In the recent years, various implementations of the Knuth-Bendix completion procedure have been developed. For an overview the reader may consult e.g. M. Hermann, C. Kirchner and H. Kirchner [14].

## 2. Preliminaries

In this section we review some basic facts about rewriting systems.

Let $X$ be a set and let $X^*$ be the free monoid on $X$, the empty word of which will be denoted by 1. As usual, the length of a word $w \in X^*$ is denoted by $|w|$.

A *rewriting system* (or a *string-rewriting system*) on a set $X$ is a subset $R$ of $X^* \times X^*$. An element $(\ell, r) \in R$, also written $\ell \to r$, is called a *rule* of $R$. The *single-step reduction relation* on $X^*$ induced by $R$, which by abuse of notation will also be denoted by $\to$, is defined as follows:

$$u \to v \quad \text{iff} \quad \exists x, y \in X^* \quad \exists(\ell, r) \in R: \quad u = x\ell y \text{ and } v = xry.$$

Its reflexive and transitive closure $\xrightarrow{*}$ is the *reduction relation* induced by $R$, while its reflexive, symmetric and transitive closure $\xleftrightarrow{*}$ coincides with the congruence on $X^*$ generated by $R$.

A rewriting system $R$ on $X$ is called:

– *terminating*, if for any word $x \in X^*$ there is no infinite chain of single-step reductions $x \to x_1 \to x_2 \to \ldots$;

– *confluent*, if for any reductions $x \xrightarrow{*} y$ and $x \xrightarrow{*} z$ there exists $w \in X^*$ such that $y \xrightarrow{*} w$ and $z \xrightarrow{*} w$;

– *locally confluent*, if for any single-step reductions $x \to y$ and $x \to z$ there exists $w \in X^*$ such that $y \xrightarrow{*} w$ and $z \xrightarrow{*} w$;

– *complete*, if it is both terminating and confluent.

We call an irreflexive and transitive binary relation an *ordering*. If $>$ is an ordering, then $u \geq v$ means that either $u > v$ or $u = v$.

Let $>$ be an ordering on $X^*$. This ordering is called:

– *monotonic*, if it is compatible with the operation of concatenation, i.e. for all $u, v, x, y \in X^*$, if $u > v$, then also $xuy > xvy$;

– *well-founded*, if for any word $x \in X^*$ there is no infinite descending chain $x > x_1 > x_2 > \ldots$;

– *reduction*, if it is both monotonic and well-founded.

Let $R$ be a rewriting system on $X$ and let $>$ be an ordering on $X^*$. The ordering $>$ is called *compatible* with $R$, if $\ell > r$ for each rule $(\ell \to r) \in R$.

**Theorem A.** (D. S. Lankford [19]). *A rewriting system $R$ on $X$ is terminating, if and only if there exists a reduction ordering on $X^*$ which is compatible with $R$.*

As in J. Steinbach [26], we use the following abbreviations for orderings:

LO : length-reducing ordering;

LLO-L : length-plus-lexicographic ordering from the left;

LLO-R: length-plus-lexicograhic ordering from the right;

WO : weight-reducing ordering;

WLO-L : weight-plus-lexicographic ordering from the left;

WLO-R : weight plus-lexicographic ordering from the right;

RPO-L : recursive path ordering from the left;

RPO-R : recursive path ordering from the right.

A function $\varphi : X \to \mathbf{N}$ satisfying $\varphi(a) > 0$ for all $a \in X$ is called a *weight-function*. It can uniquely be extended to a homomorphism from $X^*$ to $\mathbf{N}$, which by abuse of notation will also be denoted by $\varphi$.

Let $u, v \in X^*$, let $\varphi$ be a weight-function, let $\triangleright$ be a well-founded ordering on $X$, called a *precedence* on $X$, let $>_{Lex-L}$ (resp. $>_{Lex-R}$) be the lexicographic ordering from the left (resp. from the right) on $X^*$ induced by the precedence $\triangleright$, and let $>$ denote the usual ordering on $\mathbf{N}$. Define the orderings:

$u >_{LO} v$ iff $|u| > |v|$;

$u >_{LLO-L} v$ iff either $|u| > |v|$
or $|u| = |v|$ and $u >_{Lex-L} v$;

$u >_{LLO-R} v$ iff either $|u| > |v|$
or $|u| = |v|$ and $u >_{Lex-R} v$;

$u >_{WO} v$ iff $\varphi(u) > \varphi(v)$;

$u >_{WLO-L} v$ iff either $\varphi(u) > \varphi(v)$
or $\varphi(u) = \varphi(v)$ and $u >_{Lex-L} v$;

$u >_{WLO-R} v$ iff either $\varphi(u) > \varphi(v)$
or $\varphi(u) = \varphi(v)$ and $u >_{Lex-R} v$;

$u >_{RPO-L} v$ iff either $u \neq 1,\ v = 1$
or $u = au',\ v = bv',\ a, b \in X,\ u', v' \in X^*$ and
either $a \triangleright b$ and $au' >_{RPO-L} v'$
or $a = b$ and $u' >_{RPO-L} v'$
or $u' \geq_{RPO-L} bv'$;

$u >_{RPO-R} v$ iff either $u \neq 1,\ v = 1$
or $u = u'a,\ v = v'b,\ a, b \in X,\ u', v' \in X^*$ and
either $a \triangleright b$ and $u'a >_{PRO-R} v'$
or $a = b$ and $u' >_{RPO-R} v'$
or $u' \geq_{RPO-R} v'b$.

The above orderings are reduction orderings. (See e.g. N. Dershowitz [11] or J. Steinbach [26]).

Let $(uv \to s) \in R$, $(vw \to t) \in R$ and $u, v, w$ are nonempty words. Then the word $uvw$ is called an *overlap ambiguity* of $R$. Let $(v \to s) \in R$, $(uvw \to t) \in R$ and let $u = 1$ and $w = 1$ imply $s \neq t$. Then the word $uvw$ is called an *inclusion ambiguity* of $R$. The pair of words $(sw, ut)$ or $(usw, t)$, respectively, is called a *critical pair* of $R$. A critical pair $(p, q)$ of $R$ is *resolved*, if there is a word $z \in X^*$ such that $p \xrightarrow{*} z$ and $q \xrightarrow{*} z$, *unresolved* otherwise.

**Theorem B.** *Let $R$ be a terminating rewriting system. Then the following conditions are equivalent:*

(i) *$R$ is confluent;*

(ii) *$R$ is locally confluent;*

(iii) *all critical pairs of $R$ are resolved.*

The equivalence (i) $\iff$ (ii) is due to M.H.A. Newman [24] and the equivalenve (ii) $\iff$ (iii) to D.E. Knuth and P.B. Bendix [17].

Given a semigroup $S$, a rewriting system $R$ on $X$ is called a *rewriting system for $S$*, if $sgp(X\ ;\ \ell = r$ where $(\ell \to r) \in R)$ is a semigroup presentation for $S$. Given a monoid $M$, a rewriting system $R$ on $X$ is called a *rewriting system* *for* $M$ if $mon(X\ ;\ \ell = r$ where $(\ell \to r) \in R)$ is a monoid presentation for $M$. A rewriting system for a group $G$ is a rewriting system for $G$ as a monoid.

A word $u \in X^*$ is called *R-irreducible*, if there is no single-step reduction $u \to v$ for some $v \in X^*$. The set of all $R$-irreducible words is denoted by $IRR(R)$. If $R$ is a rewriting system for a semigroup, then we denote by $IRR(R)$ the set of all $R$-irreducible words, distinct from the empty word 1. A word $v \in X^*$ is called an *R-irreducible form* of the word $u$ if $v$ is an $R$-irreducible word and $u \xrightarrow{*} v$. We denote by $S(u)$ an $R$-irreducible form of $u$.

**Theorem C.** (M.H.A. Newman [24])

(i) *If $R$ is a complete rewriting system for a semigroup $S$, then there exists exactly one $R$-irreducible word representing each element of $S$.*

(ii) *If $R$ is a complete rewriting system for a group $G$, then there exists exactly one $R$-irreducible word representing each element of $G$.*

A rewriting system $R$ on $X$ is *finite*, if both $X$ and $R$ are finite sets.

Two rewriting systems $R_1$ and $R_2$ on the same set $X$ are called *equivalent*, if they generate the same congruence on $X^*$. Let $R$ be a finite rewriting system and let $>$ be a reduction ordering which is compatible with $R$. D.E. Knuth and P.B. Bendix [17] (see also D.E. Cohen [4]) have developed a procedure for creating a complete rewriting system which is equivalent to $R$. A simplified version of the Knuth-Bendix completion procedure is as follow. For every unresolved critical pair $(p, q)$ and every corresponding pair $(S(p), S(q))$ we add a new rule $S(p) \to S(q)$, if $S(p) > S(q)$ or $S(q) \to S(p)$ if $S(q) > S(p)$. If $S(p)$ and $S(q)$ are not comparable with respect to the ordering $>$, then we extend $>$ such that either $S(p) > S(q)$ or $S(q) > S(p)$. The Knuth-Bendix completion procedure iterates this basic step. If the procedure stops, then it will create a finite complete rewriting system which is equivalent to $R$.

For some results of the author on rewriting systems, see [8,9,10].

## 3. Applying rewriting methods to semigroups

Given a semigroup $S$, $R(S)$ denotes a complete rewriting system for $S$.

In 1967 B. H. Neumann [23] introduced an enumeration method for finitely presented semigroups analogous to the Todd-Coxeter enumeration procedure for groups. In 1978 A. Jura [16] gave a proof of the Neumann's method and in 1992 E. R. Robertson and Y. Ünlü [25] described an implementation of the method. B. H. Neumann [23] gave two examples to illustrate the method. In Examples 1.1 and 1.2 we give new proofs of the B. H. Neumann's examples.

**Example 1.1.** The Neumann semigroup.

$$S = sgp\ (a, b\ ;\ b^2 = a^3,\ ba = a^2b).$$

$$R_1(S) = \{b^2 \to a^3,\ ba \to a^2b,\ a^6b \to a^3b,\ a^7 \to a^4\}.$$

Termination: RPO-L: $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R_1(S)$.

$$IRR(R_1(S)) = \{a, a^2, a^3, a^4, a^5, a^6, b, ab, a^2b, a^3b, a^4b, a^5b\}.$$

Example 1.2 below gives a finite complete rewriting system $R_2(S)$ for $S$ such that $IRR(R_2(S))$ coincides with the set of representatives given in B. H. Neumann [23].

**Example 1.2.** The Neumann semigroup.

$$S = sgp\ (a, b\ ;\ b^2 = a^3,\ ba = a^2b).$$

$$R_2(S) = \{a^3 \to b^2,\ a^2b \to ba,\ b^2a \to ab^2,\ b^3 \to aba,$$
$$bab^2 \to aba^2,\ baba \to abab\}.$$

Termination : LLO-L : $b \triangleright a$.

The orderings LLO-R, WO, RPO-L and RPO-R are not able to prove the termination of $R_2(S)$.

$$IRR(R_2(S)) = \{a, a^2, b, b^2, ab, ab^2, ba, ba^2, aba, bab, aba^2, abab\}.$$

Examples 2.1 to 2.4 below give four equivalent finite complete rewriting systems for the quaternion group $Q_8$. Note that these examples illustrate the mutual independence of the orderings LLO-L, LLO-R, RPO-L and RPO-R.

**Example 2.1.** The quaternion group.

$$Q_8 = sgp\ (a, b\ ;\ aba = b,\ bab = a).$$

$$R_1(Q_8) = \{b^2 \to a^2,\ a^5 \to a,\ ba \to a^3b,\ a^4b \to b\}.$$

Termination : RPO-L : $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R_1(Q_8)$.

$$IRR(R_1(Q_8)) = \{a, a^2, a^3, a^4, b, ab, a^2b, a^3b\}.$$

**Example 2.2.** The quaternion group.

$$Q_8 = sgp\ (a, b\ ;\ aba = b,\ bab = a).$$

$$R_2(Q_8) = \{b^2 \to a^2,\ a^5 \to a,\ ab \to ba^3,\ ba^4 \to b\}.$$

Termination : RPO-R : $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-L are not able to prove the termination of $R_2(Q_8)$.

$$IRR(R_2(Q_8)) = \{a, a^2, a^3, a^4, b, ba, ba^2, ba^3\}.$$

**Example 2.3.** The quaternion group.

$$Q_8 = sgp\ (a, b\ ;\ aba = b,\ bab = a).$$

$$R_3(Q_8) = \{aba \to b,\ bab \to a,\ b^2 \to a^2,\ a^5 \to a,\ ba^2 \to a^2b,\ a^3b \to ba\}.$$

Termination : LLO-L : $b \triangleright a$.

The orderings LLO-R, WO, RPO-L and RPO-R are not able to prove the termination of $R_3(Q_8)$.

$$IRR(R_3(Q_8)) = \{a, a^2, a^3, a^4, b, ab, a^2b, ba\}.$$

**Example 2.4.** The quaternion group.

$$Q_8 = sgp\ (a, b\ ;\ aba = b,\ bab = a).$$

$$R_4(Q_8) = \{aba \to b,\ bab \to a,\ b^2 \to a^2,\ a^5 \to a,\ a^2b \to ba^2,\ ba^3 \to ab\}.$$

Termination : LLO-R : $b \triangleright a$.

The orderings LLO-L, WO, RPO-L and RPO-R are not able to prove the termination of $R_4(Q_8)$.

$$IRR(R_4(Q_8)) = \{a, a^2, a^3, a^4, b, ba, ba^2, ab\}.$$

In 1968 R. C. Buck [2] introduced a semigroup $B$ given in terms of generators and defining relations and gave a solution to the word problem for $B$. Examples 3.1 and 3.2 below give two equivalent finite complete rewriting systems for the Buck semigroup.

**Example 3.1.** The Buck semigroup.

$$B = sgp\ (a, b\ ;\ aba = b,\ bab = a^n),$$

where $n \geq 1$.

$$R_1(B) = \{b^2 \to a^{n+1},\ a^{3n+2} \to a^n,\ ba \to a^{2n+1}b,\ a^{2n+2}b \to b\}.$$

Clearly, if $n = 1$, then $B = Q_8$ and $R_1(B) = R_1(Q_8)$.

Termination : RPO-L : $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R_1(B)$.

$$IRR(R_1(B)) = \{a, a^2, \dots, a^{3n+1}, b, ab, a^2b, \dots, a^{2n+1}b\}.$$

Example 3.2 below gives a finite complete rewriting sytem $R_2(B)$ for $B$ such that $IRR(R_2(B))$ coincides with the set of representatives given in R. C. Buck [2, p.854].

**Example 3.2.** The Buck semigroup.

$$B = sgp\ (a, b\ ;\ aba = b,\ bab = a^n),$$

where $n \geq 1$.

$$R_2(B) = \{aba \to b,\ bab \to a^n,\ b^2 \to a^{n+1},\ a^{3n+2} \to a^n,\ a^{n+1}b \to ba^{n+1},$$
$$ba^{n+2} \to a^n b,\ ba^k b \to a^{3n+3-k},\ k = 2, \dots, n\}.$$

Clearly, if $n = 1$, then $R_2(B) = R_4(Q_8)$.

Termination : WLO-R : $\varphi(a) = 1$, $\varphi(b) = 2n$, $b \triangleright a$.

The orderings LLO-L, WO, RPO-L and RPO-R are not able to prove the termination of $R_2(B)$.

$$IRR(R_2(B)) = \{a, a^2, \dots, a^{3n+1}, b, ba, ba^2, \dots, ba^{n+1}, ab, a^2b, \dots, a^n b\}.$$

In 1969 V. Dlab and B. H. Neumann [12, p.164] gave an example of one-relation semigroup $S$ which has no endomorphisms other than the identity automorphism. They have used the method of R. Croisot [7] (see also A. H. Clifford and G. B. Preston [3, pp.169-171]) to solve the word problem for $S$. Example 4 below gives a new proof of Lemma 4.2 of [12].

**Example 4.** The Dlab-Neumann semigroup.

$$S = sgp\ (a, b\ ;\ ab^2 = baba).$$

$$R(S) = \{ab^2 \to baba\}.$$

Termination : RPO-R : $b \triangleright a$.

The ordreings WLO-L, WLO-R and RPO-L are not able to prove the termination of $R(S)$.

$$IRR(R(S)) = \{b^t a^{s_1} ba^{s_2} \dots ba^{s_n}\ :\ n \geq 1,\ t \geq 0,\ s_n \geq 0,$$
$$\text{and } s_i \geq 1 \text{ for } 1 \leq i < n\}.$$

## 4. Applying rewriting methods to groups

Given a group $G$, $R(G)$ denotes a complete rewriting system for $G$.

Examples 5.1 to 5.4 below give four equivalent finite complete rewriting systems for the quaternion group $Q_8$.

**Example 5.1.** The quaternion group.

$$Q_8 = mon\ (a, b\,;\, a^4 = 1,\, b^2 = a^2,\, aba = b).$$

$$R_1(Q_8) = \{a^4 \to 1,\ \ b^2 \to a^2,\ \ ab \to ba^3\}.$$

Termination: RPO-R: $b \rhd a$.

The orderings WLO-L, WLO-R and RPO-L are not able to prove the termination of $R_1(Q_8)$.

$$IRR(R_1(Q_8)) = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}$$

Example 5.2 below gives a finite complete rewriting system $R_2(Q_8)$ for $Q_8$ such that $IRR(R_2(Q_8))$ coincides with the set of representatives given in W. Magnus, A. Karrass and D. Solitar [22, Ch.I, Sect. 1, Exercise 7] or in A. I. Kostrikin [18, Ch. 7, Sect. 3.5, Example 2].

**Example 5.2.** The quaternion group.

$$Q_8 = mon\ (a, b\ ;\ a^4 = 1,\ b^2 = a^2,\ aba = b).$$

$$R_2(Q_8) = \{a^4 \to 1,\ \ b^2 \to a^2,\ \ ba \to a^3b\}.$$

Termination : RPO-L : $b \rhd a$.

The orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R_2(Q_8)$.

$$IRR(R_2(Q_8)) = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

**Example 5.3.** The quaternion group.

$$Q_8 = mon\ (a, b\ ;\ a^4 = 1,\ b^2 = a^2,\ aba = b)$$

$$R_3(Q_8) = \{a^4 \to 1,\ \ b^2 \to a^2,\ \ aba \to b,\ \ ba^3 \to ab,\ \ a^2b \to ba^2,\ \ bab \to a\}.$$

Termination : LLO-R : $b \rhd a$.

The orderings LLO-L, RPO-L and RPO-R are not able to prove the termination of $R_3(Q_8)$.

$$IRR(R_3(Q_8)) = \{1, a, a^2, a^3, b, ba, ba^2, ab\}.$$

**Example 5.4.** The quaternion group.

$$Q_8 = mon \ (a, b \ ; \ a^4 = 1, \ b^2 = a^2, \ aba = b).$$

$$R_4(Q_8) = \{a^4 \to 1, \ b^2 \to a^2, \ aba \to b, \ a^3b \to ba, \ ba^2 \to a^2b, \ bab \to a\}.$$

Termination : LLO-L : $b \triangleright a$.

The orderings LLO-R, RPO-L and RPO-R are not able to prove the termination of $R_4(Q_8)$.

$$IRR(R_4(Q_8)) = \{1, a, a^2, a^3, b, ba, ab, a^2b\}.$$

Example 6 below gives a finite complete rewriting system for the dicyclic group $< 2, 2, n > = mon \ (a, b \ ; \ a^{2n} = 1, \ b^2 = a^n, \ aba = b)$. (See H. S. M. Coxeter and W. O. J. Moser [6, Ch. 1, Sect. 6]). Clearly, the main special cases of $G =< 2, 2, n >$ are $Q_8 =< 2, 2, 2 >$ and $T =< 2, 2, 3 >$. (We use the notation $T$ for $< 2, 2, 3 >$ in accordance with T. W. Hungerford [15, p.98]).

**Example 6.** The dicyclic group.

$$G = mon \ (a, b \ ; \ a^{2n} = 1, \ b^2 = a^n, \ ba = a^{2n-1}b).$$

where $n \geq 2$.

$$R(G) = \{a^{2n} \to 1, \ b^2 \to a^n, \ ba \to a^{2n-1}b\}.$$

Termination : RPO-L : $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R(G)$.

$$IRR(R(G)) = \{a^i b^j ; 0 \leq i < 2n, 0 \leq j \leq 1\}.$$

In 1986 Ph. LeChenadec [20, Ch.6, Sect.3] gave a finite complete rewriting system for the dihedral group $D_n$. Examples 7.1 and 7.2 below give two equivalent finite complete rewriting systems for $D_n$, distinct from the rewriting system due to LeChenadec [20]. Example 7.1 gives a finite complete rewriting system $R_1(D_n)$ for $D_n$ such that $IRR(R_1(D_n))$ coincides with the set of represenatives given in W. Magnus, A. Karrass and D. Solitar (for the case $n = 3$) [22, Ch.I, Sect.1, Exercise 6(a)], or in P. M. Cohn (for the case $n = 3$) [5, p.157].

**Example 7.1.** The dihedral group of degree $n$.

$$D_n = mon \ (a, b \ ; \ a^n = 1, \ , b^2 = 1, \ ab = ba^{n-1}),$$

where $n \geq 3$.

$$R_1(D_n) = \{a^n \to 1, \ b^2 \to 1, \ ab \to ba^{n-1}\}.$$

Termination : RPO-R : $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-L are not able to prove the termination of $R_1(D_n)$.

$$IRR(R_1(D_n)) = \{1, a, a^2, \ldots, a^{n-1}, b, ba, ba^2, \ldots, ba^{n-1}\}.$$

Example 7.2 below gives a finite complete rewriting system $R_2(D_n)$ for $D_n$ such that $IRR(R_2(D_n))$ coincides with the set of representatives given in W. Magnus, A. Karrass and D. Solitar [22, Ch.I, Sect.1, Exercise 8], or in A. I. Kostrikin [18, Ch.7. Sect.3.5, Example 1], or in T. W. Hungerford [15, Ch.I, Theorem 6.13].

**Example 7.2.** The dihedral group of degree $n$.

$$D_n = mon\ (a, b\ ;\ a^n = 1,\ b^2 = 1,\ ba = a^{n-1}b),$$

where $n \geq 3$.

$$R_2(D_n) = \{a^n \to 1,\ b^2 \to 1,\ ba \to a^{n-1}b\}.$$

Termination : RPO-L : $b \triangleright a$.

The orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R_2(D_n)$.

$$IRR(R_2(D_n)) = \{1, a, a^2, \ldots, a^{n-1}, b, ab, a^2b, \ldots, a^{n-1}b\}.$$

Example 8 below gives a finite complete rewriting system for the metacyclic group $C_n \times_\theta C_m$ (see S. MacLane and G. Birkhoff [21, Ch.13, Section 2]). Here we denote the metacyclic group by $M(n, m, k)$ or simply by $M$. Clearly, the main special cases of $M(n, m, k)$ are the dihedral group $D_n = M(n, 2, n-1)$ and the direct product of two cyclic groups $C_n$ and $C_m$ or orders $n$ and $m$, respectively, $C_n \times C_n = M(n, m, 1)$.

**Example 8.** The metacyclic group.

$$M = mon\ (a, b\ ;\ a^n = 1,\ b^m = 1,\ ba = a^k b),$$

where $n \geq 2,\ m \geq 2,\ k \geq 1$ and $k^m \equiv 1 \pmod{n}$.

$$R(M) = \{a^n \to 1,\ b^m \to 1,\ ba \to a^k b\}.$$

Clearly, if $n \geq 3,\ m = 2$ and $k = n-1$, then $R(M) = R_2(D_n)$.

Termination : RPO-L : $b \triangleright a$.

If $k \geq 2$, then the orderings WLO-L, WLO-R and RPO-R are not able to prove the termination of $R(M)$.

$$IRR(R(M)) = \{a^i b^j \; ; \; 0 \le i < n, \; 0 \le j < m\}.$$

In 1986 Ph. LeChenadec [20, Ch.6, Sect.3] gave a finite complete rewriting system for the alternating group $A_4$. Example 9 below gives a new finite complete rewriting system $R(A_4)$ for $A_4$ such that $IRR(R(A_4))$ coincides with the set of representatives given in W. Magnus, A. Karrass and D. Solitar [22, Ch.I, Sect.2, Exercise 12].

**Example 9.** The alternating group of degree 4.

$$A_4 = mon\ (a, b \; ; \; a^3 = 1, \; b^2 = 1, \; bab = a^2ba^2).$$

$$R(A_4) = \{a^3 \to 1, \; b^2 \to 1, \; bab \to a^2ba^2, \; ba^2b \to aba\}.$$

Termination: RPO-L : $b \rhd a$,
or RPO-R : $b \rhd a$,
or WO : $\varphi(a) = 1, \; \varphi(b) = 4$.

The orderings LLO-L and LLO-R are not able to prove the termination of $R(A_4)$.

$$IRR(R(A_4)) = \{1, a, a^2, b, ab, a^2b, ba, ba^2, aba, aba^2, a^2ba, a^2ba^2\}.$$

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*"Zahari Kniajeski" 81*
*6000 - Stara Zagora, BULGARIA*

*e-mail: ddekov@sz.techno-link.com*

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