Semigroup Forum Vol. 58 (1999) 395–414

RESEARCH ARTICLE

Deciding Embeddability of Partial Groupoids into Semigroups

Deko V. Dekov

Communicated by G. Lallement

1. Introduction

The embeddability problem for the class of finite partial groupoids is unsolvable, i.e., there is no algorithm for deciding whether or not a finite partial groupoid can be embedded into a semigroup. (See e.g., T. Evans [23,24]). In recent years, various sufficient conditions for embeddability of not necessarily finite partial groupoids into semigroups have been studied, and we deal in this paper mainly with classes of partial groupoids which are embeddable into a semigroup.

After preliminaries in Section 2, in Section 3 we prove (Theorem 1) that the following classes of partial groupoids are not first-order finitely axiomatizable: the class of all associative partial groupoids, the class of all partial semigroups, the class of all R-presemigroups, and the class of all S-presemigroups.

In Section 4 we start with characterizations of U-presemigroups in Theorem 2. It is based on results by M.H.A. Newman [47] and by D.E. Knuth and P.B. Bendix [34], and it yields in particular a new proof of a theorem of R. Baer [4, Section 3, Theorem 1]. Then we show (Theorem 3) that a theorem of J.R. Stallings [51] on U-pregroups is a special case of the R. Baer's theorem on U-presemigroups, just cited. We also investigate a set of partial groupoids shown to be embeddable into semigroups by P.W. Bunting, J. van Leeuwen, and D. Tamari in [8]. Apart from a new proof of this result, we state further properties of these partial groupoids (Theorem 4) and illustrate in this way several concepts occuring in our paper.

In Section 5 we introduce the use of the Knuth-Bendix completion procedure for deciding embeddability of finite partial groupoids into semigroups. In 1975 T.E. Hall [29] gave an example (attributed by T.E. Hall to C.J. Ash) of an embeddable finite semigroup amalgam which is not embeddable into a finite semigroup. Among other applications we use the Knuth-Bendix completion procedure to give a simple proof of this result.

2. Preliminaries

For any set X, we denote by X^+ the free semigroup on X, and by X^* the free monoid on X with the empty word λ as identity. The length of a word $w \in X^*$ is denoted by $|w|$.

2.1. Partial groupoids

A *partial groupoid* is a triple $G = (G, D, \mu)$ where G is a nonempty set, $D \subseteq G \times G$, and $\mu : D \longrightarrow G$ is a mapping. Let $G = (G, D, \mu)$ and $H = (H, E, \nu)$ be partial groupoids. Denote $\mu(x,y)$ by $x \cdot y$ and $\nu(x,y)$ by $x \circ y$. A *homomorphism* of G into H is a mapping $\varphi: G \to H$ such that $(x,y) \in D$ implies $(\varphi(x), \varphi(y)) \in E$ and $\varphi(x \cdot y) = \varphi(x) \circ \varphi(y)$, for all $x, y \in G$. In particular, if there is a bijective homomorphism φ of G onto H such that φ^{-1} is also a homomorphism, then the partial groupoids G and H are called *isomorphic*. We use the notation ([48])

$$\begin{aligned} (x,y)_D \quad &\text{iff} \quad (x,y) \in D\,, \\ (x_1, \ldots, x_n)_D \quad &\text{iff} \quad (x_i, x_{i+1})_D \quad \text{for} \quad i = 1, \ldots, n-1\,. \end{aligned}$$

We shall sometimes say that $x \cdot y$ is defined, instead of $(x,y)_D$. As in [8], the symbol $*$ in a multiplication table of a partial groupoid means that the corresponding product is not defined. A word $x_1 \ldots x_n$ is said to be *G-irreducible* if for $i = 1, \ldots, n-1$, $(x_i, x_{i+1}) \notin D$. We denote by $IRR(G)$ the set of all G-irreducible words.

Let $G = (G, D, \mu)$ be a partial groupoid and let $\Theta_0(G)$ be the relation on G^+ defined as follows: $(xy, \mu(x,y)) \in \Theta_0(G)$ iff $(x,y)_D$. Let $\Theta(G)$ be the congruence on G^+ generated by $\Theta_0(G)$. Then $U(G) = G^+/\Theta(G)$ is called the *universal semigroup* of the partial groupoid G.

Let G and H be partial groupoids. G is said to be *embeddable* into H if there is an injective homomorphism of G into H.

A partial groupoid $G = (G, D, \mu)$ is said to be a *relative partial subgroupoid* of a partial groupoid $H = (H, E, \nu)$ if the following conditions are satisfied:

(i) $G \subseteq H$;

(ii) $(x,y) \in D$ iff $(x,y) \in E$ and $\nu(x,y) \in G$ for all $(x,y) \in G \times G$;

(iii) $\mu(x,y) = \nu(x,y)$ for all $(x,y) \in D$.

Since each subset G of H determines uniquely a relative partial subgroupoid of (H, E, ν) (where D has to be defined by (ii) and μ by (iii)), one calls (G, D, μ) the *relative partial subgroupoid of* (H, E, ν) *determined by* $G \subseteq H$.

Concerning the last both concepts we note the following: If $G = (G, D, \mu)$ is embeddable into $H = (H, E, \nu)$, then the homomorphic image $\varphi(G)$ of G need not be a relative partial subgroupoid of H. In fact, using that each element of $\varphi(G) = G' \subseteq H$ satisfies $x' = \varphi(x)$ for a unique $x \in G$, we introduce $D' = \{(x', y') : (x,y) \in D\}$ and $\mu'(x', y') = \varphi(\mu(x,y))$ and obtain for the partial groupoid (G', D', μ') contained in (H, E, ν) instead of (i), (ii) and (iii) merely

(i′) $G' \subseteq H$;

(ii′) $(x', y') \in D' \Longrightarrow (x', y') \in E$ for all $(x', y') \in G' \times G'$;

(iii′) $\mu'(x', y') = \nu(x', y')$ for all $(x', y') \in D'$.

Let again G be a partial groupoid and $\Theta(G)$ the congruence on G^+ generated by $\Theta_0(G)$. Then $w \longrightarrow w'$ for $w, w' \in G^+$ is called a *direct move* from w to w' and an *inverse move* from w' to w if there are $u, v \in G^*$ and $x, y \in G$ such that $w = uxyv$ and $w' = u\mu(x,y)v$ (P.M. Cohn [16, p. 159], S.M. Gensemer and H.J. Weinert [27]). Moreover, $w \sim w' \pmod{\Theta(G)}$ holds for $w \neq w'$ of G^+ iff there is a sequence of words $w = w_0, w_1, \ldots, w_n = w'$ in G^+ where each step from w_i to w_{i+1} is either a direct or an inverse move. Following [33, p. 26] we also call $w = w_0, w_1, \ldots, w_n = w'$

a *sequence of elementary* $\Theta_0(G)$*-transitions from* w *to* w' and n the *length* of this sequence. In particular, for each word $w \in G^+$ of length k, $k \geq 3$, $\mathfrak{S}_G(w)$ denotes the subset of G consisting of all elements of G which can be obtained from w by a sequence of $k-1$ direct moves (R. Baer [2, Section 1, Definition 2]). If no ambiguity is possible, then we write $\mathfrak{S}(w)$ instead of $\mathfrak{S}_G(w)$.

We introduce the following conditions on a partial groupoid G:

(A0) For all $x, y, z \in G$, if $(x, y, z)_D$, $(x, y \cdot z)_D$ and $(x \cdot y, z)_D$, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

(B1) For all $x, y, z \in G$, if $(x, y, z)_D$ and $(x \cdot y, z)_D$, then $(x, y \cdot z)_D$.

(B2) For all $x, y, z \in G$, if $(x, y, z)_D$ and $(x, y \cdot z)_D$, then $(x \cdot y, z)_D$.

(C1) For all $x, y, z \in G$, if $(x, y, z)_D$ and $(x \cdot y, z) \notin D$, then $(x, y \cdot z) \notin D$, $x \cdot y = x$ and $y \cdot z = z$.

(C2) For all $x, y, z \in G$, if $(x, y, z)_D$ and $(x, y \cdot z) \notin D$, then $(x \cdot y, z) \notin D$, $x \cdot y = x$ and $y \cdot z = z$.

(D1) For all $w, x, y, z \in G$, if $(w, x, y, z)_D$, then $(w, x \cdot y)_D$ or $(x \cdot y, z)_D$.

(D2) For all $w, x, y, z \in G$, if $(x, y, z)_D$ and $(w, x \cdot y)_D$, then $(w, x)_D$ or $(x \cdot y, z)_D$.

(D3) For all $w, x, y, z \in G$, if $(w, x, y)_D$ and $(x \cdot y, z)_D$, then $(w, x \cdot y)_D$ or $(y, z)_D$.

(A_n) For all $w = x_1 \ldots x_n \in G^+$ where $n \geq 3$, $|\mathfrak{S}(w)| \leq 1$ holds.

(A) For all $n \geq 3$, condition (A_n) holds.

(P) For all $x, y \in G$, if $x \sim y \pmod{\Theta(G)}$, then $x = y$.

($\mathrm{Q_R}$) For all $x, y, z \in G$, if $xy \sim z \pmod{\Theta(G)}$, then $(x, y)_D$.

(Q) For all $x, x_1, \ldots, x_n \in G$ and all $n \geq 2$, if $x_1 \ldots x_n \sim x \pmod{\Theta(G)}$, then there exists $i \in \{1, \ldots, n-1\}$ such that $(x_i, x_{i+1})_D$.

(S) For all $u, v \in G^+$, if $u \sim v \pmod{\Theta(G)}$, then $\mathfrak{S}(u) = \mathfrak{S}(v)$.

(L) For all $u, v \in IRR(G)$, if $u \sim v \pmod{\Theta(G)}$, then $|u| = |v|$.

(U) For all $u, v \in IRR(G)$, if $u \sim v \pmod{\Theta(G)}$, then $u = v$.

Clearly, conditions (A0) and (A_3) are equivalent.

A partial groupoid G is called

- an *associative partial groupoid* if it satisfies condition (A);
- a *partial semigroup* (J.-C. Spehner [50, Definition 1]) if it is embeddable into a semigroup;
- an *R-presemigroup* if it is a relative partial subgroupoid of a semigroup;
- an *S-presemigroup* if it satisfies condition (S);
- an *L-presemigroup* if it satisfies conditions (L) and (P);
- an *U-presemigroup* if it satisfies condition (U) and thus (P).

Theorem A. *Let* G *be a partial groupoid. Then*

(i) (R. Baer [2]). *G is a partial semigroup if and only if it satisfies condition* (P);

(ii) (R. Baer [2]). *G is an R-presemigroup if and only if it satisfies conditions* (P) *and* ($\mathrm{Q_R}$);

(iii) (R. Baer [2]). *G is an S-presemigroup if and only if it satisfies conditions* (P) *and* (Q);

(iv) (R. Baer [4]). *G is an L-presemigroup if and only if it satisfies conditions* (A0), (B1), (B2), (D1), (D2), *and* (D3);

(v) (R. Baer [4]). *G is an U-presemigroup if and only if it satisfies conditions* (A0), (B1), (B2), (C1), *and* (C2). ∎

The following notations will be convenient:

A - the class of all associative partial groupoids;
P - the class of all partial semigroups;
R - the class of all R-presemigroups;
S - the class of all S-presemigroups;
L - the class of all L-presemigroups;
U - the class of all U-presemigroups.

We have $\mathbf{U} \subset \mathbf{L} \subset \mathbf{S} \subset \mathbf{R} \subset \mathbf{P} \subset \mathbf{A}$. R. Baer [2-4] has shown that all these inclusions are strict, and we refer to various examples given in Section 4 and Section 5.

Note that recently the author studied various classes of partial groupoids (see [17-21]).

2.2. Rewriting systems

Let X be a set. An arbitrary binary relation R on X^* is called a *rewriting system* (or a *string-rewriting system*) on X. An element $(\ell, r) \in R$, also written $\ell \longrightarrow r$, is called a *rule* of R. The *single-step reduction relation* on X^* induced by R, which by abuse of notation will also be denoted by $\longrightarrow$, is defined as follows:

$$u \longrightarrow v \quad \text{iff} \quad \exists\, x, y \in X^* \quad \exists\, (\ell, r) \in R: \; u = x\ell y \quad \text{and} \quad v = xry\,.$$

Its reflexive and transitive closure $\xrightarrow{*}$ is the *reduction relation* induced by R, while its reflexive, symmetric and transitive closure $\Theta(R)$ coincides with the congruence on X^* generated by R.

A rewriting system R on X is called *terminating* if there is no infinite sequence of single-step reductions. The *length-reducing* ordering $>_{\mathrm{LO}}$ on X^* is defined as follows: for all $u, v \in X^*$, $u >_{\mathrm{LO}} v$ iff $|u| > |v|$. If $\ell >_{\mathrm{LO}} r$ for all $(\ell \longrightarrow r) \in R$, then R is terminating. A *length-plus-lexicographic* ordering $>_{\mathrm{LLO}}$ on X^* is defined as follows: for all $u, v \in X^*$, $u >_{\mathrm{LLO}} v$ iff either $|u| > |v|$ or $|u| = |v|$ and v precedes u in the lexicographic ordering induced by some well-ordering on X. If $\ell >_{\mathrm{LLO}} r$ for all $(\ell \longrightarrow r) \in R$, then R is terminating.

A rewriting system R on X is called *confluent* if for any $w, u, v \in X^*$ such that $w \xrightarrow{*} u$ and $w \xrightarrow{*} v$ there exists $w' \in X^*$ such that $u \xrightarrow{*} w'$ and $v \xrightarrow{*} w'$. A rewriting system R is *complete* if it is both terminating and confluent.

Let $(uv \longrightarrow s) \in R$, $(vw \longrightarrow t) \in R$ for nonempty words $u, v, w \in X^*$. Then the word uvw is called an *overlap ambiguity* of R. Let $(v \longrightarrow s) \in R$, $(uvw \longrightarrow t) \in R$ and let $u = \lambda$ and $w = \lambda$ imply $s \neq t$. Then the word uvw is called an *inclusion ambiguity* of R. In these both cases, the pair of words (sw, ut) or (usw, t), respectively, is called a *critical pair* of R. A critical pair (p, q) of R is *resolved* if there is a word $w' \in X^*$ such that $p \xrightarrow{*} w'$ and $q \xrightarrow{*} w'$.

A word $u \in X^*$ is called *R-irreducible* if there is no single-step reduction $u \longrightarrow v$ for some $v \in X^*$. We denote by $IRR(R)$ the set of all R-irreductible words.

Theorem B. *Let R be a terminating rewriting system on X. Then the following conditions are equivalent:*

(i) *for all $u, v \in IRR(R)$, if $u \sim v \ (mod\, \Theta(R))$, then $u = v$;*

(ii) *R is complete;*

(iii) *all critical pairs of R are resolved.* ■

The equivalence (i) $\Longleftrightarrow$ (ii) is due to M.H.A. Newman [47] and the equivalence (ii) $\Longleftrightarrow$ (iii) to D.E. Knuth and P.B. Bendix [34].

Two rewriting systems R_1 and R_2 on the same set X are called *equivalent* if they generate the same congruence on X^*. A rewriting system on X is *finite* if both X and R are finite sets. Let R be a finite terminating rewriting system. D. Knuth and P. Bendix [34] have developed a procedure for creating a finite complete rewriting system which is equivalent to R. Note that for some inputs the completion procedure will never terminate. For a description of the Knuth-Bendix completion procedure see e.g. [7, 15, 30, 36].

3. The class of all partial semigroups is not finitely axiomatizable

Clearly, a partial groupoid $G = (G, D, \mu)$ can be considered as a pair $M(G) = (G, T)$ where T is the ternary relation on G defined by $(x, y, z) \in T$ if and only if $(x, y)_D$ and $\mu(x, y) = z$. In the terminology of A.I. Malcev [44], $M(G)$ is the *model corresponding to* G. If we use a first-order language L consisting of one ternary relation symbol, p say, then we can rewrite each of conditions (A0), (B1), (B2), (C1), (C2), (D1), (D2), and (D3) as a sentence of L. For example, (A0) and (B1) can be rewritten as sentences of L as follows:

A0$'$. $\forall\, x\, \forall\, y\, \forall\, x_1 \ldots \forall\, x_5 [p x_1 x_2 x \ \wedge\ p x_3 x_4 x_1 \ \wedge\ p x_4 x_2 x_5 \ \wedge\ p x_3 x_5 y \longrightarrow x = y]\,.$

B1$'$. $\forall\, x_1 \ldots \forall\, x_6\, \exists\, y [p x_2 x_3 x_1 \ \wedge\ p x_4 x_5 x_2 \ \wedge\ p x_5 x_3 x_6 \longrightarrow p x_4 x_6 y]\,.$

We denote by M the following sentence:

$$\forall\, x\, \forall\, y\, \forall\, x_1\, \forall\, x_2 [p x_1 x_2 x \ \wedge\ p x_1 x_2 y \longrightarrow x = y]\,.$$

The next proposition follows from Theorem A and Proposition I.5.10 of [33]:

Proposition 1. *The following classes of partial groupoids are first-order axiomatizable:*

(i) *The class* $\mathbf{A}$ *of all associative partial groupoids is defined by the sentence M and by an infinite set A' of sentences. For each $n \geq 3$, condition* (A_n) *rewrites as a finite set A'_n of sentences of the form*

$$(1) \qquad \forall\, x\, \forall\, y\, \forall\, x_1 \ldots \forall\, x_m [p v_1 v_2 x \ \wedge\ \ldots\ \wedge\ p v_{s-1} v_s y \longrightarrow x = y]$$

where $v_1, \ldots, v_s \in \{x_1, \ldots, x_m\}$;

(ii) *The class* $\mathbf{P}$ *of all partial semigroups is defined by an infinite set P' of sentences of the form* (1);

(iii) *The class* $\mathbf{R}$ *of all* R*-presemigroups is defined by the infinite set* P' *and by an infinite set* Q'_R *of sentences of the form*

$$\forall\, x_1 \ldots \forall\, x_m\, \exists\, y[pv_1v_2v_3 \;\wedge\; \ldots \;\wedge\; pv_{s-2}v_{s-1}v_s \longrightarrow pv_{s+1}v_{s+2}y]$$

where $v_1, \ldots, v_{s+2} \in \{x_1, \ldots, x_m\}$;

(iv) *The class* $\mathbf{S}$ *of all* S*-presemigroups is defined by the infinite sets* P' *and* Q'_R *and by an infinite set* Q' *of sentences of the form*

$$\forall\, x_1 \ldots \forall\, x_m\, \exists\, y_1 \ldots \exists\, y_k[pv_1v_2v_3 \;\wedge\; \ldots \;\wedge\; pv_{s-2}v_{s-1}v_s \\ \longrightarrow pv_{s+1}v_{s+2}y_1 \;\vee\; \ldots \;\vee\; pv_{s+2k-1}v_{s+2k}y_k]$$

where $v_1, \ldots, v_{s+2k} \in \{x_1, \ldots, x_m\}$. ∎

Note that the class of all associative partial groupoids, the class of all partial semigroups and the class of all R-presemigroups are quasivarieties in the sense of A.I. Malcev [44, 45]. The sentence M is a quasi-identity (in the sense of A.I. Malcev), the sentences of the form (1) are quasi-identities, and the sets P$'$ and Q'_R are equivalent to an infinite set of quasi-identities.

Theorem 1. *The following classes are not first-order finitely axiomatizable:*
(i) *The class* $\mathbf{A}$ *of all associative partial groupoids*;
(ii) *The class* $\mathbf{P}$ *of all partial semigroups*;
(iii) *The class* $\mathbf{R}$ *of all* R*-presemigroups*;
(iv) *The class* $\mathbf{S}$ *of all* S*-presemigroups.*

Proof. (i) We shall prove that the infinite set of first-order axioms for associative partial groupoids consisting of M and A'_n, $n \geq 3$, is not equivalent to any of its finite subsets. From this it follows that the class of all associative partial groupoids is not first-order finitely axiomatizable. We refer the reader to [6] for the fundamentals of first-order theories.

Let $n \geq 5$. We shall prove that conditions M and A'_k, $k = 3, \ldots, n-1$, do not imply A'_n. To prove this we construct a partial groupoid G_n whose model $M(G_n)$ satisfies A'_k, $k = 3, \ldots, n-1$ but not A'_n. On the set $G_n = \{a_1, \ldots, a_n,\, b_1, \ldots, b_{n-2},\, c_1, \ldots, c_{n-2},\, d,\, e\}$ we define a partial groupoid G_n by the multiplications

$$a_2 \cdot a_3 = b_1,\;\; a_1 \cdot b_1 = b_2,\;\; b_i \cdot a_{i+2} = b_{i+1},\;\; i = 2, \ldots, n-3\,,$$

$$b_{n-2} \cdot a_n = d,\;\; a_3 \cdot a_4 = c_1,\;\; c_j \cdot a_{j+4} = c_{j+1},\;\; j = 1, \ldots, n-4\,,$$

$$a_2 \cdot c_{n-3} = c_{n-2},\;\; a_1 \cdot c_{n-2} = e\,.$$

We denote $\Theta_0(G_n)$ by Θ_0, $\Theta(G_n)$ by Θ, and for each word w of G_n^+ we denote by $w\Theta$ the Θ-class of w. We have $a_i\Theta = \{a_i\}$, $i = 1, \ldots, n$. We shall examine the other Θ-classes containing an element of G_n by considering an oriented graph Γ as defined in P.M. Cohn [16, p. 159]. The vertices of Γ are the elements of G_n^+ and the arrows of Γ are the direct moves. Then the different Θ-classes are just the connected components of Γ. Now the connected components of Γ containing an element of G_n are as follows:

$$a_2a_3 \longrightarrow b_1\,, \quad a_1a_2a_3 \longrightarrow a_1b_1 \longrightarrow b_2\,,$$

$$\begin{tikzcd}[row sep=small] & a_1b_1a_4 \arrow[r] & b_2a_4 \arrow[r] & b_3 \\ a_1a_2a_3a_4 \arrow[ur] \arrow[dr] & & & \\ & a_1a_2c_1 & & \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & a_1b_1a_4\ldots a_{i+1} \arrow[r] & \ldots \arrow[r] & b_{i-1}a_{i+1} \arrow[r] & b_i \\ a_1a_2a_3a_4\ldots a_{i+1} \arrow[ur] \arrow[dr] & & & & \\ & a_1a_2c_1a_5\ldots a_{i+1} \arrow[r] & \ldots \arrow[r] & a_1a_2c_{i-2} & \end{tikzcd}$$

where $i = 4, \ldots, n-2$,

$$a_3a_4 \longrightarrow c_1\,,$$

$$a_3a_4a_5\ldots a_{j+3} \longrightarrow c_1a_5\ldots a_{j+3} \longrightarrow \ldots \longrightarrow c_{j-1}a_{j+3} \longrightarrow c_j$$

where $j = 2, \ldots, n-3$,

$$\begin{tikzcd}[row sep=small] & b_1a_4\ldots a_n & & & \\ a_2a_3a_4\ldots a_n \arrow[ur] \arrow[dr] & & & & \\ & a_2c_1a_5\ldots a_n \arrow[r] & \ldots \arrow[r] & a_2c_{n-3} \arrow[r] & c_{n-2} \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small] & a_1b_1a_4\ldots a_n \arrow[r] & b_2a_4\ldots a_n \arrow[r] & \ldots \arrow[r] & b_{n-2}a_n \arrow[r] & d \\ a_1a_2a_3a_4\ldots a_n \arrow[ur] \arrow[dr] & & & & & \\ & a_1a_2c_1a_5\ldots a_n \arrow[r] & \ldots \arrow[r] & a_1a_2c_{n-3} \arrow[r] & a_1c_{n-2} \arrow[r] & e \end{tikzcd}$$

We see that $\mathfrak{S}(a_1a_2\ldots a_n) = \{d, e\}$ and for each word w of G_n^+ distinct from $a_1a_2\ldots a_n$ we have $|\mathfrak{S}(w)| \leq 1$. Hence G_n satisfies (A_k), $k = 3, \ldots, n-1$ but not (A_n), whence $M(G_n)$ satisfies M and A'_k, $k = 3, \ldots, n-1$ but not A'_n. This completes the proof of (i).

(ii) The proof is similar to the proof of (i). We shall prove that the infinite set P' of first-order axioms for partial semigroups is not equivalent to any of its finite subsets.

Consider again the partial groupoid G_n defined in the proof of (i). We have seen that the Θ-class of $x \in G_n - \{d, e\}$ contains exactly one element of G_n, namely x. The sequence of elementary Θ_0-transitions of minimal length from d to e has length $2n-2$.

Since d and e belong to the same Θ-class, by Theorem A (i), the partial groupoid G_n is not a partial semigroup. Hence $M(G_n)$ does not satisfy P'. Let P'_0 be a finite subset of P'. For each quasi-identity $\psi \in P'$ we denote by $\ell_p(\psi)$ the number of occurences of the symbol p in ψ. Let $\ell_p(\psi) = n_\psi$, $\psi \in P'_0$. Choose $n \geq n_\psi$, for all $\psi \in P'_0$. Then $M(G_n)$ satisfies P'_0 but not P'.

(iii) The proof is similar to the proof of (ii). Consider again the partial groupoid G_n defined in the proof of (i). From the connected components of the oriented graph Γ we see that G_n satisfies condition $(\mathrm{Q_R})$, whence $M(G_n)$ satisfies Q'_R. Choose

$n \geq n_\psi$, for all $\psi \in P'_0$ where P'_0 is a finite subset of P' and n_ψ is the number of occurences of the symbol p in $\psi \in P'$. Then $M(G_n)$ satisfies Q'_R and P'_0 but not P'.

(iv) The proof is similar to the proof of (ii). Let Q'_0 be a finite subset of Q'. We construct a partial groupoid H_n whose model $M(H_n)$ satisfies P', Q'_R and Q'_0 but not Q'. Let G_n be the partial groupoid defined in the proof of (i) and let H_n be the relative partial subgroupoid of G_n defined by the subset $H_n = \{a_2, \ldots, a_n,\ b_1,\ c_1, \ldots, c_{n-2}\}$. Since $x\Theta(G_n) = x\Theta(H_n)$, for all $x \in H_n$, from the connected components of the oriented graph Γ we see that for each $x \in H_n - \{c_{n-2}\}$ the $\Theta(H_n)$-class of x contains exactly one H_n-irreducible word, namely x. But c_{n-2} and the H_n-irreducible word $b_1a_4 \ldots a_n$ belong to the same $\Theta(H_n)$-class. The sequence of elementary $\Theta_0(H_n)$-transitions of minimal length from c_{n-2} to $b_1a_4 \ldots a_n$ has length $n-1$.

Since c_{n-2} and the H_n-irreducible word $b_1a_4 \ldots a_n$ belong to the same $\Theta(H_n)$-class, the partial groupoid H_n does not satisfy condition (Q), whence $M(H_n)$ does not satisfy Q'. For each sentence $\psi \in Q'$ we denote by $\ell_{pS}(\psi)$ the number of occurences of the symbol p in ψ. Let $\ell_{pS}(\psi) = n_\psi$, $\psi \in Q'_0$. Choose $n > n_\psi$, for all $\psi \in Q'_0$. Then $M(H_n)$ satisfies P', Q'_R and Q'_0 but not Q'.

This completes the proof of Theorem 1. ∎

Notes. (i) In 1979 E.S. Lyapin [42] (see also [43, Chapter I, Section 5]) proved that, for $n \geq 3$, the conditions (A_n) are mutually independent. In the proof of Theorem 1 (i) we have defined the partial groupoid G_n since it is simpler than the E.S. Lyapin's one.

(ii) The *embeddability problem* for semigroups is solvable if there is an algorithm for deciding whether or not any finite partial groupoid is embeddable in a semigroup. In 1953 Trevor Evans [23] (see also [24, 25]) proved that the embeddability problem for semigroups is solvable if and only if the word problem for semigroups is solvable. T. Evans' result and the well-known result that the word problem for semigroups is unsolvable imply that the class of all partial semigroups is not first-order finitely axiomatizable (as noted in 1951 by T. Evans [22, p. 66]). The above proof of Theorem 1 (ii) does not use the unsolvability of the word problem for semigroups.

(iii) Let $\mathrm{A} = [\{S_i\ :\ \ i \in I\};\ U]$ be a semigroup amalgam. In 1975 G. Lallement [35] gave an example to show that no finite set of equational implications with existential quantifiers and with variables taken from card I distinct sets can serve as a necessary and sufficient condition for the amalgam A to be embeddable into a semigroup. A slight modification of G. Lallement's example can be used to prove that the class of all partial semigroups is not first-order finitely axiomatizable. Note that the partial groupoid G_n defined in the proof of Theorem 1 is simpler than that used in [35].

(iv) A partial groupoid G is called an *incomplete semigroup* (T. Evans [22]) if it satisfies conditions (A0), (B1), and (B2). If G is an R-presemigroup, then it is an incomplete semigroup (R. Baer [3, Section 3, Lemma 1]). In 1972 S.P. Gudder [28, p. 718] raised the question of whether the converse is true, and if not to characterize R-presemigroups. For an example of an incomplete semigroup which is not an R-presemigroup, see e.g., the partial groupoid $G(b)$ defined in Section 4. Theorem 1 (iii) shows that the class of all R-presemigroups cannot be characterized by any finite set of frist-order axioms.

(v) Theorem 1 (iv) is a corollary to Theorem 2 of [18], Proposition 1 of [2, Section 4], and Lemma 1 of [3, Section 3]. Note that the proof of Theorem 2 of [18] is based on a theorem of R. Baer [3, Section 4, Theorem] (Theorem 1 of [18]) while the above proof of Theorem 1 (iv) does not use R. Baer's theorem.

4. Deciding embeddability of partial groupoids

As already announced in the introduction, we give a theorem which includes R. Baer's theorem on U-presemigroups, formulated here as Theorem A (v).

Theorem 2. *Let $G = (G, D, \mu)$ be a partial groupoid. Then the following conditions are equivalent:*

(i) *G is an U-presemigroup;*

(ii) *$\Theta_0(G)$ is a complete rewriting system;*

(iii) *G satisfies conditions* (A0), (B1), (B2), (C1), *and* (C2).

Proof. For all $(xy, \mu(x,y)) \in \Theta_0(G)$ we have $|xy| > |\mu(x,y)|$, whence $\Theta_0(G)$ is a terminating rewriting system for each partial groupoid G.

Setting $R = \Theta_0(G)$ in conditions (i) and (ii) of Theorem B, we obtain conditions (i) and (ii) of Theorem 2. Hence, to complete our proof by Theorem B, we show that condition (iii) of Theorem B for $R = \Theta_0(G)$ is equivalent to condition (iii) of Theorem 2.

The rewriting system $\Theta_0(G)$ does not have any inclusion ambiguities. All overlap ambiguities of $\Theta_0(G)$ are of the form xyz where $(x, y, z)_D$. Let $x \cdot y = s$ and $y \cdot z = t$. Then the corresponding critical pairs of $\Theta_0(G)$ are of the form (sz, xt). We prove that all critical pairs of $\Theta_0(G)$ are resolved if and only if G satisfies conditions (A0), (B1), (B2), (C1), and (C2).

Suppose that $(x \cdot y, z)_D$. Then the critical pair (sz, xt) is resolved if and only if $(x, y \cdot z)_D$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, i.e., if and only if G satisfies conditions (B1) and (A0). Similarly, if $(x, y \cdot z)_D$, then the critical pair (sz, xt) is resolved if and only if G satisfies conditions (B2) and (A0).

Now suppose that $(x \cdot y, z) \notin D$. The critical pair (sz, xt) is resolved if and only if $(x, y \cdot z) \notin D$, $x \cdot y = x$ and $y \cdot z = z$, i.e., if and only if G satisfies condition (C1). Similarly, if $(x, y \cdot z) \notin D$, then the critical pair (sz, xt) is resolved if and only if G satsifies condition (C2).

This completes the proof of Theorem 2. ■

We shall give some examples of U-presemigroups.

Example 1. Any set X can be considered as an U-presemigroup with empty set of multiplications. Then X^+ is the universal semigroup of X.

Example 2. Let $\{S_i : i \in I\}$ be a family of semigroups such that $S_i \cap S_j = \emptyset$ for all $i, j \in I$ with $i \neq j$. We define a partial groupoid G on the set $\bigcup\{S_i : i \in I\}$ in which a product of two elements is defined if and only if they both belong to the same S_i and their product is then taken as their product in S_i. Then the partial groupoid G is an U-presemigroup and the universal semigroup of G is the free product of the semigroups S_i, $i \in I$.

Example 3. Let $\{S_i : i \in I\}$ be a family of semigroups, let U be a subsemigroup of S_i for all $i \in I$ and let $S_i \cap S_j = U$ for all $i, j \in I$ with $i \neq j$. The semigroup amalgam $\mathrm{A} = [\{S_i : i \in I\}; U]$ determines a partial groupoid G(A) (see [14, Section 9.4]). A description of semigroup amalgams whose partial groupoids are U-presemigroups is due to E.S. Lyapin [38, 39, 40] (for the case $|I| = 2$). For a short proof see [19, Section 5]. As a special case, if G(A) satisfies condition (A0) and U is an ideal of S_i for all $i \in I$, then G(A) is an U-presemigroup.

Example 4. Let $\{S_i : i \in I\}$ be a family of semigroups, let $\{U_{ij} : i, j \in I,\ i \neq j\}$ be a family of semigroups and let $S_i \cap S_j = U_{ij}$ for all $i, j \in I$ with $i \neq j$. Then $\mathrm{A_g} = [\{S_i : i \in I\}; \{U_{ij} : i, j \in I,\ i \neq j\}]$ is called a *generalized semigroup amalgam* (E.S. Lyapin [41]). The generalized semigroup amalgam $\mathrm{A_g}$ determines a partial groupoid $\mathrm{G(A_g)}$ on the set $\bigcup\{S_i : i \in I\}$ in which a product of two elements is defined if and only if they both belong to the same S_i and their product is then taken as their product in S_i. In 1992 L.V. Lobodina [37, Theorem] proved that if the partial groupoid $\mathrm{G(A_g)}$ satisfies condition (A0) and U_{ij} is an ideal of S_i and of S_j for all $i, j \in I$ with $i \neq j$, then $\mathrm{G(A_g)}$ is a partial semigroup. One can easily verify that $\mathrm{G(A_g)}$ is actually an U-presemigroup. This gives a new proof of L.V. Lobodina's theorem.

Example 5. In 1978 K. Byleen, J. Meakin, and F. Pastijn [10] introduced the four-spiral semigroup Sp_4. Let G be the partial groupoid whose multiplication table is

G	a	b	c	d
a	a	b	$*$	$*$
b	a	b	b	$*$
c	$*$	c	c	d
d	d	$*$	c	d

.

Then G is an U-presemigroup whose universal semigroup is Sp_4.

Example 6. Let G be the partial groupoid on the set $\{1, p, q\}$ defined by the multiplications $1 \cdot 1 = 1$, $1 \cdot p = p \cdot 1 = p$, $1 \cdot q = q \cdot 1 = q$, $p \cdot q = 1$. Then G is an U-presemigroup whose universal semigroup is the bicyclic semigroup $C(p, q)$.

Example 7. In 1980 K. Byleen, J. Meakin, and F. Pastijn [11] introduced a family of semigroups $\bar{A}(\alpha, \beta)$, $\alpha \geq 1$, $\beta \geq 1$, which may be considered to be generalizations of the bicyclic semigroup $C(p, q)$ which is isomorphic to $\bar{A}(1, 1)$. Let $G(\alpha, \beta)$ be the partial groupoid on the set $\{1, p_1, \ldots, p_\alpha,\ q_1, \ldots, q_\beta\}$ defined by the multiplications $1 \cdot 1 = 1$, $1 \cdot p_j = p_j \cdot 1$, $1 \cdot q_k = q_k \cdot 1 = q_k$, $p_j \cdot q_k = 1$, $j \in \{1, \ldots, \alpha\}$, $k \in \{1, \ldots, \beta\}$. Then $G(\alpha, \beta)$ is an U-presemigroup whose universal semigroup is $\bar{A}(\alpha, \beta)$.

We introduce the following conditions on a partial groupoid G:

(G1) There exists an identity element $1 \in G$ such that for all $x \in G$, $(x, 1)_D$, $(1, x)_D$ and $x \cdot 1 = 1 \cdot x = x$.

(G2) For each $x \in G$ there exists $x^{-1} \in G$ such that $(x, x^{-1})_D$, $(x^{-1}, x)_D$ and $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

(G3) For all $x, y, z \in G$, if $(x, y, z)_D$ and $y \neq 1$, then $(x \cdot y, z)_D$.

(G4) For all $x, y, z \in G$, if $(x, y, z)_D$ and $y \neq 1$, then $(x, y \cdot z)_D$.

We say that an U-presemigroup is an U*-pregroup* if it satisfies conditions (G1) and (G2). Note that if G is an U-pregroup, then the universal semigroup $U(G)$ of G is actually a group (R. Baer [2, Section 4, Proposition 1]).

Theorem 3 below was proven by R.J. Stallings [51] (see also F. Rimlinger [49]). It can be proven in the same way as Theorem 2, i.e., one can show that Theorem 3 is a special case of Theorem B. Here we shall give a proof of Theorem 3 as a corollary to Theorem A (v).

Theorem 3. *Let $G = (G, D, \mu)$ be a partial groupoid. Then G is an U-pregroup if and only if it satisfies conditions* (A0), (G1), (G2), (G3), *and* (G4).

Proof. Suppose that G satisfies (G1) and (G2). We shall prove that G satisfies (C1) if and only if it satisfies (G3).

Suppose that G satisfies (C1). Suppose $(x, y, z)_D$ and $y \neq 1$. Suppose, be way of contradiction, that G does not satisfy (G3), i.e., suppose that $(x \cdot y, z) \notin D$. Then, by (C1), $x \cdot y = x$, whence, by (G1) and (G2), $y = 1$. A contradiction. Hence (G3) holds. Now suppose that G satisfies (G3). Suppose $(x, y, z)_D$ and $(x \cdot y, z) \notin D$. We have two cases: either $y = 1$ or $y \neq 1$. In the first case (C1) holds, by (G1). In the second case (G3) implies $(x \cdot y, z)_D$. A contradiction. Hence (C1) holds.

Similarly, if G satisfies (G1) and (G2), then G satisfies (C2) if and only if it satisfies (G4).

Next we shall prove that conditions (G1), (G3), and (G4) imply (B1) and (B2). Suppose $(x, y, z)_D$. We have two cases: either $y = 1$ or $y \neq 1$. In the first case (G1) implies (B1) and (B2). In the second case (G3) and (G4) imply (B1) and (B2).

We use Theorem A (v) to complete the proof of Theorem 3. ∎

P.W. Bunting, J. van Leeuwen, and D. Tamari have shown in [8] that the following partial groupoids $G(x)$ are partial semigroups:

$G(a)$	a	b	c
a	$*$	$*$	b
b	$*$	c	$*$
c	b	$*$	a

$G(b)$	a	b	c
a	$*$	$*$	b
b	$*$	a	$*$
c	b	$*$	b

$G(c)$	a	b	c
a	$*$	$*$	b
b	$*$	c	$*$
c	b	$*$	b

$G(d)$	a	b	c
a	$*$	c	b
b	c	$*$	a
c	b	a	$*$

$G(e)$	a	b	c
a	a	$*$	c
b	b	c	$*$
c	c	$*$	a

$G(f)$	a	b	c
a	a	b	c
b	b	a	$*$
c	c	$*$	a

$G(g)$	a	b	c
a	a	b	c
b	b	a	$*$
c	c	$*$	b

$G(h)$	a	b	c
a	b	$*$	$*$
b	$*$	c	$*$
c	$*$	$*$	a

$G(i)$	a	b	c
a	$*$	$*$	a
b	c	c	b
c	b	b	c

$G(j)$	a	b	c
a	a	$*$	a
b	b	c	b
c	c	b	c

$G(k)$	a	b	c
a	a	$*$	a
b	c	b	c
c	c	b	c

$G(\ell)$	a	b	c
a	b	$*$	$*$
b	$*$	a	a
c	$*$	a	a

$G(m)$	a	b	c
a	a	a	a
b	a	c	$*$
c	a	$*$	b

$G(n)$	a	b	c
a	$*$	$*$	a
b	$*$	$*$	b
c	b	a	$*$

$G(o)$	a	b	c
a	$*$	b	c
b	c	b	c
c	b	b	c

$G(p)$	a	b	c
a	a	$*$	c
b	$*$	c	c
c	c	c	c

$G(2)$	a	b
a	b	$*$
b	$*$	a

.

We give a new proof of these results of [8] and further information about the considered partial semigroups:

Theorem 4. (i) *All partial groupoids $G(x)$ introduced above are partial semigroups*;

(ii) $G(f), G(j), G(k), G(o)$, *and* $(G(p)$ *are U-presemigroups*;

(iii) $G(n)$ *is an L-presemigroup but not an U-presemigroup*;

(iv) $G(b)$ *and* $G(c)$ *are not R-presemigroups*;

(v) *The remaining 9 partial semigroups are R-presemigroups, but not S-presemigroups.*

Proof. By the aid of Theorem A, one verifies in a straightforward manner that the partial groupoids $G(f)$, $(G(j)$, $G(k)$, $G(o)$, and $G(p)$ are U-presemigroups and $G(n)$ is an L-presemigroup.

Let $S(b)$ and $S(c)$ be the semigroups whose multiplication tables are:

$S(b)$	a	b	c	d
a	b	d	b	a
b	d	a	d	b
c	b	d	b	a
d	a	b	a	d

$S(c)$	a	b	c	d
a	b	d	b	c
b	d	c	d	b
c	b	d	b	c
d	c	b	c	d

.

Let $G_R(b)$ and $G_R(c)$ be the relative partial subgroupoids of $S(b)$ and $S(c)$, respectively, defined by the subset $\{a, b, c\}$. If we delete the multiplication $a \cdot a = b$ from the multiplication tables of $G_R(b)$ and $G_R(c)$, then we obtain the multiplication tables of $G(b)$ and $G(c)$, respectively. Hence $G(b)$ and $G(c)$ are partial semigroups.

Henceforth we denote by $C_n = \{1, a, a^2, \ldots, a^{n-1}\}$ the cyclic group of order n. Let $G_R(e)$ be the relative partial subgroupoid of C_4 defined by the subset $\{1, a, a^2\}$. If we delete the multiplication $1 \cdot a = a$ from the multiplication table of $G_R(e)$, then we obtain a partial groupoid which we denote by $G'(e)$. The mapping of $G'(e)$ onto $G(e)$ given by $1 \mapsto a,\ \ a \mapsto b,\ \ a^2 \mapsto c$ is an isomorphism. Hence $G(e)$ is a partial semigroup.

We prove that the other partial groupoids are R-presemigroups. Let $S(i)$ and $S(\ell)$ be the semigroups whose multiplication tables are:

$$
\begin{array}{c|cccc}
S(i) & a & b & c & d \\ \hline
a & d & d & a & a \\
b & c & c & b & b \\
c & b & b & c & c \\
d & a & a & d & d
\end{array}
\qquad\qquad
\begin{array}{c|cccc}
S(\ell) & a & b & c & d \\ \hline
a & b & d & d & a \\
b & d & a & a & b \\
c & d & a & a & b \\
d & a & b & b & d
\end{array}
.
$$

Then $G(i)$ and $G(\ell)$ are the relative partial subgroupoids of $S(i)$ and $S(\ell)$, respectively, defined by the subset $\{a, b, c\}$. Let $G_R(2)$ be the relative partial subgroupoid of C_3 defined by the subset $\{a, a^2\}$. Then the mapping of $G_R(2)$ onto $G(2)$ given by $a^2 \mapsto a,\ \ a \mapsto b$ is an isomorphism. Let C_3^0 denote the semigroup obtained from C_3 by adjoining a zero element 0 and let $G_R(m)$ be the relative partial subgroupoid of G_3^0 defined by the subset $\{0, a, a^2\}$. Then the mapping of $G_R(m)$ onto $G(m)$ given by $0 \mapsto a,\ \ a \mapsto b,\ \ a^2 \mapsto c$ is an isomorphism. Let $G_R(a)$ be the relative partial subgroupoid of C_5 defined by the subset $\{a, a^3, a^4\}$. Then the mapping of $G_R(a)$ onto $G(a)$ given by $a \mapsto a,\ \ a^4 \mapsto b,\ \ a^3 \mapsto c$ is an isomorphism. Let $G_R(d)$ be the relative partial subgroupoid of $C_2 \oplus C_2 = \{1, a, b, ab\}$ defined by the subset $\{a, b, ab\}$. Then the mapping of $G_R(d)$ onto $G(d)$ given by $a \mapsto a,\ \ b \mapsto b,\ \ ab \mapsto c$ is an isomorphism. Let $G_R(g)$ be the relative partial subgroupoid of C_4 defined by the subset $\{1, a, a^2\}$. Then the mapping of $G_R(g)$ onto $G(g)$ given by $1 \mapsto a,\ \ a^2 \mapsto b,\ \ a \mapsto c$ is an isomorphism. Finally, let $G_R(h)$ be the relative partial subgroupoid of C_7 defined by the subset $\{a, a^2, a^4\}$. Then the mapping of $G_R(h)$ onto $G(h)$ given by $a \mapsto a,\ \ a^2 \mapsto b,\ \ a^4 \mapsto c$ is an isomorphism. This completes the proof of (i).

Next we show that the partial groupoid $G(n)$ is not an U-presemigroup. Indeed, we have $(a, c, a)_D$ and $(a, c \cdot a) \notin D$ but $c \cdot a \neq a$. Hence $G(n)$ does not satisfy condition (C2), whence, by Theorem A (v), $G(n)$ is not an U-presemigroup.

The partial groupoids $G(b)$ and $G(e)$ are not R-presemigroups. We prove that $G(b)$ is not an R-presemigroup. We have

$$
ccaca \begin{array}{l} \nearrow\ baca \longrightarrow bba \longrightarrow aa \\[1em] \searrow\ ccab \longrightarrow cbb \longrightarrow ca \longrightarrow b \end{array}
$$

so that $aa \sim b \pmod{\Theta(G(b))}$ but aa is a $G(b)$-irreducible word. Hence $G(b)$ does not satisfy condition $(\mathrm{Q_R})$, whence, by Theorem A (ii), $G(b)$ is not an R-presemigroup. Similarly, for the partial groupoid $G(e)$ we have

$$
bbbbb \begin{array}{l} \nearrow\ cbbb \longrightarrow ccb \longrightarrow ab \\[1em] \searrow\ bbbc \longrightarrow bcc \longrightarrow ba \longrightarrow b \end{array}
.
$$

Hence $G(e)$ is not an R-presemigroup.

The remaining partial groupoids $G(x)$, $\quad x \in \{a, c, d, g, h, i, \ell, m\}$ and $G(2)$ are R-presemigroups but not S-presemigroups. We have proved that these partial groupoids are R-presemigroups, except for $G(c)$. We prove that $G(c)$ is actually an R-presemigroup. To prove this, we use the above defined semigroup $S(c)$. If we delete the multiplication $a \cdot a = b$ from the multiplication table of $S(c)$, then we obtain a partial semigroup which we denote by $G'(c)$. We denote the universal semigroup of $G'(c)$ by $U(c)$, and $\Theta(G'(c))$ by Θ. Since $G'(c)$ is a partial semigroup, by Theorem A (i), the Θ-classes $a\Theta$, $b\Theta$ and $c\Theta$ are mutually disjoint. Let $G'_R(c)$ be the relative partial subgroupoid of $U(c)$ defined by the subset $\{a\Theta, b\Theta, c\Theta\}$. The mapping of $G'_R(c)$ onto $G(c)$ given by $a\Theta \mapsto a$, $\; b\Theta \mapsto b$, $\; c\Theta \mapsto c$ is an isomorphism. Hence $G(c)$ is an R-presemigroup.

We prove that $G(2)$ is not an S-presemigroup. We have

$$\begin{tikzcd} & baa \arrow[r] & bb \arrow[r] & a \\ aaaa \arrow[ur] \arrow[dr] & & & \\ & aba & & \end{tikzcd}$$

so that $aba \sim a \pmod{\Theta(G(2))}$ but aba is a $G(2)$-irreducible word. Hence $G(2)$ does not satisfy condition (Q), whence, by Theorem A (iii), $G(2)$ is not an S-presemigroup. Similarly, one can prove that $G(x)$, $\quad x \in \{a, c, d, g, h, i, \ell, m\}$ is not an S-presemigroup. This completes the proof of Theorem 4. ∎

Notes. (i) We refer the reader to A.H. Clifford [12, 13] (or to A.E. Evseev [26, Section 5]) for the definition of a warp. For any semigroup S we denote by $E(S)$ the set of all idempotents of S. If $E(S) \neq \emptyset$, then the relative partial subgroupoid of S defined by the subset $E(S)$ is called a *partial band* (G.R. Baird [5]). Every partial band is a warp (A.H. Clifford [13, Theorem 1.1]). In 1974 A.H. Clifford [12] raised the question of whether the converse is true. The partial groupoid $G(k)$ can serve as a simple example of a warp which is not a partial band. One can verify that $G(k)$ is a warp. Suppose, by way of contradiction, that $G(k)$ is a partial band, i.e., suppose that there exists a semigroup S such that $E(S) = \{a, b, c\}$ and $G(k)$ is the relative partial subgroupoid of S defined by the subset $\{a, b, c\}$. Since $abab = ab$ in S, ab is an idempotent of S, whence the product $a \cdot b$ is defined in $G(k)$, contrary to the definition of $G(k)$. Hence $G(k)$ is not a partial band.

(ii) We say that a partial groupoid G is a T-*presemigroup* if its multiplication table can be completed to a multiplication table of a semigroup.

In 1974 E.S. Lyapin [40, p. 142] (see also E.S. Lyapin and A.E. Evseev [43, p. 112, Example 2]) gave an example of an R-presemigroup which is not a T-presemigroup. The partial groupoid $G(2)$ can serve as another example. We have proved that $G(2)$ is an R-presemigroup. We prove that $G(2)$ is not a T-presemigroup. Suppose, by way of contradiction, that $G(2)$ is a T-presemigroup. Then either $a \cdot b = a$ or $a \cdot b = b$. In both cases we have $\mathfrak{S}(aab) = \{a, b\}$, contrary to the hypothesis that $G(2)$ is a T-presemigroup.

In 1991 E.S. Lyapin and A.E. Evseev [43, p. 112, Example 3] gave an example of a partial semigroup which is not an R-presemigroup and not a T-presemigroup. The partial groupoid $G(e)$ can serve as another example. We have proved that $G(e)$ is not an R-presemigroup. We prove that $G(e)$ is not a T-presemigroup. Suppose, by way of contradiction, that $G(e)$ is a T-presemigroup. Then either (i) $b \cdot c = a$, or (ii) $b \cdot c = b$, or (iii) $b \cdot c = c$. In all three cases we have $|\mathfrak{S}(bbc)| = 2$, contrary to the hypothesis that $G(e)$ is a T-presemigroup.

(iii) In 1966 S.I. Adjan [1] proved that a semigroup defined by a cycle-free presentation is embeddable in a group. S.I. Adjan gave an example [1, Chapter 2, Example 3] which shows that the condition is not necessary. Independently, in 1963 G.C. Bush [9] (see also P.M. Higgins [31, p. 185]) gave another example to show that the Adjan's condition is not necessary. We can use the partial groupoid $G(2)$ to construct a simple example. Indeed, the semigroup presentation $\mathbb{P} = sgp(a, b;\ aa = b,\ bb = a)$ has a left cycle and a right cycle, but the semigroup defined by $\mathbb{P}$ is actually a group, namely, the cyclic group of order 3. Clearly, if we use a full groupoid instead of a partial one, then the simplest example is the presentation $\mathbb{P}_1 = sgp(a;\ aa = a)$ of the one-element semigroup which is simultaneously the one-element group. Obviously the presentation $\mathbb{P}_1$ has a left cycle and a right cycle.

(iv) Bunting, van Leeuwen, and Tamari [8, p. 598] stated that the partial groupoid $G(p)$ is not a T-pesemigroup. (For the definition of a T-presemigroup see note (ii) above). The multiplication table of $G(p)$ can be completed to a multiplication table of a semigroup by defining $a \cdot b = b \cdot a = c$, whence $G(p)$ is a T-presemigroup which indicates that it should be some mistake in the computer program used in [8].

5. Applying the Knuth-Bendix completion procedure

In this section we introduce the use of the Knuth-Bendix completion procedure for deciding embeddability of finite partial groupoids into semigroups. We illustrate the use of the Knuth-Bendix completion procedure by means of two examples. First we consider the partial groupoid determined by the Ash's semigroup amalgam. Secondly, we consider again the 17 Bunting, van Leeuwen, and Tamari's partial groupoids.

It is known (B.H. Neumann [46, p. 532]) that a finite group amalgam is always embeddable in a finite group. In 1964 J.M. Howie [32, p. 55] raised the question of whether there exists a finite semigroup amalgam which is embeddable in a semigroup but not in a finite semigroup. In 1975 T.E. Hall [29, p. 377] gave an example (attributed by T.E. Hall to C.J. Ash) to show that the answer is in the positive. Here we shall give a simple proof of this result.

Let G be the partial groupoid whose multiplication table is

G	0	e	f	a	b	g	x	y
0	0	0	0	0	0	0	0	0
e	0	e	0	a	0	0	x	0
f	0	0	f	0	b	g	0	y
a	0	0	a	0	e	$*$	$*$	$*$
b	0	b	0	f	0	$*$	$*$	$*$
g	0	0	g	$*$	$*$	g	0	y
x	0	0	x	$*$	$*$	x	0	e
y	0	y	0	$*$	$*$	0	g	0

.

Let S, T, and U be the relative partial subgroupoids of G defined by the subsets $\{0, e, f, a, b\}$, $\{0, e, f, g, x, y\}$, and $\{0, e, f\}$, respectively. Then S, T, and U are semigroups such that $S \cap T = U$. The multiplication tables of S and T are given in [33, p. 252]. Clearly, G is the partial groupoid determined by the semigroup amalgam $\mathrm{A} = [S, T; U]$. The partial groupoid G is embeddable in a semigroup since A is an inverse semigroup amalgam (T.E. Hall [29]; see also J.M. Howie [33, Chapter VII, Section 4]).

We shall use the Knuth-Bendix completion procedure to give a short proof that G is a partial semigroup. The rewriting system $\Theta_0(G)$ is not complete. If we apply the Knuth-Bendix completion procedure to $\Theta_0(G)$, then we obtain the following complete rewriting system:

$$\Theta_0(H) = \Theta_0(G) \cup \{ax \longrightarrow 0,\ \ bg \longrightarrow 0,\ \ by \longrightarrow 0,\ \ ga \longrightarrow 0,\ \ xa \longrightarrow 0,\ \ yb \longrightarrow 0\}.$$

Next we use $\Theta_0(H)$ to extend the multiplications of G by defining $a \cdot x = 0,\ \ b \cdot g = 0,\ \ b \cdot y = 0,\ \ g \cdot a = 0,\ \ x \cdot a = 0,\ \ y \cdot b = 0$. We denote by H the obtained partial groupoid. By Theorem 2, H is an U-presemigroup. Clearly, G embeds in H, whence G is a partial semigroup.

We prove that the partial groupoid G is not embeddable in a finite semigroup. The rewriting systems $\Theta_0(G)$ and $\Theta_0(H)$ are equivalent, i.e., they generate the same congruence on G^+. Hence the universal semigroup $U(G)$ of G coincides with the universal semigroup $U(H)$ of H. We denote $\Theta(G) = \Theta(H)$ by Θ. Since H is an U-presemigroup, the Θ-classes $e\Theta$, $xb\Theta$, and $ay\Theta$ are mutually disjoint. Let E be the relative partial subgroupoid of $U(G)$ defined by the subset $\{e\Theta,\ xb\Theta,\ ay\Theta\}$. Let F be the relative partial subgroupoid of the bicyclic semigroup $C(p,q)$ defined by the subset $\{1, p, q\}$. Then the mapping of E onto F given by $e\Theta \mapsto 1,\ xb\Theta \mapsto p,\ ay\Theta \mapsto q$ is an isomorphism. Hence $U(G)$ contains a copy of $C(p,q)$, whence, by Corollary 1.32 of [14], G cannot be embedded in a finite semigroup.

Consider again the Bunting, van Leeuwen, and Tamari's partial groupoids $G(x)$ of Section 4. For each of them we denote the rewriting system $\Theta_0(G(x))$ by $\Theta_0(x)$. For example, $\Theta_0(2) = \{aa \longrightarrow b,\ bb \longrightarrow a\}$. Define the rewriting systems

$$\begin{aligned}
R_1(2) &= \{ab \longrightarrow ba\},\\
R_1(a) &= \{ab \longrightarrow ba,\ \ aa \longrightarrow cb,\ \ bc \longrightarrow cb,\ \ cba \longrightarrow c\},\\
R_1(b) &= \{ab \longrightarrow cb,\ \ ba \longrightarrow cb,\ \ bc \longrightarrow cb,\ \ aa \longrightarrow b\},\\
R_1(c) &= \{ab \longrightarrow cb,\ \ ba \longrightarrow cb,\ \ bc \longrightarrow cb\},\\
R_1(d) &= \{aa \longrightarrow cc,\ \ bb \longrightarrow cc,\ \ ccc \longrightarrow c\},\\
R_1(e) &= \{bc \longrightarrow cb,\ \ ab \longrightarrow b\},\\
R_1(g) &= \{bc \longrightarrow cb\},\\
R_1(h) &= \{ab \longrightarrow ba,\ \ ac \longrightarrow ca,\ \ bc \longrightarrow cb\},\\
R_1(i) &= \{aa \longrightarrow ab\},\\
R_1(\ell) &= \{ab \longrightarrow ca,\ \ ac \longrightarrow ca,\ \ ba \longrightarrow ca\},\\
R_1(m) &= \{bc \longrightarrow cb\},\\
R_1(n) &= \{aa \longrightarrow ab,\ \ ba \longrightarrow bb\}.
\end{aligned}$$

Put $\Lambda_0 = \{f, j, k, o, p\}$ and $\Lambda_1 = \{a, b, c, d, e, g, h, i, \ell, m, n, 2\}$ and denote $R(x) = \Theta_0(x) \cup R_1(x),\ \ x \in \Lambda_1$.

Proposition 2. *For $x \in \Lambda_0$, the rewriting system $\Theta_0(x)$ is complete. For $x \in \Lambda_1$, the rewriting system $R(x)$ is a Knuth-Bendix completion of the rewriting system $\Theta_0(x)$.*

Proof. For $x \in \Lambda_0$, we use the length-reducing ordering on $\{a,b,c\}^*$.For $x \in \Lambda_1$, we use the length-plus-lexicographic ordering on $\{a,b,c\}^*$ induced by the ordering $a > b > c$. The rest of the proof is left to the reader or the reader's favourite completion program. ■

Now we give a second proof of Theorem 4 (i), based on Proposition 2.

For $x \in \Lambda_0$, Porposition 2 and Theorem 2 imply that $G(x)$ is an U-presemigroup. Let $x \in \Lambda_1$. Since each element of $G(x)$ is a $R(x)$-irreducible word and since, by Proposition 2, $R(x)$ is a complete rewriting system, by Theorem B, distinct elements of $G(x)$ belong to distinct $\Theta(R(x))$-classes, and consequently, to distinct $\Theta(G(x))$-classes. Hence, by Theorem A (i), $G(x)$ is a partial semigroup.

Notes. Let C_n be the cyclic group of order n and let $U(x)$ be the universal semigroup of $G(x)$. It is easy to see that $U(f)$, $U(c)$, $U(n)$, $U(o)$, and $U(p)$ are infinite semigroups. Actually $U(f)$ is an infinite group, namely $U(f) \cong C_2 * C_2$. The universal semigroups of the remaining partial groupoids are finite, so that, by using the corresponding complete rewriting system, we can easily write their multiplication tables. The rewriting system $R(2)$ has three $R(2)$-irreducible words (distinct form the empty word λ), namely a, b, and ba, so that $U(2)$ is a semigroup of order 3. Set $ba = c$. Then $U(2)$ has multiplication table

$U(2)$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

.

We can easily write the above multiplication table by using the complete rewriting system $R(2)$. E.g., $ac = aba \longrightarrow baa \longrightarrow bb \longrightarrow a$. The mapping of $U(2)$ onto C_3 given by $a \mapsto a, \quad b \mapsto a^2, \quad c \mapsto 1$ is an isomorphism. Similarly, $U(a) \cong C_5, \quad U(d) \cong C_2 \oplus C_2, \quad U(e) \cong U(g) \cong C_4, \quad U(h) \cong C_7, \quad U(m) \cong C_3^0$. Let $S(b)$, $S(i)$, and $S(\ell)$ are the semigroups whose multiplication tables are given in Section 4. Then $U(b) \cong S(b)$, $U(i) \cong S(i)$, and $U(\ell) \cong S(\ell)$. If we set $ab = d$, then the multiplication tables of $U(j)$ and $U(k)$ are:

$U(j)$	a	b	c	d
a	a	d	a	d
b	b	c	b	c
c	c	b	c	b
d	d	a	d	a

$U(k)$	a	b	c	d
a	a	d	a	d
b	c	b	c	b
c	c	b	c	b
d	a	d	a	d

.

Acknowledgements

The author would like to thank the referee for various improvements in the paper. The author would also like to thank Professors D. Skordev, I. Dimovski, and S. Dodunekov for their help and encouragement.

References

[1] Adjan, S. I., "Defining relations and algorithmic problems for groups and semigroups", Trudy Mat. Inst. Steklov **vol. 85**, Nauka, Moscow, 1966; English translation, Proc. Steklov Inst. Math. **vol. 85**, Amer. Math. Soc., Providence, R.I., 1967.

[2] Baer, R., *Free sums of groups and their generalizations. An analysis of the associative law*, Amer. J. Math. **71** (1949), 706–742.

[3] Baer, R., *Free sums of groups and their generalizations II*, Amer. J. Math. **72** (1950), 625–646.

[4] Baer, R., *Free sums of groups and their generalizations III*, Amer. J. Math. **72** (1950), 647–670.

[5] Baird, G. R., *On semigroups and uniform partial bands*, Semigroup Forum **4** (1972), 185–188.

[6] Barwise, J., editor, "Handbook of Mathematical Logic", North-Holland, Amsterdam, 1977.

[7] Benninghofen, B., S. Kemmerich, and M. M. Richter, "Systems of Reductions", Lecture Notes in Computer Science **vol. 277**, Springer-Verlag, Berlin, 1987.

[8] Bunting, P. W., J. van Leeuwen, and D. Tamari, *Deciding associativity for partial multiplication tables of order 3*, Math. of Computation **32** (1978), 593–605.

[9] Bush, G. C., *Note on an embedding theorem of Adyan*, Proc. Amer. Math. Soc. **14** (1963), 597-599.

[10] Byleen, K., J. Meakin, and F. Pastijn, *The fundamental four-spiral semigroup*, J. Algebra **54** (1978), 6–26.

[11] Byleen, K., J. Meakin, and F. Pastijn, *The double four-spiral semigroup*, Simon Stevin, **54** (1980), no. 2, 75–105.

[12] Clifford, A. H., "The fundamental representation of a regular semigroup", Math. Dept. of Tulane University, 1974.

[13] Clifford, A. H., *The fundamental representation of a regular semigroup*, Semigroup Forum **10** (1975), 84–92.

[14] Clifford, A. H., and G. B. Preston, "The Algebraic Theory of Semigroups", Mathematical Surveys, no. 7, Amer. Math. Soc., Providence, R.I., **vol. I**, 1961, **vol. II**, 1967.

[15] Cohen, D. E., *String rewriting. A survey for group theorists*, Geometric Group Theory, vol. 1 (Niblo, G.A., and Roller, M.A., editors), Cambridge University Press, Cambridge, 1993, 37–47.

[16] Cohn, P. M., "Universal Algebra", Harper and Row, New York, 1965.

[17] Dekov, D. V., *The embedding of semigroup amalgams*, J. Algebra **141** (1991), 158–161.

[18] Dekov, D. V., *The class of all S-pregroups is not finitely axiomatizable*, Proc. Amer. Math. Soc. **115** (1992), 895–897.

[19] Dekov, D. V., *Free products with amalgamation of semigroups*, Semigroup Forum **46** (1993), 54–61.

[20] Dekov, D. V., *HNN extensions of semigroups*, Semigroup Forum **49** (1994), 83–87.

[21] Dekov, D. V., *Embeddability and the word problem*, J. Symbolic Logic **60** (1995), 1194–1198.

[22] Evans, T., *The word problem for abstract algebras*, J. London Math. Soc. **26** (1951), 64–71.

[23] Evans, T., *Embeddability and the word problem*, J. London Math. Soc. **28** (1953), 76–80.

[24] Evans, T., *Word problems*, Bull. Amer. Math. Soc. **84** (1978), 789–802.

[25] Evans, T., *Some solvable word problems*, Word Problems II (Adian, S.I., Boone, W.W., and Higman, G., eds.), North-Holland, Amsterdam, 1980, 87–100.

[26] Evseev, A. E., *A survey of partial groupoids*, Properties of Semigroups (Lyapin, E.S., ed.), Leningrad. Gos. Ped. Inst., Leningrad, 1984, 39-76 (Russian); English translation in Amer. Math. Soc. Transl. (2) **139** (1988), 43–67.

[27] Gensemer, S. H., and H. J. Weinert, *On the embedding of partial groupoids into semigroups*, Bayreuther Math. Schr. **28** (1989), 139–163.

[28] Gudder, S. P., *Partial algebraic structures associated with orthomodular posets*, Pacific J. Math. **41** (1972), 717–730.

[29] Hall, T. E., *Free products with amalgamation of inverse semigroups*, J. Algebra **34** (1975), 375–385.

[30] Hermiller, S. M., *Rewriting systems for Coxeter groups*, J. Pure Appl. Algebra **92** (1994), 137-148.

[31] Higgins, P. M., "Techniques of Semigroup Theory", Oxford University Press, Oxford, 1992.

[32] Howie, J. M., *The embedding of semigroup amalgams*, Quart. J. Math. Oxford (2) **15** (1964), 55–68.

[33] Howie, J. M., "An Introduction to Semigroup Theory", Academic Press, London, 1976.

[34] Knuth, D. E., and P. B. Bendix, *Simple word problems in universal algebra*, Computational Problems in Abstract Algebra (Leech, J., ed.) Pergamon Press, Oxford, 1970, 263–297.

[35] Lallement, G., *Amalgamated products of semigroups: The embedding problem*, Trans. Amer. Math. Soc. **206** (1975), 375–394.

[36] LeChenadec, Ph., "Canonical Forms in Finitely Presented Algebras", Pitman, London and Wiley, New York, 1986.

[37] Lobodina, L. V., *The embedding of generalized semigroup amalgams with ideal intersections*, Partitions and Homomorphisms of Semigroups (Lyapin, E.S., ed.), Ross. Gos. Ped. Inst., St.-Petersburg, 1992, 88–99 (Russian).

[38] Lyapin, E. S., *The independence of subsemigroups of a semigroup*, Dokl. Akad. Nauk SSSR **185** (1969), 1229-1231 (Russian); English translation, Soviet Math. Dokl. **10** (1969), 492–494.

[39] Lyapin, E. S., *Intersections of independent subsemigroups of a semigroup*, Izv. Vyssh. Uchebn. Zaved. Matematika, no. 4 (95) (1970), 67–73 (Russian).

[40] Lyapin, E. S., *Problems of the theory of semigroup extensions of partial groupoids*, Modern Algebra, vyp. 1 (Lyapin, E.S., ed.), Leningrad. Gos. Ped. Inst., Leningrad, 1974, 130–145 (Russian).

[41] Lyapin, E. S., *Independent semigroup extensions of partial groupoids*, Modern Algebra, vyp. 2 (Lyapin, E.S., ed.), Leningrad. Gos. Ped. Inst., Leningrad, 1974, 55–71 (Russian).

[42] Lyapin, E. S., *Weak associativity of partial operations*, Semigroup Varieties and Semigroups of Endomophisms (Lyapin, E.S., ed.), Leningrad. Gos. Ped. Inst., Leningrad, 1979, 95–112 (Russian).

[43] Lyapin, E. S., and A. E. Evseev, "Partial Groupoids", Ross. Gos. Ped. Inst., St.-Petersburg, 1991 (Russian).

[44] Malcev, A. I., *A few remarks on quasivarieties of algebraic systems*, Algebra i Logika **5** (1966), no. 3, 3-9 (Russian); English translation in Malcev, A.I., "The Metamathematics of Algebraic Systems", North-Holland, Amsterdam, 1971, 416–421.

[45] Malcev, A. I., "Algebraic Systems", Nauka, Moscow, 1970 (Russian); English translation, Springer-Verlag, New York, 1973.

[46] Neumann, B. H., *An essay on free products of groups with amalgamations*, Philos. Trans. Royal Soc. London, Ser. A, **246** (1954), 503–554.

[47] Newman, M. H. A., *On theories with a combinatorial definition of "equivalence"*, Ann. of Math. **43** (1942), 223–243.

[48] Rimlinger, F. S., "Pregroups and Bass-Serre Theory", Mem. Amer. Math. Soc., **no. 361**, Amer. Math. Soc., Providence, R.I., 1987.

[49] Rimlinger, F. S., *The structure of Promislow's continuous free product*, Arboreal Group Theory (Alperin, R.C., ed.), Springer-Verlag, New York, 1991, 345–353.

[50] Spehner, J.-C., *Le demi-groupe librement engendré par un groupoide partiel G et l'image homomorphe associative maximale de G*, C.R. Acad. Sci. Paris **274A** (1972), 940–943.

[51] Stallings, J. R., *A remark about the description of free products of groups*, Proc. Cambridge Philos. Soc. **62** (1966), 129–134.

Zahari Knjazeski 81
6000 Stara Zagora
Bulgaria

Received November 5, 1996
and in final form July 14, 1997