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EMBEDDABILITY AND THE WORD PROBLEM

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§**1. Introduction.** Let $\mathscr{V}$ be a finitely presented variety with operations Ω and let $\mathscr{V}^*$ be the variety having the same set of operations Ω but defined by the empty set of identities. A *partial* $\mathscr{V}^*$*-algebra* $\mathscr{P} = (P, \overline{\Omega})$ is a set P with a set of mappings $\overline{\Omega}$ containing for each n-ary operation f of Ω a mapping $\bar{f}: D \to P$, where $D \subseteq P^n$. An *incomplete* $\mathscr{V}$*-algebra* is a partial $\mathscr{V}^*$-algebra $\mathscr{P}$ which satisfies the defining identities of $\mathscr{V}$, insofar as they can be applied to the partial operations of $\mathscr{P}$ (Trevor Evans [4, p. 67]). We call an incomplete $\mathscr{V}$-algebra a *partial Evans* $\mathscr{V}$*-algebra* if it can be embedded in a member of the variety $\mathscr{V}$.

If the class of all partial Evans $\mathscr{V}$-algebras is (first-order) finitely axiomatizable, then the word problem for the variety $\mathscr{V}$ is solvable. (Evans [4, 5]). In 1953 Evans [5, p. 79] raised the question of whether the converse is true. In this paper we show that the answer is in the negative.

Let **CSg** denote the variety of commutative semigroups. We call an incomplete **CSg**-algebra an *incomplete commutative semigroup* and we call a partial Evans **CSg**-algebra a *partial Evans commutative semigroup*. It is known (A. I. Malcev [9]; see also Evans [6]) that the variety of commutative semigroups has solvable word problem. We show (Theorem 1) that the class of all partial Evans commutative semigroups is not finitely axiomatizable. Therefore the solvability of the word problem for the variety of commutative semigroups does not imply finite axiomatizability of the class of all partial Evans commutative semigroups.

In the proof of Theorem 1 we construct a partial groupoid G_n, $n \geq 4$. If $n = 4$, then G_4 coincides with the partial groupoid constructed by S. H. Gensemer and H. J. Weinert in Example 6.2 of [7].

§**2. Preliminaries.** A *partial groupoid* is a triple $G = (G, D, \mu)$, where G is a nonempty set, $D \subseteq G \times G$, and $\mu: D \to G$ is a map. Let $G = (G, D, \mu)$ and $H = (H, E, \nu)$ be partial groupoids. Denote $\mu(x, y)$ by $x \cdot y$ and $\nu(x, y)$ by $x * y$. A *homomorphism* of G into H is a map $\varphi: G \to H$ such that $(x, y) \in D$ implies $(\varphi(x), \varphi(y)) \in E$ and $\varphi(x \cdot y) = \varphi(x) * \varphi(y)$ for all $x, y \in G$.

Let $G = (G, D, \mu)$ be a partial groupoid and let $F(G)$ be the free semigroup on the set G. Let $\pi_0(G)$ be the relation on $F(G)$ consisting of the pairs (xy, yx) for all $x, y \in G$. Let $\rho_0(G)$ be the relation on $F(G)$ consisting of the pairs $(xy, \mu(x, y))$ for all $(x, y) \in D$. Set $\theta_0(G) = \pi_0(G) \cup \rho_0(G)$. Let $\theta(G)$ be the congruence on

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$F(G)$ generated by $\theta_0(G)$. Then $U(G) = F(G)/\theta(G)$ is the *universal commutative semigroup* of the partial groupoid G.

A partial groupoid $G = (G, D, \mu)$ (we denote $\mu(x, y)$ by $x \cdot y$) is said to be an *incomplete commutative semigroup* if it satisfies the following axioms:

A1. For all $x, y, z \in G$, if $(x, y) \in D$, $(y, z) \in D$ and $(x \cdot y, z) \in D$, then $(x, y \cdot z) \in D$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

A2. For all $x, y, z \in G$, if $(x, y) \in D$, $(y, z) \in D$ and $(x, y \cdot z) \in D$, then $(x \cdot y, z) \in D$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

A3. For all $x, y \in G$, if $(x, y) \in D$, then $(y, x) \in D$ and $x \cdot y = y \cdot x$.

Let G and H be partial groupoids. G is said to be *embeddable* into H if there is a one-to-one homomorphism of G into H. A partial groupoid G is said to be *partial commutative semigroup* if it is embeddable in a commutative semigroup. An incomplete commutative semigroup G is said to be a *partial Evans commutative semigroup* if it is a partial commutative semigroup.

THEOREM A (T. Evans [4]). *An incomplete commutative semigroup G is a partial Evans commutative semigroup if and only if it is embeddable in its universal commutative semigroup, i.e., if and only if $x \sim y \pmod{\theta(G)}$ implies $x = y$, for all $x, y \in G$.*

We shall examine the congruence $\theta(G)$ by considering elementary $\theta_0(G)$-transitions as defined in J. M. Howie [8, §I.5]. An elementary $\theta_0(G)$-transition $w = uxyv \to u\mu(x, y)v = w'$, where $u, v \in F^1(G)$ and $(xy, \mu(x, y)) \in \rho_0(G)$, is called a *direct move* from w to w'. Further, an elementary $\theta_0(G)$-transition $w = uxyv \to uyxv = w'$, where $u, v \in F^1(G)$ and $(xy, yx) \in \pi_0(G)$, is called a *permutation step* from w to w'. If $x, y \in G$, then $x \sim y \pmod{\theta(G)}$ if and only if either $x = y$ or for some $n \geq 1$ there is a sequence

$$\tau\colon x = w_o \to w_1 \to \cdots \to w_n = y$$

of elementary $\theta_0(G)$-transitions connecting x to y. ([8, Proposition I.5.10]).

Clearly, a partial groupoid $G = (G, D, \mu)$ can be considered as a pair $G^* = (G, R)$, where R is the ternary relation on G defined by $(x, y, z) \in R$ if and only if $(x, y) \in D$ and $\mu(x, y) = z$. In the terminology of A. I. Malcev [10], G^* is the *model corresponding to G*.

If we use a first-order language L consisting of one ternary relation symbol, p say, we can rewrite each of axioms A1, A2, and A3 as a sentence of L as follows:

A1′. $\forall x \forall y \forall z \forall x_1 \forall y_1 \forall z_1 [pxyx_1 \wedge pyzy_1 \wedge px_1zz_1 \to pxy_1z_1]$.

A2′. $\forall x \forall y \forall z \forall x_1 \forall y_1 \forall z_1 [pxyx_1 \wedge pyzy_1 \wedge pxy_1z_1 \to px_1zz_1]$.

A3′. $\forall x \forall y \forall z [pxyz \to pyxz]$.

We can describe the class of all incomplete commutative semigroups as a quasivariety (in the sense of Malcev [10]; see also [11, Chapter V]) defined by the quasi-identities A1′, A2′, A3′ and the following quasi-identity:

A0. $\forall x \forall y \forall x_1 \forall x_2 [px_1x_2x \wedge px_1x_2y \to x = y]$.

The following theorem is a corollary to [1, §3, Theorem 3].

THEOREM B. *The class of all partial commutative semigroups is a quasivariety (in the sense of Malcev [10, 11]) defined by an infinite set B of quasi-identities of the form*

$$\forall x \forall y \forall x_1 \cdots \forall x_m [p(x, y, x_1, \ldots, x_m) \wedge \cdots \wedge p(x, y, x_1, \ldots, x_m) \to x = y].$$

From Theorem B it follows that the class of all partial Evans commutative semigroups is a quasivariety defined by the quasi-identities A0, A1′, A2′, A3′ and the infinite set of quasi-identities B.

§3. The class of all partial Evans commutative semigroups is not finitely axiomatizable.

THEOREM 1. *The class of all partial Evans commutative semigroups is not finitely axiomatizable.*

PROOF. We shall prove that the infinite set of first-order axioms for partial Evans commutative semigroups consisting of A0, A1′, A2′, A3′and B is not equivalent to any of its finite subsets. From this it follows that the class of all partial Evans commutative semigroups is not finitely axiomatizable. We refer the reader to [2] for the fundamentals of first-order theories.

Let B_0 be a finite subset of B. We shall prove that A0, A1′, A2′, A3′ and B_0 do not imply B. To prove this we construct a partial groupoid G_n whose model G_n^* satisfies A0, A1′, A2′, A3′ and B_0 but not B.

Let $n \geq 4$. On the set $G_n = \{a_0, a_1, \dots, a_n, b_1, \dots, b_{n-2}, c_1, \dots, c_{n-2}, d, e, f\}$ we define a partial groupoid G_n by the multiplications

$$a_1 \cdot a_2 = a_2 \cdot a_1 = b_1,$$

$$b_i \cdot a_{i+2} = a_{i+2} \cdot b_i = b_{i+1}, \qquad i = 1, \dots, n-3,$$

$$b_{n-2} \cdot a_n = a_n \cdot b_{n-2} = d, \qquad a_0 \cdot a_3 = a_3 \cdot a_0 = c_1, \qquad c_1 \cdot a_2 = a_2 \cdot c_1 = c_2,$$

$$c_k \cdot a_{k+2} = a_{k+2} \cdot c_k = c_{k+1}, \qquad k = 2, \dots, n-3,$$

$$c_{n-2} \cdot a_n = a_n \cdot c_{n-2} = e, \qquad a_1 \cdot a_n = a_n \cdot a_1 = f, \qquad a_0 \cdot a_n = a_n \cdot a_0 = f.$$

By construction G_n satisfies A3. It is routine to verify that G_n satisfies A1 and A2.

We denote $\theta_0(G_n)$ by θ_0, $\theta(G_n)$ by θ, $\rho_0(G_n)$ by ρ_0, and for each word w of $F(G_n)$ we denote by $w\theta$ the θ-class of w. Let S_m denote the symmetric group on $\{1, \dots, m\}$. Clearly, for each word $w = x_1 x_2 \cdots x_m \in F(G_n)$ and each permutation $\sigma \in S_m$ we have $\sigma(w) \in w\theta$, where $\sigma(w)$ denotes $x_{\sigma 1} x_{\sigma 2} \cdots x_{\sigma m}$. We have $a_i\theta = \{a_i\}$, for $i = 0, 1, \dots, n$, and $f\theta = \{f, a_1 a_n, a_n a_1, a_0 a_n, a_n a_0\}$. We shall examine the θ-classes $b_i\theta$ and $c_i\theta$ $(i = 1, \dots, n-2)$ by considering an oriented graph Γ as defined in P. M. Cohn [3, p. 159]. The vertices of Γ are the elements of $F(G_n)$, and the arrows of Γ are the direct moves and the permutation steps. The different θ-classes are just the connected components of Γ. For the sake of simplicity we shall exhibit only parts of connected components, namely, only the direct moves defined by the following multiplications: $a_1 \cdot a_2 = b_1$, $b_i \cdot a_{i+2} = b_{i+1}$, $i = 1, \dots, n-3$, $a_0 \cdot a_3 = c_1$, $c_1 \cdot a_2 = c_2$, and $c_k \cdot a_{k+2} = c_{k+1}$, $k = 2, \dots, n-3$. We shall not exhibit the other direct moves or permutation steps. Any connected component of Γ containing b_i or c_i can easily be exhibited in full from its part shown below. Under this convention the connected components of Γ containing b_i and c_i $(i =$

$1, \ldots, n-2)$ are as follows:

$$a_1 a_2 \to b_1,$$
$$a_1 \cdots a_{k+1} \to b_1 a_3 \cdots a_{k+1} \to \cdots \to b_{k-1} a_{k+1} \to b_k,$$

where $k = 2, \ldots, n-2$,

$$a_0 a_3 \to c_1, \qquad a_0 a_3 a_2 \to c_1 a_2 \to c_2,$$
$$a_0 a_3 a_2 a_4 \cdots a_{k+1} \to c_1 a_2 a_4 \cdots a_{k+1} \to \cdots \to c_{k-1} a_{k+1} \to c_k,$$

where $k = 3, \ldots, n-2$.

Each of the above connected components of Γ contains exactly one element of G_n. But d and e belong to the same θ-class. For an arbitrary sequence of elementary θ_0-transitions we say that the number of elementary ρ_0-transitions in the sequence is the *length* of the sequence. Denote

$$T = \{\sigma(a_1 \ldots a_n) \colon \sigma \in S_n\},$$
$$U = \{\sigma(f a_2 \ldots a_{n-1}) \colon \sigma \in S_{n-1}\},$$
$$V = \{\sigma(a_0 a_2 \ldots a_n) \colon \sigma \in S_n\}.$$

Each sequence of elementary θ_0-transitions of minimal length from d to e has the form

$$d \to \cdots \to w \to \cdots \to w' \to \cdots \to w'' \to \cdots \to e,$$

where $w \in T$, $w' \in U$, $w'' \in V$, each of the subsequences $d \to \cdots \to w$ and $w'' \to \cdots \to e$ has length $n-1$, and each of the subsequences $w \to \cdots \to w'$ and $w' \to \cdots \to w''$ has length 1. Consequently, each sequence of minimal length from d to e (and hence each sequence of minimal length from e to d) has length $2n$.

Since d and e belong to the same θ-class, by Theorem A, the incomplete commutative semigroup G_n is not a partial Evans commutative semigroup. Hence G_n^* does not satisfy B. For each quasi-identity $\psi \in B$ we denote by $l_p(\psi)$ the number of occurrences of the symbol p in ψ. Let $l_p(\psi) = n_\psi$, $\psi \in B_0$. Choose $n \geq n_\psi$, for all $\psi \in B_0$. Then G_n^* satisfies A0, A1′, A2′, A3′ and B_0, but not B. This completes the proof of Theorem 1. $\square$

REFERENCES

[1] H. Andréka and I. Németi, *Generalization of the concept of variety and quasivariety to partial algebras through category theory*, *Dissertationes Mathematicae (Rozprawy Matematyczne)* no. 204 (1983).

[2] J. Barwise (editor), *Handbook of mathematical logic*, North-Holland, Amsterdam, 1977.

[3] P. M. Cohn, *Universal algebra*, Harper and Row, New York, 1965.

[4] T. Evans, *The word problem for abstract algebras*, *Journal of the London Mathematical Society*, vol. 26 (1951), pp. 64–71.

[5] ———, *Embeddability and the word problem*, *Journal of the London Mathematical Society*, vol. 28 (1953), pp. 76–80.

[6] ———, *Some connections between residual finiteness, finite embeddability and the word problem*, *Journal of the London Mathematical Society*, ser. 2, vol. 1 (1969), pp. 399–403.

[7] S. H. GENSEMER and H. J. WEINERT, *On the embedding of partial groupoids into semigroups*, ***Bayreuther Mathematische Schriften***, vol. 28 (1989), pp. 139–163.

[8] J. M. HOWIE, ***An introduction to semigroup theory***, Academic Press, London, 1976.

[9] A. I. MALCEV, *On homomorphisms onto finite groups*, ***Ivanovskiĭ Gosudarstvennyĭ Pedagogicheskiĭ Institut Uchenye Zapiski***, vol. 18 (1958), pp. 49–60; English translation, ***American Mathematical Society Translations*** ser. 2, vol. 119 (1983), pp. 67–79.

[10] ———, *A few remarks on quasivarieties of algebraic systems*, ***Algebra i Logika***, vol. 5 (1966), no. 3, pp. 3–9; English translation, in A. I. Malcev, ***The metamathematics of algebraic systems***, North-Holland, Amsterdam, 1971, pp. 416–421.

[11] ———, ***Algebraic systems***, "Nauka", Moscow, 1970 (Russian); English translation, Springer-Verlag, New York, 1973.

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