MR2600450 (2011b:34107) 34C23 (34C07 37G15)

Iliev, Iliya D. (BG-AOS); **Li, Chengzhi [Li, Cheng Zhi[1]]** (PRC-BJ-MAM);
Yu, Jiang [Yu, Jiang[1]] (PRC-JTU)

Bifurcations of limit cycles in a reversible quadratic system with a center, a saddle and two nodes. (English summary)

Commun. Pure Appl. Anal. **9** (2010), *no. 3,* 583–610.

For a class of planar reversible quadratic systems with a center, a saddle and two nodes, the authors prove that the cyclicity of the period annulus around the center is 2 under a small 1-parameter quadratic perturbation. The approach involving complete elliptic integrals and a computer algebra system (in this case, MAPLE) is by now classic.

Reviewed by *Bourama Toni*

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109 (1994), 223–273. MR1273302 (95b:34047)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2600970 (2011c:35509) 35Q53 (35B10 35B35 35C07 37K45)

Hakkaev, Sevdzhan (BG-SHUMI); **Iliev, Iliya D.** (BG-AOS-IMI);
Kirchev, Kiril (BG-AOS-IMI)

Stability of periodic traveling waves for complex modified Korteweg-de Vries equation. (English summary)

J. Differential Equations **248** (2010), *no. 10,* 2608–2627.

The authors prove the existence and stability of periodic traveling-wave solutions for the complex modified Korteweg-de Vries equation

$$u_t + 6|u|^2 u_x + u_{xxx} = 0,$$

where u is a complex-valued function of $(x, t) \in \mathbb{R}^2$. They also prove that the initial value problem for this equation is locally ill-posed for initial data in the periodic space H^s with $s < \frac{1}{2}$ in the sense that the data-to-solution mapping is not uniformly continuous.

Reviewed by *Wengu Chen*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2525189 (2010h:34074) 34C07 (34C05 37G15)
Gautier, Sébastien (F-TOUL3); **Gavrilov, Lubomir** (F-TOUL3); **Iliev, Iliya D.** (BG-AOS)
Perturbations of quadratic centers of genus one. (English summary)
Discrete Contin. Dyn. Syst. **25** (2009), *no. 2,* 511–535.

Summary: "We propose a program for finding the cyclicity of period annuli of quadratic systems with centers of genus one. As a first step, we classify all such systems and determine the essential one-parameter quadratic perturbations which produce the maximal number of limit cycles. We compute the associated Poincaré-Pontryagin-Melnikov functions whose zeros control the number of limit cycles. To illustrate our approach, we determine the cyclicity of the annuli of two particular reversible systems."

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quadratic systems, Proc. Roy. Soc. Edinburgh - A, **127** (1997), 1207–1217. MR1489433 (99c:34051)
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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2526806 (2010h:34075) 34C07 (34C05 34C08 37C10)
Gavrilov, Lubomir (F-TOUL3-IM); **Iliev, Iliya D.** (BG-AOS)

Quadratic perturbations of quadratic codimension-four centers. (English summary)

J. Math. Anal. Appl. **357** (2009), *no. 1*, 69–76.

In the space of all quadratic systems, the systems with a center form a union of four irreducible affine algebraic sets in the space of all quadratic systems. Among them is the codimension-four set with a center, a node and a rational first integral.

Up to now nothing was known about the number of limit cycles in the generic codimension-four case.

The authors consider quadratic perturbations of such systems. They prove that the cyclicity of the open period annulus surrounding the center in this case is less than or equal to eight.

They first use an explicit change of coordinates in such a way that the first integral defines generically elliptic curves in the new coordinates. Thus the Poincaré-Pontryagin-Melʹnikov function is a complete elliptic integral. However, this function involves a differential of the third kind with residues depending algebraically on the value of the first integral, and not only polynomially.

The authors show that the Abelian integrals defined by any quadratic perturbation satisfy a Picard-Fuchs equation and they give it explicitly. It involves Picard-Fuchs operators; the solution space of one of them is a Chebyshev space.

They prove the cyclicity result by several steps. In particular they make a beautiful reduction of this equation, assuming that there is a nonvanishing solution, to a very simple linear equation. Thus they can conclude that if the right-hand side has k zeros on the interval, then the solutions have at most $k+2$ zeros on this interval.

The Petrov method in the complex domain applies to show a more general result than announced and to finish the proof.

Reviewed by *Michèle Pelletier*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2451299 (2009m:35429) 35Q53 (34C25 34L40 35B10 35Q51 76B15 76E15)
Hakkaev, Sevdzhan (BG-SHUMI); **Iliev, Iliya D.** (BG-AOS-IMI);
Kirchev, Kiril (BG-AOS-IMI)
Stability of periodic travelling shallow-water waves determined by Newton's equation. (English summary)
J. Phys. A **41** (2008), *no. 8,* 085203, 31 *pp*.

Considering the nonlinear evolution equation

$$u_t + (a(u))_x - u_{xxt} = \left(b'(u)\frac{u_x^2}{2} + b(u)u_{xx}\right)_x$$

where $a, b\colon \mathbb{R} \to \mathbb{R}$ are smooth functions and $a(0) = 0$, the existence and stability of periodic travelling wave solutions of the form

$$u = \varphi(x - vt)$$

are studied. The above equation includes generalized Benjamin-Bona-Mahony and Camassa-Holm equations. Orbital stability of the travelling wave solutions is analyzed using the abstract results of M. G. Grillakis, J. Shatah and W. A. Strauss [J. Funct. Anal. **74** (1987), no. 1, 160–197; MR0901236 (88g:35169)] and the Floquet theory for periodic eigenvalue problems.

Reviewed by *Muthusamy Lakshmanan*

MR2183522 (2006h:34065) 34C07 (34C08 37C10 37C27 58K10)
Gavrilov, Lubomir (F-TOUL3-LM); **Iliev, Iliya D.** (BG-AOS)
The displacement map associated to polynomial unfoldings of planar Hamiltonian vector fields. (English summary)
Amer. J. Math. **127** (2005), *no. 6,* 1153–1190.

In this paper, the authors study the displacement map $\mathcal{P}_\varepsilon(t) - t$ associated to small one-parameter polynomial unfoldings of polynomial Hamiltonian vector fields in the plane of the form

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon P(x,y,\varepsilon), \\ \dot{y} = -H_x(x,y) + \varepsilon Q(x,y,\varepsilon), \end{cases}$$

where H, P and Q are real polynomials in x, y with P and Q depending analytically on a small real parameter ε. The leading term, the generating function $M(t)$, has an analytic continuation in the complex plane and the real zeroes of $M(t)$ correspond to the limit cycles bifurcating from the periodic orbits of the Hamiltonian flow. A geometric description of the monodromy group of $M(t)$ is given and some sufficient conditions for $M(t)$ to satisfy a differential equation of Fuchs or Picard-Fuchs type are formulated.

Reviewed by *Chun-Gen Liu*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2109479 (2005h:34078) 34C07 (34C08 34C23 37G15)
Iliev, Iliya D. (BG-AOS); **Li, Chengzhi [Li, Cheng Zhi[1]]** (PRC-BJ-MAM);
Yu, Jiang [Yu, Jiang[1]] (PRC-JTU)

Bifurcations of limit cycles from quadratic non-Hamiltonian systems with two centres and two unbounded heteroclinic loops. (English summary)

Nonlinearity **18** (2005), *no. 1,* 305–330.

The authors investigate the bifurcation of limit cycles of a specific class of integrable non-

Hamiltonian systems under small quadratic perturbations. After a Poincaré transformation and a change of time, the system becomes a reversible cubic Hamiltonian system in which the perturbation remains polynomial. Then classical methods allow the authors to compute the number of zeros of abelian integrals. Thus the authors obtain complete results about the distribution and the number of limite cycles bifurcating from the two period annuli and they draw the bifurcation diagram.

Reviewed by *Michèle Pelletier*

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

MR2021617 (2004i:34077) 34C08
Gavrilov, Lubomir (F-TOUL3-LM); **Iliev, Iliya D.** (BG-AOS)
Complete hyperelliptic integrals of the first kind and their non-oscillation. (English summary)
Trans. Amer. Math. Soc. **356** (2004), *no. 3,* 1185–1207 (*electronic*).

In this paper the authors consider the problem of finding the zeros of the following complete abelian integral

$$I(h)=\int_{\delta(h)} f(x,y)\,dx+g(x,y)\,dy,\ h\in\Sigma,$$

where $f,\ g\in\mathbb{R}[x,y]$ are real polynomials, and $\delta(h)\subset\{(x,y)\in\mathbb{R}^2\colon H(x,y)=h\}$ is a continuous family of ovals for $h\in\Sigma$. Σ is a maximal open interval in $\mathbb{R}$ on which the continuous family of ovals $\delta(h)$ exists. Here $H\in\mathbb{R}[x,y]$ is a real polynomial. This problem was called the "weakened 16th Hilbert problem" by Arnold.

The problem is called hyperelliptic if $H(x,y)=y^2+P(x)$. Progress in solving the weakened 16th Hilbert problem has only considered the "elliptic case" (the complex algebraic curve $\{H=h\}$ is of genus at most one). It was known that, in several cases, the vector space $\mathcal{A}_{H,d}$ of abelian integrals of degree d polynomials along the ovals of H obeys the so-called Chebyshev property (the number of the zeros of each integral is smaller than the dimension of the vector space $\mathcal{A}_{H,d}$).

In this relation Arnold asked whether the g-dimensional vector space of abelian integrals

$$(1)\qquad I(h)=\int_{\delta(h)}\frac{(\alpha_0+\alpha_1 x+\cdots+\alpha_{g-1}x^{g-1})dx}{y},\ h\in\Sigma,$$

where $\delta(h)\subset\{(x,y)\in\mathbb{R}^2\colon y^2+P(x)=h\}$, $g=[\frac{1}{2}(\deg P-1)]>1$, is Chebyshev.

In this paper the authors prove an interesting result: when $g=2$ and $\deg P=5$, exceptional families of ovals $\{\delta(h)\}$ exist such that every abelian integral of the form

$$I(h)=\int_{\delta(h)}\frac{(\alpha_0+\alpha_1 x)dx}{y},\ \alpha_0^2+\alpha_1^2\neq 0$$

has at most one simple zero on the interval Σ. The authors also give a negative answer to the initial question posed by Arnold: it is proved that there exist abelian integrals of the form (1) with exactly $[\frac{3}{2}g]-1$ zeros in a neighborhood of the origin.

Reviewed by *Chun-Gen Liu*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1973284 (2004b:34078) 34C08 (34C05 34C07 37C25)
Gavrilov, Lubomir (F-TOUL3-LM); **Iliev, Iliya D.** (BG-AOS)
Two-dimensional Fuchsian systems and the Chebyshev property. (English summary)
J. Differential Equations **191** (2003), *no. 1,* 105–120.

Fuchsian systems of the form $I(h) = A(h)I'(h)$ for $I = (I_1, I_2)^\top$ and $A(h) = A_0 + hA_1$ are considered. Such systems that arise in the case $I_{1,2}$ are abelian integrals of the form $\int_{H=h} \omega_{1,2}$, where H is a polynomial (or rational) Hamiltonian function and $\omega_{1,2}$ is a polynomial (respectively, rational) 1-form.

The following assumptions are imposed: the matrix A_1 has distinct real eigenvalues, $\det A(h)$ has distinct zeroes $h_0 < h_1$ and the identity $(\det A)' \equiv \operatorname{tr} A$ holds, and $I(h)$ is analytic near h_0. (These assumptions imply that the characteristic exponents of the Fuchsian system near $h_{0,1}$ are $\alpha = 0$ and $\alpha = 1$.)

The spaces $V_s = \{J(h) = P(h)I_1(h) + Q(h)I_2(h)\colon P, Q \in \mathbb{R}[h],\ |J(h)| < \mathrm{const} \cdot |h|^s, s \to \infty\}$ are introduced.

The main result of the paper states that the space V_s is Chebyshev with accuracy κ (which is explicitly calculated) in the domain $\mathbb{C} \setminus [h_1, \infty)$; this means that any J has at most $\dim V_s + \kappa - 1$ zeroes there.

Application of this theorem to cases with $H = y^2 + x^2 - x^3$, $y^2 + x^2 - xy^2$, $\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + xy^2$, $y^2 + x^2 \pm x^4$, $y^2 + x^2 \pm x^2y^2$, $(y^2 - 2x^2 + x)/x^3$ allows one to re-prove some known results as well as to obtain some new estimates for the number of limit cycles appearing in perturbations of the corresponding Hamiltonian systems.

Reviewed by *Henryk Żołądek*

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MR1924712 (2003f:34056) 34C05 (34C07 37G15)

Gavrilov, Lubomir (F-TOUL3-LM); **Iliev, Iliya D.** (BG-AOS)

Bifurcations of limit cycles from infinity in quadratic systems. (English summary)

Canad. J. Math. **54** (2002), *no. 5,* 1038–1064.

The paper is devoted to a study of limit cycles of quadratic planar vector fields in the cases where these cycles escape to infinity as a parameter of perturbation tends to zero.

After suitable changes of variables and parameters, the problem is eventually reduced to a perturbation of a time-reversible system of the form $\dot{z} = -iz + Az^2 + Bz\overline{z} + C\overline{z}^2$. Here, either (i) $A = -2, B = 2, C = 0$, (ii) $A = -1, B = 2, C = 1/3$, (iii) $A = 2a^2 + 2b^2 + a - 1, B = 2\ (2 + a = a^2 - b^2), C = a + 1$, (iv) $A = C = -1, B = 2$, or (v) $A = 1, B = 0, C = 1/3$.

The analysis of limit cycles is reduced to a study of the zeroes of corresponding Abelian integrals. This analysis is performed for the Bogdanov-Takens system (v), for the isochronous center $A = 5, B = 2, C = -3$, for the linear-like system (i) $\dot{z} = -iz(1 + 4\,\mathrm{Im}\, z)$ and for the case (ii). In all these cases the bound ≤ 2 for the number of limit cycles is proved.

In all but the isochronous center case, the result follows from earlier results. In the isochronous center case the investigation is subtle; it uses such tools such as Picard-Fuchs equations, hypergeometric functions and bifurcation diagrams of zeroes in the complex domain.

Reviewed by *Henryk Żołądek*

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MR1804952 (2003c:34029) 34C05 (34C07 37J99)
Gavrilov, Lubomir (F-TOUL3-LM); **Iliev, Iliya D.** (BG-AOS)
Second-order analysis in polynomially perturbed reversible quadratic Hamiltonian systems. (English summary)
Ergodic Theory Dynam. Systems **20** (2000), *no. 6,* 1671–1686.

In this paper the authors study small polynomial perturbations

$$X_H + \varepsilon Y$$

of the Hamiltonian vector field X_H, $H = 1/2(x^2 + y^2) - 1/3x^3 + axy^2$, $a \in (-1/2, 0)$. This Hamiltonian vector field has a center and a saddle and is reversible.

The displacement function develops as $d(h, \varepsilon) = \varepsilon M_1 + \varepsilon^2 M_2(h) + o(\varepsilon^2)$.

The authors study the maximal number of isolated zeros of d for ε small in the case when $M_1 \equiv 0$ and $M_2 \not\equiv 0$. They prove that if degree of Y is n, then this number is $2(n-1)$.

In the particular case $n = 2$, they prove more. That is, they prove that there is a full neighbourhood $\mathcal{U}$ of X_H in the space of quadratic vector fields such that any $X \in \mathcal{U}$ has at most two limit cycles.

Reviewed by *Pavao Mardešić*

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MR1749682 (2001a:34055) 34C07 (34C23 34C25)

Iliev, Iliya D. (BG-AOS)

On the limit cycles available from polynomial perturbations of the Bogdanov-Takens Hamiltonian. (English summary)

Israel J. Math. **115** (2000), 269–284.

Summary: "The displacement map related to small polynomial perturbations of the planar Hamiltonian system $dH = 0$ is studied in the elliptic case $H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3$. An estimate of the number of isolated zeros for each of the successive Mel′nikov functions $M_k(h),\ k = 1, 2, \cdots$, is given in terms of the order k and the maximal degree n of the perturbation. This sets up an upper bound to the number of limit cycles emerging from the periodic orbits of the Hamiltonian system under polynomial perturbations."

Reviewed by *Pavao Mardešić*

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MR1705462 (2000d:34054) 34C05
Iliev, Iliya D. (BG-AOS)
The number of limit cycles due to polynomial perturbations of the harmonic oscillator.
Math. Proc. Cambridge Philos. Soc. **127** (1999), *no. 2,* 317–322.

Consider a polynomial vector field (of degree n) of the type

$$\dot{x} = y + \varepsilon f(x, y; \varepsilon), \quad \dot{y} = -x + \varepsilon g(x, y; \varepsilon)$$

which depends analytically on the small parameter ε. The equation for limit cycles can be written in the form $\varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots = 0$ where $M_j(h)$ are the Mel′nikov functions (with the argument h parametrizing the ovals of the Hamilton function).

The author proves that if M_k is the first nonzero Mel′nikov function, then it has no more than $[k(n-1)/2]$ zeroes (and there are $\leq [k(n-1)/2]$ limit cycles). For the cases $k = 1, 2, 3,\ n$ arbitrary and $n = 3,\ k < 6$ there are examples of perturbations with $[k(n-1)/2]$ limit cycles. In the case $n = 3,\ k = 6$ five cycles are obtained.

Reviewed by *Henryk Żołądek*

MR1691076 (2000i:34064) 34C07 (34C08 34C23 37G15)
Iliev, I. D. (BG-AOS); **Perko, L. M.** (1-NAZ)
Higher order bifurcations of limit cycles. (English summary)
J. Differential Equations **154** (1999), *no. 2,* 339–363.

In this paper the authors study one-parameter families of asymmetrically perturbed symmetric Hamiltonian systems of the form $\dot{x} = y,\ \dot{y} = \pm(x \pm x^3) + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y$, with $\lambda_i = \lambda_i(\varepsilon)$. They claim that for any choice of the signs $\pm$, at most two limit cycles bifurcate from any periodic annulus of the system. The proof is done by studying higher order Mel′nikov functions M_k associated to the problem and showing that they can always be written as a linear combination of three Abelian integrals. Next the Picard-Fuchs equations of these Abelian integrals are examined.

The authors correct a sign mistake in a previous paper of J. M. Jebrane and H. Żołądek [Adv. in Appl. Math. **15** (1994), no. 1, 1–12; MR1260295 (94k:34069)] leading to a wrong bound on the number of limit cycles in that paper.

However, the paper under review is incomplete. In fact it treats only (higher order) Melʹnikov functions and not the displacement function itself. Probably, using R. Roussarie's results [Bol. Soc. Brasil. Mat. **17** (1986), no. 2, 67–101; MR0901596 (88i:34061); Nonlinearity **2** (1989), no. 1, 73–117; MR0980858 (90m:58169)], it can be shown that all limit cycles are controlled by zeros of Melʹnikov functions in the neighborhood of a homoclinic loop, but there is no analogous general structure theory for a neighborhood of a heteroclinic loop joining two saddle points and for the figure-eight separatrix.

Note also that a uniform bound for cyclicity of analytic one-parameter families in general does not give cyclicity of a multi-parameter family. This point is not stressed in the paper.

Reviewed by *Pavao Mardešić*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

Citations
From References: 11
From Reviews: 1

MR1668195 (2000b:34045) 34C05 (34C23 37C27 37G15)
Iliev, I. D. (BG-AOS)
On second order bifurcations of limit cycles. (English summary)
J. London Math. Soc. (2) **58** (1998), *no. 2,* 353–366.

A formula is given for the second order Mel′nikov function for a general planar system at a periodic orbit where the first Mel′nikov function vanishes. This formula is applied to the problem of the continuation of periodic orbits for polynomial perturbations of the planar Hamiltonian system $H(x,y) = y^2/2 - U(x)$ where U is a polynomial function. In particular, a method is introduced for representing the second order Mel′nikov function for this case in terms of Abelian integrals. This result is then used to prove that at most two of the periodic orbits surrounding the center at the origin of the Hamiltonian system $\dot{x} = y$, $\dot{y} = -x - x^3$ continue into the perturbed system $\dot{x} = y$, $\dot{y} = -x - x^3 + \lambda_1 y + \lambda_2 x^2 + \lambda_3 xy + \lambda_4 x^2 y$. The proof uses a clever application of Picard-Fuchs equations.

Reviewed by *Carmen Chicone*

Citations
From References: 25
From Reviews: 6

MR1660361 (99j:34036) 34C05 (58F21)
Horozov, Emil (BG-SOFI); **Iliev, Iliya D.** (BG-AOS)
Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians. (English summary)
Nonlinearity **11** (1998), *no. 6,* 1521–1537.

In this interesting paper the authors give an explicit linear upper bound $B(3,n) = 5n + 15$ for the maximal number $Z(3,n)$ of zeros of Abelian integrals $I(h) = \int_{\delta(h)} g(x,y)dx + f(x,y)dy$, where $\delta(h) \subset H^{-1}(h)$ is an oval of a cubic polynomial $H(x,y)$ and f, g are polynomials of degree n. This result should be related to an unpublished result of G. S. Petrov and A. Khovanskiĭ, who considered the number of zeros $Z(H,n)$ of Abelian integrals of a 1-form $\omega = gdx + fdy$ of degree n along ovals of a given polynomial H of degree m and proved the existence of an upper bound for $Z(H,n)$ of the form $B(H,n) = a_1(\deg(H))n + a_0(H)$.

The result of Khorozov and Iliev is obtained by studying the Picard-Fuchs system of order 4 satisfied by basic Abelian integrals, i.e. integrals forming a basis of the space of Abelian integrals considered as a $\mathbf{C}[h]$ module. It is shown that the second order derivatives of these basic Abelian integrals belong to a two-dimensional module. This allows the authors essentially to reduce the study of a 4-dimensional system to a two-dimensional one and then use standard Riccati equation arguments. The reason why this is possible here lies in the geometry: the generic fibre of a generic cubic Hamiltonian is a torus with three points deleted.

In the paper a very interesting conjecture is also formulated stating that the maximal number $Z(m,n)$ of zeros of Abelian integrals of 1-forms of degree n along ovals of a polynomial of degree m should be bounded by (the dimension of the space of Hamiltonians)+(dimension of the space of integrals) -1. Here the dimension of the space of Hamiltonians means the dimension of the orbit space under the action of the group of affine coordinate changes of the space of all degree m Hamiltonians.

Reviewed by *Pavao Mardešić*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1612784 (99a:34082) 34C05 (34C23)

Iliev, Iliya D. (BG-AOS)

Perturbations of quadratic centers. (English summary)

Bull. Sci. Math. **122** (1998), *no. 2,* 107–161.

Since early in this century, conditions characterizing the occurrence of a center in a system of quadratic polynomial differential equations in the plane have been known. Of more recent interest is the question of the number of limit cycles which can be made to bifurcate from either the singularity itself or the period annulus surrounding it under small quadratic perturbation. The current article is a contribution to this question. In it, the author first of all constructs a list of the essential quadratic perturbations which will create the maximal number of limit cycles. He then finds an upper bound for the number of limit cycles which can be produced from the period annulus in terms of the number of zeros of certain Mel′nikov functions. Several special cases are further examined in more detail.

Reviewed by *Douglas S. Shafer*

MR1489433 (99c:34051) 34C05 (34A20)

Iliev, Iliya D. (BG-AOS)

Inhomogeneous Fuchs equations and the limit cycles in a class of near-integrable quadratic systems. (English summary)

Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), *no. 6,* 1207–1217.

The system $\dot{z} = -iz + 4z^2 + 2|z|^2 - 2\overline{z}^2$ belongs to the intersection of two strata of the set of quadratic systems with a center, the stratum of codimension 4 usually denoted by Q_4 and that of reversible systems. The author shows that a small quadratic perturbation can create at most 3 limit cycles in the open annulus filled by closed orbits surrounding the center.

Since the first Mel′nikov function is identically zero in this case, the author computes the second Mel′nikov function. It turns out to be a linear combination of 3 abelian integrals and one non-abelian. These 4 integrals together satisfy a system of linear differential equations which contains a subsystem of order 2. The proof uses the division-differentiation algorithm coupled with asymptotic estimates at the endpoints of the interval parametrizing the closed orbits in the annulus.

Reviewed by *Dmitri Novikov*

MR1401409 (97k:34039) 34C05 (34C23 34C37 58F21)
Iliev, I. D. (BG-AOS)
Higher-order Melnikov functions for degenerate cubic Hamiltonians. (English summary)
Adv. Differential Equations **1** (1996), *no. 4,* 689–708.

Summary: "It is shown that, in general, the first four Mel′nikov functions have to be taken into account in order to obtain definitive results concerning limit cycles in quadratic perturbations of Hamiltonian systems in the plane with degenerate cubic Hamiltonians. As an application, a complete proof of the result that no more than two limit cycles can bifurcate out of homoclinic loops of quadratic Hamiltonian systems is given."

Reviewed by *Shi Qing Zhang*

MR1392404 (97g:34043) 34C05 (58F21)
Iliev, Iliya D. (BG-AOS)
The cyclicity of the period annulus of the quadratic Hamiltonian triangle. (English summary)
J. Differential Equations **128** (1996), *no. 1,* 309–326.

Summary: "The paper studies quadratic Hamiltonian centers surrounded by a separatrix contour having the form of a triangle. It is proved that in this situation the cyclicity of the period annulus under quadratic perturbations is equal to three."

Reviewed by *Robert Roussarie*