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Codes and Designs in Polynomial Metric Spaces

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In the current work we consider codes and designs with a small number of distances in some metric spaces. The simplest such codes are the equidistant codes which are among the first objects to be studied in Coding theory. Admitting two or more distances, however, allows for a much greater variety of structures. The spaces which are studied in the work are the unit sphere in the n-dimensional Euclidean space, $\mathbb{R}^n$, and some finite linear spaces with Hamming metric.

Chapter 1. Introduction

The first chapter introduces basic definitions from Coding Theory and classical results for spherical codes and designs, which can be obtained through linear programming.

Definition 1.1 Code is every finite non-empty set of points in a metric space.

Definition 1.2 If a code is also a subspace, we will call it **linear**.

Definition 1.3 Minimal distance of a code is the minimal distance between two different points in the code.

These definitions naturally lead to the following main parameters of a code C − its cardinality, $M = |C|$, dimension n and the distances between the elements of C, in particular the smallest distance d. Often instead of minimal distance, out of convenience, we will work with the maximal inner product s. We will call codes with such parameters (n, M, d) or (n, M, s)-codes. The first and most natural question that arises is what are the interconnection between those parameters, the so called **fundamental problem of coding theory**.

Problem 1.1 For fixed n and d, find $M(n, d)$, the biggest natural M, such that an (n, M, d)-code exists.

Problem 1.2 For fixed n and M, find $d(n, M)$, the biggest number such that an (n, M, d)-code exists.

Let $\mathbb{M}$ be a finite metric space with a diameter $D_{\mathbb{M}}$ and finite measure $\mu_{\mathbb{M}}$. Let the Hilbert space $\mathcal{L}_2(\mathbb{M})$ be a direct sum of countably many pairwise orthogonal subspaces V_i:

$$\mathcal{L}_2(\mathbb{M}) = V_0 \oplus V_1 \oplus ...$$

Let $r_i = dim(V_i)$ and the subspaces V_i be such that there exist polynomials of one variable with real coefficients G_i of degree i, $i = 0, 1, \dots,$ for which

$$G_i(t_{\mathbb{M}}(d_{\mathbb{M}}(x, y))) = \frac{1}{r_i} \sum_{j=1}^{r_i} v_{ij}(x)\overline{v_{ij}(y)},$$

where the polynomials v_{ij}, $j = 1, 2, \dots, r_i$ form an orthonormal basis of V_i and $t_{\mathbb{M}} : [0, D_{\mathbb{M}}] \to [-1, 1]$ is a continuous decreasing function for which $t_{\mathbb{M}}(0) = 1$ and $t_{\mathbb{M}}(D_{\mathbb{M}}) = -1$. Then we call $\mathbb{M}$ **polynomial metric space**, and the polynomials G_i - **zonal spherical functions** [11].

This definition is not constructive and is difficult to use for the classification of such spaces, but it provides all the properties that we need for the methods of linear programming. The infinite spaces which satisfy it are the Euclidean sphere and some projective spaces. The finite polynomial metric spaces do not have a full classification yet.

Let us consider the standard n-dimensional Euclidean space $\mathbb{R}^n$ with distance and inner product for arbitrary points $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ respectively:

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2},$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

With $\mathbb{S}^{n-1}$ we denote the unit sphere in $\mathbb{R}^n$,

$$\mathbb{S}^{n-1} = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \quad | \quad x_1^2 + x_2^2 + ... + x_n^2 = 1\}.$$

On it we have the following connection between distance and inner product:

$$d(x,y) = \sqrt{2(1 - \langle x, y \rangle)},$$

$$\langle x, y \rangle = 1 - \frac{d(x,y)^2}{2}.$$

Definition 1.4 Let $C \in \mathbb{S}^{n-1}$ be a nonempty set with cardinality M, and maximal inner product s. Then C is a (n, M, s)-spherical code.

The fundamental problem of Coding theory, formulated for spherical codes, asks to find $A(n,s)$ and $s(n,M)$, where

$$A(n,s) = \max\{M : \exists(n, M, s) - \text{spherical code}\},$$

$$s(n,M) = \min\{s : \exists(n, M, s) - \text{spherical code}\}.$$

$A(n,s)$ is known for non-positive s, where it is obtained through classical constructions ([43, 44], see also [25, 48] and the references there). For positive values of s only some special cases are known.

Definition 1.5 [27] An n-dimensional spherical code C is called spherical τ-design, if for every polynomial of n variables with degree of at most τ we have

$$\int_{\mathbb{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x).$$

Here $\mu(x)$ is the standard Lebesgue measure, normalized with $\mu(\mathbb{S}^{-1}) = 1$.

The points of C can be considered as nodes of a quadrature formula on $\mathbb{S}^{n-1}$ with equal weights $1/|C|$ and algebraic degree of accuracy τ, while the inner products define a quadrature formula over $[-1, 1]$, again with degree of accuracy τ. This shows the importance of τ and of searching codes with large τ.

Definition 1.8 The maximal τ, for which a given code is also a τ-design is called **strength** of the design.

For fixed dimension n the Gegenbauer polynomials are defined with the following recurrent relation

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)} - iP_{i-1}^{(n)}$$

with initial conditions $P_0^{(n)}(t) = 1$ and $P_1^{(n)}(t) = t$.

Gegenbauer polynomials are orthogonal over $[-1,1]$ with respect to the following inner product

$$\langle f, g\rangle = c_n \int_{-1}^{1} f(t)g(t)(1-t^2)^{\frac{n-3}{2}}\,dt.$$

Furthermore, the orthogonality relation can be expressed explicitly as:

$$\left\langle P_i^{(n)}(t), P_j^{(n)}(t)\right\rangle = \delta_{ij}\frac{\Gamma(n-1)}{2^{n-2}\Gamma(\frac{n-1}{2})^2} = \frac{\delta_{ij}}{c_n},$$

where δ_{ij} is the Kroneker delta. The constant c_n is used for normalization. If

$$f(t) = \sum_{i=0}^{k} a_i t^i \in \mathbb{R}[t],$$

then $f(t)$ can be uniquely presented (due to orthogonality) in the following way

$$f(t) = f_0 P_0^{(n)}(t) + f_1 P_1^{(n)}(t) + \cdots + f_k P_k^{(n)}(t).$$

For the coefficients we have

$$f_i = c_n \int_{-1}^{1} f(t)P_i^{(n)}(t)(1-t^2)^{\frac{n-3}{2}}\,dt. \tag{1}$$

We denote f_0 for the polynomials t^k, $k \geq 0$ in the following way:

$$t^k = b_k + \sum_{i=1}^{k} P_i^{(n)}(t).$$

An explicit formula for them is:

$$b_{2j} = \frac{(2j-1)!!}{n(n+2)\cdots(n+2j-2))},$$

For odd k the coefficient is 0, which follows directly from (1).

The Gegenbauer polynomials are zonal spherical functions, thus the following equality holds for them ([41]):

$$P_i^{(n)}(\langle x, y\rangle) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_{ij}(x)v_{ij}(y), \tag{2}$$

where $x, y \in \mathbb{S}^{n-1}$ and $r_i = \dim(Harm(i))$ is the dimension of the space of homogeneous harmonic polynomials (over $\mathbb{S}^{n-1}$) of degree i, and $v_{ij}(x)$, $j = 1, 2, \ldots, r_i$ form an orthonormal basis of that space. A classic result [46] is

$$r_i = \left(\begin{array}{c} n+i-1 \\ i \end{array} \right) - \left(\begin{array}{c} n+i-3 \\ i-2 \end{array} \right) = \frac{n+2i-2}{i} \left(\begin{array}{c} n+i-3 \\ i-1 \end{array} \right).$$

For the methods we employ the number and arrangement of the roots of the Gegenbauer polynomials and other polynomials close to them are of major importance.

Lemma 1.1 The polynomial $P_k^{(n)}(t)$ has exactly k different real roots in the interval $[-1, 1]$.

We will denote these roots in the following way:

$$-1 < t_{k,1} < t_{k,2} < \cdots < t_{k,k} = 1.$$

Gegenbauer polynomials are a special case of the polynomials of Jacobi $P_i^{(\alpha,\beta)}(t)$ with parameters $\alpha = \beta = (n-3)/2$ [45]. Along with them, for obtaining bounds on spherical codes we use the so-called adjacent polynomials:

$$P_i^{a,b}(t) = \frac{P_i^{(a+(n-3)/2, b+(n-3)/2)}(t)}{P_i^{(a+(n-3)/2, b+(n-3)/2)}(1)}.$$

They are also orthogonal in the interval $[-1, 1]$ with respect to the following weight

$$(1-t)^{a+(n-3)/2}(1+t)^{b+(n-3)/2}.$$

Furthermore, $P_i^{1,1}(t)$ is also a Gegenbauer polynomial with dimension $n+2$:

$$P_i^{1,1}(t) = \frac{P_i^{1+(n-3)/2, 1+(n-3)/2}(t)}{P_i^{1+(n-3)/2, 1+(n-3)/2}(1)} = P_i^{(n+2)}(t).$$

Let $t_k^{a,b}$ is the largest root of $P_k^{a,b}(t)$, and $t_0^{-1,1} := -1$. Then we have:

Lemma 1.2 [37, 38] For every k the following inequalities hold

$$t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1} < t_{k-1}^{0,1}, \quad t_{k,k}^{1,0} < t_{k,k}.$$

The addition formula (2), which connects the Gegenbauer polynomials and the harmonic polynomials allows us to prove the following main theorem[38].

Theorem 1.1 For every spherical code $C \in \mathbb{S}^{n-1}$ and every real polynomial $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ the following equality holds:

$$|C|f(1) + \sum_{x,y \in C, x \neq y} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^{k} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} v_{ij}(x) \right)^2. \quad (3)$$

The sums on the right side of (3) are of particular importance for studying spherical codes.

Definition 1.9 [12, 20] For a given code $C \in \mathbb{S}^{n-1}$ and natural numbers i,

$$M_i(C) = \frac{1}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x \in C} v_{ij}(x) \right)^2$$

are called $i-$th**moments** of C. We will note that $M_i \geq 0$ for every $i \geq 1$. We also have the following property of spherical designs:

Theorem 1.2 [12] A code $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design, $\tau \geq 1$, if and only if $M_i(C) = 0$ for $i = 1, 2, \ldots, \tau$.

Furthermore if a code C is **antipodal**, which means $C = -C$, then $M_{2i-1}(C) = 0$ for every i.

Working with moments can sometimes improve the linear programming bounds, as we'll see in the next chapter.

Linear programming bounds are based on finding suitable polynomials for the following theorem, which can directly be obtained from (3).

Theorem 1.3 [27, 33] If C is a (n, M, s)-spherical code, for which $n \geq 3$, $s \in [-1, 1)$ and the polynomial $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ is such that

- $f(t) \leq 0 \quad \forall t \in [-1, s]$,
- $f_0 > 0$ and $f_i \geq 0 \quad \forall i > 0$.

Then

$$A(n, s) \leq \frac{f(1)}{f_0}.$$

Analogically, (3) and Theorem allow us to find a general bound for the minimal cardinality of a spherical τ-design when n and τ are fixed,

$$B(n, \tau) := \min\{|C| : C \subset \mathbb{S}^{n-1} \text{ is a spherical } \tau\text{-design}\}.$$

Theorem 1.4 [27] If $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design, for which $n \geq 3$, $\tau \geq 1$ and the polynomial $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ is such that:

- $f(t) \geq 0 \quad \forall t \in [-1, 1]$,
- $f_0 > 0$ and $f_i \leq 0 \quad \forall i > \tau$.

Then $|C| \geq \frac{f(1)}{f_0}$ and thus:

$$B(n, \tau) \geq \frac{f(1)}{f_0}.$$

In a similar way Yudin obtains bounds for the energies of spherical codes and designs. More precise version of Yudin's results is given in chapter 3. We call potential the following:

Definition 1.10 For a given function(potential) $h : [-1, 1] \to [0, +\infty]$, we call *energy* (or potential energy; or h-energy) of a spherical code (or τ-design) $C \subset \mathbb{S}^{n-1}$ the following sum

$$E(n, C; h) := \sum_{x,y \in C, x \neq y} h(\langle x, y\rangle).$$

Following [24] and [20, 19] we will consider only absolutely monotone potentials h - $h^{(k)}(t) \geq 0$ for every $k \geq 0$ and $t \in [-1, 1)$. The classical example for such a potential is the Riesz s-potential $h(t) = 1/(2(1-t))^s$. Other frequently used potentials are:

- Newton potential - $h(t) = (2-2t)^{\frac{-(n-2)}{2}}$;
- Log potential - $h(t) = \log(2-2t)$;
- Gauss potential - $h(t) = e^{2(1-t)}$.

For fixed n and τ Delsarte, Goethals and Seidel [27] use the linear programming bound with the polynomials

$$d_\tau^{(n)}(t) = \begin{cases} (t+1)\left(P_{k-1}^{1,1}(t)\right)^2, & \text{if } \tau = 2k-1, \\ \left(P_k^{1,0}(t)\right)^2, & \text{if } \tau = 2k. \end{cases}$$

This leads to the following bound:

$$B(n,\tau) \geq D(n,\tau) = \begin{cases} 2\begin{pmatrix} n+k-2 \\ n-1 \end{pmatrix}, & \text{if } \tau = 2k-1, \\ \begin{pmatrix} n+k-1 \\ n-1 \end{pmatrix} + \begin{pmatrix} n+k-2 \\ n-1 \end{pmatrix}, & \text{if } \tau = 2k. \end{cases} \tag{4}$$

Designs that reach this bound are called tight. The bound (4) is attained in all dimensions in the following cases: for $\tau = 1$ by two antipodal points; for $\tau = 2$ by the 2-design formed by the vertices ($n+1$ in total) of the regular n-dimensional simplex; for $\tau = 3$ by the 3-design, formed by the vertices of an orthonormal basis and their antipodal points. Apart from these cases there are only 8 known tight designs (with $n \geq 3$ and $\tau \geq 4$).

All these cases have been described by Delsarte-Goethals-Seidel in [27] and are unique up to isometry. Results for non-existence have been obtained by Banai-Damerel[4, 5], Banai-Sloen [6] and Boyvalenkov[1, 10] (see also [2, 3]).

The Levenshtein bound uses series of polynomials in different sub-intervals of $[-1, 1]$ in the following way. Let:

$$I_m = \begin{cases} \left[t_{k-1}^{1,1}, t_k^{1,0}\right], & \text{if } m = 2k-1, \\ \left[t_k^{1,0}, t_k^{1,1}\right], & \text{if } m = 2k. \end{cases}$$

and let:

$$f_m^{(n,s)}(t) = \begin{cases} (t-s)\left(T_{k-1}^{1,0}(t,s)\right)^2, & \text{if } m = 2k-1, \\ (t+1)(t-s)\left(T_{k-1}^{1,1}(t,s)\right)^2, & \text{if } m = 2k. \end{cases}$$

Here $T_k^{a,b}(t,s)$ are the so called **kernels of adjacent polynomials** or **kernels of Christoffel–Darboux**

$$T_k^{a,b}(t,s) = \sum_{i=0}^{k} r_i^{a,b} P_i^{(a+(n-3)/2, b+(n-3)/2)}(t) P_i^{(a+(n-3)/2, b+(n-3)/2)}(s)$$

where:

$$r_i^{a,b} = \frac{2i+a+b+n-2}{i+a+b+n-2} \cdot \frac{\binom{i+a+b+n-2}{i}\binom{i+a+(n-3)/2}{i}}{\binom{i+b+(n-3)/2}{i}}.$$

This leads to the following bound:

$$A(n,s) \le L(n,s) = \begin{cases} L_{2k-1}(n,s) = \binom{k+n-3}{k-1}\left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)}\right] \\ \text{for } s \in I_{2k-1}, \\ L_{2k}(n,s) = \binom{k+n-2}{k}\left[\frac{2k+n-1}{n-1} - \frac{(1+s)\left(P_k^{(n)}(s) - P_{k+1}^{(n)}(s)\right)}{(1-s)\left(P_k^{(n)}(s) + P_{k+1}^{(n)}(s)\right)}\right] \\ \text{for } s \in I_{2k}. \end{cases}$$

The function $L(n,s)$ is continuous with respect to s and in the end points of the intervals I_m we have equalities that connect it with the Delsarte-Goethals-Seidel bounds in the following way:

$$L_{2k-2}\left(n, t_{k-1}^{1,1}\right) = L_{2k-1}\left(n, t_{k-1}^{1,1}\right) = D(n, 2k-1) = 2\binom{n+k-2}{n-1}, \tag{5}$$

$$\begin{aligned} L_{2k-1}\left(n, t_k^{1,0}\right) = L_{2k}\left(n, t_k^{1,0}\right) = D(n,2k) = \\ = \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}. \end{aligned} \tag{6}$$

We are interested in the subintervals of $[-1,1)$ and the connected quadrature formulas. Let $\alpha_0, \ldots, \alpha_{k-1}$ be the roots of

$$P_k^{1,0}(t)P_{k-1}^{1,0}(s) - P_k^{1,0}(s)P_{k-1}^{1,0}(t) = 0$$

and $\beta_0 = -1$, and $\beta_1, \ldots, \beta_k$ be the roots of

$$P_k^{1,1}(t)P_{k-1}^{1,1}(s) - P_k^{1,1}(s)P_{k-1}^{1,1}(t) = 0.$$

Levenshtein proves [39, 41], that they are all different and are into $[-1,1]$. Furthermore, he also proves the following formula of Gauss-Jacobi type:

$$f_0 = \begin{cases} \dfrac{f(1)}{L_{2k-1}(n,s)} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i), & \text{if } m = 2k-1, \\ \dfrac{f(1)}{L_{2k}(n,s)} + \sum_{i=0}^{k} \gamma_i f(\beta_i), & \text{if } m = 2k. \end{cases} \tag{7}$$

for every polynomial of degree at most m.

Chapter 2. Maximal antipodal codes with few distances

The linear programming method is essentially algebraic. Working with combinatorial properties is a different approach that plays an important role in the study of codes and designs with parameters close to the optimal. In Chapter 2 of the dissertation we consider antipodal codes, for which these combinatorial properties carry additional information. The results in it are published in cite BD17 and are obtained by the classical approach of studying derived codes.

Definition 2.1 With $A(C)$ we denote the set of the inner products between different elements of a spherical code C.

Definition 2.2 For a fixed point $x \in C$ and $t \in [-1,1)$ we denote

$$A_t(x) = |\{y \in C : \langle x, y\rangle = t\}|.$$

We will call **distance distribution of C with respect to x** the system of real numbers

$$(A_t(x) : t \in [-1,1)\,, \exists y \in C, \langle x, y\rangle = t),$$

and the numbers in $A_t(x)$ **- elements of the distribution**.

In the case of antipodal code, we have the following properties:

$$A_{-1}(x) = 1 \quad \forall x \in C,$$

$$A_t(x) = A_{-t}(x) \quad \forall x \in C, \quad \forall t \in [-1,1],$$

$$\sum_{t\in(-1,1)} A_t(x) = |C| - 2 \quad \forall x \in C.$$

Definition 2.3 Let $C \in \mathbb{S}^{n-1}$ be a spherical code, $x \in C$ and α is a number for which $A_\alpha(x) > 0$. Then the set

$$C_\alpha(x) = \{y \in C : \langle x, y\rangle = \alpha\}$$

is called a derived code of C.

It's easy to see that after a suitable homothety $C_\alpha(x)$ becomes a $(n-1, |C_\alpha(x)|, s')$-code, where s' will be defined below.

Theorem 2.1 [27] All inner products of $C_\alpha(x)$ belong to the set

$$I_{\alpha,x} = \left\{ \frac{\beta - \alpha^2}{1-\alpha^2} : \beta \in A(C) \right\} \cap [-1,1).$$

In particular, the maximal inner product s' of $C_\alpha(x)$ is the largest number in $I_{\alpha,x}$, so $s' = \frac{s-\alpha^2}{1-\alpha^2}$.

Other properties of the derived codes are:

Theorem 2.2 [27] If C is a spherical τ-design, then $C_\alpha(x)$ is a spherical $(\tau - \ell + 1)$-design, where $\ell = |I_{\alpha,x}|$.

Theorem 2.3[27] Let C be a spherical τ-design with $|C| = M$ and let the number of different inner products between different points of C be q. Then if $q \leq \tau + 1$, the elements of the distance distribution $A_t(x)$ are not dependent on the choice of x and can be expressed as functions of n, M and the inner products themselves.

Codes and designs with this property are called **regular** and we will omit the argument x when we work with their distance distribution A_t, which we will call distance distribution of the code.

Antipodal codes which have inner products only -1 and $\pm s$, where $s \in (0,1)$ define sets of lines (one through every pair of points), called **equiangular lines**. They are introduced for the first time in 1948 by Haantjes [30]. They are also called tight frames, and a special case is known as SIC-POVM (symmetric, informationally complete, positive operator-valued measure), which finds application in quantum cryptography[28].

Definition 2.4 For given $n \geq 3$ and $s \in (0,1)$ we denote

$$M_n(s) := \max\{|C| : C \subset \mathbb{S}^{n-1} \text{ is antipodal and } A(C) = \{-1, \pm s\}\}.$$

One of the classic results for equiangular lines is the so-called Lloyd-type Theorem [8], which states that if $C \subset \mathbb{S}^{n-1}$ is antipodal, $A(C) = \{-1, \pm s\}$ and $|C| > 2n$ ($M_s(n) > 2n$), then $s = \frac{1}{2\ell+1}$, where ℓ is an integer.

Bounds for equiangular lines have been obtained by Barg and Yu [8]. We obtain one of their results by a direct proof, based on the integer values of the distance distribution.

Theorem 2.4 If n, k and ℓ are such that

$$P_{2k}^{(n)}(\frac{1}{2\ell+1}) < 0,$$

then

$$M_{2\ell+1}(n) \leq 2 - \frac{2}{P_{2k}^{(n)}(\frac{1}{2\ell+1})}.$$

For $k = 1$ we have $P_2^{(n)}(t) = \frac{nt^2-1}{n-1}$, from which follows the next bound.

$$M_{2\ell+1}(n) \leq \frac{8n\ell(\ell+1)}{(2\ell+1)^2 - n}, \tag{8}$$

It is valid for $n < (2\ell+1)^2$. The bound (8) is also known as relative bound [42].

For $k = 2$ we have $P_4^{(n)}(t) = \frac{(n+2)(n+4)t^4 - 6(n+2)t^2 + 3}{n^2-1}$, from which we obtain

$$M_{2\ell+1}(n) \leq \frac{2(n-2)((2\ell+1)^4(n+2) + 6(2\ell+1)^2 - n - 4)}{6(2\ell+1)^2(n+2) - 3(2\ell+1)^4 - (n+2)(n+4)}.$$

This bound is valid when

$$6(2\ell+1)^2(n+2) - 3(2\ell+1)^4 - (n+2)(n+4) > 0.$$

This bound is better than the relative for $n \geq 96$ and all ℓ. It can be expected that this tendency will continue with higher degrees being better for higher dimensions, but numerical experiments show that the $k = 2$ bound remains optimal (within the confines of Theorem) at least until $n = 900$.

The first of the two natural follow-ups of the study of equiangular lines is the study of antipodal codes $C \subset \mathbb{S}^{n-1}$ with cardinality $M = |C|$ and inner products -1, $\pm s$ and 0, equivalently $A(C) = \{-1, 0, \pm s\}$.

For these codes we obtain:

Theorem 2.5 If C is antipodal (n, M, s)-spherical code with inner products -1, $\pm s$ and 0, and we also have $s^2 < 3/(n+2)$, then

$$M \leq \frac{2n(n+2)(1-s^2)}{3 - s^2(n+2)}.$$

The proof is obtained through the main identity and the polynomial $f(t) = t^2(t^2 - s^2)$. By studying the distance distribution we obtain the following theorems:

Theorem 2.6 If C is antipodal (n, M, s)-spherical code with inner products -1, $\pm s$ and 0, which meets the bound **Theorem 2.5**, then s is rational.

Theorem 2.7 If C is antipodal (n, M, s)-spherical code with inner products -1, $\pm s$ and 0, which is spherical 3-design, and $k \geq 2$ is such that

$$P_{2k}^{(n)}(s) + (ns^2 - 1)P_{2k}^{(n)}(0) < 0,$$

then

$$M \leq \frac{n\left(2ns + (1-2s^2)P_{2k}^{(n)}(0) - P_{2k}^{(n)}(s)\right)}{\left|P_{2k}^{(n)}(s) + (ns^2-1)P_{2k}^{(n)}(0)\right|}. \quad (9)$$

The second natural extension is to add a second angle, to study "biangular" lines. They are also of certain interest, mainly engineering, and have been the topic of several articles [29, 32].

In this case the analogy of the relative bound has the following form:

Theorem 2.8 Let $C \subset \mathbb{S}^{n-1}$ be an antipodal (n, M, s_2)-code with inner products -1, $\pm s_1$ and $\pm s_2$, where $0 < s_1 < s_2 < 1$. Suppose that the following inequalities hold:

$$s_1^2 s_2^2 + \frac{3-(n+2)(s_1^2+s_2^2)}{n(n+2)} > 0,$$

$$6-(n+4)(s_1^2+s_2^2) > 0.$$

Then

$$M \leq \frac{n(n+2)(1-s_1^2)(1-s_2^2)}{n(n+2)s_1^2 s_2^2-(n+2)(s_1^2+s_2^2)+3}.$$

As in the previous case we can present a bound expressed with the angles.

Theorem 2.9 Let $C \subset \mathbb{S}^{n-1}$ be an antipodal (n, M, s_2)-code with inner products -1, $\pm s_1$ and $\pm s_2$, where $0 < s_1 < s_2 < 1$, which is also a spherical 5-design, $k \geq 2$ and

$$(1-ns_1^2)P_{2k}^{(n)}(s_1)+(1-ns_2^2)P_{2k}^{(n)}(s_2) < 0.$$

Then

$$M \leq \frac{2n\left((1-s_1^2)P_{2k}^{(n)}(s_1)+(1-s_2^2)P_{2k}^{(n)}(s_2)+s_2^2-s_1^2\right)}{\left|(1-ns_1^2)P_{2k}^{(n)}(s_1)+(1-ns_2^2)P_{2k}^{(n)}(s_2)\right|}.$$

Chapter 3. Upper bounds for the energy of spherical designs with relatively low cardinality

In Chapter 3 we focus on the problem of finding upper bounds of the h-energy of spherical τ-designs with M points over $\mathbb{S}^{n-1}$, where M is close to the DGS bound(4). We base our approach and make parallels to the results for the energies of spherical designs obtained by Boyvalenkov et al. in [19, 20]. The results have been published originally in [14].

Our main result shows that spherical designs are, in a certain way energy-effective - all designs with a relatively small cardinality have h-energy in a very narrow interval. Furthermore the upper bound results are very close to those obtained in [19, 20] and are also valid for all absolutely monotone potentials.

In [11] Boyvalenkov, Danev and Boumova note that Levenshtein's formula can be used for the study of spherical designs designs when $L_\tau(n, s)$ is replaced by the cardinality M of a hypothetical spherical τ-design. This allows us to consider $M = L_\tau(n, s)$ as a function of the points and the coefficients in the formula, if τ is chosen such that $M \in (D(n, \tau), D(n, \tau+1))$.

Let's introduce the following notation:

$$u(n, M, \tau) := \sup\{u(C) : C \subset \mathbb{S}^{n-1} \text{ is } \tau\text{-design}, |C| = M\},$$

where $u(C) := \max\{\langle x, y\rangle : x, y \in C, x \neq y\}$, and

$$\ell(n, M, \tau) := \inf\{\ell(C) : C \subset \mathbb{S}^{n-1} \text{ is } \tau\text{-design}, |C| = M\},$$

where $\ell(C) := \min\{\langle x, y\rangle : x, y \in C, x \neq y\}$.

For every n, τ and $M \in (D(n,\tau), D(n,\tau+1))$ there exist non-trivial bounds for $u(n,M,\tau)$, obtained by Boyvalenkov, Boumova and Danev[11] and Boumova, Boyvalenkov, Kulina and Stoyanova[9].

Boyvalenkov et al in [19] introduce the following general technique for obtaining upper bounds for the energy of spherical codes with given dimension cardinality and strength by using linear programming.

Theorem 3.2 [19] Let n, τ, and $M \geq D(n,\tau)$ be integer numbers. Let $h : [-1,1] \to [0,+\infty]$ and $I \in [-1,1)$ and $g(t) = \sum_{i=0}^{\deg(g)} g_i P_i^{(n)}(t)$ is a polynomial with real coefficients for which:

(D1) $g(t) \geq h(t)$ for $t \in I$;

(D2) The coefficients $g(t)$ in its Gegenbauer expansion satisfy $g_i \leq 0$ for $i \geq \tau+1$.

If $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design, for which $|C| = M$ and $\langle x,y\rangle \in I$ then for every two points $x, y \in C$. Then $E(n,C;h) \leq M(g_0 M - g(1))$. In particular, if $[\ell(n,M,\tau), u(n,M,\tau)] \subseteq I$, then

$$\mathcal{U}(n,M,\tau;h) \leq M(g_0 M - g(1)). \tag{10}$$

The proof of the theorem is similar to the proof of the classical linear programming bounds theorems.

In [20] and [19] Boyvalenkov et al. obtain lower bounds for the energy of spherical codes and designs, which is universal in the context of Levenshtein [41]. For spherical designs the bound has the following form.

Let's introduce the following notation:

$$\mathcal{L}(n,M,\tau;h) := \inf\{E(n,C;h) : |C| = M,\ C \subset \mathbb{S}^{n-1} \text{ is } \tau\text{-design}\}. \tag{11}$$

Then we have the theorem:

Theorem 3.3 [19] If $n \geq 3$, τ, and $M \in [D(n,\tau), D(n,\tau+1))$ are integers and $h : [-1,1] \to [0,+\infty]$ is absolutely monotone. Then:

$$\mathcal{L}(n,M,\tau;h) \geq \begin{cases} M^2 \displaystyle\sum_{i=0}^{k-1} \rho_i h(\alpha_i), & \text{if } \tau = 2k-1, \\ M^2 \sum_{i=0}^{k} \gamma_i h(\beta_i), & \text{if } \tau = 2k \end{cases}. \tag{12}$$

Let $n \geq 3$, τ, and $M \in (D(n,\tau), D(n,\tau+1))$ be integer numbers and $h : [-1,1] \to [0,+\infty]$ - absolutely monotone. In the current chapter we present two polynomial constructions which satisfy (D1) and (D2) of Theorem and thus present valid bounds for $\mathcal{U}(n,M,\tau;h)$.

In order to improve the bounds, we use Hermite interpolation, but not in $[-1,1]$. We use $[-1,u]$ instead, where u is a valid upper bound for the maximal inner product.

Case 1. $\tau = 2k-1$ Consider the polynomial $g(t)$, Hermite interpolant of h in the roots -1 and $t^{1,1}_{k-1,i}$, $i = 1,2,\ldots,k-1$, of $d^{(n)}_{2k-1}(t)$ (DGS polynomial 4)

Case 2. $\tau = 2k$. Consider the polynomial $g(t)$, Hermite interpolant of h in the roots $t^{1,1}_{k-1,i}$, $i = 1,2,\ldots,k-1$, of the polynomial $d^{(n)}_{2k}(t)$

These bounds can be also written as a weighted sum of values of $h(t)$, by using Levenshtein's quadrature rule.

Theorem 3.4 Let $n \geq 3$, τ and $M \in (D(n,\tau), D(n,\tau+1))$ be integers and $h : [-1,1] \to [0,+\infty]$ - absolutely monotone. Then:

$$\frac{\mathcal{U}(n,M,\tau;h)}{M^2} \leq \begin{cases} g_0\left(1 - \frac{D(n,2k-1)}{M}\right) + \frac{D(n,2k-1)}{M}\sum_{i=0}^{k-1}\gamma_i h(t^{1,1}_{k-1,i}), & \tau = 2k-1 \\ g_0\left(1 - \frac{D(n,2k)}{M}\right) + \frac{D(n,2k)}{M}\sum_{i=0}^{k-1}\rho_i h(t^{1,0}_{k,i+1}), & \tau = 2k \end{cases} \tag{13}$$

(here $t^{1,1}_{k-1,0} := -1$ for odd τ).

The first of the two ways in which we present the bound uses the values of h, the second will use the universal lower bound instead and the parameters $\rho_i, \alpha_i, \gamma_i$ and β_i.

Theorem 3.5 Let $n \geq 3$, τ, and $M \in (D(n,\tau), D(n,\tau+1))$ be integers and $h : [-1,1] \to [0,+\infty]$ - absolutely monotone. Then the following holds:

$$\frac{\mathcal{U}(n,M,\tau;h)}{M^2} \leq \begin{cases} \frac{ULB}{M^2} + \sum_{i=0}^{k-1}\rho_i(g(\alpha_i) - h(\alpha_i)), & \text{if } \tau = 2k-1 \\ \frac{ULB}{M^2} + \sum_{i=0}^{k}\gamma_i(g(\beta_i) - h(\beta_i)), & \text{if } \tau = 2k, \end{cases} \tag{14}$$

where ULB is the universal lower bound (12) and ρ_i, α_i, γ_i, and β_i are the same as in (12).

We also provide asymptotic result for the behaviour of the bound.

Theorem 3.8 If n and M tend to infinity with the following relation $M = \lambda n + o(1)$, where $\lambda \in (1,2)$ is a constant, then:

$$h(0) + \frac{h(1-\lambda) - \lambda h(0)}{M(\lambda-1)} \leq \frac{\mathcal{L}(n,M,2;h)}{M^2} \leq \frac{\mathcal{U}(n,M,2;h)}{M^2}$$
$$\leq h(0) + \frac{h(\lambda-1) + (\lambda-2)h(0) - (\lambda^2 - 3\lambda + 3)h'(0)}{M(\lambda-1)} + o(M^{-2}).$$

The proof is by direct computation.

Chapter 4. On q-ary codes with distances d and $d+1$

Chapters 4 and 5 present the results of the work with D.V. Zinoviev and V.A. Zinoviev [15, 17] on q-ary codes in Hamming space with two distances. With the main focus being on distances which are close to each other, both in the linear and the general case. Families of such codes are considered, based on connections with various classical structures such as equidistant codes. General bounds for the cardinality are also presented.

The motivation to study codes with two adjacent distances comes from two sources. First, the question how much richer the structure of these codes would be compared to the good equidistant codes in the same spaces. Second, due to their close distances, they will be codes with almost constant energy under amplitude-phase modulation, which is potential practical application.

In previous chapters, we used Gegenbauer polynomials and their properties due to their connection with harmonic polynomials and thus - the existence of the main identity that leads to the linear programming bounds. They are, however, only relevant for the n-dimensional sphere. For q-ary codes analogical functions have the Krawtchouk polynomials.

For fixed n and q, normalized Krawtchouk polynomials are defined as

$$Q_i^{(n,q)}(t) = \frac{1}{r_i} K_i^{(n,q)}(d),\ d = \frac{n(1-t)}{2},\ r_i = (q-1)^i \binom{n}{i},$$

where

$$K_i^{(n,q)}(d) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{d}{j} \binom{n-d}{i-j}.$$

are the standard Krawtchouk polynomials [34, 45]; see also [40]. We shall work only with $Q_i^{(n,q)}(t)$.

We have considered two types of bounds - bounds stemming from linear programming and bounds stemming from a bijection between codes in $\mathbb{Q}^n$ and codes in $\mathbb{S}^{n(q-1)-1}$. In Chapter 4 we consider the case where distances are adjacent and in the next - the general case of d and $d+\delta$.

Let

$$A_q(n;\{d,d+1\}) = \max\{|C| : C \text{ is } (n,|C|,\{d,d+1\})\text{-code}\},$$

be the maximal cardinality of a code in Q^n with distances d and $d+1$.

The classical results for two-distance codes are Delsarte's harmonic bound[26] and it's improvement by Barg and Musin [7]. Both, however, do not reflect well the specifics of the adjacent distances and are relatively inaccurate in this case.

The classical linear programming bound for q-ary codes [26, 40] has the following form in our case.

Theorem 4.6 Let $n \geq q \geq 2$ and $f(t)$ be a polynomial with real coefficients of degree at most $m \leq n$ for which the following conditions hold:

(A1) $f(t) \leq 0$ for every $t \in \{1-2d/n, 1-2(d+1)/n\}$;

(A2) The Krawtchouk coefficients in $f(t) = \sum_{i=0}^{m} f_i Q_i^{(n,q)}(t)$ are non-negative for every i.

Then $A_q(n; \{d, d+1\}) \leq f(1)/f_0$. If the bound is attained for a $(n, N, \{d, d+1\})_q$-code$C$ and polynomial $f(t)$, then $f(1-2(d+i)/n) = 0$, $i = 0, 1$ and $f_i M_i(C) = 0$.

Here again

$$M_i(C) = \sum_{x,y \in C} Q_i^{(n,q)}(1 - 2d(x,y)/n)$$

is the i-th moment of C, but defined for q-ary codes in the same manner.

If we use $f(t) = t - 1 + 2d/n$ in Theorem we obtain

$$\frac{f(1)}{f_0} = \frac{2d}{2d-n},$$

valid for $f_0 > 0$. This is the classic Plotkin bound.

If we use quadratic polynomials we obtain:

Theorem 4.8 If $d \geq (n-1)(q-1)/q$, then

$$A_q(n; \{d, d+1\}) \leq \frac{q^2 d(d+1)}{n^2(q-1)^2 - n(q-1)(2dq+q-1) + dq^2(d+1)}. \tag{15}$$

If the bound (22) is obtained by a $(n, N, \{d, d+1\})_q$-code$C$, then$M_2(C) = 0$ and $M_1(C)(dq - (n-1(q-1))) = 0$.

This bound is attained in some cases. For example

$$A_q(n; \{d, d+1\}) \leq q^2$$

for $d = n-1$ is met for $(q, n) = (3,3), (3,4), (4,5)$, and $(5,6)$. All such cases are marked with $d2$ in the attached tables. By studying the moments sometimes one can achieve additional improvements. Those cases are denoted with n.

Other polynomials also give good bounds, such as $f(t) = 1 + (q-1)nQ_{(n(q-1)+1)/q}^{(n,q)}(t)$ and $f(t) = 1 + \frac{n+2}{2} Q_{n/2}^{(n,2)}(t) + \frac{n}{2} Q_{1+n/2}^{(n,2)}(t)$. These cases will be denoted with a in the tables for upper and lower bounds.

For the construction of good codes with two adjacent distances pivotal role have the equidistant codes. The are one of the classic objects in Coding theory on which we have a vast number of results, i.e. the Johnson bound and many examples of codes with good parameters.

Following [17] we describe the following constructions:

- Sphere of radius 1, with or without parity check
- Equidistant code with an arbitrary column added or removed
- Concatenations of equidistant code with a code that has two adjacent distances

- Codes based on Latin squares
- Codes based on matrices

An interesting way to obtain bounds for q-ary codes is through a bijection between q-ary code and a spherical code on $\mathbb{S}^{(q-1)n-1}$. This bijection is well known and through it and the results of Larman, Rogers and Seidel [36] we can obtain the following bound for $A_q(n,\{d,d{+}1\})$.

Theorem 4.13 If $d > (\sqrt{2(q-1)n}-1)/2$, then

$$A_q(n,\{d,d+1\}) \le 2(q-1)n+1.$$

This bound is usually better than a linear programming bound obtained through the simplex method, for sufficiently large n and middle values of d . The smallest examples are $(q,n,d) = (2,13,4)$, $(q,n,d) = (3,9,3)$, $(q,n,d) = (4,8,3)$ and $(q,n,d) = (5,7,4)$.

In the dissertation we have summarized the results for $q = 2,3,4,5$. For the lower bound we the best between the computer-generated examples and codes coming from the aforementioned constructions.

Upper bounds are chosen as the best from the linear programming bounds obtained with the simplex method, with ad-hoc polynomials (denoted by $d2$, n and a), through the best possible value for $A_q(n,d)$, taken from [23] (denoted by $*$), and through spherical codes (Theorem - denoted by sc).

Computer-generated codes were obtained through a random walk with backtrack. For $q = 2$ this allowed results for up to $n = 26$, for $q = 3$ - up to $n = 14$ and $q = 4$ - up to $n = 12$.

Chapter 5. Upper linear programming bounds for codes with two non-adjacent distances

Chapter 5 presents a generalisation of Chapter 4 where distances d and $d+\delta$ are studied for small values of δ. These codes we denote as $(n,N,\{d,d+\delta\})_q$-codes. The results have been published in [16, 18].

The linear programming bound has the following form:

Theorem 5.1 Let $n \ge q \ge 2$ and $f(t)$ be a polynomial with real coefficients which satisfies the following conditions:

(A1) $f(t) \le 0$ for $t \in \{1-2d/n, 1-2(d+\delta)/n\}$;

(A2) The Krawtchouk coefficients f_i are non-negative for every $i \ge 1$.

Then

$$A_q(n;\{d,d+\delta\}) \le \frac{f(1)}{f_0}. \tag{16}$$

If a $(n,N,\{d,d+\delta\})_q$-code C attains the bound for a given polynomial $f(t)$, then $f(1-2(d+i)/n) = 0$, $i = 0,\delta$, when there are elements of C at a distance of $d+i$, $i = 0,\delta$, and $f_iM_i(C) = 0$, where M_i is the i-th moment of C.

The proof is analogous to the proofs of the other theorems of this type.

Again for $f(t) = t - 1 + 2d/n$ we have the Plotkin bound, which is attained for many cases with large d. Using quadratic polynomials we can get the following result for the parameters of a q-ary code with two distances.

Theorem 5.2 If C is a q-ary two-weight $(n, N, \{w_1, w_2\})_q$-code, which is also an orthogonal array with strength $t \geq 2$, (thus M is a multiple of q^2) and if $u_i = n - w_i,\ i = 1, 2$ then the following holds:

- The length n satisfies the following equation:

$$n^2 - n\left(Q_1(u_1 + u_2 - 1) + 1\right) + Q_2 u_1 u_2 = 0, \tag{17}$$

$$Q_1 = q\,\frac{N - q}{N - q^2},\quad Q_2 = q^2\,\frac{N - 1}{N - q^2}. \tag{18}$$

- If $w_2 = n$ (equivalently $u_2 = 0$), then the length n of C satisfies

$$n = \frac{Q_1(d + 1) - 1}{Q_1 - 1}, \tag{19}$$

  where $d = w_1$ is the minimal distance in C.

This theorem also has a combinatorial proof, proposed by D. Zinoviev and V. Zinoviev.

With a quadratic polynomial we can also obtain a bound that's identical with the bound of Helleseth-Klove-Levenshtein for codes with known minimal and maximal distance [31], as well as the bound for $k = 1$ in Theorem 5.2 in [22].

Theorem 5.3 If

$$q(2d + \delta) \geq 2nq + 2 - 2n - q, \tag{20}$$

$$n(q - 1)(nq - n + 1) + nq(2d + \delta) > q^2(2nd + n\delta - d^2 - d\delta), \tag{21}$$

then

$$A_q(n, \{d, d + \delta\}) \leq \frac{d(d + \delta)q^2}{n(q - 1)(nq - n + 1) - q^2(2nd + n\delta - d^2 - d\delta) + nq(2d + \delta)}. \tag{22}$$

If a given code C with parameters $(n, N, \{d, d + \delta\})_q$ attains the bound then the code is also an orthogonal array with strength 2.

As in Chapter 2, if the right hand side of (22) is an integer, we can find the distance distribution of C, which gives several new non-existence results such as : $A_2(12, \{6, 10\}) \leq 19$ instead of 20, $A_2(20, \{10, 14\}) \leq 27$ instead of 28, and $A_2(16, \{8, 14\}) \leq 27$ instead of 28. we denote these cases with (22).

As in the case $\delta = 1$, we have the following theorem, due to the connection to spherical codes:

Theorem 5.4 Let $\frac{d}{d+\delta} = \frac{r}{s}$ in lowest terms. Then if $s - r \geq 2$ (in particular when $\mathrm{GCD}(d, d+\delta) = 1$), we have

$$A_q(n, \{d, d+\delta\}) \leq 2(q-1)n + 1.$$

For several infinite series of codes we can prove nonexistence or find their cardinality directly, through combinatorial arguments and the triangle inequality.

Lemma 5.1 Codes $C \subset E_2^n$ with distances d and $d+\delta$ do not exist of:

(a) d and $d+\delta$ are both odd;

(b) d is odd, $d+\delta$ is even and $n < (3d-\delta)/2$;

(c) d is odd, $d+\delta$ is even and $d < \delta$;

(d) $d+\delta = n$ and $n \neq 2d$;

(e) $d+\delta = n-1$ and $2d > n+1$.

Lemma 5.2 For $q = 2$, if d is odd and $|C| > 4$, then

$$A_2(n, \{d, 2d\}) = 1 + \left[\frac{n}{d}\right].$$

Lemma 5.3 $A_3(n, \{1, 3\}) = 6$ for every $n \geq 4$.

As in the case for $q = 1$ we have gathered the results for $q = 2, 3, 4, 5$ in table form, with lower bounds presenting the better result between computer-generated codes and the described constructions.

Computer-generated codes are obtained by a modification of the same software, described in the previous chapter.

Upper bounds are the best between the linear programming bounds obtained through the general simplex method (denoted with lp), through distance distribution arguments (dd), through ad-hoc polynomials ($d2$), best known bound for $A_q(n, d)$, taken from [23] ($*$), and the theorem for spherical codes (sc).

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New Results

In the current work, the following new results have been described:

- New bounds for the cardinality of antipodal spherical codes with few distances, motivated by the study of equiangular lines.
- Lloyd-type results for rationality of the inner products of maximal antipodal codes with few distances.
- Upper bounds for the energies of spherical designs with cardinality close to the Delsarte-Goethals-Seidel bound
- Asymptotic bounds for the energies of spherical 2-designs
- Algorithm for constructing codes with two adjacent or close distances through random search with backtracking
- Constructions of q-ary codes with two adjacent distances, based on algebraic or combinatorial structures
- Constructions of q-ary codes with two close distances, based on algebraic or combinatorial structures
- Bounds on the cardinality and nonexistence results for q-ary codes with two adjacent distances, based on linear programming, bijection with spherical codes and others
- Bounds on the cardinality and nonexistence results for q-ary codes with two close distances, based on linear programming, bijection with spherical codes and others

Publications in the scope of the dissertation

1. Boyvalenkov, P., Delchev, K.. On maximal antipodal spherical codes with few distances. *Electronic Notes in Discrete Mathematics* 57, Elsevier, 2017, P.85-90. SJR (Scopus): 0.262

2. Boyvalenkov, P., Delchev, K., Zinoviev, D., Zinoviev, V.. Codes with two distances: d and $d+1$. *Proceedings of 16th International workshop on Algebraic and Combinatorial Coding Theory*, Kaliningrad, Russia, Sep 2-8, 2018, P. 40-45, 2018.

3. Boyvalenkov, P., Delchev, K., Zinoviev, D., Zinoviev, V.. On q-ary codes with two distances d and $d+1$. *Problems of Information Transmission* 56, 1, Springer, 2020, 33-44. SJR (Scopus): 0.506, JCR-IF (Web of Science): 0.593

4. Boyvalenkov, P., Delchev, K., Zinoviev, D., Zinoviev, V.. On two-weight codes. *Discrete Mathematics*, 344, 5, paper no. 112318, Elsevier, 2021, SJR (Scopus): 0.824, JCR-IF (Web of Science): 0.77

5. Boyvalenkov, P., Delchev, K., Jourdain, M.. Upper energy bounds for spherical designs of relatively small cardinalities. *Discrete and Computational Geometry* 65, 1, Springer, 2021, P. 244–260, SJR (Scopus): 0.611, JCR-IF (Web of Science): 0.621

6. Boyvalenkov, P., Delchev, K., Zinoviev, D., Zinoviev, V.. On two-weight (linear and non-linear) codes, *Proceedings of 17th International workshop on Algebraic and Combinatorial Coding Theory*, Bulgaria (on-line), Oct 11-17, 2020, 40-45

Citations of the publications

1. Boyvalenkov, P., Delchev, K.. On maximal antipodal spherical codes with few distances. *Electronic Notes in Discrete Mathematics* 57, Elsevier, 2017.

is cited in:

1 Womersly, R., "Efficient Spherical Designs with Good Geometric Properties", In: Dick J., Kuo F., Woźniakowski H. (eds) *Contemporary Computational Mathematics - A Celebration of the 80th Birthday of Ian Sloan*, Springer, Cham, 2018, P. 1243-1285.

2 Ganzhinov, M., Szöllősi, F., Biangular lines revisited, 2019 https://arxiv.org/abs/1910.05950, accessed: 02.02.2021.

2. Boyvalenkov, P., Delchev, K., Zinoviev, D., Zinoviev, V.. Codes with two distances: d and d+1. *Proceedings of 16th International workshop on Algebraic and Combinatorial Coding Theory, Kaliningrad, Russia, Sep 2-8, 2018*, 40-45, 2018.

is cited in:

3 Landjev, I., Rousseva, A., Storme, L., On linear codes of almost constant weight and the related arcs, *Comp. Rend. Acad. Bulg. Sci.* 72(12), 2019, P. 1626-1633.

3. Boyvalenkov, P., Delchev, K., Zinoviev, D., Zinoviev, V.. On q-ary codes with two distances d and d+1. *Problems of Information Transmission* 56, 1, Springer, 2020, 33-44.

is cited in:

4 Landjev, I., Rousseva, A., Storme, L., On linear codes of almost constant weight and the related arcs, *Comp. Rend. Acad. Bulg. Sci.* 72(12), 2019, P. 1626-1633.