R E V I E W

on a Dissertation

for obtaining the educational and scientific degree "Doctor"

Research area: 4. Natural Sciences, Mathematics and Informatics,

Professional field: 4.6. Informatics and Computer Science

Author: Konstantin Vassilev Delchev

Title: Codes and Designs in Polynomial Metric Spaces

Referee: Prof. Stefka Hristova Bouyuklieva
Faculty of Mathematics and Informatics,
St. Cyril and St. Methodius University of Veliko Tarnovo

I am a member of the scientific panel for this procedure according to order No. 48/26.03.2021 of the Director of Institute of Mathematics and Informatics, Bulgarian Academy of Sciences. As a member of the scientific panel, I have received all the administrative and scientific documents required by the Act on the Development of the Academic Staff in the Republic of Bulgaria (ADASRB), the Rules for its implementation and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at IMI and BAS.

1. Personal data.

Konstantin Delchev graduated in mathematics at Sofia University "St. Kliment Ohridski" in 2015. The same year he was enrolled in the PhD program in Informatics and Computer Science at the Department of Mathematical Foundations of Informatics of Institute of Mathematics and Informatics. He is very active in the project activity, as well as in the organization of competitions for pupils and students. Konstantin Delchev is the deputy national organizer of the national competition "Young Talents". He has been responsible for the training of the Bulgarian team for the EUCYS international competition since 2011. He himself participated in the European competition in 2003. Unfortunately, the CV he presented does not contain information about other competitions and contests in which the dissertation participated as a student..

2. Relevance of the topic.

The dissertation of Konstantin Delchev is devoted to research in Algebraic Coding Theory, related to the construction of different types of codes, as well as to the study and improvement of bounds for some basic parameters of these codes. Basically, two large families of codes are considered, namely spherical codes and block codes with two distances.

Spherical codes are sets of points in the unit sphere in the n-dimensional Euclidean space. The author's research is devoted to antipodal codes (i.e. codes C for which $C = -C$) with a small number of distances (Chapter 2), as well as to spherical designs with relatively small cardinality (Chapter 3). There is a group of mathematicians in Bulgaria who do research on spherical codes and designs. Several dissertations have already been defended in this field, the first of which is by Prof. Boyvalenkov, and the others are under his supervision.

The codes with two distances considered in the dissertation are block codes over an alphabet with q elements. The motivation is to consider a family of codes that have good characteristics and useful structure (for practical and theoretical applications) and optimal values of the considered parameters, having in mind the very detailed research on codes with a fixed distance between every two codewords (equidistant codes). The author proposes constructions and methods to prove upper bounds for the cardinality of codes with two distances d and $d + \delta$, $\delta \geq 1$, as a special attention in Chapter 4 is paid to the case $\delta = 1$.

3. General characterization of the dissertation.

The dissertation consists of 84 pages and is structured in five chapters and references of 100 titles. The first chapter called “Introduction” presents basic notions and classical results about spherical codes and designs, obtained by the linear programming. In my opinion, it would be better if Chapter 1 has another title, but before that to have a real introduction to the dissertation that gives a general idea of its content and structure.

Chapter 2 is devoted to the antipodal spherical codes, and the results are obtained by examining their derivative codes. Techniques related to linear programming are used, as well as some combinatorial properties of these codes. Bounds for antipodal spherical codes with several possible distances are obtained. Theorems on the maximal possible cardinality of such codes in the cases when the different inner products are $(1) - 1$ and $\pm s$; $(2) - 1, 0$ and $\pm s$;

and $(3) - 1, \pm s_1$and $\pm s_2$, are proved. The spherical codes consists of points (vectors) in the Euclidean space $\mathbb{R}^n$, so there is a relationship between the distance and the inner product, namely $d(x,y) = \sqrt{2(1 - \langle x,y \rangle)}, x,y \in \mathbb{R}^n$. It is more convenient for the researchers to use the inner product due to its relation with the theory of orthogonal polynomials. This chapter follows the publication of Konstantin Delchev with his advisor in the scientific journal *Electronic Notes in Discrete Mathematics* from 2017.

In Chapter 3 the author considers bounds on the energies of spherical designs with relatively small cardinality. The spherical designs are a special class of spherical codes that can be applied in approximation theory, in statistics for experimental design, in combinatorics, etc. The author has given three equivalent definitions but here I present the following taken from R. H. Hardin and N. J. A. Sloane: A set of M points is called a spherical τ -design if the integral of any polynomial of degree at most τ over the sphere is equal to the average value of the polynomial over the set of M points. Given a value of M, one wishes to choose the M points so as to maximize τ. Gegenbauer polynomials are very important in the research on spherical designs. They can be defined recursively as follows:

$$(i+n-2)P_{i+1}^{(n)}(t) = (2i+n-2)tP_i^{(n)} - iP_{i-1}^{(n)}, \qquad P_0^{(n)}(t) = 1, P_1^{(n)}(t) = t.$$

The h-energy of a spherical code (in particular a $\tau -$ design) C is defined as

$$E(n,C;h) = \sum_{x,y \in C, x \neq y} h(\langle x,y \rangle),$$

where $h: [-1,1] \to [0,+\infty]$. The author derives upper bounds for the potential energy of spherical designs of cardinality close to the Delsarte–Goethals–Seidel bound. These bounds are obtained by linear programming with use of the Hermite interpolating polynomial of the potential function in suitable nodes. Numerical computations show that the results are quite close to certain lower energy bounds confirming that spherical designs are, in a sense, energy efficient.

The last two chapters are devoted to another type of codes, namely block codes over an alphabet with q elements. They are based on the joint research of the author with his supervisor prof. Boyvalenkov and the colleagues from Russian Academy of Sciences V. Zinoviev and D. Zinoviev. Without loss of generality, we can take the alphabet to be $Q = \{0,1,\dots,q-1\}$. These codes are subsets of Q^n for a given length n. I have the following remark to the author: If we want Q^n to be a linear space, it is necessary for the alphabet Q to be a field. If we consider nonlinear codes, this is not necessary but for linear codes q must be a prime power and the considered alphabet must be a field. In the literature linear codes over rings are also studied (for example $Q = \mathbb{Z}_q$ for an arbitrary positive integer q), but if such a ring is not a field, Q^n is not a linear space and then the definition for a linear code is a little bit different.

In Chapter 4 the author considers block codes with two distances d and d+1. Several constructions of such codes are given. Upper bounds for the maximum cardinality of such codes are derived. Tables of lower and upper bounds for small q and n are presented. I did not understand why linear codes are denoted by $[n, M, d]$ instead of $[n, k, d]$ $(M = q^k)$. In the paper [34] the standard notation is used. I noticed that the author did not include in the dissertation the statements for linear codes obtained by constructions 4, 5 and 6, as well as the results from Section 4 of [34], related to linear codes. I suppose that this part of the research has been done by the Russian coauthors. If so, it would be better not to address the issue of linearity in the dissertation at all.

In Section 4.3 upper bounds for the cardinality of codes with two distances are presented. This section has a title *Upper bounds for codes with two distances*, as the option for two consecutive distances is presented in Subsection 4.3.1. This does not seem to me to be well thought out, because in practice only the case of two consecutive distances is considered in this section. Constructions for codes of the considered type are derived in Section 4.5. Section 4.6 presents tables with upper and lower bounds for the cardinality of codes with two consecutive distances over alphabets with 2, 3, 4 and 5 elements for $n \leq 18, d \leq 13$. The table for $q = 2$ is not complete as it is in [34].

Chapter 5 is a kind of extension of the previous chapter, as the author considers again block codes with two distances but these distances are d and $d + \delta$ for different natural numbers δ.

The dissertation is well balanced in content and follows the papers published during the doctoral program. I have already mentioned my remark on the introduction. The text is relatively well formed, but there are some inaccuracies in the terminology. The references of 100 titles shows a very good knowledge of Konstantin Delchev about spherical codes and the combinatorial structures used.

4. Contributions and importance of the results obtained

I accept and approve the submitted contributions, indicated by the author. I would highlight the following contributions:

- About the spherical codes:

    - New bounds for the antipodal spherical codes with few distances are derived.
    - Upper bounds for the potential energy of spherical designs with a given cardinality are proved.
- About the block codes with two distances:
    - Several constructions of codes with two distances are given.
    - Upper bounds for the maximum cardinality of codes with two consecutive distances are proved.
    - Necessary conditions for existence of codes with two given distances are derived.

5. Publications and citations

Konstantin Delchev's dissertation is based on 6 publications. All of them are written in English and published in international scientific journals and in proceedings of international conferences. Four of the articles are with SJR, and three of them have a JCR-IF from Web of Science.

The first serious scientific paper of Delchev is co-authored with his supervisor. The two publications in the proceedings from the international conferences ACCT in 2018 and 2020, as well as two of the journal papers, are co-authored by P. Boyvalenkov, V. Zinoviev and D. Zinoviev. The latest paper, published in 2021, has two co-authors - P. Boyvalenkov and the graduate student Matthieu Jourdain, who was an intern at IMI. My opinion is that the contribution of Konstantin Delchev in the joint publications is equal.

The results of this dissertation were reported at two conferences on algebraic and combinatorial coding theory ACCT'2018 in Russia and ACCT'2020, held online. Konstantin Delchev has presented a list of four citations.

6. The author's summary

The author's summary is made in accordance with the rules and accurately and completely assesses the main results obtained in the thesis.

7. Personal impressions

I have known Konstantin Delchev for about ten years, since he participated in the annual seminar on Coding Theory. About 4-5 years ago I had to work with him on the preparation of a project documents for the Bulgarian Science Fund. I was impressed by his paperwork skills and his responsiveness. Delchev is an intelligent young man with diverse interests, who has already gained experience in organizing various events, preparing young people to participate in Olympiads and competitions, as well as in applying for research and educational projects.

8. Comments and Recommendations

The dissertation follows the papers published by Delchev, as the first chapter presents the main definitions, theorems and classical results related to spherical codes. The title also suggests that the original idea was for the entire dissertation to be devoted to spherical codes, but the topic of codes with two distances was later developed. Therefore, the introduction to this topic is only in Chapter 4 and does not present the classic results for this type of codes.

I have already made a few remarks on terminology and notations. I could add to the list of such remarks, for example on page 43 in the definition of weight should be written "non-zero coordinates", on the same page instead of "codes over Q" it says "codes over Q^n“, it is said about points in a code instead of codewords, etc.

9. Conclusion

The presented dissertation satisfies all the criteria and indicators of the law and the regulations in Bulgaria. After I familiarized myself with the presented dissertation, the importance of the research and the scientific and applied contributions contained therein, I give an overall positive assessment to the applicant **Konstantin Vassilev Delchev** to obtain the scientific degree "Doctor" in

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17.05.2021 г. Reviewer:

/Prof. Stefka Bouyuklieva/