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SYMPLECTIC TOPOLOGY, NON-COMMUTATIVE GEOMETRY, AND MIRROR SYMMETRY

DISSERTATION

for awarding of the scientific degree

"Doctor of Science"

Higher education area: 4. Natural Sciences, Mathematics and Informatics

Professional Field: 4.5. Mathematics

Scientific specialty: "Geometry and Topology"

Sofia, 2023

Contents

Introduction.

Homological Mirror Symmetry is a new direction in Modern mathematics. The main purpose of this disseration is developing Homological Mirror Symmetry for the benefit of classical birational geometry - showing that generic four dimensional cubic and other Fano manifolds are not rational.

In fact the main idea in this dissertaion is to use ideas of theoretical physics in order to solve classical problems in Algebraic G‘qeometry.

Our considerations are based on two major parts of Theoretical Physics

1) **Conformal Field Theory** - a quantum field theory that is invariant under conformal transformations.

2) **Mirror Symmetry** - in the Hori Vafa interpretation and its categorical upgrade made by Kontsevich.

The development of Conformal Field Theory begins with the two-dimensional case. It is consolidted within the 1983 article by Belavin, Polyakov and Zamolodchikov.

In the two-dimensional quantum theory we have the Witt algebra of infinitesimal conformal transformations which is centrally extended, with a central charge and other renormalization charges - spectra of dimensions. Alexander Zamolodchikov has proven the Zamolodchikov C-theorem which in particular tells us that renormalization group flow in two dimensions is irreversible.

Computing the charges of Conformal Field theories is a chanlenging exercise in general. In the case of massive theories one can use geometry in order to compute them. The theory of spectra of singularities was developed in a parallel way to the theory of central charges. In fact it was developed in the same city - in Moscow - by Arnold and Varchenko. The spectra of singularities corresponds to the charges of conformal field theories and the Zamolodchikov C-theorem is the semicontinuity theorem in the theory of spectra of singularities. The full corresponcence between charges of Conformal Field theories spectra of singularity and assymptotics of solutions of ODE was indicated by Vafa and Cecotti in the nineties.

In this disseration we connect the above correspondence with the Homological Mirror Symmetry - the second building block of theoretical physics we use. The main body of the dissertaion is developing the Homological Mirror Symmetry. Mirror Symmetry started as a theory allowing counting curves - done by physicists. For us that counting of curves - Gromov-Witten theory - is a tool which leads to higher structures and as a result higher order applications.

The point we take is that from the non-commutative geometry perspective Birational Geometry (derived categories) is mirror to theory of singularities (the category of vanishing cycles).

We start with the very simple case - rational surfaces where birational geometry is rather easy. First we develop the categorical foundations of Homological Mirror Symmetry. We establish the fact that the birational transfornmations lead to creation of new singularities on

the mirror site. We extend this correspondence in general - this is the main content of the first part of the disseratation.

In the second part we develop the theory of Noncommutative Hodge Structures. We also show that the quantum differential equation and its assymptotics corresponds to the spectrum of singularities of the Landau-Ginzburg model.

The above developments lead to combining both sides of Homological Mirror Symmetry and creates new birational invariants. Several spectacular applications are discussed at the end.

Part I

Constructions of mirrors

Mirror symmetry for weighted projective planes and their noncommutative deformations.

1 Homological mirror symmetry

The phenomenon of Mirror Symmetry, in its "classical" version, was first observed for Calabi-Yau manifolds, and mathematicians were introduced to it through a series of remarkable papers [98, 45, 244, 253, 71, 173], ... Some very strong conjectures have been made about its topological interpretation – e.g. the Strominger-Yau-Zaslow conjecture. In a different direction, the framework of mirror symmetry was extended by Batyrev, Givental, Hori, Vafa, etc. to the case of Fano manifolds.

In this chapter, we approach mirror symmetry for Fano manifolds from the point of view suggested by the work of Kontsevich and his remarkable Homological Mirror Symmetry (HMS) conjecture [151]. We extend the previous investigations in the following two directions:

- Building on recent works by Seidel [214], Hori and Vafa [122] (see also an earlier paper by Witten [254]), we prove HMS for some Fano manifolds, namely weighted projective lines and planes, and Hirzebruch surfaces. This extends, at a greater level of generality, a result of Seidel [215] concerning the case of the usual $\mathbb{CP}^2$.

- We obtain the first explicit description of the extension of HMS to noncommutative deformations of Fano algebraic varieties.

In the long run, the goal is to explore in greater depth the fascinating ties brought forth by HMS between complex algebraic geometry and symplectic geometry, hoping that the currently more developed algebro-geometric methods will open a fine opportunity for obtaining new interesting results in symplectic geometry. We first describe the results of this chapter in some more detail.

Most of the classical works on string theory deal with the case of $N = 2$ superconformal sigma models with a Calabi-Yau target space. In this situation the corresponding field theory has two topologically twisted versions, the A- and B-models, with D-branes of types A and B respectively. Mirror symmetry interchanges these two classes of D-branes. In mathematical terms, the category of B-branes on a Calabi-Yau manifold X is the derived category of coherent sheaves on X, $\mathbf{D}^b(\mathrm{coh}(X))$. The so-called (derived) Fukaya category $D\mathcal{F}(Y)$ has been proposed as a candidate for the category of A-branes on a Calabi-Yau manifold Y; in short this is a category whose objects are Lagrangian submanifolds equipped with flat vector bundles. The HMS Conjecture claims that if two Calabi Yau manifolds X and Y are mirrors to each other then $\mathbf{D}^b(\mathrm{coh}(X))$ is equivalent to $D\mathcal{F}(Y)$.

Physicists also consider more general $N = 2$ supersymmetric field theories and the corresponding D-branes; among these, two families of theories are of particular interest to us: on one hand, sigma models with a Fano variety as target space, and on the other hand, $N = 2$ Landau-Ginzburg models. Mirror symmetry relates the former with a certain subclass of the latter. In particular, B-branes on a Fano variety are described by the derived category of coherent sheaves, and under mirror symmetry they correspond to the A-branes of a mirror Landau-Ginzburg model. These A-branes are described by a suitable analogue of the Fukaya category, namely the derived category of Lagrangian vanishing cycles.

In order to demonstrate this feature of mirror symmetry, we use a procedure introduced by Batyrev [25], Givental [92], Hori and Vafa [122], which we will call the *toric mirror ansatz*. Starting from a complete intersection Y in a toric variety, this procedure yields a description of an affine subset of its mirror Landau-Ginzburg model (to obtain a full description of the mirror it is usually necessary to consider a partial (fiberwise) compactification) – an open symplectic manifold (X, ω) and a symplectic fibration $W : X \to \mathbb{C}$ (see e.g. [134]).

Following ideas of Kontsevich [152] and Hori-Iqbal-Vafa [120], Seidel rigorously defined (in the case of non-degenerate critical points) a derived category of Lagrangian vanishing cycles $D(\mathrm{Lag}_{\mathrm{vc}}(W))$ [214], whose objects represent A-branes on $W : X \to \mathbb{C}$.

In the case of Fano manifolds the statement of the HMS conjecture is the following:

Conjecture 1.1 *The category of A-branes $D(\mathrm{Lag}_{\mathrm{vc}}(W))$ is equivalent to the derived category of coherent sheaves (B-branes) on Y.*

We will prove this conjecture for various examples.

There is also a parallel statement of HMS relating the derived category of B-branes on $W : X \to \mathbb{C}$, whose definition was suggested by Kontsevich and carried out algebraically in [194], and the derived Fukaya category of Y. Since very little is known about these Fukaya categories, we will not discuss the details of this statement in the present chapter. Our hope in this direction is that algebro-geometric methods will allow us to look at Fukaya categories from a different perspective.

The case we will be mainly concerned with in this chapter is that of the weighted projective plane $\mathbb{CP}^2(a,b,c)$ (where a,b,c are coprime positive integers). Its mirror is the affine hypersurface $X = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, equipped with an exact symplectic form ω and the superpotential $W = x + y + z$. Our main theorem is:

Theorem 1.2 *HMS holds for* $\mathbb{CP}^2(a,b,c)$ *and its noncommutative deformations.*

Namely, we show that the derived category of coherent sheaves (B-branes) on the weighted projective plane $\mathbb{CP}^2(a,b,c)$ is equivalent to the derived category of vanishing cycles (A-branes) on the affine hypersurface $X \subset (\mathbb{C}^*)^3$. Moreover, we also show that this mirror correspondence between derived categories can be extended to toric noncommutative deformations of $\mathbb{CP}^2(a,b,c)$ where B-branes are concerned, and their mirror counterparts, non-exact deformations of the symplectic structure of X where A-branes are concerned.

Observe that weighted projective planes are rigid in terms of commutative deformations, but have a one-dimesional moduli space of toric noncommutative deformations ($\mathbb{CP}^2$ also has some other noncommutative deformations, see §6.2). We expect a similar phenomenon to hold in many cases where the toric mirror ansatz applies. An interesting question will be to extend this correspondence to the case of general noncommutative toric vareties.

We will also consider some other examples besides weighted projective planes, in order to demonstrate the ubiquity of HMS:

- as a warm-up example, we give a proof of HMS for weighted projective lines (a result also announced by D. van Straten in [245]).
- we also discuss HMS for Hirzebruch surfaces $\mathbb{F}_n$. For $n \geq 3$, the canonical class is no longer negative ($\mathbb{F}_n$ is not Fano), and HMS does not hold directly, because some modifications of the toric mirror ansatz are needed, as already noticed in [120]. The direct application of the ansatz produces a Landau-Ginzburg model whose derived category of vanishing cycles is identical to that on the mirror of the weighted projective plane $\mathbb{CP}^2(1,1,n)$. In order to make the HMS conjecture work we need to restrict ourselves to an open subset in the target space X of this Landau-Ginzburg model.
- we will also outline an idea of the proof of HMS (missing only some Floer-theoretic arguments about certain moduli spaces of pseudo-holomorphic discs) for some higher-dimensional Fano manifolds, e.g. $\mathbb{CP}^3$.

A word of warning is in order here. We do not describe completely and do not make use of the full potential of the toric mirror ansatz in this chapter. Indeed we do not compactify and desingularize the open manifold X. Compactification and desingularization procedures will be addressed in full detail in future papers [17, 18] dealing with the cases of more general Fano manifolds and manifolds of general type, where these extra steps are needed in order to exhibit the whole category of D-branes of the Landau-Ginzburg model. In this chapter we work with specific examples for which compactification and desingularization are not needed (conjecturally this is the case for all toric varieties). However there are two principles which are readily apparent from these specific examples:

- noncommutative deformations of Fano manifolds are related to variations of the cohomology class of the symplectic form on the mirror Landau-Ginzburg models;
- even in the toric case, a fiberwise compactification of the Landau-Ginzburg model is required in order to obtain general noncommutative deformations. The noncompact case then arises as a limit where the symplectic form on the compactified fiber acquires poles along the compactification divisor.

Moreover there are two features of HMS for toric varieties, which become apparent in this chapter and which we would like to emphasize:

- it is important to think of singular toric varieties as smooth quotient stacks. As a consequence of the work of Cox [64] this characterization is possible in many cases;
- as suggested by our specific examples, we would like to conjecture that the derived category of coherent sheaves over a smooth toric quotient stack is always generated by an exceptional collection of line bundles.

The chapter is organized as follows. In Section 2 we give a detailed description of derived categories of coherent sheaves over weighted projective spaces and some of their noncommutative deformations. After recalling the definition of the weighted projective space $\mathbb{P}(\overline{\mathbf{a}})$ as a quotient stack, we describe the category of coherent sheaves over $\mathbb{P}(\overline{\mathbf{a}})$ and its noncommutative deformations $\mathbb{P}_\theta(\overline{\mathbf{a}})$, and describe explicitly generating exceptional collections for $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta(\overline{\mathbf{a}})))$ (Theorem 2.12 and Corollary 2.27). This is a novel result, and we believe that it suggests a procedure that applies to many other examples of noncommutative toric varieties. We also discuss derived categories of coherent sheaves over Hirzebruch surfaces.

In Section 3 we introduce the category of Lagrangian vanishing cycles associated to a Lefschetz fibration, and outline the main steps involved in its determination; to illustrate the definitions, we treat the case of the mirror of a weighted projective line. After this warm-up, in Section 4 we turn to our main examples, namely the Landau-Ginzburg models mirror to weighted projective planes and their non-exact symplectic deformations. More precisely we start by studying the vanishing cycles and their intersection properties, which allows us to determine all the morphisms in $\mathrm{Lag}_{\mathrm{vc}}$ (Lemma 4.3). Next we study moduli spaces of pseudo-holomorphic discs in the fiber in order to determine Floer products (Lemmas 4.4–4.5); this gives formulas for compositions of morphisms and higher products in $\mathrm{Lag}_{\mathrm{vc}}$ (the latter turn out to be identically zero). Finally, after a discussion of Maslov index and grading, we establish an explicit correspondence between deformation parameters on both sides (noncommutative deformation of the weighted projective plane, and complexified Kähler class on the mirror) and complete the proof of Theorem 1.2.

Section 5 deals with the case of mirrors to Hirzebruch surfaces, showing how their categories of Lagrangian vanishing cycles relate to those of mirrors to weighted projective planes $\mathbb{CP}^2(n,1,1)$. In particular we prove HMS for $\mathbb{F}_n$ when $n \in \{0,1,2\}$, and show how for $n \geq 3$ a certain degenerate limit of the Landau-Ginzburg model singles out a full subcategory of $\mathrm{Lag}_{\mathrm{vc}}$ whose derived category is equivalent to that of coherent sheaves on the Hirzebruch surface.

Finally, in Section 6 we make various observations and concluding remarks, related to the following directions for future research:

- HMS for Del Pezzo surfaces, and for higher-dimensional weighted projective spaces (cf. §6.1 for a discussion of the case of $\mathbb{CP}^3$);
- HMS for general (non toric) noncommutative deformations (cf. §6.2 for a discussion of the case of $\mathbb{CP}^2$);
- the "other side" of HMS – relating derived Fukaya categories to derived categories of B-branes on the mirror Landau-Ginzburg model.

Another topic that will be investigated in a forthcoming paper [18] is HMS for products: our considerations for $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ suggest a certain product formula on both sides of HMS – if we consider two manifolds Y_1, Y_2 with mirror Landau-Ginzburg models (X_1, W_1) and (X_2, W_2), then the mirror of $Y_1 \times Y_2$ is simply $(X_1 \times X_2, W_1 + W_2)$, and we have the following general conjecture:

Conjecture 1.3 $D(\mathrm{Lag}_{\mathrm{vc}}(W_1 + W_2))$ *is equivalent to the product* $D(\mathrm{Lag}_{\mathrm{vc}}(W_1) \otimes \mathrm{Lag}_{\mathrm{vc}}(W_2))$.

More precisely, the vanishing cycles of $W_1 + W_2$ are in one-to-one correspondence with pairs of vanishing cycles of W_1 and W_2, and it can be checked (cf. §6.3) that

$$\mathrm{Hom}_{\mathrm{Lag}_{\mathrm{vc}}(W_1+W_2)}((A_1, A_2), (B_1, B_2)) \simeq \mathrm{Hom}_{\mathrm{Lag}_{\mathrm{vc}}(W_1)}(A_1, B_1) \otimes \mathrm{Hom}_{\mathrm{Lag}_{\mathrm{vc}}(W_2)}(A_2, B_2).$$

The conjecture asserts that Floer products behave in the expected manner with respect to these isomorphisms.

2 Weighted projective spaces

2.1 Weighted projective spaces as stacks

We start by reviewing definitions from the theory of weighted projective spaces.

Let $\mathbb{C}$ be a base field. Let $a_0, \ldots, a_n$ be positive integers. Define the graded algebra $S = S(a_0, \ldots, a_n)$ to be the polynomial algebra $\mathbb{C}[x_0, \ldots, x_n]$ graded by $\deg x_i = a_i$. Classically the projective variety $\mathbf{Proj}\, S$ is called the weighted projective space with weights $a_0, \dots, a_n$ and is denoted by $\mathbf{P}(a_0, \dots, a_n)$. Consider the action of the algebraic group $\mathbf{G}_m = \mathbb{C}^*$ on the affine space $\mathbf{A}^{n+1}$ given in some affine coordinates $x_0, \dots, x_n$ by the formula

$$\lambda(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n). \tag{2.1}$$

In geometric terms, the weighted projective space $\mathbf{P}(a_0, \dots, a_n)$ is the quotient variety $(\mathbf{A}^{n+1} \backslash \mathbf{0}) \big/ \mathbf{G}_m$ under the induced action of the group $\mathbf{G}_m$.

The variety $\mathbf{P}(a_0, \dots, a_n)$ is a rational n-dimensional projective variety, singular in general, whose affine charts $x_i \neq 0$ are isomorphic to $\mathbf{A}^n \big/ \mathbb{Z}_{a_i}$. For example, the variety $\mathbf{P}(1, 1, n)$ is the projective cone over a twisted rational curve of degree n in $\mathbf{P}^n$.

Denote by $\overline{\mathbf{a}}$ the vector $(a_0, \dots, a_n)$ and write $\mathbf{P}(\overline{\mathbf{a}})$ instead $\mathbf{P}(a_0, \dots, a_n)$ for brevity.

There is also another way to define the quotient of the action above: in the category of stacks. The quotient stack

$$\left[(\mathbf{A}^{n+1} \backslash \mathbf{0}) \big/ \mathbf{G}_m\right]$$

will be denoted by $\mathbb{P}(\overline{\mathbf{a}})$ and will also be called the weighted projective space. The stack $\mathbb{P}(\overline{\mathbf{a}})$ is smooth, and from many points of view it is a more natural object than $\mathbf{P}(\overline{\mathbf{a}})$.

We now review the notion of an algebraic stack as needed to understand our main example – weighted projective spaces. A detailed treatment of algebraic stacks can be found in [164] and [88].

There are two ways of thinking about an algebraic stack:

a) as a category $\mathcal{X}$, with additional properties;

b) as a presentation $R \rightrightarrows U$, with R and U schemes, R determining an equivalence relation on U.

From the categorical point of view a stack is a category $\mathcal{X}$ fibered in groupoids $p : \mathcal{X} \to \mathrm{Sch}$ over the category Sch of $\mathbb{C}$-schemes, satisfying two descent (sheafy) properties in the étale topology. An algebraic stack has to satisfy some additional representability conditions. For the precise definition see [164, 88].

Any scheme $X \in \mathrm{Sch}$ defines a category $\mathrm{Sch}\,/X$: its objects are pairs (S, ϕ) with $\{S \xrightarrow{\phi} X\}$ a map in Sch, and a morphism from (S, ϕ) to (T, ψ) is a morphism $f : T \to S$ such that $\phi f = \psi$. The category $\mathrm{Sch}\,/X$ comes with a natural functor to Sch. Thus, any scheme is an algebraic stack.

Another example, the most important one for us, comes from an action of an algebraic group G on a scheme X. The quotient stack $[X/G]$ is defined to be the category whose objects are those G-torsors (principal homogeneous right G-schemes) $\mathcal{G} \to S$ which are locally trivial in the étale topology, together with a G-equivariant map from $\mathcal{G}$ to X.

In order to work with coherent sheaves on a stack it is convenient to use an atlas for the stack. We describe very briefly groupoid presentations (or atlases) of algebraic stacks. A pair of schemes R and U with morphisms s, t, e, m, i, satisfying certain group-like properties, is called a groupoid in Sch or an algebraic groupoid. For any scheme S the morphisms $s, t : R \to U$ ("source" and "target") determine two maps from the set $\mathrm{Hom}(S, R)$ to the set $\mathrm{Hom}(S, U)$. A quick way to state all relations between s, t, e, m, i is to say that the induced morphisms make the "objects" $\mathrm{Hom}(S, U)$ and "morphisms" $\mathrm{Hom}(S, R)$ into a category in which all arrows are invertible. We will denote an algebraic groupoid by $R \rightrightarrows U$ (the two arrows being the source and target maps), omitting the notations for e, m, and i.

Any scheme X determines a groupoid $X \rightrightarrows X$, whose morphisms are identity maps. The main example for us is the transformation groupoid associated to an algebraic group action $X \times G \to X$, which provides an atlas for the quotient stack $[X/G]$. The transformation groupoid $X \times G \rightrightarrows X$ is defined by

$$s(x, g) = x, \quad t(x, g) = x{\cdot}g, \quad m((x, g), (x{\cdot}g, h)) = (x, g{\cdot}h), \quad e(x) = (x, e_G), \quad i(x, g) = (x{\cdot}g, g^{-1}).$$

If $R \rightrightarrows U$ is a presentation for a stack $\mathcal{X}$, giving a coherent sheaf on $\mathcal{X}$ is equivalent to giving a coherent sheaf $\mathcal{F}$ on U, together with an isomorphism $s^*\mathcal{F} \xrightarrow{\sim} t^*\mathcal{F}$ on R satisfying a cocycle condition on $R \underset{t,U,s}{\times} R$. In particular, for a quotient stack $[X/G]$ the category of coherent sheaves is equivalent to the category of G-equivariant sheaves on X due to effective descent for strictly flat morphisms of algebraic stacks (see, e.g., [164], Thm. 13.5.5). Applying this fact to weighted projective spaces, we obtain that

$$\mathrm{coh}(\mathbb{P}(\overline{\mathbf{a}})) \cong \mathrm{coh}_{\overline{\mathbf{a}}}^{\mathbf{G}_m}(\mathbf{A}^{n+1} \backslash \mathbf{0}), \tag{2.2}$$

where $\mathrm{coh}_{\overline{\mathbf{a}}}^{\mathbf{G}_m}(\mathbf{A}^{n+1} \backslash \mathbf{0})$ is the category of $\mathbf{G}_m$-equivariant coherent sheaves on $(\mathbf{A}^{n+1} \backslash \mathbf{0})$ with respect to the action given by rule (2.1).

2.2 Coherent sheaves on weighted projective spaces

Let $A = \bigoplus_{i \geq 0} A_i$ be a finitely generated graded algebra. Denote by $\mathrm{mod}(A)$ the category of finitely generated right A-modules and by $\mathrm{gr}(A)$ the category of finitely generated graded right A-modules in which morphisms are the homomorphisms of degree zero. Both are abelian categories.

Denote by $\mathrm{tors}(A)$ the full subcategory of $\mathrm{gr}(A)$ which consists of those graded A-modules which have finite dimension over $\mathbb{C}$.

Definition 2.1 *Define the category* $\mathrm{qgr}(A)$ *to be the quotient category* $\mathrm{gr}(A)/\,\mathrm{tors}(A)$. *The objects of* $\mathrm{qgr}(A)$ *are the objects of the category* $\mathrm{gr}(A)$ *(we denote by* $\widetilde{M}$ *the object in* $\mathrm{qgr}(A)$ *which corresponds to a module* M*). The morphisms in* $\mathrm{qgr}(A)$ *are defined to be*

$$\mathrm{Hom}_{\mathrm{qgr}}(\widetilde{M}, \widetilde{N}) = \varinjlim_{M'} \mathrm{Hom}_{\mathrm{gr}}(M', N),$$

where M' *runs over all submodules of* M *such that* M/M' *is finite dimensional over* $\mathbb{C}$.

The category $\mathrm{qgr}(A)$ is an abelian category and there is a shift functor on it: for a given graded module $M = \bigoplus_{i\geq 0} M_i$ the shifted module $M(p)$ is defined by $M(p)_i = M_{p+i}$, and the induced shift functor on the quotient category $\mathrm{qgr}(A)$ sends $\widetilde{M}$ to $\widetilde{M}(p) = \widetilde{M(p)}$.

Similarly, we can consider the category $\mathrm{Gr}(A)$ of all graded right A-modules. It contains the subcategory $\mathrm{Tors}(A)$ of torsion modules. Recall that a module M is called torsion if for any element $x \in M$ one has $xA_{\geq s} = 0$ for some s, where $A_{\geq s} = \bigoplus_{i\geq s} A_i$. We denote by $\mathrm{QGr}(A)$ the quotient category $\mathrm{Gr}(A)/\mathrm{Tors}(A)$. It is clear that the intersection of the categories $\mathrm{gr}(A)$ and $\mathrm{Tors}(A)$ in the category $\mathrm{Gr}(A)$ coincides with $\mathrm{tors}(A)$. In particular, the category $\mathrm{QGr}(A)$ contains $\mathrm{qgr}(A)$ as a full subcategory. Sometimes it is convenient to work with $\mathrm{QGr}(A)$ instead of $\mathrm{qgr}(A)$.

In the case when the algebra $A = \bigoplus_{i\geq 0} A_i$ is a commutative graded algebra generated over $\mathbb{C}$ by its degree one component (which is assumed to be finite dimensional) J.-P. Serre [220] proved that the category of coherent sheaves $\mathrm{coh}(X)$ on the projective variety $X = \mathbf{Proj}\, A$ is equivalent to the category $\mathrm{qgr}(A)$. Such an equivalence also holds for the category of quasicoherent sheaves on X and the category $\mathrm{QGr}(A) = \mathrm{Gr}(A)/\mathrm{Tors}(A)$.

This theorem can be extended to general finitely generated commutative algebras if we work at the level of quotient stacks.

Let $S = \bigoplus_{p=0}^{\infty} S_p$ be a commutative graded $\mathbb{C}$-algebra which is connected, i.e. $S_0 = \mathbb{C}$. The grading on S induces an action of the group $\mathbf{G}_m$ on the affine scheme $\mathbf{Spec}\, S$. Let $\mathbf{0}$ be the closed point of $\mathbf{Spec}\, S$ that corresponds to the ideal $S_+ = S_{\geq 1} \subset S$. This point is invariant under the action.

Definition 2.2 *Denote by* $\mathbb{P}\mathrm{roj}\, S$ *the quotient stack* $\left[(\mathbf{Spec}\, S \backslash \mathbf{0})/\mathbf{G}_m\right]$.

There is a natural map $\mathbb{P}\mathrm{roj}\, S \to \mathbf{Proj}\, S$, which is an isomorphism when the algebra S is generated by its degree one component S_1.

Proposition 2.3 *Let* $S = \oplus_{i\geq 0} S_i$ *be a graded finitely generated algebra. Then the category of (quasi)coherent sheaves on the quotient stack* $\mathbb{P}\mathrm{roj}\,(S)$ *is equivalent to the quotient category* $\mathrm{qgr}(S)$ (*resp.* $\mathrm{QGr}(S)$).

Proof. Let $\mathbf{0}$ be the closed point on the affine scheme $\mathbf{Spec}\, S$ which corresponds to the maximal ideal $S_+ \subset S$. Denote by U the scheme $(\mathbf{Spec}\, S \backslash \mathbf{0})$. We know that the category of (quasi)coherent sheaves on the stack $\mathbb{P}\mathrm{roj}\, S$ is equivalent to the category of $\mathbf{G}_m$-equivariant (quasi)coherent sheaves on U. The category of (quasi)coherent sheaves on U is equivalent to the quotient of the category of (quasi)coherent sheaves on $\mathbf{Spec}\, S$ by the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$. This is also true for the categories of $\mathbf{G}_m$-equivariant sheaves. But the category of (quasi)coherent $\mathbf{G}_m$-equivariant sheaves on

$\mathbf{Spec}\, S$ is just the category $\mathrm{gr}(S)$ (resp. $\mathrm{Gr}(S)$) of graded modules over S, and the subcategory of (quasi)coherent sheaves with support on $\mathbf{0}$ coincides with the subcategory $\mathrm{tors}(S)$ (resp. $\mathrm{Tors}(S)$). Thus, we obtain that $\mathrm{coh}(\mathbb{P}\mathrm{roj}\, S)$ is equivalent to the quotient category $\mathrm{qgr}(S) = \mathrm{gr}(S)/\mathrm{tors}(S)$ (and $\mathrm{Qcoh}(\mathbb{P}\mathrm{roj}\, S)$ is equivalent to $\mathrm{QGr}(S) = \mathrm{Gr}(S)/\mathrm{Tors}(S)$). $\square$

Corollary 2.4 *The category of (quasi)coherent sheaves on the weighted projective space* $\mathbb{P}(\overline{\mathbf{a}})$ *is equivalent to the category* $\mathrm{qgr}(S(a_0, \ldots, a_n))$ (*resp.* $\mathrm{QGr}(S(a_0, \ldots, a_n))$).

We conclude this section by giving the definition of noncommutative weighted projective spaces and the categories of coherent sheaves on them. Consider a matrix $\theta = (\theta_{ij})$ of dimension $(n+1) \times (n+1)$ with entries $\theta_{ij} \in \mathbb{C}^*$ for all i, j. The set of all such matrices will be denoted by $\mathrm{M}(n+1, \mathbb{C}^*)$. Consider the graded algebra $S_\theta = S_\theta(a_0, \ldots, a_n)$ generated by elements $x_i, i = 0, \ldots, n$ of degree a_i and with relations

$$\theta_{ij} x_i x_j = \theta_{ji} x_j x_i$$

for all i and j. This algebra is a noncommutative deformation of the algebra $S(a_0, \ldots, a_n)$. It can be easily checked that the algebra S_θ depends only on the matrix θ^{an}, with entries

$$\theta^{\mathrm{an}}_{ij} := \theta_{ij}\theta_{ji}^{-1} \qquad \text{for all} \quad 0 \le i, j \le n. \tag{2.3}$$

Thus, if $(\theta')^{\mathrm{an}} = \theta^{\mathrm{an}}$ for two matrices θ' and θ, then $S_{\theta'} \cong S_\theta$.

As before, denote by $\mathrm{qgr}(S_\theta)$ the quotient category $\mathrm{gr}(S_\theta)/\mathrm{tors}(S_\theta)$, where $\mathrm{gr}(S_\theta)$ is the category of finitely generated graded right S_θ-modules and $\mathrm{tors}(A)$ is the full subcategory of $\mathrm{gr}(S_\theta)$ consisting of graded modules of finite dimension over $\mathbb{C}$.

Corollary 2.4 suggests that the category $\mathrm{qgr}(S_\theta)$ should be considered as the category of coherent sheaves on a noncommutative weighted projective space. We will denote this space by $\mathbb{P}_\theta(\overline{\mathbf{a}})$ and will write $\mathrm{coh}(\mathbb{P}_\theta)$ instead $\mathrm{qgr}(S_\theta)$. Similarly, the category of quasi-coherent sheaves $\mathrm{Qcoh}(\mathbb{P}_\theta)$ is defined as the quotient $\mathrm{QGr}(S_\theta) = \mathrm{Gr}(S_\theta)/\mathrm{Tors}(S_\theta)$.

2.3 Cohomological properties of coherent sheaves on $\mathbb{P}_\theta(\overline{\mathbf{a}})$

In this section we discuss properties of categories of coherent sheaves on the noncommutative weighted projective spaces $\mathbb{P}_\theta(\overline{\mathbf{a}})$. Note that the usual commutative weighted projective space is a particular case of the noncommutative one, when θ is the matrix with all entries equal to 1.

All algebras $S_\theta(a_0, \ldots, a_n)$ are noetherian. This follows from the fact that they are Ore extensions of commutative polynomial algebras (see for example [178]). For the same reason the algebras $S_\theta(a_0, \ldots, a_n)$ have finite right (and left) global dimension, which is equal to $(n+1)$ (see [178], p. 273). Recall that the global dimension of a ring A is the minimal number d (if it exists) such that for any two modules M and N we have $\mathrm{Ext}^{d+1}_A(M, N) = 0$.

The notion of a regular algebra was introduced in [9]. As we will see below, regular algebras have many good properties. More details can be found in [11].

Definition 2.5 *A graded algebra A is called regular of dimension d if it satisfies the following conditions:*

(1) *A has global dimension d,*

(2) *A has polynomial growth, i.e. $\dim A_p \leq cp^{\delta}$ for some $c, \delta \in \mathbb{R}$,*

(3) *A is Gorenstein, meaning that $\mathrm{Ext}^i_A(\mathbb{C}, A) = 0$ if $i \neq d$, and $\mathrm{Ext}^d_A(\mathbb{C}, A) = \mathbb{C}(l)$ for some l. The number l is called the Gorenstein parameter.*

Here Ext_A stands for the Ext functor in the category of right modules $\mathrm{mod}(A)$.

Proposition 2.6 *The algebra $S_\theta(a_0, \ldots, a_n)$ is a noetherian regular algebra of global dimension $n+1$. The Gorenstein parameter l of this algebra is equal to the sum $\sum\limits_{i=0}^{n} a_i$.*

Proof. Property (1) holds, as for all Ore extensions of commutative polynomial algebras. Property (2) holds because our algebras have the same growth as ordinary polynomial algebras. Property (3) follows from the following Koszul resolution of the right module $\mathbb{C}_{S_\theta}$

$$
\begin{aligned}
(2.4)\quad 0 \to S_\theta(-\sum_{i=0}^{n} a_i) \to \bigoplus_{i_0<\ldots<i_{n-1}} S_\theta(-\sum_{j=0}^{n-1} a_{i_j}) \to \cdots \\
\cdots \to \bigoplus_{i_0<i_1} S_\theta(-a_{i_0} - a_{i_1}) \to \bigoplus_{i=0}^{n} S_\theta(-a_i) \to S_\theta \to \mathbb{C}_{S_\theta} \to 0,
\end{aligned}
$$

and the fact that the transposed complex is a resolution of the left module ${}_{S_\theta}\mathbb{C}$, shifted to the degree $l = \sum a_i$. The explicit formula for the differentials in the complex (2.4) will be given later (see (2.8)). □

Denote by $\mathcal{O}(i)$ the object $\widetilde{S_\theta(i)}$ in the category $\mathrm{coh}(\mathbb{P}_\theta) = \mathrm{qgr}(S_\theta)$. Consider the sequence $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$. It can be checked that the following properties hold true:

(a) For any coherent sheaf $\mathcal{F}$ there are integers $k_1, \ldots, k_s$ and an epimorphism

$$\bigoplus_{i=1}^{s} \mathcal{O}(-k_i) \to \mathcal{F}.$$

(b) For every epimorphism $\mathcal{F} \to \mathcal{G}$ the induced map $\mathrm{Hom}(\mathcal{O}(-n), \mathcal{F}) \to \mathrm{Hom}(\mathcal{O}(-n), \mathcal{G})$ is surjective for $n \gg 0$.

A sequence which satisfies such conditions will be called ample. It is proved in [11] that the sequence $\{\mathcal{O}(i)\}$ is ample in $\mathrm{qgr}(A)$ for any graded right noetherian $\mathbb{C}$-algebra A if it satisfies the extra condition:

$$(\chi_1): \quad \dim_{\mathbb{C}} \mathrm{Ext}^1_A(\mathbb{C}, M) < \infty$$

for any finitely generated graded A-module M.

This condition can be verified for all noetherian regular algebras (see [11], Theorem 8.1). In particular, the sequence $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$ in the category $\mathrm{coh}(\mathbb{P}_\theta)$ is ample.

For any sheaf $\mathcal{F}\in\mathrm{qgr}(A)$ we can define a graded module $\Gamma(\mathcal{F})$ by the rule:

$$\Gamma(\mathcal{F}) := \bigoplus_{i\geq 0}\mathrm{Hom}(\mathcal{O}(-i),\mathcal{F})$$

It is proved in [11] that for any noetherian algebra A that satisfies the condition (χ_1) the correspondence Γ is a functor from $\mathrm{qgr}(A)$ to $\mathrm{gr}(A)$ and the composition of Γ with the natural projection $\pi:\mathrm{gr}(A)\longrightarrow\mathrm{qgr}(A)$ is isomorphic to the identity functor (see [11], § 3,4).

We formulate next a result about the cohomology of sheaves on noncommutative weighted projective spaces. This result is proved in [11] (Theorem 8.1) for a general regular algebra and parallels the commutative case.

Proposition 2.7 *Let $S_\theta = S_\theta(a_0,\ldots,a_n)$ be the algebra of the noncommutative weighted projective space $\mathbb{P}_\theta=\mathbb{P}_\theta(\overline{\mathbf{a}})$. Then*

1) *The cohomological dimension of the category $\mathrm{coh}(\mathbb{P}_\theta(\overline{\mathbf{a}}))$ is equal to n, i.e. for any two coherent sheaves $\mathcal{F},\mathcal{G}\in\mathrm{coh}(\mathbb{P}_\theta)$ the space $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ vanishes if $i>n$.*

2) *There are isomorphisms*

$$H^p(\mathbb{P}_\theta,\mathcal{O}(k)) = \begin{cases}(S_\theta)_k & \text{for } p=0,\ k\geq 0\\ (S_\theta)^*_{-k-l} & \text{for } p=n,\ k\leq -l\\ 0 & \text{otherwise}\end{cases} \tag{2.5}$$

This proposition and the ampleness of the sequence $\{\mathcal{O}(i)\}$ imply the following corollary.

Corollary 2.8 *For any sheaf $\mathcal{F}\in\mathrm{coh}(\mathbb{P}_\theta)$ and for all sufficiently large $i\gg 0$ we have $H^k(\mathbb{P}_\theta,\mathcal{F}(i))=0$ for all $k>0$.*

Proof. The group $H^k(\mathbb{P}_\theta,\mathcal{F}(i))$ coincides with $\mathrm{Ext}^k(\mathcal{O}(-i),\mathcal{F})$. Let k be the maximal integer (it exists because the global dimension is finite) such that for some $\mathcal{F}$ there exists arbitrarily large i such that $\mathrm{Ext}^k(\mathcal{O}(-i),\mathcal{F})\neq 0$. Assume that $k\geq 1$. Choose an epimorphism $\bigoplus_{j=1}^{s}\mathcal{O}(-k_j)\to\mathcal{F}$. Let $\mathcal{F}_1$ denote its kernel. Then for $i>\max\{k_j\}$ we have $\mathrm{Ext}^{>0}(\mathcal{O}(-i),\bigoplus_{j=1}^{s}\mathcal{O}(-k_j))=0$, hence $\mathrm{Ext}^k(\mathcal{O}(-i),\mathcal{F})\neq 0$ implies $\mathrm{Ext}^{k+1}(\mathcal{O}(-i),\mathcal{F}_1)\neq 0$. This contradicts the assumption of the maximality of k. □

One of the useful properties of commutative smooth projective varieties is the existence of the dualizing sheaf. Recall that a sheaf ω_X is called dualizing if for any $\mathcal{F}\in\mathrm{coh}(X)$ there are natural isomorphisms of $\mathbb{C}$-vector spaces

$$H^i(X,\mathcal{F})\cong\mathrm{Ext}^{n-i}(\mathcal{F},\omega_X)^*,$$

where $*$ denotes the $\mathbb{C}$–dual space. The Serre duality theorem asserts the existence of a dualizing sheaf for smooth projective varieties. In this case the dualizing sheaf is a line bundle and coincides with the sheaf of differential forms Ω_X^n of top degree.

Since the definition of ω_X is given in abstract categorical terms, it can be extended to the noncommutative case as well. More precisely, we will say that $\mathrm{qgr}(A)$ satisfies classical *Serre duality* if there is an object $\omega \in \mathrm{qgr}(A)$ together with natural isomorphisms

$$\mathrm{Ext}^i(\mathcal{O}, -) \cong \mathrm{Ext}^{n-i}(-, \omega)^*.$$

Our noncommutative varieties $\mathbb{P}_\theta(\overline{\mathbf{a}})$ satisfy classical Serre duality, with dualizing sheaves being $\mathcal{O}(-l)$, where $l = \sum a_i$ is the Gorenstein parameter for $S_\theta(a_0, \ldots, a_n)$. This follows from the paper [255], where the existence of a dualizing sheaf in $\mathrm{qgr}(A)$ has been proved for a class of algebras which includes all noetherian regular algebras. In addition, the authors of [255] showed that the dualizing sheaf coincides with $\widetilde{A}(-l)$, where l is the Gorenstein parameter for A.

There is a reformulation of Serre duality in terms of bounded derived categories [35]. A *Serre functor* in the bounded derived category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ is by definition an exact autoequivalence S of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ such that for any objects $X, Y \in \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ there is a bifunctorial isomorphism

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(Y, SX)^*.$$

Serre duality can be reinterpreted as the existence of a Serre functor in the bounded derived category.

2.4 Exceptional collection on $\mathbb{P}_\theta(\overline{\mathbf{a}})$

For many reasons it is more natural to work not with the abelian category of coherent sheaves but with its bounded derived category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$. The purpose of this section is to describe the bounded derived category of coherent sheaves on the noncommutative weighted projective spaces in the terms of exceptional collections.

First, we briefly recall the definition of the bounded derived category for an abelian category $\mathbb{A}$. We start with the category $\mathbf{C}^b(\mathbb{A})$ of bounded differential complexes

$$M^\bullet = (0 \longrightarrow \cdots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \xrightarrow{d^{p+1}} M^{p+2} \longrightarrow \cdots \longrightarrow 0), \quad M^p \in \mathbb{A}, \quad p \in \mathbb{Z}, \quad d^2 = 0.$$

A morphism of complexes $f : M^\bullet \longrightarrow N^\bullet$ is called null-homotopic if $f^p = d_N h^p + h^{p+1} d_M$ for all $p \in \mathbb{Z}$ and some family of morphisms $h^p : M^p \longrightarrow N^{p-1}$. Now the *homotopy category* $\mathbf{H}^b(\mathbb{A})$ is defined as a category with the same objects as $\mathbf{C}^b(\mathbb{A})$, whereas morphisms in $\mathbf{H}^b(\mathbb{A})$ are equivalence classes $\overline{f}$ of morphisms of complexes modulo null-homotopic morphisms. A morphism of complexes $s : N^\bullet \to M^\bullet$ is called a *quasi-isomorphism* if the induced morphisms $H^p s : H^p(N^\bullet) \to H^p(M^\bullet)$ are isomorphisms for all $p \in \mathbb{Z}$. Denote by Σ the class of all quasi-isomorphisms. The bounded derived category $\boldsymbol{D}^b(\mathbb{A})$ is now defined as the localization of $\mathbf{H}^b(\mathbb{A})$ with respect to the class Σ of all quasi-isomorphisms. This means that the derived category has the same objects as the homotopy category $\mathbf{H}^b(\mathbb{A})$, and that morphisms in the derived category are given by left fractions $s^{-1} \circ f$ with $s \in \Sigma$.

Remark 2.9 *For any full subcategory $\mathcal{E} \subset \mathbb{A}$ one can construct the homotopy category $\mathbf{H}^b(\mathcal{E})$ and a functor $\mathbf{H}^b(\mathcal{E}) \to \boldsymbol{D}^b(\mathbb{A})$. In some cases, for example when $\mathbb{A}$ is the abelian category of modules over an algebra A of finite global dimension and $\mathcal{E}$ is the subcategory of projective modules, this functor $\mathbf{H}^b(\mathcal{E}) \to \boldsymbol{D}^b(\mathbb{A})$ is an equivalence of triangulated categories.*

Second, we recall the notion of an exceptional collection.

Definition 2.10 *An object E of a $\mathbb{C}$-linear triangulated category $\mathcal{D}$ is said to be* exceptional *if $\operatorname{Hom}(E, E[k]) = 0$ for all $k \neq 0$, and $\operatorname{Hom}(E, E) = \mathbb{C}$.*

An ordered set of exceptional objects $\sigma = (E_0, \ldots E_n)$ is called an exceptional collection *if $\operatorname{Hom}(E_j, E_i[k]) = 0$ for $j > i$ and all k. The exceptional collection σ is called* strong *if it satisfies the additional condition $\operatorname{Hom}(E_j, E_i[k]) = 0$ for all i, j and for $k \neq 0$.*

Definition 2.11 *An exceptional collection $(E_0, \ldots, E_n)$ in a category $\mathcal{D}$ is called* full *if it generates the category $\mathcal{D}$, i.e. the minimal triangulated subcategory of $\mathcal{D}$ containing all objects E_i coincides with $\mathcal{D}$. We write in this case*

$$\mathcal{D} = \langle E_0, \ldots, E_n \rangle .$$

Consider the bounded derived category of coherent sheaves $\boldsymbol{D}^b(\operatorname{coh}(\mathbb{P}_\theta))$. We prove that this category has an exceptional collection which is strong and full. In this case we will say that the noncommutative weighted projective space $\mathbb{P}_\theta$ possesses a full strong exceptional collection.

Theorem 2.12 *For any noncommutative weighted projective space $\mathbb{P}_\theta(\overline{\mathbf{a}})$ and for any $k \in \mathbb{Z}$ the ordered set $\sigma(k) = (\mathcal{O}(k), \ldots, O(k + l - 1))$, where $l = \sum a_i$ is the Gorenstein parameter of S_θ, forms a full strong exceptional collection in the category $\boldsymbol{D}^b(\operatorname{coh}(\mathbb{P}_\theta))$.*

Proof. It follows directly from Proposition 2.7 that the collection $\sigma(k)$ is exceptional and strong. To prove that the collection is full let us consider the triangulated subcategory $\mathcal{D} \subset \boldsymbol{D}^b(\operatorname{coh}(\mathbb{P}_\theta))$ generated by the collection $\sigma(k)$. The exact sequence (2.4) induces the exact sequence

$$(2.6) \quad 0 \to \mathcal{O}(-\sum_{i=0}^{n} a_i) \to \bigoplus_{i_0 < \ldots < i_{n-1}} \mathcal{O}(-\sum_{j=0}^{n-1} a_{i_j}) \to \cdots$$
$$\cdots \to \bigoplus_{i_0 < i_1} \mathcal{O}(-a_{i_0} - a_{i_1}) \to \bigoplus_{i=0}^{n} \mathcal{O}(-a_i) \to \mathcal{O} \to 0.$$

Shifting it by $k + l$ one obtains that the object $\mathcal{O}(k + l)$ also belongs to $\mathcal{D}$ and repeating this procedure deduce that $\mathcal{O}(i)$ for all i belongs to $\mathcal{D}$. Assume that $\mathcal{D}$ does not coincide with $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ and take an object U which does not belong to $\mathcal{D}$. It is proved in [34] (Theorem 3.2) that the subcategory $\mathcal{D}$ is *admissible*, i.e. the natural embedding functor $\mathcal{D} \hookrightarrow \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ has right and left adjoint functors. Denote by j the right adjoint and complete the canonical map $jU \longrightarrow U$ to a distinguished triangle

$$jU \longrightarrow U \longrightarrow C \longrightarrow jU[1].$$

It follows from adjointness that for any object $V \in \mathcal{D}$ the space $\mathrm{Hom}(V, C)$ vanishes. The object C is a bounded complex of coherent sheaves. Denote by $H^k(C)$ the leftmost nontrivial cohomology of the complex C. The ampleness of the sequence $\{\mathcal{O}(i)\}_{i\in\mathbb{Z}}$ guarantees that for sufficiently large i the space $\mathrm{Hom}(\mathcal{O}(-i), H^k(C))$ is nontrivial. This implies that $\mathrm{Hom}(\mathcal{O}(-i)[-k], C)$ is nontrivial, which contradicts to the fact that the object $\mathcal{O}(-i)[-k]$ belongs to $\mathcal{D}$. $\square$

The strong exceptional collection on the ordinary projective space $\mathbb{P}^n$ was constructed by Beilinson in [27]. This question for the weighted projective spaces was considered in [19].

Definition 2.13 *The* algebra of the strong exceptional collection $(E_0, \ldots, E_n)$ *is the algebra of endomorphisms of the object* $\overset{n}{\underset{i=0}{\oplus}} E_i$. *Denote by* $\mathcal{E}$ *the sheaf* $\overset{l-1}{\underset{i=0}{\oplus}} \mathcal{O}(i)$ *and by* B *the algebra of the collection* $(\mathcal{O}, \ldots, \mathcal{O}(l-1))$ *on the noncommutative weighted projective space* $\mathbb{P}_\theta$, *i.e.* $B = \mathrm{End}(\mathcal{E})$.

The algebra B is a finite dimensional algebra over $\mathbb{C}$. Denote by mod–B the category of finitely generated right modules over B. For any coherent sheaf $\mathcal{F} \in \mathrm{coh}(\mathbb{P}_\theta)$ the space $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ has a structure of right B-module. Denote by P_i the modules $\mathrm{Hom}(\mathcal{E}, \mathcal{O}(i))$ for $i = 0, \ldots, (l-1)$. All these are projective B-modules and $B = \overset{l-1}{\underset{i=0}{\oplus}} P_i$. The algebra B has l primitive idempotents $e_i, i = 0, \ldots, l-1$ such that $1_B = e_0 + \cdots + e_{l-1}$ and $e_i e_j = 0$ if $i \neq j$. The right projective modules P_i coincide with $e_i B$. The morphisms between them can be easily described since

$$\mathrm{Hom}(P_i, P_j) = \mathrm{Hom}(e_i B, e_j B) \cong e_j B e_i \cong \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) = (S_\theta)_{j-i}.$$

Moreover, the algebra B has finite global dimension. This follows from the fact that any right (and left) module M has a finite projective resolution consisting of the projective modules P_i. Indeed the map

$$\bigoplus_{i=0}^{l-1} \mathrm{Hom}(P_i, M) \otimes P_i \longrightarrow M$$

is surjective and there are no non-trivial homomorphisms from P_{l-1} to the kernel of this map. Iterating this procedure we get a finite resolution of M.

Sometimes it is useful to represent the algebra B as a category $\mathfrak{B}$ which has l objects, say $v_0, \dots, v_{l-1}$, and morphisms defined by

$$\mathrm{Hom}(v_i, v_j) \cong \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)) \cong (S_\theta)_{j-i}$$

with the natural composition law. Thus $B = \bigoplus_{0 \le i,j \le l-1} \mathrm{Hom}(v_i, v_j)$.

The algebra B is a basis algebra. This means that the quotient of B by the radical $\mathrm{rad}(B)$ is isomorphic to the direct sum of l copies of the field $\mathbb{C}$. The category mod–B has l irreducible modules which will be denoted Q_i, $i = 0, \dots, l-1$, and $\overset{l-1}{\underset{i=0}{\oplus}} Q_i = B/\mathrm{rad}(B)$. The modules Q_i are chosen so that $\mathrm{Hom}(P_i, Q_j) \cong \delta_{i,j}\,\mathbb{C}$.

Our next topic is the notion of **mutation** in an exceptional collection. Let $\sigma = (E_0, \dots, E_n)$ be an exceptional collection in a triangulated category $\mathcal{D}$. Consider a pair (E_i, E_{i+1}) and the canonical maps

$$\mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \longrightarrow E_{i+1} \qquad \text{and} \qquad E_i \longrightarrow \mathrm{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1},$$

where by definition

$$\mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k(E_i, E_{i+1}) \otimes E_i[-k],$$
$$\mathrm{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1} = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^{-k}(E_i, E_{i+1}) \otimes E_{i+1}[-k]$$

(recall that the tensor product of a vector space V with an object X may be considered as the direct sum of $\dim V$ copies of the object X).

We define objects LE_{i+1} and RE_i as the objects obtained from the distinguished triangles

$$LE_{i+1} \longrightarrow \mathrm{Hom}^\bullet(E_i, E_{i+1}) \otimes E_i \longrightarrow E_{i+1},$$
$$E_i \longrightarrow \mathrm{Hom}^\bullet(E_i, E_{i+1})^* \otimes E_{i+1} \longrightarrow RE_i.$$

The object LE_{i+1} (resp. RE_i) is called by **left (right) mutation** of E_{i+1} (resp. E_i) in the collection σ. It can be checked that the objects LE_{i+1} and RE_i are exceptional and, moreover, the two collections

$$L_i\sigma = (E_0, \dots, E_{i-1}, LE_{i+1}, E_i, E_{i+2}, \dots, E_n)$$
$$R_i\sigma = (E_0, \dots, E_{i-1}, E_{i+1}, RE_i, E_{i+2}, \dots, E_n)$$

are exceptional as well. These collections are called left and right mutations of the collection σ in the pair (E_i, E_{i+1}). Consider R_i and L_i as operations on the set of all exceptional collections in the category $\mathcal{D}$. It is easy to see that they are mutually inverse, i.e. $R_i L_i = 1$. Moreover, L_i (resp. R_i) satisfy the Artin braid group relations:

$$L_i L_{i+1} L_i = L_{i+1} L_i L_{i+1}, \qquad R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$$

(see [34, 95]).

Denote by $L^{(k)}E_i$ with $k \leq i$ the result of k left mutations of the object E_i in the collection σ. Analogously for right mutations.

Definition 2.14 *The exceptional collection* $(L^{(n)}E_n, L^{(n-1)}E_{n-1}, \dots E_0)$ *is called the* left dual collection *for the collection* $(E_0, \dots, E_n)$. *Analogously, the* right dual collection *is defined as* $(E_n, RE_{n-1}, \dots, R^{(n)}E_0)$.

Example 2.15 *For example, let us consider the full exceptional collection* $(P_0, \dots, P_{l-1})$ *in the category* $\boldsymbol{D}^b(\mathrm{mod}-B)$, *consisting of the projective* B*-modules* P_i. *It can be shown (e.g. [34], Lemma 5.6) that the irreducible modules* $Q_i, 0 \leq i < l$ *can be expressed as*

$$Q_i \cong L^{(i)}P_i[i].$$

Thus, the left dual for the exceptional collection $(P_0, \dots, P_{l-1})$ *coincides with the collection* $(Q_{l-1}[1-l], \dots, Q_0)$.

2.5 A description of the derived categories of coherent sheaves on $\mathbb{P}_\theta(\overline{\mathbf{a}})$

The natural isomorphisms $\mathrm{Hom}(P_i, P_j) \cong \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j))$, which are direct consequences of the construction of the algebra B, allow us to construct a functor $\bar{F} : \mathbf{H}^b(\mathbb{P}) \longrightarrow \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$, where $\mathbb{P}$ is the full subcategory of the category of right modules mod–B consisting of finite direct sums of the projective modules $P_i, i = 0, \dots, l-1$. The functor $\bar{F}$ sends P_i to $\mathcal{O}(i)$ and any bounded complex of projective modules to the corresponding complex of $\mathcal{O}(i), i = 0, \dots, l-1$. It follows from Remark 2.9 that the functor $\bar{F}$ induces a functor

$$F : \boldsymbol{D}^b(\text{mod–}B) \longrightarrow \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta)).$$

Theorem 2.16 *The functor* $F : \boldsymbol{D}^b(\text{mod–}B) \longrightarrow \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ *is an equivalence of the derived categories.*

Since the exceptional collection $(\mathcal{O}, \dots, \mathcal{O}(l-1))$ generates the category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ it is sufficient to check that the functor F is fully faithful. We know that for any $0 \leq i, j \leq l-1$ and any k there are isomorphisms

$$\mathrm{Hom}(P_i, P_j[k]) \xrightarrow{\sim} \mathrm{Hom}(FP_i, FP_j[k]) = \mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(j)[k]).$$

Since $P_i, i = 0, \dots, l-1$ generate $\boldsymbol{D}^b(\text{mod–}B)$, the proof of the theorem is a consequence of the following lemma.

Lemma 2.17 *Let* $\mathbb{A}$ *be abelian category and* $\mathcal{D}$ *be a triangulated category. Let* $F : \boldsymbol{D}^b(\mathbb{A}) \longrightarrow \mathcal{D}$ *be an exact functor and let* $\{E_i\}_{i \in I}$ *be a set of objects of* $\boldsymbol{D}^b(\mathbb{A})$ *which generates* $\boldsymbol{D}^b(\mathbb{A})$ *(i.e.*

the minimal full triangulated subcategory of $\boldsymbol{D}^b(\mathbb{A})$ containing all E_i coincides with $\boldsymbol{D}^b(\mathbb{A})$). Assume that the maps

$$\mathrm{Hom}(E_i, E_j[k]) \longrightarrow \mathrm{Hom}(FE_i, FE_j[k])$$

are isomorphisms for all $i, j \in I$ and any $k \in \mathbb{Z}$. Then the functor F is fully faithful.

Proof. This lemma is known and results from dévissage (e.g. [112],10.10, [147]4.2). We first consider the full subcategory $\mathcal{C} \in \boldsymbol{D}^b(\mathbb{A})$ which consists of all objects X such that the maps

$$\mathrm{Hom}(X, E_i[k]) \xrightarrow{\sim} \mathrm{Hom}(FX, FE_i[k])$$

are isomorphisms for all $i \in I$ and all $k \in \mathbb{Z}$. The category $\mathcal{C}$ is a triangulated subcategory, because it is closed with respect to the translation functor and, for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1],$$

if X and Y belong to $\mathcal{C}$, then Z belongs too. The last statement is a consequence of the five lemma, i.e., since the morphisms f_1, f_2, f_4, f_5 in the diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(Y[1], E_i) & \longrightarrow & \mathrm{Hom}(X[1], E_i) & \longrightarrow & \mathrm{Hom}(Z, E_i) & \longrightarrow & \\
{\scriptstyle f_1}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & \big\downarrow{\scriptstyle f_3} & & \\
\mathrm{Hom}(FY[1], FE_i) & \longrightarrow & \mathrm{Hom}(FX[1], FE_i) & \longrightarrow & \mathrm{Hom}(FZ, FE_i) & \longrightarrow & \\
 & & & \longrightarrow & \mathrm{Hom}(Y, E_i) & \longrightarrow & \mathrm{Hom}(X, E_i) \\
 & & & & \big\downarrow{\scriptstyle f_4} & & \big\downarrow{\scriptstyle f_5} \\
 & & & \longrightarrow & \mathrm{Hom}(FY, FE_i) & \longrightarrow & \mathrm{Hom}(FX, FE_i)
\end{array}$$

are isomorphisms, the morphism f_3 is an isomorphism too. The subcategory $\mathcal{C}$ contains the objects E_i and, hence, coincides with $\boldsymbol{D}^b(\mathbb{A})$. Now consider the full subcategory $\mathcal{B} \subset \boldsymbol{D}^b(\mathbb{A})$ consisting of all objects X such that the map

$$\mathrm{Hom}(Y, X[k]) \xrightarrow{\sim} \mathrm{Hom}(FY, FX[k])$$

is an isomorphism for every object $Y \in \boldsymbol{D}^b(\mathbb{A})$ and all $k \in \mathbb{Z}$. By the same argument as above the subcategory $\mathcal{B}$ is triangulated and contains all E_i. Therefore, it coincides with $\boldsymbol{D}^b(\mathbb{A})$. This proves the lemma and completes the proof of the Theorem. □

There is also a right adjoint to F, namely a functor $G : \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta)) \longrightarrow \boldsymbol{D}^b(\mathrm{mod}-B)$. To construct it we have to consider the functor

$$\mathrm{Hom}(\mathcal{E}, -) : \mathrm{Qcoh}(\mathbb{P}_\theta) \longrightarrow \mathrm{Mod}-B$$

where $\mathrm{Mod}-B$ is the category of all right modules over B. Since $\mathrm{Qcoh}(\mathbb{P}_\theta)$ has enough injectives and has finite global dimension there is a right derived functor

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, -) : \boldsymbol{D}^b(\mathrm{Qcoh}(\mathbb{P}_\theta)) \longrightarrow \boldsymbol{D}^b(\mathrm{Mod}-B).$$

$\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ is equivalent to the full subcategory $\boldsymbol{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(\mathbb{P}_\theta))$ of $\boldsymbol{D}^b(\mathrm{Qcoh}(\mathbb{P}_\theta))$ whose objects are complexes with cohomologies in $\mathrm{coh}(\mathbb{P}_\theta)$. Moreover, the functor $\mathbf{R}\mathrm{Hom}(\mathcal{E}, -)$ sends an object of $\boldsymbol{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(\mathbb{P}_\theta))$ to an object of the subcategory $\boldsymbol{D}^b_{\mathrm{mod}}(\mathrm{Mod}-B)$, which is also equivalent to $\boldsymbol{D}^b(\mathrm{mod}-B)$. This gives us a functor

$$G = \mathbf{R}\mathrm{Hom}(\mathcal{E}, -) : \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta)) \longrightarrow \boldsymbol{D}^b(\mathrm{mod}-B).$$

The functor G is right adjoint to F, and it is an equivalence of categories as well.

In the end of this paragraph we describe an equivalence relation $\theta \sim \theta'$ on the space of all matrices θ with $\theta_{ij} \in \mathbb{C}^*$ for all i, j under which the noncommutative weighted projective spaces $\mathbb{P}_\theta$ and $\mathbb{P}_{\theta'}$ have equivalent abelian categories of coherent sheaves. It was mentioned above that the graded algebras S_θ depend only on the matrix θ^{an} defined by the rule (2.3). However, it can also happen that two different algebras S_θ and $S_{\theta'}$ produce isomorphic algebras B_θ and $B_{\theta'}$.

Proposition 2.18 *Let $(m_0, \ldots, m_n) \in (\mathbb{C}^*)^{(n+1)}$ be any vector with non-zero entries. Suppose that two matrices $\theta, \theta' \in \mathrm{M}(n+1, \mathbb{C}^*)$ are related by the formula*

$$\theta'_{ij} = \theta_{ij} \cdot m_i^{a_j}. \tag{2.7}$$

Then the algebras $B_{\theta'}$ and B_θ are isomorphic.

Proof. Consider the category $\mathfrak{B}_{\theta'}$ and its autoequivalence τ which acts by identity on the objects and acts on the spaces $\mathrm{Hom}(v_i, v_j)$ as the multiplication by $(m_i)^{(j-i)}$. There is a natural basis of the spaces $\mathrm{Hom}(v_i, v_j)$ which is induced by the monomial basis $x_{i_0} \cdots x_{i_k}$, $0 \le i_0 \le \cdots \le i_k \le n$ of $S_{\theta'}$. The transformation of this basis under the equivalence τ gives us a new basis in which the category $\mathfrak{B}_{\theta'}$ coincides with the category $\mathfrak{B}_\theta$ equipped with its natural basis coming from the monomial basis of S_θ. The equivalence of the categories $\mathfrak{B}_{\theta'}$ and $\mathfrak{B}_\theta$ implies an isomorphism of the algebras $B_{\theta'}$ and B_θ. $\square$

If now the algebras $B_{\theta'}$ and B_θ are isomorphic, then the composition of the functors

$$\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_{\theta'})) \xrightarrow{G_{\theta'}} \boldsymbol{D}^b(\mathrm{mod}-B_{\theta'}) \cong \boldsymbol{D}^b(\mathrm{mod}-B_\theta) \xrightarrow{F_\theta} \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$$

is an equivalence of derived categories. This equivalence evidently takes a sheaf $\mathcal{O}(i)$, $0 \le i \le l-1$ on $\mathbb{P}_{\theta'}$ to the sheaf $\mathcal{O}(i)$ on $\mathbb{P}_\theta$. Using the resolution (2.6) it can be easily checked that this functor takes $\mathcal{O}(i)$ to $\mathcal{O}(i)$ for all $i \in \mathbb{Z}$. Now, it follows from the ampleness condition on $\{\mathcal{O}(i)\}$ and Corollary 2.8 that the functor sends the subcategory $\mathrm{coh}(\mathbb{P}_{\theta'})$ to $\mathrm{coh}(\mathbb{P}_\theta)$ and induces an equivalence $\mathrm{coh}(\mathbb{P}_{\theta'}) \cong \mathrm{coh}(\mathbb{P}_\theta)$. We just proved:

Corollary 2.19 *If the matrices θ' and θ are connected by the relation (2.7) then the noncommutative weighted projective spaces $\mathbb{P}_{\theta'}(\overline{\mathbf{a}})$ and $\mathbb{P}_{\theta}(\overline{\mathbf{a}})$ have equivalent abelian categories of coherent sheaves* $\mathrm{coh}(\mathbb{P}_{\theta'})$ *and* $\mathrm{coh}(\mathbb{P}_{\theta})$.

In the case $n = 1$, it follows immediately that for any $\theta, \theta' \in M(2, \mathbb{C}^*)$ the categories $\mathrm{coh}(\mathbb{P}_{\theta}(a_0, a_1))$ and $\mathrm{coh}(\mathbb{P}_{\theta'}(a_0, a_1))$ are equivalent.

Next consider the case $n = 2$. For any matrix $\theta \in M(3, \mathbb{C}^*)$ denote the expression

$$(\theta_{01}^{an})^{a_2}(\theta_{12}^{an})^{a_0}(\theta_{20}^{an})^{a_1} = (\theta_{01})^{a_2}(\theta_{12})^{a_0}(\theta_{20})^{a_1}(\theta_{10})^{-a_2}(\theta_{21})^{-a_0}(\theta_{02})^{-a_1}$$

by $q(\theta)$. Now, the result of Proposition 2.18 can be written in the following form.

Corollary 2.20 *Let $n = 2$ and let θ' and θ be two matrices from* $\mathrm{M}(3, \mathbb{C}^*)$. *If $q(\theta') = q(\theta)$ then the abelian categories* $\mathrm{coh}(\mathbb{P}_{\theta'}(a_0, a_1, a_2))$ *and* $\mathrm{coh}(\mathbb{P}_{\theta}(a_0, a_1, a_2))$ *are equivalent.*

2.6 DG algebras and Koszul duality. The aim of this section is to give another description of the derived category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$. It was shown above that this category is equivalent to the derived category $\boldsymbol{D}^b(\mathrm{mod}-B)$. We introduce a finite dimensional differential $\mathbb{Z}$-graded algebra (DG algebra) $C_\theta^\bullet$ and prove that the category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ is equivalent to the derived category of $C_\theta^\bullet$.

This new description of the derived category in terms of the DG-algebra $C_\theta^\bullet$ naturally yields an exceptional collection (Corollary 2.27), which is essentially the (left) dual of the collection described in Theorem 2.12, cf. the discussion at the end of §2.4.

We recall here that a DG algebra over $\mathbb{C}$ is a graded associative $\mathbb{C}$–algebra

$$R = \bigoplus_{p \in \mathbb{Z}} R^p$$

with a differential d of degree $+1$ such that

$$d(rs) = (dr)s + (-1)^p r(ds)$$

for all $r \in R^p, s \in R$. We will suppose that R is noetherian as a graded algebra.

A right DG module over a DG algebra is a graded right R–module $M = \bigoplus_{p \in \mathbb{Z}} M^p$ with a differential ∇ of degree 1 such that

$$\nabla(mr) = (\nabla m)r + (-1)^p m dr$$

for all $m \in M^p$ and $r \in R$.

A morphism of DG R-modules $f : M \longrightarrow N$ is called null-homotopic if $f = d_N h + h d_M$, where $h : M \longrightarrow N$ is a morphism of the underlying graded R-modules which is homogeneous of degree -1. The homotopy category $\mathbf{H}^b(R)$ is defined as a category which has all finitely generated DG R-modules as objects, and whose morphisms are the equivalence classes $\overline{f}$ of morphisms of DG R-modules modulo null-homotopic morphisms. A morphism of DG R-modules $s : M \to N$ is called a quasi-isomorphism if the induced morphism $H^*s : H^*(M) \to H^*(N)$ is an isomorphism of graded vector spaces. Now, by definition, the derived category $\boldsymbol{D}^b(R)$ is the localization

$$\boldsymbol{D}^b(R) := \mathbf{H}^b(R)\left[\Sigma^{-1}\right],$$

where Σ is the class of all quasi-isomorphisms. It can be checked that there are canonical isomorphisms

$$\mathrm{Hom}_{\boldsymbol{D}^b(R)}(R, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{H}^b(R)}(R, M) \xrightarrow{\sim} H^0 M$$

for each DG R-module M.

Any ordinary $\mathbb{C}$-algebra A can be considered as the DG algebra $A^\bullet$ with $A^0 = A$ and $A^p = 0$ for all $p \neq 0$. In this case the derived category of the DG algebra $\boldsymbol{D}^b(A^\bullet)$ identifies with the bounded derived category of finitely generated right A-modules, i.e. $\boldsymbol{D}^b(A^\bullet) \cong \boldsymbol{D}^b(\mathrm{mod}-A)$. For a detailed exposition of the facts about derived categories of DG algebras, see [147, 148].

Now denote by B_s the algebra $B/\,\mathrm{rad}(B)$ and consider it as a right B-module, isomorphic to the sum $\overset{l-1}{\underset{i=0}{\oplus}} Q_i$ of all irreducibles. Introduce the finite dimensional DG algebra

$$\mathrm{Ext}^\bullet_B(B_s, B_s) = \underset{p\in\mathbb{Z}}{\oplus} \mathrm{Ext}^p_B(B_s, B_s)$$

with the natural composition law and trivial differential. In what follows we give a precise description of this DG algebra and prove the existence of an equivalence

$$\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta)) \cong \boldsymbol{D}^b(\mathrm{Ext}^\bullet_B(B_s, B_s)),$$

which gives the promised description of the category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$.

Let us introduce a graded DG algebra $\Lambda^\bullet = \Lambda^\bullet(a_0, \ldots, a_n)$. As a DG algebra it is the skew-symmetric algebra with trivial differential which is generated by skew-commutative elements y_i, $i = 0, \ldots, l-1$ of degree 1, i.e.

$$\Lambda^\bullet = \bigoplus_{p=0}^{n+1} \Lambda^p,$$

where $y_i \in \Lambda^1$ with the relations $y_i y_j = -y_j y_i$ for all $0 \le i, j \le n$.

The additional grading on the DG algebra $\Lambda^\bullet(a_0, \ldots, a_n)$ is defined by putting $y_i \in \Lambda^\bullet_{-a_i}$. Thus $\Lambda^\bullet(a_0, \ldots, a_n)$ is just a bigraded skew-symmetric algebra

$$\Lambda^\bullet(a_0, \ldots, a_n) = \bigoplus_{p,i\in\mathbb{Z}} \Lambda^p_i$$

with generators $y_i \in \Lambda^1_{-a_i}$. For any $(n+1)\times(n+1)$-matrix θ we also can define a graded DG algebra $\Lambda^\bullet_\theta(a_0, \ldots, a_n)$ as the DG algebra with trivial differential and generated by elements $y_i \in (\Lambda_\theta)^1_{-a_i}$, $i = 0, \ldots, n$ with the relations

$$\theta_{ij} y_i y_j + \theta_{ji} y_j y_i = 0$$

for all $0 \le i, j \le n$.

Consider the following complex $\mathrm{Com}^\bullet$ of right S_θ-modules

$$\mathrm{Com}^\bullet := 0 \to S_\theta(-\sum_{i=0}^{n} a_i) \to \bigoplus_{i_0<\ldots<i_{n-1}} S_\theta(-\sum_{j=0}^{n-1} a_{i_j}) \to \cdots \\ \cdots \to \bigoplus_{i_0<i_1} S_\theta(-a_{i_0}-a_{i_1}) \to \bigoplus_{i=0}^{n} S_\theta(-a_i) \to S_\theta \to 0, \tag{2.8}$$

in which the differentials are defined componentwise as follows: for any set $I = \{i_0, \ldots i_k\}$ the differential sends the generator of $S_\theta(-\sum_{i\in I} a_i)$ to the sum of the elements

$$(-1)^s \left(\prod_{i\in I} \theta_{ii_s}\right) x_{i_s}$$

of $S_\theta\big(-\sum_{i\in(I\setminus i_s)} a_i\big)$, for $0 \le s \le k$. With this we see that the complex $\mathrm{Com}^\bullet$ is a free resolution of the right S_θ-module $\mathbb{C}_{S_\theta}$.

Now we define a structure of left DG module over the DG algebra $\Lambda_\theta^\bullet$ on the complex $\mathrm{Com}^\bullet$, such that the element y_j takes the generator of $S_\theta(-\sum_{i\in I} a_i)$ to the generator of $S_\theta(-\sum_{i\in(I\setminus i_s)} a_i)$ with coefficient

$$(-1)^s \prod_{i\in I} \theta_{i_s i}$$

if $j = i_s \in I = \{i_0, \ldots, i_k\}$, and takes it to zero if $j \notin I$. It can be checked that this action is well defined and makes the complex $\mathrm{Com}^\bullet$ a DG $\Lambda_\theta^\bullet$-S_θ-bimodule.

Remark 2.21 *It is not difficult to see that the complex* $\mathrm{Com}^\bullet$ *as a graded* $\Lambda_\theta^\bullet$-S_θ*-bimodule (i.e. without differential) is isomorphic to* $(\Lambda_\theta^\bullet)^* \underset{\mathbb{C}}{\otimes} S_\theta$, *where* $(\Lambda_\theta^\bullet)^*$ *is* $\mathrm{Hom}_{\mathbb{C}}(\Lambda_\theta^\bullet, \mathbb{C})$.

Definition 2.22 *Define a DG category* $\mathfrak{C}_\theta$ *(actually graded category, because all differentials are trivial) as a DG category with* l *objects, say* $w_0, \ldots, w_{l-1}$, *and the spaces of morphisms between which are the complexes*

$$\mathrm{Hom}^\bullet(w_j, w_i) \cong (\Lambda_\theta^\bullet)_{i-j}$$

with the natural composition law induced by that of the DG algebra $\Lambda_\theta^\bullet$.

It follows from the definition of the DG algebra $\Lambda_\theta^\bullet$ that

$$\mathrm{Hom}^\bullet(w_j, w_i) = 0 \qquad \text{when} \quad j < i.$$

Definition 2.23 *Define the DG algebra $C_\theta^\bullet$ as the DG algebra of the DG category $\mathfrak{C}_\theta$, i.e.*

$$C_\theta^\bullet := \bigoplus_{0\le i,j\le l-1} \mathrm{Hom}^\bullet(w_j, w_i).$$

The quotient of this DG algebra by its radical is isomorphic to $\mathbb{C}^{\oplus l}$. In particular the DG algebra $C_\theta^\bullet$, similarly to the algebra B, has l irreducible DG modules in degree 0. Moreover, as a right DG $C_\theta^\bullet$-module the algebra $C_\theta^\bullet$ is a direct sum

$$C_\theta^\bullet = \bigoplus_{i=0}^{l-1} H_i, \qquad \text{where} \quad H_i = \bigoplus_{0\le j\le l-1} \mathrm{Hom}^\bullet(w_j, w_i),$$

and the direct summands H_i are homotopically projective right DG $C_\theta^\bullet$-modules. Recall that a DG module H is called homotopically projective if, for any acyclic DG module N, $\mathrm{Hom}(H, N) = 0$ in the homotopy category (see e.g. [147, 148]).

Let us construct a DG $C_\theta^\bullet$-B-bimodule $X^\bullet$, obtained from the DG $\Lambda_\theta^\bullet$-S_θ-bimodule $\mathrm{Com}^\bullet$ by the formula

$$X^\bullet = \bigoplus_{0\le i,j\le l-1} X^\bullet(i,j), \qquad \text{with} \quad X^\bullet(i,j) \cong \mathrm{Com}^\bullet_{j-i}$$

where $\mathrm{Com}^\bullet_{j-i}$ is the degree $(j-i)$ component of the graded complex $\mathrm{Com}^\bullet$. In particular, $X^\bullet(i,j) = 0$ when $i > j$ and $X^\bullet(i,i) \cong \mathbb{C}$ for all i. The structure of DG $C_\theta^\bullet$-B-bimodule on $X^\bullet$ comes from the structure of DG $\Lambda_\theta^\bullet$-S_θ-bimodule on $\mathrm{Com}^\bullet$. The bimodule $X^\bullet$ is quasi-isomorphic to $\mathbb{C}^{\oplus l}$, and it is quasi-isomorphic to $B/\operatorname{rad}(B)$ as a right B-module and to $C_\theta^\bullet/\operatorname{rad}(C_\theta^\bullet)$ as a left DG $C_\theta^\bullet$-module. This fact allows us to say that the DG algebra $C_\theta^\bullet$ is the Koszul dual to the algebra B.

Remark 2.24 *It follows from Remark* 2.21 *that $X^\bullet$ as a graded $C_\theta^\bullet$-B-bimodule (i.e. without differential) is isomorphic to*

$$\bigoplus_{i=0}^{l-1} H_i^* \otimes P_i,$$

where H_i^* are the left DG $C_\theta^\bullet$-modules $\mathrm{Hom}_{\mathbb{C}}(H_i, \mathbb{C})$. *In other words, as a graded $C_\theta^\bullet$-B-bimodule $X^\bullet$ is isomorphic to $C_\theta^{\bullet *} \otimes_{\mathbb{C}^{\oplus l}} B$.*

For any right DG $C_\theta^\bullet$-module N, the tensor product $N \otimes_{\mathbb{C}} X^\bullet$ is naturally a complex of right B-modules, in which the module structure is given by the action of B on $X^\bullet$, and the grading and differential are given by

$$(N \otimes_{\mathbb{C}} X^\bullet)^k = \bigoplus_{p+q=k} N^p \otimes_{\mathbb{C}} X^q, \qquad d(n \otimes x) = (dn) \otimes x + (-1)^p n \otimes dx$$

for all $n \in N^p, x \in X^\bullet$. The $\mathbb{C}$-submodule generated by all differences $nc \otimes x - m \otimes cx$ is closed under the differential and under multiplication by any element of B. So the quotient by this submodule, which we denote by $N \otimes_{C_\theta^\bullet} X^\bullet$, is a well-defined complex of right B-modules.

For any complex M of right B-modules we define a right DG $C_\theta^\bullet$-module

$$\mathcal{H}om_B(X^\bullet, M)^k = \prod_{p-q=k} \mathrm{Hom}_B(X^q, M^p), \qquad (df)(x) = d(f(x)) - (-1)^n f(dx).$$

In this way we get a pair of adjoint functors $(-)\otimes_{C_\theta^\bullet} X^\bullet$ and $\mathcal{H}om_B(X^\bullet, -)$ between homotopy categories, which induce a pair of adjoint functors on the level of derived categories as well:

$$\overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet : \boldsymbol{D}^b(C_\theta^\bullet) \longrightarrow \boldsymbol{D}^b(\mathrm{mod}\text{-}B), \qquad \mathbf{R}\mathrm{Hom}_B(X^\bullet, -) : \boldsymbol{D}^b(\mathrm{mod}\text{-}B) \longrightarrow \boldsymbol{D}^b(C_\theta^\bullet).$$

Moreover, since $X^\bullet$ is a projective finitely generated right B-module and a flat left $C_\theta^\bullet$-module, both functors $(-)\otimes_{C_\theta^\bullet} X^\bullet$ and $\mathcal{H}om_B(X^\bullet, -)$ between homotopy categories preserve acyclicity. Hence, the derived functors in this case are defined by the same formulas. For more information about derived functors see e.g. [147].

Theorem 2.25 *The functors* $\overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet$ *and* $\mathbf{R}\mathrm{Hom}_B(X^\bullet, -)$ *are equivalences of triangulated categories.*

Proof. It is evident that the first functor $\overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet$ takes $C_\theta^\bullet$ as a right DG $C_\theta^\bullet$-module to $X^\bullet$ as a right B-module which is isomorphic to $B_s = \overset{l-1}{\underset{i=0}{\oplus}} Q_i$ in the derived category $\boldsymbol{D}^b(\mathrm{mod}\text{-}B)$. On the other hand, it follows from Remark 2.24 and the equalities $\mathrm{Hom}_B(P_i, Q_j) = \delta^i_j\mathbb{C}$ that the latter functor, $\mathbf{R}\mathrm{Hom}_B(X^\bullet, -)$, takes the module $B_s = \overset{l-1}{\underset{i=0}{\oplus}} Q_i$ to the free DG module $C_\theta^\bullet = \bigoplus_{i=0}^{l-1} H_i$ and takes Q_i to H_i for any $0 \le i \le l-1$. Thus, the composition functor $\mathbf{R}\mathrm{Hom}_B(X^\bullet, -) \overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet$ sends B_s to itself and it also sends all direct summands Q_i to Q_i. The adjunction maps

$$\mathbf{R}\mathrm{Hom}_B(X^\bullet, Q_i) \overset{\mathbf{L}}{\otimes}_{C_\theta^\bullet} X^\bullet \longrightarrow Q_i$$

cannot be trivial, hence they are isomorphisms for all i. Therefore, we obtain isomorphisms

$$\mathrm{Hom}_B(Q_i, Q_j[k]) \xrightarrow{\sim} \mathrm{Hom}_{C_\theta^\bullet}(\mathbf{R}\mathrm{Hom}_B(X^\bullet, Q_i), \mathbf{R}\mathrm{Hom}_B(X^\bullet, Q_j)[k]) \cong \mathrm{Hom}(H_i, H_j[k])$$

for any $0 \le i, j \le l-1$ and all $k \in \mathbb{Z}$.

Since $Q_i, i = 0, \ldots, l-1$ generate the derived category $\boldsymbol{D}^b(\mathrm{mod}\text{-}B)$, Lemma 2.17 implies that the functor

$$\mathbf{R}\mathrm{Hom}_B(X^\bullet, -) : \boldsymbol{D}^b(\mathrm{mod}\text{-}B) \longrightarrow \boldsymbol{D}^b(C_\theta^\bullet)$$

is fully faithful.

Consider the triangulated subcategory $\mathcal{D}$ of $\boldsymbol{D}^b(C_\theta^\bullet)$ generated by H_i, $i = 0, \dots, l-1$. By Remark 2.24 $X^\bullet$ as a graded $C_\theta^\bullet$-B-bimodule is isomorphic to $\bigoplus_{i=0}^{l-1} H_i^* \otimes P_i$, and hence, the dual to $X^\bullet$ over $\mathbb{C}$ gives a resolution of $C_\theta^\bullet / \operatorname{rad}(C_\theta^\bullet)$ in terms of H_i. Therefore, the subcategory $\mathcal{D}$ contains all irreducible DG modules and coincides with the whole $\boldsymbol{D}^b(C_\theta^\bullet)$. Thus, H_i, $i = 0, \dots, l-1$ generate the category $\boldsymbol{D}^b(C_\theta^\bullet)$, and the functor $\mathbf{R}\mathrm{Hom}_B(X^\bullet, -)$ is an equivalence of the derived categories. $\square$

Corollary 2.26 *There is an isomorphism of DG algebras*

$$C_\theta^\bullet \cong \bigoplus_{0 \le i,j \le l-1} \mathrm{Ext}^\bullet(Q_i, Q_j).$$

The assertion of the Corollary is clear now, because the functor $\overset{\mathbf{L}}{\otimes}$, which is an equivalence, sends $C_\theta^\bullet$ to $B_s = \overset{l-1}{\underset{i=0}{\oplus}} Q_i$.

Corollary 2.27 *The derived category of coherent sheaves $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}_\theta))$ on the noncommutative weighted space $\mathbb{P}_\theta$ is equivalent to the derived category $\boldsymbol{D}^b(C_\theta^\bullet)$.*

2.7 Hirzebruch surfaces $\mathbb{F}_n$ The surfaces $\mathbb{F}_n$ are minimal rational surfaces defined as the projectivizations $\mathbf{Proj}\,(\mathcal{O} \oplus \mathcal{O}(-n))$ of the vector bundles $\mathcal{O} \oplus \mathcal{O}(-n)$ over $\mathbb{P}^1$. The surface $\mathbb{F}_n$ has a $(-n)$-section that will be denoted by s. There is a simple connection between $\mathbb{F}_n$ and the weighted projective plane $\mathbf{P}(1,1,n)$, namely the latter can be obtained from $\mathbb{F}_n$ by contracting the $(-n)$section s. In this way $\mathbb{F}_n$ is a resolution of the singularity of $\mathbf{P}(1,1,n)$. Thus, we have two different resolutions of the singularity of $\mathbf{P}(1,1,n)$:

$$\begin{tikzcd} \mathbb{F}_n \arrow[dr] & & \mathbb{P}(1,1,n) \arrow[dl] \\ & \mathbf{P}(1,1,n) & \end{tikzcd}$$

For this reason the derived categories of coherent sheaves on $\mathbb{F}_n$ and on $\mathbb{P}(1,1,n)$ are closely related to each other. We will show that for $n \ge 2$ there is a fully faithful functor

$$MK_n : \boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_n)) \longrightarrow \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}(1,1,n)))$$

and will give its description.

Denote by f the class of the fiber of $\mathbb{F}_n$ in the Picard group. Since $\mathbb{F}_n$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ the derived category of coherent sheaves on $\mathbb{F}_n$ has an exceptional collection of length 4 (see [193]). More precisely, we have

Proposition 2.28 *The collection* $\sigma = (\mathcal{O}, \mathcal{O}(f), \mathcal{O}(s+nf), \mathcal{O}(s+(n+1)f))$ *is a full strong exceptional collection on* $\mathbb{F}_n$*. The derived category* $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_n))$ *is equivalent to the derived category* $\boldsymbol{D}^b(\mathrm{mod}\text{-}F(n))$*, where* $F(n)$ *is the algebra of the exceptional collection* σ*.*

Denote by U the two dimensional vector space $H^0(\mathbb{F}_n, \mathcal{O}(f))$. From the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(s+nf) \longrightarrow \mathcal{O}_s \longrightarrow 0$$

we find that $H^0(\mathbb{F}_n, \mathcal{O}(s+nf))$ is the direct sum of the space S^nU and a one-dimensional space. Analogously, we can check that $H^0(\mathbb{F}_n, \mathcal{O}(s+(n+1)f))$ is isomorphic to $S^nU \oplus U$.

On the other hand, we know that the weighted projective plane $\mathbb{P}(1,1,n)$ has an exceptional collection

$$(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n), \mathcal{O}(n+1))\,.$$

Denote the algebra of this exceptional collection by $B(1,1,n)$. It follows from Proposition 2.7 that the space $H^0(\mathbb{P}(1,1,n), \mathcal{O}(1))$ is isomorphic to U, $H^0(\mathbb{P}(1,1,n), \mathcal{O}(n))$ is isomorphic to the direct sum of S^nU and a one-dimensional space, and $H^0(\mathbb{P}(1,1,n), \mathcal{O}(n+1))$ is isomorphic to $S^nU \oplus U$. This implies that the algebra of the exceptional collection $(\mathcal{O}, \mathcal{O}(f), \mathcal{O}(s+nf), \mathcal{O}(s+(n+1)f))$ on $\mathbb{F}_n$ is isomorphic to the algebra of the exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1))$ on $\mathbb{P}(1,1,n)$.

Thus, the algebra of endomorphisms of the projective $B(1,1,n)$-module

$$M = P_0 \oplus P_1 \oplus P_n \oplus P_{n+1}$$

coincides with $F(n)$, which makes M a $F(n)$-$B(1,1,n)$-bimodule. The natural functor

$$(-) \overset{\mathbf{L}}{\otimes}_{F(n)} M : \boldsymbol{D}^b(\mathrm{mod}\text{-}F(n)) \longrightarrow \boldsymbol{D}^b(\mathrm{mod}\text{-}B(1,1,n))$$

takes the free module $F(n)$ to M, and there are isomorphisms

$$\mathrm{Hom}_{F(n)}(F(n), F(n)[k]) \xrightarrow{\sim} \mathrm{Hom}_{B(1,1,n)}(M, M[k]).$$

Since the direct summands of $F(n)$ generate the derived category $\boldsymbol{D}^b(\mathrm{mod}\text{-}F(n))$, Lemma 2.17 guarantees that the functor $(-) \overset{\mathbf{L}}{\otimes}_{F(n)} M$ is fully faithful. Using the descriptions of the derived categories of coherent sheaves on $\mathbb{F}_n$ and $\mathbb{P}(1,1,n)$ in terms of the exceptional collections, we obtain the following theorem.

Theorem 2.29 *The functor*

$$MK_n : \boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_n)) \longrightarrow \boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}(1,1,n)))$$

induced by $(-) \overset{\mathbf{L}}{\otimes}_{F(n)} M$ *is fully faithful.*

3 Categories of Lagrangian vanishing cycles

3.1 The category of vanishing cycles of an affine Lefschetz fibration We begin this section by briefly reviewing Seidel's construction of a Fukaya-type A_∞-category associated to a symplectic Lefschetz fibration [214, 215, 217], following a proposal of Kontsevich [152]. For an account of the underlying physics, the reader is referred to the work of Hori et al [120].

Let (X, ω) be an open symplectic manifold, and let $f : X \to \mathbb{C}$ be a symplectic Lefschetz fibration, i.e. a C^∞ complex-valued function with isolated non-degenerate critical points $p_1, \dots, p_r$ near which f is given in local complex coordinates by $f(z_1, \dots, z_n) = f(p_i) + z_1^2 + \dots + z_n^2$, and whose fibers are symplectic submanifolds of X. Fix a regular value λ_0 of f, and consider an arc $\gamma \subset \mathbb{C}$ joining λ_0 to a critical value $\lambda_i = f(p_i)$. Using the horizontal distribution given by the symplectic orthogonal to the fibers of f, we can transport the Lagrangian vanishing cycle at p_i along the arc γ to obtain a Lagrangian disc $D_\gamma \subset X$ fibered above γ, whose boundary is an embedded Lagrangian sphere L_γ in the fiber $\Sigma_0 = f^{-1}(\lambda_0)$. When the fibers of f are non-compact, parallel transport along the horizontal distribution is not always well-defined; we will always assume that the symplectic form ω satisfies the conditions required to make the construction valid. The Lagrangian disc D_γ is called the *Lefschetz thimble* over γ, and its boundary L_γ is the vanishing cycle associated to the critical point p_i and to the arc γ.

Let $\gamma_1, \dots, \gamma_r$ be a collection of arcs in $\mathbb{C}$ joining the reference point λ_0 to the various critical values of f, intersecting each other only at λ_0, and ordered in the clockwise direction around p_0. Each arc γ_i gives rise to a Lefschetz thimble $D_i \subset X$, whose boundary is a Lagrangian sphere $L_i \subset \Sigma_0$. After a small perturbation we can always assume that these spheres intersect each other transversely inside Σ_0.

Definition 3.1 (Seidel) *The directed category of vanishing cycles* $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ *is an* A_∞-*category (over a coefficient ring* R*) with* r *objects* $L_1, \dots, L_r$ *corresponding to the vanishing cycles (or more accurately to the thimbles); the morphisms between the objects are given by*

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; R) = R^{|L_i \cap L_j|} & \text{if } i < j \\ R \cdot id & \text{if } i = j \\ 0 & \text{if } i > j; \end{cases}$$

and the differential m_1*, composition* m_2 *and higher order products* m_k *are defined in terms of Lagrangian Floer homology inside* Σ_0*. More precisely,*

$$m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \dots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$$

is trivial when the inequality $i_0 < i_1 < \dots < i_k$ *fails to hold (i.e. it is always zero in this case, except for* m_2 *where composition with an identity morphism is given by the obvious formula).*

When $i_0 < \cdots < i_k$, m_k is defined by fixing a generic ω-compatible almost-complex structure on Σ_0 and counting pseudo-holomorphic maps from a disc with $k+1$ cyclically ordered marked points on its boundary to Σ_0, mapping the marked points to the given intersection points between vanishing cycles, and the portions of boundary between them to $L_{i_0}, \dots, L_{i_k}$ respectively.

While the general definition of Lagrangian Floer homology is a very delicate task [81], we will only consider cases where most of the technical considerations can be skipped. For example, Seidel considers the case where the symplectic form ω is exact ($\omega = d\theta$ for some 1-form θ) and the L_i are exact Lagrangian submanifolds in Σ_0 (i.e. $\theta_{|L_i} = dg_i$ is also exact). Here, we assume instead that the restricted symplectic form $\omega_{|\Sigma_0}$ is exact and that the homotopy groups $\pi_2(\Sigma_0)$ and $\pi_2(\Sigma_0, L_i)$ are trivial. The first condition prevents the bubbling of pseudo-holomorphic spheres, while the second one prevents the bubbling of pseudo-holomorphic discs in the definition of Lagrangian Floer homology. Therefore, the moduli spaces of pseudo-holomorphic maps involved in the definition of $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ have well-defined fundamental classes.

Another assumption that we will make concerns the Maslov class, which we will assume to vanish over L_i. In fact, we will restrict ourselves to the case where X and Σ_0 are affine Calabi-Yau manifolds, and the spheres L_i can be lifted to graded Lagrangian submanifolds of Σ_0, e.g. by fixing a holomorphic volume form on Σ_0 and choosing a real lift of the phase $\exp(i\phi) = \Omega_{|L_i}/vol_{L_i} : L_i \to S^1$. This makes it possible to define a $\mathbb{Z}$-grading (by Maslov index) on the Floer complexes $CF^*(L_i, L_j; R)$, as will be discussed later (see also [214]).

For simplicity, Seidel uses $R = \mathbb{Z}/2$ as coefficient ring in his definition; however the moduli spaces considered below are orientable, so it is possible to assign a sign ± 1 to each pseudo-holomorphic curve and hence define Floer homology over $\mathbb{Z}$. We will further extend the coefficient ring to $R = \mathbb{C}$, and count the contribution of each pseudo-holomorphic disc $u : (D^2, \partial D^2) \to (\Sigma_0, \bigcup L_i)$ in the moduli space with a coefficient of the form $\pm \exp(-2\pi \int_{D^2} u^*\omega)$. Weighting by area is irrelevant in the case of exact Lagrangian vanishing cycles considered by Seidel, where it does not affect at all the structure of the category: indeed, the symplectic areas can then be expressed in terms of the primitives g_i of θ over L_i, and can be eliminated from the description simply by a rescaling of the chosen bases of the Floer complexes (considering the basis $\{\exp\left(g_i(p) - g_j(p)\right) p,\ p \in L_i \cap L_j\}$ of $CF^*(L_i, L_j)$). On the contrary, in the non-exact case it is important to incorporate this weighting by area into the definition.

Hence, given two intersection points $p \in L_i \cap L_j$, $q \in L_j \cap L_k$ ($i < j < k$), we have by definition

$$m_2(p, q) = \sum_{\substack{r \in L_i \cap L_k \\ \deg r = \deg p + \deg q}} \left(\sum_{[u] \in \mathcal{M}(p,q,r)} \pm \exp(-2\pi \int_{D^2} u^*\omega) \right) r$$

where $\mathcal{M}(p, q, r)$ is the moduli space of pseudo-holomorphic maps u from the unit disc to M (equipped with a generic ω-compatible almost-complex structure) such that $u(1) = p$, $u(\mathrm{j}) = q$, $u(\mathrm{j}^2) = r$ (where $\mathrm{j} = \exp(\frac{2i\pi}{3})$), and mapping the portions of unit circle $[1, \mathrm{j}]$, $[\mathrm{j}, \mathrm{j}^2]$, $[\mathrm{j}^2, 1]$ to

L_i, L_j and L_k respectively. The other products are defined similarly.

It is worth mentioning that this definition of Floer homology over the complex numbers is in fact essentially equivalent to the use of coefficients in a Novikov ring, since in both cases the main goal is to keep track of (relative) homology classes.

Although the category $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ depends on the chosen ordered collection of arcs $\{\gamma_i\}$, Seidel has obtained the following result [214]:

Theorem 3.2 (Seidel) *If the ordered collection $\{\gamma_i\}$ is replaced by another one $\{\gamma_i'\}$, then the categories $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ and $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i'\})$ differ by a sequence of mutations.*

Hence, the category naturally associated to the Lefschetz fibration f is not the finite directed category defined above, but rather a (bounded) *derived* category, obtained from $\mathrm{Lag}_{\mathrm{vc}}(f, \{\gamma_i\})$ by considering twisted complexes of formal direct sums of Lagrangian vanishing cycles, and adding idempotent splittings and formal inverses of quasi-isomorphisms. It is a classical result that, if two categories are related by mutations, then their derived categories are equivalent; hence the derived category $D(\mathrm{Lag}_{\mathrm{vc}}(f))$ only depends on the Lefschetz fibration f rather than on the choice of an ordered system of arcs [214].

We finish this overview with a couple of remarks. In "usual" Fukaya categories, objects are pairs consisting of a compact Lagrangian submanifold and a flat connection on some complex vector bundle defined over it. In the case of the category associated to a Lefschetz fibration, the objects are vanishing cycles, or perhaps more accurately, the Lefschetz thimbles bounded by the vanishing cycles. Since the thimbles are contractible, all flat vector bundles over them are trivial, which eliminates the need to consider Floer homology with twisted coefficients. This ceases to be true in presence of a non-trivial B-field, but even then the equivalence class of the connection is entirely determined by the thimble. Another related issue is the choice of a spin structure on the vanishing cycles in order to fix the orientation on the moduli spaces: in the one-dimensional case that will be of interest to us, each vanishing cycle admits two distinct spin structures ($H^1(S^1, \mathbb{Z}/2) = \mathbb{Z}/2$). However we must always consider the spin structure which extends to the thimble, i.e. the non-trivial one.

The reader is referred to Seidel's papers [214, 215] for various examples – we will focus specifically on the Landau-Ginzburg models mirror to weighted projective spaces and Hirzebruch surfaces.

3.2 Structure of the proof of Theorem 1.2 Derived categories of coherent sheaves on the weighted projective planes $\mathbb{P}^2(a, b, c)$ and their noncommutative deformations $\mathbb{P}^2_\theta(a, b, c)$ have been described in Section 2. Hence, to prove Theorem 1.2, we need to find a similar description of the derived categories of Lagrangian vanishing cycles on the mirror Landau-Ginzburg models.

Recall that the mirror to $\mathbb{P}^2_\theta(a, b, c)$ is (X, W), where X is the affine hypersurface $\{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, equipped with an exact (for the commutative case) or non-exact (for the noncommutative case) symplectic form, and the superpotential $W = x + y + z$.

By construction, categories of Lagrangian vanishing cycles for Lefschetz fibrations always admit full exceptional collections. Indeed, for any choice of arcs $\{\gamma_i\}$ the objects L_i of $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ form a generating exceptional collection of the derived category. Hence, in view of Theorem 2.12 and Corollary 2.27, all we need to do is exhibit a set of arcs $\{\gamma_i\}$ for which $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is equivalent to one of the categories $\mathfrak{B}$ or $\mathfrak{C}_\theta$ introduced in §2 (it turns out that the latter choice is slightly easier to achieve).

Recall from Corollary 2.27 that $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{P}^2_\theta(a,b,c)))$ is equivalent to the derived category of the DG-algebra $C^\bullet_\theta$ associated to the finite DG-category $\mathfrak{C}_\theta$ which has $l = a+b+c$ objects $w_0, \dots, w_{l-1}$, with morphisms between them given by the complexes

$$\mathrm{Hom}^\bullet(w_j, w_i) \cong (\Lambda^\bullet_\theta)_{i-j}$$

with the natural composition law induced by that of the deformed exterior algebra $\Lambda^\bullet_\theta$ on three generators of degrees $-a, -b, -c$, with relations of the form $\theta_{ij} y_i y_j + \theta_{ji} y_j y_i$ where $\theta \in M(3, \mathbb{C}^*)$ (see §2.6). Moreover, by Corollary 2.20, this category depends only on the quantity

$$q(\theta) = (\theta_{01})^c (\theta_{12})^a (\theta_{20})^b (\theta_{10})^{-c} (\theta_{21})^{-a} (\theta_{02})^{-b}.$$

From a practical viewpoint, the cyclic group $\mathbb{Z}/(a+b+c)$ acts by diagonal multiplication on X, and the superpotential $W = x + y + z$ is equivariant with respect to this action. The $(a+b+c)$ critical values of W form a single orbit under this action (see §4.2). In order to exploit this symmetry, it is therefore natural to choose the smooth fiber $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and an ordered system of arcs $\gamma_i \subset \mathbb{C}$ $(i = 0, \dots, a+b+c-1)$ consisting of straight line segments from the origin to the various critical values λ_i.

With this understood, Theorem 1.2 follows immediately from Corollary 2.27 and the following statement:

Theorem 3.3 $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is a DG category, and it is equivalent to* $\mathfrak{C}_\theta$ *for any* $\theta \in \mathrm{M}(3, \mathbb{C}^*)$ *such that* $q(\theta) = \exp(2\pi i [B + i\omega] \cdot [T])$*, where* $[B + i\omega] \in H^2(X, \mathbb{C})$ *is the complexified Kähler class, and* $[T]$ *is the generator of* $H_2(X, \mathbb{Z})$*.*

The proof of Theorem 3.3 consists of several steps, carried out in the various subsections of §4. First, as a prerequisite to the determination of the vanishing cycles, one needs a convenient description of the reference fiber Σ_0. This is done by considering the projection to the first coordinate axis, $\pi_x : \Sigma_0 \to \mathbb{C}^*$, which makes Σ_0 a $(b+c)$-fold covering of $\mathbb{C}^*$ branched in $(a+b+c)$ points (Lemma 4.1). With this understood, it becomes fairly easy to identify the vanishing cycles associated to the arcs γ_j, at least in the special case where the symplectic form is anti-invariant under complex conjugation (which implies its exactness). Indeed, this assumption implies that the vanishing cycles L_j are *Hamiltonian isotopic* (and hence equivalent from the point of view of Floer theory) to the double lifts via π_x of certain arcs $\delta_j \subset \mathbb{C}^*$ (Lemma 4.2) which can be described explicitly (Figure 5).

With an explicit description of the vanishing cycles at hand, it becomes possible to understand the Floer complexes $CF^*(L_i, L_j)$, by studying the intersections between L_i and L_j for all $0 \le i < j < a+b+c$. Using the projection to the first coordinate, these correspond to certain specific intersections between the arcs δ_i and δ_j in $\mathbb{C}^*$, as dictated by the combinatorics of the branched covering π_x. Such a description is given by Lemma 4.3, from which it follows readily that $CF^*(L_i, L_j) \simeq (\Lambda^\bullet_\theta)_{i-j}$ for all i, j.

The next step is to study the Floer differentials and products in $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ by counting pseudo-holomorphic maps from $(D^2, \partial D^2)$ to $(\Sigma_0, \bigcup L_i)$. This is done by searching for immersed polygonal regions in Σ_0 with boundary contained in $\bigcup L_i$, or equivalently, images of such regions under the projection π_x (see §4.4). In our case, it turns out that the only possible contributions come from triangular regions in Σ_0; hence, the Floer differential m_1 and the higher compositions $(m_k)_{k\ge 3}$ are identically zero (Lemmas 4.3 and 4.4) for purely topological reasons, while the Floer product m_2 has a particularly simple structure (Lemma 4.5). In particular, the A_∞-category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is actually a DG category with trivial differential.

The grading in $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is determined by the Maslov indices of intersection points. Since the Maslov class vanishes, each L_i can be lifted to a *graded Lagrangian* submanifold of Σ_0 by choosing a real lift of its phase function (see §4.5). The degree of a given intersection point $p \in L_i \cap L_j$ is then determined by the difference between the phases of L_i and L_j at p. Although the determination of phases is the most technical part of the argument, it actually presents little conceptual difficulty, and after some calculations one readily checks that the grading of morphisms in $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is the expected one. Namely, the "generating" morphisms corresponding to the generators of the deformed exterior algebra $\Lambda^\bullet_\theta$ have degree 1, and their pairwise products have degree 2 (cf. Lemma 4.7).

The argument is then completed by determining more precisely the structure coefficients for the Floer product m_2, which depend on the symplectic areas of the various pseudo-holomorphic discs and on the choice of consistent orientations of the moduli spaces (see §4.6). In the case where the symplectic form is anti-invariant under complex conjugation, the argument is greatly simplified by symmetry considerations, and the Floer products obey the anticommutation rules of an (undeformed) exterior algebra (Lemma 4.8) – recall that complex conjugation anti-invariance implies exactness of the symplectic form. In the non-exact case or in presence of a non-zero B-field, there is no simple method for determining the symplectic areas of the various pseudo-holomorphic discs involved in the definition of m_2. However the deformation of the category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is governed by a single parameter (analogous to the quantity $q(\theta)$ introduced in Corollary 2.20), for which a simple topological interpretation can be found, involving only the evaluation of $[B + i\omega]$ on the generator of $H_2(X, \mathbb{Z})$ (Lemmas 4.9 and 4.10).

This provides the desired characterization of the category of Lagrangian vanishing cycles, and Theorem 3.3 becomes an easy corollary of Lemmas 4.3–4.10. The only subtle point is that the objects of the category $\mathfrak{C}_\theta$ are numbered "backwards" (because the generators of $\Lambda^\bullet_\theta$ are assigned *negative* degrees), so the equivalence of categories actually takes the objects $L_0, \dots, L_{a+b+c-1}$ of $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ to the objects $w_{a+b+c-1}, \dots, w_0$ of $\mathfrak{C}_\theta$.

3.3 Mirrors of weighted projective lines As a warm-up example, we prove HMS for the

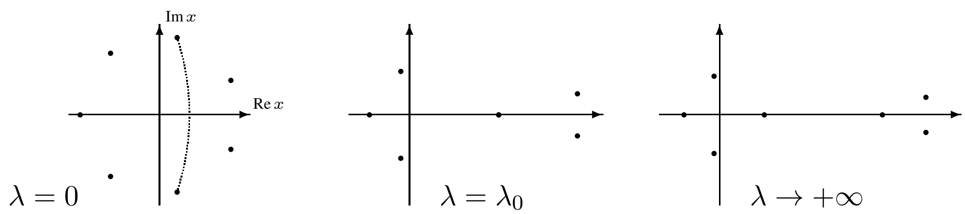


Figure 1: The fiber of W for $\lambda \in \mathbb{R}_+$ ($(a, b) = (4, 3)$)

weighted projective lines $\mathbb{CP}^1(a, b)$, where a, b are mutually prime positive integers (see also [215] and [245]). The argument is an extremely simplified version of that outlined in §3.2. Indeed, the mirror Landau-Ginzburg model is the curve $X = \{x^a y^b = 1\} \subset (\mathbb{C}^*)^2$ equipped with the superpotential $W = x + y$, whose generic fiber is just a finite set of $a + b$ points; so most of the considerations that arise in the case of weighted projective planes are irrelevant here (in particular the symplectic structure on X plays no role whatsoever, which is consistent with the fact that the category $\mathrm{coh}(\mathbb{P}_\theta(a, b))$ does not depend on θ).

More precisely, the fiber of W above a point $\lambda \in \mathbb{C}$ is

$$W^{-1}(\lambda) = \{(x, \lambda - x) \in (\mathbb{C}^*)^2,\ x^a(\lambda - x)^b = 1\},$$

which consists of $a + b$ distinct points, unless $P(x) = x^a(\lambda - x)^b - 1$ has a double root. Since

$$P'(x) = \Big(\frac{a}{x} - \frac{b}{\lambda - x}\Big)(P(x) + 1),$$

a root of P is a double root if and only if $x = \frac{a}{a+b}\lambda$; hence a double root exists if and only if $P(\frac{a}{a+b}\lambda) = 0$, i.e.

$$\lambda^{a+b} = \frac{(a + b)^{a+b}}{a^a b^b}. \tag{3.1}$$

Let λ_0 be the positive real root of this equation, and let $\lambda_j = \lambda_0 \zeta^{-j}$ where $\zeta = \exp(\frac{2\pi i}{a+b})$: then the critical values of W are exactly $\lambda_0, \dots, \lambda_{a+b-1}$. We choose $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and consider the ordered system of arcs $\gamma_0, \dots, \gamma_{a+b-1}$, where $\gamma_j \subset \mathbb{C}$ is a straight line segment joining the origin to λ_j. With this understood, we have the following result, which implies that HMS holds for $\mathbb{CP}^1(a, b)$:

Theorem 3.4 $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is a DG category, equivalent to* $\mathfrak{C}_\theta$ *for any* $\theta \in \mathrm{M}(2, \mathbb{C}^*)$.

In order to prove Theorem 3.4, we study the vanishing cycles of the superpotential W and their intersection properties. To start with, observe that W is equivariant with respect to the diagonal action of the cyclic group $\mathbb{Z}/(a+b)$. Therefore, the vanishing cycles $L_j \subset \Sigma_0$ (which are Lagrangian 0-spheres, i.e. pairs of points) form a single $\mathbb{Z}/(a+b)$-orbit, and $L_j = \zeta^{-j} \cdot L_0$.

In order to determine L_0, we study how the fiber $W^{-1}(\lambda)$ varies as λ increases along the positive real axis (see Figure 1). For $\lambda = 0$, the fiber Σ_0 consists of $a + b$ points whose first coordinates are the roots of the equation $x^{a+b} = (-1)^b$ (these form a $\mathbb{Z}/(a+b)$-orbit, hence the points of Σ_0 can naturally be identified with the elements of $\mathbb{Z}/(a+b)$ up to a translation). As λ increases towards λ_0, two complex conjugate points of the fiber converge towards each other, and become real points for $\lambda > \lambda_0$. By considering the situation for $\lambda \to +\infty$, where the solutions of $x^a(\lambda - x)^b = 1$ split into two groups, one consisting of a roots near the origin, and the other consisting of b roots near λ, one easily checks that the vanishing cycle L_0 consists of the two points of Σ_0 with first coordinate $x = \exp(\pm\frac{i\pi\, b}{a+b})$.

Hence, for a suitable identification of the fiber Σ_0 with $\mathbb{Z}/(a+b)$, the vanishing cycle associated to the arc $\gamma_0 = [0, \lambda_0]$ is $L_0 = \{0, b\}$. It follows immediately that $L_j = \zeta^{-j} \cdot L_0 = \{-j, b-j\}$ for all $j = 0, 1, \dots, a+b-1$.

Given $0 \le i < j < a+b$, the vanishing cycles L_i and L_j intersect if and only if the subsets $\{-i, b-i\}$ and $\{-j, b-j\}$ of $\mathbb{Z}/(a+b)$ have non-empty intersection, i.e. if $j = i+a$ or $j = i+b$. Therefore, we have:

Lemma 3.5 *The direct sum $\bigoplus_{i<j} CF^*(L_i, L_j)$ is a free module of total rank $(a+b)$ over the coefficient ring, generated by the intersection points*

$$x_i \in CF^*(L_i, L_{i+a}) \quad (0 \le i < b) \qquad \text{and} \qquad y_i \in CF^*(L_i, L_{i+b}) \quad (0 \le i < a).$$

Because Σ_0 is a discrete set, all pseudo-holomorphic curves in Σ_0 must be constant maps. However, each point of Σ_0 occurs exactly once as an intersection between two vanishing cycles (there are no triple intersections), which implies that the Floer differentials and products are trivial. Therefore, we have:

Lemma 3.6 *The differentials and products $m_k,\ k \ge 1$ in the A_∞-category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ are all identically zero, with the exception of the obvious ones $m_2(\cdot, id)$ and $m_2(id, \cdot)$.*

This of course greatly simplifies the argument, eliminating the need for many of the arguments required in the case of higher-dimensional weighted projective spaces. At this point, our only remaining task is to determine the Maslov indices of the various intersection points, by choosing graded Lagrangian lifts of the vanishing cycles. A word of warning is in order here: because we are actually dealing with graded Lagrangian submanifolds in a Calabi-Yau 0-fold, the argument is very specific (see §2 of [215] for a discussion of graded Lagrangian submanifolds of 0-dimensional symplectic manifolds) and does not give a good intuition of the higher-dimensional case.

Lemma 3.7 *There exists a natural choice of gradings for which $\deg(x_i) = \deg(y_i) = 1$.*

Proof. View the curve $X = \{x^a y^b = 1\} \subset (\mathbb{C}^*)^2$ as a complex manifold. The holomorphic volume form $d\log x \wedge d\log y$ on $(\mathbb{C}^*)^2$ induces a $(1,0)$-form Ω on X, characterized by the property that it is the restriction to X of a 1-form (which we also call Ω) such that $\Omega \wedge d(x^a y^b) = d\log x \wedge d\log y$, i.e., using the fact that $x^a y^b = 1$ along X,

$$\Omega \wedge \Big(\frac{a}{x}\, dx + \frac{b}{y}\, dy\Big) = \frac{dx \wedge dy}{xy}.$$

Outside of the branch points of W, the 1-form Ω can be expressed as $\Theta\, dW$, for some meromorphic function Θ with simple poles at the branch points. The above equation becomes $\Theta(\frac{b}{y}-\frac{a}{x})=\frac{1}{xy}$, i.e. $\Theta=(bx-ay)^{-1}=((a+b)x-aW)^{-1}$. In particular, near $\Sigma_0=W^{-1}(0)$, we have $\arg\Theta=-\arg x$.

The complex-valued function Θ is (up to scaling by a positive real factor) the natural holomorphic volume form induced by Ω on the 0-dimensional manifold $\Sigma_0 = W^{-1}(0)$. Let $L_0=\{p_-,p_+\}$, where the x-coordinate of $p_\pm$ is $x_\pm=\exp(\pm\frac{i\pi b}{a+b})$. The *phase* of L_0 is the function $\phi_{L_0}: L_0\to\mathbb{R}/\pi\mathbb{Z}$ defined by

$$\phi_{L_0}(p_\pm)=\arg\Theta(p_\pm)=\mp\frac{\pi b}{a+b}.$$

Note that an orientation on L_0 determines a lift of ϕ_{L_0} to a $\mathbb{R}/2\pi\mathbb{Z}$-valued function; in order to define the Maslov index, we need to view L_0 as a *graded Lagrangian* submanifold, i.e. to choose a real lift $\tilde\phi_{L_0}: L_0\to\mathbb{R}$ of the phase function. Although there is *a priori* a $\mathbb{Z}^2$-space of such choices, one has to restrict oneself to only those lifts which are compatible with a graded Lagrangian lift of the Lefschetz thimble D_0 (which reduces the space of choices to $\mathbb{Z}$, as expected since vanishing cycles are only defined up to shifts). If we orient D_0 from p_- towards p_+, then the phase of D_0 (the function $\phi_{D_0}: D_0\to\mathbb{R}/2\pi\mathbb{Z}$ defined by $\phi_{D_0}(p)=\arg\Omega(v)$ for any $p\in D_0$ and $v\in T_pD_0-\{0\}$ compatible with the orientation) has the property that

$$\phi_{D_0}(p_-)=\frac{\pi b}{a+b}\ \text{ and }\ \phi_{D_0}(p_+)=\frac{\pi a}{a+b}.$$

Moreover, it is easy to check that $\phi_{D_0}(p)\in(0,\pi)$ for all $p\in D_0$ (because $\Omega=\frac{1}{b}d\log x$, and $\arg x$ is monotonically increasing along D_0). Hence, there exists a graded Lagrangian lift of D_0 for which the phase function takes values in $(0,\pi)$, which means that we can choose a graded lift of L_0 by setting

$$\tilde\phi_{L_0}(p_-)=\frac{\pi b}{a+b}\ \text{ and }\ \tilde\phi_{L_0}(p_+)=\frac{\pi a}{a+b}.$$

Arguing similarly for the other vanishing cycles (or using the $\mathbb{Z}/(a+b)$-equivariance), we can choose graded lifts of $L_j=\{p_{j,-},p_{j,+}\}$ (where $\arg x_{j,\pm}=\frac{1}{a+b}(\pm\pi b-2\pi j)$) by setting

$$\tilde\phi_{L_j}(p_{j,-})=\frac{\pi(b+2j)}{a+b}\ \text{ and }\ \tilde\phi_{L_j}(p_{j,+})=\frac{\pi(a+2j)}{a+b}.$$

Now, the degree of the morphism x_j, corresponding to $p_{j,+}=p_{j+a,-}\in L_j\cap L_{j+a}$, is given by the difference of phases:

$$\deg x_j=\frac{1}{\pi}(\tilde\phi_{L_{j+a}}(p_{j+a,-})-\tilde\phi_{L_j}(p_{j,+}))=\frac{b+2(j+a)}{a+b}-\frac{a+2j}{a+b}=1.$$

Similarly for y_j:

$$\deg y_j = \frac{1}{\pi}(\tilde{\phi}_{L_{j+b}}(p_{j+b,+}) - \tilde{\phi}_{L_j}(p_{j,-})) = \frac{a+2(j+b)}{a+b} - \frac{b+2j}{a+b} = 1.$$

□

Theorem 3.4 now follows immediately from Lemmas 3.5–3.7; as in the case of weighted projective planes, the only difference between the DG-categories $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ and $\mathfrak{C}_\theta$ is that the objects of $\mathfrak{C}_\theta$ are numbered “backwards”, so the equivalence of categories takes the objects $L_0, \dots, L_{a+b-1}$ of $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ to the objects $w_{a+b-1}, \dots, w_0$ of $\mathfrak{C}_\theta$.

4 Mirrors of weighted projective planes

4.1 The mirror Landau-Ginzburg model and its fiber Σ_0 The mirror to the weighted projective plane $\mathbb{CP}^2(a,b,c)$ is the affine hypersurface $X = \{x^a y^b z^c = 1\} \subset (\mathbb{C}^*)^3$, equipped with the superpotential $W = x+y+z$, and a symplectic form ω that we leave unspecified for the moment. During most of the argument, we will assume ω to be anti-invariant under complex conjugation (which implies exactness) and invariant under the diagonal action of the cyclic group $\mathbb{Z}/(a+b+c)$, but these assumptions will be weakened at the end. Of course, since X is non-compact, we also need to choose ω in such a way as to ensure that the Lefschetz thimbles and vanishing cycles considered below are well-defined. It is easy to check that, among many other possibilities, a symplectic form such as

$$\omega = i \sum_{i,j=1}^{3} a_{ij} \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_j}{\bar{z}_j}$$

(where (a_{ij}) is a positive definite Hermitian matrix, with real coefficients if we require complex conjugation anti-invariance) generates a horizontal distribution for which parallel transport is well-defined, because, with respect to the induced Kähler metric, X is complete and the gradient vector of W has norm bounded from below outside of a compact set.

Topologically, X is just a complex torus $(\mathbb{C}^*)^2$, at least if $\delta = gcd(a,b,c) = 1$; otherwise X is disconnected, and each of its δ components is a complex torus.

For each $\lambda \in \mathbb{C}$, the fiber $\Sigma_\lambda = W^{-1}(\lambda) \subset X$ is an affine curve given by the equation $x^a y^b (\lambda - x - y)^c = 1$; this curve is smooth unless λ is one of the $a+b+c$ critical values of W. We will view Σ_λ as a branched covering of $\mathbb{C}^*$, by projecting to the x axis (this choice is arbitrary, and we will occasionally use the symmetry between the variables x, y, z in the argument). For a generic value of $x \in \mathbb{C}^*$, the polynomial $x^a y^b (\lambda - x - y)^c - 1$ of degree $b+c$ in the variable y admits $b+c$ distinct simple roots; therefore, the projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$ is a

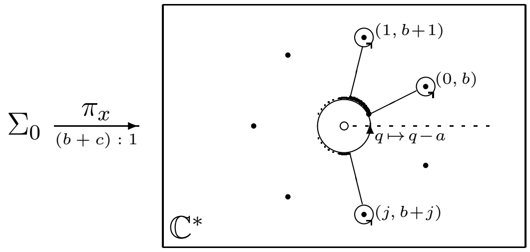


Figure 2: The projection $\pi_x : \Sigma_0 \to \mathbb{C}^*$ (of degree $b+c$, with $a+b+c$ branch points)

$(b+c)$-fold covering. The branch points of π_x are those values of x for which there is a double root, i.e. a value of y such that $P(y) = x^a y^b (\lambda - x - y)^c = 1$ and $P'(y) = 0$. Since

$$\frac{P'(y)}{P(y)} = \frac{b}{y} - \frac{c}{\lambda - x - y},$$

the condition $P'(y) = 0$ implies that $cy = b(\lambda - x - y)$, i.e. $y = \frac{b}{b+c}(\lambda - x)$. Substituting into the equation of Σ_λ, we obtain the equation

$$x^a (\lambda - x)^{b+c} = \frac{(b+c)^{b+c}}{b^b\, c^c} \tag{4.1}$$

for the branch points of π_x. Since this is a polynomial equation of degree $a+b+c$, for a generic value of λ there are $a+b+c$ distinct branch points, all of which are simple (i.e. isolated non-degenerate critical points of π_x).

In the remainder of this section, we set $\lambda = 0$, and describe the curve Σ_0 in detail, by computing the monodromy of the $(b+c)$-fold branched covering $\pi_x : \Sigma_0 \to \mathbb{C}^*$ around the origin and around its $a+b+c$ branch points.

Lemma 4.1 *The fiber of $\pi_x : \Sigma_0 \to \mathbb{C}^*$ can be identified with $\mathbb{Z}/(b+c)$ in such a way that the monodromy of π_x around the origin in $\mathbb{C}^*$ is given by $q \mapsto q - a$, and the monodromies around the $a+b+c$ branch points are given by the transpositions $(j, j+b)$, $0 \le j < a+b+c$ (see Figure 2).*

To understand this statement, first observe that, when $x = \epsilon e^{i\theta}$ is close to 0, the $b+c$ roots of the equation

$$x^a y^b (-x-y)^c = 1 \tag{4.2}$$

lie close to those of the equation

$$(-1)^c y^{b+c} = \epsilon^{-a} e^{-ia\theta}.$$

Hence, we can choose an identification of the fiber of π_x above a small real positive value $x = \epsilon$ (or any other $\epsilon e^{i\theta}$ fixed in advance) with the cyclic group $\mathbb{Z}/(b+c)$ in a manner compatible with the cyclic ordering of the points. Moreover, varying θ from 0 to 2π, we obtain that the monodromy of π_x around the origin is given by the translation $q \mapsto q - a$ in $\mathbb{Z}/(b+c)$ (i.e., the permutation sending the root y_q of $x^a y^b(-x-y)^c = 1$ to y_{q-a}).

Next, consider a critical value of π_x, i.e. a root x_0 of (4.1) for $\lambda = 0$, and the radial half-line ℓ through x_0, i.e. the set of all $x \in \mathbb{C}^*$ with argument equal to $\theta_0 = \arg x_0$. Moving x along ℓ starting from a point $x_* = \epsilon e^{i\theta_0}$ close to the origin, two of the $b+c$ roots of (4.2) become equal to each other as x approaches x_0; this determines the monodromy of π_x around x_0, namely a transposition in the symmetric group S_{b+c} acting on a fiber of π_x. We claim that, identifying the fiber $\pi_x^{-1}(x_*)$ with $\mathbb{Z}/(b+c)$ as above, this transposition exchanges two elements q_0 and $q_0 + b$. This can be seen as follows.

Assume for simplicity that $b + c$ is even and that x_0 is the positive real root of (4.1) for $\lambda = 0$; the general case is handled similarly, inserting factors $e^{i\theta_0}$ where needed. For $x \to 0$, as explained above, the $b + c$ roots of (4.2) are close to those of

$$y^{b+c} = (-1)^c x^{-a},$$

i.e. $b+c$ evenly spaced points on a circle (Figure 3, left). As x increases, two complex conjugate roots $y, \bar{y}$ approach the real axis and eventually become equal for $x = x_0$ (Figure 3, center), so that there are two additional real roots for $x > x_0$. As $x \to +\infty$, the roots of (4.2) are divided into two groups, b roots close to the origin, approximated by those of

$$y^b = (-1)^c x^{-(a+c)},$$

and c roots close to $-x$, corresponding to values of $z = -x - y$ close to the origin and approximated by the roots of

$$z^c = (-1)^b x^{-(a+b)}$$

(Figure 3, right). Hence, identifying the fiber of π_x for x small with $\mathbb{Z}/(b + c)$ in a manner compatible with the cyclic ordering, the two points which merge for $x = x_0$ (the vanishing cycle of π_x at x_0) differ from each other by exactly b (this can also be checked by numerical experimentation).

The above argument gives us that the monodromy around one of the branch points x_0 of π_x, e.g. the branch point located on the positive real axis or immediately above it, is a transposition

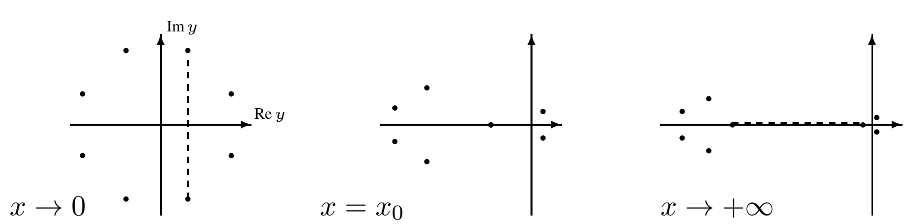


Figure 3: The roots of $x^a y^b(-x-y)^c = 1$ for $x \in \mathbb{R}_+$ ($(a, b, c) = (1, 3, 5)$)

$(q_0, q_0 + b)$; changing the identification between the reference fiber of π_x above x_* and the cyclic group $\mathbb{Z}/(b+c)$ if necessary, we can assume that $q_0 = 0$.

We now find the monodromy around the other branch points of π_x. For this purpose, observe that the group $G = \mathbb{Z}/(a+b+c)$ acts on X by $(x,y,z) \mapsto (x\zeta^j, y\zeta^j, z\zeta^j)$, where $\zeta = \exp(\frac{2\pi i}{a+b+c})$, and that this action preserves Σ_0, mapping the fiber of π_x above x to the fiber above $x\zeta^j$. Hence, denoting by y', y'' the two points of the fiber above $x_* = \epsilon e^{i\theta_0}$ which converge to each other as x moves radially outwards to x_0 (those labelled 0 and b), we know that the two points of the fiber above $x_*\zeta^j$ which converge to each other as x moves radially outwards to $x_0\zeta^j$ are $y'\zeta^j$ and $y''\zeta^j$. We now transport these two values of y from the fiber $\pi_x^{-1}(x_*\zeta^j)$ to $\pi_x^{-1}(x_*)$ along the arc $x(t) = x_* e^{2\pi i t}$ for $t \in [0, \frac{j}{a+b+c}]$. Approximating the $b+c$ points of $\pi_x^{-1}(\epsilon e^{i\theta})$ by the roots of $(-1)^c y^{b+c} = \epsilon^{-a} e^{-ia\theta}$, the parallel transport along the considered arc induces a multiplication by $\exp(2\pi i \frac{a}{b+c} \frac{j}{a+b+c})$. Observing that

$$\zeta^j \exp(2\pi i \tfrac{j\,a}{(b+c)(a+b+c)}) = \exp(2\pi i \tfrac{j}{b+c}),$$

we obtain that the two points of $\pi_x^{-1}(x_*)$ which become equal as x is moved first counterclockwise around the origin and then radially outwards to $x_0\zeta^j$ are those which correspond to the elements j and $b+j$ of $\mathbb{Z}/(b+c)$. Hence, the monodromy of π_x around $x_0\zeta^j$ (joining x_* to $x_0\zeta^j$ in the prescribed way) is the transposition $(j, b+j)$, which completes the proof of Lemma 4.1. By the way, note that the comparison between the values $j = 0$ and $j = a+b+c$ is consistent with our determination of the monodromy around $x = 0$.

4.2 The vanishing cycles Now that the fiber Σ_0 is well-understood, we compute the vanishing cycles of the Lefschetz fibration $W : X \to \mathbb{C}$ by studying the degeneration of Σ_λ as λ approaches a critical value of W.

The curve Σ_λ becomes singular when two branch points of the projection $\pi_x : \Sigma_\lambda \to \mathbb{C}^*$ merge with each other, giving rise to a nodal point. This occurs whenever (4.1) admits a double root. Considering the logarithmic derivative of the left-hand side, we obtain the relation $\frac{a}{x} - \frac{b+c}{\lambda - x} = 0$, which leads to $x = \frac{a}{a+b+c}\lambda$ for a double root of (4.1), and substituting we obtain the equation

$$\lambda^{a+b+c} = \frac{(a+b+c)^{a+b+c}}{a^a\, b^b\, c^c} \tag{4.3}$$

for the $a+b+c$ critical values of W (this equation can also be obtained directly).

For symmetry and for simplicity, we will choose the smooth curve $\Sigma_0 = W^{-1}(0)$ as our reference fiber of the Lefschetz fibration $W : X \to \mathbb{C}$, and we will choose straight line segments for the arcs γ_j joining the origin to the various critical values $\lambda_j = \lambda_0 \zeta^{-j}$ of W $(0 \le j < a+b+c)$, where λ_0 is the real positive root of (4.3) and $\zeta = \exp(\frac{2\pi i}{a+b+c})$. Hence, in order to construct the category of Lagrangian vanishing cycles of W, we need to understand how the smooth fiber Σ_0 above the reference point 0 degenerates to the nodal curve Σ_{λ_j} when λ moves radially from 0 to λ_j.

We first consider the motion of the branch points of π_x as λ increases along the positive real axis from 0 to the critical value λ_0. For each value of λ, the $a+b+c$ branch points are given by the roots of (4.1). When $\lambda = 0$, they all lie on a circle centered at the origin, as represented in Figure 2. As $\lambda \to \lambda_0$, two complex conjugate branch points converge to each other, so that for $\lambda = \lambda_0$ the equation (4.1) has a double root $x = \frac{a}{a+b+c}\lambda_0$ on the positive real axis (Figure 4, center). Finally, for $\lambda \to +\infty$, the roots of (4.1) split into two groups, one of a points close to the origin that can be approximated by the roots of $x^a = K_{b,c}\lambda^{-(b+c)}$ (where $K_{b,c} = b^{-b}c^{-c}(b+c)^{b+c}$), and one of $b+c$ points close to λ for which $\xi = \lambda - x$ can be approximated by the roots of $\xi^{b+c} = K_{b,c}\lambda^{-a}$ (Figure 4, right). Hence, it can be checked that the two branch points of $\pi_x : \Sigma_0 \to \mathbb{C}^*$ which merge for $\lambda \to \lambda_0$ are those with argument $\arg x = \pm\frac{b+c}{a+b+c}\pi$, and that the projection to $\mathbb{C}^*$ of the corresponding vanishing cycle is an arc δ_0 which is symmetric with respect to the real axis, intersects it only once in its positive part, and remains everywhere inside the circle containing the critical values of π_x (Figure 4, left).

More precisely, the above discussion gives us a *topological* description of the vanishing cycle $L_0 \subset \Sigma_0$, up to homotopy. Namely, two of the $b+c$ lifts to Σ_0 of the arc $\delta_0 \subset \mathbb{C}^*$ have common end points (the ramification points of π_x lying above the end points of δ_0), and their union forms a closed loop L_0' in Σ_0. This loop is a topological vanishing cycle, i.e. it shrinks to a point in Σ_λ when $\lambda \to \lambda_0$, but a priori it is only homotopic to the symplectic vanishing cycle L_0 (obtained by parallel transport using the symplectic connection).

The actual position of the vanishing cycle L_0 inside Σ_0 depends on the choice of the symplectic form ω on X; for a given ω it can be calculated numerically (and it can be checked that for "reasonable" choices of ω, L_0 and L_0' intersect all other vanishing cycles in the same manner). However, this calculation is unnecessary for our purposes. Indeed, if we endow X with a symplectic form that is anti-invariant by complex conjugation, then the vanishing cycle L_0 is invariant by complex conjugation, i.e. complex conjugation maps L_0 to itself in an orientation-preserving manner, and the same is true of L_0'. Since L_0 and L_0' are homotopic to each other in Σ_0, their (oriented) invariance under complex conjugation is sufficient to imply that they are *Hamiltonian isotopic*, which means that for the purpose of determining categories of vanishing cycles, L_0 and L_0' are interchangeable.

If we deform ω to a non-exact form, complex conjugation invariance is lost. The intersection patterns between vanishing cycles remain the same for small deformations (and can be forced to remain the same even for large deformations by performing suitable Hamiltonian

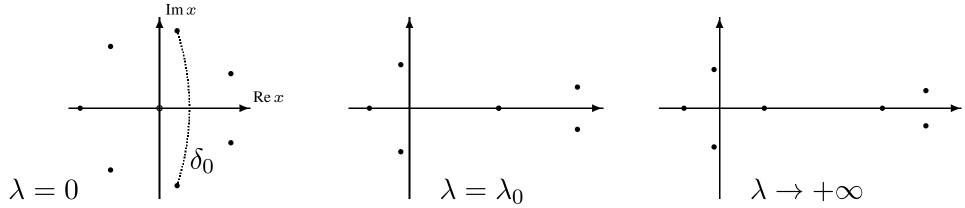
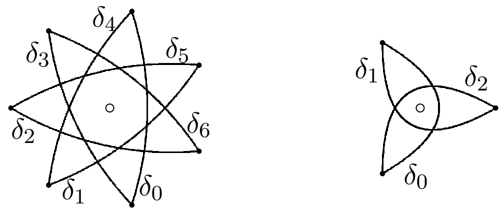


Figure 4: The branch points of π_x for $\lambda \in \mathbb{R}_+$ $((a,b,c) = (4,2,1))$



$(a,b,c) = (4,2,1)$ $(a,b,c) = (1,1,1)$

Figure 5: The vanishing cycles $L_j \subset \Sigma_0$

isotopies), but the calculation of the coefficient assigned to a given pseudo-holomorphic curve involves its symplectic area and hence requires one to work with the actual vanishing cycles rather than their topological approximations. Hence, we may obtain non-trivial deformations of the category of vanishing cycles; however, these deformations only amount to modifications of the structure constants of the products m_k, rather than changes in the Floer complexes themselves or in the types of pseudo-holomorphic curves that may arise.

In any case, except at the very end of the argument, we will always be considering symplectic forms that are anti-invariant under complex conjugation, in which case the approximation of L_0 by L'_0 is legitimate.

We now consider the other vanishing cycles L_j of the Lefschetz fibration W. Recall that the group $G = \mathbb{Z}/(a+b+c)$ acts on X, in a manner that preserves Σ_0; moreover, $W : X \to \mathbb{C}$ is G-equivariant. If we assume the symplectic form ω to be G-invariant, the symplectic connection and the associated parallel transport will also be G-equivariant. Therefore, since the arc $\gamma_j \subset \mathbb{C}$ joining the origin to $\lambda_j = \lambda_0\zeta^{-j}$ is the image of γ_0 by the action of ζ^{-j} (where $\zeta = \exp(\frac{2\pi i}{a+b+c})$), the same is true of the corresponding Lefschetz thimbles, and hence of the vanishing cycles in Σ_0. This gives us a description of L_j for all values of j. As in the case of L_0, we will consider, rather than L_j itself, a loop $L'_j \subset \Sigma_0$ which is homotopic to L_j and can be obtained as a double lift via $\pi_x : \Sigma_0 \to \mathbb{C}^*$ of an embedded arc $\delta_j \subset \mathbb{C}^*$. The loop L'_j is defined to be the image of L'_0 by the action of ζ'_j, which means that δ_j is the image of δ_0 by a rotation of angle $-\frac{2\pi j}{a+b+c}$. If, in addition to its G-invariance, ω is assumed to be anti-invariant under complex conjugation, then L'_j is Hamiltonian isotopic to L_j, so we can work with L'_j instead of L_j.

Hence, to summarize the above discussion, we have the following lemma:

Lemma 4.2 *The vanishing cycles* $L_j \subset \Sigma_0$ $(0 \le j < a+b+c)$ *are homotopic (and, if* ω *is invariant under the action of* $\mathbb{Z}/(a+b+c)$ *and anti-invariant under complex conjugation, Hamiltonian isotopic) to closed loops* $L'_j \subset \Sigma_0$ *which project by* π_x *to arcs* $\delta_j \subset \mathbb{C}^*$ *as represented in Figure* 5 *(the end points of* δ_j *are the branch points of* π_x *for which* $\arg x = -2\pi\frac{j}{a+b+c} \pm \pi\frac{b+c}{a+b+c}$*).*

In the following sections, we assume that ω is $\mathbb{Z}/(a+b+c)$-invariant and anti-invariant under complex conjugation, and we implicitly identify L_j with L'_j.

4.3 The Floer complexes The objects of the category $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_j\})$ are described by Lemma 4.2; we now determine its morphisms by studying the intersections between the closed loops $L_j \subset \Sigma_0$. This simply involves looking carefully at Figures 2 and 5 in order to determine, among the intersections between δ_i and δ_j, which ones lift to intersections between L_i and L_j.

Lemma 4.3 *The direct sum $\bigoplus_{i<j} CF^*(L_i,L_j)$ is a free module of total rank $3(a+b+c)$ over the coefficient ring, generated by the following intersection points:*

$$\begin{array}{llll} x_i \in CF^*(L_i,L_{i+a}) & (0\le i<b+c), & \bar{x}_i \in CF^*(L_i,L_{i+b+c}) & (0\le i<a), \\ y_i \in CF^*(L_i,L_{i+b}) & (0\le i<a+c), & \bar{y}_i \in CF^*(L_i,L_{i+a+c}) & (0\le i<b), \\ z_i \in CF^*(L_i,L_{i+c}) & (0\le i<a+b), & \bar{z}_i \in CF^*(L_i,L_{i+a+b}) & (0\le i<c). \end{array}$$

Moreover, the Floer differential is trivial, i.e. $m_1=0$.

To determine $CF^*(L_i,L_j)$ for given $0\le i<j<a+b+c$, one must look for intersection points between the projected arcs δ_i and δ_j. The arcs δ_i and δ_j intersect only if $j-i\le b+c$ or $j-i\ge a$; in all other cases, $\delta_i\cap\delta_j=\emptyset$ and hence $CF^*(L_i,L_j)=0$. More precisely, $\delta_i\cap\delta_j$ contains one point if $j-i\le b+c$, and one point if $j-i\ge a$; if both conditions hold simultaneously, then $|\delta_i\cap\delta_j|=2$ (see Lemma 4.2 and Figure 5). Moreover, if equality holds ($j-i=b+c$ or $j-i=a$), then the corresponding intersection occurs at an end point of δ_i and δ_j, i.e. a branch point of π_x. In this case, the intersection of δ_i and δ_j always lifts to a transverse intersection of L_i and L_j, at the corresponding critical point of π_x; this accounts for the generators x_i and $\bar{x}_i$ mentioned in the statement of Lemma 4.3.

When $j-i<b+c$ or $j-i>a$, we need to consider the structure of the branched covering π_x in order to determine whether intersections between δ_i and δ_j lift to intersections between L_i and L_j. Call p_i the branch point of π_x with argument $\arg x=-2\pi\frac{j}{a+b+c}-\pi\frac{b+c}{a+b+c}$, which is an end point of δ_i, and define similarly p_j. When $j-i<b+c$, consider the corresponding intersection point $q\in\delta_i\cap\delta_j$, and use the arcs joining p_j to q in δ_j and q to p_i in δ_i to define an arc $\eta\subset\mathbb{C}^*$ joining p_j to p_i, with a rotation angle of $2\pi\frac{j-i}{a+b+c}$ around the origin. It follows from Lemma 4.1 (cf. also Figure 2) that, over a neighborhood of η, we can consistently label the sheets of the covering π_x by elements of $\mathbb{Z}/(b+c)$, in such a way that the monodromies around the branch points p_i and p_j are transpositions of the form (k_i,k_i+b) and (k_j,k_j+b), with $k_i-k_j=j-i$. Hence, near the point q, the vanishing cycle L_i lies in the two sheets of π_x labelled k_i and k_i+b, and similarly for L_j; the intersections of L_i with L_j above q correspond to the elements of $\{k_i,k_i+b\}\cap\{k_j,k_j+b\}$. Since $0<k_i-k_j=j-i<b+c$, this intersection is empty unless $k_i=k_j+b \bmod b+c$, i.e. $j-i=b$, which corresponds to the generator y_i of the Floer complex, or $k_j=k_i+b \bmod b+c$, i.e. $j-i=c$, which corresponds to the generator z_i. When $j-i>a$, one proceeds similarly, introducing an arc in $\mathbb{C}^*$ joining p_j to p_i through the relevant intersection point q' of δ_i with δ_j, with a rotation angle of $2\pi(\frac{j-i}{a+b+c}-1)$ around the origin. The sheets of π_x containing L_i and L_j above the intersection point q' are now labelled $k_i',k_i'+b$ and $k_j',k_j'+b$, with k_i' and k_j' two constants in $\mathbb{Z}/(b+c)$ such that $k_i'-k_j'=j-i-(a+b+c)=j-i-a \bmod b+c$. Therefore, the two cases where L_i and L_j

intersect above q' are when $i + j = a + b$, which corresponds to the generator z_i' of the Floer complex, and when $i + j = a + c$, which corresponds to y_i'.

At this point it is worth observing that, for generic values of (a, b, c), each Floer complex $CF^*(L_i, L_j)$ has total rank at most one, so that the Floer differential is necessarily zero. However, for specific values of (a, b, c) we may have numerical coincidences leading to more than one intersection between two vanishing cycles; the most striking example is that of the usual projective plane, $(a, b, c) = (1, 1, 1)$, for which $|L_i \cap L_j| = 3\ \forall i < j$ (cf. Figure 5). Nonetheless, even in these cases, the Floer differential vanishes, because L_i and L_j always realize the minimal geometric intersection number between closed loops in their homotopy classes, as can be checked by enumerating the various posible cases. This minimality of intersection implies that Σ_0 contains no non-constant immersed disc with boundary in $L_i \cup L_j$, and hence that the Floer differential vanishes.

Another way to prove the vanishing of the Floer differential is to endow Σ_0 and $\mathbb{C}^*$ with almost-complex structures which make the projection π_x holomorphic, and to observe that the projection to $\mathbb{C}^*$ of a pseudo-holomorphic disc in Σ_0 with boundary in $L_i \cup L_j$ is a pseudo-holomorphic disc in $\mathbb{C}^*$ with boundary in $\delta_i \cup \delta_j$. If $|\delta_i \cap \delta_j| = 1$, the maximum principle implies that the projected pseudo-holomorphic disc is a constant map, and hence that the disc in Σ_0 is contained in a fiber of π_x, which implies that it is also constant. If $|\delta_i \cap \delta_j| = 2$, one reaches the same conclusion by observing the respective positions of the two intersection points in $\mathbb{C}^*$ (a non-constant disc would have to pass through the origin). As before, one concludes that the absence of non-trivial pseudo-holomorphic discs makes the Floer differential identically zero, which completes the proof of Lemma 4.3.

4.4 The product structures The aim of this section is to prove the following results concerning the category $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_j\})$:

Lemma 4.4 *The higher products m_k $(k \geq 3)$ are all identically zero.*

Lemma 4.5 *There exist non-zero constants $\alpha_{uv,i}$ such that*

$$\begin{aligned} m_2(x_i, y_{i+a}) &= \alpha_{xy,i}\,\bar{z}_i, & m_2(x_i, z_{i+a}) &= \alpha_{xz,i}\,\bar{y}_i,\\ m_2(y_i, z_{i+b}) &= \alpha_{yz,i}\,\bar{x}_i, & m_2(y_i, x_{i+b}) &= \alpha_{yx,i}\,\bar{z}_i,\\ m_2(z_i, x_{i+c}) &= \alpha_{zx,i}\,\bar{y}_i, & m_2(z_i, y_{i+c}) &= \alpha_{zy,i}\,\bar{x}_i. \end{aligned}$$

All other compositions (except those involving identity morphisms) vanish.

These results follow from a careful observation of the boundary structure of a pseudo-holomorphic disc in Σ_0 with boundary in $\bigcup L_j$. Endow Σ_0 with any almost-complex structure, and let $u : D^2 \to \Sigma_0$ be a pseudo-holomorphic map from the disc with $k + 1 \geq 3$ marked points on its boundary to Σ_0, mapping each segment on the boundary to an arc in one of the Lagrangian submanifolds L_j. Each "corner" of the image of u corresponds to an intersection point between two of the vanishing cycles, and as such it corresponds to a generator of the Floer complex.

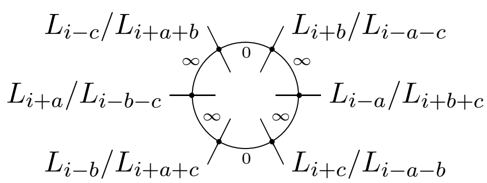


Figure 6: The intersections of L_i with the other vanishing cycles

According to Lemma 4.3, we can classify the generators of the Floer complex into three families, those of *type* x (corresponding to generators $x_i, \bar{x}_i$), those of *type* y (generators $y_i, \bar{y}_i$), and those of *type* z (generators $z_i, \bar{z}_i$). Moreover, observe that the total intersection of each L_i with all other vanishing cycles consists of 6 points, two of each type: depending on the value of i, L_i is either the source of the morphism x_i or the target of $\bar{x}_{i-b-c}$, and it is either the source of $\bar{x}_i$ or the target of x_{i-a}; similarly for types y and z.

The manner in which these points are arranged along the loop L_i can be seen easily by looking at Figure 5 and recalling the discussion in the previous section. Recall that L_i passes through two branch points of π_x, which split it into two halves (lifts of δ_i lying in different sheets of π_x). One of these branch points corresponds to x_i or $\bar{x}_{i-b-c}$, while the other corresponds to $\bar{x}_i$ or x_{i-a}. In between them, we have, on one half of L_i, one intersection of type y (either y_i or $\bar{y}_{i-a-c}$) and one of type z (either $\bar{z}_i$ or z_{i-c}); on the other half of L_i, we have similarly one intersection of type y (either $\bar{y}_i$ or y_{i-b}) and one of type z (either z_i or $\bar{z}_{i-a-b}$). This structure is summarized in Figure 6.

An important property is that, for every one of the six portions of L_i delimited by these intersection points, one of the two immediately adjacent components of $\Sigma_0 - \bigcup L_j$ (on either side of L_i) is unbounded (it is denoted by 0 or ∞ on Figure 6 depending on whether its image under π_x contains the origin or the point at infinity in $\mathbb{C}^*$). These unbounded components form an alternating pattern around L_i, changing side (left or right) every time one of the intersection points is crossed.

On the other hand, the image of the pseudo-holomorphic map u cannot intersect any of the unbounded components of $\Sigma_0 - \bigcup L_j$, because otherwise the maximum principle would imply that the image of u is unbounded. This imposes very strong constraints on the behavior of u along the boundary of the disc. Namely, consider two consecutive marked points ("corners"), such that the portion of boundary ("edge") in between them is mapped to an arc η (oriented according to the boundary orientation of the unit disc) contained in the vanishing cycle L_i. Then, η is exactly one of the six portions of L_i delimited by its intersections with the other vanishing cycles, and its orientation is determined by the requirement that the component of $\Sigma_0 - \bigcup L_j$ immediately to the left of η be bounded (see Figure 6). Moreover, the local behavior of u at an end point p of η is "convex", i.e. u locally maps into only one of the four regions delimited locally by the two vanishing cycles meeting at p. In other words, the boundary of $\mathrm{Im}(u)$ is an oriented piecewise smooth curve $\theta \subset \bigcup L_j$ which always turns left at every intersection point it encounters. This boundary behavior has several important consequences.

Lemma 4.6 *Among any three consecutive corners of the image of u, there is always exactly one of each type x, y, z.*

Proof. Observe that two consecutive corners of the image of u are necessarily of different types (because two adjacent intersections of L_i with other vanishing cycles are always of different types). Let p, q, r be three consecutive corners of the image of u, such that the edge from p to q lies in a vanishing cycle L_i and the edge from q to r lies in a vanishing cycle L_j. The knowledge of the types of the points p and q completely determines them, which in turn determines the type of r. For example, if p is of type y and q is of type z, then on the diagram of Figure 6 the edge joining them is the lowermost portion of L_i; in particular the edge from p to q is adjacent to an unbounded component of Σ_0 whose image under π_x contains the origin. Considering the intersection diagram for L_j (similar to Figure 6), the point q can be located by comparison with the diagram for L_i (in our example, q is the point to the upper left of the diagram). Moreover, the direction from which θ reaches q can be determined by identifying the unbounded component to which it is adjacent (in our example, the component whose image under π_x contains the origin, so θ reaches q from the innermost side of the diagram); since θ turns left at q, this determines the edge from q to r and hence the type of r (in our example, r is the left-most point on the intersection diagram, and hence of type x). It can be checked easily that in all six cases, the type of r is different from those of p and q. □

Next, recall that by definition the successive edges of the image of u lie inside vanishing cycles $L_{i_0}, L_{i_1}, \dots, L_{i_k}$ with $i_0 < i_1 < \cdots < i_k$ (see Definition 3.1), and observe that following θ at a corner of u leads from a vanishing cycle L_i to another vanishing cycle L_j, with $i < j$ if and only if the intersection point is x_i, y_i or z_i, and $i > j$ if and only if the intersection point is $\bar{x}_j$, $\bar{y}_j$ or $\bar{z}_j$ (see Figure 6). Therefore, all corners of u but one correspond to generators of the Floer complexes among $\{x_i, y_i, z_i\}$, while the last corner (between the edge on L_{i_k} and the edge on L_{i_0}) correspond to a generator among $\{\bar{x}_i, \bar{y}_i, \bar{z}_i\}$.

With this observation, Lemma 4.4 follows immediately from Lemma 4.6. Indeed, assume that there exists a pseudo-holomorphic map u from a disc with $k+1$ marked points to Σ_0, with edges lying in vanishing cycles $L_{i_0}, L_{i_1}, \dots, L_{i_k}$ ($0 \le i_0 < i_1 < \cdots < i_k < a+b+c$), contributing to the product m_k, for some $k \ge 3$. Among the first three corners of u, one is among the generators x_i, one is among the y_i, and one is among the z_i. Therefore, $i_3 = i_0 + a + b + c$, which contradicts the inequality $i_3 < a + b + c$. Hence the moduli spaces of pseudo-holomorphic curves involved in the definition of m_k are all empty for $k \ge 3$, which implies that $m_k = 0$.

Lemma 4.5 also follows immediately at this point: in the case of a pseudo-holomorphic map u from a disc with 3 marked points, the three corners p, q, r are all of different types (by Lemma 4.6), and the first two corners p, q correspond to generators among $\{x_i, y_i, z_i\}$ while the last one r corresponds to a generator among $\{\bar{x}_i, \bar{y}_i, \bar{z}_i\}$. Therefore, p and q completely determine r, and moreover it is easy to check from the above discussion and from Figures 5 and 6 that the image of the pseudo-holomorphic map u is also uniquely determined by the pair (p, q). For example, if p is of type x and q is of type y, then necessarily there exists $i < c$ such that $p = x_i$, $q = y_{i+a}$, and $r = \bar{z}_i$; moreover, it is easy to check (see Lemma 4.2 and Figure 5) that the moduli space determining the coefficient of $\bar{z}_i$ in $m_2(x_i, y_{i+a})$ consists of a single curve, regular, whose image $T_{xy,i}$ is the triangular region of Σ_0 delimited by arcs joining p, q, r in the vanishing cycles L_i, L_{i+a}, L_{i+a+b}. Therefore, we have $m_2(x_i, y_{i+a}) = \alpha_{xy,i}\,\bar{z}_i$, where $\alpha_{xy,i} = \pm\exp(-\mathrm{Area}(T_{xy,i}))$. The situation is the same in all other cases.

Remark. The $a + b + c$ triangles $T_{xy,i}$ ($i < c$), $T_{yz,i}$ ($i < a$), $T_{zx,i}$ ($i < b$) are all related to each other via the action of the cyclic group $\mathbb{Z}/(a + b + c)$. Indeed, the diagonal multiplication by a power of $\zeta = \exp(\frac{2\pi i}{a+b+c})$ induces a permutation of the vanishing cycles and of the intersection points, preserving the cyclic ordering of the L_i and the types of their intersection points, and hence mapping every triangle in Σ_0 with boundary in $\bigcup L_i$ to another such triangle. A similar description holds for the triangles $T_{yx,i}$, $T_{zy,i}$, $T_{xz,i}$.

4.5 Maslov index and grading The aim of this section is to define a $\mathbb{Z}$-grading on the Floer complexes $CF^*(L_i, L_j)$, and to compute the degree of the various generators. Using the triviality of the canonical bundles of Σ_0 and X, it is easy to prove (by considering the Lefschetz thimbles) that the Maslov class of L_i is trivial, and hence that it is possible to lift each vanishing cycle to a *graded Lagrangian* submanifold of Σ_0, that we denote again by L_i. This lets us associate a degree to each generator of the Floer complex.

Lemma 4.7 *There exists a natural choice of gradings, for which* $\deg(x_i) = \deg(y_i) = \deg(z_i) = 1$ *and* $\deg(\bar{x}_i) = \deg(\bar{y}_i) = \deg(\bar{z}_i) = 2$.

Assume for simplicity that the symplectic form ω is compatible with the standard complex structure of Σ_0 inherited from that of $(\mathbb{C}^*)^3$, which allows us to define explicitly a holomorphic volume form Ω on Σ_0 (i.e., a non-vanishing holomorphic 1-form). Then, given an oriented Lagrangian submanifold $L \subset \Sigma_0$, the *phase* of L is the function $\phi_L : L \to \mathbb{R}/2\pi\mathbb{Z}$ whose value at every point is the argument of the (non-zero) complex number obtained by evaluating Ω on an oriented volume element in L (in the 1-dimensional case, $\phi_L(x) = \arg\Omega(v)$ for v a tangent vector to L at x defining the orientation of L). The Maslov class is the 1-cocycle representing the obstruction to lift ϕ_L to a real-valued function; if it vanishes, then L can be lifted to a *graded Lagrangian* submanifold, i.e. we can choose a real-valued lift of the phase, $\tilde{\phi}_L : L \to \mathbb{R}$. In the 1-dimensional case, the relationship between Maslov index and phase is very simple: given a transverse intersection point p between two graded Lagrangians $L, L' \subset \Sigma_0$, the Maslov index of $p \in CF^*(L, L')$ is equal to the smallest integer greater than $\frac{1}{\pi}(\phi_{L'}(p) - \phi_L(p))$.

The holomorphic volume form Ω on Σ_0 can be defined from the standard holomorphic volume form $\Omega_0 = d\log x \wedge d\log y \wedge d\log z$ on $(\mathbb{C}^*)^3$ by taking residues first along the hypersurface X of equation $x^a y^b z^c = 1$ and then along the level set $W = 0$. We can characterize Ω as follows: Ω is the restriction to Σ_0 of a 1-form (that we denote again by Ω) such that $\Omega \wedge dW \wedge d(x^a y^b z^c) = \Omega_0$, i.e. (using the fact that $x^a y^b z^c = 1$ along X)

$$\Omega \wedge (dx + dy + dz) \wedge (\frac{a}{x}\,dx + \frac{b}{y}\,dy + \frac{c}{z}\,dz) = \frac{dx \wedge dy \wedge dz}{xyz}.$$

(In fact the 1-form Ω determined in this way may differ from the "usual" one by a real positive factor, irrelevant for our purposes). At this point it is easy to see why the Maslov class of L_i is trivial: indeed, $\Omega \wedge dW$ extends to a non-vanishing $(2,0)$-form on X, whose phase over the Lefschetz thimble D_i admits a real lift; because W maps D_i to an embedded arc, the phase of $\Omega \wedge dW$ over the boundary of D_i and the phase of Ω over L_i differ by a constant term, so that the latter also admits a real lift.

At every point of Σ_0 except for the branch points of π_x, the 1-form Ω can be expressed as $\Theta\, dx$, for some meromorphic function Θ over Σ_0 (with simple poles at the branch points of π_x). The above equation becomes: $\Theta(\frac{c}{z} - \frac{b}{y}) = \frac{1}{xyz}$, which determines Θ. At this point, the most direct method of determination of the phases of the vanishing cycles L_i at their intersection points (and hence of the corresponding Maslov indices) involves computer calculations; however we will attempt to give a sketch of a geometric argument.

If we restrict ourselves to the domain where x is very small, then we have $y \simeq -z$, so that $\Theta \simeq \frac{1}{(b+c)xy}$. Therefore, $\arg\Theta \simeq -\arg x - \arg y$ in this region of Σ_0. Hence, the calculations are simplified if we can deform the vanishing cycles L_i in such a way that the intersection points of a given type (y or z) occur close to the origin in $\mathbb{C}^*$. Of course this process preserves gradings and Maslov indices only if the intersection pattern between the relevant vanishing cycles is not affected by the deformation. We consider a deformation where L_i is replaced by a loop $\tilde{L}_i \subset \Sigma_0$, obtained as a double lift of a piecewise smooth arc $\tilde{\delta}_i \subset \mathbb{C}^*$ joining two branch points of π_x (a deformation of δ_i with fixed end points). The arc $\tilde{\delta}_0$ consists of three line segments, two joining the end points $p, \bar{p} \in \mathrm{crit}(\pi_x)$ to two complex conjugate points $q, \bar{q}$ very close to the origin, and such that $0 < \mathrm{Re}\, q \ll \mathrm{Im}\, q \ll 1$. The other arcs $\tilde{\delta}_i$ are obtained from $\tilde{\delta}_0$ by the action of $\mathbb{Z}/(a+b+c)$ (see Figure 7).

Assuming that $b < a + c$, this deformation can be carried out for intersections of type y without affecting the intersection pattern between L_i and L_{i+b} or L_{i+a+c}, and in such a way that the intersection occurs in the central portion of $\tilde{\delta}_i$ (see Figure 7). The same is true for intersections of type z when $c < a + b$. If we choose $a \geq b \geq c$ then these two assumptions hold, so we can use this method to determine the degrees of $y_i, z_i, \bar{y}_i, \bar{z}_i$.

We start by considering the portion of $\tilde{L}_0$ lying above the central segment in $\tilde{\delta}_0$ (joining q to $\bar{q}$). Recall that, for x small, the $b + c$ sheets of the covering π_x (i.e. the $b + c$ roots of $x^a y^b (-x-y)^c = 1$) can be approximated by the roots of $y^{b+c} = (-1)^c x^{-a}$. Hence, the possible values for the argument of y are $\arg y \simeq -\frac{a}{b+c} \arg x + \pi \frac{c}{b+c} \bmod \frac{2\pi}{b+c}$. It follows from Lemma 4.1 that the two sheets of π_x containing $\tilde{L}_0$ are those where $\arg y \simeq -\frac{a}{b+c} \arg x + \epsilon\pi \frac{c}{b+c}$, for

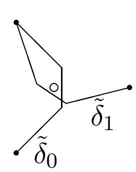


Figure 7: The deformed cycles $\tilde{L}_j$ ($(a, b, c) = (1, 1, 1)$)

$\epsilon = \pm 1$. Hence, we have $\arg\Theta \simeq \frac{a-b-c}{b+c}\arg x - \epsilon\pi\frac{c}{b+c}$. We choose to orient $\tilde{L}_0$ in such a way that its projection goes counterclockwise around the origin in the sheet corresponding to $\epsilon = 1$, and clockwise in the sheet corresponding to $\epsilon = -1$. With this understood, since the projection of oriented tangent vector to $\tilde{L}_0$ is positively proportional to ϵi, we obtain the following formula for the phase of the central portion of $\tilde{L}_0$, modulo 2π:

$$\phi(\tilde{L}_0) \simeq \frac{a-b-c}{b+c}\arg x + \epsilon\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big). \tag{4.4}$$

We choose a lift of $\tilde{L}_0$ (and hence also L_0 via the isotopy between them) as a graded Lagrangian by setting the (real-valued) phase of $\tilde{L}_0$ to be given by (4.4), choosing the determination of $\arg x$ with the smallest absolute value; checking that the choices made in the two portions of $\tilde{L}_0$ corresponding to $\epsilon = \pm 1$ are consistent with each other is a tedious task, best left to a computer program.

The phase of $\tilde{L}_j = \zeta^{-j}\cdot\tilde{L}_0$ is easily deduced from the above calculations for $\tilde{L}_0$. Indeed, the above formula for Θ implies that the value of $\arg\Theta$ at the point $\zeta^{-j}\cdot p$ differs from that at the point p by $4\pi\frac{j}{a+b+c}$. On the other hand, the argument of the x component of the tangent vector to $\tilde{L}_j$ at $\zeta^{-j}\cdot p$ differs from that of the tangent vector to $\tilde{L}_0$ at p by $-2\pi\frac{j}{a+b+c}$. Therefore, (4.4) implies that

$$\phi(\tilde{L}_j) \simeq \frac{a-b-c}{b+c}\Big(\arg x + \frac{2\pi j}{a+b+c}\Big) + \epsilon\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi j}{a+b+c},$$

or equivalently

$$\phi(\tilde{L}_j) \simeq \frac{a-b-c}{b+c}\arg x + \epsilon\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi j\, a}{(a+b+c)(b+c)}. \tag{4.5}$$

This formula can also be obtained directly by observing that the two sheets of π_x containing $\tilde{L}_j$ are those where $\arg y \simeq -\frac{a}{b+c}\arg x - 2\pi\frac{j}{b+c} + \epsilon\pi\frac{c}{b+c}$, for $\epsilon = \pm 1$, by Lemmas 4.1 and 4.2. As in the case of $\tilde{L}_0$, we choose a lift of $\tilde{L}_j$ whose (real-valued) phase is given by (4.5), using the determination of $\arg x$ closest to $-2\pi\frac{j}{a+b+c}$.

We are now in a position to compare the phases of two vanishing cycles at one of their intersection points. Consider an intersection point between $\tilde{L}_i$ and $\tilde{L}_{i+b}$, corresponding to the intersection y_i between L_i and L_{i+b}. Comparing the values of $\arg y$ on both vanishing cycles, it is easy to see that the intersection occurs in the $\epsilon = 1$ part of L_i and in the $\epsilon = -1$ part of L_{i+b}. Therefore, (4.5) yields that, at the intersection point,

$$\phi(\tilde{L}_{i+b}) - \phi(\tilde{L}_i) \simeq -2\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi\, b\, a}{(a+b+c)(b+c)} = \pi - \frac{2\pi\, b}{a+b+c},$$

which is between 0 and π since we have assumed that $b < a+c$. Therefore, we have $\deg y_i = 1$. Similarly, the intersection between $\tilde{L}_i$ and $\tilde{L}_{i+c}$ corresponding to z_i occurs in the $\epsilon = -1$ part of $\tilde{L}_i$ and the $\epsilon = 1$ part of $\tilde{L}_{i+c}$, so that (4.5) yields

$$\phi(\tilde{L}_{i+c}) - \phi(\tilde{L}_i) \simeq 2\Big(\frac{\pi}{2} - \frac{\pi\, c}{b+c}\Big) + \frac{2\pi\, c\, a}{(a+b+c)(b+c)} = \pi - \frac{2\pi\, c}{a+b+c},$$

which is also between 0 and π since $c < a+b$. Therefore, $\deg z_i = 1$. In the case of $\bar{y}_i$, things are similar, but with one new subtlety: in accordance with the above prescriptions, the determinations of $\arg x$ at the intersection point to be used for $\tilde{L}_i$ and $\tilde{L}_{i+a+c}$ differ by 2π. Therefore, from (4.5) we now get (taking $\epsilon = -1$ for $\tilde{L}_i$ and $+1$ for $\tilde{L}_{i+a+c}$)

$$\phi(\tilde{L}_{i+a+c}) - \phi(\tilde{L}_i) \simeq -2\pi\,\frac{a-b-c}{b+c} + 2\Big(\frac{\pi}{2} - \frac{\pi\,c}{b+c}\Big) + \frac{2\pi\,(a+c)\,a}{(a+b+c)(b+c)} = \pi + \frac{2\pi\,b}{a+b+c},$$

which is between π and 2π; therefore, $\deg \bar{y}_i = 2$. Similarly, for $\bar{z}_i$ one finds that

$$\phi(\tilde{L}_{i+a+b}) - \phi(\tilde{L}_i) \simeq -2\pi\,\frac{a-b-c}{b+c} - 2\Big(\frac{\pi}{2} - \frac{\pi\,c}{b+c}\Big) + \frac{2\pi\,(a+b)\,a}{(a+b+c)(b+c)} = \pi + \frac{2\pi\,c}{a+b+c},$$

which is also between π and 2π, so that $\deg \bar{z}_i = 2$.

Finally, the degrees of x_i and $\bar{x}_i$ can be deduced from those of the intersections of types y and z by considering e.g. the triangles $T_{xy,i}$, which gives that $\deg x_i + \deg y_{i+a} = \deg \bar{z}_i$, and hence $\deg x_i = 1$, and $T_{yz,i}$, which gives that $\deg y_i + \deg z_{i+b} = \deg \bar{x}_i$, and hence $\deg \bar{x}_i = 2$. This completes the proof of Lemma 4.7.

4.6 The exterior algebra structure

The aim of this section is to determine the coefficients appearing in Lemma 4.5, by studying the orientations of the moduli spaces of pseudo-holomorphic curves and the symplectic areas of their images ($T_{xy,i}, \dots$).

Lemma 4.8 *If the symplectic form ω is anti-invariant under complex conjugation and invariant under the action of $\mathbb{Z}/(a+b+c)$, then there exists a constant $\alpha \in \mathbb{C}^*$ such that $\alpha_{xy,i} = \alpha_{yz,i} = \alpha_{zx,i} = \alpha$ and $\alpha_{yx,i} = \alpha_{zy,i} = \alpha_{xz,i} = -\alpha$ for all i. Therefore, $m_2(x_i, y_{i+a}) = -m_2(y_i, x_{i+b})$, $m_2(y_i, z_{i+b}) = -m_2(z_i, y_{i+c})$, and $m_2(z_i, x_{i+c}) = -m_2(x_i, z_{i+a})$.*

The coefficients $\alpha_{xy,i}, \dots$ are determined up to sign by the symplectic areas of the triangular regions $T_{xy,i}, \dots$ inside Σ_0. To simplify notations, define

$$T_i = \begin{cases} T_{xy,i} & \text{if } 0 \le i < c, \\ T_{zx,i-c} & \text{if } c \le i < b+c, \\ T_{yz,i-b-c} & \text{if } b+c \le i < a+b+c, \end{cases} \qquad \text{and } T'_i = \begin{cases} T_{xz,i} & \text{if } 0 \le i < b, \\ T_{yx,i-b} & \text{if } b \le i < b+c, \\ T_{zy,i-b-c} & \text{if } b+c \le i < a+b+c, \end{cases}$$

so that T_i and T'_i are the two triangles having either x_i or $\bar{x}_{i-b-c}$ as one of their vertices. We similarly define α_i and α'_i to be the coefficients associated to T_i and T'_i in the formula giving m_2, namely $\alpha_i = \pm\exp(-\mathrm{Area}(T_i))$ and $\alpha'_i = \pm\exp(-\mathrm{Area}(T'_i))$. Then, as observed at the end of §4.4, the invariance properties of ω imply that the $a+b+c$ triangles T_i form a single orbit under the action of $\mathbb{Z}/(a+b+c)$, with $\zeta^{-q} \cdot T_i = T_{i+q}$, and similarly for the other triangles T'_i, with $\zeta^{-q} \cdot T'_i = T'_{i+q}$. Moreover, complex conjugation exchanges these two families of triangular regions, by mapping T_i to T'_{b+c-i} (see Figure 5). It follows that all of these triangles have the same symplectic area, and therefore that the various constants α_i and α'_i are all equal up to sign.

In order to identify the signs, one needs to orient the relevant moduli spaces of pseudo-holomorphic discs in some consistent way, which requires the choice of a spin structure over each Lagrangian L_i. As explained at the end of §3.1, we need to endow each L_i with the spin structure which extends to the corresponding thimble, i.e. the non-trivial one.

We now describe a convenient rule for determining the correct signs in the one-dimensional case, due to Seidel [217]. We start with the case of trivial spin structures. Then to each intersection point $p \in L_i \cap L_j$ ($i < j$) one can associate an orientation line $\mathcal{O}_p$. This orientation line is canonically trivial when $\deg p$ is even, whereas in the odd degree case, a choice of trivialization of $\mathcal{O}_p$ is equivalent to a choice of orientation of the line T_pL_j. If one considers a pseudo-holomorphic map $u : D^2 \to \Sigma_0$ contributing to m_k, whose image is a polygonal region with $k+1$ vertices $p_0, \dots, p_k$, then the corresponding sign factor is actually an element of the tensor product $\Lambda = \mathcal{O}_{p_0} \otimes \cdots \otimes \mathcal{O}_{p_k}$. We can define a preferred trivialization of Λ by choosing, at each vertex of odd degree, the orientation of the vanishing cycle which agrees with the positive orientation on the boundary of the image of u. The sign factor associated to u is then equal to $+1$ with respect to this trivialization of Λ (or -1 with respect to the other trivialization). In the presence of non-trivial spin structures, this rule needs to be modified as follows: fix a marked point on each L_i carrying a non-trivial spin structure (distinct from its intersection points with the other vanishing cycles); then the sign associated to u is affected by a factor of -1 for each marked point that the boundary of u passes through [217].

It is worth mentioning that, while it is clear from the above construction that the individual sign factors fail to be canonical and depend on some choices, the various possibilities yield equivalent categories, since the coefficients of Floer homology and Floer products simply differ by the conjugation action of some diagonal matrix with ± 1 coefficients.

In our case, we choose trivializations of the orientation lines as follows: for every intersection point $p \in L_i \cap L_j$ of degree 1 (i.e., one of x_i, y_i, z_i), we orient T_pL_j consistently with the boundary orientation of the single triangular region among $T_0, \dots, T_{a+b+c-1}$ having p among its vertices. If we consider trivial spin structures, then with this convention the sign factor associated to each triangle T_i is by definition equal to $+1$. In the case of T'_i, at each of the two vertices of degree 1 the chosen trivialization of T_pL_j disagrees with the boundary orientation of the triangular region, so that for trivial spin structures we get a sign factor of $(-1)^2 = +1$ again. Since we need to consider non-trivial spin structures, we must introduce a marked point on each L_i; for example, we choose this marked point in the portion of L_i that corresponds to the top-most edge on Figure 6. With this choice, the boundary of each T'_i passes through exactly one marked point (between the vertex of type z and that of type y), while the boundary of T_i does not meet any marked point. Therefore, with these conventions, the sign factors are $+1$ for all T_i and -1 for all T'_i; this completes the proof of Lemma 4.8.

4.7 Non-exact symplectic forms and non-commutative deformations

The purpose of this section is to describe the effect on the category of Lagrangian vanishing cycles of W of relaxing the assumptions made above on the symplectic form, losing in particular its exactness. In order to make the vanishing cycle construction well-defined, we will keep assuming that ω induces a complete Kähler metric on X and that the gradient of W with respect to this metric is bounded from below outside of a compact set. For example, choosing a 3×3 positive definite Hermitian matrix (a_{ij}), we can endow X with the symplectic form

$$\omega = i \sum_{i,j=1}^{3} a_{ij} \frac{dz_i}{z_i} \wedge \frac{d\bar{z}_j}{\bar{z}_j}.$$

Observe that $H_2(X, \mathbb{Z}) \simeq \mathbb{Z}$ is generated by the torus $T = \{(x, y, z) \in X,\ |x| = |y| = |z| = 1\}$ (for simplicity we assume $gcd(a, b, c) = 1$). An easy calculation shows that

$$[\omega] \cdot [T] = 4\pi^2 i \,(a\,(a_{23} - a_{32}) + b\,(a_{31} - a_{13}) + c\,(a_{12} - a_{21})). \tag{4.6}$$

Many other choices of symplectic form are equally acceptable, and it is important to mention that the most sensible course of action in presence of a non-explicit symplectic form is to search for a *topological* interpretation of the category of Lagrangian vanishing cycles, involving only topological quantities such as the cohomology class of ω.

In comparison to the restrictive situation considered above, the vanishing cycles L_j remain in the same smooth isotopy classes, because one can continuously deform from one symplectic structure to the other. Hence, the vanishing cycles are smoothly isotopic to the loops $L'_j \subset \Sigma_0$ introduced in §4.2, but not necessarily Hamiltonian isotopic to them. Nonetheless, because the ends of the non-compact Riemann surface Σ_0 all have infinite volume, we can easily deform L'_j into loops $L''_j \subset \Sigma_0$ that are Hamiltonian isotopic to the vanishing cycles, without modifying the pattern of the intersections between them. More precisely, recall from §4.2 that each L'_j is the double lift via $\pi_x : \Sigma_0 \to \mathbb{C}^*$ of an arc joining two branch points of π_x. Then, by "pulling" a suitable portion of one of the two lifts towards an end of Σ_0 (either towards infinity or towards zero in the x-axis projection), we can make L'_j sweep through an arbitrarily large amount of symplectic area to obtain the desired L''_j, without affecting the intersection points with the other vanishing cycles.

Since the vanishing cycles are Hamiltonian isotopic to the loops L''_j, we may use L''_j instead of the actual vanishing cycles in order to determine the category $D(\mathrm{Lag}_{\mathrm{vc}}(W))$. Hence, the symplectic deformation does not affect in any way the generators of the Floer complexes and the types of pseudo-holomorphic maps to be considered. The only significant change has to do with the coefficients assigned to the various pseudo-holomorphic discs appearing in the definition of m_2, as the symplectic areas of the various triangular regions T_i and T'_i ($i = 0, \dots, a+b+c-1$) inside Σ_0 may now take more or less arbitrary values instead of all being equal to each other. Because the description of ω and of the vanishing cycles is not explicit, it is hopeless (and useless) to calculate the individual coefficients α_i and α'_i. However, we can state the following result:

Lemma 4.9 *Lemmas* 4.3–4.7 *remain valid in the more general case of an arbitrary symplectic form inducing a complete Kähler metric on X for which $|\nabla W|$ is bounded from below at infinity. Moreover, the structure constants for the composition m_2 are related by the identity*

$$\frac{\prod_{i=0}^{a+b+c-1}\alpha_i}{\prod_{i=0}^{a+b+c-1}\alpha'_i}=\frac{\prod\alpha_{xy,i}\prod\alpha_{yz,i}\prod\alpha_{zx,i}}{\prod\alpha_{yx,i}\prod\alpha_{zy,i}\prod\alpha_{xz,i}}=(-1)^{a+b+c}\exp(-2\pi[\omega]\cdot[T]).$$

The assumption of completeness of the induced Kähler metric can be dropped if we have some other way of ensuring that the vanishing cycles are well-defined and that the deformation from L'_j to L''_j can be carried out without introducing new intersection points. In fact, the invariance of Floer homology under Hamiltonian isotopies essentially implies that the introduction of new intersection points in the deformation does not have any particular impact on the derived category, so the only thing that matters is actually the well-definedness of the vanishing cycles.

Although Lemma 4.9 seems to give only very partial information about the constants α_i and α'_i, it actually completely determines the category $D(\mathrm{Lag_{vc}}(W))$. Indeed, simply by rescaling the generators of the Floer complexes we can modify the coefficients α_i and α'_i almost at will: for example, replacing x_i with $\lambda\, x_i$ has the effect of simultaneously multiplying α_i and α'_i by λ^{-1}; similarly, rescaling the generator y_i simultaneously affects α_{i-a} (or α_{i+b+c}) and α'_{i+b}. Still assuming $gcd(a,b,c)=1$, it is not hard to check that the only quantity left invariant by all rescalings of the generators is the ratio $\prod\alpha_i/\prod\alpha'_i$, which is therefore sufficient to characterize the derived category. This observation that the symplectic deformations of $D(\mathrm{Lag_{vc}}(W))$ are governed by a single parameter is naturally related to the fact that the second Betti number of X is equal to 1.

Proof. [Proof of Lemma 4.9] The key observation to be made here is that the boundary of the 2-chain $C=\sum T_i-\sum T'_i\subset\Sigma_0$ is exactly $\partial C=-\sum L_i$ (for a suitable choice of orientation of the L_i). Indeed, looking at Figure 6, each of the six portions of L_i arises exactly once as an edge of one of the triangular regions, and the boundary orientation of the triangular region is the "clockwise" orientation of L_i in the case of $T_0,\dots,T_{a+b+c-1}$, and the "counterclockwise" orientation in the case of $T'_0,\dots,T'_{a+b+c-1}$. Recalling that each vanishing cycle L_i bounds a Lefschetz thimble D_i in X, we can build a 2-cycle $\tilde C\subset X$ by capping C with these $a+b+c$ Lagrangian discs. Next, observe that the sign factors arising from the orientations of the moduli spaces remain the same as in §4.6, and that $\int_{D_i}\omega=0$, so that

$$\frac{\prod\alpha_i}{\prod\alpha'_i}=(-1)^{a+b+c}\frac{\prod\exp(-2\pi\int_{T_i}\omega)}{\prod\exp(-2\pi\int_{T'_i}\omega)}=(-1)^{a+b+c}\exp(-2\pi\int_C\omega)=(-1)^{a+b+c}\exp(-2\pi[\omega]\cdot[\tilde C]).$$

Hence, the last step in the proof is to show that $[\tilde C]$ and $[T]$ are the same elements of $H_2(X,\mathbb{Z})\simeq\mathbb{Z}$. A simple way to achieve this is to compute the intersection pairing of $\tilde C$ with the relative cycle $R=\{(x,y,z)\in X,\ x,y,z\in\mathbb{R}^+\}$, which intersects T transversely once at the point $(1,1,1)$.

To understand how R intersects $\tilde{C}$, we compare the values of W over R and over $\tilde{C}$. By construction, $\tilde{C}$ is the union of the 2-chain $C \subset \Sigma_0$, over which W vanishes identically, and the various Lefschetz thimbles D_j, which W maps to straight line segments joining the origin to the critical values λ_j. On the other hand, the restriction to R of $W = x + y + z$ is a proper function which takes real positive values. With respect to the standard complex structure, R is totally real and W is holomorphic, so any critical point of $W_{|R}$ is also a critical point of W, and in particular the minimum of W over R is a critical value of W. Indeed, a simple computation shows that the minimum of W over R is exactly $(a+b+c)(a^a b^b c^c)^{-1/(a+b+c)} = \lambda_0$, achieved at the critical point p_0 of W corresponding to the critical value λ_0.

It follows that the only point where $\tilde{C}$ and R intersect is p_0. Moreover, by considering the local model near p_0, it is easy to check that this intersection is transverse, since the Hessian of W at p_0 restricts to the tangent space $T_{p_0}D_0$ as a negative definite real quadratic form, and to $T_{p_0}R$ as a positive definite real quadratic form. Therefore the intersection number between $\tilde{C}$ and R is equal to 1 (for a suitable choice of orientation that we will not discuss here), and it follows that $[\tilde{C}] = [T]$ in $H_2(X, \mathbb{Z})$. □

4.8 B-fields and complexified deformations

So far we have identified a real one-parameter family of deformations of the category of Lagrangian vanishing cycles of W. To extend this to a complex family of deformations, we need to introduce a non-trivial B-field, i.e. a closed 2-form $B \in \Omega^2(X, \mathbb{R})$. The presence of a B-field affects Fukaya categories by modifying the nature of the objects to be considered: namely, one should consider pairs consisting of a Lagrangian submanifold and a vector bundle over it equipped with a *projectively flat* (rather than flat) connection with curvature equal to $-2\pi i B \otimes \mathrm{Id}$ (depending on conventions, the factor of 2π is sometimes omitted).

In our case, we are considering Lagrangian vanishing cycles $L_j \simeq S^1$ arising as boundaries of the Lefschetz thimbles D_j. Since $\dim L_j = 1$, over L_j every bundle is trivial and every connection is flat; moreover, we can safely restrict ourselves to the case of line bundles. However, the presence of the B-field results in a nontrivial holonomy. By Stokes' theorem, if a U(1)-connection $\nabla_j = d + i\alpha_j$ is the restriction to L_j of a U(1)-connection with curvature $-2\pi i B$ over D_j, then the holonomy of ∇_j around L_j is given by $\mathrm{hol}_{\nabla_j}(L_j) = \exp(\int_{L_j} i\alpha_j) = \exp(\int_{D_j} i\, d\alpha_j) = \exp(-2\pi i \int_{D_j} B)$. Since this property characterizes the connection ∇_j uniquely up to gauge, we can drop the line bundle and the connection from the notation when considering the objects (L_j, E_j, ∇_j) of $\mathrm{Lag_{vc}}(W, \{\gamma_j\})$.

However, we do need to take the holonomy of ∇_j into account when computing the twisted Floer differential and compositions m_k, since the weight attributed to a given pseudo-holomorphic disc $u : (D^2, \partial D^2) \to (\Sigma_0, \bigcup L_j)$ is modified by a factor corresponding to the holonomy along its boundary, and becomes $\pm\, \mathrm{hol}(u(\partial D^2))\, \exp(2\pi i \int_{D^2} u^*(B + i\omega))$. More precisely, for each intersection point $p \in L_i \cap L_j$ we need to fix an isomorphism between the fibers $(E_i)_{|p}$ and $(E_j)_{|p}$; then it becomes possible to define the holonomy along the closed loop $u(\partial D^2)$ using the parallel transport induced by ∇_j from one "corner" of u to the next one, and the chosen isomorphism at each corner.

In this context, we now have the following result characterizing $D(\mathrm{Lag}_{\mathrm{vc}}(W))$:

Lemma 4.10 *Lemmas* 4.3–4.7 *remain valid for an arbitrary symplectic form inducing a complete Kähler metric on X for which $|\nabla W|$ is bounded from below at infinity, and an arbitrary B-field. Moreover, the structure constants for the composition m_2 are related by the identity*

$$\frac{\prod_{i=0}^{a+b+c-1}\alpha_i}{\prod_{i=0}^{a+b+c-1}\alpha'_i} = \frac{\prod\alpha_{xy,i}\prod\alpha_{yz,i}\prod\alpha_{zx,i}}{\prod\alpha_{yx,i}\prod\alpha_{zy,i}\prod\alpha_{xz,i}} = (-1)^{a+b+c}\exp(2\pi i[B+i\omega]\cdot[T]).$$

Proof. We again consider the 2-chain $C = \sum T_i - \sum T'_i \subset \Sigma_0$, with boundary $\partial C = -\sum L_j$, and the 2-cycle $\tilde{C} \subset X$ obtained by capping C with the Lagrangian discs D_j. We now have:

$$\begin{aligned}\frac{\prod\alpha_i}{\prod\alpha'_i} = \frac{(-1)^{a+b+c}}{\prod \mathrm{hol}_{\nabla_j}(L_j)}\,\frac{\prod\exp(2\pi i\int_{T_i}B+i\omega)}{\prod\exp(2\pi i\int_{T'_i}B+i\omega)} &= \frac{(-1)^{a+b+c}}{\prod\exp(\int_{D_j}-2\pi i B)}\exp(2\pi i\int_C B+i\omega)\\ &= (-1)^{a+b+c}\exp(2\pi i[B+i\omega]\cdot[\tilde{C}]).\end{aligned}$$

This completes the proof since $[\tilde{C}] = [T]$. □

It is interesting to observe that this statement reinterpretes the quantity $\prod\alpha_i/\prod\alpha'_i$ in purely topological terms, thus avoiding the pitfall of having to compute the individual coefficients attached to the various pseudo-holomorphic discs in Σ_0. This outcome is rather unsurprising since, whereas the individual coefficients α_i and α'_i are heavily dependent on a number of arbitrary choices, the underlying derived category of Lagrangian vanishing cycles is expected to depend only on the meaningful parameters – in our case, the cohomology class $[B+i\omega]$.

We would like to suggest that this feature reflects a general *principle*. Namely, the various structure coefficients of the Floer differentials and products involved in the definition of the category $\mathrm{Lag}_{\mathrm{vc}}(W)$ depend on many choices and have no precise meaning in general. However, different sets of values of the structure coefficients may become equivalent after a suitable rescaling of the generators of the Floer complexes or other similarly benign operations. Hence, we can reduce to a much smaller set of parameters (certain combinations of the individual Floer coefficients) that actually govern the structure of the category. Then, we expect the following statement to hold in much greater generality than the examples studied here:

Property 4.11 *The structure of the derived category of Lagrangian vanishing cycles is governed by deformation parameters which are all of the form $\exp(2\pi i[B+i\omega]\cdot[C_j])$ for suitable 2-cycles $C_j \subset X$.*

This is of course ultimately related to the fact that Floer homology and Floer products can be defined over *Novikov rings*, counting pseudo-holomorphic discs with coefficients that reflect relative homology classes rather than actual symplectic areas; the version with complex coefficients that we used here is then recovered from the version with Novikov ring coefficients by evaluation at the point $[B+i\omega]$.

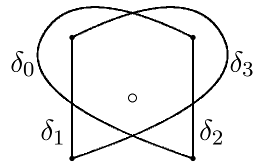


Figure 8: The vanishing cycles for $\mathbb{F}_0$

5 Hirzebruch surfaces

We now consider the case of Hirzebruch surfaces $\mathbb{F}_n$, for which the mirror Landau-Ginzburg model consists of $X = (\mathbb{C}^*)^2$ equipped with a superpotential of the form

$$W = x + y + \frac{a}{x} + \frac{b}{x^n y}$$

for some non-zero constants a, b. For simplicity we will only consider the case of an exact symplectic form. Since different values of the constants a, b lead to mutually isotopic exact symplectic Lefschetz fibrations, the actual choices do not matter (we can e.g. assume $a = b = 1$ or any other convenient choice).

5.1 The case of $\mathbb{F}_0$ and $\mathbb{F}_1$

The first two Hirzebruch surfaces $\mathbb{F}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{F}_1$ (i.e., $\mathbb{CP}^2$ blown up at one point) need to be considered separately.

Proposition 5.1 *When $n = 0$, there exists a system of arcs $\{\gamma_i\}$ such that $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is equivalent to the full subcategory of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_0))$ whose objects are $\mathcal{O}$, $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$, $\mathcal{O}(1,1)$. Therefore, $D(\mathrm{Lag}_{\mathrm{vc}}(W)) \simeq \boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_0))$.*

Proof. The four critical values of $W = x + y + \frac{a}{x} + \frac{b}{y}$ are $\pm 2a^{1/2} \pm 2b^{1/2}$. Up to an exact deformation which does not affect the category of Lagrangian vanishing cycles, we can choose $a > b > 0$, and assume the symplectic form to be anti-invariant under reflection about the imaginary axis $(x, y) \mapsto (-\bar{x}, -\bar{y})$. We choose $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and join it to the singular fibers by considering arcs γ_i that pass below the real axis in $\mathbb{C}$, so that the clockwise ordering of the critical values agrees with their natural ordering $-2a^{1/2} - 2b^{1/2} < -2a^{1/2} + 2b^{1/2} < 2a^{1/2} - 2b^{1/2} < 2a^{1/2} + 2b^{1/2}$. The projection π_x to the x variable realizes Σ_0 as a double cover of $\mathbb{C}^*$ branched at four points, and the vanishing cycles L_i can be represented as double lifts of the arcs $\delta_i \subset \mathbb{C}^*$ shown in Figure 8.

It follows that $\mathrm{Hom}(L_1, L_2) = 0$, while $\mathrm{Hom}(L_0, L_1)$, $\mathrm{Hom}(L_2, L_3)$, $\mathrm{Hom}(L_0, L_2)$, and $\mathrm{Hom}(L_1, L_3)$ are two-dimensional; label the corresponding intersection points $L_0 \cap L_1 = \{s, t\}$, $L_2 \cap L_3 = \{s', t'\}$, $L_0 \cap L_2 = \{u, v\}$, $L_1 \cap L_3 = \{u', v'\}$. Finally, $\mathrm{Hom}(L_0, L_3)$ has rank

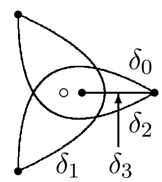


Figure 9: The vanishing cycles for $\mathbb{F}_1$

4. By considering the triangular regions delimited by the vanishing cycles in Σ_0, and using the symmetry of the configuration with respect to $(x,y)\mapsto(-\bar{x},-\bar{y})$, we can easily show that $m_2(s,u')=m_2(s',u)$, $m_2(t,u')=m_2(t',u)$, $m_2(s,v')=m_2(s',v)$, and $m_2(t,v')=m_2(t',v)$; these four elements of $\mathrm{Hom}(L_0,L_3)$ are proportional to the generators. All other products vanish ($m_k=0$ for $k\neq 2$). Finally, gradings can be chosen so that all morphisms have degree 0 (the verification is left to the reader).

Therefore, the category $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ is indeed equivalent to the full subcategory of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_0))$ whose objects are $\mathcal{O}$, $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$, $\mathcal{O}(1,1)$, as can be seen by thinking of (s,t) and (u,v) as homogeneous coordinates on the two factors of $\mathbb{F}_0=\mathbb{CP}^1\times\mathbb{CP}^1$. Since these four line bundles form a full strong exceptional collection generating $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_0))$, the result follows. $\square$

Alternatively, Proposition 5.1 can also be obtained as a direct corollary of a general product formula for categories of Lagrangian vanishing cycles of Lefschetz fibrations of the form $(X_1\times X_2, W_1+W_2)$ ([18], cf. also §6.3).

Proposition 5.2 *When $n=1$, there exists a system of arcs $\{\gamma_i\}$ such that $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ is equivalent to the full subcategory of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_1))$ whose objects are $\mathcal{O}$, $\pi^*(T_{\mathbb{P}^2}(-1))$, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $\mathcal{O}_E$ (where E is the exceptional curve and $\pi:\mathbb{F}_1\to\mathbb{CP}^2$ is the blow-up map). Therefore, $D(\mathrm{Lag}_{\mathrm{vc}}(W))\simeq\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_1))$.*

Proof. We choose $a=b=1$, and equip X with a symplectic form that is anti-invariant under complex conjugation. Let $(\lambda_i)_{0\le i\le 3}$ be the four critical values of $W=x+y+\frac{1}{x}+\frac{1}{xy}$, ordered clockwise around the origin so that $\mathrm{Im}(\lambda_0)>0$, $\lambda_1\in\mathbb{R}_+$, $\mathrm{Im}(\lambda_2)<0$, and $\lambda_3\in\mathbb{R}_-$. We choose $\Sigma_0=W^{-1}(0)$ as reference fiber, and choose the arcs γ_i joining 0 to λ_i to be straight lines. The projection π_x to the x variable realizes Σ_0 as a double cover of $\mathbb{C}^*$ branched at four points, and the vanishing cycles L_i can be represented as double lifts of the arcs $\delta_i\subset\mathbb{C}^*$ shown in Figure 9.

The corresponding category of vanishing cycles can then be studied explicitly. In fact, much of the work has already been carried out in §4, since the situation for L_0,L_1,L_2 is rigorously identical (including grading and orientation issues) to that previously considered for the three vanishing cycles of the Lefschetz fibration mirror to $\mathbb{CP}^2$. While the choice of grading used in §4 yields morphisms in degrees 1 and 2, a different choice of gradings (shifting L_1 by 1 and L_2 by 2) ensures that all morphisms between L_0,L_1,L_2 have degree 0. This readily implies that a category equivalent to the derived category of $\mathbb{CP}^2$ can be realized inside $D(\mathrm{Lag}_{\mathrm{vc}}(W))$ as a full subcategory, with the exceptional collection L_0, L_1, L_2 corresponding to the exceptional collection $\mathcal{O}, T_{\mathbb{P}^2}(-1), \mathcal{O}(1)$ dual to the standard one. (This claim can of course also be verified "by hand" following the same outline of argument as in §4).

From Figure 9 it is clear that $\mathrm{Hom}(L_0, L_3)$ and $\mathrm{Hom}(L_2, L_3)$ are one-dimensional, (call their generators p_0 and p_2), while $\mathrm{Hom}(L_1, L_3)$ has rank 2 (call its generators q and q'). To be consistent with the notation of §4, call x_0, y_0, z_0 (resp. x_1, y_1, z_1; resp. $\bar{x}, \bar{y}, \bar{z}$) the generators of $\mathrm{Hom}(L_0, L_1)$ (resp. $\mathrm{Hom}(L_1, L_2)$; resp. $\mathrm{Hom}(L_0, L_2)$). Then, looking at the various pseudo-holomorphic discs in Σ_0 (including a constant one at the triple intersection of L_0, L_2, L_3), we have: $m_2(x_0, q) = m_2(x_0, q') = 0$, $m_2(y_0, q) = \alpha\, p_0$, $m_2(y_0, q') = 0$, $m_2(z_0, q) = 0$, $m_2(z_0, q') = \alpha'\, p_0$, $m_2(x_1, p_2) = 0$, $m_2(y_1, p_2) = -\alpha\, q'$, $m_2(z_1, p_2) = \alpha'\, q$, $m_2(\bar{x}, p_2) = p_0$, $m_2(\bar{y}, p_2) = m_2(\bar{z}, p_2) = 0$ (for some non-zero constants α, α'). Moreover, for a suitable choice of grading of L_3 it can be checked that all morphisms have degree 0.

It is then easy to check that these formulas correspond exactly to the composition formulas in the full subcategory of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_1))$ whose objects are the pull-backs $\mathcal{O}$, $\pi^*(T_{\mathbb{P}^2}(-1))$, $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, and the structure sheaf $\mathcal{O}_E$ of the exceptional curve (If one follows the analogy suggested by the notation between the morphisms from L_0 to L_2 and the homogeneous coordinates on $\mathbb{CP}^2$, then the blow-up point is located at $(1:0:0)$). The result follows. $\square$

5.2 Other Hirzebruch surfaces

For larger values of n, the situation becomes different:

Lemma 5.3 *If $n \geq 2$, then the Lefschetz fibrations over $(\mathbb{C}^*)^2$ defined by $W = x + y + \frac{1}{x} + \frac{1}{x^n y}$ and $\tilde{W} = x + y + \frac{1}{x^n y}$ are isotopic. Therefore, $D(\mathrm{Lag}_{\mathrm{vc}}(W)) \simeq D(\mathrm{Lag}_{\mathrm{vc}}(\tilde{W})) \simeq$ $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$.*

Proof. Consider the maps $W_a = x + y + \frac{a}{x} + \frac{1}{x^n y}$ for $a \in [0,1]$. The key observation is that the $n+2$ critical points of W_a remain distinct and stay in a compact subset of $(\mathbb{C}^*)^2$. Indeed, the critical points of W_a are the solutions of

$$\begin{cases} 1 - \frac{a}{x^2} - \frac{n}{x^{n+1}y} = 0 \\ 1 - \frac{1}{x^n y^2} = 0, \end{cases}$$

i.e.

$$y = nx^{1-n}(x^2 - a)^{-1}, \text{ and } x^{n-2}(x^2-a)^2 - n^2 = 0.$$

It is easy to check that for $|a| \leq 1$ the roots of this equation satisfy $1 \leq |x| \leq \sqrt{n+1}$. It follows that $|x^2 - a| = n|x|^{1-\frac{n}{2}}$ is bounded between two positive constants, and hence that $y = nx^{1-n}(x^2-a)^{-1} = (x^2-a)/nx$ is also bounded between two positive constants independently of a. Hence the critical points of W_a remain inside a compact subset of $(\mathbb{C}^*)^2$. Moreover, the polynomial $P(x) = x^{n-2}(x^2-a)^2 - n^2$ always has simple roots when $|a| \leq 1$, since the roots of $P'(x) = x^{n-3}(x^2-a)((n+2)x^2 - (n-2)a)$ are 0, $\pm\sqrt{a}$, and $\pm\sqrt{\frac{n-2}{n+2}a}$, where P never vanishes. In fact, even though this is not necessary for the argument, the critical values of W_a also remain distinct throughout the deformation, since at a critical point we have $W_a = \frac{n+2}{n}x + \frac{n-2}{n}\frac{a}{x}$, which as a function of x is injective over $\{|x| \geq 1\}$.

Therefore, W_a defines an exact symplectic Lefschetz fibration on $(\mathbb{C}^*)^2$ for all $a \in [0,1]$, which allows us to match the vanishing cycles of $W_1 = W$ with those of $W_0 = \tilde{W}$. The resulting categories of vanishing cycles differ at most by a deformation of the structure coefficients of the compositions m_2, but since the isotopy is through *exact* Lagrangian vanishing cycles, we need not worry about those (see also the argument for Lemma 4.9).

We can therefore conclude that $D(\mathrm{Lag}_{\mathrm{vc}}(W)) \simeq D(\mathrm{Lag}_{\mathrm{vc}}(\tilde{W}))$. Since $((\mathbb{C}^*)^2, \tilde{W})$ is exactly the mirror to $\mathbb{CP}^2(n,1,1)$ studied at length in §4, our result for weighted projective planes implies that this category is also equivalent to $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$. $\square$

For $n = 2$, it is well-known that $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_2)) \simeq \boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(2,1,1)))$, so we get the expected result. However, for $n \geq 3$ this is no longer true. Namely, the fully faithful functor MK_n constructed in §2.7 allows us to view the category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_n))$ as a full subcategory of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$, generated by the exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1))$. It is therefore a natural question to ask whether this subcategory can be singled out on the mirror side, by selecting 4 of the $n+2$ critical points of W. It turns out that this is indeed the case. Our first result in this direction is the following:

Lemma 5.4 *For $n \geq 3$, in the limit $b \to 0$, $n-2$ of the critical values of the superpotentials $W_b = x + y + \frac{1}{x} + \frac{b}{x^n y}$ go to infinity, while the remaining four critical points stay in a bounded region.*

Proof. The x coordinates of the critical points of W_b are the solutions of

$$x^{n-2}(x^2-1)^2 - n^2 b = 0.$$

As $b \to 0$, four roots of this equation converge to ± 1, while the remaining $n-2$ converge to 0. Since at a critical point we also have $y = nbx^{1-n}(x^2-1)^{-1} = \frac{1}{n}(x - \frac{1}{x})$ and $W_b = \frac{n+2}{n}x + \frac{n-2}{n}\frac{1}{x}$, we conclude that four critical points of W_b converge to $(\pm 1, 0)$, with the corresponding critical values converging to ± 2, while the others escape to infinity. $\square$

This suggests that the deformation $b \to 0$ singles out a subcategory of $D(\mathrm{Lag}_{\mathrm{vc}}(W_b))$, obtained by restricting oneself to the preimage of a disc containing only four critical values of W_b. We start by describing the case $n = 3$.

For $n = 3$, we can study explicitly the deformation process as b changes from 1 to a value close to 0. For $b = 1$ the five critical values of W_b form a pentagon roughly centered at the origin (and can for all practical purposes be identified with the critical values of the superpotential mirror to $\mathbb{CP}^2(3,1,1)$). As b decreases along the real axis, two things happen: first, the two complex conjugate critical points with $\mathrm{Re}(W_b) > 0$ merge and turn into two real critical points; then, one of these two real critical points escapes to infinity as $b \to 0$. The process is easier to visualize if one avoids the two values of b in the interval $(0,1)$ for which two critical values of W_b coincide, by considering e.g. a deformation from $b = 1$ to $b = 0$ where

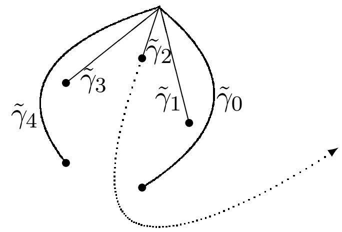


Figure 10: The deformation $b \to 0$ for $n = 3$

the imaginary part of b is kept positive. It is then easy to check that, as $b \to 0$, two critical values converge to 2 and two others converge to -2, while the fifth one escapes to infinity in the manner represented on Figure 10.

Therefore, if we consider the category of Lagrangian vanishing cycles associated to the system of arcs $\tilde{\gamma}_0, \dots, \tilde{\gamma}_4$ represented on Figure 10, the deformation $b \to 0$ singles out the full subcategory generated by the four vanishing cycles $\tilde{L}_0, \tilde{L}_1, \tilde{L}_3, \tilde{L}_4$ (where $\tilde{L}_i$ is the vanishing cycle associated to $\tilde{\gamma}_i$). The collection of arcs $\{\tilde{\gamma}_i\}$ looks very different from the collection $\{\gamma_i\}$ considered in §4, but they are related to each other by a sequence of elementary sliding transformations performed on consecutive arcs (see Figure 11).

It follows immediately from Definition 3.1 that every ordered collection of arcs yields a full exceptional collection generating $D(\mathrm{Lag}_{\mathrm{vc}}(W))$; it was shown by Seidel that (left or right) sliding operations on collections of arcs correspond to (left or right) mutations of the corresponding exceptional collections [214]. With this is mind, and identifying implicitly the critical points of W_1 with those of the superpotential mirror to $\mathbb{CP}^2(3,1,1)$, it is easy to check that the left dual to the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_4)$ associated to the arcs $\{\tilde{\gamma}_i\}$ is equivalent (up to some shifts) to the exceptional collection associated to the arcs $(\gamma_2, \gamma_3, \gamma_4, \gamma_0, \gamma_1)$. Moreover, using $\mathbb{Z}/5$-equivariance for $\mathbb{CP}^2(3,1,1)$, there exists an auto-equivalence of $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ which maps this exceptional collection to the one associated to the collection of arcs $(\gamma_0, \dots, \gamma_4)$ considered in §4.

Recall that the two exceptional collections for $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(3,1,1)))$ presented in §2 are mutually dual (cf. Example 2.15), and that Theorem 3.3 identifies the exceptional collection associated to the arcs $(\gamma_0, \dots, \gamma_4)$ with that given by Corollary 2.27. Therefore, there is an equivalence of categories which maps the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_4)$ for $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ to the exceptional collection $(\mathcal{O}, \dots, \mathcal{O}(4))$ for $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(3,1,1)))$. The full subcategory of $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ singled out by the deformation $b \to 0$ is that generated by the exceptional col-

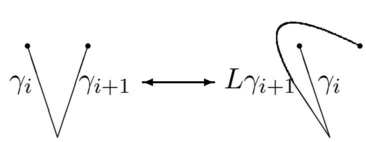


Figure 11: The (left) sliding operation $(\gamma_i, \gamma_{i+1}) \longleftrightarrow (L\gamma_{i+1}, \gamma_i)$

lection $(\tilde{L}_0, \tilde{L}_1, \tilde{L}_3, \tilde{L}_4)$, which corresponds under the above identification to the full subcategory of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(3,1,1)))$ generated by the exceptional collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(3), \mathcal{O}(4))$, which is in turn known to be equivalent to the derived category of the Hirzebruch surface $\mathbb{F}_3$ (see §2.7).

A similar analysis of the deformation $b \to 0$ can be carried out for all values of n, and leads to the following result:

Proposition 5.5 *Given any $n \geq 3$ and $R \gg 2$, and assuming that b is sufficiently close to 0, the full subcategory of $D(\mathrm{Lag}_{\mathrm{vc}}(W_b))$ arising from restriction to the open domain $\{|W_b| < R\}$ is equivalent to $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{F}_n))$.*

In order to prove this proposition we need a lemma about mutations in the standard full exceptional collection $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n+1))$ on the weighted projective plane $\mathbb{CP}^2(n,1,1)$. Let us fix a pair $(\mathcal{O}(k), \mathcal{O}(k+1))$ with $2 < k < n$. Denote by F_{k+2} the mutation of the object $\mathcal{O}(k+2)$ to the left through $\mathcal{O}(k), \mathcal{O}(k+1)$, i.e. $F_{k+2} \cong L^{(2)}\mathcal{O}(k+2)$. Performing the same mutations on $\mathcal{O}(k+3), \dots, \mathcal{O}(n+1)$ we obtain exceptional objects $F_i = L^{(2)}\mathcal{O}(i)$ for $k+2 \leq i \leq n+1$ and a new exceptional collection

$$(\mathcal{O}, \dots, \mathcal{O}(k-1), F_{k+2}, \dots, F_{n+1}, \mathcal{O}(k), \mathcal{O}(k+1))\,.$$

Denote by G_k, G_{k+1} the left mutations of $\mathcal{O}(k), \mathcal{O}(k+1)$ respectively through all F_i. We get an exceptional collection

$$(\mathcal{O}, \dots, \mathcal{O}(k-1), G_k, G_{k+1}, F_{k+2}, \dots, F_{n+1})\,.$$

Denote by $\mathcal{D}$ the triangulated subcategory of the category $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$ generated by the collection $(\mathcal{O}, \mathcal{O}(1), G_k, G_{k+1})$

Lemma 5.6 *The triangulated subcategory $\mathcal{D}$ coincides with the subcategory*

$$\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1) \rangle.$$

Proof. This Lemma is equivalent to the statement that the subcategory $\langle G_k, G_{k+1} \rangle$ coincides with the subcategory $\langle \mathcal{O}(n), \mathcal{O}(n+1) \rangle$. First, let us show that $\mathcal{O}(n)$ and $\mathcal{O}(n+1)$ belong to $\langle G_k, G_{k+1} \rangle$. Since $\mathrm{Hom}(\mathcal{O}(l), \mathcal{O}(s)) = 0$ for $l = n, n+1$ and $0 \leq s < k$, we can immediately conclude that $\mathcal{O}(n)$ and $\mathcal{O}(n+1)$ belong to $\langle G_k, G_{k+1}, F_{k+2}, \dots, F_{n+1} \rangle$. Therefore, it is sufficient to check that

$$\mathrm{Hom}^\bullet(F_i, \mathcal{O}(n)) = 0, \qquad \mathrm{Hom}^\bullet(F_i, \mathcal{O}(n+1)) = 0$$

for all $k+2 \leq i \leq n+1$.

By definition of F_i there are distinguished triangles

$$T_i \longrightarrow V_i \otimes \mathcal{O}(k+1) \longrightarrow \mathcal{O}(i), \tag{5.1}$$

$$F_i \longrightarrow W_i \otimes \mathcal{O}(k) \longrightarrow T_i, \tag{5.2}$$

with $V_i = \mathrm{Hom}(\mathcal{O}(k+1), \mathcal{O}(i))$ and $W_i = \mathrm{Hom}(\mathcal{O}(k), T_i)$. It is clear that $V_i \cong S^{i-k-1}U$, where U is the two dimensional vector space $H^0(\mathbb{CP}^2(n,1,1), \mathcal{O}(1))$. Considering the sequence of Hom's from $\mathcal{O}(k)$ to the triangle (5.1), it is easy to check that $W_i \cong S^{i-k-2}U$ (we use an isomorphism $\Lambda^2 U \cong \mathbb{C}$).

We have isomorphisms

$$\mathrm{Hom}(V_i \otimes \mathcal{O}(k+1), \mathcal{O}(n+1)) = S^{i-k-1}U^* \otimes S^{n-k}U \cong \bigoplus_{j=0}^{i-k-1} S^{n-i+1+2j}U,$$

which implies that

$$\mathrm{Hom}(T_i, \mathcal{O}(n+1)) \cong \bigoplus_{j=1}^{i-k-1} S^{n-i+1+2j}U.$$

On the other hand, there are isomorphisms

$$\mathrm{Hom}(W_i \otimes \mathcal{O}(k), \mathcal{O}(n+1)) = S^{i-k-2}U^* \otimes S^{n-k+1}U \cong \bigoplus_{j=1}^{i-k-1} S^{n-i+1+2j}U,$$

and, moreover, it can be checked that the natural morphism $\mathrm{Hom}(T_i, \mathcal{O}(n+1)) \to \mathrm{Hom}(W_i \otimes \mathcal{O}(k), \mathcal{O}(n+1))$ is an isomorphism. Hence, $\mathrm{Hom}^\bullet(F_i, \mathcal{O}(n+1)) = 0$ for all $k+2 \leq i \leq n+1$. By the same reasons $\mathrm{Hom}^\bullet(F_i, \mathcal{O}(n)) = 0$ for all $k+2 \leq i \leq n+1$. Thus the subcategory $\langle \mathcal{O}(n), \mathcal{O}(n+1) \rangle$ is contained in $\langle G_k, G_{k+1} \rangle$.

Since $\mathrm{Hom}(G_k, G_{k+1}) \cong U \cong \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n+1))$, these two categories are both equivalent to the derived category of representations of the quiver with two vertices and two arrows $\bullet \rightrightarrows \bullet$, and, as consequence, it can be easily shown that they are equivalent. $\square$

Proof. [Proof of Proposition 5.5] The argument is similar to the case $n = 3$: in the initial configuration, for $b = 1$, the $n+2$ critical values of W_b approximate a regular polygon, and can essentially be identified with the critical values of the superpotential mirror to $\mathbb{CP}^2(n,1,1)$. We label these critical values by integers from 0 to $n+1$, with 0 corresponding to the positive real critical value, and continuing counterclockwise. As the value of b is decreased towards 0, pairs of complex conjugate critical values of W_b (those labelled k and $n+2-k$, for $1 \leq k \leq \frac{n}{2}$), successively converge towards each other. For $2 \leq k < \frac{n}{2}$, the corresponding vanishing cycles are disjoint, and the two complex conjugate critical values essentially exchange their positions before escaping to infinity (with complex arguments close to $\mp\frac{k-1}{n-2}\,2\pi$) for $b \to 0$. On the other hand, for $k = 1$ the two complex conjugate critical points labelled 1 and $n+1$ merge and turn into two real critical points, one of which escapes to infinity as $b \to 0$; similarly for $k = \frac{n}{2}$ if n is even.

If instead of following the real axis we carry out the deformation $b \to 0$ with $\mathrm{Im}(b)$ small positive, then we can avoid all the values of b for which two critical values of W_b coincide, which allows us to keep track of the manner in which $n-2$ of the critical values escape to infinity. This is represented on Figure 12 (left).

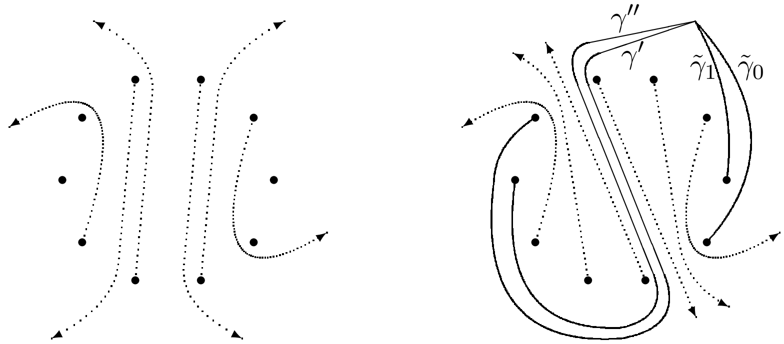


Figure 12: The deformation $b \to 0$ ($n = 8$)

Observe that the vanishing cycles at the critical points corresponding to labels in the range $1 \le k < \frac{n}{2}$ are disjoint from those at the critical points with labels in the range $\frac{n}{2} + 2 \le k \le n$. Therefore, for the purposes of determining the remaining vanishing cycles as $b \to 0$, the family of Lefschetz fibrations W_b is equivalent to one where the various critical values escape to infinity in a slightly different manner, with the critical values coming from the $\operatorname{Im} W < 0$ half-plane staying "to the left" (towards the negative real axis) of those coming from the $\operatorname{Im} W > 0$ half-plane, as pictured on Figure 12 (right).

Therefore, if we consider the category of Lagrangian vanishing cycles associated to a system of arcs containing the four arcs $\tilde{\gamma}_0, \tilde{\gamma}_1, \gamma', \gamma''$ represented on Figure 12 right, then the full subcategory singled out by the deformation $b \to 0$ is that generated by the four vanishing cycles $\tilde{L}_0, \tilde{L}_1, L', L''$ associated to these arcs. A suitable collection of arcs can be built by a sequence of sliding operations, starting from a collection $\{\tilde{\gamma}_i,\ 0 \le i \le n+1\}$ where $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are as pictured, and all the $\tilde{\gamma}_i$ remain outside of the unit disc. Identify implicitly the critical points of W_1 with those of the superpotential mirror to $\mathbb{CP}^2(n,1,1)$, and recall that sliding operations correspond to mutations. Then the left dual to the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_{n+1})$ associated to the arcs $\{\tilde{\gamma}_i\}$ is equivalent (up to some shifts) to the exceptional collection associated to the arcs $(\gamma_2, \gamma_3, \dots, \gamma_{n+1}, \gamma_0, \gamma_1)$ (using the notation of §4). Using $\mathbb{Z}/(n+2)$-equivariance, the latter is equivalent to the exceptional collection associated to the system of arcs $(\gamma_0, \dots, \gamma_{n+1})$ considered in §4.

Recall that the two exceptional collections for $\boldsymbol{D}^b(\operatorname{coh}(\mathbb{CP}^2(n,1,1)))$ presented in §2 are mutually dual (cf. Example 2.15), and that Theorem 3.3 identifies the exceptional collection associated to the arcs $(\gamma_0, \dots, \gamma_{n+1})$ with that given by Corollary 2.27. Therefore, there is an equivalence of categories which maps the exceptional collection $(\tilde{L}_0, \dots, \tilde{L}_{n+1})$ for $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ to the exceptional collection $(\mathcal{O}, \dots, \mathcal{O}(n+1))$ for $\boldsymbol{D}^b(\operatorname{coh}(\mathbb{CP}^2(n,1,1)))$.

Next, let $k = \lfloor \frac{n+3}{2} \rfloor$, so that γ' and γ'' have the same endpoints as $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ respectively. First slide $\tilde{\gamma}_{k+2}, \dots, \tilde{\gamma}_{n+1}$ to the left of $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ to obtain another system of arcs $(\tilde{\gamma}_0, \dots, \tilde{\gamma}_{k-1}, \eta_{k+2}, \dots, \eta_{n+1}, \tilde{\gamma}_k, \tilde{\gamma}_{k+1})$. Then the arcs obtained by sliding $\tilde{\gamma}_k$ and $\tilde{\gamma}_{k+1}$ to the left of $\eta_{k+2}, \dots, \eta_{n+1}$ are homotopic to γ' and γ''. This gives us a new system of arcs $(\tilde{\gamma}_0, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1}, \gamma', \gamma'', \eta_{k+2}, \dots, \eta_{n+1})$, which determines a full exceptional collection

$(\tilde{L}_0, \tilde{L}_1, \ldots, \tilde{L}_{k-1}, L', L'', \Lambda_{k+2}, \ldots, \Lambda_{n+1})$ in $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$.

By construction, the full subcategory $\langle \tilde{L}_0, \tilde{L}_1, L', L'' \rangle$ of the category $D(\mathrm{Lag}_{\mathrm{vc}}(W_1))$ is equivalent to the triangulated subcategory $\langle \mathcal{O}, \mathcal{O}(1), G_k, G_{k+1} \rangle$ of $\boldsymbol{D}^b(\mathrm{coh}(\mathbb{CP}^2(n,1,1)))$, which by Lemma 5.6 coincides with $\langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(n), \mathcal{O}(n+1) \rangle$. As seen in §2.7 this category is equivalent to the derived category of the Hirzebruch surface $\mathbb{F}_n$, which completes the proof. □

It is also possible to prove Proposition 5.5 by a direct calculation involving the monodromy of W_1, instead of using Lemma 5.6. Starting from the description of the vanishing cycles associated to the arcs γ_i in §4, one can determine first the vanishing cycles $\tilde{L}_i$ associated to $\tilde{\gamma}_i$ for all i, and then those associated to γ' and γ''. It is then possible to check that, although the vanishing cycles associated to γ' and γ'' do not quite correspond to $\tilde{L}_n$ and $\tilde{L}_{n+1}$, after sliding γ' and γ'' around each other a certain number of times one obtains two vanishing cycles that are Hamiltonian isotopic to $\tilde{L}_n$ and $\tilde{L}_{n+1}$.

6 Further remarks

6.1 Higher-dimensional weighted projective spaces Many of the arguments in §4 extend to higher-dimensional weighted projective spaces, working by induction on dimension in a manner similar to the ideas in §5 of [12]. Indeed, the mirror to the weighted projective space $\mathbb{CP}^n(a_0, \ldots, a_n)$ is the affine hypersurface $X = \{x_0^{a_0} \ldots x_n^{a_n} = 1\} \subset (\mathbb{C}^*)^{n+1}$, equipped with the superpotential $W = x_0 + \cdots + x_n$ and an exact symplectic form ω that we can choose to be invariant under the diagonal action of $\mathbb{Z}/(a_0 + \cdots + a_n)$ and anti-invariant under complex conjugation for simplicity. It is easy to check that W has $a_0 + \cdots + a_n$ critical points over X, all isolated and non-degenerate; the corresponding critical values are the roots λ_j of

$$\lambda^{a_0+\cdots+a_n} = \frac{(a_0 + \cdots + a_n)^{a_0+\cdots+a_n}}{a_0^{a_0} \ldots a_n^{a_n}}.$$

As in the two-dimensional case we use $\Sigma_0 = W^{-1}(0)$ as our reference fiber, and join it to the singular fibers of W via straight line segments $\gamma_j \subset \mathbb{C}$ joining the origin to λ_j.

In order to understand the vanishing cycles $L_j \subset \Sigma_0$, we consider as before the projection to one of the coordinate axes, for example $\pi_0 : (x_0, \ldots, x_n) \mapsto x_0$. For generic values of λ, the map $\pi_0 : \Sigma_\lambda \to \mathbb{C}^*$ defines an affine Lefschetz fibration on $\Sigma_\lambda = W^{-1}(\lambda)$, with $a_0 + \cdots + a_n$ singular fibers. These singular fibers are the preimages of the critical values of π_0 over Σ_λ, which are the roots of

$$x_0^{a_0} (\lambda - x_0)^{a_1+\cdots+a_n} = \frac{(a_1 + \cdots + a_n)^{a_1+\cdots+a_n}}{a_1^{a_1} \ldots a_n^{a_n}} \tag{6.1}$$

(compare with (4.1)). This equation acquires a double root whenever λ is one of the λ_j; the manner in which two of the roots approach each other as one moves from $\lambda = 0$ to $\lambda = \lambda_j$

along the arc γ_j defines an arc $\delta_j \subset \mathbb{C}^*$, which is a *matching path* for the Lefschetz fibration $\pi_0 : \Sigma_0 \to \mathbb{C}^*$. As in the two-dimensional case, the Lagrangian vanishing cycle $L_j \subset \Sigma_0$ is isotopic to a sphere L'_j which lies above the arc δ_j; the generic fiber of $\pi_{0|L'_j} : L'_j \to \delta_j \subset \mathbb{C}^*$ is now a Lagrangian $(n-2)$-sphere inside the fiber of π_0.

Because of the similarity between equations (6.1) and (4.1), it is easy to check that Lemma 4.2 extends almost verbatim to the higher-dimensional case, substituting a_0 for a and $a_1+\dots+a_n$ for $b+c$.

In order to determine the Floer complexes $CF^*(L_i, L_j)$, or equivalently $CF^*(L'_i, L'_j)$, we need to understand, for each point of $\delta_i \cap \delta_j$, how L'_i and L'_j intersect each other inside the corresponding fiber of π_0. Because L'_i and L'_j each arise from matching pairs of vanishing cycles of the Lefschetz fibration π_0, this can be done by studying in more detail the topology of the fiber of $\pi_0 : \Sigma_0 \to \mathbb{C}^*$ and the manner in which it degenerates as one moves from a generic value of x_0 to one of the critical values. In fact, we can use the same approach to study the vanishing cycles of $\pi_0 : \Sigma_0 \to \mathbb{C}^*$ as in the case of $W : X \to \mathbb{C}$, namely project the fiber $F_\mu = \pi_0^{-1}(\mu)$ to one of the coordinates, e.g. x_1. This gives rise to a map $\pi_1 : F_\mu \to \mathbb{C}^*$, which is again a Lefschetz fibration (whose fibers are now $(n-3)$-dimensional), with $a_1+\dots+a_n$ singular fibers corresponding to values of x_1 that solve the equation

$$\mu^{a_0} x_1^{a_1} (-\mu - x_1)^{a_2+\dots+a_n} = \frac{(a_2+\dots+a_n)^{a_2+\dots+a_n}}{a_2^{a_2}\dots a_n^{a_n}},$$

which presents a double root precisely when μ is a solution of (6.1) (for $\lambda = 0$). The process can go on similarly, considering successive restrictions to fibers and coordinate projections until we reach the easily understood case of 0-dimensional fibers; once this process is completed, it becomes possible to describe explicitly $CF^*(L'_i, L'_j)$ in terms of the available combinatorial data. The final result is the following:

Proposition 6.1 *For $i < j$, the vanishing cycles L'_i and L'_j intersect transversely, and*

$$|L'_i \cap L'_j| = \#\{I \subset \{0, \dots, n\},\ \sum_{k \in I} a_k = j - i\}.$$

Hence the Floer complex $CF^*(L'_i, L'_j)$ is naturally isomorphic to the degree $j-i$ part of the exterior algebra on $n+1$ generators of respective degrees $a_0, \dots, a_n$. Moreover, the Floer differential is trivial, i.e. $m_1 = 0$.

Instead of providing a complete proof, we simply illustrate Proposition 6.1 by considering the example of the projective space $\mathbb{CP}^3$. In that case, Σ_0 is an affine K3 surface, and $\pi_0 : \Sigma_0 \to \mathbb{C}^*$ is a fibration by affine elliptic curves, with four singular fibers. The four vanishing cycles $L'_j \subset \Sigma_0$ project to arcs $\delta_j \subset \mathbb{C}^*$ as shown on Figure 13 (left).

Using the projection π_1 to the second coordinate, we can view each of the fibers of $\pi_0 : \Sigma \to \mathbb{C}^*$ as a double cover of $\mathbb{C}^*$ branched in 3 points (Figure 13, right). To describe the monodromy of the elliptic fibration π_0, we choose a reference fiber $F_{\mu_0} = \pi_0^{-1}(\mu_0)$ for some

$\mu_0 \in \mathbb{C}^*$ close to 0 on the positive real axis. The monodromy of π_0 around the origin is the diffeomorphism of F_{μ_0} obtained by rotating the three branch points of π_1 counterclockwise by $2\pi/3$. To describe the four vanishing cycles of π_0, we join the regular value μ_0 of π_0 to each of the four critical values by considering arcs which start at μ_0, rotate clockwise around the origin from $\arg\mu = 0$ to $\arg\mu = -\frac{\pi}{4} - j\frac{\pi}{2}$ ($0 \le j \le 3$), and then go radially outwards to the corresponding critical values of π_0. The vanishing cycles $\beta_0, \dots, \beta_3$ obtained in this way are isotopic to the double lifts via $\pi_1 : F_{\mu_0} \to \mathbb{C}^*$ of the arcs shown on Figure 13 (right).

Now that the monodromy of π_0 is well-understood, it is not hard to visualize the Lagrangian spheres $L'_j \subset \Sigma_0$ lying above the arcs δ_j, and in particular their intersections. For example, $L'_0 \cap L'_1$ consists of four points, one of which is the critical point of π_0 with $\arg x_0 = \frac{3\pi}{4}$ (lying above the common end point of δ_0 and δ_1), while the three others lie in the fiber above the other point p of $\delta_0 \cap \delta_1$ (with $\arg x_0 = -\frac{\pi}{4}$), and correspond (under a suitable parallel transport operation) to the three intersections between β_1 and β_2 in F_{μ_0}. Similarly, $L'_0 \cap L'_2$ consists of 6 points (three above each point of $\delta_0 \cap \delta_2$), and so on.

Finally, we observe that there cannot be any contributions to the Floer differential m_1, for purely topological reasons. Indeed, if we consider any two intersection points $p, q \in L'_i \cap L'_j$ for some pair (i, j), and any two arcs $\gamma \subset L'_i$ and $\gamma' \subset L'_j$ joining p to q, then γ and γ' are never homotopic inside Σ_0, as easily seen by considering either $\pi_0(\gamma)$ and $\pi_0(\gamma')$ (if $\pi_0(p) \neq \pi_0(q)$), or $\pi_1(\gamma)$ and $\pi_1(\gamma')$ (if $\pi_0(p) = \pi_0(q)$).

The proof of Proposition 6.1 is essentially a careful induction on successive slices and coordinate projections, where one manages to understand the structure of the intersections between vanishing cycles by starting with a 1-dimensional slice of Σ_0 and then adding one extra dimension at a time; the main difficulty resides in setting up the induction properly and in choosing manageable notations for the many objects that appear in the proof, rather than in the actual calculations which are essentially always the same.

The next step towards understanding the category of vanishing cycles of the Lefschetz fibration $W : X \to \mathbb{C}$ would be to study the moduli spaces of pseudo-holomorphic maps from a disc with three or more marked points to Σ_0 with boundary on $\bigcup L'_j$, something which falls beyond the scope of this chapter. Nonetheless, a careful observation suggests that the main features observed in the two-dimensional case, namely the vanishing of m_k for $k \ge 3$ and the exterior algebra structure underlying m_2, should extend to the higher-dimensional case.

For example, in the case of $\mathbb{CP}^3$, we can study $m_2 : \mathrm{Hom}(L'_0, L'_1) \otimes \mathrm{Hom}(L'_1, L'_2) \to$

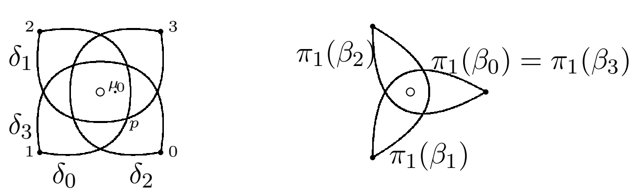


Figure 13: The case of $\mathbb{CP}^3$: images by π_0 of the vanishing cycles $L'_j \subset \Sigma_0$ of W (left), and images by π_1 of the vanishing cycles $\beta_j \subset F_{\mu_0}$ of π_0 (right)

$\mathrm{Hom}(L'_0, L'_2)$ by looking carefully at Figure 13. Let α_0 (resp. β_0) be the morphism from L'_0 to L'_1 (resp. from L'_1 to L'_2) which corresponds to their intersection at a critical point of π_0, and let $\alpha_1, \alpha_2, \alpha_3$ (resp. $\beta_1, \beta_2, \beta_3$) be the three other morphisms between these two vanishing cycles (labelling them in a consistent way with respect to the other coordinate projections). Equipping Σ_0 with an almost-complex structure for which the projection π_0 is holomorphic, pseudo-holomorphic discs project to immersed triangular regions in $\mathbb{C}^*$ with boundary on $\delta_0 \cup \delta_1 \cup \delta_2$; there are three such regions (to the upper-left, to the upper-right, and to the bottom of Figure 13 left). To start with, it is immediate from an observation of Figure 13 that $m_2(\alpha_0, \beta_0) = 0$. Next, by deforming the arcs δ_0 and δ_1 to make them lie very close to each other near their common end point, we can shrink the upper-left region to a very thin triangular sector, in which case exactly one pseudo-holomorphic map contributes to the composition of α_0 with each of $\beta_1, \beta_2, \beta_3$. It is then easy to see that composition with α_0 induces an isomorphism from $\mathrm{span}(\beta_1, \beta_2, \beta_3) \subset \mathrm{Hom}(L'_1, L'_2)$ to the subspace of $\mathrm{Hom}(L'_0, L'_2)$ spanned by the three intersections for which $\arg x_0 = \frac{\pi}{2}$. Considering the upper-right triangular region delimited by $\delta_0, \delta_1, \delta_2$ on Figure 13 left, we can conclude that the same is true for the compositions of $\alpha_1, \alpha_2, \alpha_3$ with β_0, and arguing by symmetry we can check that $m_2(\alpha_0, \beta_i) = \pm m_2(\alpha_i, \beta_0)$ for $i = 1, 2, 3$ (and, hopefully, a careful study of orientations should allow one to conclude that the signs are all negative).

By a similar argument, we can study $m_2(\alpha_i, \beta_j)$ for $1 \leq i, j \leq 3$ by shrinking the lower triangular region of Figure 13 left to a single point, which allows us to localize all the relevant intersection points and pseudo-holomorphic discs into a single fiber of π_0. The intersection pattern inside that fiber of π_0 is then described by Figure 13 right, so that things become essentially identical to the discussion carried out in the previous section for the Lefschetz fibration mirror to $\mathbb{CP}^2$ (observe the similarity between Figures 13 right and 5 right). Hence, the same argument as in the two-dimensional case shows in particular that $m_2(\alpha_i, \beta_i) = 0$ for $1 \leq i \leq 3$ and $m_2(\alpha_i, \beta_j) = \pm m_2(\alpha_j, \beta_i)$ for $1 \leq i \neq j \leq 3$.

6.2 Non-commutative deformations of $\mathbb{CP}^2$

As mentioned in the introduction, in the general case one expects the mirror to be obtained by partial (fiberwise) compactification of the Landau-Ginzburg model given by the toric mirror ansatz. While not required in the toric Fano case considered here, this fiberwise compactification allows for more freedom of deformation, since it enlarges $H^2(X, \mathbb{C})$; this sometimes makes it possible to recover more general (non-toric) noncommutative deformations of the Fano manifold. We now illustrate this by briefly discussing the case of $\mathbb{CP}^2$ (see [17] for more details and additional examples). We will show the following:

Proposition 6.2 *Non-exact symplectic deformations of the fiberwise compactified Landau-Ginzburg model $(\bar{X}, \bar{W})$ correspond to general noncommutative deformations of the projective plane.*

Moreover, we expect that there is a simple relation between the cohomology class of the symplectic form on $\bar{X}$ and the noncommutative deformation parameters for $\mathbb{CP}^2$.

Recall that a general noncommutative projective plane is defined by a graded regular algebra which is presented by 3 generators of degree one and 3 quadratic relations. All these noncommutative planes were described in the papers [10, 9], and with another point of view in [40]. It was proved in [10] that isomorphism classes of regular graded algebras of dimension 3 generated by 3 elements of degree 1 are in bijective correspondence with isomorphism classes of regular triples $\mathcal{T} = (E, \sigma, L)$, where one of the following holds:

1) $E = \mathbb{P}^2$, σ is an automorphism of $\mathbb{P}^2$, and $L = \mathcal{O}(1)$;

2) E is a divisor of degree 3 in $\mathbb{P}^2$, L is the restriction of $\mathcal{O}_{\mathbb{P}^2}(1)$, and σ is an automorphism of E such that $(\sigma^* L)^2 \cong L \otimes \sigma^{2*} L, \quad \sigma^* L \ncong L.$

The triples (and the algebras) of the first type are related to the ordinary commutative $\mathbb{P}^2$ in the sense that the category qgr of such an algebra is equivalent to the category $\mathrm{coh}(\mathbb{P}^2)$, whereas the triples of the second type are related to the nontrivial noncommutative projective planes. For example, the toric noncommutative deformations of $\mathbb{P}^2$, which were discussed above, correspond to the triples with E isomorphic to a triangle (union of three lines).

Consider now the noncommutative projective planes which correspond to triples with E isomorphic to a smooth elliptic curve. We know that sometimes the categories qgr of two different graded algebras can be equivalent. In particular, with this point of view any triple with smooth E is equivalent to a triple with the same E and such that σ is a translation by a point of E (see sect. 8 of [40]). On the other hand, according to [9](10.14), the equations defining a generic regular graded algebra, which corresponds to a triple (E, σ, L) with E a smooth elliptic curve and σ a translation, can be put into the form

$$\begin{aligned} f_1 &= cx^2 + byz + azy = 0 \\ f_2 &= axz + cy^2 + bzx = 0 \\ f_3 &= bxy + ayx + cz^2 = 0. \end{aligned}$$

This means that the DG category $\mathfrak{C}$ for these noncommutative projective planes can be described in the following way. It has three objects, say l_0, l_1, l_2, and for $i < j$ the spaces of morphisms $\mathrm{Hom}(l_i, l_j)$ are 3-dimensional, with all elements of degree $(j - i)$. There are bases $x_0, y_0, z_0 \in \mathrm{Hom}(l_0, l_1)$, $x_1, y_1, z_1 \in \mathrm{Hom}(l_1, l_2)$, $\bar{x}, \bar{y}, \bar{z} \in \mathrm{Hom}(l_0, l_2)$ in which the nontrivial compositions are given by the following formulas:

$$\begin{array}{lll} m_2(x_0, y_1) = a\bar{z}, & m_2(x_0, z_1) = b\bar{y}, & m_2(x_0, x_1) = c\bar{x}, \\ m_2(y_0, z_1) = a\bar{x}, & m_2(y_0, x_1) = b\bar{z}, & m_2(y_0, y_1) = c\bar{y}, \\ m_2(z_0, x_1) = a\bar{y}, & m_2(z_0, y_1) = b\bar{x}, & m_2(z_0, z_1) = c\bar{z}. \end{array}$$

All other compositions (except those involving identity morphisms) vanish.

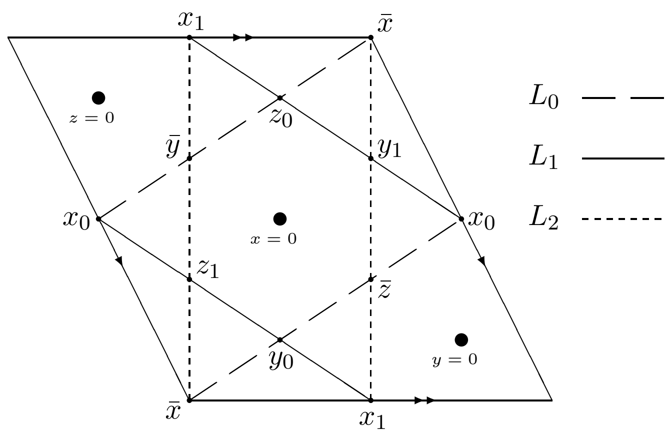


Figure 14: The vanishing cycles of the compactified mirror of $\mathbb{CP}^2$

Recall from §4 that the mirror of $\mathbb{CP}^2$ is an elliptic fibration with three singular fibers. In the affine setting, the generic fibers of $W = x + y + z$ on $X = \{xyz = 1\}$ are tori with three punctures, but it is possible to compactify X partially into an elliptic fibration $\bar{W} : \bar{X} \to \mathbb{C}$ whose fibers are closed curves; unlike what happens in more complicated (non-toric) examples, this does not introduce any extra critical points.

The generic fiber of $\bar{W}$ and the three vanishing cycles are as represented on Figure 14 (compare with Figure 5 right, which represents the images by π_x of the same vanishing cycles; see also Figure 2 of [215]); the bold dots represent the intersections of the fiber with the compactification divisor.

While it is easy to see that m_k remains trivial for $k \neq 2$, the compactification modifies the product m_2 in the category $\mathrm{Lag}_{\mathrm{vc}}(\bar{W}, \{\gamma_i\})$ by introducing an infinite number of immersed triangular regions with boundary in $L_0 \cup L_1 \cup L_2$. This induces a deformation of the product structure, and the uncompactified case considered in §4 now arises as a limiting situation in which the areas of the hexagonal regions containing the intersections with the compactification divisor tend to infinity.

For example, the product $m_2(x_0, y_1)$ remains a multiple of $\bar{z}$, but the relevant coefficient is now a sum of infinitely many contributions, corresponding to immersed triangles in which the edge joining x_0 to y_1 is an arbitrary immersed arc between these two points inside L_1. The convergence of the series $\sum_i \pm \exp(-2\pi \,\mathrm{area}(T_i))$ follows directly from the fact that the area grows quadratically with the number of times that the x_0y_1 edge wraps around L_1. Similarly, $m_2(y_0, x_1)$ is a multiple of $\bar{z}$ as in the uncompactified case, but with a coefficient now given by the sum of an infinite series of contributions; and similarly for $m_2(y_0, z_1)$ and $m_2(y_1, z_0)$, which remain multiples of $\bar{x}$, and for $m_2(z_0, x_1)$ and $m_2(x_0, z_1)$, which are proportional to $\bar{y}$.

The important new feature of the compactified Landau-Ginzburg model is that $m_2(x_0, x_1)$ is now a multiple of $\bar{x}$ (with a coefficient that may be zero or non-zero depending on the choice of the cohomology class of the symplectic form); since there are again infinitely many immersed triangular regions with vertices $x_0, x_1, \bar{x}$ (the smallest two of which are embedded and easily visible on Figure 14), the relevant coefficient is the sum of an infinite series.

Observe that the two embedded triangles are to be counted with opposite signs (the differences in orientations at the two vertices of degree 1 cancel each other, while the non-triviality of the spin structures and the complementarity of the sides result in a total of three sign changes, see §4.6); hence, in the "symmetric" case where the six triangular regions delimited by $L_0 \cup L_1 \cup L_2$ have equal areas, these two contributions cancel each other. The same is true of the other (immersed) triangles with vertices $x_0, x_1, \bar{x}$, which arise in similarly cancelling pairs. Hence, in the symmetric situation, we end up having $m_2(x_0, x_1) = 0$ as in §4; however in the general case $m_2(x_0, x_1)$ can still be a non-zero multiple of $\bar{x}$. There are similar statements for $m_2(y_0, y_1)$ and $m_2(z_0, z_1)$, which are multiples of $\bar{y}$ and $\bar{z}$ respectively (and also vanish in the symmetric case).

6.3 HMS for products

Let $W_1 : X_1 \to \mathbb{C}$ and $W_2 : X_2 \to \mathbb{C}$ be two Lefschetz fibrations, with critical points respectively p_i, $1 \le i \le r$ and q_j, $1 \le j \le s$, and associated critical values $\lambda_i = W_1(p_i)$ and $\mu_j = W_2(q_j)$. Then $W = W_1 + W_2 : X_1 \times X_2 \to \mathbb{C}$ is a Lefschetz fibration with rs critical points (p_i, q_j), and associated critical values $W(p_i, q_j) = \lambda_i + \mu_j$ (we will assume that these are pairwise distinct and nonzero).

For all $t \in \mathbb{C}$, the fiber $M_t = W^{-1}(t) \subset X_1 \times X_2$ can be viewed as the total space of a fibration $\phi_t : M_t \to \mathbb{C}$ given by $\phi_t(p, q) = W_1(p)$, with fiber $\phi_t^{-1}(\lambda) = W_1^{-1}(\lambda) \times W_2^{-1}(t-\lambda)$. The $r+s$ critical values of ϕ_t are $\lambda_1, \dots, \lambda_r$ and $t - \mu_1, \dots, t - \mu_s$. If t varies along an arc γ joining 0 to $\lambda_i + \mu_j$, the critical value $t - \mu_j$ of ϕ_t converges to the critical value λ_i by following the arc $\gamma - \mu_j$. Hence, the vanishing cycle $L_\gamma \subset M_0$ associated to the arc γ is a *fibered* Lagrangian sphere, mapped by ϕ_0 to the arc $\tilde{\gamma} = \gamma - \mu_j$ joining the critical values $-\mu_j$ and λ_i of ϕ_0.

More precisely, the fiber of ϕ_0 above an interior point of $\tilde{\gamma}$ is symplectomorphic to the product $\Sigma_1 \times \Sigma_2$ of the smooth fibers of W_1 and W_2, and its intersection with the vanishing cycle L_γ is a product of two Lagrangian spheres $S_i \times T_j \subset \Sigma_1 \times \Sigma_2$, where S_i and T_j correspond to vanishing cycles of W_1 and W_2 associated to the critical values λ_i and μ_j respectively. Above the end points of $\tilde{\gamma}$, the product $S_i \times T_j$ collapses to either $\{p_i\} \times T_j$ (above $\tilde{\gamma}(1) = \lambda_i$) or $S_i \times \{q_j\}$ (above $\tilde{\gamma}(0) = -\mu_j$). Denoting by n_i the complex dimension of X_i, a model for the topology of the restriction of ϕ_0 to L_γ is given by the map $\phi : S^{n_1+n_2-1} \to [0, 1]$ defined over the unit sphere in $\mathbb{R}^{n_1+n_2}$ by $(x_1, \dots, x_{n_1}, x_{n_1+1}, \dots, x_{n_1+n_2}) \mapsto x_1^2 + \dots + x_{n_1}^2$.

Up to a suitable isotopy we can assume that the critical values λ_i all have the same imaginary part, and $0 < \mathrm{Im}(\lambda_i) \ll \mathrm{Re}(\lambda_1) \ll \dots \ll \mathrm{Re}(\lambda_r)$ (so that line segments joining the origin to λ_i form an ordered collection that can be used to define $\mathrm{Lag}_{\mathrm{vc}}(W_1)$). Similarly, assume that μ_j all have the same real part, and $0 < \mathrm{Re}(\mu_j) \ll \mathrm{Im}(\mu_s) \ll \dots \ll \mathrm{Im}(\mu_1)$. Then there is a natural way to choose arcs γ_{ij}, $1 \le i \le r$, $1 \le j \le s$, joining the origin to $\lambda_i + \mu_j$, with both real and imaginary parts monotonically increasing, in such a way that the lexicographic ordering of the labels ij coincides with the clockwise ordering of the arcs γ_{ij} around the origin.

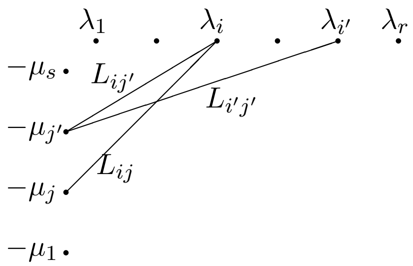


Figure 15: The vanishing cycles of $W = W_1 + W_2 : X_1 \times X_2 \to \mathbb{C}$

The arcs $\tilde{\gamma}_{ij}$ to which the vanishing cycles $L_{ij} \subset M_0$ project under ϕ_0 are then as shown in Figure 15.

In this situation, we have the following result, which gives supporting evidence for Conjecture 1.3:

Proposition 6.3 *The vanishing cycles L_{ij} of W are in one-to-one correspondence with pairs of vanishing cycles (S_i, T_j) of W_1 and W_2, and*

$$\mathrm{Hom}_{\mathrm{Lag_{vc}}(W_1+W_2)}(L_{ij}, L_{i'j'}) \simeq \mathrm{Hom}_{\mathrm{Lag_{vc}}(W_1)}(S_i, S_{i'}) \otimes \mathrm{Hom}_{\mathrm{Lag_{vc}}(W_2)}(T_j, T_{j'}).$$

Proof. [Sketch of proof] For $i < i'$ and $j < j'$, the intersections between L_{ij} and $L_{i'j'}$ localize into a single smooth fiber of ϕ_0, whose intersection with L_{ij} is $S_i \times T_j$ while the intersection with $L_{i'j'}$ is $S_{i'} \times T_{j'}$ (up to isotopy in general, but by suitably modifying the fibrations W_1 and W_2 to make them trivial over large open subsets and by choosing the arcs γ_{ij} carefully we can make this hold strictly). Therefore, in this case intersections points between L_{ij} and $L_{i'j'}$ correspond to pairs of intersections between S_i and $S_{i'}$ and between T_j and $T_{j'}$, so $\mathrm{Hom}(L_{ij}, L_{i'j'}) \simeq \mathrm{Hom}(S_i, S_{i'}) \otimes \mathrm{Hom}(T_j, T_{j'})$. After choosing suitable trivializations of the canonical bundles (so that the phase of L_{ij} at an intersection point can easily be compared with the sums of the phases of S_i and T_j), it becomes easy to check that this isomorphism is compatible with gradings.

When $i = i'$ and $j < j'$ the intersections between L_{ij} and $L_{ij'}$ lie in a singular fiber of ϕ_0 (of the form $W_1^{-1}(\lambda_i) \times \Sigma_2$), inside which L_{ij} and $L_{ij'}$ identify with $\{p_i\} \times S_j$ and $\{p_i\} \times S_{j'}$ respectively (see Figure 15); recalling that $\mathrm{Hom}(S_i, S_i) = \mathbb{C}$ by definition, we obtain the desired formula. Similarly for $L_{ij} \cap L_{i'j}$ when $i < i'$ and $j = j'$. Finally, the case $i = i'$ and $j = j'$ is trivial.

In all other cases, there are no morphisms from L_{ij} to $L_{i'j'}$. Indeed, if either $i > i'$ or $i = i'$ and $j > j'$ then (i, j) follows (i', j') in the lexicographic ordering, so there are no morphisms from L_{ij} to $L_{i'j'}$. The only remaining case is when $i < i'$ and $j > j'$; in that case the triviality of $\mathrm{Hom}(L_{ij}, L_{i'j'})$ follows from the fact $L_{ij} \cap L_{i'j'} = \emptyset$ (because the projections $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}_{i'j'}$ are disjoint). $\square$

In order to prove Conjecture 1.3, one needs to achieve a better understanding of pseudo-holomorphic discs in M_0 with boundary in $\bigcup L_{ij}$. This is most easily done in the case of low-dimensional examples such as the mirror to $\mathbb{CP}^1 \times \mathbb{CP}^1$ (already studied in a different manner in §5.1), or more generally any situation where the fibers are 0-dimensional, because the description then becomes purely combinatorial. Another piece of supporting evidence is the following

Lemma 6.4 *When $i < i' < i''$ and $j < j' < j''$, the composition $m_2 : \mathrm{Hom}(L_{ij}, L_{i'j'}) \otimes \mathrm{Hom}(L_{i'j'}, L_{i''j''}) \to \mathrm{Hom}(L_{ij}, L_{i''j''})$ is expressed (up to homotopy) in terms of compositions in $\mathrm{Lag}_{\mathrm{vc}}(W_1)$ and $\mathrm{Lag}_{\mathrm{vc}}(W_2)$ by the formula $m_2(s \otimes t, s' \otimes t') = m_2(s, s') \otimes m_2(t, t')$.*

Proof. [Sketch of proof] After deforming the fibrations W_1 and W_2 and the arcs γ_{ij}, $\gamma_{i'j'}$, $\gamma_{i''j''}$ (hence "up to homotopy" in the statement), we can assume that all intersections between L_{ij}, $L_{i'j'}$ and $L_{i''j''}$ occur in a portion of M_0 where the fibration ϕ_0 is trivial. Choose an almost-complex structure which is locally a product in $\phi_0^{-1}(U) \simeq U \times \Sigma_1 \times \Sigma_2 \subset M_0$. Then every pseudo-holomorphic disc with boundary in $L_{ij} \cup L_{i'j'} \cup L_{i''j''}$ contributing to m_2 projects under ϕ_0 to the same triangular region in U (the unique triangular region with boundary in $\tilde{\gamma}_{ij} \cup \tilde{\gamma}_{i'j'} \cup \tilde{\gamma}_{i''j''}$, which we can assume to be arbitrarily small), while the projections to the factors Σ_1 and Σ_2 are exactly those pseudo-holomorphic discs which contribute to $m_2 : \mathrm{Hom}(S_i, S_{i'}) \otimes \mathrm{Hom}(S_{i'}, S_{i''}) \to \mathrm{Hom}(S_i, S_{i''})$ and $m_2 : \mathrm{Hom}(T_j, T_{j'}) \otimes \mathrm{Hom}(T_{j'}, T_{j''}) \to \mathrm{Hom}(T_j, T_{j''})$. □

Other parts of Conjecture 1.3 are also accessible to similar methods. However, the general situation is quite subtle, partly because the definition of higher compositions in a product of two A_∞-categories is more complicated than one might think, but also because one has to deal with more complicated moduli spaces of pseudo-holomorphic discs.

Mirror symmetry for Del Pezzo surfaces: Vanishing cycles and coherent sheaves.

1 Mirrors of Fano varieties

The phenomenon of mirror symmetry has been studied extensively in the case of Calabi-Yau manifolds (where it corresponds to a duality between $N = 2$ superconformal sigma models), but also manifests itself in more general situations. For example, a sigma model whose target space is a Fano variety is expected to admit a mirror, not necessarily among sigma models, but in the more general context of *Landau-Ginzburg models*.

For us, a Landau-Ginzburg model is simply a pair (M, W), where M is a non-compact manifold (carrying a symplectic structure and/or a complex structure), and W is a complex-valued function on M called *superpotential*. The general philosophy is that, when a Landau-Ginzburg model (M, W) is mirror to a Fano variety X, the complex (resp. symplectic) geometry of X corresponds to the symplectic (resp. complex) geometry of the *critical points* of W.

We place ourselves in the context of *homological mirror symmetry*, where mirror symmetry is interpreted as an equivalence between certain triangulated categories naturally associated to a mirror pair [151]. In our case, B-branes on a Fano variety are described by its derived category of coherent sheaves, and under mirror symmetry they correspond to the A-branes of a mirror Landau-Ginzburg model. These A-branes are described by a suitable analogue of the Fukaya category for a symplectic fibration, namely the *derived category of Lagrangian vanishing cycles*. A rigorous definition of this category has been proposed by Seidel [214] in the case where the critical points of the superpotential are isolated and non-degenerate, following ideas of Kontsevich [152] and Hori, Iqbal, Vafa [120].

Therefore, for a Fano variety X and a mirror Landau-Ginzburg model $W : M \to \mathbb{C}$, the homological mirror symmetry conjecture can be formulated as follows:

Conjecture 1.1 *The derived category of Lagrangian vanishing cycles* $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$ *is equivalent to the derived category of coherent sheaves* $\boldsymbol{D}^b(\mathrm{coh}(X))$.

Remark 1.2 *Homological mirror symmetry also predicts another equivalence of derived categories. Namely, viewing now X as a symplectic manifold and M as a complex manifold, the derived category of B-branes of the Landau-Ginzburg model $W : M \to \mathbb{C}$, which was defined algebraically in [134, 194] following ideas of Kontsevich, should be equivalent to the derived Fukaya category of X. This aspect of mirror symmetry will be addressed in a further paper; for now, we focus exclusively on Conjecture* 1.1.

One of the first examples for which Conjecture 1.1 has been verified is that of $\mathbb{CP}^2$ and its mirror Landau-Ginzburg model which is the elliptic fibration with three singular fibers determined by the superpotential $W_0 = x+y+1/xy$ on $(\mathbb{C}^*)^2$ (or rather a fiberwise compactification of this fibration), see [215], [16]. Other examples of surfaces for which the derived category of coherent sheaves has been shown to be equivalent to the derived category of Lagrangian vanishing cycles of a mirror Landau-Ginzburg model include weighted projective planes, Hirzebruch surfaces [16], and toric blow-ups of $\mathbb{CP}^2$ [243]. For all these examples, the toric structure plays a crucial role in determining the geometry of the mirror Landau-Ginzburg model.

Our goal in this chapter is to consider the case of a Del Pezzo surface X_K obtained by blowing up $\mathbb{CP}^2$ at a set K of $k \leq 8$ points (this is never toric as soon as $k \geq 4$). Our proposal is that a mirror of X_K can be constructed in the following manner. Observe that the elliptic fibration with three singular fibers determined by the superpotential $W_0 = x + y + 1/xy$ on $(\mathbb{C}^*)^2$ (i.e., the mirror of $\mathbb{CP}^2$) admits a natural compactification to an elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$ in which the fiber above infinity consists of nine rational components (see §3.1 for details). Consider a deformation of $\overline{W_0}$ to another elliptic fibration $\overline{W_k} : \overline{M} \to \mathbb{CP}^1$, such that k of the 9 critical points in the fiber $\overline{W_0}^{-1}(\infty)$ are displaced towards finite values of the superpotential. Let

$$M_k = \overline{M} \setminus \overline{W_k}^{-1}(\infty),$$

and denote by $W_k : M_k \to \mathbb{C}$ the restriction of $\overline{W_k}$ to M_k. In the generic case, W_k is an elliptic fibration with $k+3$ nodal fibers, while $\overline{W_k}^{-1}(\infty)$ is a singular fiber with $9-k$ rational components. Although we will focus on the Del Pezzo case, this construction also provides a mirror in some borderline situations. For example, it can be applied without modification to the case where $\mathbb{CP}^2$ is blown up at $k = 9$ points which lie at the intersection of two elliptic curves (the fiber $\overline{W_k}^{-1}(\infty)$ is then a smooth elliptic curve).

There are two aspects to the geometry of M_k. Viewing M_k as a *complex* manifold (a Zariski open subset of a rational elliptic surface), its complex structure is closely related to the set of critical values of W_k, which has to be chosen in accordance with a given symplectic structure on X_K. A generic choice of the symplectic structure on X_K (for which there are no homologically nontrivial Lagrangian submanifolds) determines a complex structure on M_k

for which the $k+3$ critical values of W_k are all distinct (leading to a very simple category of B-branes). In the opposite situation, which we will not consider here, if we equip X_K with a symplectic form for which there are homologically nontrivial Lagrangian submanifolds, then some of the critical values of W_k become equal, and the topology of the singular fibers may become more complicated.

The *symplectic* geometry of M_k is more important to us. Since $H^2(M_k, \mathbb{C}) \simeq \mathbb{C}^{k+2}$, the symplectic form ω on M_k, or rather its complexified variant $B + i\omega$, depends on $k+2$ moduli parameters. As we will see in §4, these parameters completely determine the derived category of Lagrangian vanishing cycles of W_k; the actual positions of the critical values are of no importance, as long as the critical points of W_k remain isolated and non-degenerate (see Lemma 3.2). This means that we shall not concern ourselves with the complex structure on M_k; in fact, a compatible almost-complex structure is sufficient for our purposes, which makes the problem of deforming the elliptic fibration $\overline{W_0}$ in the prescribed manner a non-issue.

To summarize, we have:

Construction 1.3 *Given a Del Pezzo surface X_K obtained by blowing up $\mathbb{CP}^2$ at k points, the mirror Landau-Ginzburg model is an elliptic fibration $W_k : M_k \to \mathbb{C}$ with $k+3$ nodal singular fibers, which has the following properties:*

(i) *the fibration W_k compactifies to an elliptic fibration $\overline{W_k}$ over $\mathbb{CP}^1$ in which the fiber above infinity consists of $9-k$ rational components;*

(ii) *the compactified fibration $\overline{W_k}$ can be obtained as a deformation of the elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$ which compactifies the mirror to $\mathbb{CP}^2$.*

Moreover, the manifold M_k is equipped with a symplectic form ω and a B-field B, whose cohomology classes are determined by the set of points K in an explicit manner as discussed in §5.

Our main result is the following:

Theorem 1.4 *Given any Del Pezzo surface X_K obtained by blowing up $\mathbb{CP}^2$ at k points, there exists a complexified symplectic form $B+i\omega$ on M_k for which $\boldsymbol{D}^b(\mathrm{coh}(X_K)) \cong \boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$.*

The *mirror map*, i.e. the relation between the cohomology class $[B+i\omega] \in H^2(M_k, \mathbb{C})$ and the positions of the blown up points in $\mathbb{CP}^2$, can be described explicitly (see Proposition 5.1).

On the other hand, not every choice of $[B+i\omega] \in H^2(M_k, \mathbb{C})$ yields a category equivalent to the derived category of coherent sheaves on a Del Pezzo surface. There are two reasons for this. First, certain specific choices of $[B+i\omega]$ correspond to deformations of the complex structure of X_K for which the surface contains a -2-curve, which causes the anticanonical class to no longer be ample. There are many ways in which this can occur, but perhaps the simplest one corresponds to the case where a same point is blown up twice, i.e. we first blow up $\mathbb{CP}^2$ at $k-1$ generic points and then blow up a point on one of the exceptional curves. We then say that X_K is obtained from $\mathbb{CP}^2$ by blowing up k points, two of which are infinitely close, and call this a "simple degeneration" of a Del Pezzo surface. In this case again we have:

Theorem 1.5 *If X_K is a blowup of $\mathbb{CP}^2$ at k points, two of which are infinitely close, and a simple degeneration of a Del Pezzo surface, then there exists a complexified symplectic form $B + i\omega$ on M_k for which $\boldsymbol{D}^b(\mathrm{coh}(X_K)) \cong \boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$.*

More importantly, deformations of the symplectic structure on M_k need not always correspond to deformations of the complex structure on X_K (observe that $H^2(M_k, \mathbb{C})$ is larger than $H^1(X_K, TX_K)$). The additional deformation parameters on the mirror side can however be interpreted in terms of *noncommutative deformations* of the Del Pezzo surface X_K (i.e., deformations of the derived category $\boldsymbol{D}^b(\mathrm{coh}(X_K))$). In this context we have the following theorem, which generalizes the result obtained in [16] for the case of $\mathbb{CP}^2$:

Theorem 1.6 *Given any noncommutative deformation of the Del Pezzo surface X_K, there exists a complexified symplectic form $B + i\omega$ on M_k for which the deformed derived category $\boldsymbol{D}^b(\mathrm{coh}(X_{K,\mu}))$ is equivalent to $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$. Conversely, for a generic choice of $[B + i\omega] \in H^2(M_k, \mathbb{C})$, the derived category of Lagrangian vanishing cycles $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ is equivalent to the derived category of coherent sheaves of a noncommutative deformation of a Del Pezzo surface.*

The mirror map is again explicit, i.e. the parameters which determine the noncommutative Del Pezzo surface can be read off in a simple manner from the cohomology class $[B + i\omega]$.

Remark 1.7 *The key point in the determination of the mirror map is that the parameters which determine the composition tensors in $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ can be expressed explicitly in terms of the cohomology class $[B + i\omega]$ (see §4.3). A remarkable feature of these formulas is that they can be interpreted in terms of* theta functions *on a certain elliptic curve (see §4.5). As a consequence, our description of the mirror map also involves theta functions (see §5).*

The rest of the chapter is organized as follows. In §2 we describe the bounded derived categories of coherent sheaves on Del Pezzo surfaces, their simple degenerations, and their noncommutative deformations. In §3 we describe the topology of the elliptic fibration M_k and its vanishing cycles. In §4.1 we recall Seidel's definition of the derived category of Lagrangian vanishing cycles of a symplectic fibration, and in the rest of §4 we determine $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$. Finally in §5 we compare the two viewpoints, describe the mirror map, and prove the main theorems.

2 Derived categories of coherent sheaves on blowups of $\mathbb{CP}^2$

The purpose of this section is to give a description of the bounded derived categories of coherent sheaves on Del Pezzo surfaces, their simple degenerations, and their noncommutative deformations. We always work over the field of complex numbers $\mathbb{C}$.

2.1 Del Pezzo surfaces and blowups of the projective plane at distinct points

Definition 2.1 *A smooth projective surface S is called a Del Pezzo surface if the anticanonical sheaf $\mathcal{O}_S(-K_S)$ is ample (i.e., a Del Pezzo surface is a Fano variety of dimension 2).*

The Kodaira vanishing theorem and Serre duality give us immediately that for any Del Pezzo surface

$$\begin{aligned} &H^1(S,\mathcal{O}(-mK_S)) = 0 \qquad \text{for all } m \in \mathbb{Z},\\ &H^2(S,\mathcal{O}(-mK_S)) = 0 \qquad \text{for all } m \geq 0,\\ &H^2(S,\mathcal{O}(-mK_S)) = H^0(S,\mathcal{O}((m+1)K_S)) \qquad \text{for all } m \in \mathbb{Z}. \end{aligned}$$

In particular, we obtain that $H^1(S,\mathcal{O}_S) \cong H^2(S,\mathcal{O}_S) = 0$, and $H^0(S,\mathcal{O}(mK_S)) = 0$ for all $m > 0$. By the Castelnuovo-Enriques criterion any Del Pezzo surface is rational.

Let S be a Del Pezzo surface. The integer K_S^2 is called the *degree* of S and will be denoted by d. The Noether formula gives a relation between the degree and the rank of the Picard group of a Del Pezzo surface: $d = K_S^2 = 10 - \operatorname{rk}\operatorname{Pic} S \leq 9$.

We can also introduce another integer number which is called the *index* of S. This is the maximal $r > 0$ such that $\mathcal{O}(-K_S) = \mathcal{O}(rH)$ for some divisor H. The inequality $d \leq 9$ implies that $r \leq 3$.

Now recall the classification of Del Pezzo surfaces.

If $r = 3$, then $S \cong \mathbb{P}^2$ is the projective plane and $d = 9$. If $r = 2$, then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the quadric and $d = 8$. The other Del Pezzo surfaces are not minimal and can be obtained by blowing up the projective plane $\mathbb{P}^2$. More precisely, if S is a Del Pezzo surface of index $r = 1$, then it has degree $1 \leq d \leq 8$ and S is a blowup of the projective plane $\mathbb{P}^2$ at $k = 9 - d$ distinct points. The ampleness of the anticanonical class requires that in this set no three points lie on a line, and no six points lie on a conic; moreover, if $k = 8$ the eight points are not allowed to lie on an irreducible cubic which has a double point at one of these points. Conversely, any surface which is a blowup of the projective plane at a set of $k \leq 8$ different points satisfying these constraints is a Del Pezzo surface of degree $d = 9 - k$. All these facts are well-known and can be found in any textbook on surfaces (see e.g. [99]).

Denote by $\boldsymbol{D}^b(\operatorname{coh}(S))$ the bounded derived category of coherent sheaves on S. It is known that the bounded derived category of coherent sheaves on any Del Pezzo surface has a full exceptional collection, which makes it possible to establish an equivalence between the category $\boldsymbol{D}^b(\operatorname{coh}(S))$ and the bounded derived category of finitely generated modules over the algebra of the exceptional collection ([193], see also [160]). This is a particular case of a more general statement about derived categories of blowups.

First, recall the notion of exceptional collection.

Definition 2.2 *An object* E *of a* $\mathbb{C}$*-linear triangulated category* $\mathcal{D}$ *is said to be* exceptional *if* $\mathrm{Hom}(E, E[k]) = 0$ *for all* $k \neq 0$*, and* $\mathrm{Hom}(E, E) = \mathbb{C}$*. An ordered set of exceptional objects* $\sigma = (E_0, \dots E_n)$ *is called an* exceptional collection *if* $\mathrm{Hom}(E_j, E_i[k]) = 0$ *for* $j > i$ *and all* k*. The exceptional collection* σ *is said to be* strong *if it satisfies the additional condition* $\mathrm{Hom}(E_j, E_i[k]) = 0$ *for all* i, j *and for* $k \neq 0$*.*

Definition 2.3 *An exceptional collection* $(E_0, \dots, E_n)$ *in a category* $\mathcal{D}$ *is called* full *if it generates the category* $\mathcal{D}$*, i.e. the minimal triangulated subcategory of* $\mathcal{D}$ *containing all objects* E_i *coincides with* $\mathcal{D}$*. In this case we say that* $\mathcal{D}$ *has a semiorthogonal decomposition of the form*

$$\mathcal{D} = \langle E_0, \dots, E_n \rangle .$$

The most studied example of an exceptional collection is the sequence of invertible sheaves $\langle \mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$ on the projective space $\mathbb{P}^n$ ([27]). In particular, this exceptional collection on the projective plane $\mathbb{P}^2$ has length 3.

Definition 2.4 *The algebra of a strong exceptional collection* $\sigma = (E_0, \dots, E_n)$ *is the algebra of endomorphisms* $B(\sigma) = \mathrm{End}(\mathcal{E})$ *of the object* $\mathcal{E} = \bigoplus_{i=0}^{n} E_i$*.*

Assume that the triangulated category $\mathcal{D}$ has a full strong exceptional collection $(E_0, \dots, E_n)$ and B is the corresponding algebra. Denote by $\mathrm{mod}\, B$ the category of finitely generated right modules over B. There is a theorem according to which if $\mathcal{D}$ is an *enhanced triangulated category* in the sense of Bondal and Kapranov [37], then it is equivalent to the bounded derived category $\boldsymbol{D}^b(\mathrm{mod}\, B)$. This equivalence is given by the functor $\mathbf{R}\mathrm{Hom}(\mathcal{E}, -)$ (see [37]).

For example, if $\mathcal{D} \cong \boldsymbol{D}^b(\mathrm{coh}(X))$ is the bounded derived category of coherent sheaves on a projective variety X, then it is enhanced. Actually, the category of quasi-coherent sheaves Qcoh has enough injectives, and $\boldsymbol{D}^b(\mathrm{coh}(X))$ is equivalent to the full subcategory $\boldsymbol{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(X)) \subset \boldsymbol{D}^b(\mathrm{Qcoh}(X))$ whose objects are complexes with cohomologies in $\mathrm{coh}(X)$.

Assume that X is smooth and $(E_0, \dots, E_n)$ is a strong exceptional collection on X. The object $\mathcal{E} = \bigoplus_{i=0}^{n} E_i$ defines the derived functor

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, -) : \boldsymbol{D}^+(\mathrm{Qcoh}(X)) \longrightarrow \boldsymbol{D}^+(\mathrm{Mod}\, B),$$

where $\mathrm{Mod}\, B$ is the category of all right modules over B. Moreover, the functor $\mathbf{R}\mathrm{Hom}(\mathcal{E}, -)$ sends objects of $\boldsymbol{D}^b_{\mathrm{coh}}(\mathrm{Qcoh}(X))$ to objects of the subcategory $\boldsymbol{D}^b_{\mathrm{mod}}(\mathrm{Mod}\, B)$, which is also equivalent to $\boldsymbol{D}^b(\mathrm{mod}\, B)$. This gives us a functor

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, -) : \boldsymbol{D}^b(\mathrm{coh}(X)) \longrightarrow \boldsymbol{D}^b(\mathrm{mod}\, B).$$

The objects E_i for $i = 0, \dots, n$ are mapped to the projective modules $P_i = \mathrm{Hom}(\mathcal{E}, E_i)$. Moreover, $B = \bigoplus_{i=0}^{n} P_i$. The algebra B has $n+1$ primitive idempotents $e_i,\ i = 0, \dots, n$ such that $1_B = e_0 + \cdots + e_n$ and $e_i e_j = 0$ if $i \neq j$. The right projective modules P_i coincide with $e_i B$. The morphisms between them can be easily described since

$$\mathrm{Hom}(P_i, P_j) = \mathrm{Hom}(e_i B, e_j B) \cong e_j B e_i \cong \mathrm{Hom}(E_i, E_j).$$

This yields an equivalence between the triangulated subcategory of $\boldsymbol{D}^b(\mathrm{coh}(X))$ generated by the collection $\langle E_0, \dots, E_n \rangle$ and the derived category $\boldsymbol{D}^b(\mathrm{mod}\, B)$. Here we use the fact that the algebra B has a finite global dimension and any right (and left) module M has a finite projective resolution consisting of the projective modules P_i with $i = 0, \dots, n$. Finally, if the collection $(E_0, \dots, E_n)$ is full, then we obtain an equivalence between $\boldsymbol{D}^b(\mathrm{coh}(X))$ and $\boldsymbol{D}^b(\mathrm{mod}\, B)$.

Sometimes it is useful to represent the algebra B as a category $\mathfrak{B}$ which has $n+1$ objects, say $v_0, \dots, v_n$, and morphisms defined by the rule $\mathrm{Hom}(v_i, v_j) \cong \mathrm{Hom}(E_i, E_j)$ with the natural composition law. Thus $B = \bigoplus_{0 \le i,j \le n} \mathrm{Hom}(v_i, v_j)$.

Theorem 2.5 [193, 160] *Let $\pi : X_K \to \mathbb{P}^2$ be a blowup of the projective plane $\mathbb{P}^2$ at a set $K = \{p_1, \dots, p_k\}$ of any k distinct points, and let $l_1, \dots, l_k$ be the exceptional curves of the blowup. Let (F_0, F_1, F_2) be a full strong exceptional collection of vector bundles on $\mathbb{P}^2$. Then the sequence*

$$(\pi^* F_0, \pi^* F_1, \pi^* F_2, \mathcal{O}_{l_1}, \dots, \mathcal{O}_{l_k}) \,, \tag{2.1}$$

where the $\mathcal{O}_{l_i}$ are the structure sheaves of the exceptional -1-curves l_i, is a full strong exceptional collection on X_K. Moreover, the sheaves $\mathcal{O}_{l_i}$ and $\mathcal{O}_{l_j}$ are mutually orthogonal for all $i \neq j$.

In particular, there is an equivalence

$$\boldsymbol{D}^b(\mathrm{coh}(X_K)) \cong \boldsymbol{D}^b(\mathrm{mod}\, B_K), \tag{2.2}$$

*where B_K is the algebra of homomorphisms of the exceptional collection (*2.1*).*

There are no restrictions on the set of points $K = \{p_1, \dots, p_k\}$ in this theorem and, in particular, we do not need to assume that X_K is a Del Pezzo surface.

We can easily describe the space of morphisms from $\pi^* F_i$ to the sheaf $\mathcal{O}_{l_j}$, since it is naturally identified with the space that is dual to the fiber of the vector bundle F_i at the point $p_j \in \mathbb{P}^2$, i.e.

$$\mathrm{Hom}_{X_K}(\pi^* F_i, \mathcal{O}_{l_j}) \cong \mathrm{Hom}_{\mathbb{P}^2}(F_i, \mathcal{O}_{p_j}).$$

There are various standard exceptional collections on the projective plane. One of them is the collection of line bundles $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2))$, another is the collection $(\mathcal{O}, \mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{O}(1))$, where $\mathcal{T}_{\mathbb{P}^2}$ is the tangent bundle on $\mathbb{P}^2$. The latter choice is the most convenient for us. It is easy to see that

$$\mathrm{Hom}(\mathcal{O}, \mathcal{T}_{\mathbb{P}^2}(-1)) \cong \mathrm{Hom}(\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{O}(1)) \cong V \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) \cong \Lambda^2 V \cong V^*,$$

where V is the 3-dimensional vector space whose projectivization $\mathbb{P}(V)$ is the given projective plane $\mathbb{P}^2$.

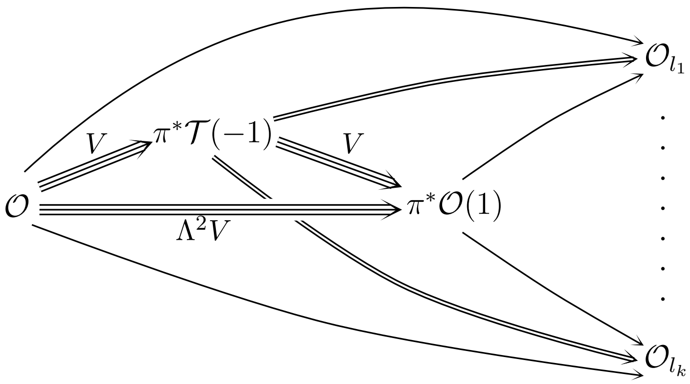


Figure 1: The quiver $\mathfrak{B}_K$ for a blowup of $\mathbb{P}^2$ at k distinct points.

Let us consider the blowup X_K of the projective plane $\mathbb{P}(V)$ at a set $K = \{p_1, \ldots, p_k\}$ of k distinct points, and the exceptional collection

$$\sigma = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_1}, \ldots, \mathcal{O}_{l_k}). \tag{2.3}$$

Let $\mathfrak{B}_K(\sigma)$ be the category of homomorphisms of this exceptional collection (see Figure 1). Then the surface X_K can be recovered from the category $\mathfrak{B}_K(\sigma)$ by means of the following procedure.

Denote by S_j the 2-dimensional space of homomorphisms from $\pi^*\mathcal{T}_{\mathbb{P}^2}(-1)$ to $\mathcal{O}_{l_j}$ and denote by U_j the 1-dimensional space of homomorphisms from $\mathcal{O}(1)$ to $\mathcal{O}_{l_j}$. The composition law in the category $\mathfrak{B}_K(\sigma)$ gives a map from $U_j \otimes V$ to S_j. The kernel of this map is a 1-dimensional subspace $V_j \subset V$, which defines a point $p_j \in \mathbb{P}(V)$. In this way, we can determine all the points $p_1, \ldots, p_k \in \mathbb{P}(V)$ and completely recover the surface X_K starting from the category $\mathfrak{B}_K(\sigma)$.

Remark 2.6 *Exceptional objects and exceptional collections on Del Pezzo surfaces are well-studied objects. First, any exceptional object of the derived category is isomorphic to a sheaf up to translation. Second, any exceptional sheaf can be included in a full exceptional collection. Third, any full exceptional collection can be obtained from a given one by a sequence of natural operations on exceptional collections called mutations. All these facts can be found in the paper [160].*

2.2 Simple degenerations of Del Pezzo surfaces

We now look at some simple degenerations of the situation considered above, namely when two points, for example p_1 and p_2, converge to each other and finally coincide. More precisely, this means that we first blow up a point p and after that we blow up some point p' on the -1-curve which is the pre-image of p under the first blowup. This operation is sometimes called a blowup at two "infinitely close" points; more precisely, it corresponds to blowing up a subscheme of length 2 supported at p. In this case, the pre-image $\pi^{-1}(p)$ consists of two rational curves meeting at one point. One of them is a -1-curve which we denote by l', and the other is a -2-curve which we denote by l. The curve l is the proper transform of the exceptional curve of the first blow up performed at the point $p \in \mathbb{P}^2$.

In this paragraph, we consider the situation where the surface X_K is the blowup of the projective plane $\mathbb{P}^2$ at a subscheme K which is supported at a set of $k-1$ points $\{p, p_3, \ldots, p_k\}$ and has length 2 at the point p. In this case the surface X_K is not a Del Pezzo surface, because it possesses a -2-curve l. However, it follows from general results about blowups that $\boldsymbol{D}^b(\mathrm{coh}(X_K))$ still possesses a full exceptional collection [193].

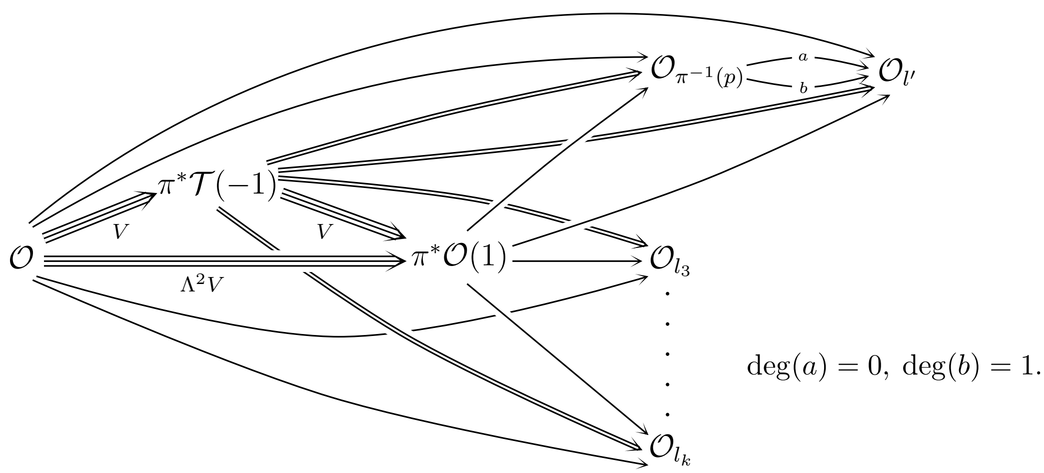


Figure 2: The cohomology algebra of the DG-quiver $\mathfrak{B}_K(\tau)$ for a blowup of $\mathbb{P}^2$ with two infinitely close points.

Proposition 2.7 *Let X_K be the blowup of $\mathbb{P}^2$ at a subscheme K supported at a set of $k-1$ points $\{p, p_3, \ldots, p_k\}$ and with length 2 at the point p. Then the sequence*

$$\tau = \left(\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\pi^{-1}(p)}, \mathcal{O}_{l'}, \mathcal{O}_{l_3}, \ldots, \mathcal{O}_{l_k}\right) \tag{2.4}$$

is a full exceptional collection on X_K.

As before we can see that the sheaves $\mathcal{O}_{l_i}$ and $\mathcal{O}_{l_j}$ are mutually orthogonal for all $i \neq j$, and each $\mathcal{O}_{l_i}$ is orthogonal to both $\mathcal{O}_{l'}$ and $\mathcal{O}_{\pi^{-1}(p)}$. However, the collection τ is not strong, because there are non-trivial morphisms from $\mathcal{O}_{\pi^{-1}(p)}$ to $\mathcal{O}_{l'}$ in degrees 0 and 1. More precisely,

$$\mathrm{Hom}(\mathcal{O}_{\pi^{-1}(p)}, \mathcal{O}_{l'}) \cong \mathbb{C} \qquad \text{and} \qquad \mathrm{Ext}^1(\mathcal{O}_{\pi^{-1}(p)}, \mathcal{O}_{l'}) \cong \mathbb{C}.$$

Denote by a and b two morphisms from $\mathcal{O}_{\pi^{-1}(p)}$ to $\mathcal{O}_{l'}$ of degrees 0 and 1 respectively. It is easy to see that composition with the morphism a gives isomorphisms between the spaces $\mathrm{Hom}(F, \mathcal{O}_{\pi^{-1}(p)})$ and $\mathrm{Hom}(F, \mathcal{O}_{l'})$ for any element F of the exceptional collection τ (see Figure 2).

Two approaches can be used to obtain an analogue of equivalence (2.2) for this situation. The first possibility is to associate to the non-strong exceptional collection τ a differential graded algebra of homomorphisms, and obtain an equivalence between the derived category of coherent sheaves and the derived category of finitely generated (right) DG-modules over the DG-algebra of homomorphisms of the exceptional collection. (One could also try to work in the framework of A_∞-algebras, which might be more appropriate here considering that the mirror situation involves an A_∞-category with non-zero m_3, see §4.4).

Another approach is to change the exceptional collection τ to another one which is strong. There are natural operations on exceptional collections which are called mutations and which allow us to obtain new exceptional collections starting from a given one.

We omit the definition of mutations, which is classical and can be found in many places. However, we note that the left mutation of the exceptional collection (2.4) in the pair $(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\pi^{-1}(p)})$ gives us a new exceptional collection

$$\tau' = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{M}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l'}, \mathcal{O}_{l_3}, \ldots, \mathcal{O}_{l_k}) \tag{2.5}$$

where $\mathcal{M}$ is the line bundle on X_K which is the kernel of the surjection $\pi^*\mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_{\pi^{-1}(p)}$. This new exceptional collection τ' is strong.

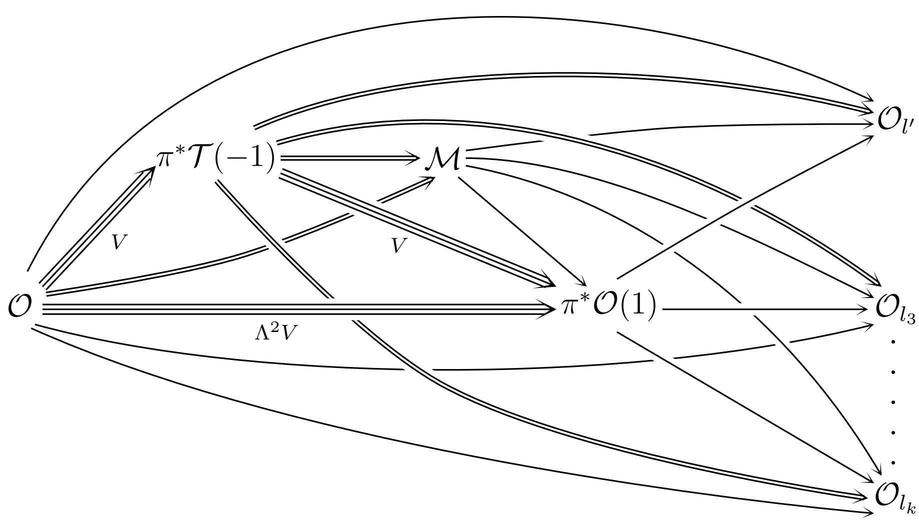


Figure 3: The quiver $\mathfrak{B}_K(\tau')$ for a blowup of $\mathbb{P}^2$ with two infinitely close points.

In fact, we can also consider the same left mutation when the blown up points are all distinct, and obtain in that case as well a strong exceptional collection

$$\sigma' = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \mathcal{M}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_2}, \mathcal{O}_{l_3}, \ldots, \mathcal{O}_{l_k}),$$

which behaves very much like τ'. The distinguishing feature of the case where we blow up the point p twice is that in the exceptional collection τ' the composition map

$$\mathrm{Hom}(\mathcal{M}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathrm{Hom}(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l'}) \longrightarrow \mathrm{Hom}(\mathcal{M}, \mathcal{O}_{l'}) \tag{2.6}$$

is identically zero, whereas for σ' (i.e., when the points of K are distinct) the corresponding composition is non-trivial. In this sense, the mutation allows us to give a simple description of the behaviour of the category under the degeneration process where two points of K converge to each other. Namely, the algebra $B_K(\tau')$ of homomorphisms of the exceptional collection τ' is obtained as a degeneration of the algebra of homomorphisms of the exceptional collection σ' in which the composition (2.6) becomes zero.

Proposition 2.8 *Let X_K be the blowup of $\mathbb{P}^2$ at a subscheme K supported at a set of $k-1$ points $\{p, p_3, \ldots, p_k\}$ and with length 2 at the point p. Then there is an equivalence $\boldsymbol{D}^b(\mathrm{coh}(X_K)) \cong \boldsymbol{D}^b(\mathrm{mod}\, B_K(\tau'))$, where $B_K(\tau')$ is the algebra of homomorphisms of the exceptional collection τ'.*

In this context, the surface X_K can again be recovered from the category $\mathfrak{B}_K(\tau')$. Namely, the points $p, p_3, \ldots, p_k$ can be determined by the same method as above. To recover X_K, we also have to determine the position of the point p' on the exceptional curve of the blowup of the point p. This is equivalent to finding a tangent direction at the point p. Consider the kernel of the composition map

$$\mathrm{Hom}(\mathcal{O}, \mathcal{M}) \otimes \mathrm{Hom}(\mathcal{M}, \mathcal{O}_{l'}) \longrightarrow \mathrm{Hom}(\mathcal{O}, \mathcal{O}_{l'}).$$

It is a one-dimensional subspace of $\mathrm{Hom}(\mathcal{O}, \mathcal{M})$. The image of this subspace in the space $V^* = \mathrm{Hom}(\mathcal{O}, \pi^*\mathcal{O}(1))$ determines a line in the projective space $\mathbb{P}(V)$ which passes through the point p and hence a tangent direction at p.

2.3 Noncommutative deformations of Del Pezzo surfaces

As before, let X_K be the blowup of the projective plane at a set K of k distinct points. Consider the strong exceptional collection

$$\sigma = (\mathcal{O}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_1}, \ldots, \mathcal{O}_{l_k}).$$

By the discussion in §2.1, the derived category of coherent sheaves $\boldsymbol{D}^b(\mathrm{coh}(X_K))$ is equivalent to the category of finitely generated (right) modules over the algebra B_K of homomorphisms of σ. The algebra B_K can also be represented by the category $\mathfrak{B}_K$ associated to the exceptional collection σ (see Figure 1).

The category $\mathfrak{B}_K$ has strictly more deformations than the surface X_K. We saw above that the surface X_K can be reconstructed from the category $\mathfrak{B}_K$, and that the deformation of the surface X_K is controlled by the variation of the set $K \subset \mathbb{P}^2$.

A general deformation of the category $\mathfrak{B}_K$ can be viewed as the category of an exceptional collection on a noncommutative deformation of the surface X_K. In other words, if $\mathfrak{B}_{K,\mu}$ is a deformation of the category $\mathfrak{B}_K$ then the bounded derived category $\boldsymbol{D}^b(\operatorname{mod} B_{K,\mu})$ of finitely generated (right) modules over the algebra $B_{K,\mu}$ will be viewed as the derived category of coherent sheaves on a noncommutative surface $X_{K,\mu}$. Any such noncommutative surface can be represented as the blowup of a noncommutative plane $\mathbb{P}^2_\mu$ at some set K consisting of k of its "points". This procedure is discussed in detail in [246].

In the rest of this section, we describe the deformations of the category $\mathfrak{B}_K$. Recall that a deformation of a category is, by definition, a deformation of its composition law. We proceed in two steps. The first step is to describe the deformations of the subcategory $\mathfrak{B}(\sigma_0)$ associated to the subcollection $\sigma_0 = (\mathcal{O}, \pi^*\mathcal{T}(-1), \pi^*\mathcal{O}(1))$. This subcategory $\mathfrak{B}(\sigma_0)$ is the category of homomorphisms of the full strong exceptional collection $(\mathcal{O}, \mathcal{T}(-1), \mathcal{O}(1))$ on $\mathbb{P}^2$. Therefore, considering a deformation of the subcategory $\mathfrak{B}(\sigma_0)$ we obtain a noncommutative deformation $\mathbb{P}^2_\mu$ of the projective plane. The second step is to describe the deformations of all other compositions in the category $\mathfrak{B}_K$. These deformations correspond to variations of the set of "points" K on the noncommutative projective plane $\mathbb{P}^2_\mu$.

$$
\begin{tikzcd}
 & F_1 \arrow[dr, Rrightarrow, "V"] & \\
F_0 \arrow[ur, Rrightarrow, "U"] \arrow[rr, Rrightarrow, "W"'] & & F_2
\end{tikzcd}
$$

Figure 4: The quiver $\mathfrak{B}_\mu$ for a noncommutative $\mathbb{P}^2_\mu$.

Noncommutative deformations of the projective plane have been described in [10], [40]. Any deformation of the category $\mathfrak{B}(\sigma_0)$ is a category with three ordered objects F_0, F_1, F_2 and with three-dimensional spaces of homomorphisms from F_i to F_j when $i < j$ (see Figure 4). Any such category $\mathfrak{B}_\mu$ is determined by the composition tensor $\mu : V \otimes U \to W$. We will consider only the nondegenerate (geometric) case, where the restrictions $\mu_u = \mu(\cdot, u) : V \to W$ and $\mu_v = \mu(v, \cdot) : U \to W$ have rank at least two for all nonzero elements $u \in U$ and $v \in V$, and the composition of μ with any nonzero linear form on W is a bilinear form of rank at least two on $V \otimes U$. The equations $\det \mu_u = 0$ and $\det \mu_v = 0$ define closed subschemes $\Gamma_U \subset \mathbb{P}(U)$ and $\Gamma_V \subset \mathbb{P}(V)$. Namely, up to projectivization the set of points of Γ_U (resp. Γ_V) consists of all $u \in U$ (resp. $v \in V$) for which the rank of μ_u (resp. μ_v) is equal to 2. It is easy to see that the correspondence which associates to a vector $v \in V$ the kernel of the map $\mu_v : U \to W$ defines an isomorphism between Γ_V and Γ_U. Moreover, under these circumstances Γ_V is either the entire projective plane $\mathbb{P}(V)$ or a cubic in $\mathbb{P}(V)$. If $\Gamma_V = \mathbb{P}(V)$ then μ is isomorphic to the tensor $V \otimes V \to \Lambda^2 V$, i.e. we get the usual projective plane $\mathbb{P}^2$.

Thus, the non-trivial case is the situation where Γ_V is a cubic, i.e. an elliptic curve which we now denote by E. This elliptic curve comes equipped with two embeddings into the projective planes $\mathbb{P}(U)$ and $\mathbb{P}(V)$ respectively; by restriction of $\mathcal{O}(1)$ these embeddings determine two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ of degree 3 over E, and it can be checked that $\mathcal{L}_1 \neq \mathcal{L}_2$. This construction has a converse:

Construction 2.9 *The tensor μ can be reconstructed from the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$. Namely, the spaces U, V are isomorphic to $H^0(E, \mathcal{L}_1)^*$ and $H^0(E, \mathcal{L}_2)^*$ respectively, and the space W is dual to the kernel of the canonical map*

$$H^0(E, \mathcal{L}_1) \otimes H^0(E, \mathcal{L}_2) \longrightarrow H^0(E, \mathcal{L}_1 \otimes \mathcal{L}_2),$$

which induces the tensor $\mu : V \otimes U \longrightarrow W$.

The details of these constructions and statements can be found in [10], [40].

Remark 2.10 *Note that we can also consider a triple $(E, \mathcal{L}_1, \mathcal{L}_2)$ such that $\mathcal{L}_1 \cong \mathcal{L}_2$. Then the procedure described above produces a tensor μ with $\Gamma_V \cong \mathbb{P}(V)$, which defines the usual commutative projective plane. Thus, in this particular case the tensor μ does not depend on the curve E.*

Now we describe the deformations of the other compositions in the category $\mathfrak{B}_K$. Given a category $\mathfrak{B}_\mu$ of the form described above, corresponding to a noncommutative projective plane $\mathbb{P}^2_\mu$, and given a set $K = \{p_1, \ldots, p_k\}$ of k points on the elliptic curve E, we can construct a category $\mathfrak{B}_{K,\mu}$ in the following manner. A point $p_j \in E \subset \mathbb{P}(U)$ determines a one-dimensional subspace of U, generated by a vector $u_j \in U$. The map $\mu_{u_j} : V \to W$ has rank 2; denote by v_j a non-zero vector in its kernel. The category $\mathfrak{B}_{K,\mu}$ is then constructed from the category $\mathfrak{B}_\mu$ by adding k mutually orthogonal objects $\mathcal{O}_{l_j}$ for $j = 1, \ldots, k$, and defining the spaces of morphisms by the rule

$$\mathrm{Hom}(F_2, \mathcal{O}_{l_j}) = \mathbb{C}, \qquad \mathrm{Hom}(F_1, \mathcal{O}_{l_j}) = V/\operatorname{Ker}\mu_{u_j} = V/\langle v_j \rangle, \qquad \mathrm{Hom}(F_0, \mathcal{O}_{l_j}) = W/\mathrm{Im}\,\mu_{v_j}.$$

The two composition tensors involving $\mathrm{Hom}(F_2, \mathcal{O}_{l_j})$ are defined in the obvious manner as suggested by the notation. The only non-obvious composition is the map $V/\langle v_j \rangle \otimes U \to W/\mathrm{Im}\,\mu_{v_j}$, which is by definition induced by the tensor $\mu : V \otimes U \longrightarrow W$.

Conversely, if we consider a category $\mathfrak{B}_{K,\mu}$ which is a deformation of $\mathfrak{B}_K$ and an extension of the category $\mathfrak{B}_\mu$, then the kernel of the composition map

$$\mathrm{Hom}(F_2, \mathcal{O}_{l_j}) \otimes V \longrightarrow \mathrm{Hom}(F_1, \mathcal{O}_{l_j})$$

defines a one-dimensional subspace $\langle v_j \rangle \subset V$. The map μ_{v_j} must have rank 2, since otherwise μ_{v_j} would be an isomorphism and the composition map $\mathrm{Hom}(F_2, \mathcal{O}_{l_j}) \otimes W \longrightarrow \mathrm{Hom}(F_0, \mathcal{O}_{l_j})$ would vanish identically, which by assumption is not the case. Therefore, the objects $\mathcal{O}_{l_j}$ correspond to points on the curve E.

Thus, any category $\mathfrak{B}_{K,\mu}$ is defined by the data $(E, \mathcal{L}_1, \mathcal{L}_2, p_1, \ldots, p_k)$, where E is a cubic, $\mathcal{L}_1, \mathcal{L}_2$ are line bundles of degree 3 on E, and $p_1, \ldots, p_k$ is a set of distinct points on E. If $\mathcal{L}_1 \cong \mathcal{L}_2$, then we obtain the category $\mathfrak{B}_K$ related to a blowup of the usual commutative projective plane. In the general case, the bounded derived category $\boldsymbol{D}^b(\operatorname{mod} B_{K,\mu})$ of finite rank modules over the algebra $B_{K,\mu}$ is viewed as the derived category of coherent sheaves on the non-commutative surface $X_{K,\mu}$, which is a blowup of k points on the non-commutative projective plane $\mathbb{P}^2_\mu$.

A standard approach to noncommutative geometry is to determine a noncommutative variety either by an abelian category of (quasi)coherent sheaves on it or by a noncommutative (graded) algebra which is considered as its (homogeneous) coordinate ring. The question of how to define the abelian category of coherent sheaves on Del Pezzo surfaces and on other blowups of surfaces is discussed in the paper [246]. We briefly describe one of the possible approaches. It is very important to note that the category $\boldsymbol{D}^b(\operatorname{mod} B_{K,\mu})$ possess a Serre functor S, i.e. an additive autoequivalence for which there are bi-functorial isomorphisms

$$\operatorname{Hom}(X, SY) \xrightarrow{\sim} \operatorname{Hom}(Y, X)^*$$

for any $X, Y \in \boldsymbol{D}^b(\operatorname{mod} B_{K,\mu})$. In the case of the bounded derived category of finite rank modules over a finite dimensional algebra of finite homological dimension, the Serre functor is the functor which takes a complex of modules $M^\bullet$ to the complex $\mathbf{R}\operatorname{Hom}_{B_{K,\mu}}(M, B_{K,\mu})^*$. The Serre functor is an exact autoequivalence.

Now we can take the projective module P_0 (corresponding to $\mathcal{O}$, see the discussion after Definition 2.4) and consider the sequence of objects $R_m = S^m[-2m]P_0$ for all $m \in \mathbb{Z}$. Let us consider the subcategory $\mathbb{A} \subset \boldsymbol{D}^b(\operatorname{mod} B_{K,\mu})$ consisting of all objects F such that

$$\operatorname{Hom}(R_m, F[i]) = 0 \quad \text{for all } i \neq 0 \text{ and sufficiently large } m \gg 0.$$

If the category $\mathbb{A}$ is abelian and its bounded derived category $\boldsymbol{D}^b(\mathbb{A})$ is equivalent to $\boldsymbol{D}^b(\operatorname{mod} B_{K,\mu})$ then $\mathbb{A}$ can be considered as the category of coherent sheaves on the noncommutative surface $X_{K,\mu}$, and $X_{K,\mu}$ can be called a noncommutative Del Pezzo surface.

The reason of such a definition of the abelian category of coherent sheaves on a noncommutative Del Pezzo surface is inspired by the commutative case. In the commutative case the Serre functor is isomorphic to the functor $\otimes\mathcal{O}(K)[2]$, where $\mathcal{O}(K)$ is the canonical line bundle. Hence, for usual commutative surfaces the objects R_m are isomorphic to the invertible sheaves $\mathcal{O}(mK)$. Since for a Del Pezzo surface X the anticanonical sheaf $\mathcal{O}(-K)$ is ample, we have $H^i(X, F(-mK)) = 0$ for all $i \neq 0$ and any coherent sheaf F when m is sufficiently large. This property makes it possible to separate pure coherent sheaves from other complexes of coherent sheaves.

We can also consider the graded space $A = \bigoplus_{p=0}^{\infty} \operatorname{Hom}(R_0, R_{-p})$ and can endow it with the structure of a graded algebra using the isomorphisms $\operatorname{Hom}(R_0, R_{-p}) \cong \operatorname{Hom}(R_i, R_{i-p})$ given by the functors $S^i[-2i]$ for all $i \in \mathbb{Z}$. This algebra can be considered as the homogeneous coordinate ring of a noncommutative Del Pezzo surface. It seems that such rings are noncommutative deformations of homogeneous commutative coordinate rings of usual Del Pezzo surfaces.

In any case, these remarks about abelian categories of coherent sheaves on noncommutative Del Pezzo surfaces will not be needed in the rest of the argument. We will only use the description of the bounded derived category of coherent sheaves on the noncommutative blowup $X_{K,\mu}$ in terms of finite rank modules over the algebra $B_{K,\mu}$, i.e. we state an equivalence of triangulated categories

$$\boldsymbol{D}^b(\mathrm{coh}(X_{K,\mu})) \cong \boldsymbol{D}^b(\mathrm{mod}\, B_{K,\mu}). \tag{2.7}$$

3 The mirror Landau-Ginzburg models

3.1 Compactification of the mirror of $\mathbb{CP}^2$

As mentioned in the introduction, the mirror of $\mathbb{CP}^2$ is an elliptic fibration with 3 singular fibers, determined by (a fiberwise compactification of) the superpotential $W_0 = x + y + 1/xy$ on $(\mathbb{C}^*)^2$. This Landau-Ginzburg model compactifies naturally to an elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$, which we now describe.

Compactifying $(\mathbb{C}^*)^2$ to $\mathbb{CP}^2$, we can view W_0 as the quotient of the two homogeneous degree 3 polynomials $P_0 = X^2Y + XY^2 + Z^3$ and $P_\infty = XYZ$, which define a pencil of cubics with three base points of multiplicities respectively 4, 4, and 1. Namely, the cubic C_0 defined by P_0 intersects the line $X = 0$ at $(0 : 1 : 0)$ (with multiplicity 3), the line $Y = 0$ at $(1 : 0 : 0)$ (with multiplicity 3), and the line $Z = 0$ at $(0 : 1 : 0)$, $(1 : 0 : 0)$ and $(1 : -1 : 0)$. Blow up $\mathbb{CP}^2$ three times successively at the point where the cubic C_0 and the line $X = 0$ (or their proper transforms) intersect each other, i.e. first at the point $(0 : 1 : 0)$, and then twice at suitable points of the exceptional divisors (see Figure 5). Similarly, blow up three times the intersection of the cubic C_0 with the line $Y = 0$.

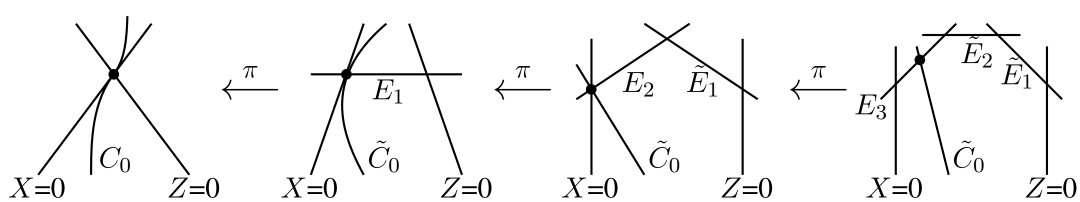


Figure 5: The successive blowups at $(0 : 1 : 0)$.

Let $\tilde{C}_0$ be the proper transform of C_0 under these blowups, and let $\tilde{C}_\infty$ be the configuration of 9 rational curves formed by the proper transforms of the three coordinate lines and the exceptional divisors of the six blowups (so, in Figure 5, all components other than $\tilde{C}_0$ are eventually part of $\tilde{C}_\infty$). Then $\tilde{C}_0$ and $\tilde{C}_\infty$ intersect transversely at three smooth points, and define a pencil of elliptic curves representing the anticanonical class in $\mathbb{CP}^2$ blown up six times. The complement of $\tilde{C}_\infty$ identifies with $(\mathbb{C}^*)^2$, and the restriction of the $\mathbb{CP}^1$-valued map defined by the pencil to this open subset coincides with W_0. Blowing up the three points where $\tilde{C}_0$ and $\tilde{C}_\infty$ intersect, we obtain a rational elliptic surface $\overline{M}$, and the pencil becomes an elliptic fibration $\overline{W_0} : \overline{M} \to \mathbb{CP}^1$, which provides a natural compactification of $W_0 : (\mathbb{C}^*)^2 \to \mathbb{C}$.

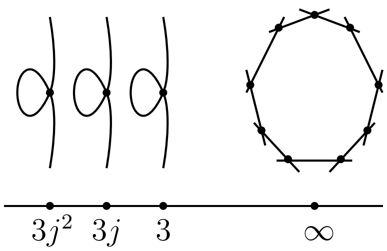


Figure 6: The singular fibers of $\overline{W_0}$.

The meromorphic function $\overline{W_0}$ has 12 isolated non-degenerate critical points. Three of them are the pre-images of the points $(1:1:1)$, $(j:j:1)$, and $(j^2:j^2:1)$ ($j = e^{2i\pi/3}$), and correspond to the three critical points of W_0 in $(\mathbb{C}^*)^2$ (with associated critical values 3, $3j$, and $3j^2$). The nine other critical points all lie in the fiber above infinity: they are the nodes of the reducible configuration $\tilde{C}_\infty$ (see Figure 6).

This compactification process can also be described in a more symmetric manner by viewing $(\mathbb{C}^*)^2$ as the surface $\{xyz = 1\} \subset (\mathbb{C}^*)^3$, and $W_0 = x + y + z$. Compactifying $(\mathbb{C}^*)^3$ to $\mathbb{CP}^3$ leads one to consider the cubic surface $\{XYZ = T^3\} \subset \mathbb{CP}^3$, which presents A_2 singularities at the three points $(1:0:0:0)$, $(0:1:0:0)$, and $(0:0:1:0)$. After blowing up $\mathbb{CP}^3$ at these three points, we obtain a smooth cubic surface, in which the hyperplane sections $\tilde{C}_0 = \{X + Y + Z = 0\}$ and $\tilde{C}_\infty = \{T = 0\}$ define a pencil of elliptic curves with three base points. As before, $\tilde{C}_0$ is a smooth elliptic curve, and $\tilde{C}_\infty$ is a configuration of 9 rational curves (the proper transforms of the three coordinate lines where the singular cubic surface intersects the plane $T = 0$, and the six -2-curves arising from the resolution of the singularities). Blowing up the three points of $\tilde{C}_0 \cap \tilde{C}_\infty$, we again obtain a rational elliptic surface, and an elliptic fibration with 12 isolated critical points, 9 of which lie in the fiber above infinity (as in Figure 6).

3.2 The vanishing cycles of $\overline{W_0}$

Each singular fiber of $\overline{W_0}$ is obtained from the regular fiber by collapsing a certain number of *vanishing cycles*, and the monodromy of the fibration around a singular fiber is given by a product of Dehn twists along these vanishing cycles. In this section, we determine the homology classes of the vanishing cycles associated to the critical points of $\overline{W_0}$.

More precisely, consider the fiber $\Sigma_0 = \overline{W_0}^{-1}(0)$, which is a smooth elliptic curve (in fact, the proper transform of the curve called $\tilde{C}_0$ in §3.1), and consider the following ordered collection of arcs $(\gamma_i)_{0\le i\le 3}$ joining the origin to the various critical values of $\overline{W_0}$: γ_0, γ_1, γ_2 are straight line segments joining the origin to $\lambda_0 = 3$, $\lambda_1 = 3j^2$, and $\lambda_2 = 3j$ respectively, and γ_3 is the straight line $e^{i\pi/3}\mathbb{R}_+$ joining the origin to $\lambda_3 = \infty$.

Using parallel transport (with respect to an arbitrary horizontal distribution) along the arc γ_i, we can associate a vanishing cycle to each critical point $p \in \overline{W_0}^{-1}(\lambda_i)$; this vanishing cycle is well-defined up to isotopy, and in particular we can consider its homology class in $H_1(\Sigma_0,\mathbb{Z}) \simeq \mathbb{Z}^2$ (well-defined up to a choice of orientation). If we fix a symplectic structure on $\overline{M}$ for which the fibers of $\overline{W_0}$ are symplectic submanifolds, then we have a canonical horizontal distribution (given by the symplectic orthogonal to the fiber), which allows us to consider the vanishing cycles as Lagrangian submanifolds of Σ_0, well-defined up to *Hamiltonian isotopy*; in §4 this will be of utmost importance, but for now we ignore the symplectic structure and only view $\overline{W_0}$ as a topological fibration.

Lemma 3.1 *In terms of a suitable basis $\{a,b\}$ of $H_1(\Sigma_0,\mathbb{Z})$, the vanishing cycles L_0, L_1, L_2 associated to the critical values $\lambda_0, \lambda_1, \lambda_2$ (and the arcs $\gamma_0, \gamma_1, \gamma_2$) represent the classes $[L_0] = -2a-b$, $[L_1] = -a+b$, and $[L_2] = a+2b$, respectively; and the vanishing cycles $L_3, \dots, L_{11}$ associated to the nine critical points in the fiber at infinity represent the class $[L_3] = \cdots = [L_{11}] = a+b$.*

Proof. The vanishing cycles L_0, L_1, L_2 are exactly those of the mirror of $\mathbb{CP}^2$, and are well-known (cf. e.g. [215] or [16]). In particular it is known that, choosing a suitable homology basis $\{a,b\}$ for $H_1(\Sigma_0,\mathbb{Z})$, and fixing appropriate orientations of L_0, L_1, L_2, we have $[L_0] = -2a-b$, $[L_1] = -a+b$, and $[L_2] = a+2b$ (cf. e.g. Figure 14 in [16]).

We now consider the 9 critical points in the fiber at infinity. It is clear that $L_3, \dots, L_{11}$ admit disjoint representatives, and hence are all homologous. Their homology class can be determined by considering the monodromy of the elliptic fibration $\overline{W_0}$, which is given by the product of the positive Dehn twists along the vanishing cycles. Considering the action on $H_1(\Sigma_0,\mathbb{Z})$, and still using the basis $\{a,b\}$ considered above, the monodromies around the critical values $\lambda_0, \lambda_1, \lambda_2$ are given by

$$\tau_0 = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and } \tau_2 = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix},$$

while the monodromy around the fiber at infinity is given by τ^9, where τ is the positive Dehn twist along $[L_3] = \cdots = [L_{11}]$. On the other hand, because the arcs $\gamma_0, \dots, \gamma_3$ are ordered clockwise around the origin, we have $\tau_0\tau_1\tau_2\tau^9 = 1$. Therefore,

$$\tau^9 = \begin{pmatrix} -8 & 9 \\ -9 & 10 \end{pmatrix},$$

and considering $\mathrm{Ker}(\tau^9 - 1)$ we obtain $[L_3] = \cdots = [L_{11}] = a+b$. $\square$

Proof. [Alternative proof] (compare with §4.2 of [16]). Recall that $\overline{M}$ is obtained from $\mathbb{CP}^2$ by successive blowups of the base points of the pencil of cubics defined by $P_0 = X^2Y + XY^2 + Z^3$ and $P_\infty = XYZ$. Consider the ruled surface F obtained by blowing up $\mathbb{CP}^2$ just once at the point $(0:1:0)$: the projection $(X:Y:Z) \mapsto (X:Z)$ naturally extends into a fibration $\pi_x : F \to \mathbb{CP}^1$, of which the exceptional divisor is a section. For $\lambda \in \mathbb{CP}^1$, denote by $\hat{C}_\lambda$ the proper transform of the plane cubic C_λ defined by $P_0 - \lambda P_\infty$, which is also the image of $\overline{W_0}^{-1}(\lambda)$ under the natural projection $p : \overline{M} \to F$.

The restriction $\pi_{x,\lambda}$ of π_x to $\hat{C}_\lambda$ has degree two, and for $\lambda \notin \mathrm{crit}(\overline{W_0})$ its four branch points are associated to distinct critical values in $\mathbb{CP}^1$, namely zero and the three roots of the equation $x(\lambda - x)^2 = 4$. Indeed, since C_λ always has an order 3 tangency with the line $X = 0$ at $(0:1:0)$, $\hat{C}_\lambda$ is always tangent to the fiber $\pi_x^{-1}(0)$. The three other branch points are the critical points of the projection to the first coordinate on $(\mathbb{C}^*)^2 \cap C_\lambda = \{(x,y) \in (\mathbb{C}^*)^2,\ xy(\lambda - x - y) = 1\}$; viewing $xy(\lambda - x - y) = 1$ as a quadratic equation in the variable y, the discriminant is $x(\lambda - x)^2 - 4$.

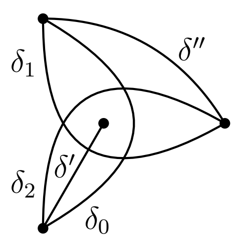


Figure 7: The projections of the vanishing cycles of $\overline{W_0}$

As λ tends to λ_i ($i \in \{0,1,2\}$), two of the critical values of $\pi_{x,\lambda}$ converge to each other; keeping track of the manner in which these critical values coalesce when λ varies from 0 to λ_i along the arc γ_i, we obtain an arc $\delta_i \subset \mathbb{CP}^1$, with end points in $\mathrm{crit}(\pi_{x,0})$ (see Figure 7). The lift of δ_i under the double cover $\pi_{x,0}$ is (up to homotopy) the vanishing cycle L_i (note that the projection $p : \overline{M} \to F$ allows us to implicitly identify $\hat{C}_\lambda$ with $\overline{W_0}^{-1}(\lambda)$ for $\lambda \neq \infty$).

Similarly, the behavior of the critical values of $\pi_{x,\lambda}$ as λ tends to infinity describes the degeneration of $\hat{C}_\lambda$ to the singular configuration $\hat{C}_\infty$, which consists of two sections and two fibers of $\pi_x : F \to \mathbb{CP}^1$ (the fibers above 0 and ∞, the exceptional section, and the pre-image of the line $Y = 0$). Namely, as λ tends to infinity along the arc γ_3, the critical value with argument $-2\pi/3$ approaches zero, while the two other roots of $x(\lambda - x)^2 - 4$ tend to infinity. The manner in which pairs of critical values coalesce is encoded by the arcs δ' and δ'' in Figure 7, and the four vanishing cycles associated to the degeneration are essentially the lifts under $\pi_{x,0}$ of closed loops which bound regular neighborhoods of the arcs δ' and δ''; they all represent the same homotopy class inside $\hat{C}_0$.

Recall that $\overline{W_0}^{-1}(\infty) \simeq \tilde{C}_\infty$ is obtained from $\hat{C}_\infty$ by repeatedly blowing up two of the nodes. Taking pre-images under these blowup operations, the vanishing cycles associated to the two other nodes of $\hat{C}_\infty$ are naturally identified with two of the nine vanishing cycles $L_3, \ldots, L_{11}$

associated to the fiber at infinity of $\overline{W_0}$. In particular, these vanishing cycles represent the same homology class in $H_1(\Sigma_0, \mathbb{Z}) \simeq H_1(\hat{C}_0, \mathbb{Z})$ as the lifts of δ' and δ''.

It is then easy to check that, for suitable choices of orientations, we have $[L_0] \cdot [L_1] = [L_0] \cdot [L_2] = [L_1] \cdot [L_2] = -3$, $[L_0] \cdot [L_{3+i}] = [L_2] \cdot [L_{3+i}] = -1$, and $[L_1] \cdot [L_{3+i}] = -2$, which completes the proof of Lemma 3.1. □

3.3 The vanishing cycles of (M_k, W_k)

Recall from the introduction that our proposal for the mirror of a Del Pezzo surface X_K obtained from $\mathbb{CP}^2$ by blowing up $k \leq 8$ generic points is an elliptic fibration $W_k : M_k \to \mathbb{C}$, obtained by deforming the fibration $\overline{W_0}$ to another elliptic fibration $\overline{W_k} : \overline{M} \to \mathbb{CP}^1$, and considering the restriction to $M_k = \overline{M} \setminus \overline{W_k}^{-1}(\infty)$. More precisely, remember that $\overline{W_k}$ has $3 + k$ irreducible nodal fibers corresponding to critical values $\lambda_0, \dots, \lambda_{k+2} \in \mathbb{C}$, of which the first three correspond naturally to the irreducible nodal fibers of $\overline{W_0}$, while the k other finite critical values correspond to the deformation of critical points in $\overline{W_0}^{-1}(\infty)$ towards finite values of the superpotential. Meanwhile, $\overline{W_k}^{-1}(\infty)$ is a singular fiber with $9 - k$ components.

While the precise locations of the critical values λ_i are closely related to the complex structure on M_k, they are essentially irrelevant from the point of view of symplectic topology and categories of vanishing cycles. Indeed, if we consider a family $(M_{k,t}, W_{k,t})$ of fibrations indexed by a real parameter t, with the property that for all t the critical points of $W_{k,t}$ are isolated and non-degenerate, then the vanishing cycles remain the same for all values of t, up to smooth isotopies inside the reference fiber. For this reason, we do not need to make a specific choice of λ_i. To fix ideas, let us say that λ_0 is close to 3, λ_1 is close to $3j^2$, λ_2 is close to $3j$, and λ_i is close to infinity for $i \geq 3$; we again choose the origin as base point, and note that the smooth elliptic curve $W_k^{-1}(0)$ is diffeomorphic to $\overline{W_0}^{-1}(0)$, so we implicitly identify them and again call our reference fiber Σ_0. We also choose an ordered collection of arcs γ_i joining the origin to λ_i which lie close to those considered in §3.2, thus ensuring that the homology classes $[L_0], \dots, [L_{k+2}] \in H_1(\Sigma_0, \mathbb{Z})$ of the corresponding vanishing cycles remain those given by Lemma 3.1.

Fixing a symplectic form ω_k on M_k (compatible with W_k, i.e. restricting positively to the fibers), the vanishing cycles $L_0, \dots, L_{k+2}$ associated to the arcs $\gamma_0, \dots, \gamma_{k+2}$ naturally become *Lagrangian submanifolds* of the reference fiber $(\Sigma_0, \omega_{k|\Sigma_0})$ (cf. e.g. [7, 214, 217]). Indeed, the symplectic form defines a natural horizontal distribution outside of the critical points of W_k, given by the symplectic orthogonal to the fiber. Using this horizontal distribution, parallel transport induces symplectomorphisms between the smooth fibers, and the vanishing cycle L_i is by definition the set of points in the reference fiber Σ_0 for which parallel transport along γ_i converges to the critical point in the fiber $W_k^{-1}(\lambda_i)$. It is also useful to consider the *Lefschetz thimble* D_i, which is the set of points swept out by parallel transport of L_i above γ_i; by construction, D_i is a Lagrangian disk in (M_k, ω_k), fibered above the arc γ_i, and $\partial D_i = L_i$.

We recall the following classical result (we provide a proof for completeness):

Lemma 3.2 *A deformation of the system of arcs $\{\gamma_i\}$ by an isotopy in* $\mathrm{Diff}(\mathbb{C}, \mathrm{crit}(W_k))$ *affects* *the vanishing cycles L_i by Hamiltonian isotopies; moreover, the same property holds if the symplectic fibration (M_k, ω_k, W_k) is deformed in a manner such that the cohomology class $[\omega_k]$ remains constant and the critical points of W_k remain isolated and non-degenerate.*

Proof. We first consider a deformation of the system of arcs $\{\gamma_i\}$, based at a regular value $\lambda_* \in \mathbb{C} \setminus \mathrm{crit}(W_k)$ (in our case the origin), to an isotopic system of arcs $\{\gamma'_i\}$ based at a regular value λ'_*. This means that we are given an arc $\delta : [0,1] \to \mathbb{C} \setminus \mathrm{crit}(W_k)$ joining λ_* to λ'_*, and continuous families of arcs $\{\gamma_{i,t}\}$, $0 \le t \le 1$, with $\gamma_{i,0} = \gamma_i$ and $\gamma_{i,1} = \gamma'_i$, such that $\gamma_{i,t}$ joins the regular value $\delta(t)$ to the critical value λ_i, and $\{\gamma_{i,t}\}_{0 \le i \le k+2}$ is an ordered collection of arcs for all $t \in [0,1]$. The vanishing cycles L'_i associated to the arcs γ'_i live inside $\Sigma'_* = W_k^{-1}(\lambda'_*)$, while the original vanishing cycles L_i associated to γ_i are submanifolds of $\Sigma_* = W_k^{-1}(\lambda_*)$. However, we claim that the isotopy induces a symplectomorphism $\phi : \Sigma_* \to \Sigma'_*$ with the property that $\phi(L_i)$ and L'_i are mutually Hamiltonian isotopic for all i; this is the meaning of the statement of the lemma.

Namely, parallel transport along the arc δ (using the horizontal distribution described above) induces a symplectomorphism ϕ from $\Sigma_* = W_k^{-1}(\lambda_*)$ to $\Sigma'_* = W_k^{-1}(\lambda'_*)$. For all $t \in [0,1]$ we can consider the vanishing cycle $L_{i,t} \subset W_k^{-1}(\delta(t))$ associated to the arc $\gamma_{i,t}$, and its image $L'_{i,t} \subset \Sigma'_*$ under the symplectomorphism induced by parallel transport along $\delta([t,1])$. The family $L'_{i,t}$, $t \in [0,1]$ defines a smooth isotopy from $L'_{i,0} = \phi(L_i)$ to $L'_{i,1} = L'_i$ through Lagrangian submanifolds of Σ'_*. Moreover, each vanishing cycle $L_{i,t} \subset W_k^{-1}(\delta(t))$ bounds a Lagrangian thimble $D_{i,t}$, and the cylinder $C_{i,t}$ swept by $L_{i,t}$ under parallel transport along $\delta([t,1])$ is also Lagrangian. By continuity, the relative cycles $D_{i,t} \cup C_{i,t}$ (with boundary $L'_{i,t}$) all represent the same relative homotopy class in $\pi_2(M_k, \Sigma'_*)$, and since $D_{i,t}$ and $C_{i,t}$ are Lagrangian they all have zero symplectic area. This implies that the Lagrangian submanifolds $L'_{i,t}$, $t \in [0,1]$ are all Hamiltonian isotopic inside Σ'_*; in particular, $\phi(L_i)$ and L'_i are Hamiltonian isotopic.

We now consider a symplectic fibration $W'_k : (M_k, \omega'_k) \to \mathbb{C}$ which is isotopic to W_k through an isotopy $W_{k,t} : (M_k, \omega_{k,t}) \to \mathbb{C}$ that preserves the cohomology class of the symplectic form (i.e., $[\omega_{k,t}] = [\omega_k]$ for all $t \in [0,1]$). We assume that each $W_{k,t}$ has isolated non-degenerate critical points. This allows us to deform the system of arcs $\{\gamma_i\}$ through a family $\{\gamma_{i,t}\}$ with end points at the critical values of $W_{k,t}$; for $t = 1$ we obtain a system of arcs $\{\gamma'_i\}$ based at a regular value λ'_* of W'_k. By Moser's theorem, there exists a continuous family of symplectomorphisms ϕ_t from $(M_k, \omega_{k,t})$ to (M_k, ω'_k), or rather, since these are non-compact manifolds, from open subsets of $(M_k, \omega_{k,t})$ to an open subset of (M_k, ω'_k); however, after "enlarging" (M_k, ω'_k) by adding to ω'_k the pullback of a suitable area form on $\mathbb{C}$, which affects neither the symplectic structure on the fibers nor the parallel transport symplectomorphisms, we can ensure that ϕ_t is defined over an arbitrarily large open subset of M_k, which is good enough for our purposes. Moreover, by a relative version of Moser's argument, we can also ensure that ϕ_t maps the reference fiber of $W_{k,t}$ to the reference fiber of W'_k, and in particular that $\phi = \phi_0$ maps $\Sigma_* = W_k^{-1}(\lambda_*)$ to $\Sigma'_* = W_k'^{-1}(\lambda'_*)$.

We now claim that $\phi(L_i) \subset \Sigma'_*$ is Hamiltonian isotopic to the vanishing cycle L'_i of W'_k associated to the arc γ'_i. Indeed, by considering the images under ϕ_t of the vanishing cycles $L_{i,t}$

associated to the arcs $\gamma_{i,t}$, we obtain a smooth isotopy from $\phi(L_i)$ to L'_i through Lagrangian submanifolds of Σ'_*. Moreover, the thimbles D'_i and $\phi(D_i)$ represent the same relative homotopy class (as can be seen by considering the images by ϕ_t of the thimbles $D_{i,t}$ associated to $\gamma_{i,t}$), and both are Lagrangian with respect to ω'_k, which again implies that $\phi(L_i)$ and L'_i are Hamiltonian isotopic. □

3.4 A basis of $H_2(M_k)$

The manifold M_k is simply connected, and its second Betti number is equal to $k+2$. A $\mathbb{Q}$-basis of $H_2(M_k)$ is given by considering the homology class of the fiber of W_k, $[\Sigma_0]$, and $k+1$ classes $[\bar{C}], [\bar{C}_0], \dots, [\bar{C}_{k-1}]$ constructed from the vanishing cycles L_i and Lefschetz thimbles D_i in the following manner.

By Lemma 3.1 we have $[L_1] = [L_0] + [L_2]$ in $H_1(\Sigma_0, \mathbb{Z})$, so there exists a 2-chain C in Σ_0 such that $\partial C = -L_0 + L_1 - L_2$. Then

$$\bar{C} = C + D_0 - D_1 + D_2$$

is a 2-cycle in M_k. Note that $[\bar{C}]$ is in fact the image of the generator of $H_2((\mathbb{C}^*)^2, \mathbb{Z}) \simeq \mathbb{Z}$ under the inclusion map (see the proof of Lemma 4.9 in [16]).

Similarly, for $0 \le i < k$ we have $3\,[L_{3+i}] = [L_2] - [L_0]$ in $H_1(\Sigma_0, \mathbb{Z})$, so there exists a 2-chain C_i in Σ_0 such that $\partial C_i = 3\,L_{3+i} + L_0 - L_2$, and we can consider the 2-cycle

$$\bar{C}_i = C_i - 3\,D_{3+i} - D_0 + D_2$$

in M_k. We also introduce 2-chains $\Delta_{i,j}$ ($i, j \in \{0, \dots, k-1\}$) in Σ_0 such that $\partial \Delta_{i,j} = L_{3+j} - L_{3+i}$, and the corresponding 2-cycles

$$\bar{\Delta}_{i,j} = \Delta_{i,j} + D_{3+i} - D_{3+j}.$$

We can choose C_i and $\Delta_{i,j}$ in such a way that $C_j - C_i = 3\,\Delta_{i,j}$ (and hence $[\bar{C}_j] - [\bar{C}_i] = 3\,[\bar{\Delta}_{i,j}]$ in $H_2(M_k)$).

To summarize the discussion, the vanishing cycles L_i and the 2-chains C, C_i, $\Delta_{i,j}$ are represented on Figures 8–9 (compare with Figure 2 in [215] and with [243]).

4 Categories of vanishing cycles

4.1 Definition

As proposed by Kontsevich [152] and Hori-Iqbal-Vafa [120], the category of A-branes associated to a Landau-Ginzburg model $W : (M, \omega) \to \mathbb{C}$ is a Fukaya-type category which

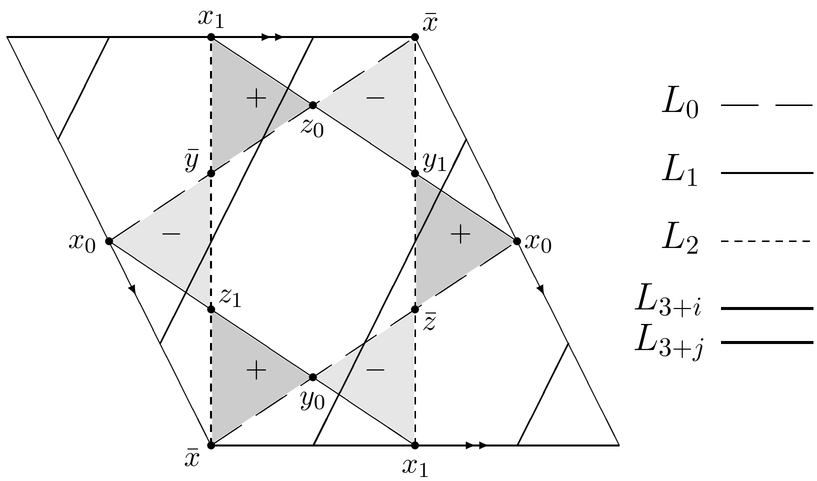


Figure 8: The vanishing cycles of W_k and the chain C

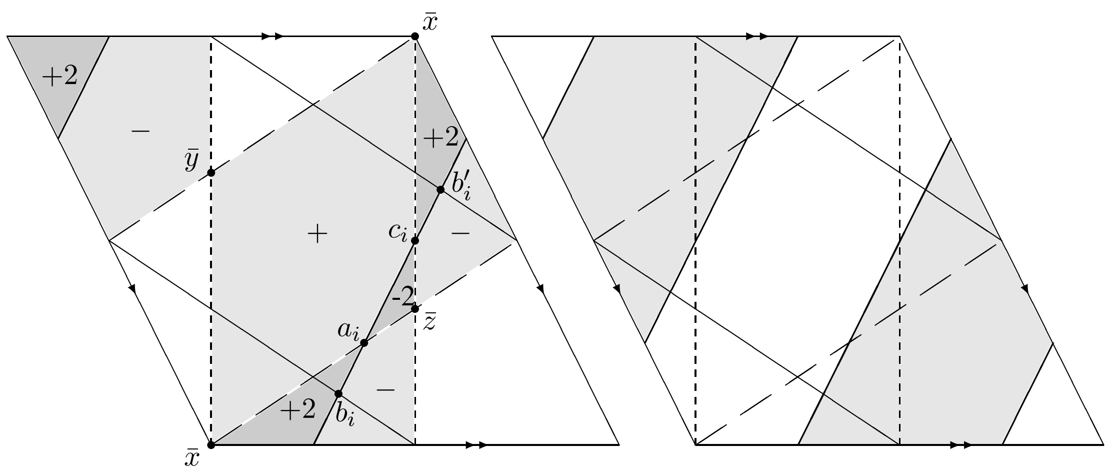


Figure 9: The chains C_i (left) and $\Delta_{i,j}$ (right)

contains not only compact Lagrangian submanifolds of M but also certain non-compact Lagrangians whose ends fiber in a specific way above half-lines in $\mathbb{C}$. In the case where the critical points of W are isolated and non-degenerate, this category admits an exceptional collection whose objects are Lagrangian thimbles associated to the critical points. Following the formalism introduced by Seidel [214], [217], we view it as the derived category of a finite directed A_∞-category $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ associated to an ordered collection of arcs $\{\gamma_i\}$. We briefly recall the definition; the reader is referred to [214, 217] and to §3.1 of [16] for details.

Consider a symplectic fibration $W : (M, \omega) \to \mathbb{C}$ with isolated non-degenerate critical points, and assume for simplicity that the critical values $\lambda_0, \dots, \lambda_r$ of W are distinct. Pick a regular value λ_* of W, and choose a collection of arcs $\gamma_0, \dots, \gamma_r \subset \mathbb{C}$ joining λ_* to the various critical values of W, intersecting each other only at λ_*, and ordered in the clockwise direction around λ_*. Consider the horizontal distribution defined by the symplectic form: by parallel transport along the arc γ_i, we obtain a Lagrangian thimble D_i and a vanishing cycle $L_i = \partial D_i \subset \Sigma_*$ (where $\Sigma_* = W^{-1}(\lambda_*)$). After a small perturbation we can always assume that the vanishing cycles L_i intersect each other transversely inside Σ_*. The following definition is motivated by the observation that the intersection theory of the Lagrangian thimbles $D_i \subset M$ is closely related to that of the vanishing cycles L_i inside Σ_* [214]:

Definition 4.1 (Seidel) *The directed category of vanishing cycles* $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ *is an* A_∞*-category (over a coefficient ring* R*) with objects* $L_0, \dots, L_r$ *corresponding to the vanishing cycles (or more accurately to the thimbles); the morphisms between the objects are given by*

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; R) = R^{|L_i \cap L_j|} & \text{if } i < j \\ R \cdot id & \text{if } i = j \\ 0 & \text{if } i > j; \end{cases}$$

and the differential m_1*, composition* m_2 *and higher order products* m_k *are defined in terms of Lagrangian Floer homology inside* Σ_**. More precisely,*

$$m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$$

is trivial when the inequality $i_0 < i_1 < \cdots < i_k$ *fails to hold (i.e. it is always zero in this case, except for* m_2 *where composition with an identity morphism is given by the obvious formula). When* $i_0 < \cdots < i_k$*,* m_k *is defined by fixing a generic* ω*-compatible almost-complex structure on* Σ_* *and counting pseudo-holomorphic maps from a disk with* $k+1$ *cyclically ordered marked points on its boundary to* Σ_**, mapping the marked points to the given intersection points between vanishing cycles, and the portions of boundary between them to* $L_{i_0}, \dots, L_{i_k}$ *respectively.*

This definition calls for several clarifications. First of all, in our case Σ_* is a smooth elliptic curve and the vanishing cycles are homotopically non-trivial closed loops, we have $\pi_2(\Sigma_*) = 0$ and $\pi_2(\Sigma_*, L_i) = 0$; hence, we need not be concerned by bubbling issues in the definition of the Floer differential and products. In fact, the pseudo-holomorphic disks in Σ_* that we have to consider are nothing but immersed polygonal regions bounded by the vanishing cycles, satisfying a local convexity condition at each corner point.

Also, the Maslov class vanishes identically, so we have a well-defined $\mathbb{Z}$-grading by Maslov index on the Floer complexes $CF^*(L_i, L_j; R)$ once we choose graded Lagrangian lifts of the vanishing cycles. Since in our case $c_1(\Sigma_*) = 0$, we can do this by considering a nowhere vanishing 1-form $\Omega \in \Omega^1(\Sigma_*, \mathbb{C})$ and choosing a real lift of the phase function $\phi_i = \arg(\Omega_{|L_i}) : L_i \to S^1$ for each vanishing cycle. The degree of a given intersection point $p \in L_i \cap L_j$ is then determined by the difference between the phases of L_i and L_j at p.

Our next remark is that the pseudo-holomorphic disks appearing in Definition 4.1 are counted with appropriate weights, and with signs determined by choices of orientations of the relevant moduli spaces. The orientation is determined by the choice of a spin structure for each vanishing cycle L_i; in our case this spin structure must extend to the thimble, so it is necessarily the non-trivial one. In the one-dimensional case there is a convenient recipe for determining the correct sign factors, due to Seidel [217]. As will be clear from the discussion in §4.2 below, we will only be interested in the specific case where all morphisms have even degree and all spin structures are non-trivial. The sign rule can then be summarized as follows: pick a marked point on each L_i, distinct from the intersections with the other vanishing cycles; then the sign associated to a pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_*, \cup L_i)$ is $(-1)^{\nu(u)}$, where $\nu(u)$ is the number of marked points that the boundary of u passes through ([217], see also §4.6 of [16]).

Finally, the weight attributed to each pseudo-holomorphic map u keeps track of its relative homology class, which makes it possible to avoid convergence problems. The usual approach favored by mathematicians is to work over a Novikov ring, which keeps track of the relative homology class by introducing suitable formal variables. To remain closer to the physics, we use $\mathbb{C}$ as our coefficient ring, and assign weights according to the symplectic areas; this is in fact equivalent to working over the Novikov ring and specializing at the cohomology class of the symplectic form.

The weight formula is simplest when there is no B-field; in that case, we consider untwisted Floer theory, since any flat unitary bundle over the thimble D_i is trivial and hence restricts to L_i as the trivial bundle. We then count each map $u : (D^2, \partial D^2) \to (\Sigma_*, \cup L_i)$ with a coefficient $(-1)^{\nu(u)} \exp(-2\pi \int_{D^2} u^*\omega)$. (The normalization factor 2π is purely a matter of conventions, and is sometimes omitted in the literature; here we include it for convenience). Hence, given two intersection points $p \in L_i \cap L_j$, $q \in L_j \cap L_k$ ($i < j < k$), we have by definition

$$m_2(p,q) = \sum_{\substack{r \in L_i \cap L_k \\ \deg r = \deg p + \deg q}} \left(\sum_{[u] \in \mathcal{M}(p,q,r)} (-1)^{\nu(u)} \exp(-2\pi \int_{D^2} u^*\omega) \right) r$$

where $\mathcal{M}(p,q,r)$ is the moduli space of pseudo-holomorphic maps u from the unit disk to Σ_* (equipped with a generic ω-compatible almost-complex structure) such that $u(1) = p$, $u(\mathrm{j}) = q$, $u(\mathrm{j}^2) = r$ (where $\mathrm{j} = \exp(\frac{2i\pi}{3})$), and mapping the portions of unit circle $[1, \mathrm{j}]$, $[\mathrm{j}, \mathrm{j}^2]$, $[\mathrm{j}^2, 1]$ to L_i, L_j and L_k respectively. The other products are defined similarly. (Observe that Seidel's definition [214] does not involve any weights; this is because he only considers exact Lagrangian submanifolds in exact symplectic manifolds, in which case the symplectic areas are entirely determined by the primitives of the Liouville form and can be eliminated by considering suitably rescaled bases of the Floer complexes.)

In presence of a B-field, the weights are modified by the fact that we now consider *twisted* Floer homology. Indeed, each thimble D_i now comes equipped with a trivial complex line bundle $E_i = \underline{\mathbb{C}}$ and a connection ∇_i with curvature $-2\pi i B$, so its boundary L_i is equipped with the restricted bundle and the restricted connection, whose holonomy is $\mathrm{hol}_{\nabla_i}(L_i) = \exp(-2\pi i \int_{D_i} B)$ by Stokes' theorem. Since this property characterizes the connection ∇_i uniquely up to gauge, we can drop the line bundle and the connection from the notation when considering the objects (L_i, E_i, ∇_i) of $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$. Nonetheless, the holonomy of

∇_i modifies the weights attributed to the pseudo-holomorphic disks in the definition of the twisted Floer differentials and compositions. Namely, the weight attributed to a given pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_*, \cup L_i)$ is modified by a factor corresponding to the holonomy along its boundary, and becomes

$$(-1)^{\nu(u)} \operatorname{hol}(u(\partial D^2)) \exp(2\pi i \int_{D^2} u^*(B + i\omega)).$$

More precisely, we fix trivializations of the line bundles E_i, so that for each intersection point $p \in L_i \cap L_j$ we have a preferred isomorphism between the fibers $(E_i)_{|p}$ and $(E_j)_{|p}$; then it becomes possible to define the holonomy along the closed loop $u(\partial D^2)$ using the parallel transport induced by ∇_i from one "corner" of u to the next one, and the chosen isomorphism at each corner.

Although the category $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ depends on the chosen ordered collection of arcs $\{\gamma_i\}$, Seidel has obtained the following result [214] (for the exact case, but the proof extends to our situation):

Theorem 4.2 (Seidel) *If the ordered collection $\{\gamma_i\}$ is replaced by another one $\{\gamma_i'\}$, then the categories $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ and $\mathrm{Lag_{vc}}(W, \{\gamma_i'\})$ differ by a sequence of mutations.*

Hence, the category naturally associated to the fibration W is not the finite A_∞-category defined above, but rather a (bounded) *derived* category, obtained from $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ by considering twisted complexes of formal direct sums of Lagrangian vanishing cycles, and adding idempotent splittings and formal inverses of quasi-isomorphisms (see [152] and §5 of [214]). If two categories differ by mutations, then their derived categories are equivalent; hence the derived category $\boldsymbol{D}^b(\mathrm{Lag_{vc}}(W))$ depends only on the symplectic topology of W and not on the choice of an ordered system of arcs [214].

For the examples we consider, the A_∞-category $\mathrm{Lag_{vc}}(W, \{\gamma_i\})$ will in fact be an honest category (see below); the bounded derived category $\boldsymbol{D}^b(\mathrm{Lag_{vc}}(W))$ is then by definition the bounded derived category of finite rank modules over the algebra associated to this category.

4.2 Objects and morphisms

We now determine the categories $\mathrm{Lag_{vc}}(W_k, \{\gamma_i\})$ associated to the Landau-Ginzburg models (M_k, W_k) mirror to Del Pezzo surfaces and the systems of arcs $\{\gamma_i\}$ introduced in §3.3. We start with the objects and morphisms.

Recall that W_k has $k + 3$ isolated critical points, giving rise to $k + 3$ vanishing cycles $L_0, \dots, L_{k+2}$ in the reference fiber $\Sigma_0 \simeq W_k^{-1}(0)$. The homology classes of these vanishing cycles have been determined in §3 and are given by Lemma 3.1; these determine the vanishing cycles up to Lagrangian isotopy.

The derived category of vanishing cycles is not affected if we modify some of the vanishing cycles by Hamiltonian isotopies (more precisely, a Hamiltonian isotopy induces a chain map on the Floer complexes, which yields a quasi-isomorphism between the finite A_∞-categories of vanishing cycles). Hence, equipping the elliptic curve Σ_0 with a compatible flat metric, we can identify Σ_0 with the quotient of $\mathbb{C}$ by a lattice, and represent the vanishing cycles L_i by closed geodesics parallel to those represented in Figure 8.

Assume that the cohomology class of the symplectic form ω_k on M_k is generic (or more precisely, with the notations of §3.4, that $[\omega_k]\cdot[\bar{\Delta}_{i,j}]$ is never an integer multiple of $[\omega_k]\cdot[\Sigma_0]$). Then the geodesics L_i are all distinct, and their intersections are as pictured in Figure 8, so we have:

Lemma 4.3 *The geometric intersection numbers between the vanishing cycles are:*
- $|L_0\cap L_1| = |L_0\cap L_2| = |L_1\cap L_2| = 3$*;*
- *for* $0\le i<k$, $|L_0\cap L_{3+i}| = |L_2\cap L_{3+i}| = 1$ *and* $|L_1\cap L_{3+i}| = 2$*;*
- *for* $0\le i<j<k$, $|L_{3+i}\cap L_{3+j}| = 0$ *as soon as* $[\omega_k]\cdot[\Sigma_0] \not| \, [\omega_k]\cdot[\bar{\Delta}_{i,j}]$.

In the rest of this section, unless otherwise specified we always assume that the vanishing cycles L_i are represented by distinct closed geodesics.

As in [16], we denote by x_0, y_0, z_0 (resp. x_1, y_1, z_1 and $\bar{x}, \bar{y}, \bar{z}$) the generators of $\mathrm{Hom}(L_0, L_1)$ (resp. $\mathrm{Hom}(L_1, L_2)$ and $\mathrm{Hom}(L_0, L_2)$) corresponding to the intersection points represented in Figure 8. Moreover, we denote by a_i (resp. b_i, b'_i and c_i) the generators of $\mathrm{Hom}(L_0, L_{3+i})$ (resp. $\mathrm{Hom}(L_1, L_{3+i})$ and $\mathrm{Hom}(L_2, L_{3+i})$) corresponding to the intersection points between these vanishing cycles (see Figure 9).

Lemma 4.4 *For suitable choices of graded lifts of the vanishing cycles, all the morphisms in* $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ *have degree* 0*.*

Proof. Equip Σ_0 with a compatible flat metric and with a constant holomorphic 1-form Ω. Taking geodesic representatives of the vanishing cycles, the phase functions $\phi_i = \arg(\Omega_{|L_i}) : L_i \to \mathbb{R}/2\pi\mathbb{Z}$ are constant, and we can normalize Ω so that it takes real negative values on the oriented tangent space to L_0, i.e. $\phi_0 = \pi$. Then it is possible to choose real lifts $\tilde{\phi}_i \in \mathbb{R}$ of the phases in such a way that $\pi = \tilde{\phi}_0 > \tilde{\phi}_1 > \tilde{\phi}_2 > \tilde{\phi}_3 = \cdots = \tilde{\phi}_{k+2} > 0$ (see Figure 8 and recall the orientations chosen in Lemma 3.1). In the 1-dimensional case, the relationship between Maslov index and phase is very simple: given a transverse intersection point p between two graded Lagrangians $L, L' \subset \Sigma_0$, the Maslov index of $p \in CF^*(L, L')$ is equal to the smallest integer greater than $\frac{1}{\pi}(\tilde{\phi}_{L'}(p) - \tilde{\phi}_L(p))$. Since we only consider the Floer complexes $CF^*(L_i, L_j)$ for $i<j$, which implies that $\tilde{\phi}_j - \tilde{\phi}_i \in (-\pi, 0)$ at every intersection point, for these choices of graded Lagrangian lifts of the vanishing cycles all morphisms in $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ have degree 0. □

Since each product m_j shifts degree by $2-j$, it follows immediately that the A_∞-category $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ is actually an honest category:

Corollary 4.5 $m_j = 0$ *for all* $j \ne 2$*.*

Hence, the final step of the argument is a careful study of the various immersed triangular regions bounded by the vanishing cycles in Σ_0 and their contributions to m_2.

4.3 Compositions As before we assume that the Lagrangian vanishing cycles are realized by distinct closed geodesics in the flat torus Σ_0, and we determine the contributions to m_2 of the various immersed triangular regions in $(\Sigma_0, \cup L_i)$. We use the notations introduced in §4.2 for the intersection points, and those introduced in §3.4 for various 2-chains in Σ_0 and the corresponding 2-cycles in M_k. We also introduce the following notations:

Definition 4.6 *Let* $q_C = \exp(2\pi i\,[B+i\omega]\cdot[\bar{C}])$ *and* $q_F = \exp(2\pi i\,[B+i\omega]\cdot[\Sigma_0])$*, and define*

$$\zeta_+ = \sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n+1)/2},\quad \zeta_- = \sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n-1)/2},\quad \zeta_0 = \sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{3n(n-1)/2}.$$

Since ω is a symplectic form on Σ_0, we have $|q_F| = \exp(-2\pi\,[\omega]\cdot[\Sigma_0]) < 1$, which ensures the convergence of the series ζ_+, ζ_- and ζ_0.

Proposition 4.7 *There exist constants* $\alpha_{xy}, \alpha_{yx}, \alpha_{yz}, \alpha_{zy}, \alpha_{zx}, \alpha_{xz} \in \mathbb{C}$ *such that*

$$\begin{aligned} m_2(x_0, y_1) &= \alpha_{xy}\bar{z}, &\qquad m_2(y_0, x_1) &= \alpha_{yx}\bar{z},\\ m_2(y_0, z_1) &= \alpha_{yz}\bar{x}, &\qquad m_2(z_0, y_1) &= \alpha_{zy}\bar{x},\\ m_2(z_0, x_1) &= \alpha_{zx}\bar{y}, &\qquad m_2(x_0, z_1) &= \alpha_{xz}\bar{y}, \end{aligned}$$

and these constants satisfy the relation

$$\frac{\alpha_{xy}\alpha_{yz}\alpha_{zx}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -q_C \left(\frac{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n+1)/2}}{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n-1)/2}} \right)^3 = -q_C \left(\frac{\zeta_+}{\zeta_-}\right)^3. \tag{4.1}$$

Remark 4.8 *The quantity appearing in the right-hand side of* (4.1) *can be understood in terms of certain theta functions; see* §4.5 *for details.*

Before giving the proof, we make an observation which will be useful throughout this section. The geodesics L_i are not necessarily those pictured in Figure 8, but they are parallel to them. So we can deform (in a non-Hamiltonian manner) the configuration of vanishing cycles to that of Figure 8, and all intersection points and relative 2-cycles in $(\Sigma_0, \cup L_i)$ can be followed through the deformation. Hence, immersed triangular regions in $(\Sigma_0, \cup L_i)$ are in one to one correspondence with those in the configuration of Figure 8 (but of course the deformation does not preserve areas). Moreover, we can choose the deformation in such a way that the relative 2-cycles C and C_i in Σ_0 deform to those represented on Figures 8–9 (rather than to 2-cycles which differ by a multiple of the fundamental cycle of Σ_0).

Proof. [Proof of Proposition 4.7] The composition $m_2(x_0, y_1)$ is the sum of an infinite series of contributions, corresponding to all immersed triangular regions in Σ_0 with corners at the intersection points x_0, y_1, and one of the points in $L_0 \cap L_2$. By the above remark we can enumerate these regions by looking at Figure 8. Considering the side which lies on L_1, it is then easy to see that for every homotopy class of arc joining x_0 to y_1 inside L_1 there is a unique such immersed triangular region, and the third vertex is always $\bar{z}$.

These various regions can be labelled by integers $n \in \mathbb{Z}$ in such a way that, denoting by $T_{xy,n}$ the corresponding 2-chains in Σ_0, we have $\partial T_{xy,n} - \partial T_{xy,n'} = (n-n')(-L_0 + L_1 - L_2)$ for all $n, n' \in \mathbb{Z}$. We can choose the integer labels in such a way that, after deforming to the configuration in Figure 8, $T_{xy,0}$ becomes the smallest triangle with vertices $x_0, y_1, \bar{z}$. (So, in Figure 8, $T_{xy,-1}$ is the immersed region bounded by the portions of $L_0 \cup L_1 \cup L_2$ which do not belong to $\partial T_{xy,0}$; and all the other $T_{xy,n}$ have edges which wrap more than once around the vanishing cycles).

By comparing $\partial T_{xy,n}$ and $\partial T_{xy,0}$, it is clear that the 2-chain represented by $T_{xy,n}$ can be expressed in the form $T_{xy,n} = T_{xy,0} + nC + \phi(n)\Sigma_0$ for some $\phi(n) \in \mathbb{Z}$. Moreover, by looking at Figure 8 one easily checks that $\phi(n) = \frac{1}{2}n(3n+1)$. (So e.g. $T_{xy,-1} = T_{xy,0} - C + \Sigma_0$, and $T_{xy,1} = T_{xy,0} + C + 2\Sigma_0$). Let $\psi_{xy} \in \mathbb{C}$ be the coefficient of the contribution of $T_{xy,0}$ to $m_2(x_0, y_1)$. Then, by comparing the symplectic areas and boundary holonomies for $T_{xy,n}$ and $T_{xy,0}$, one easily checks that the contribution of $T_{xy,n}$ is equal to

$$(-1)^n \exp\left(2\pi i \, [B + i\omega] \cdot \left(n[\bar{C}] + \frac{n(3n+1)}{2}[\Sigma_0]\right)\right) \psi_{xy} = (-1)^n \, q_C^n \, q_F^{n(3n+1)/2} \, \psi_{xy}.$$

In this expression the sign factor $(-1)^n$ is due to the non-triviality of the spin structures (observe that $\partial C = -L_0 + L_1 - L_2$ passes once through each of the three marked points on L_0, L_1, L_2); the total holonomy of the flat connections ∇_i along $\partial T_{xy,n} - \partial T_{xy,0} = n \, \partial C$ is $\exp(2\pi i \, n \int_{D_0 - D_1 + D_2} B)$ by Stokes' theorem; and the integral of $B + i\omega$ over $T_{xy,n}$ differs from that over $T_{xy,0}$ by the amount $n \int_C (B + i\omega) + \frac{1}{2}n(3n+1) \, [B + i\omega] \cdot [\Sigma_0]$.

Summing over $n \in \mathbb{Z}$, and using the notation introduced in Definition 4.6, we obtain

$$\alpha_{xy} = \zeta_+ \, \psi_{xy}.$$

The calculations of $m_2(y_0, z_1)$ and $m_2(z_0, x_1)$ are exactly identical, and lead to similar expressions. Namely, denote by ψ_{yz} (resp. ψ_{zx}) the contribution of the triangular region $T_{yz,0}$ (resp. $T_{zx,0}$) which, after deforming to the configuration in Figure 8, corresponds to the smallest triangle with vertices y_0, z_1, $\bar{x}$ (resp. z_0, x_1, $\bar{y}$). Then one easily checks by the same argument as above that $\alpha_{yz} = \zeta_+ \, \psi_{yz}$ and $\alpha_{zx} = \zeta_+ \, \psi_{zx}$.

Next we consider the composition $m_2(y_0, x_1)$, which is again the sum of an infinite series of contributions from triangular regions $T_{yx,n}$, $n \in \mathbb{Z}$, which all have vertices y_0, x_1, $\bar{z}$. We can choose the labels in such a way that, after deforming to the configuration in Figure 8, $T_{yx,0}$ becomes the smallest such triangle, and $T_{yx,n} = T_{yx,0} + nC + \frac{1}{2}n(3n-1)\Sigma_0$. Denoting by ψ_{yx} the coefficient associated to $T_{yx,0}$, it is easy to check by the same argument as above that the contribution of $T_{yx,n}$ is equal to $(-1)^n \, q_C^n \, q_F^{n(3n-1)/2} \, \psi_{yx}$, so that

$$\alpha_{yx} = \zeta_- \psi_{yx}.$$

Similarly, with the obvious notations we have $\alpha_{zy} = \zeta_- \psi_{zy}$ and $\alpha_{xz} = \zeta_- \psi_{xz}$. Finally, observe that

$$\frac{\psi_{xy}\psi_{yz}\psi_{zx}}{\psi_{yx}\psi_{zy}\psi_{xz}} = -q_C.$$

Indeed, $T_{xy,0} + T_{yz,0} + T_{zx,0} - T_{yx,0} - T_{zy,0} - T_{xz,0} = C$ (cf. Figure 8). Therefore, comparing the weights associated to these various triangles, the weighting by area gives a factor of $\exp(2\pi i \int_C B + i\omega)$, while the holonomy along the boundary $\partial C = -L_0 + L_1 - L_2$ is equal to $\exp(2\pi i \int_{D_0-D_1+D_2} B)$, and finally the minus sign is due to the orientation conventions, since ∂C passes once through each of the three marked points on the vanishing cycles. Hence

$$\frac{\alpha_{xy}\alpha_{yz}\alpha_{zx}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = \frac{\psi_{xy}\psi_{yz}\psi_{zx}\,\zeta_+^3}{\psi_{yx}\psi_{zy}\psi_{xz}\,\zeta_-^3} = -q_C\left(\frac{\zeta_+}{\zeta_-}\right)^3.$$

□

Remark 4.9 *If $[\omega+iB]\cdot[\bar{C}] = 0$, then $q_C = 1$ and the ratio between $\alpha_{xy}\alpha_{yz}\alpha_{zx}$ and $\alpha_{yx}\alpha_{zy}\alpha_{xz}$ becomes equal to -1 irrespective of the value of q_F; this corresponds to a classical (commutative) Del Pezzo surface.*

Moreover, in the limit where $[\omega]\cdot[\Sigma_0] \to \infty$, we have $q_F = 0$ and the ratio becomes $-q_C$, which corresponds to the toric case studied in [16].

Proposition 4.10 *There exist constants $\alpha_{xx}, \alpha_{yy}, \alpha_{zz} \in \mathbb{C}$ such that*

$$m_2(x_0,x_1) = \alpha_{xx}\bar{x}, \quad m_2(y_0,y_1) = \alpha_{yy}\bar{y}, \quad m_2(z_0,z_1) = \alpha_{zz}\bar{z},$$

and these constants satisfy the relation

$$\frac{\alpha_{xx}\alpha_{yy}\alpha_{zz}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -\frac{q_F}{q_C}\left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n\,q_C^n\,q_F^{3n(n-1)/2}}{\sum_{n\in\mathbb{Z}}(-1)^n\,q_C^n\,q_F^{n(3n-1)/2}}\right)^3 = -\frac{q_F}{q_C}\left(\frac{\zeta_0}{\zeta_-}\right)^3.$$

Proof. The argument is similar to the proof of Proposition 4.7. The immersed triangular regions which contribute to $m_2(x_0,x_1)$ all have vertices $\bar{x}$ as their third vertex, and can be indexed by integers $n \in \mathbb{Z}$ in a manner such that $\partial T_{xx,n} - \partial T_{xx,n'} = (n-n')\,\partial C$ for all $n, n' \in \mathbb{Z}$. We can choose the integer labels in such a way that, after deforming to the standard configuration, $T_{xx,0}$ and $T_{xx,1} = T_{xx,0} + C$ are the two embedded triangles with vertices x_0, x_1, $\bar{x}$ visible on Figure 8. It is then easy to check that $T_{xx,n} = T_{xy,0} + nC + \frac{3}{2}n(n-1)\Sigma_0$. Hence, denoting by ψ_{xx} the coefficient associated to $T_{xx,0}$, we have

$$\alpha_{xx} = \zeta_0\psi_{xx},$$

by the same argument as in previous calculations. Similarly, with the obvious notations, we have $\alpha_{yy} = \zeta_0\psi_{yy}$ and $\alpha_{zz} = \zeta_0\psi_{zz}$. Moreover, $T_{xx,0} + T_{yy,0} + T_{zz,0} - T_{yx,0} - T_{zy,0} - T_{xz,0} = \Sigma_0 - C$, which implies (by the same argument as above) that

$$\frac{\psi_{xx}\psi_{yy}\psi_{zz}}{\psi_{yx}\psi_{zy}\psi_{xz}} = -\frac{q_F}{q_C}.$$

Therefore

$$\frac{\alpha_{xx}\alpha_{yy}\alpha_{zz}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = \frac{\psi_{xx}\psi_{yy}\psi_{zz}\,\zeta_0^3}{\psi_{yx}\psi_{zy}\psi_{xz}\,\zeta_-^3} = -\frac{q_F}{q_C}\left(\frac{\zeta_0}{\zeta_-}\right)^3.$$

□

When $q_F = 0$ (in particular in the toric case) we have $\alpha_{xx}\alpha_{yy}\alpha_{zz} = 0$, as in [16]. The same conclusion also holds when $q_C = 1$ (the commutative case). In fact, when $q_C = 1$ each of the constants α_{xx}, α_{yy}, α_{zz} is zero, since in that case we have $\zeta_0 = 0$ (because the terms corresponding to n and $1-n$ in the series defining ζ_0 exactly cancel each other).

Definition 4.11 *Let* $q_i = \exp(2\pi i\,[B + i\omega]\cdot[\bar{C}_i])$, *and define*

$$\zeta_{i,+} = \sum_{n\in\mathbb{Z}}(-1)^n\, q_i^n\, q_F^{n(3n+1)/2},\quad \zeta_{i,-} = \sum_{n\in\mathbb{Z}}(-1)^n\, q_i^n\, q_F^{n(3n-1)/2},\quad \zeta_{i,0} = \sum_{n\in\mathbb{Z}}(-1)^n\, q_i^n\, q_F^{3n(n-1)/2}.$$

Proposition 4.12 *There exist constants* $\beta_{\bar{x},i}, \beta_{\bar{y},i}, \beta_{\bar{z},i} \in \mathbb{C}$ *such that*

$$m_2(\bar{x}, c_i) = \beta_{\bar{x},i} a_i, \quad m_2(\bar{y}, c_i) = \beta_{\bar{y},i} a_i, \quad m_2(\bar{z}, c_i) = \beta_{\bar{z},i} a_i,$$

and these constants satisfy the relations

$$\frac{\beta_{\bar{z},i}^2\,\alpha_{xy}\alpha_{zz}}{\beta_{\bar{x},i}\beta_{\bar{y},i}\,\alpha_{zy}\alpha_{xz}} = \left(\frac{\zeta_{i,-}}{\zeta_-}\right)^2 \frac{\zeta_+\,\zeta_0}{\zeta_{i,+}\,\zeta_{i,0}},$$

$$\frac{\beta_{\bar{x},i}^2\,\alpha_{yz}\alpha_{xx}}{\beta_{\bar{y},i}\beta_{\bar{z},i}\,\alpha_{xz}\alpha_{yx}} = -q_i\left(\frac{\zeta_{i,+}}{\zeta_-}\right)^2 \frac{\zeta_+\,\zeta_0}{\zeta_{i,0}\,\zeta_{i,-}}, \quad \textit{and} \quad \frac{\beta_{\bar{y},i}^2\,\alpha_{zx}\alpha_{yy}}{\beta_{\bar{z},i}\beta_{\bar{x},i}\,\alpha_{yx}\alpha_{zy}} = -\frac{q_F}{q_i}\left(\frac{\zeta_{i,0}}{\zeta_-}\right)^2 \frac{\zeta_+\,\zeta_0}{\zeta_{i,-}\,\zeta_{i,+}},$$

where ζ_+, ζ_-, ζ_0, $\zeta_{i,+}$, $\zeta_{i,-}$, $\zeta_{i,0}$, q_i *and* q_F *are as in Definitions* 4.6 *and* 4.11.

Proof. As before, the constants $\beta_{\bar{x},i}, \beta_{\bar{y},i}, \beta_{\bar{z},i}$ are the sums of infinite series corresponding to all immersed triangular regions with vertices at a_i, c_i, and one of $\bar{x}, \bar{y}, \bar{z}$. For example the coefficient $\beta_{\bar{z},i}$ associated to composition $m_2(\bar{x}, c_i)$ is the sum of an infinite series of contributions associated to triangular regions $T_{\bar{z},i,n}$, $n \in \mathbb{Z}$. The integer labels can be chosen so that $\partial T_{\bar{z},i,n} - \partial T_{\bar{z},i,n'} = (n - n')\partial C_i$ and, after deforming to the configuration in Figure 9, $T_{\bar{z},i,0}$ becomes the smallest triangle with vertices $\bar{z}$, a_i, c_i (i.e., the triangle which appears with coefficient -2 in the 2-chain C_i). Then one easily checks that $T_{\bar{z},i,n} = T_{\bar{z},i,0} + nC_i + \frac{1}{2}n(3n-1)\Sigma_0$. Therefore, denoting by $\psi_{\bar{z},i}$ the coefficient associated to $T_{\bar{z},i,0}$, the same argument as in the previous calculations yields the formula

$$\beta_{\bar{z},i} = \zeta_{i,-}\psi_{\bar{z},i}.$$

Similarly, denote by $T_{\bar{x},i,n}$, $n \in \mathbb{Z}$, the immersed triangles contributing to $m_2(\bar{x}, c_i)$, in such a way that $\partial T_{\bar{x},i,n} - \partial T_{\bar{x},i,n'} = (n - n')\partial C_i$, and $T_{\bar{x},i,0}$ corresponds to the smallest triangle with vertices $\bar{x}$, a_i, c_i in Figure 9 (i.e. the triangle which appears with coefficient $+2$ in the 2-chain

C_i). Then $T_{\bar{x},i,n} = T_{\bar{x},i,0} + nC_i + \frac{1}{2}n(3n+1)\Sigma_0$. Therefore, denoting by $\psi_{\bar{x},i}$ the contribution of $T_{\bar{x},i,0}$, we have $\beta_{\bar{x},i} = \zeta_{i,+}\psi_{\bar{x},i}$.

Finally, labelling the triangles with vertices $\bar{y}$, a_i, c_i by integers in such a way that $T_{\bar{y},i,0}$ and $T_{\bar{y},i,1} = T_{\bar{y},i,0} + C_i$ correspond to the negative and positive parts of C_i respectively, it is easy to check that $T_{\bar{y},i,n} = T_{\bar{y},i,0} + nC_i + \frac{3}{2}n(n-1)\Sigma_0$, so denoting by $\psi_{\bar{y},i}$ the contribution of $T_{\bar{y},i,0}$ we have $\beta_{\bar{y},i} = \zeta_{i,0}\psi_{\bar{y},i}$. It follows that

$$\frac{\beta_{\bar{z},i}^2\,\alpha_{xy}\alpha_{zz}}{\beta_{\bar{x},i}\beta_{\bar{y},i}\,\alpha_{zy}\alpha_{xz}} = \frac{\psi_{\bar{z},i}^2\,\psi_{xy}\psi_{zz}}{\psi_{\bar{x},i}\psi_{\bar{y},i}\,\psi_{zy}\psi_{xz}}\,\frac{\zeta_{i,-}^2\,\zeta_+\,\zeta_0}{\zeta_{i,+}\,\zeta_{i,0}\,\zeta_-^2}.$$

Moreover, the 2-chains $2\,T_{\bar{z},i,0} + T_{xy,0} + T_{zz,0}$ and $T_{\bar{x},i,0} + T_{\bar{y},i,0} + T_{zy,0} + T_{xz,0}$ are equal, which implies that $\psi_{\bar{z},i}^2\,\psi_{xy}\psi_{zz} = \psi_{\bar{x},i}\psi_{\bar{y},i}\,\psi_{zy}\psi_{xz}$ and completes the proof of the first identity.

The arguments are the same for

$$\frac{\beta_{\bar{x},i}^2\,\alpha_{yz}\alpha_{xx}}{\beta_{\bar{y},i}\beta_{\bar{z},i}\,\alpha_{xz}\alpha_{yx}} = \frac{\psi_{\bar{x},i}^2\,\psi_{yz}\psi_{xx}}{\psi_{\bar{y},i}\psi_{\bar{z},i}\,\psi_{xz}\psi_{yx}}\,\frac{\zeta_{i,+}^2\,\zeta_+\,\zeta_0}{\zeta_{i,0}\,\zeta_{i,-}\,\zeta_-^2},$$

observing that $2\,T_{\bar{x},i,0} + T_{yz,0} + T_{xx,0} - T_{\bar{y},i,0} - T_{\bar{z},i,0} - T_{xz,0} - T_{yx,0} = C_i$ (for which the corresponding weight is $-q_i$), and for

$$\frac{\beta_{\bar{y},i}^2\,\alpha_{zx}\alpha_{yy}}{\beta_{\bar{z},i}\beta_{\bar{x},i}\,\alpha_{yx}\alpha_{zy}} = \frac{\psi_{\bar{y},i}^2\,\psi_{zx}\psi_{yy}}{\psi_{\bar{z},i}\psi_{\bar{x},i}\,\psi_{yx}\psi_{zy}}\,\frac{\zeta_{i,0}^2\,\zeta_+\,\zeta_0}{\zeta_{i,-}\,\zeta_{i,+}\,\zeta_-^2},$$

observing that $2\,T_{\bar{y},i,0} + T_{zx,0} + T_{yy,0} - T_{\bar{z},i,0} - T_{\bar{x},i,0} - T_{yx,0} - T_{zy,0} = \Sigma_0 - C_i$ (for which the corresponding weight is $-q_F/q_i$). □

Corollary 4.13 *The constants $\beta_{\bar{x},i}$, $\beta_{\bar{y},i}$, $\beta_{\bar{z},i}$ satisfy the relations:* $\dfrac{\beta_{\bar{z},i}^3}{\beta_{\bar{x},i}^3}\,\dfrac{\alpha_{xy}\alpha_{yx}\alpha_{zz}}{\alpha_{yz}\alpha_{zy}\alpha_{xx}} = -\dfrac{1}{q_i}\Big(\dfrac{\zeta_{i,-}}{\zeta_{i,+}}\Big)^3$, $\dfrac{\beta_{\bar{x},i}^3}{\beta_{\bar{y},i}^3}\,\dfrac{\alpha_{yz}\alpha_{zy}\alpha_{xx}}{\alpha_{zx}\alpha_{xz}\alpha_{yy}} = \dfrac{q_i^2}{q_F}\Big(\dfrac{\zeta_{i,+}}{\zeta_{i,0}}\Big)^3$, *and* $\dfrac{\beta_{\bar{y},i}^3}{\beta_{\bar{z},i}^3}\,\dfrac{\alpha_{zx}\alpha_{xz}\alpha_{yy}}{\alpha_{xy}\alpha_{yx}\alpha_{zz}} = -\dfrac{q_F}{q_i}\Big(\dfrac{\zeta_{i,0}}{\zeta_{i,-}}\Big)^3$.

Proposition 4.14 *For all $0 \le i,j < k$ we have the identities*

$$\frac{\beta_{\bar{y},i}\,\beta_{\bar{z},j}}{\beta_{\bar{y},j}\,\beta_{\bar{z},i}} = \tilde{q}_{i,j}\,\frac{\zeta_{i,0}\,\zeta_{j,-}}{\zeta_{j,0}\,\zeta_{i,-}},\quad \frac{\beta_{\bar{z},i}\,\beta_{\bar{x},j}}{\beta_{\bar{z},j}\,\beta_{\bar{x},i}} = \tilde{q}_{i,j}\,\frac{\zeta_{i,-}\,\zeta_{j,+}}{\zeta_{j,-}\,\zeta_{i,+}},\quad \textit{and}\quad \frac{\beta_{\bar{x},i}\,\beta_{\bar{y},j}}{\beta_{\bar{x},j}\,\beta_{\bar{y},i}} = \tilde{q}_{i,j}^{-2}\,\frac{\zeta_{i,+}\,\zeta_{j,0}}{\zeta_{j,+}\,\zeta_{i,0}},$$

where $\tilde{q}_{i,j} = \exp(2\pi i\,[B+i\omega]\cdot[\bar{\Delta}_{i,j}])$, and $\zeta_{i,+}$, $\zeta_{i,-}$, $\zeta_{i,0}$ are as in Definition 4.11.

Proof. We claim that $T_{\bar{y},i,0} + T_{\bar{z},j,0} - T_{\bar{y},j,0} - T_{\bar{z},i,0} = \Delta_{i,j}$. Indeed, consider first a situation in which L_{3+i} lies in the position represented in Figure 9, and L_{3+j} lies close to it, but is slightly shifted towards the lower-right direction. Then the intersection points a_j and c_j lie close to a_i and c_i, and following the triangular regions through the small deformation which takes L_{3+i} to L_{3+j}, we easily see that $T_{\bar{z},j,0}$ is obtained by slightly truncating $T_{\bar{z},i,0}$ on its L_{3+i} side. Similarly,

$T_{\bar{y},j,0}$ is obtained by slightly truncating $T_{\bar{y},i,0}$, and since $\Delta_{i,j}$ is simply the thin strip in between L_{3+i} and L_{3+j} the claim follows.

The same property remains true if L_{3+i} and L_{3+j} are further apart from each other. This can be checked explicitly for example in the configuration of Figure 8, where $\Delta_{i,j}$ is as pictured on Figure 9 (right). (In this configuration the deformation from L_{3+i} to L_{3+j} passes through $\bar{y}$ and $\bar{z}$, so the triangles $T_{\bar{z},i,0}$ and $T_{\bar{z},j,0}$ lie on opposite sides of $\bar{z}$, and similarly for $T_{\bar{y},i,0}$ and $T_{\bar{y},j,0}$; this latter triangle is now the small region to the lower-right of $\bar{y}$ on Figure 8).

As a consequence, we have the identity

$$\frac{\psi_{\bar{y},i}\psi_{\bar{z},j}}{\psi_{\bar{y},j}\psi_{\bar{z},i}} = \tilde{q}_{i,j},$$

which implies the first formula in the proposition. The two other formulas are proved similarly, using the equalities $T_{\bar{z},i,0} + T_{\bar{x},j,0} - T_{\bar{z},j,0} - T_{\bar{x},i,0} = \Delta_{i,j}$ and $T_{\bar{x},i,0} + T_{\bar{y},j,0} - T_{\bar{x},j,0} - T_{\bar{y},i,0} = -2\,\Delta_{i,j}$. □

Remark 4.15 *The various ratios computed in Propositions* 4.7–4.14 *are intrinsic quantities attached to the symplectic geometry of W_k, i.e. they are invariant under Hamiltonian deformations, irrespective of whether the vanishing cycles are represented by geodesics or not. Equivalently, they are invariant under rescalings of the chosen generators of the morphism spaces in $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$. On the other hand, if we allow ourselves to use the fact that the vanishing cycles are geodesics in a flat torus, we can also compute some interesting non-intrinsic quantities (i.e., quantities which depend on a particular choice of scaling of the generators).*

For example, the invariance of L_0, L_1, L_2 under the translation of the torus which maps x_0 to y_0 (and y_0 to z_0, z_0 to x_0) implies that, for suitable choices of the marked points associated to the spin structures and of the isomorphisms between lines used to calculate boundary holonomies, $\alpha_{xy} = \alpha_{yz} = \alpha_{zx}$, $\alpha_{yx} = \alpha_{xy} = \alpha_{xz}$, and $\alpha_{xx} = \alpha_{yy} = \alpha_{zz}$. In fact, going over the calculations in the proofs of Propositions 4.7 *and* 4.10, *and observing that, in terms of areas and boundary holonomies, the contributions of $T_{xy,0} - T_{yx,0}$ and $T_{xx,0} - T_{yx,0}$ are equivalent to those of $\frac{1}{3}C$ and $\frac{1}{3}(\Sigma_0 - C)$ respectively, one easily checks that there exists a constant $s \neq 0$ such that*

$$\begin{aligned} \alpha_{xy} = \alpha_{yz} = \alpha_{zx} &= s\, q_C^{1/3}\, \zeta_+, \\ \alpha_{xx} = \alpha_{yy} = \alpha_{zz} &= s\, q_F^{1/3} q_C^{-1/3}\, \zeta_0, \\ \alpha_{yx} = \alpha_{zy} = \alpha_{xz} &= -s\, \zeta_-, \end{aligned} \tag{4.2}$$

where by definition $q_C^{1/3} = \exp(\frac{2\pi i}{3}[B + i\omega]\cdot[\bar{C}])$ and $q_F^{1/3} = \exp(\frac{2\pi i}{3}[B + i\omega]\cdot[\Sigma_0])$. Similarly, for suitable choices we have

$$\beta_{\bar{x},i} = s_i\, q_i^{1/3}\, \zeta_{i,+}, \quad \beta_{\bar{y},i} = s_i\, q_F^{1/3}\, q_i^{-1/3}\, \zeta_{i,0}, \quad \text{and } \beta_{\bar{z},i} = -s_i\, \zeta_{i,-}, \tag{4.3}$$

where s_i is a non-zero constant and $q_i^{1/3} = \exp(\frac{2\pi i}{3}[B + i\omega]\cdot[\bar{C}_i])$.

The formulas (4.2) and (4.3) are only valid in the flat case, when the complexified symplectic form on Σ_0 is translation-invariant and the vanishing cycles are geodesics; however, in the general case we can always modify our choices of generators of the various morphism spaces by suitable scaling factors (or equivalently, modify the vanishing cycles by certain Hamiltonian isotopies) in order to make these formulas hold. It is therefore these simpler formulas that we will use in order to determine the mirror map in §5 below.

4.4 Simple degenerations

In this section we consider the situation where the symplectic area of one of the 2-cycles $\bar{\Delta}_{i,j}$ becomes a multiple of that of the fiber Σ_0. The vanishing cycles L_{3+i} and L_{3+j} are then Hamiltonian isotopic to each other in Σ_0, and hence cannot be represented by disjoint geodesics anymore. However we can still represent L_{3+i} by a closed geodesic, and L_{3+j} by a small generic Hamiltonian perturbation of L_{3+i}, intersecting it transversely in two points. These two intersection points have Maslov indices 0 and 1 respectively (if we choose the same graded lifts as previously), and for this configuration we have:

Lemma 4.16 *If there exist integers $n \in \mathbb{Z}$ and $i < j$ such that $[\omega] \cdot [\bar{\Delta}_{i,j}] = n\,[\omega] \cdot [\Sigma_0]$, then $\mathrm{Hom}(L_{3+i}, L_{3+j})$ is graded isomorphic to $H^*(S^1) \otimes \mathbb{C}$. Moreover, the differential*

$$m_1 : \mathrm{Hom}^0(L_{3+i}, L_{3+j}) \to \mathrm{Hom}^1(L_{3+i}, L_{3+j})$$

is zero if $[B] \cdot [\bar{\Delta}_{i,j}] \in \mathbb{Z} + n\,[B] \cdot [\Sigma_0]$, and an isomorphism otherwise.

Proof. The only contributions to m_1 come from the two disks D' and D'' bounded by L_{3+i} and L_{3+j}. The 2-chain $D' - D''$ in Σ_0 has symplectic area zero, and is in fact given by $D' - D'' = \Delta_{i,j} - n\Sigma_0$. Hence we can compare the coefficients ψ' and ψ'' associated to these two disks by the same argument as in §4.3. Namely, ψ' and ψ'' differ by a sign factor, a holonomy factor, and an area factor.

In this case the sign factor is -1 (the sign rule for odd degree morphisms is slightly more subtle than that for even degree morphisms [217]; here we can see directly that the signs for D' and D'' have to be different since the untwisted Floer homology of L_{3+i} and L_{3+j} is non-trivial); the holonomy factor is the total holonomy along $\partial(D' - D'') = L_{3+j} - L_{3+i}$, i.e. $\exp(2\pi i \int_{D_{3+i}-D_{3+j}} B)$; and the area factor is $\exp(2\pi i \int_{D'-D''} B + i\omega)$. It follows that

$$\psi' = -\exp\bigl(2\pi i\,[B + i\omega] \cdot ([\bar{\Delta}_{i,j}] - n[\Sigma_0])\bigr)\,\psi'',$$

since $D' - D'' + D_{3+i} - D_{3+j} = \bar{\Delta}_{i,j} - n\Sigma_0$. Since m_1 is determined by the sum $\psi' + \psi''$, we conclude that $m_1 = 0$ if and only if $[B + i\omega] \cdot ([\bar{\Delta}_{i,j}] - n[\Sigma_0])$ is an integer. □

In other words, if $[B + i\omega] \cdot [\bar{\Delta}_{i,j}] \in \mathbb{Z} \oplus ([B + i\omega] \cdot [\Sigma_0])\,\mathbb{Z}$, then (L_{3+i}, ∇_{3+i}) and (L_{3+j}, ∇_{3+j}) are essentially identical, and we have a non-cancelling pair of extra morphisms of degrees 0 and 1 from L_{3+i} to L_{3+j}; this mirrors the situation in which $\mathbb{CP}^2$ is blown up twice at infinitely close points, in which case there is a rational -2-curve and the derived category of coherent sheaves is richer than in the generic case. In all other situations the intersection points between L_{3+i} and L_{3+j}, if any, are killed by the twisted Floer differential (even when L_{3+i} and L_{3+j} are Hamiltonian isotopic).

Remark 4.17 *It is important to note that, due to the presence of immersed convex polygonal regions with two edges on $L_0 \cup L_1 \cup L_2$ and two edges on $L_{3+i} \cup L_{3+j}$ (with a corner at the intersection point of Maslov index 1), we have to consider not only the Floer differential m_1, but also the higher-order composition m_3. For example, when L_{3+i} and L_{3+j} are Hamiltonian isotopic the composition*

$$m_3 : \mathrm{Hom}(L_0, L_2) \otimes \mathrm{Hom}(L_2, L_{3+i}) \otimes \mathrm{Hom}^1(L_{3+i}, L_{3+j}) \longrightarrow \mathrm{Hom}(L_0, L_{3+j})$$

is in general non-zero (and similarly with L_1 instead of L_0 or L_2).

As in §2.2, it is possible to describe things in a simpler and more unified manner by considering a suitable mutation of the exceptional collection $(L_0, \dots, L_{k+2})$. Assume for simplicity that the two vanishing cycles which may coincide are L_3 and L_4, while the others are represented by distinct geodesics. Then we can modify the system of arcs $\{\gamma_i\}$ considered so far to a new ordered system of arcs $\{\gamma'_i\}$ such that $\gamma'_i = \gamma_i$ for $i \not\in \{2,3\}$, $\gamma'_3 = \gamma_2$, and γ'_2 connects the origin to $\lambda_3 \approx \infty$ along the negative real axis. This gives rise to a new category $\mathrm{Lag_{vc}}(W_k, \{\gamma'_i\})$, in which all objects but one can be identified with the objects $L_i,\ i \neq 3$ of $\mathrm{Lag_{vc}}(W_k, \{\gamma_i\})$; thus, we denote by $L_0, L_1, L', L_2, L_4, \dots, L_{k+2}$ the objects of $\mathrm{Lag_{vc}}(W_k, \{\gamma'_i\})$. The morphisms and compositions not involving L' are as in $\mathrm{Lag_{vc}}(W_k, \{\gamma_i\})$.

The new vanishing cycle L' is Hamiltonian isotopic to the image of L_3 under the positive Dehn twist along L_2. In particular, with the notations of Lemma 3.1, and for a suitable choice of orientation, its homology class is $[L'] = [L_2] - [L_3] = b$. Choosing a geodesic representative, we have $|L_0 \cap L'| = 2$, $|L_1 \cap L'| = 1$, $|L' \cap L_2| = 1$, and $|L' \cap L_{3+i}| = 1$ for $i \geq 1$, and all morphisms in $\mathrm{Lag_{vc}}(W_k, \{\gamma'_i\})$ have degree 0.

Because L' is Hamiltonian isotopic to the image of L_3 under the Dehn twist along L_2, the fiber Σ_0 contains a 2-chain Δ' with $\partial\Delta' = L' + L_4 - L_2$ and such that $\int_{\Delta'} \omega = \int_{\Delta_{3,4}} \omega$. Capping off Δ' with the appropriate Lefschetz thimbles, we obtain a 2-cycle $\bar{\Delta}'$ in M_k, with $[\bar{\Delta}'] = [\bar{\Delta}_{3,4}]$ in $H_2(M_k, \mathbb{Z})$. The composition

$$\mathrm{Hom}(L', L_2) \otimes \mathrm{Hom}(L_2, L_4) \longrightarrow \mathrm{Hom}(L', L_4)$$

corresponds to an infinite series of triangular immersed regions in Σ_0, of which in general two are embedded. The case where the symplectic area of Δ' is a multiple of that of the fiber corresponds precisely to the situation where the two embedded triangular regions have equal symplectic areas. In general, the immersed triangles contributing to the composition can be labelled T'_n, $n \in \mathbb{Z}$, in such a way that $T'_n = T'_0 + n\Delta' + \frac{1}{2}n(n-1)\Sigma_0$. Arguing as before, one easily shows that the composition is given by the contribution of T'_0 multiplied by the factor

$$\sum_{n\in\mathbb{Z}} (-1)^n\, q'^{\,n}\, q_F^{n(n-1)/2}, \quad \text{where } q' = \exp(2\pi i[B + i\omega] \cdot [\bar{\Delta}']) = \tilde{q}_{3,4}.$$

This multiplicative factor vanishes if and only if $q' = q_F^k$ for some $k \in \mathbb{Z}$ (an easy way to see this is to view this factor as a theta function, see below), i.e. iff $[B + i\omega] \cdot [\bar{\Delta}'] \in \mathbb{Z} \oplus ([B + i\omega] \cdot [\Sigma_0])\mathbb{Z}$. Hence, as in §2.2 the mutation makes it possible to avoid dealing with a non-trivial differential, and provides an alternative description in which the simple degeneration corresponds to one of the composition maps becoming identically zero.

4.5 Modular invariance and theta functions

In this section we study the modularity properties of the category $\mathrm{Lag}_{\mathrm{vc}}(W_k, \{\gamma_i\})$ with respect to some of the parameters governing deformations of the complexified symplectic structure, and the relation with theta functions.

Proposition 4.18 *Consider two complexified symplectic forms $\kappa = B + i\omega$ and $\kappa' = B' + i\omega'$ on M_k, such that $[\kappa'] \cdot [\Sigma_0] = [\kappa] \cdot [\Sigma_0]$ and $[\kappa'] - [\kappa] \in H^2(M_k, \mathbb{Z}) \oplus (\kappa \cdot [\Sigma_0])\, H^2(M_k, \mathbb{Z})$. Then the categories $\mathrm{Lag}_{\mathrm{vc}}(W_k, \kappa, \{\gamma_i\})$ and $\mathrm{Lag}_{\mathrm{vc}}(W_k, \kappa', \{\gamma_i\})$ are equivalent.*

Proof. First consider the situation where $\omega' = \omega$, and $B' = B + d\chi$ for some 1-form χ. Then the vanishing cycles L_i remain the same, but the associated flat connections differ, and we can e.g. take $\nabla'_i = \nabla_i - 2\pi i\chi$. Then the contribution of a pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_0, \cup L_i)$ is actually the same in both cases, since the holonomy term changes by $\exp(-2\pi i \int_{u(\partial D^2)} \chi)$, while the weight factor changes by $\exp(2\pi i \int_{D^2} u^* d\chi) = \exp(2\pi i \int_{u(\partial D^2)} \chi)$. So, in the more general situation where $[B' - B] \in H^2(M_k, \mathbb{Z})$ and $[B' - B] \cdot [\Sigma_0] = 0$ (still assuming $\omega' = \omega$), after modifying B by an exact term we can assume that B and B' coincide over Σ_0, and that the integral of $B' - B$ over each thimble D_i is a multiple of 2π. In this situation the vanishing cycles L_i are the same, and the associated flat connections are gauge equivalent (since their holonomies differ by multiples of 2π), so the corresponding twisted Floer theories are identical.

Next, consider the situation where $[B+i\omega]$ changes by an integer multiple of $[B+i\omega]\cdot[\Sigma_0]$. After adding an exact term to $\kappa = B + i\omega$ (which does not affect the category of vanishing cycles by Lemma 3.2 and by the above remark), we can assume that κ and κ' coincide over Σ_0, and that the relative cohomology class of $\kappa' - \kappa$ is an element of $(\kappa \cdot [\Sigma_0])H^2(M_k, \Sigma_0; \mathbb{Z})$.

Let D_i and D'_i be the thimbles associated to the arc γ_i and to the symplectic forms ω and ω' respectively. The integrality assumption on $\kappa' - \kappa$ implies that there exists an integer $n_i \in \mathbb{Z}$ such that $\int_{D_i} \kappa' = n_i[\kappa] \cdot [\Sigma_0] + \int_{D_i} \kappa$. Since D_i and D'_i can be deformed continuously into each other (by deforming the horizontal distribution), there exists a 2-chain K_i in Σ_0 such that $[D_i + K_i - D'_i] = 0$ in $H_2(M_k)$. Then $\int_{K_i} \omega = \int_{K_i} \omega' = -\int_{D_i} \omega' = -n_i[\omega] \cdot [\Sigma_0]$. Since the symplectic area of the 2-chain $K_i \subset \Sigma_0$ is an integer multiple of that of the fiber, the two vanishing cycles $L'_i = \partial D'_i$ and $L_i = \partial D_i$ are mutually Hamiltonian isotopic in Σ_0, and hence we can assume that $L'_i = L_i$. Moreover, in $H_2(M_k, L_i)$ we have $[D'_i] = [D_i] - n_i[\Sigma_0]$. Therefore, $\int_{D'_i} B' = \int_{D_i} B' - n_i \int_{\Sigma_0} B' = (\int_{D_i} B + n_i[B] \cdot [\Sigma_0]) - n_i[B] \cdot [\Sigma_0] = \int_{D_i} B$. So the flat connections ∇_i and ∇'_i have the same holonomy, which implies that (L_i, ∇_i) and (L'_i, ∇'_i) behave identically for twisted Floer theory. □

This property explains the invariance of the structure coefficients (α_{xy}, etc.) under certain changes of variables. More precisely, one easily checks that $\zeta_+(q_C q_F^3, q_F) = -q_C^{-1} q_F^{-2} \zeta_+(q_C, q_F)$, $\zeta_-(q_C q_F^3, q_F) = -q_C^{-1} q_F^{-1} \zeta_-(q_C, q_F)$, and $\zeta_0(q_C q_F^3, q_F) = -q_C^{-1} \zeta_0(q_C, q_F)$. This implies that the quantities considered in Propositions 4.7 and 4.10 are invariant under the change of variables $(q_C, q_F) \mapsto (q_C q_F^3, q_F)$; a closer examination shows that the individual constants α_{xy}, etc. are also invariant under this change of variables.

On the other hand, one easily checks that $\zeta_+(q_C q_F, q_F) = -q_C^{-1} \zeta_0(q_C, q_F)$, $\zeta_0(q_C q_F, q_F) = \zeta_-(q_C, q_F)$, and $\zeta_-(q_C q_F, q_F) = \zeta_+(q_C, q_F)$, which may seem surprising at first. The reason is that this change of variables corresponds to a non-Hamiltonian deformation of e.g. L_1 which sweeps exactly once through the entire fiber Σ_0. This deformation preserves the intersection points, but induces a non-trivial permutation of their labels: namely, $x_0,\ y_0,\ z_0$ become $y_0,\ z_0,\ x_0$ respectively, and $x_1,\ y_1,\ z_1$ become $z_1,\ x_1,\ y_1$ respectively. Thus, for example, $\alpha_{xy}(q_C, q_F) = \alpha_{yx}(q_C q_F, q_F) = \alpha_{zz}(q_C q_F^2, q_F)$ (and similarly for the other coefficients).

Another way to understand these invariance properties is to relate the functions ζ_+, ζ_-, and ζ_0 to theta functions. Recall that the ordinary theta function is an analytic function defined by

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z),$$

where $z \in \mathbb{C}$ and $\tau \in \mathcal{H}$ (here $\mathcal{H}$ is the upper half-plane $\{\operatorname{Im}\tau > 0\}$). This function is quasiperiodic with respect to the lattice $\Lambda_\tau \subset \mathbb{C}$ generated by 1 and τ, and its behavior under translation by an element of the lattice is given by the formula

$$\theta(z + u\tau + v, \tau) = \exp(-\pi i u^2 \tau - 2\pi i u z)\theta(z,\tau).$$

The zeros of the theta function are the infinite set $\left\{ z = (n+\frac{1}{2}) + (m+\frac{1}{2})\tau \mid n, m \in \mathbb{Z} \right\}$.

Here we consider theta functions with rational characteristics $a, b \in \mathbb{Q}$, defined by

$$\theta_{a,b}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i (n+a)^2 \tau + 2\pi i (n+a)(z+b)).$$

Let us introduce new variables $q = \exp(\pi i \tau)$ and $w = \exp(\pi i z)$. Now the following three θ-functions play a very important role in our considerations:

$$\begin{aligned}
\theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau) &= \exp(\tfrac{i\pi}{2})\, q^{3/4} \sum_{n\in\mathbb{Z}} (-1)^n w^{6n+3} q^{3n^2+3n}, \\
\theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau) &= \exp(\tfrac{i\pi}{6})\, q^{1/12} \sum_{n\in\mathbb{Z}} (-1)^n w^{6n+1} q^{3n^2+n}, \\
\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau) &= \exp(-\tfrac{i\pi}{6})\, q^{1/12} \sum_{n\in\mathbb{Z}} (-1)^n w^{6n-1} q^{3n^2-n}.
\end{aligned}$$

The zero set of the function $\theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau)$ is $\left\{\frac{n}{3} + m\tau \mid n, m \in \mathbb{Z}\right\}$, while the zero sets of the functions $\theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau)$ and $\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau)$ are

$$\left\{\tfrac{n}{3} + (m+\tfrac{1}{3})\tau \mid n, m \in \mathbb{Z}\right\} \quad \text{and} \quad \left\{\tfrac{n}{3} + (m-\tfrac{1}{3})\tau \mid n, m \in \mathbb{Z}\right\}$$

respectively. These three theta functions can be viewed as holomorphic sections of a line bundle of degree 3 on the elliptic curve $E = \mathbb{C}/\Lambda_\tau$; considering the zero sets, we see that this line bundle is $\mathbb{L} = \mathcal{O}_E(3 \cdot (0))$. These three sections of $\mathbb{L}$ determine an embedding of the elliptic curve $E = \mathbb{C}/\Lambda_\tau$ into the projective plane, given by

$$z \mapsto (\theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau) : \theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau) : \theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau)).$$

Observe that the two functions

$$\theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau)\,\theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau)\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau) \quad \text{and} \quad \theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau)^3 + \theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau)^3 + \theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau)^3$$

coincide up to a constant multiplicative factor, since they both correspond to holomorphic sections of the line bundle $\mathbb{L}^{\otimes 3}$ over E, and an easy calculation shows that they have the same zero set $\{\frac{n}{3} + \frac{m}{3}\tau \mid n, m \in \mathbb{Z}\}$. Therefore, the image of the above embedding of E into $\mathbb{P}^2$ is the cubic given by the equation

$$(A^3 + B^3 + C^3)XYZ - ABC(X^3 + Y^3 + Z^3) = 0,$$

where (A, B, C) are the values of the three theta functions at any given point of $\mathbb{C}/\Lambda_\tau$ (not in $\frac{1}{3}\Lambda_\tau$).

Consider the function

$$\left(\frac{\theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau)}{\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau)}\right)^3 = -\left(\frac{\sum_{n\in\mathbb{Z}} (-1)^n\, w^{6n+1}\, q^{n(3n+1)}}{\sum_{n\in\mathbb{Z}} (-1)^n\, w^{6n-1}\, q^{n(3n-1)}}\right)^3.$$

Substituting $q^2 = q_F$ and $w^6 = q_C$, one easily checks that this coincides with the expression which appears in Proposition 4.7,

$$\frac{\alpha_{xy}\alpha_{yz}\alpha_{zx}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -q_C \left(\frac{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n+1)/2}}{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n-1)/2}}\right)^3.$$

Similarly,

$$\left(\frac{\theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau)}{\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau)}\right)^3 = q^2 \left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n+3} q^{3n^2+3n}}{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-1} q^{3n^2-n}}\right)^3 = -q^2 \left(\frac{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-3} q^{3n^2-3n}}{\sum_{n\in\mathbb{Z}}(-1)^n w^{6n-1} q^{3n^2-n}}\right)^3.$$

After the same substitution $q^2 = q_F$ and $w^6 = q_C$, this coincides with the expression given in Proposition 4.10,

$$\frac{\alpha_{xx}\alpha_{yy}\alpha_{zz}}{\alpha_{yx}\alpha_{zy}\alpha_{xz}} = -\frac{q_F}{q_C} \left(\frac{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{3n(n-1)/2}}{\sum_{n\in\mathbb{Z}} (-1)^n\, q_C^n\, q_F^{n(3n-1)/2}}\right)^3.$$

Similarly, in the case where (4.2) holds, one easily checks that

$$\begin{aligned} \alpha_{xy} = \alpha_{yz} = \alpha_{zx} &= \tilde{s}\, e^{-2i\pi/3}\, \theta_{\frac{1}{6},\frac{1}{2}}(3z_0, 3\tau), \\ \alpha_{xx} = \alpha_{yy} = \alpha_{zz} &= \tilde{s}\, \theta_{\frac{1}{2},\frac{1}{2}}(3z_0, 3\tau), \\ \alpha_{yx} = \alpha_{zy} = \alpha_{xz} &= \tilde{s}\, e^{2i\pi/3}\, \theta_{\frac{5}{6},\frac{1}{2}}(3z_0, 3\tau), \end{aligned} \tag{4.4}$$

where $\tau = [B + i\omega] \cdot [\Sigma_0]$, $z_0 = \frac{1}{3}[B + i\omega] \cdot [\bar{C}]$, and $\tilde{s} = e^{i\pi/2}\, q_F^{-1/24}\, q_C^{1/6}\, s \neq 0$. Similar interpretations can be made for the quantities considered in Propositions 4.12–4.14 and in (4.3).

5 Proof of the main theorems

The derived categories considered in §2 depend on an elliptic curve E, two degree 3 line bundles $\mathcal{L}_1, \mathcal{L}_2$ over E, and k points $p_1, \dots, p_k$ on E. Meanwhile, the categories considered in §4 depend on a cohomology class $[B + i\omega] \in H^2(M_k, \mathbb{C})$. We now show how to relate these two sets of parameters.

Fix the cohomology class $[B+i\omega] \in H^2(M_k, \mathbb{C})$, and consider the category $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ studied in §4. With the notations of §3.4, assume that $[\omega] \cdot [\bar{\Delta}_{i,j}]$ is not an integer multiple of $[\omega] \cdot [\Sigma_0]$ for any $i, j \in \{0, \dots, k-1\}$. Then $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k))$ admits a full strong exceptional collection $(L_0, \dots, L_{k+2})$, whose properties have been studied in §4. In particular, the objects and morphisms in this exceptional collection are the same as for the exceptional collection $\sigma = (\mathcal{O}_{X_K}, \pi^*\mathcal{T}_{\mathbb{P}^2}(-1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{l_1}, \dots, \mathcal{O}_{l_k})$ considered in §2 for the derived category of coherent sheaves on a (possibly noncommutative) Del Pezzo surface. Hence, our goal is now to compare the composition laws and show that, for a suitable choice of the parameters $(E, \mathcal{L}_1, \mathcal{L}_2, K)$, the algebra of homomorphisms of the exceptional collection $(L_0, \dots, L_{k+2})$ is isomorphic to the algebra $B_{K,\mu}$ considered in §2. More precisely, we claim:

Proposition 5.1 *Let E be the elliptic curve $\mathbb{C}/\Lambda_\tau$, where $\tau = [B + i\omega] \cdot [\Sigma_0]$, realized as a plane cubic via the embedding $j : E \to \mathbb{P}^2$ given by $z \mapsto (\vartheta_+(z) : \vartheta_0(z) : \vartheta_-(z))$, where*

$$\vartheta_+(z) = e^{-2i\pi/3}\theta_{\frac{1}{6},\frac{1}{2}}(3z, 3\tau), \quad \vartheta_0(z) = \theta_{\frac{1}{2},\frac{1}{2}}(3z, 3\tau), \quad \textit{and } \vartheta_-(z) = e^{2i\pi/3}\theta_{\frac{5}{6},\frac{1}{2}}(3z, 3\tau).$$

Let $z_0 = \frac{1}{3}[B + i\omega] \cdot [\bar{C}]$, and for $i \in \{0, \dots, k-1\}$ let $p_i = \frac{1}{3}[B + i\omega] \cdot [\bar{C}_i]$. Finally, let $\mathcal{L}_1 = \mathcal{O}_E(3 \cdot (-z_0))$ and $\mathcal{L}_2 = \mathcal{O}_E(3 \cdot (0))$. Then the algebra of homomorphisms of the exceptional collection $(L_0, \dots, L_{k+2})$ is isomorphic to $B_{K,\mu}$, where μ is determined by $(E, \mathcal{L}_1, \mathcal{L}_2)$ via Construction 2.9 and $K = \{j(z_0 + p_0), \dots, j(z_0 + p_{k-1})\}$.

Proof. After a suitable rescaling of the chosen bases of the morphism spaces (or just by deforming to the situation where the fiber is flat and the vanishing cycles are geodesics), we can assume that the compositions of morphisms between the objects $L_0, \dots, L_{k+2}$ are given by the formulas (4.2) and (4.3). We identify the vector spaces $U = \mathrm{Hom}(L_0, L_1)$, $V = \mathrm{Hom}(L_1, L_2)$, and $W = \mathrm{Hom}(L_0, L_2)$ with $\mathbb{C}^3$ by considering the bases (x_0, y_0, z_0), (x_1, y_1, z_1), and $(\bar{x}, \bar{y}, \bar{z})$. The composition tensor $\mu : V \otimes U \to W$ is determined by the three constants $a = \alpha_{xy} = \alpha_{yz} = \alpha_{zx}$, $b = \alpha_{xx} = \alpha_{yy} = \alpha_{zz}$, and $c = \alpha_{yx} = \alpha_{zy} = \alpha_{xz}$. In particular, given an element $v = (X, Y, Z) \in V$, the composition map $\mu_v = \mu(v, \cdot) : U \to W$ is given by the matrix

$$\begin{pmatrix} \alpha_{xx}X & \alpha_{yz}Z & \alpha_{zy}Y \\ \alpha_{xz}Z & \alpha_{yy}Y & \alpha_{zx}X \\ \alpha_{xy}Y & \alpha_{yx}X & \alpha_{zz}Z \end{pmatrix} = \begin{pmatrix} bX & aZ & cY \\ cZ & bY & aX \\ aY & cX & bZ \end{pmatrix} \tag{5.1}$$

which has rank 2 precisely when

$$\det(\mu_v) = (a^3 + b^3 + c^3)\, XYZ - abc(X^3 + Y^3 + Z^3) = 0. \tag{5.2}$$

By (4.4), the constants a, b, c are (up to a non-zero constant factor) the values of the theta functions $\vartheta_+, \vartheta_0, \vartheta_-$ at the point z_0. Therefore, by the discussion in §4.5, there are two possibilities:

1. if $z_0 \in \frac{1}{3}\Lambda_\tau$, then $abc = 0$ and μ_v always has rank 2; as explained in §2.3 this corresponds to a commutative situation;
2. if $z_0 \notin \frac{1}{3}\Lambda_\tau$, then (5.2) defines a cubic $\Gamma_V \subset \mathbb{P}(V) = \mathbb{P}^2$, and this cubic is precisely the image of the embedding j.

The same situation holds for μ_u; interestingly, under the chosen identifications of $\mathbb{P}(U)$ and $\mathbb{P}(V)$ with $\mathbb{P}^2$, the two subschemes $\Gamma_U \subset \mathbb{P}(U)$ and $\Gamma_V \subset \mathbb{P}(V)$ determined by the equations $\det(\mu_u) = 0$ and $\det(\mu_v) = 0$ coincide exactly. However, with this description, the isomorphism $\sigma : \Gamma_V \to \Gamma_U$ which takes v to the point of Γ_U corresponding to $\mathrm{Ker}\, \mu_v$ is *not* the identity map. Here the reader is referred to the discussion on pp. 37–38 of [10], which we follow loosely.

Given a point $v = (X : Y : Z) \in \Gamma_V$, the kernel of μ_v can be obtained as the cross-product of any two of the rows of the matrix (5.1). Taking e.g. the first two rows, we obtain that the corresponding point of Γ_U is

$$\sigma(X : Y : Z) = (a^2XZ - bcY^2 : c^2YZ - abX^2 : b^2XY - acZ^2). \tag{5.3}$$

Observe that j maps the origin to $(1 : 0 : -1) \in \Gamma_V$, and that the corresponding point in Γ_U is $\sigma(1 : 0 : -1) = (a : b : c) = j(z_0)$. Hence, considering only the situation where $\Gamma_U \simeq \Gamma_V \simeq E$, and identifying E with Γ_V by means of the embedding j, the identification of E with Γ_U is given by the embedding $\sigma \circ j$, which is the composition of j with the translation by z_0. Therefore, the line bundles on E induced by the two inclusions of E into $\mathbb{P}(U)$ and $\mathbb{P}(V)$ are respectively $(\sigma \circ j)^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_E(3 \cdot (-z_0)) = \mathcal{L}_1$ and $j^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{O}_E(3 \cdot (0)) = \mathcal{L}_2$. It then follows from the discussion in §2.3 that the composition tensor μ corresponds to the data

$(E, \mathcal{L}_1, \mathcal{L}_2)$. This remains true even when $z_0 \in \frac{1}{3}\Lambda_\tau$, since in that case we have $\mathcal{L}_1 \simeq \mathcal{L}_2$ and the composition tensor associated to the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$ is that of the usual projective plane (see Remark 2.10).

Next we consider the composition $\mathrm{Hom}(L_2, L_{3+i}) \otimes W \longrightarrow \mathrm{Hom}(L_0, L_{3+i})$. Choosing generators of the lines $\mathrm{Hom}(L_2, L_{3+i})$ and $\mathrm{Hom}(L_0, L_{3+i})$ we can view this map as a linear form on W. In the given basis of W, this linear form is given by $\big(\beta_{\bar{x},i}\,,\,\beta_{\bar{y},i}\,,\,\beta_{\bar{z},i}\big)$, which by (4.3) coincides up to a non-zero constant factor with

$$\big(\vartheta_+(p_i)\,,\,\vartheta_0(p_i)\,,\,\vartheta_-(p_i)\big).$$

On the other hand we know from §2.3 that the kernel of this linear form should be exactly $\mathrm{Im}\,\mu_{v_i}$, where $v_i \in \Gamma_V$ is the point being blown up.

For any $v = (X : Y : Z) \in \Gamma_V$, the projection $W \to W/\mathrm{Im}\,\mu_v$ is a linear form given up to a scaling factor by the dot product of any two columns of the matrix (5.1). Taking e.g. the first two columns, we obtain that the expression of this linear form relatively to our chosen basis of W is

$$\big(c^2XZ - abY^2\,,\,a^2YZ - bcX^2\,,\,b^2XY - acZ^2\big).$$

Interestingly, if we assume that $(X : Y : Z) = \sigma(\tilde{X} : \tilde{Y} : \tilde{Z})$, where σ is the transformation given by (5.3), then this expression simplifies to a scalar multiple of $(\tilde{X}\,,\,\tilde{Y}\,,\,\tilde{Z})$. Hence, we conclude that $v_i = \sigma(j(p_i)) = j(z_0 + p_i)$. □

Remark 5.2 *At this point the reader may legitimately be concerned that, since the homology classes $[\bar{C}]$ and $[\bar{C}_i]$ are canonically defined only up to a multiple of $[\Sigma_0]$, and since $[B]$ is only defined up to an element of $H^2(M_k, \mathbb{Z})$, the points z_0 and p_i of E are canonically determined only up to translations by elements of $\frac{1}{3}\Lambda_\tau$. However, the line bundle $\mathcal{L}_1 = \mathcal{O}_E(3\cdot(-z_0))$ is not affected by this ambiguity in the determination of z_0, and neither are the relative positions of the points p_i, since the quantity $p_j - p_i = [B + i\omega]\cdot[\bar{\Delta}_{i,j}]$ is well-defined up to an element of Λ_τ. Moreover, a simultaneous translation of all the blown up points by an element of $\frac{1}{3}\Lambda_\tau$ amounts to an automorphism of the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$, which does not actually affect the category. (From the point of view of the embedding j, this automorphism simply permutes the homogeneous coordinates X, Y, Z and multiplies them by cubic roots of unity; this is consistent with the observation made after the proof of Proposition* 4.18*).*

Theorems 1.4 and 1.6 now follow directly from the discussion. Namely, in the case of a blowup of $\mathbb{CP}^2$ at a set $K = \{p_0, \ldots, p_{k-1}\}$ of k distinct points (Theorem 1.4), we consider a cubic curve $E \subset \mathbb{CP}^2$ which contains all the points of K, and view it as an elliptic curve $\mathbb{C}/\Lambda_\tau$ for some $\tau \in \mathbb{C}$ with $\mathrm{Im}\,\tau > 0$. This allows us to view the points p_i as elements of $\mathbb{C}/\Lambda_\tau$ (well-defined up to a simultaneous translation of all p_i by an element of $\frac{1}{3}\Lambda_\tau$, since the origin can be chosen at any of the flexes of E; however by Remark 5.2 this does not matter for our construction). Then we equip M_k with a complexified symplectic structure such that $[B + i\omega]\cdot[\Sigma_0] = \tau$, $[B + i\omega]\cdot[\bar{C}] = 0$, and $[B + i\omega]\cdot[\bar{C}_i] = 3p_i$. The existence of such a $B + i\omega$ follows from a standard result about symplectic structures on Lefschetz fibrations:

Proposition 5.3 (Gompf) *Given any cohomology class $[\zeta] \in H^2(M_k, \mathbb{R})$ such that $[\zeta] \cdot [\Sigma_0] > 0$, the manifold M_k admits a symplectic structure in the cohomology class $[\zeta]$, for which the fibers of W_k are symplectic submanifolds.*

Proof. The map $W_k : M_k \to \mathbb{C}$ is a Lefschetz fibration, and the argument given in the proof of [96, Theorem 10.2.18] can be adapted in a straightforward manner to this situation, even though the base of the fibration is not compact. (Alternatively, one can also work with the compactified fibration $\overline{W_k} : \overline{M} \to \mathbb{CP}^1$). The symplectic form ω constructed by this argument lies in the cohomology class $t[\zeta] + W_k^*([\mathrm{vol}_\mathbb{C}])$ for some constant $t > 0$; since the area form on $\mathbb{C}$ is exact, we have $[\omega] = t[\zeta]$, and scaling ω by a constant factor we obtain the desired result. □

By Proposition 5.1 the algebra of homomorphisms of the exceptional collection $(L_0, \ldots, L_{k+2})$ is then isomorphic to B_K, which implies that $\boldsymbol{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W_k)) \cong \boldsymbol{D}^b(\mathrm{mod}\, B_K) \cong \boldsymbol{D}^b(\mathrm{coh}(X_K))$.

In the case of a noncommutative blowup of $\mathbb{P}^2$ (Theorem 1.6), consider the triple $(E, \mathcal{L}_1, \mathcal{L}_2)$ associated to the underlying noncommutative $\mathbb{P}^2$, and view again E as a quotient $\mathbb{C}/\Lambda_\tau$. Choose z_0 (well-defined up to an element of $\frac{1}{3}\Lambda_\tau$) such that $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1} \simeq \mathcal{O}_E(3\cdot(z_0) - 3\cdot(0)) \in \mathrm{Pic}^0(E)$. As explained in §2.3, the blown up points must all lie in $\Gamma_V \subset \mathbb{P}(V)$, and under the identification $\Gamma_V \simeq E$ they can be viewed as elements $p_i \in \mathbb{C}/\Lambda_\tau$. Equip M_k with a complexified symplectic structure such that $[B+i\omega]\cdot[\Sigma_0] = \tau$, $[B+i\omega]\cdot[\bar{C}] = 3z_0$, and $[B+i\omega]\cdot[\bar{C}_i] = 3(p_i - z_0)$. By Proposition 5.1 the algebra of homomorphisms of the exceptional collection $(L_0, \ldots, L_{k+2})$ is then isomorphic to $B_{K,\mu}$, which yields the desired equivalence of categories.

Theorem 1.5 is proved similarly, working with the mutated exceptional collections τ' (introduced in §2.2) and $(L_0, L_1, L', L_2, L_4, \ldots, L_{k+2})$ (introduced in §4.4). The details are left to the reader.

Remark 5.4 *The construction carried out for Theorem 1.4 also applies to some limit situations in which X_K is actually not a Del Pezzo surface. For example, the argument applies equally well to the situation where $\mathbb{CP}^2$ is blown up at nine points which lie at the intersection of two elliptic curves. In this case the mirror is an elliptic fibration over $\mathbb{C}$ for which the compactification has a smooth fiber at infinity. Compared to that of $\mathbb{CP}^2$ ($k = 0$), this extreme case where $k = 9$ lies at the opposite end of the spectrum that we consider.*

Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces.

1 Mirrors of Landau-Ginzburg models

A number of recent results [136], [218], [73], [3], [103] suggest that the phenomenon of mirror symmetry is not restricted to Calabi-Yau or Fano manifolds. Indeed, while mirror symmetry was initially formulated as a duality between Calabi-Yau manifolds, it was already suggested in the early works of Givental and Batyrev that Fano manifolds also exhibit mirror symmetry. The counterpart to the presence of a nontrivial first Chern class is that the mirror of a compact Fano manifold is not a compact manifold, but rather a *Landau-Ginzburg model*, i.e. a (non-compact) Kähler manifold equipped with a holomorphic function called *superpotential*. A physical explanation of this phenomenon and a number of examples have been given by Hori and Vafa [122]. From a mathematical point of view, Hori and Vafa's construction amounts to a toric duality, and can also be applied to varieties of general type [57], [139], [136], [103].

The Strominger-Yau-Zaslow (SYZ) conjecture [228] provides a geometric interpretation of mirror symmetry for Calabi-Yau manifolds as a duality between (special) Lagrangian torus fibrations. In the language of Kontsevich's homological mirror symmetry [151], the SYZ conjecture reflects the expectation that the mirror can be realized as a moduli space of certain objects in the Fukaya category of the given manifold, namely, a family of Lagrangian tori equipped with rank 1 local systems. Note that this homological perspective eliminates the requirement of finding special Lagrangian fibrations, at the cost of privileging one side of mirror symmetry: in the Calabi-Yau case, the framework we follow produces a degenerating family Y^0 of complex manifolds (B-side) starting with a Lagrangian torus fibration on a symplectic manifold X^0 (A-side).

Outside of the Calabi-Yau situation, homological mirror symmetry is still expected to hold [152], but the Lagrangian tori bound holomorphic discs, which causes their Floer theory to be obstructed; the mirror superpotential can be interpreted as a weighted count of these holomorphic discs [119], [13], [14], [82]. We call such a mirror a *B-side Landau-Ginzburg model*.

In the Calabi-Yau case, mirror symmetry is expected to be involutive; i.e when the symplectic form on X^0 is in fact a Kähler form for some degenerating family of complex structures then the mirror Y should be equipped with its own Kähler form which is mirror to these complex structures. Involutivity should hold beyond the Calabi-Yau situation, but requires making sense of a class of potential functions on symplectic manifolds, called *A-side Landau-Ginzburg models*, which have well defined Fukaya categories. The idea for such a definition goes back to Kontsevich [152], and was studied in great depth by Seidel in [217] in the special case of Lefschetz fibrations.

Remark 1.1 The general theory of Fukaya categories $\mathcal{F}(X, W^\vee)$ of A-side Landau-Ginzburg models is still under development in different contexts [5], [2], [4]; we shall specifically point out where it is being used in this chapter. In fact, we will also need to consider *twisted* versions of A-side Landau-Ginzburg models, where objects of the Fukaya category carry *relatively spin* structures with respect to a background class in $H^2(X, \mathbb{Z}/2)$ (rather than spin structures); see Section 7.

On manifolds of general type (or more generally, whose first Chern class cannot be represented by an effective divisor), the SYZ approach to mirror symmetry seems to fail at first glance due to the lack of a suitable Lagrangian torus fibration. The idea that allows one to overcome this obstacle is to replace the given manifold with another closely related space which does carry an appropriate SYZ fibration. Thus, we make the following definition:

Definition 1.2 *We say that a B-side Landau-Ginzburg model (Y, W) is* SYZ mirror *to a Kähler manifold X (resp. an A-side Landau-Ginzburg model $(X, W^\vee)$) if there exists an open dense subset X^0 of X, and a Lagrangian torus fibration $\pi : X^0 \to B$, such that the following properties hold:*

1. *Y is a completion of a moduli space of unobstructed torus-like objects of the Fukaya category $\mathcal{F}(X^0)$ (resp. $\mathcal{F}(X^0, W^\vee)$) containing those objects which are supported on the fibers of π;*
2. *the function W restricts to the superpotential induced by the deformation of $\mathcal{F}(X^0)$ to $\mathcal{F}(X)$ (resp. $\mathcal{F}(X^0, W^\vee)$ to $\mathcal{F}(X, W^\vee)$) for these objects.*

We say that (Y, W) is a generalized SYZ mirror *of X if (after shifting W by a suitable additive constant) it is an SYZ mirror of a (suitably twisted) A-side Landau-Ginzburg model with Morse-Bott superpotential, whose critical locus is isomorphic to X.*

The last part of the definition is motivated by the expectation that the Fukaya category of a Morse-Bott superpotential, twisted by a background class which accounts for the non-triviality of the normal bundle to the critical locus, is equivalent (up to an additive constant shift in the curvature term, which accounts for exceptional curves through the critical locus) to the Fukaya category of the critical locus; see Corollary 7.8 and Proposition 7.10.

Definition 1.2 and the construction of moduli spaces of objects of the Fukaya category are clarified in Section 2 and Appendix A. To understand the first condition in the case of an A-side Landau-Ginzburg model, it is useful to note that every object of the Fukaya category $\mathcal{F}(X^0)$ of compact Lagrangians also defines an object of $\mathcal{F}(X^0, W^\vee)$ since the objects of the latter are Lagrangians satisfying admissibility properties outside a compact set and such properties trivially hold for compact Lagrangians. Hence the fibers of π automatically define objects of $\mathcal{F}(X^0, W^\vee)$; we shall enlarge this space by considering certain non-compact Lagrangians in X^0 which can be seen as limits of compact Lagrangians.

Remark 1.3 It is important to note that, even in the absence of superpotentials, the assertion that Y^0 is SYZ mirror to X^0 may not imply that the Fukaya category of X^0 is equivalent to the derived category of Y^0; at a basic level, the example of the Kodaira surface mentioned in [1] shows that there may in general be an analytic gerbe on Y^0 so that the Fukaya category of X^0 is in fact mirror to sheaves twisted by this gerbe. Beyond the Calabi-Yau situation, a complete statement of homological mirror symmetry for SYZ mirrors would have to consider further deformations of the derived category of sheaves by (holomorphic) polyvector fields on Y. The superpotential W should be thought of as the leading order term of this deformation corresponding to discs of Maslov index 2.

More fundamentally, our construction of the analytic completion relies on choices, and it is expected that different choices will given rise to different mirrors. Indeed, this phenomenon would provide a mirror symmetry explanation for the existence of derived equivalent varieties which are birational. Nonetheless, as completely arbitrary choices of completions give rise to varieties which are not derived equivalent (e.g. a blowup), the task of passing from our SYZ miror statement to homological mirror symmetry would require a more careful understanding of the completions that we have introduced. This chapter begins this task by explaining how some of the points that we add should correspond to objects of the Fukaya category supported by immersed or non-compact Lagrangians (see Remark A.12).

In this chapter we use this perspective to study mirror symmetry for hypersurfaces (and complete intersections) in toric varieties. If H is a smooth hypersurface in a toric variety V, then one simple way to construct a closely related Kähler manifold with effective first Chern class is to blow up the product $V \times \mathbb{C}$ along the codimension 2 submanifold $H \times 0$. By a result of Bondal and Orlov [39], the derived category of coherent sheaves of the resulting manifold X admits a semi-orthogonal decomposition into subcategories equivalent to $D^b\mathrm{Coh}(H)$ and $D^b\mathrm{Coh}(V \times \mathbb{C})$; and ideas similar to those of [227] can be used to study the Fukaya category of X, as we explain in Section 7 (cf. Corollary 7.8). Thus, finding a mirror to X is, for many purposes, as good as finding a mirror to H. Accordingly, our main results concern SYZ mirror symmetry for X and, by a slight modification of the construction, for H. Along the way we also obtain descriptions of SYZ mirrors to various related spaces. These results provide a geometric foundation for mirror constructions that have appeared in the recent literature [57], [139], [136], [218], [221], [3], [103].

We focus primarily on the case where V is affine, and other cases which can be handled with the same techniques. The general case requires more subtle arguments in enumerative geometry, which should be the subject of further investigation.

1.1 Statement of the results Our main result can be formulated as follows (see §3 for the details of the notations).

Let $H = f^{-1}(0)$ be a smooth nearly tropical hypersurface (cf. §3.1) in a (possibly non-compact) toric variety V of dimension n, and let X be the blow-up of $V \times \mathbb{C}$ along $H \times 0$, equipped with an S^1-invariant Kähler form ω_ϵ for which the fibers of the exceptional divisor have sufficiently small area $\epsilon > 0$ (cf. §3.2).

Let Y be the toric variety defined by the polytope $\{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R} \,|\, \eta \geq \varphi(\xi)\}$, where φ is the tropicalization of f. Let $w_0 = -T^\epsilon + T^\epsilon v_0 \in \mathcal{O}(Y)$, where T is the Novikov parameter and v_0 is the toric monomial with weight $(0, \dots, 0, 1)$, and set $Y^0 = Y \setminus w_0^{-1}(0)$. Finally, let $W_0 = w_0 + w_1 + \cdots + w_r \in \mathcal{O}(Y)$ be the *leading-order superpotential* of Definition 3.10, namely the sum of w_0 and one toric monomial w_i ($1 \leq i \leq r$) for each irreducible toric divisor of V (see Definition 3.10). We assume:

Assumption 1.4 $c_1(V) \cdot C > \max(0, H \cdot C)$ *for every rational curve* $C \simeq \mathbb{P}^1$ *in* V.

This includes the case where V is an affine toric variety as an important special case. Under this assumption, our main result is the following:

Theorem 1.5 *Under Assumption* 1.4*, the* B*-side Landau-Ginzburg model* (Y^0, W_0) *is SYZ mirror to* X.

In the general case, the mirror of X differs from (Y^0, W_0) by a correction term which is of higher order with respect to the Novikov parameter (see Remark 6.3).

Equipping X with an appropriate superpotential, given by the affine coordinate of the $\mathbb{C}$ factor, yields an A-side Landau-Ginzburg model whose singularities are of Morse-Bott type. Up to twisting by a class in $H^2(X, \mathbb{Z}/2)$, this A-side Landau-Ginzburg model can be viewed as a stabilization of the sigma model with target H.

Theorem 1.6 *Assume* V *is affine, and let* $W_0^H = -v_0 + w_1 + \cdots + w_r \in \mathcal{O}(Y)$ *(see Definition* 3.10*). Then the* B*-side Landau-Ginzburg model* (Y, W_0^H) *is a generalized SYZ mirror of* H.

Unlike the other results stated in this introduction, this theorem strictly speaking relies on the assumption that Fukaya categories of Landau-Ginzburg models satisfy certain properties for which we do not provide complete proofs. In Section 7, we give sketches of the proofs of these results, and indicate the steps which are missing from our argument.

A result similar to Theorem 1.6 can also be obtained from the perspective of mirror duality between toric Landau-Ginzburg models [122], [57], [136], [103]. However, the toric approach is much less illuminating, because geometrically it works at the level of the open toric strata in the relevant toric varieties (the total space of $\mathcal{O}(-H) \to V$ on one hand, and Y on the other hand), whereas the interesting geometric features of these spaces lie entirely within the toric divisors.

Theorem 1.5 relies on a mirror symmetry statement for open Calabi-Yau manifolds which is of independent interest. Consider the conic bundle

$$X^0 = \{(\mathbf{x}, y, z) \in V^0 \times \mathbb{C}^2 \mid yz = f(\mathbf{x})\}$$

over the open stratum $V^0 \simeq (\mathbb{C}^*)^n$ of V, where f is again the defining equation of the hypersurface H. The conic bundle X^0 sits as an open dense subset inside X, see Remark 3.5. Then we have:

Theorem 1.7 *The open Calabi-Yau manifold Y^0 is SYZ mirror to X^0.*

In the above statements, and in most of this chapter, we view X or X^0 as a symplectic manifold, and construct the SYZ mirror Y^0 (with a superpotential) as an algebraic moduli space of objects in the Fukaya category of X or X^0. This is the same direction considered e.g. in [218], [73], [3]. However, one can also work in the opposite direction, starting from the symplectic geometry of Y^0 and showing that it admits X^0 (now viewed as a complex manifold) as an SYZ mirror. For completeness we describe this converse construction in Section 8 (see Theorem 8.4); similar results have also been obtained independently by Chan, Lau and Leung [51].

The methods we use apply in more general settings as well. In particular, the assumption that V be a toric variety is not strictly necessary – it is enough that SYZ mirror symmetry for V be sufficiently well understood. As an illustration, in Section 11 we derive analogues of Theorems 1.5–1.7 for *complete intersections*.

1.2 A reader's guide The rest of this chapter is organized as follows.

First we briefly review (in Section 2) the SYZ approach to mirror symmetry, following [13], [14]. Then in Section 3 we introduce notation and describe the protagonists of our main results, namely the spaces X and Y and the superpotential W_0.

In Section 4 we construct a Lagrangian torus fibration on X^0, similar to those previously considered by Gross [101], [102] and by Castaño-Bernard and Matessi [46], [47]. In Section 5 we study the Lagrangian Floer theory of the torus fibers, which we use to prove Theorem 1.7. In Section 6 we consider the partial compactification of X^0 to X, and prove Theorem 1.5. Theorem 1.6 is then proved in Section 7.

In Section 8 we briefly consider the converse construction, namely we start from a Lagrangian torus fibration on Y^0 and recover X^0 as its SYZ mirror.

Finally, some examples illustrating the main results are given in Section 9, while Sections 10 and 11 discusses various generalizations, including to hypersurfaces in abelian varieties (Theorem 10.4) and complete intersections in toric varieties (Theorem 11.1).

2 Review of SYZ mirror symmetry

In this section, we briefly review SYZ mirror symmetry for Kähler manifolds with effective anticanonical class; the reader is referred to [13], [14] for basic ideas about SYZ, and to Appendix A for technical details.

2.1 Lagrangian torus fibrations and SYZ mirrors

In first approximation, the Strominger-Yau-Zaslow conjecture [228] states that mirror pairs of Calabi-Yau manifolds carry mutually dual Lagrangian torus fibrations (up to "instanton corrections"). A reformulation of this statement in the language of homological mirror symmetry [151] is that a mirror of a Calabi-Yau manifold can be constructed as a moduli space of suitable objects in its Fukaya category (namely, the fibers of an SYZ fibration, equipped with rank 1 local systems); and vice versa. In Appendix A, we explain how ideas of Fukaya [80] yield a precise construction of such a mirror space from local rigid analytic charts glued via the equivalence relation which identifies objects that are quasi-isomorphic in the Fukaya category.

We consider an open Calabi-Yau manifold of the form $X^0 = X \setminus D$, where (X, ω, J) is a Kähler manifold of complex dimension n and $D \subset X$ is an anticanonical divisor (reduced, with normal crossing singularities). X^0 can be equipped with a holomorphic n-form Ω (with simple poles along D), namely the inverse of the defining section of D. The restriction of Ω to an oriented Lagrangian submanifold $L \subset X^0$ is a nowhere vanishing complex-valued n-form on L; the complex argument of this n-form determines the *phase* function $\arg(\Omega_{|L}) : L \to S^1$. Recall that L is said to be *special Lagrangian* if $\arg(\Omega_{|L})$ is constant; a weaker condition is to require the vanishing of the *Maslov class* of L in X^0, i.e. we require the existence of a lift of $\arg(\Omega_{|L})$ to a real-valued function. (The choice of such a real lift then makes L a *graded Lagrangian*, and yields $\mathbb{Z}$-gradings on Floer complexes.)

The main input of the construction of the SYZ mirror of the open Calabi-Yau manifold X^0 is a Lagrangian torus fibration $\pi : X^0 \to B$ (with appropriate singularities) whose fibers have trivial Maslov class. (Physical considerations suggest that one should expect the fibers of π to be special Lagrangian, but such fibrations are hard to produce.)

The base B of the Lagrangian torus fibration π carries a natural real affine structure (with singularities along the locus B^{sing} of singular fibers), i.e. $B \setminus B^{sing}$ can be covered by a set of coordinate charts with transition functions in $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$. A smooth fiber $L_0 = \pi^{-1}(b_0)$ and a collection of loops $\gamma_1, \dots, \gamma_n$ forming a basis of $H_1(L_0, \mathbb{Z})$ determine an affine chart centered at b_0 in the following manner: given $b \in B \setminus B^{sing}$ close enough to b_0, we can isotope

L_0 to $L = \pi^{-1}(b)$ among fibers of π. Under such an isotopy, each loop γ_i traces a cylinder Γ_i with boundary in $L_0 \cup L$; the affine coordinates associated to b are then the symplectic areas $(\int_{\Gamma_1} \omega, \dots, \int_{\Gamma_n} \omega)$.

In the examples we will consider, "most" fibers of π do not bound nonconstant holomorphic discs in X^0; we call such Lagrangians *tautologically unobstructed.* Recall that a (graded, spin) Lagrangian submanifold L of X^0 together with a unitary rank one local system ∇ determines an object (L, ∇) of the Fukaya category $\mathcal{F}(X^0)$ [81] whenever certain counts of holomorphic discs cancel; this condition evidently holds if there are no non-constant discs. Thus, given an open subset $U \subset B \setminus B^{sing}$ such that all the fibers in $\pi^{-1}(U)$ are tautologically unobstructed, the moduli space of objects (L, ∇) where $L \subset \pi^{-1}(U)$ is a fiber of π and ∇ is a unitary rank 1 local system on L yields an open subset $U^\vee \subset Y^0$ of the SYZ mirror of X^0.

A word is in order about the choice of coefficient field. A careful definition of Floer homology involves working over the *Novikov field* (here over complex numbers),

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \,\middle|\, c_i \in \mathbb{C},\ \lambda_i \in \mathbb{R},\ \lambda_i \to +\infty \right\}. \tag{2.1}$$

Recall that the *valuation* of a non-zero element of Λ is the smallest exponent λ_i that appears with a non-zero coefficient; the above-mentioned local systems are required to have holonomy in the multiplicative subgroup

$$U_\Lambda = \left\{ c_0 + \textstyle\sum c_i T^{\lambda_i} \in \Lambda \,\middle|\, c_0 \neq 0 \text{ and } \lambda_i > 0 \right\}$$

of *unitary elements* (or units) of the Novikov field, i.e. elements whose valuation is zero. The local system $\nabla \in \mathcal{H}om(\pi_1(L), U_\Lambda)$ enters into the definition of Floer-theoretic operations by contributing holonomy terms to the weights of holomorphic discs: a rigid holomorphic disc u with boundary on Lagrangians (L_i, ∇_i) is counted with a weight

$$T^{\omega(u)} \mathrm{hol}(\partial u), \tag{2.2}$$

where $\omega(u)$ is the symplectic area of the disc u, and $\mathrm{hol}(\partial u) \in U_\Lambda$ is the total holonomy of the local systems ∇_i along its boundary. (Thus, local systems are conceptually an exponentiated variant of the "bounding cochains" used by Fukaya et al [81], [82]). Gromov compactness ensures that all structure constants of Floer-theoretic operations are well-defined elements of Λ.

Thus, in general the SYZ mirror of X^0 is naturally an analytic space defined over Λ. However, it is often possible to obtain a complex mirror by treating the Novikov parameter T as a numerical parameter $T = e^{-2\pi t}$ with $t > 0$ sufficiently large; of course it is necessary to assume the convergence of all the power series encountered. The local systems are then taken to be unitary in the usual sense, i.e. $\nabla \in \mathcal{H}om(\pi_1(L), S^1)$, and the weight of a rigid holomorphic disc, still given by (2.2), becomes a complex number. The complex manifolds obtained by varying the parameter t are then understood to be mirrors to the family of Kähler manifolds $(X^0, t\omega)$.

To provide a unified treatment, we denote by $\mathbb{K}$ the coefficient field (Λ or $\mathbb{C}$), by $U_{\mathbb{K}}$ the subgroup of unitary elements (either U_Λ or S^1), and by $\mathrm{val}: \mathbb{K} \to \mathbb{R}$ the valuation (in the case of complex numbers, $\mathrm{val}(z) = -\frac{1}{2\pi t}\log|z|$).

Consider as above a contractible open subset $U \subset B \setminus B^{sing}$ above which all fibers of π are tautologically unobstructed, a reference fiber $L_0 = \pi^{-1}(b_0) \subset \pi^{-1}(U)$, and a basis $\gamma_1, \ldots, \gamma_n$ of $H_1(L_0, \mathbb{Z})$. A fiber $L = \pi^{-1}(b) \subset \pi^{-1}(U)$ and a local system $\nabla \in \mathcal{H}om(\pi_1(L), U_{\mathbb{K}})$ determine a point of the mirror, $(L, \nabla) \in U^\vee \subset Y^0$. Identifying implicitly $H_1(L, \mathbb{Z})$ with $H_1(L_0, \mathbb{Z})$, the local system ∇ is determined by its holonomies along the loops $\gamma_1, \ldots, \gamma_n$, while the fiber L is determined by the symplectic areas of the cylinders $\Gamma_1, \ldots, \Gamma_n$. This yields natural coordinates on $U^\vee \subset Y^0$, identifying it with an open subset of $(\mathbb{K}^*)^n$ via

$$(L, \nabla) \mapsto (z_1, \ldots, z_n) = \left(T^{\int_{\Gamma_1}\omega}\,\nabla(\gamma_1), \ldots, T^{\int_{\Gamma_n}\omega}\,\nabla(\gamma_n)\right). \tag{2.3}$$

One feature of Floer theory that is conveniently captured by this formula is the fact that, in the absence of instanton corrections, the non-Hamiltonian isotopy from L_0 to L is formally equivalent to equipping L_0 with a *non-unitary* local system for which $\mathrm{val}(\nabla(\gamma_i)) = \int_{\Gamma_i}\omega$.

The various regions of B over which the fibers are tautologically unobstructed are separated by *walls* (real hypersurfaces in B, or thickenings of real hypersurfaces) of *potentially obstructed* fibers (i.e. which bound non-constant holomorphic discs), across which the local charts of the mirror (as given by (2.3)) need to be glued together in an appropriate manner to account for “instanton corrections”.

The discussion preceding Equation (12.4) makes precise the idea that we can embed the moduli space of Lagrangians equipped with unitary local systems in an analytic space obtained by gluing coordinate charts coming from non-unitary systems. This will be the first step in the construction of the mirror manifold as a completion of the moduli space of Lagrangians.

Consider a potentially obstructed fiber $L = \pi^{-1}(b)$ of π, where $b \in B \setminus B^{sing}$ lies in a wall that separates two tautologically unobstructed chambers. By deforming this fiber to a nearby chamber, we obtain a bounding cochain (with respect to the Floer differential) for the moduli space of holomorphic discs with boundary on L. While local systems on L define objects of $\mathcal{F}(X^0)$, the quasi-isomorphism type of such objects depends on the choice of bounding cochain, which in our case amounts to a choice of this isotopy. In the special situation we are considering, we use this argument to prove in Lemma A.13 that any unitary local system on L can be represented by a non-unitary local system on a fiber lying in a tautologically unobstructed chamber. This implies that the space obtained by gluing the mirrors of the chambers contains the analytic space corresponding to all unitary local systems on smooth fibers of π.

The gluing maps for the mirrors of nearby chambers are given by wall-crossing formulae, with instanton corrections accounting for the disc bubbling phenomena that occur as a Lagrangian submanifold is isotoped across a wall of potentially obstructed Lagrangians (see [13] for an informal discussion, and Appendix A.1 for the relation with the invariance proof of Floer cohomology in this setting [81], [80]). Specifically, consider a Lagrangian isotopy $\{L_t\}_{t\in[0,1]}$ whose end points are tautologically unobstructed and lie in adjacent chambers. Assume that all nonconstant holomorphic discs bounded by the Lagrangians L_t in X^0 represent a single relative homotopy class $\beta \in \pi_2(X^0, L_t)$ (we implicitly identify these groups with each other by means of the isotopy), or its multiples (for non-simple discs). The weight associated to the class β defines a regular function

$$z_\beta = T^{\omega(\beta)} \nabla(\partial\beta) \in \mathcal{O}(U_i^\vee),$$

in fact a monomial in the coordinates $(z_1, \ldots, z_n)$ of (2.3). In this situation, assuming its transversality, the moduli space $\mathcal{M}_1(\{L_t\}, \beta)$ of all holomorphic discs in the class β bounded by L_t as t varies from 0 to 1, with one boundary marked point, is a closed $(n-1)$-dimensional manifold, oriented if we fix a spin structure on L_t. Thus, evaluation at the boundary marked point (combined with identification of the submanifolds L_t via the isotopy) yields a cycle $C_\beta = \mathrm{ev}_*[\mathcal{M}_1(\{L_t\}, \beta)] \in H_{n-1}(L_t)$. The instanton corrections to the gluing of the local coordinate charts (2.3) are then of the form

$$z_i \mapsto (h(z_\beta))^{C_\beta \cdot \gamma_i} z_i, \tag{2.4}$$

where $h(z_\beta) = 1 + z_\beta + \cdots \in \mathbb{Q}[[z_\beta]]$ is a power series recording the (virtual) contributions of multiple covers of the discs in the class β. In practice, we shall only use the weaker property that these transformations are of the form

$$z_i \mapsto h_i(z_\beta) z_i, \tag{2.5}$$

where $h_i(z_\beta) \in 1 + z_\beta\mathbb{Q}[[z_\beta]]$.

In the examples we consider in this chapter, there are only finitely many walls in B, and the above considerations are sufficient to construct the SYZ mirror of X^0 out of instanton-corrected gluings of local charts. In general, intersections between walls lead, via a "scattering" phenomenon, to an infinite number of higher-order instanton corrections; it is conjectured that these Floer-theoretic corrections can be determined using the machinery developed by Kontsevich-Soibelman [157], [158] and Gross-Siebert [104], [106].

Remark 2.1 We have discussed how to construct the analytic space Y^0 ("B-model") from the symplectic geometry of X^0 ("A-model"). When Y^0 makes sense as a complex manifold (i.e., assuming convergence), one also expects it to carry a natural Kähler structure for which the A-model of Y^0 is equivalent to the B-model of X^0. We will however not emphasize this feature of mirror symmetry.

2.2 The superpotential In the previous section we explained the construction of the SYZ mirror Y^0 of an open Calabi-Yau manifold $X^0 = X \setminus D$, where D is an anticanonical divisor in a Kähler manifold (X, ω, J), equipped with a Lagrangian torus fibration $\pi : X^0 \to B$. We now turn to mirror symmetry for X itself.

The Fukaya category of X is a deformation of that of X^0: the Floer cohomology of Lagrangian submanifolds of X^0, when viewed as objects of $\mathcal{F}(X)$, is deformed by the presence of additional holomorphic discs that intersect the divisor D. Let L be a Lagrangian fiber of the SYZ fibration $\pi : X^0 \to B$: since the Maslov class of L in X^0 vanishes, the Maslov index of a holomorphic disc bounded by L in X is equal to twice its algebraic intersection number with D. Following Fukaya, Oh, Ohta, and Ono [81] we associate to L and a rank 1 local system ∇ over it the *obstruction*

$$\mathfrak{m}_0(L,\nabla) = \sum_{\beta\in\pi_2(X,L)\setminus\{0\}} z_\beta(L,\nabla)\,\mathrm{ev}_*[\mathcal{M}_1(L,\beta)] \in C^*(L;\mathbb{K}), \tag{2.6}$$

where $z_\beta(L,\nabla) = T^{\omega(\beta)}\nabla(\partial\beta)$ is the weight associated to the class β, and $\mathcal{M}_1(L,\beta)$ is the moduli space of holomorphic discs with one boundary marked point in (X,L) representing the class β. In the absence of bubbling, one can achieve regularity, and $[\mathcal{M}_1(L,\beta)]$ can be defined as the fundamental class of the manifold $\mathcal{M}_1(L,\beta)$. To consider a more general situation, we appeal to the work of Fukaya, Oh, Ohta, and Ono who define such a potential for Lagrangian fibers in toric manifolds in [82]. While the examples we consider are not toric, their construction applies more generally whenever the moduli spaces of stable holomorphic discs with non-positive Maslov index contribute trivially to the Floer differential. The situation is therefore simplest when the divisor D is nef, or more generally when the following condition holds:

Assumption 2.2 *Every rational curve $C \simeq \mathbb{P}^1$ in X has non-negative intersection number $D\cdot C \geq 0$.*

Consider first the case of a Lagrangian submanifold L which is tautologically unobstructed in X^0. By positivity of intersections, the minimal Maslov index of a non-constant holomorphic disc with boundary on L is 2 (when $\beta\cdot D = 1$). Gromov compactness implies that the chain $\mathrm{ev}_*[\mathcal{M}_1(L,\beta)]$ is actually a cycle, of dimension $n-2+\mu(\beta) = n$, i.e. a scalar multiple $n(L,\beta)[L]$ of the fundamental class of L; whereas the evaluation chains for $\mu(\beta) > 2$ have dimension greater than n and we discard them. Thus (L,∇) is *weakly unobstructed*, i.e.

$$\mathfrak{m}_0(L,\nabla) = W(L,\nabla)\,e_L$$

is a multiple of the unit in $H^0(L,\mathbb{K})$, which is Poincaré dual to the fundamental class of L. More generally, Assumption 2.2 excludes discs of negative Maslov index, while the vanishing of the contribution of discs of Maslov index 0 is explained in Appendix A.2.

Given an open subset $U \subset B \setminus B^{sing}$ over which the fibers of π are tautologically unobstructed in X^0, the coordinate chart $U^\vee \subset Y^0$ considered in the previous section now parametrizes weakly unobstructed objects $(L = \pi^{-1}(b),\nabla)$ of $\mathcal{F}(X)$, and the *superpotential*

$$W(L,\nabla) = \sum_{\substack{\beta\in\pi_2(X,L)\\ \beta\cdot D=1}} n(L,\beta)\,z_\beta(L,\nabla) \tag{2.7}$$

is a regular function on $U^\vee$. The superpotential represents a curvature term in Floer theory: the differential on the Floer complex of a pair of weakly unobstructed objects (L,∇) and (L',∇') squares to $(W(L',\nabla')-W(L,\nabla))\,\mathrm{id}$. In particular, the family Floer cohomology [79] of an unobstructed Lagrangian submanifold of X with the fibers of the SYZ fibration over U is expected to yield no longer an object of the derived category of coherent sheaves over $U^\vee$ but rather a *matrix factorization* of the superpotential W.

In order to construct the mirror of X globally, we again have to account for instanton corrections across the walls of potentially obstructed fibers of π. As before, these corrections are needed in order to account for the bubbling of holomorphic discs of Maslov index 0 as one crosses a wall, and encode weighted counts of such discs. Under Assumption 2.2, positivity of intersection implies that all the holomorphic discs of Maslov index 0 are contained in X^0; therefore the instanton corrections are exactly the same for X as for X^0, i.e. the moduli space of objects of $\mathcal{F}(X)$ that we construct out of the fibers of π is again Y^0 (the SYZ mirror of X^0).

A key feature of the instanton corrections is that the superpotential defined by (2.7) naturally glues to a regular function on Y^0; this is because, by construction, the gluing via wall-crossing transformations identifies quasi-isomorphic objects of $\mathcal{F}(X)$, for which the obstructions $\mathfrak{m}_0$ have to match, as explained in Corollary A.11. Thus, the mirror of X is the B-side Landau-Ginzburg model (Y^0,W), where Y^0 is the SYZ mirror of X^0 and $W\in\mathcal{O}(Y^0)$ is given by (2.7). (However, see Remark 1.3).

Remark 2.3 The regularity of the superpotential W is a useful feature for the construction of the SYZ mirror of X^0. Namely, rather than directly computing the instanton corrections by studying the enumerative geometry of holomorphic discs in X^0, it is often easier to determine them indirectly, by considering either X or some other partial compactification of X^0 (satisfying Assumption 2.2), computing the mirror superpotential in each chamber of $B\setminus B^{sing}$, and matching the expressions on either side of a wall via a coordinate change of the form (2.4).

When Assumption 2.2 fails, the instanton corrections to the SYZ mirror of X might differ from those for X^0 (hence the difference between the mirrors might be more subtle than simply equipping Y^0 with a superpotential). However, this only happens if the (virtual) counts of Maslov index 0 discs bounded by potentially obstructed fibers of π in X differ from the corresponding counts in X^0. Fukaya-Oh-Ohta-Ono have shown that this issue never arises for toric varieties [82, Corollary 11.5]. In that case, the deformation of the Fukaya category which occurs upon (partially) compactifying X^0 to X (due to the presence of additional holomorphic discs) is accurately reflected by the deformation of the mirror B-model given by the superpotential W (i.e., considering matrix factorizations rather than the usual derived category).

Unfortunately, the argument of [82] does not adapt immediately to our setting; thus for the time being we only consider settings in which Assumption 2.2 holds. This will be the subject of further investigation.

The situation is in fact symmetric: just as partially compactifying X^0 to X is mirror to equipping Y^0 with a superpotential, equipping X^0 or X with a superpotential is mirror to partially compactifying Y^0. One way to justify this claim would be to switch to the other direction of mirror symmetry, reconstructing X^0 as an SYZ mirror of Y^0 equipped with a suitable Kähler structure (cf. Remark 2.1). However, in simple cases this statement can also be understood directly. The following example will be nearly sufficient for our purposes (in Section 7 we will revisit and generalize it):

Example 2.4 Let $X^0 = \mathbb{C}^*$, whose mirror $Y^0 \simeq \mathbb{K}^*$ parametrizes objects (L, ∇) of $\mathcal{F}(X^0)$, where L is a simple closed curve enclosing the origin (up to Hamiltonian isotopy) and ∇ is a unitary rank 1 local system on L. The natural coordinate on Y^0, as given by (2.3), tends to zero as the area enclosed by L tends to infinity. Equipping X^0 with the superpotential $W(x) = x$, the Fukaya category $\mathcal{F}(X^0, W)$ also contains "admissible" non-compact Lagrangian submanifolds, i.e. properly embedded Lagrangians whose image under W is only allowed to tend to infinity in the direction of the positive real axis. Denote by L_∞ a properly embedded arc which connects $+\infty$ to itself by passing around the origin (and encloses an infinite amount of area). An easy calculation in $\mathcal{F}(X^0, W)$ shows that $\mathrm{HF}^*(L_\infty, L_\infty) \simeq H^*(S^1; \mathbb{K})$; so L_∞ behaves Floer cohomologically like a torus. In particular, L_∞ admits a one-parameter family of deformations in $\mathcal{F}(X^0, W)$; these are represented by equipping L_∞ with a bounding cochain in $\mathrm{HF}^1(L_\infty, L_\infty) = \mathbb{K}$ of sufficiently large valuation (with our conventions, the valuation of 0 is $+\infty$). Given a point $cT^\lambda \in \mathbb{K}$, the Floer differential on the Floer complex of (L_∞, cT^λ) with another Lagrangian counts, in addition to the usual strips, triangles with one boundary puncture converging to a time 1 chord of an appropriate Hamiltonian (equal to a positive multiple of $\mathrm{Re}(x)$ near $+\infty$) with ends on L_∞ (this is the implementation of the Fukaya category $\mathcal{F}(X^0, W)$ appearing in [219]); these triangles are counted with Novikov weights equal to their topological energy.

Except for the case $c = 0$, these additional objects of the Fukaya category turn out to be isomorphic to simple closed curves (enclosing the origin) with rank 1 local systems. More precisely, let L_λ be the fiber enclosing an additional amount of area $\lambda \in \mathbb{R}$ compared to a suitable reference Lagrangian L_0, and ∇_c the local system with holonomy c. (Fixing a Liouville 1-form θ, we choose L_0 so that $\int_{L_0} \theta$ is equal to the action $\mathcal{A}$ of the Hamiltonian chord from L_∞ to itself; so $\int_{L_\lambda} \theta = \mathcal{A} + \lambda$.) Then an easy computation shows that the pairs (L_∞, cT^λ) and (L_λ, ∇_c) represent quasi-isomorphic objects of $\mathcal{F}(\mathbb{C}^*, W)$. Thus, in $\mathcal{F}(\mathbb{C}^*, W)$ the previously considered moduli space of objects contains an additional point L_∞; this naturally extends the mirror from $Y^0 \simeq \mathbb{K}^*$ to $Y \simeq \mathbb{K}$, and the coordinate coming from identifying bounding cochains on L_∞ with local systems on closed curves defines an analytic structure near this point.

Alternatively, one can geometrically recover the Lagrangians L_λ (together with a trivial noncompact component which is quasi-isomorphic to zero) as self-surgeries of the immersed Lagrangian obtained by deforming L_∞ to a curve with one self-intersection, enclosing the same amount of area as L_λ. This self-intersection corresponds to a generator in $HF^1(L_\infty, L_\infty)$, giving rise to a bounding cochain. The Floer-theoretic isomorphisms between bounding cochains on admissible Lagrangians and embedded Lagrangians then become an instance of the surgery formula of [83].

3 Notations and constructions

3.1 Hypersurfaces near the tropical limit

Let V be a (possibly non-compact) toric variety of complex dimension n, defined by a fan $\Sigma_V \subseteq \mathbb{R}^n$. We denote by $\sigma_1, \dots, \sigma_r$ the primitive integer generators of the rays of Σ_V. We consider a family of smooth algebraic hypersurfaces $H_\tau \subset V$ (where $\tau \to 0$), transverse to the toric divisors in V, and degenerating to the "tropical" limit. Namely, in affine coordinates $\mathbf{x} = (x_1, \dots, x_n)$ over the open stratum $V^0 \simeq (\mathbb{C}^*)^n \subset V$, H_τ is defined by an equation of the form

$$f_\tau = \sum_{\alpha \in A} c_\alpha \tau^{\rho(\alpha)} \mathbf{x}^\alpha = 0, \tag{3.1}$$

where A is a finite subset of the lattice $\mathbb{Z}^n$ of characters of the torus V^0, $c_\alpha \in \mathbb{C}^*$ are arbitrary constants, and $\rho : A \to \mathbb{R}$ satisfies a certain convexity property.

More precisely, f_τ is a section of a certain line bundle $\mathcal{L}$ over V, determined by a convex piecewise linear function $\lambda : \Sigma_V \to \mathbb{R}$ with integer linear slopes. (Note that $\mathcal{L}$ need not be ample; however the convexity assumption forces it to be nef.) The polytope P associated to $\mathcal{L}$ is the set of all $v \in \mathbb{R}^n$ such that $\langle v, \cdot \rangle + \lambda$ takes everywhere non-negative values; more concretely, $P = \{v \in \mathbb{R}^n \,|\, \langle \sigma_i, v \rangle + \lambda(\sigma_i) \geq 0 \ \forall 1 \leq i \leq r\}$. It is a classical fact that the integer points of P give a basis of the space of sections of $\mathcal{L}$. The condition that H_τ be transverse to each toric stratum of V is then equivalent to the requirement that $A \subseteq P \cap \mathbb{Z}^n$ intersects nontrivially the closure of each face of P (i.e., in the compact case, A should contain every vertex of P).

Consider a polyhedral decomposition $\mathcal{P}$ of the convex hull $\mathrm{Conv}(A) \subseteq \mathbb{R}^n$, whose set of vertices is exactly $\mathcal{P}^{(0)} = A$. We will mostly consider the case where the decomposition $\mathcal{P}$ is *regular*, i.e. every cell of $\mathcal{P}$ is congruent under the action of $GL(n, \mathbb{Z})$ to a standard simplex. We say that $\rho : A \to \mathbb{R}$ is *adapted* to the polyhedral decomposition $\mathcal{P}$ if it is the restriction to A of a convex piecewise linear function $\bar{\rho} : \mathrm{Conv}(A) \to \mathbb{R}$ whose maximal domains of linearity are exactly the cells of $\mathcal{P}$.

Definition 3.1 *The family of hypersurfaces $H_\tau \subset V$ has a* maximal degeneration *for $\tau \to 0$ if it is given by equations of the form* (3.1) *where ρ is adapted to a regular polyhedral decomposition $\mathcal{P}$ of* $\mathrm{Conv}(A)$.

The logarithm map $\mathrm{Log}_\tau : \mathbf{x} = (x_1, \dots, x_n) \mapsto \frac{1}{|\log \tau|}(\log |x_1|, \dots, \log |x_n|)$ maps H_τ to its *amoeba* $\Pi_\tau = \mathrm{Log}_\tau(H_\tau \cap V^0)$; it is known [180], [201] that, for $\tau \to 0$, the amoeba $\Pi_\tau \subset \mathbb{R}^n$ converges to the *tropical hypersurface* $\Pi_0 \subset \mathbb{R}^n$ defined by the tropical polynomial

$$\varphi(\xi) = \max \{ \langle \alpha, \xi \rangle - \rho(\alpha) \,|\, \alpha \in A \} \tag{3.2}$$

(namely, Π_0 is the set of points where the maximum is achieved more than once). Combinatorially, Π_0 is the dual cell complex of $\mathcal{P}$; in particular the connected components of $\mathbb{R}^n \setminus \Pi_0$ can be naturally labelled by the elements of $\mathcal{P}^{(0)} = A$, according to which term achieves the maximum in (3.2).

Example 3.2 The toric variety $V = \mathbb{P}^1 \times \mathbb{P}^1$ is defined by the fan $\Sigma \subseteq \mathbb{R}^2$ whose rays are generated by $\sigma_1 = (1,0)$, $\sigma_2 = (0,1)$, $\sigma_3 = (-1,0)$, $\sigma_4 = (0,-1)$. The piecewise linear function $\lambda : \Sigma \to \mathbb{R}$ with $\lambda(\sigma_1) = \lambda(\sigma_2) = 0$, $\lambda(\sigma_3) = 3$, and $\lambda(\sigma_4) = 2$ defines the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,2)$, whose associated polytope is $P = \{(v_1, v_2) \in \mathbb{R}^2 : \ 0 \leq v_1 \leq 3,\ 0 \leq v_2 \leq 2\}$. Let $A = P \cap \mathbb{Z}^2$. The regular decomposition of P shown in Figure 1 (left) is induced by the function $\rho : A \to \mathbb{R}$ whose values are given in the figure. The corresponding tropical hypersurface $\Pi_0 \subseteq \mathbb{R}^2$ is shown in Figure 1 (right); Π_0 is the limit of the amoebas of a maximally degenerating family of smooth genus 2 curves $H_\tau \subset V$ as $\tau \to 0$.

When the toric variety V is non-compact, P is unbounded, and the convex hull of A is only a proper subset of P. For instance, Figure 1 also represents a maximally degenerating family of smooth genus 2 curves in $V^0 \simeq (\mathbb{C}^*)^2$ (where now $P = \mathbb{R}^2$).

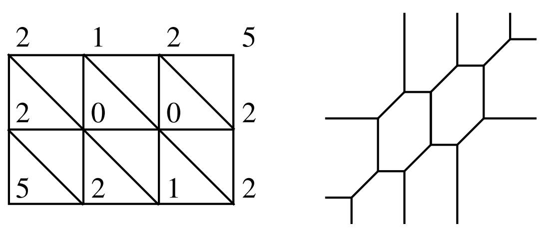


Figure 1: A regular decomposition of the polytope for $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,2)$, and the corresponding tropical hypersurface.

We now turn to the symplectic geometry of the situation we just considered. Assume that V is equipped with a complete toric Kähler metric, with Kähler form ω_V. The torus $T^n = (S^1)^n$ acts on (V, ω_V) by Hamiltonian diffeomorphisms; we denote by $\mu_V : V \to \mathbb{R}^n$ the corresponding moment map. It is well-known that the image of μ_V is a convex polytope $\Delta_V \subset \mathbb{R}^n$, dual to the fan Σ_V. The preimage of the interior of Δ_V is the open stratum $V^0 \subset V$; over V^0 the logarithm map Log_τ and the moment map μ_V are related by some diffeomorphism $g_\tau : \mathbb{R}^n \xrightarrow{\sim} \mathrm{int}(\Delta_V)$.

For a fixed Kähler form ω_V, the diffeomorphism g_τ gets rescaled by a factor of $|\log \tau|$ as τ varies; in particular, the moment map images $\mu_V(H_\tau) = \overline{g_\tau(\Pi_\tau)} \subseteq \Delta_V$ of a degenerating family of hypersurfaces collapse towards the boundary of Δ_V as $\tau \to 0$. This can be avoided by considering a varying family of Kähler forms $\omega_{V,\tau}$, obtained from the given ω_V by symplectic inflation along all the toric divisors of V, followed by a rescaling so that $[\omega_{V,\tau}] = [\omega_V]$ is independent of τ. (To be more concrete, one could e.g. consider a family of toric Kähler forms which are multiples of the standard complete Kähler metric of $(\mathbb{C}^*)^n$ over increasingly large open subsets of V^0.)

Throughout this chapter, we will consider smooth hypersurfaces that are close enough to the tropical limit, namely hypersurfaces of the form considered above with τ sufficiently close to 0. The key requirement we have for "closeness" to the tropical limit is that the amoeba should lie in a sufficiently small neighborhood of the tropical hypersurface Π_0, so that the complements have the same combinatorics. Since we consider a single hypersurface rather than the whole family, we will omit τ from the notation.

Definition 3.3 *A smooth hypersurface $H = f^{-1}(0)$ in a toric variety V is* nearly tropical *if it is a member of a maximally degenerating family of hypersurfaces as above, with the property that the amoeba $\Pi = \mathrm{Log}(H) \subset \mathbb{R}^n$ is entirely contained inside a neighborhood of the tropical hypersurface Π_0 which retracts onto Π_0.*

In particular, each element $\alpha \in A$ determines a non-empty open component of $\mathbb{R}^n \setminus \Pi$; we will (abusively) refer to it as the component over which the monomial of f with weight α dominates.

We equip V with a toric Kähler form ω_V of the form discussed above, and denote by μ_V and Δ_V the moment map and its image. Let $\delta > 0$ be a constant such that a standard symplectic tubular neighborhood U_H of H of size δ embeds into V and the complement of the moment map image $\mu_V(U_H)$ has a non-empty component for each element of A (i.e. for each monomial in f).

Remark 3.4 The assumption that the degeneration is maximal is made purely for convenience, and to ensure that the toric variety Y constructed in §3.3 below is smooth. However, all of our arguments work equally well in the case of non-maximal degenerations.

3.2 Blowing up

Our main goal is to study SYZ mirror symmetry for the blow-up X of $V \times \mathbb{C}$ along $H \times 0$, equipped with a suitable Kähler form.

Recalling that the defining equation f of H is a section of a line bundle $\mathcal{L} \to V$, the normal bundle to $H \times 0$ in $V \times \mathbb{C}$ is the restriction of $\mathcal{L} \oplus \mathcal{O}$, and we can construct explicitly X as a hypersurface in the total space of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to V \times \mathbb{C}$. Namely, the defining section of $H \times 0$ projectivizes to a section $\mathbf{s}(\mathbf{x}, y) = (f(\mathbf{x}) : y)$ of $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$ over the complement of $H \times 0$; and X is the closure of the graph of $\mathbf{s}$. In other terms,

$$X = \{(\mathbf{x}, y, (u : v)) \in \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \mid f(\mathbf{x})v = yu\}. \tag{3.3}$$

In this description it is clear that the projection $p : X \to V \times \mathbb{C}$ is a biholomorphism outside of the exceptional divisor $E = p^{-1}(H \times 0)$.

The S^1-action on $V \times \mathbb{C}$ by rotation of the $\mathbb{C}$ factor preserves $H \times 0$ and hence lifts to an S^1-action on X. This action preserves the exceptional divisor E, and acts by rotation in the standard manner on each fiber of the $\mathbb{P}^1$-bundle $p_{|E} : E \to H \times 0$. In coordinates, we can write this action in the form:

$$e^{i\theta} \cdot (\mathbf{x}, y, (u : v)) = (\mathbf{x}, e^{i\theta} y, (u : e^{i\theta} v)). \tag{3.4}$$

Thus, the fixed point set of the S^1-action on X consists of two disjoint strata: the proper transform $\tilde{V}$ of $V \times 0$ (corresponding to $y = 0,\ v = 0$ in the above description), and the section $\tilde{H}$ of p over $H \times 0$ given by the line subbundle $\mathcal{O}$ of the normal bundle (i.e., the point $(0 : 1)$ in each fiber of $p_{|E}$).

The open stratum $V^0 \times \mathbb{C}^*$ of the toric variety $V \times \mathbb{C}$ carries a holomorphic $(n+1)$-form $\Omega_{V\times\mathbb{C}} = i^{n+1} \prod_j d\log x_j \wedge d\log y$, which has simple poles along the toric divisor $D_{V\times\mathbb{C}} = (V \times 0) \cup (D_V \times \mathbb{C})$ (where $D_V = V \setminus V^0$ is the union of the toric divisors in V). The pullback $\Omega = p^*(\Omega_{V\times\mathbb{C}})$ has simple poles along the proper transform of $D_{V\times\mathbb{C}}$, namely the anticanonical divisor $D = \tilde{V} \cup p^{-1}(D_V \times \mathbb{C})$. The complement $X^0 = X \setminus D$, equipped with the S^1-invariant holomorphic $(n+1)$-form Ω, is an open Calabi-Yau manifold.

Remark 3.5 $X \setminus \tilde{V}$ corresponds to $v \neq 0$ in (3.3), so it is isomorphic to an affine conic bundle over V, namely the hypersurface in the total space of $\mathcal{O} \oplus \mathcal{L}$ given by

$$\{(\mathbf{x}, y, z) \in \mathcal{O} \oplus \mathcal{L} \mid f(\mathbf{x}) = yz\}. \tag{3.5}$$

Further removing the fibers over D_V, we conclude that X^0 is a conic bundle over the open stratum $V^0 \simeq (\mathbb{C}^*)^n$, given again by the equation $\{f(\mathbf{x}) = yz\}$.

We equip X with an S^1-invariant Kähler form ω_ϵ for which the fibers of the exceptional divisor have a sufficiently small area $\epsilon > 0$. Specifically, we require that $\epsilon \in (0, \delta/2)$, where δ is the size of the standard tubular neighborhood of H that embeds in (V, ω_V). The most natural way to construct such a Kähler form would be to equip $\mathcal{L}$ with a Hermitian metric, which determines a Kähler form on $\mathbb{P}(\mathcal{L} \oplus \mathcal{O})$ and, by restriction, on X; on the complement of E the resulting Kähler form is given by

$$p^*\omega_{V\times\mathbb{C}} + \frac{i\epsilon}{2\pi} \partial\bar{\partial} \log(|f(\mathbf{x})|^2 + |y|^2), \tag{3.6}$$

where $\omega_{V\times\mathbb{C}}$ is the product Kähler form on $V \times \mathbb{C}$ induced by the toric Kähler form ω_V on V and the standard area form of $\mathbb{C}$.

However, from a symplectic perspective the blowup operation amounts to deleting from $V \times \mathbb{C}$ a standard symplectic tubular neighborhood of $H \times 0$ and collapsing its boundary (an S^3-bundle over H) onto E by the Hopf map. Thus, X and $V \times \mathbb{C}$ are symplectomorphic away from neighborhoods of E and $H \times 0$; to take full advantage of this, we will choose ω_ϵ in such a way that the projection $p : X \to V \times \mathbb{C}$ is a symplectomorphism away from a neighborhood of the exceptional divisor. Namely, we set

$$\omega_\epsilon = p^*\omega_{V\times\mathbb{C}} + \frac{i\epsilon}{2\pi}\partial\bar{\partial}\left(\chi(\mathbf{x},y)\log(|f(\mathbf{x})|^2+|y|^2)\right), \tag{3.7}$$

where χ is a suitably chosen S^1-invariant smooth cut-off function supported in a tubular neighborhood of $H \times 0$, with $\chi = 1$ near $H \times 0$. It is clear that (3.7) defines a Kähler form provided ϵ is small enough; specifically, ϵ needs to be such that a standard symplectic neighborhood of size ϵ of $H \times 0$ can be embedded (S^1-equivariantly) into the support of χ. For simplicity, we assume that χ is chosen so that the following property holds:

Property 3.6 *The support of χ is contained inside $p^{-1}(U_H \times B_\delta)$, where $U_H \subset V$ is a standard symplectic δ-neighborhood of H and $B_\delta \subset \mathbb{C}$ is the disc of radius δ.*

Remark 3.7 ω_ϵ lies in the same cohomology class $[\omega_\epsilon] = p^*[\omega_{V\times\mathbb{C}}] - \epsilon[E]$ as the Kähler form defined by (3.6), and is equivariantly symplectomorphic to it.

3.3 The mirror B-side Landau-Ginzburg model Using the same notations as in the previous section, we now describe a B-side Landau-Ginzburg model which we claim is SYZ mirror to X (with the Kähler form ω_ϵ, and relatively to the anticanonical divisor D).

Recall that the hypersurface $H \subset X$ has a defining equation of the form (3.1), involving toric monomials whose weights range over a finite subset $A \subset \mathbb{Z}^n$, forming the vertices of a polyhedral complex $\mathcal{P}$ (cf. Definition 3.1).

We denote by Y the (noncompact) $(n+1)$-dimensional toric variety defined by the fan $\Sigma_Y = \mathbb{R}_{\geq 0}\cdot(\mathcal{P}\times\{1\}) \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R}$. Namely, the integer generators of the rays of Σ_Y are the vectors of the form $(-\alpha, 1)$, $\alpha \in A$, and the vectors $(-\alpha_1, 1), \dots, (-\alpha_k, 1)$ span a cone of Σ_Y if and only if $\alpha_1, \dots, \alpha_k$ span a cell of $\mathcal{P}$.

Dually, Y can be described by a (noncompact) polytope $\Delta_Y \subseteq \mathbb{R}^{n+1}$, defined in terms of the tropical polynomial $\varphi : \mathbb{R}^n \to \mathbb{R}$ associated to H (cf. (3.2)) by

$$\Delta_Y = \{(\xi,\eta) \in \mathbb{R}^n \oplus \mathbb{R} \mid \eta \geq \varphi(\xi)\}. \tag{3.8}$$

Remark 3.8 The polytope Δ_Y also determines a Kähler class $[\omega_Y]$ on Y. While in this chapter we focus on the A-model of X and the B-model of Y, it can be shown that the family of complex structures on X obtained by blowing up $V \times \mathbb{C}$ along the maximally degenerating family $H_\tau \times 0$ (cf. §3.1) corresponds to a family of Kähler forms asymptotic to $|\log\tau|[\omega_Y]$ as $\tau \to 0$.

Remark 3.9 Even though deforming the hypersurface H inside V does not modify the symplectic geometry of X, the topology of Y depends on the chosen polyhedral decomposition $\mathcal{P}$ (i.e., on the combinatorial type of the tropical hypersurface defined by φ). However, the various possibilities for Y are related to each other by crepant birational transformations, and hence are expected to yield equivalent B-models. (The A-model of Y, on the other hand, is affected by these birational transformations and does depend on the tropical polynomial φ, as explained in the previous remark.)

The facets of Δ_Y correspond to the maximal domains of linearity of φ. Thus the irreducible toric divisors of Y are in one-to-one correspondence with the connected components of $\mathbb{R}^n \setminus \Pi_0$, and the combinatorics of the toric strata of Y can be immediately read off the tropical hypersurface Π_0 (see Example 3.12 below).

It is advantageous for our purposes to introduce a collection of affine charts on Y indexed by the elements of A (i.e., the facets of Δ_Y, or equivalently, the connected components of $\mathbb{R}^n \setminus \Pi_0$).

For each $\alpha \in A$, let $Y_\alpha = (\mathbb{K}^*)^n \times \mathbb{K}$, with coordinates $\mathbf{v}_\alpha = (v_{\alpha,1}, \dots, v_{\alpha,n}) \in (\mathbb{K}^*)^n$ and $v_{\alpha,0} \in \mathbb{K}$ (as before, $\mathbb{K}$ is either Λ or $\mathbb{C}$). Whenever $\alpha, \beta \in A$ are connected by an edge in the polyhedral decomposition $\mathcal{P}$ (i.e., whenever the corresponding components of $\mathbb{R}^n \setminus \Pi_0$ share a top-dimensional facet, with primitive normal vector $\beta - \alpha$), we glue Y_α to Y_β by the coordinate transformations

$$\begin{cases} v_{\alpha,i} = v_{\beta,0}^{\beta_i - \alpha_i}\, v_{\beta,i} \qquad (1 \le i \le n), \\ v_{\alpha,0} = v_{\beta,0}. \end{cases} \tag{3.9}$$

These charts cover the complement in Y of the codimension 2 strata (as Y_α covers the open stratum of Y and the open stratum of the toric divisor corresponding to α). In terms of the standard basis of toric monomials indexed by weights in $\mathbb{Z}^{n+1}$, $v_{\alpha,0}$ is the monomial with weight $(0, \dots, 0, 1)$, and for $i \ge 1$ $v_{\alpha,i}$ is the monomial with weight $(0, \dots, -1, \dots, 0, -\alpha_i)$ (the i-th entry is -1).

Denoting by T the Novikov parameter (treated as an actual complex parameter when $\mathbb{K} = \mathbb{C}$), and by v_0 the common coordinate $v_{\alpha,0}$ for all charts, we set

$$w_0 = -T^\epsilon + T^\epsilon v_0. \tag{3.10}$$

With this notation, the above coordinate transformations can be rewritten as

$$v_{\alpha,i} = \big(1 + T^{-\epsilon} w_0\big)^{\beta_i - \alpha_i}\, v_{\beta,i}, \qquad 1 \le i \le n.$$

More generally, for $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we set $\mathbf{v}_\alpha^m = v_{\alpha,1}^{m_1} \dots v_{\alpha,n}^{m_n}$. Then

$$\mathbf{v}_\alpha^m = (1 + T^{-\epsilon} w_0)^{\langle \beta - \alpha, m \rangle} \mathbf{v}_\beta^m. \tag{3.11}$$

We shall see that w_0 and the transformations (3.11) have a natural interpretation in terms of the enumerative geometry of holomorphic discs in X.

Next, recall from §3.1 that the inward normal vectors to the facets of the moment polytope Δ_V associated to (V, ω_V) are the primitive integer generators $\sigma_1, \dots, \sigma_r$ of the rays of Σ_V. Thus, there exist constants $\varpi_1, \dots, \varpi_r \in \mathbb{R}$ such that

$$\Delta_V = \{u \in \mathbb{R}^n \mid \langle \sigma_i, u\rangle + \varpi_i \geq 0 \ \ \forall 1 \leq i \leq r\}. \tag{3.12}$$

Then for $i = 1, \dots, r$ we set

$$w_i = T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i} \tag{3.13}$$

where $\alpha_i \in A$ is chosen to lie on the facet of P defined by σ_i, i.e. so that $\langle \sigma_i, \alpha_i\rangle$ is minimal. Hence, by the conditions imposed in §3.1, $\langle \sigma_i, \alpha_i\rangle + \lambda(\sigma_i) = 0$, where $\lambda : \Sigma_V \to \mathbb{R}$ is the piecewise linear function defining $\mathcal{L} = \mathcal{O}(H)$. By (3.11), the choice of α_i satisfying the required condition is irrelevant: in all cases $\mathbf{v}_{\alpha_i}^{\sigma_i}$ is simply the toric monomial with weight $(-\sigma_i, \lambda(\sigma_i)) \in \mathbb{Z}^n \oplus \mathbb{Z}$. Moreover, this weight pairs non-negatively with all the rays of the fan Σ_Y, therefore w_i defines a regular function on Y.

With all the notation in place, we can at last make the following definition, which clarifies the statements of Theorems 1.5 and 1.6:

Definition 3.10 *We denote by Y^0 the complement of the hypersurface $D_Y = w_0^{-1}(0)$ in the toric $(n+1)$-fold Y, and define the* leading-order superpotential

$$W_0 = w_0 + w_1 + \dots + w_r = -T^{\epsilon} + T^{\epsilon} v_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i} \in \mathcal{O}(Y). \tag{3.14}$$

We also define

$$W_0^H = -v_0 + w_1 + \dots + w_r = -v_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i} \in \mathcal{O}(Y). \tag{3.15}$$

Remark 3.11 Since there are no convergence issues, we can think of (Y^0, W_0) and (Y, W_0^H) either as B-side Landau-Ginzburg models defined over the Novikov field or as one-parameter families of complex B-side Landau-Ginzburg models defined over $\mathbb{C}$.

Example 3.12 When H is the genus 2 curve of Example 3.2, the polytope Δ_Y has 12 facets (2 of them compact and the 10 others non-compact), corresponding to the 12 components of $\mathbb{R}^n \setminus \Pi_0$, and intersecting exactly as pictured on Figure 1 right. The edges of the figure correspond to the configuration of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s along which the toric divisors of the 3-fold Y intersect.

Label the irreducible toric divisors by $D_{a,b}$ ($0 \leq a \leq 3$, $0 \leq b \leq 2$), corresponding to the elements $(a, b) \in A$. Then the leading-order superpotential W_0 consists of five terms: $w_0 = -T^{\epsilon} + T^{\epsilon} v_0$, where v_0 is the toric monomial of weight $(0, 0, 1)$, which vanishes with multiplicity 1 on each of the 12 toric divisors; and up to constant factors, w_1 is the toric monomial with weight $(-1,0,0)$, which vanishes with multiplicity a on $D_{a,b}$; w_2 is the toric monomial with weight $(0,-1,0)$, vanishing with multiplicity b on $D_{a,b}$; w_3 is the monomial with weight $(1,0,3)$, with multiplicity $(3-a)$ on $D_{a,b}$; and w_4 is the monomial with weight $(0,1,2)$, with multiplicity $(2-b)$ on $D_{a,b}$. In particular, the compact divisors $D_{1,1}$ and $D_{2,1}$ are components of the singular fiber $\{W_0 = -T^\epsilon\} \subset Y^0$ (which also has a third, non-compact component); and similarly for $\{W_0^H = 0\} \subset Y$.

(In general the order of vanishing of w_i on a given divisor is equal to the intersection number with Π_0 of a semi-infinite ray in the direction of $-\sigma_i$ starting from a generic point in the relevant component of $\mathbb{R}^n \setminus \Pi_0$.)

This example does not satisfy Assumption 1.4, and in this case the actual mirror of X differs from (Y^0, W_0) by higher-order correction terms. On the other hand, if we consider the genus 2 curve with 10 punctures $H \cap V^0$ in the open toric variety $V^0 \simeq (\mathbb{C}^*)^2$, which does fall within the scope of Theorem 1.5, the construction yields the same toric 3-fold Y, but now we simply have $W_0 = w_0$ (resp. $W_0^H = -v_0$).

4 Lagrangian torus fibrations on blowups of toric varieties

As in §3.2, we consider a smooth nearly tropical hypersurface $H = f^{-1}(0)$ in a toric variety V of dimension n, and the blow-up X of $V \times \mathbb{C}$ along $H \times 0$, equipped with the S^1-invariant Kähler form ω_ϵ given by (3.7). Our goal in this section is to construct an S^1-invariant Lagrangian torus fibration $\pi : X^0 \to B$ (with appropriate singularities) on the open Calabi-Yau manifold $X^0 = X \setminus D$, where D is the proper transform of the toric anticanonical divisor of $V \times \mathbb{C}$. (Similar fibrations have been previously considered by Gross [101], [102] and by Castaño-Bernard and Matessi [46], [47].) The key observation is that S^1-invariance forces the fibers of π to be contained in the level sets of the moment map of the S^1-action. Thus, we begin by studying the geometry of the reduced spaces.

4.1 The reduced spaces The S^1-action (3.4) on X is Hamiltonian with respect to the Kähler form ω_ϵ given by (3.7), and its moment map $\mu_X : X \to \mathbb{R}$ can be determined explicitly. Outside of the exceptional divisor, we identify X with $V \times \mathbb{C}$ via the projection p, and observe that $\mu_X(\mathbf{x}, y) = \int_{D(\mathbf{x},y)} \omega_\epsilon$, where $D(\mathbf{x}, y)$ is a disc bounded by the orbit of $(\mathbf{x}, y)$, namely the total transform of $\{\mathbf{x}\} \times D^2(|y|) \subset V \times \mathbb{C}$. (We normalize μ_X so that it takes the constant value 0 over the proper transform of $V \times 0$; also, our convention differs from the usual one by a factor of 2π.)

Hence, for given $\mathbf{x}$ the quantity $\mu_X(\mathbf{x}, y)$ is a strictly increasing function of $|y|$. Moreover, applying Stokes' theorem we find that

$$\mu_X(\mathbf{x},y) = \pi|y|^2 + \frac{\epsilon}{2}|y|\frac{\partial}{\partial|y|}\left(\chi(\mathbf{x},y)\log(|f(\mathbf{x})|^2+|y|^2)\right). \tag{4.1}$$

In the regions where χ is constant this simplifies to:

$$\mu_X(\mathbf{x},y) = \begin{cases} \pi|y|^2 + \epsilon\,\dfrac{|y|^2}{|f(\mathbf{x})|^2+|y|^2} & \text{where } \chi\equiv 1 \text{ (near } E\text{)},\\ \pi|y|^2 & \text{where } \chi\equiv 0 \text{ (away from } E\text{)}. \end{cases} \tag{4.2}$$

(Note that the first expression extends naturally to a smooth function over E.)

The critical points of μ_X are the fixed points of the S^1-action. Besides $\tilde{V} = \mu_X^{-1}(0)$, the fixed points occur along $\tilde{H}$, which lies in the level set $\mu_X^{-1}(\epsilon)$; in particular, all the other level sets of μ_X are smooth. Since for any given $\mathbf{x}$ the moment map μ_X is a strictly increasing function of $|y|$, each level set of μ_X intersects $p^{-1}(\{\mathbf{x}\}\times\mathbb{C})$ along a single S^1-orbit. Hence, for $\lambda > 0$, the natural projection to V (obtained by composing p with projection to the first factor) yields a natural identification of the reduced space $X_{red,\lambda} = \mu_X^{-1}(\lambda)/S^1$ with V.

For $\lambda \gg \epsilon$, $\mu_X^{-1}(\lambda)$ is disjoint from the support of the cut-off function χ, and the reduced Kähler form $\omega_{red,\lambda}$ on $X_{red,\lambda} \cong V$ coincides with the toric Kähler form ω_V. As λ becomes closer to ϵ, $\omega_{red,\lambda}$ differs from ω_V near H but remains cohomologous to it. At the critical level $\lambda = \epsilon$, the reduced form $\omega_{red,\epsilon}$ is singular along H (but its singularities are fairly mild, see Lemma B.1). Finally, for $\lambda < \epsilon$ the Kähler form $\omega_{red,\lambda}$ differs from ω_V in a tubular neighborhood of H, inside which the normal direction to H has been symplectically *deflated*. In particular, one easily checks that

$$[\omega_{red,\lambda}] = [\omega_V] - \max(0,\epsilon-\lambda)[H]. \tag{4.3}$$

Our goal is to exploit the toric structure of V to construct families of Lagrangian tori in $X_{red,\lambda}$. The Kähler form $\omega_{red,\lambda}$ on $X_{red,\lambda} \cong V$ is not T^n-invariant near H; in fact it isn't even smooth along H for $\lambda = \epsilon$. However, there exist (smooth) toric Kähler forms $\omega'_{V,\lambda}$, depending piecewise smoothly on λ, with $[\omega'_{V,\lambda}] = [\omega_{red,\lambda}]$; see (13.5) for an explicit construction. The following result will be proved in Appendix B.

Lemma 4.1 *There exists a family of homeomorphisms $(\phi_\lambda)_{\lambda\in\mathbb{R}_+}$ of V such that:*

1. *ϕ_λ preserves the toric divisor $D_V \subset V$;*
2. *the restriction of ϕ_λ to V^0 is a diffeomorphism for $\lambda \neq \epsilon$, and a diffeomorphism outside of H for $\lambda = \epsilon$;*
3. *ϕ_λ intertwines the reduced Kähler form $\omega_{red,\lambda}$ and the toric Kähler form $\omega'_{V,\lambda}$;*
4. *$\phi_\lambda = \mathrm{id}$ at every point whose T^n-orbit is disjoint from the support of χ;*

5. ϕ_λ depends on λ in a continuous manner, and smoothly except at $\lambda = \epsilon$.

The diffeomorphism (singular along H for $\lambda = \epsilon$) ϕ_λ given by Lemma 4.1 yields a preferred Lagrangian torus fibration on the open stratum $X^0_{red,\lambda} = (\mu_X^{-1}(\lambda) \cap X^0)/S^1$ of $X_{red,\lambda}$ (naturally identified with V^0 under the canonical identification $X_{red,\lambda} \cong V$), namely the preimage by ϕ_λ of the standard fibration of $(V^0, \omega'_{V,\lambda})$ by T^n-orbits:

Definition 4.2 *We denote by $\pi_\lambda : X^0_{red,\lambda} \to \mathbb{R}^n$ the composition $\pi_\lambda = \mathrm{Log} \circ \phi_\lambda$, where* Log : $V^0 \cong (\mathbb{C}^*)^n \to \mathbb{R}^n$ *is the logarithm map $(x_1, \dots, x_n) \mapsto \frac{1}{|\log \tau|}(\log |x_1|, \dots, \log |x_n|)$, and $\phi_\lambda : (X_{red,\lambda}, \omega_{red,\lambda}) \to (V, \omega'_{V,\lambda})$ is as in Lemma* 4.1.

Remark 4.3 By construction, the natural affine structure (see §2.1) on the base of the Lagrangian torus fibration π_λ identifies it with the interior of the moment polytope $\Delta_{V,\lambda}$ associated to the cohomology class $[\omega'_{V,\lambda}] = [\omega_{red,\lambda}] \in H^2(V, \mathbb{R})$.

4.2 A Lagrangian torus fibration on X^0 We now assemble the Lagrangian torus fibrations π_λ on the reduced spaces into a (singular) Lagrangian torus fibration on X^0:

Definition 4.4 *We denote by $\pi : X^0 \to B = \mathbb{R}^n \times \mathbb{R}_+$ the map which sends the point $x \in \mu_X^{-1}(\lambda) \cap X^0$ to $\pi(x) = (\pi_\lambda(\bar{x}), \lambda)$, where $\bar{x} \in X^0_{red,\lambda}$ is the S^1-orbit of x.*

The map π is continuous, and smooth away from $\lambda = \epsilon$. The fiber of π above $(\xi, \lambda) \in B$ is obtained by lifting the Lagrangian torus $\pi_\lambda^{-1}(\xi) \subset X_{red,\lambda}$ to $\mu_X^{-1}(\lambda)$ and "spinning" it by the S^1-action.

Away from the fixed points of the S^1-action, $\mu_X^{-1}(\lambda)$ is a coisotropic manifold with isotropic foliation given by the S^1-orbits. Moreover, the S^1-bundle $\mu_X^{-1}(\lambda) \to X_{red,\lambda}$ is topologically trivial for $\lambda > \epsilon$ (setting $y \in \mathbb{R}_+$ gives a global section), trivial over the complement of H for $\lambda = \epsilon$, and the circle bundle associated to the line bundle $\mathcal{O}(-H)$ for $\lambda < \epsilon$; in any case, its restriction to a fiber of π_λ is topologically trivial. The fibers of π_λ are smooth Lagrangian tori (outside of H when $\lambda = \epsilon$, which corresponds precisely to the S^1-fixed points); therefore, we conclude that the fibers of π are smooth Lagrangian tori unless they contain fixed points of the S^1-action.

The only fixed points occur for $\lambda = \epsilon$, when $\mu_X^{-1}(\lambda)$ contains the stratum of fixed points $\tilde{H}$. The identification of the reduced space with V maps $\tilde{H}$ to the hypersurface H, so the singular fibers map to

$$B^{sing} = \Pi' \times \{\epsilon\} \subset B, \tag{4.4}$$

where $\Pi' = \pi_\epsilon(H \cap V^0) \subset \mathbb{R}^n$ is essentially the amoeba of the hypersurface H (up to the fact that π_ϵ differs from the logarithm map by ϕ_ϵ). The fibers above the points of B^{sing} differ from the regular fibers in that, where a smooth fiber $\pi^{-1}(\xi, \lambda) \simeq T^{n+1}$ is a trivial S^1-bundle over

$\pi_\lambda^{-1}(\xi) \simeq T^n \subset V^0$, for $\lambda = \epsilon$ some of the S^1 fibers (namely those which lie over points of H) are collapsed to points.

Because the fibration π has non-trivial monodromy around B^{sing}, the only globally defined affine coordinate on B is the last coordinate λ (the moment map of the S^1-action); other affine coordinates are only defined over subsets of $B \setminus B^{sing}$, i.e. in the complement of certain cuts. Our preferred choice for such a description relates the affine structure on B to the moment polytope $\Delta_V \times \mathbb{R}_+$ of $V \times \mathbb{C}$. Namely, away from a tubular neighborhood of $\Pi' \times (0, \epsilon)$ the Lagrangian torus fibration π coincides with the standard toric fibration on $V \times \mathbb{C}$:

Proposition 4.5 *Outside of the support of χ (a tubular neighborhood of the exceptional divisor E), the Kähler form ω_ϵ is equal to $p^*\omega_{V\times\mathbb{C}}$, and the moment map of the S^1-action is the standard one $\mu_X(\mathbf{x}, y) = \pi|y|^2$. Moreover, outside of $\pi(\operatorname{supp}\chi)$, the fibers of the Lagrangian fibration π are standard product tori, i.e. they are the preimages by p of the orbits of the T^{n+1}-action in $V \times \mathbb{C}$.*

Proof. The first statement follows immediately from formulas (3.7) and (4.1). The second one is then a direct consequence of the manner in which π was constructed and condition (3) in Lemma 4.1. $\square$

Recall that the support of χ is constrained by Property 3.6. Thus, the fibration π is standard (coincides with the standard toric fibration on $V \times \mathbb{C}$) over a large subset $B^{std} = (\mathbb{R}^n \times \mathbb{R}_+) \setminus (\operatorname{Log}(U_H) \times (0, \delta))$ of B. Since $\omega_\epsilon = p^*\omega_{V\times\mathbb{C}}$ over $\pi^{-1}(B^{std})$, we conclude that over B^{std} the affine structure of B agrees with that for the standard toric fibration of $V \times \mathbb{C}$, i.e. as an affine manifold B^{std} can be naturally identified with the complement of $\mu_V(U_H) \times (0, \delta)$ inside $\operatorname{int}(\Delta_V) \times \mathbb{R}_+$.

This description of the affine structure on $B \setminus B^{sing}$ can be extended from B^{std} to the complement of a set of codimension 1 cuts. Recall from §2.1 that the affine coordinates of $b \in B \setminus B^{sing}$ relative to some reference point b_0 are given by the symplectic areas of certain relative 2-cycles $(\Gamma_1, \dots, \Gamma_{n+1})$ with boundary on $\pi^{-1}(b) \cup \pi^{-1}(b_0)$; the above identification of B^{std} with a subset of $\Delta_V \times \mathbb{R}_+$ arises from taking the boundaries of Γ_i to be (homologous to) orbits of the various S^1 factors of the T^{n+1}-action on $V \times \mathbb{C}$.

When b and b_0 have the same last coordinate $\lambda > \epsilon$, we can choose $\Gamma_1, \dots, \Gamma_n$ to be contained in $\mu_X^{-1}(\lambda)$, and obtained as the lifts of relative 2-cycles $\Gamma_{i,red}$ in $X_{red,\lambda}$ with boundary on fibers of π_λ; we can fix such lifts by requiring that $y \in \mathbb{R}_+$ on Γ_i. Since $\int_{\Gamma_i} \omega_\epsilon = \int_{\Gamma_{i,red}} \omega_{red,\lambda}$, the affine structure on the level set $\mathbb{R}^n \times \{\lambda\} \subset B$ is the same as that on the base of the fibration π_λ on the reduced space $X_{red,\lambda}$, which can be identified via the diffeomorphism ϕ_λ with the standard toric fibration on $(V, \omega'_{V,\lambda})$. For $\lambda > \epsilon$ we have $[\omega_{red,\lambda}] = [\omega'_{V,\lambda}] = [\omega_V]$, so the base is naturally identified with the interior of the moment polytope Δ_V; moreover, this identification is consistent with our previous description of the affine structure over B^{std}, since in that region the various Kähler forms agree pointwise.

In other terms, over $\mathbb{R}^n \times (\epsilon, \infty) \subset B$, the affine structure is globally a product $\operatorname{int}(\Delta_V) \times (\epsilon, \infty)$ of the affine structure on the moment polytope of (V, ω_V) and the interval (ϵ, ∞), in a manner that extends the previous description over B^{std}.

For $\lambda < \epsilon$, the affine structure on $\mathbb{R}^n \times \{\lambda\} \subset B$ can be described similarly, by choosing relative 2-cycles $\Gamma_{i,red}$ in $X_{red,\lambda}$ with boundary on fibers of π_λ and lifting them to relative 2-cycles Γ'_i in $\mu_X^{-1}(\lambda)$. Since the lifts may intersect the exceptional divisor E, we cannot require $y \in \mathbb{R}_+$ as in the case $\lambda > \epsilon$. Instead, we use the monomial $\mathbf{x}^{\alpha_0}$ for some $\alpha_0 \in A$ to fix a trivialization of $\mathcal{L} = \mathcal{O}(H)$ over V^0, and choose the lifts so that $\mathbf{x}^{-\alpha_0} z = \mathbf{x}^{-\alpha_0} f(\mathbf{x})/y \in \mathbb{R}_+$ on Γ'_i. Since $\int_{\Gamma'_i} \omega_\epsilon = \int_{\Gamma_{i,red}} \omega_{red,\lambda}$, the affine structure on the level set $\mathbb{R}^n \times \{\lambda\} \subset B$ is again identical to that on the base of the fibration π_λ on $X_{red,\lambda}$, or equivalently via ϕ_λ, the standard toric fibration on $(V, \omega'_{V,\lambda})$. Thus, the affine structure identifies $\mathbb{R}^n \times \{\lambda\} \subset B$ with the interior of the moment polytope $\Delta_{V,\lambda}$ associated to the Kähler class $[\omega'_{V,\lambda}] = [\omega_{red,\lambda}] = [\omega_V] - \max(0, \epsilon - \lambda)[H]$. However, this description is no longer consistent with that previously given for B^{std}, because the boundary of Γ'_i does not represent the expected homology class in $\pi^{-1}(b)$.

Specifically, assume b_0 and $b \in (\mathbb{R}^n \setminus \mathrm{Log}(U_H)) \times \{\lambda\}$ lie in the connected components corresponding to α_0 and $\alpha \in A$ respectively. Then the boundary of Γ'_i in $\pi^{-1}(b_0)$ does represent the homology class of the orbit of the i-th S^1-factor, while the boundary in $\pi^{-1}(b)$ differs from it by $\alpha_i - \alpha_{0,i}$ times the orbit of the last S^1-factor. Moreover,

$$\int_{\Gamma_{i,red}} \omega_V - \int_{\Gamma_{i,red}} \omega_{red,\lambda} = (\epsilon - \lambda)(\Gamma_{i,red} \cdot H) = (\epsilon - \lambda)(\alpha_i - \alpha_{0,i}).$$

This formula also gives the difference between the ω_ϵ-areas of the relative cycles Γ'_i and the relative cycles $\Gamma_i \subset \pi^{-1}(B^{std})$ previously used to determine affine coordinates over B^{std}. Hence, the affine coordinates determined by the relative cycles Γ'_i differ from those constructed previously over B^{std} by a shear

$$(4.5) \qquad (\zeta_1, \ldots, \zeta_n, \lambda) \mapsto \big(\zeta_1 + (\epsilon - \lambda)(\alpha_1 - \alpha_{0,1}), \ldots, \zeta_n + (\epsilon - \lambda)(\alpha_n - \alpha_{0,n}), \lambda\big)$$

or more succinctly, $(\zeta, \lambda) \mapsto \big(\zeta + (\epsilon - \lambda)(\alpha - \alpha_0), \lambda\big)$.

More globally, over $\mathbb{R}^n \times (0, \epsilon) \subset B$ the affine structure can be identified (using the relative cycles Γ'_i to define coordinates) with a piece of the moment polytope for the total space of the line bundle $\mathcal{O}(-H)$ over V (equipped with a toric Kähler form in the class $[\omega_V] - \epsilon[H]$), consistent with the fact that the normal bundle to $\tilde{V}$ inside X is $\mathcal{O}(-H)$; but this description is not consistent with the one we have given over B^{std}.

On the other hand, the shears (4.5) map the complement of the amoeba of H in $\Delta_{V,\lambda}$ to the complement of a standard $(\epsilon - \lambda)$-neighborhood of the amoeba of H in Δ_V. Thus, making cuts along the projection of the exceptional divisor, we can extend the affine coordinates previously described over B^{std}, and identify the affine structure on $B \setminus (\Pi' \times (0, \epsilon])$ with an open subset of $\mathrm{int}(\Delta_V) \times \mathbb{R}_+$, obtained by deleting an $(\epsilon - \lambda)$-neighborhood of the amoeba of H from $\mathrm{int}(\Delta_V) \times \{\lambda\}$ for all $\lambda \in (0, \epsilon]$.

This is the picture of B that we choose to emphasize, depicting it as the complement of a set of "triangular" cuts inside $\Delta_V \times \mathbb{R}_+$; see Figure 2.

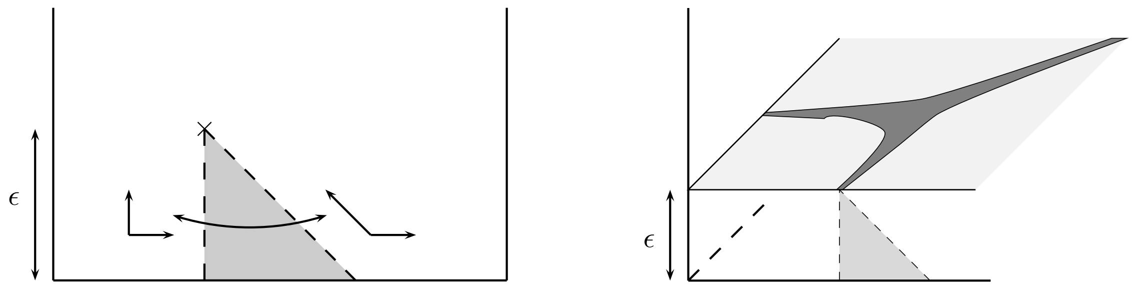


Figure 2: The base of the Lagrangian torus fibration $\pi : X^0 \to B$. Left: $H = \{\text{point}\} \subset \mathbb{CP}^1$. Right: $H = \{x_1 + x_2 = 1\} \subset \mathbb{C}^2$.

Remark 4.6 While the fibration we construct is merely Lagrangian, it is very reasonable to conjecture that in fact X^0 carries an S^1-invariant *special Lagrangian* fibration over B. The holomorphic $(n+1)$-form $\Omega = p^*\Omega_{V\times\mathbb{C}}$ on X^0 is S^1-invariant, and induces a holomorphic n-form on the reduced space $X^0_{red,\lambda}$, which turns out to coincide with the standard toric form $\Omega_V = i^n \prod_j d\log x_j$. Modifying the construction of the fibration $\pi_\lambda : X^0_{red,\lambda} \to \mathbb{R}^n$ so that its fibers are special Lagrangian with respect to Ω_V would then be sufficient to ensure that the fibers of π are special Lagrangian with respect to Ω. In dimension 1 this is easy to accomplish by elementary methods. In higher dimensions, making π_λ special Lagrangian requires the use of analysis, as the deformation of product tori in V^0 (which are special Lagrangian with respect to $\omega'_{V,\lambda}$ and Ω_V) to tori which are special Lagrangian for $\omega_{red,\lambda}$ and Ω_V is governed by a first-order elliptic PDE [179] (see also [128, §9] or [13, Prop. 2.5]). If one were to argue as in the proof of Lemma 4.1 (cf. Appendix B), the 1-forms used to construct ϕ_λ should be chosen not only to satisfy the usual condition for Moser's trick, but also to be co-closed with respect to a suitable rescaling of the Kähler metric induced by $\omega_{t,\lambda}$. When $V = (\mathbb{C}^*)^n$ this does not seem to pose any major difficulties, but in general it is not obvious that one can ensure the appropriate behavior along the toric divisors.

5 SYZ mirror symmetry for X^0

In this section we apply the procedure described in §2 to the Lagrangian torus fibration $\pi : X^0 \to B$ of §4 in order to construct the SYZ mirror to the open Calabi-Yau manifold X^0 and prove Theorem 1.7. The key observation is that, by Proposition 4.5, most fibers of π are mapped under the projection p to standard product tori in the toric variety $V \times \mathbb{C}$; therefore, the holomorphic discs bounded by these fibers can be understood by reducing to the toric case, which is well understood (see e.g. [56]).

Proposition 5.1 *The fibers of $\pi : X^0 \to B$ which bound holomorphic discs in X^0 are those which intersect the subset $p^{-1}(H \times \mathbb{C})$.*

Moreover, the simple holomorphic discs in X^0 bounded by such a fiber contained in $\mu_X^{-1}(\lambda)$ have Maslov index 0 and symplectic area $|\lambda - \epsilon|$, and their boundary represents the homology class of an

S^1-orbit if $\lambda > \epsilon$ and its negative otherwise.

Proof. Let $L \subset X^0$ be a smooth fiber of π, contained in $\mu_X^{-1}(\lambda)$ for some $\lambda \in \mathbb{R}_+$, and let $u : (D^2, \partial D^2) \to (X^0, L)$ be a holomorphic disc with boundary in L. Denote by L' the projection of L to V (i.e., the image of L by the composition p_V of p and the projection to the first factor). The restriction of p_V to $\mu_X^{-1}(\lambda)$ coincides with the quotient map to the reduced space $X_{red,\lambda} \simeq V$; thus, L' is in fact a fiber of π_λ, i.e. a Lagrangian torus in $(V^0, \omega_{red,\lambda})$, smoothly isotopic to a product torus inside $V^0 \simeq (\mathbb{C}^*)^n$.

Since the relative homotopy group $\pi_2(V^0, L') \simeq \pi_2((\mathbb{C}^*)^n, (S^1)^n)$ vanishes, the holomorphic disc $p_V \circ u : (D^2, \partial D^2) \to (V^0, L')$ is necessarily constant. Hence the image of the disc u is contained inside a fiber $p_V^{-1}(\mathbf{x})$ for some $\mathbf{x} \in V^0$.

If $\mathbf{x} \notin H$, then $p_V^{-1}(\mathbf{x}) \cap X^0 = p^{-1}(\{\mathbf{x}\} \times \mathbb{C}^*) \simeq \mathbb{C}^*$, inside which $p_V^{-1}(\mathbf{x}) \cap L$ is a circle centered at the origin (an orbit of the S^1-action). The maximum principle then implies that the map u is necessarily constant.

On the other hand, when $\mathbf{x} \in H$, $p_V^{-1}(\mathbf{x}) \cap X^0$ is the union of two affine lines intersecting transversely at one point: the proper transform of $\{\mathbf{x}\} \times \mathbb{C}$, and the fiber of E above $\mathbf{x}$ (minus the point where it intersects $\tilde{V}$). Now, $p_V^{-1}(\mathbf{x}) \cap L$ is again an S^1-orbit, i.e. a circle inside one of these two components (depending on whether $\lambda > \epsilon$ or $\lambda < \epsilon$); either way, $p_V^{-1}(\mathbf{x}) \cap L$ bounds exactly one non-constant embedded holomorphic disc in X^0 (and all of its multiple covers). The result follows. $\square$

Denote by $B^{reg} \subset B$ the set of those fibers of π which do not intersect $p^{-1}(U_H \times \mathbb{C})$. From Property 3.6 and Propositions 4.5 and 5.1, we deduce:

Corollary 5.2 *The fibers of π above the points of B^{reg} are tautologically unobstructed in X^0, and project under p to standard product tori in $V^0 \times \mathbb{C}$.*

With respect to the affine structure, $B^{reg} = (\mathbb{R}^n \setminus \mathrm{Log}(U_H)) \times \mathbb{R}_+$ is naturally isomorphic to $(\Delta_V \setminus \mu_V(U_H)) \times \mathbb{R}_+$.

Definition 5.3 *The* chamber U_α *is the connected component of B^{reg} over which the monomial of weight α dominates all other monomials in the defining equation of H.*

Remark 5.4 By construction, the complement of $\mathrm{Log}(U_H)$ is a deformation retract of the complement of the amoeba of H inside $\mathbb{R}^n$; so the set of tautologically unobstructed fibers of π retracts onto $B^{reg} = \bigsqcup U_\alpha$.

As explained in §2.1, U_α determines an affine coordinate chart $U_\alpha^\vee$ for the SYZ mirror of X^0, with coordinates of the form (2.3).

Specifically, fix a reference point $b^0 \in U_\alpha$, and observe that, since $L^0 = \pi^{-1}(b^0)$ is the lift of an orbit of the T^{n+1}-action on $V \times \mathbb{C}$, its first homology carries a preferred basis $(\gamma_1, \dots, \gamma_n, \gamma_0)$ consisting of orbits of the various S^1 factors. Consider $b \in U_\alpha$, with coordinates $(\zeta_1, \dots, \zeta_n, \lambda)$ (here we identify $U_\alpha \subset B^{reg}$ with a subset of the moment polytope $\Delta_V \times \mathbb{R}_+ \subset \mathbb{R}^{n+1}$ for the T^{n+1}-action on $V \times \mathbb{C}$), and denote by $(\zeta_1^0, \dots, \zeta_n^0, \lambda^0)$ the coordinates of b^0. Then the valuations of the coordinates given by (2.3), i.e., the areas of the cylinders $\Gamma_1, \dots, \Gamma_n, \Gamma_0$ bounded by L^0 and $L = \pi^{-1}(b)$, are $\zeta_1 - \zeta_1^0, \dots, \zeta_n - \zeta_n^0$, and $\lambda - \lambda^0$ respectively. In order to eliminate the dependence on the choice of L^0, we rescale each coordinate by a suitable power of T, and equip $U_\alpha^\vee$ with the coordinate system

$$(L,\nabla)\mapsto(v_{\alpha,1},\dots,v_{\alpha,n},w_{\alpha,0})=\Big(T^{\zeta_1}\nabla(\gamma_1),\,\dots,\,T^{\zeta_n}\nabla(\gamma_n),\,T^{\lambda}\nabla(\gamma_0)\Big).\tag{5.1}$$

(Compare with (2.3), noting that $\zeta_i=\zeta_i^0+\int_{\Gamma_i}\omega_\epsilon$ and $\lambda=\lambda^0+\int_{\Gamma_0}\omega_\epsilon$.)

As in §3.3, we set $\mathbf{v}_\alpha=(v_{\alpha,1},\dots,v_{\alpha,n})$, and for $m\in\mathbb{Z}^n$ we write $\mathbf{v}_\alpha^m=v_{\alpha,1}^{m_1}\dots v_{\alpha,n}^{m_n}$. Moreover, we write w_0 for $w_{\alpha,0}$; this is *a priori* ambiguous, but we shall see shortly that the gluings between the charts preserve the last coordinate.

The "naive" gluings between these coordinate charts (i.e., those which describe the geometry of the space of (L,∇) up to Hamiltonian isotopy without accounting for instanton corrections) are governed by the global affine structure of $B\setminus B^{sing}$. Their description is instructive, even though it is not necessary for our argument.

For $\lambda>\epsilon$ the affine structure is globally that of $\Delta_V\times(\epsilon,\infty)$. Therefore, (5.1) makes sense and is consistent with (2.3) even when b does not lie in U_α; thus, for $\lambda>\epsilon$ the naive gluing is the identity map ($\mathbf{v}_\alpha=\mathbf{v}_\beta$, and $w_{\alpha,0}=w_{\beta,0}$).

On the other hand, for $\lambda\in(0,\epsilon)$ we argue as in §4.2 (cf. equation (4.5) and the preceding discussion). When $b=(\zeta_1,\dots,\zeta_n,\lambda)$ lies in a different chamber U_β from that containing the reference point b^0 (i.e., U_α), the intersection number of a cylinder Γ_i' constructed as previously with the exceptional divisor E is equal to $\beta_i-\alpha_i$, and its symplectic area differs from $\zeta_i-\zeta_i^0$ by $(\beta_i-\alpha_i)(\epsilon-\lambda)$. Moreover, due to the monodromy of the fibration, the bases of first homology used in U_α and U_β differ by $\gamma_i\mapsto\gamma_i+(\beta_i-\alpha_i)\gamma_0$ for $i=1,\dots,n$. Thus, for $\lambda<\epsilon$ the naive gluing between the charts $U_\alpha^\vee$ and $U_\beta^\vee$ corresponds to setting

$$v_{\alpha,i}=T^{-(\beta_i-\alpha_i)(\epsilon-\lambda)}\nabla(\gamma_0)^{\beta_i-\alpha_i}v_{\beta,i}=(T^{-\epsilon}w_0)^{\beta_i-\alpha_i}v_{\beta,i},\qquad 1\le i\le n.$$

The naive gluing formulas for the two cases ($\lambda>\epsilon$ and $\lambda<\epsilon$) are inconsistent. As seen in §2.1, this is not unexpected: the actual gluing between the coordinate charts $\{U_\alpha^\vee\}_{\alpha\in A}$ differs from these formulas by instanton corrections which account for the bubbling of holomorphic discs as L is isotoped across a wall of potentially obstructed fibers.

Given a potentially obstructed fiber $L\subset\mu_X^{-1}(\lambda)$, the simple holomorphic discs bounded by L are classified by Proposition 5.1. For $\lambda>\epsilon$, the symplectic area of these discs is $\lambda-\epsilon$, and their boundary loop represents the class $\gamma_0\in H_1(L)$ (the orbit of the S^1-action), so the corresponding weight is $T^{\lambda-\epsilon}\nabla(\gamma_0)$ $(=T^{-\epsilon}w_0)$; while for $\lambda<\epsilon$ the symplectic area is $\epsilon-\lambda$ and the boundary loop represents $-\gamma_0$, so the weight is $T^{\epsilon-\lambda}\nabla(\gamma_0)^{-1}$ $(=T^{\epsilon}w_0^{-1})$. As explained in §2.1, we therefore expect the instanton corrections to the gluings to be given by power series in $(T^{-\epsilon}w_0)^{\pm1}$.

While the direct calculation of the multiple cover contributions to the instanton corrections would require sophisticated machinery, Remark 2.3 provides a way to do so by purely elementary techniques. Namely, we study the manner in which counts of Maslov index 2 discs in partial compactifications of X^0 vary between chambers. The reader is referred to Example 3.1.2 of [14] for a simple motivating example (corresponding to the case where $H=\{\text{point}\}$ in $V=\mathbb{C}$).

Recall that a point of $U_\alpha^\vee$ corresponds to a pair (L,∇), where $L=\pi^{-1}(b)$ is the fiber of π above some point $b\in U_\alpha$, and ∇ is a unitary rank 1 local system on L. Given a partial compactification X' of X^0 (satisfying Assumption 2.2), (L,∇) is a weakly unobstructed object of $\mathcal{F}(X')$, i.e. $\mathfrak{m}_0(L,\nabla)=$

$W_{X'}(L,\nabla)\,e_L$, where $W_{X'}(L,\nabla)$ is a weighted count of Maslov index 2 holomorphic discs bounded by L in X'. Varying (L,∇), these weighted counts define regular functions on each chart $U_\alpha^\vee$, and by Corollary A.11, they glue into a global regular function on the SYZ mirror of X^0.

We first use this idea to verify that the coordinate $w_0 = w_{\alpha,0}$ is preserved by the gluing maps, by interpreting it as a weighted count of discs in the partial compactification X_+^0 of X^0 obtained by adding the open stratum $\tilde{V}^0$ of the divisor $\tilde{V}$.

Lemma 5.5 *Let $X_+^0 = p^{-1}(V^0\times\mathbb{C}) = X^0\cup\tilde{V}^0 \subset X$. Then any point (L,∇) of $U_\alpha^\vee$ defines a weakly unobstructed object of $\mathcal{F}(X_+^0)$, with*

$$W_{X_+^0}(L,\nabla) = w_{\alpha,0}. \tag{5.2}$$

Proof. Let $u:(D^2,\partial D^2)\to(X_+^0,L)$ be a holomorphic disc in X_+^0 with boundary on L whose Maslov index is 2. The image of u by the projection p is a holomorphic disc in $V^0\times\mathbb{C}\simeq(\mathbb{C}^*)^n\times\mathbb{C}$ with boundary on the product torus $p(L) = S^1(r_1)\times\cdots\times S^1(r_0)$. Thus, the first n components of $p\circ u$ are constant by the maximum principle, and we can write $p\circ u(z) = (x_1,\dots,x_n,r_0\gamma(z))$, where $|x_1| = r_1,\dots,|x_n| = r_n$, and $\gamma: D^2\to\mathbb{C}$ maps the unit circle to itself. Moreover, the Maslov index of u is twice its intersection number with $\tilde{V}$. Therefore γ is a degree 1 map of the unit disc to itself, i.e. a biholomorphism; so the choice of $(x_1,\dots,x_n)$ determines u uniquely up to reparametrization.

We conclude that each point of L lies on the boundary of a unique Maslov index 2 holomorphic disc in X_+^0, namely the preimage by p of a disc $\{\mathbf{x}\}\times D^2(r_0)$. These discs are easily seen to be regular, by reduction to the toric case [56]; their symplectic area is λ (by definition of the moment map μ_X, see the beginning of §4.1), and their boundary represents the homology class $\gamma_0\in H_1(L)$ (the orbit of the S^1-action on X). Thus, their weight is $T^{\omega(u)}\nabla(\partial u) = T^\lambda\nabla(\gamma_0) = w_{\alpha,0}$, which completes the proof. $\square$

Lemma 5.5 implies that the local coordinates $w_{\alpha,0}\in\mathcal{O}(U_\alpha^\vee)$ glue to a globally defined regular function w_0 on the mirror of X^0 (hence we drop α from the notation).

Next, we consider monomials in the remaining coordinates $\mathbf{v}_\alpha$. First, let $\sigma\in\mathbb{Z}^n$ be a primitive generator of a ray of the fan Σ_V, and denote by D_σ^0 the open stratum of the corresponding toric divisor in V. We will presently see that the monomial $\mathbf{v}_\alpha^\sigma$ is related to a weighted count of discs in the partial compactification X'_σ of X^0 obtained by adding $p^{-1}(D_\sigma^0\times\mathbb{C})$:

$$X'_\sigma = p^{-1}((V^0\cup D_\sigma^0)\times\mathbb{C})\setminus\tilde{V}\subset X. \tag{5.3}$$

Let $\varpi\in\mathbb{R}$ be the constant such that the corresponding facet of Δ_V has equation $\langle\sigma,u\rangle+\varpi = 0$, and let $\alpha_{min}\in A$ be such that $\langle\sigma,\alpha_{min}\rangle$ is minimal.

Lemma 5.6 *Any point (L,∇) of $U_\alpha^\vee$ $(\alpha\in A)$ defines a weakly unobstructed object of $\mathcal{F}(X'_\sigma)$, with*

$$W_{X'_\sigma}(L,\nabla) = (1+T^{-\epsilon}w_0)^{\langle\alpha-\alpha_{min},\sigma\rangle}T^{\varpi}\mathbf{v}_\alpha^\sigma. \tag{5.4}$$

Proof. After performing dual monomial changes of coordinates on V^0 and on $U_\alpha^\vee$ (i.e., replacing the coordinates $(x_1,\dots,x_n)$ by $(\mathbf{x}^{\tau_1},\dots,\mathbf{x}^{\tau_n})$ where $\langle\sigma,\tau_i\rangle = \delta_{i,1}$, and $(v_{\alpha,1},\dots,v_{\alpha,n})$ by $(\mathbf{v}_\alpha^\sigma,\dots)$), we can reduce to the case where $\sigma = (1,0,\dots,0)$, and $V^0\cup D_\sigma^0\simeq\mathbb{C}\times(\mathbb{C}^*)^{n-1}$.

With this understood, let $u:(D^2,\partial D^2)\to(X'_\sigma,L)$ be a Maslov index 2 holomorphic disc with boundary on L. The composition of u with the projection p is a holomorphic disc in $(V^0\cup D_\sigma^0)\times\mathbb{C}\simeq$

$\mathbb{C} \times (\mathbb{C}^*)^{n-1} \times \mathbb{C}$ with boundary on the product torus $p(L) = S^1(r_1) \times \cdots \times S^1(r_0)$. Thus, all the components of $p \circ u$ except for the first and last ones are constant by the maximum principle. Moreover, since the Maslov index of u is twice its intersection number with D^0_σ, the first component of $p \circ u$ has a single zero, i.e. it is a biholomorphism from D^2 to the disc of radius r_1. Therefore, up to reparametrization we have $p \circ u(z) = (r_1 z, x_2, \ldots, x_n, r_0\gamma(z))$, where $|x_2| = r_2, \ldots, |x_n| = r_n$, and $\gamma : D^2 \to \mathbb{C}$ maps the unit circle to itself.

A further constraint is given by the requirement that the image of u be disjoint from $\tilde{V}$ (the proper transform of $V \times 0$). Thus, the last component $\gamma(z)$ is allowed to vanish only when $(r_1 z, x_2, \ldots, x_n) \in H$, and its vanishing order at such points is constrained as well. We claim that the intersection number k of the disc $\mathbb{D} = D^2(r_1) \times \{(x_2, \ldots, x_n)\}$ with H is equal to $\langle \alpha - \alpha_{min}, \sigma \rangle$. Indeed, with respect to the chosen trivialization of $\mathcal{O}(H)$ over V^0, near $p_V(L)$ the dominating term in the defining section of H is the monomial $\mathbf{x}^\alpha$, whose values over the circle $S^1(r_1) \times \{(x_2, \ldots, x_n)\}$ wind $\alpha_1 = \langle \alpha, \sigma \rangle$ times around the origin; whereas near D^0_σ (i.e., in the chambers which are unbounded in the direction of $-\sigma$) the dominating terms have winding number $\langle \alpha_{min}, \sigma \rangle$. Comparing these winding numbers we obtain that $k = \langle \alpha - \alpha_{min}, \sigma \rangle$.

Assume first that $(x_2, \ldots, x_n)$ are generic, in the sense that $\mathbb{D}$ intersects H transversely at k distinct points $(r_1 a_i, x_2, \ldots, x_n)$, $i = 1, \ldots, k$ (with $a_i \in D^2$). The condition that u avoids $\tilde{V}$ implies that γ is allowed to have at most simple zeroes at $a_1, \ldots, a_k$. Denote by $I \subseteq \{1, \ldots, k\}$ the set of those a_i at which γ does have a zero, and let

$$\gamma_I(z) = \prod_{i \in I} \frac{z - a_i}{1 - \bar{a}_i z}.$$

Then γ_I maps the unit circle to itself, and its zeroes in the disc are the same as those of γ, so that $\gamma_I^{-1}\gamma$ is a holomorphic function on the unit disc, without zeroes, and mapping the unit circle to itself, i.e. a constant map. Thus $\gamma(z) = e^{i\theta}\gamma_I(z)$, and

$$p \circ u(z) = (r_1 z, x_2, \ldots, x_n, r_0 e^{i\theta}\gamma_I(z)) \tag{5.5}$$

for some $I \subseteq \{1, \ldots, k\}$ and $e^{i\theta} \in S^1$. We conclude that there are 2^k holomorphic discs of Maslov index 2 in (X'_σ, L) whose boundary passes through a given generic point of L. It is not hard to check that these discs are all regular, using e.g. the same argument as in the proof of Lemma 7 in [15]. Succinctly: observing that u does not intersect $\tilde{H}$, projection to V decomposes (via a short exact sequence) the Cauchy-Riemann operator for u into a $\bar{\partial}$ operator on the trivial holomorphic line bundle with trivial real boundary condition (along the fibers of the projection), and the $\bar{\partial}$ operator for the "standard" disc $\mathbb{D}$ in $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$ (which itself splits into a direct sum of line bundles and is easily checked to be surjective); this implies surjectivity.

When the disc $\mathbb{D}$ is not transverse to H, we can argue in exactly the same manner, except that $a_1, \ldots, a_k \in D^2$ are no longer distinct; and γ may have a multiple zero at a_i as long as its order of vanishing does not exceed the multiplicity of $(r_1 a_i, x_2, \ldots, x_n)$ as an intersection of $\mathbb{D}$ with H. We still conclude that $p \circ u$ is of the form (5.5). These discs are not all distinct (or regular), but we can argue by continuity as follows. There are diffeomorphisms arbitrarily C^∞-close to identity which fix a neighborhood of H and map $S^1(r_1) \times \{(x_2, \ldots, x_n)\}$ to a nearby circle $S^1(r'_1) \times \{(x'_2, \ldots, x'_n)\}$ contained in a generic fiber. The moduli space of holomorphic discs with respect to the pullback of the standard complex structure by such a diffeomorphism is canonically identified with the moduli space of holomorphic discs for the standard complex structure with boundary on the nearby generic fiber.

This provides an explicit regularization of the moduli space, and we conclude that the enumeration of holomorphic discs is as in the transverse case (i.e., discs which can be written in the form (5.5) in more than one way should be counted with a multiplicity equal to the number of such expressions.)

All that remains is to calculate the weights (2.2) associated to the holomorphic discs we have identified. Denote by $(\zeta_1, \dots, \zeta_n, \lambda)$ the affine coordinates of $\pi(L) \in U_\alpha$ introduced above, and consider a disc given by (5.5) with $|I| = \ell \in \{0, \dots, k\}$. Then the relative homology class represented by $p \circ u(D^2)$ in $\mathbb{C} \times (\mathbb{C}^*)^{n-1} \times \mathbb{C} \subset V \times \mathbb{C}$ is equal to $[D^2(r_1) \times \{pt\}] + \ell[\{pt\} \times D^2(r_0)]$. By elementary toric geometry, the symplectic area of the disc $D^2(r_1) \times \{pt\}$ with respect to the toric Kähler form $\omega_{V\times\mathbb{C}}$ is equal to $\langle \sigma, \mu_V\rangle + \varpi = \zeta_1 + \varpi$, while that of $\{pt\} \times D^2(r_0)$ is equal to λ. Thus, the symplectic area of the disc $p \circ u(D^2)$ with respect to $\omega_{V\times\mathbb{C}}$ is $\zeta_1 + \varpi + \ell\lambda$. The disc we are interested in, $u(D^2) \subset X'_\sigma$, is the proper transform of $p \circ u(D^2)$ under the blowup map; since its intersection number with the exceptional divisor E is equal to $|I| = \ell$, we conclude that

$$\int_{D^2} u^*\omega_\epsilon = \left(\int_{D^2} (p \circ u)^*\omega_{V\times\mathbb{C}}\right) - \ell\epsilon = \zeta_1 + \varpi + \ell(\lambda - \epsilon). \tag{5.6}$$

On the other hand, the degree of $\gamma_{I|S^1} : S^1 \to S^1$ is equal to $|I| = \ell$, so in $H_1(L, \mathbb{Z})$ we have $[u(S^1)] = \gamma_1 + \ell\gamma_0$. Thus the weight of u is

$$T^{\omega_\epsilon(u)}\nabla(\partial u) = T^{\zeta_1+\varpi+\ell(\lambda-\epsilon)}\nabla(\gamma_1)\nabla(\gamma_0)^\ell = (T^{-\epsilon}w_0)^\ell T^\varpi v_{\alpha,1}.$$

Summing over the $\binom{k}{\ell}$ families of discs with $|I| = \ell$ for each $\ell = 0, \dots, k$, we find that

$$W_{X'_\sigma}(L, \nabla) = \sum_{\ell=0}^{k} \tbinom{k}{\ell}(T^{-\epsilon}w_0)^\ell\, T^\varpi v_{\alpha,1} = (1 + T^{-\epsilon}w_0)^k T^\varpi v_{\alpha,1}.$$

□

Next we extend Lemma 5.6 to the case of general monomials in the coordinates $\mathbf{v}_\alpha$. Let σ be *any* primitive element of $\mathbb{Z}^n$, and denote again by α_{min} an element of A such that $\langle \alpha_{min}, \sigma\rangle$ is minimal. Denote by $V'_\sigma = V^0 \cup D^0_\sigma$ the toric partial compactification of V^0 obtained by adding a single toric divisor D^0_σ in the direction of the ray $-\sigma$. The hypersurface H^0 admits a natural partial compactification H'_σ inside V'_σ.

We claim that H'_σ is smooth for τ sufficiently small in (3.1). Indeed, rescaling f_τ by a factor of $\mathbf{x}^{-\alpha_{min}}$ if necessary, we can assume without loss of generality that $\langle \alpha_{min}, \sigma\rangle = 0$. Then f_τ extends to a regular function on V'_σ, whose restriction to D^0_σ is again a maximally degenerating family of Laurent polynomials, associated to the regular polyhedral decomposition $\mathcal{P} \cap \sigma^\perp$ of the convex hull of $A \cap \sigma^\perp$. This implies that for sufficiently small τ the restriction of f_τ to D^0_σ vanishes transversely; the smoothness of H'_σ follows.

By blowing up $V'_\sigma \times \mathbb{C}$ along $H'_\sigma \times 0$ and removing the proper transform of $V'_\sigma \times 0$, we obtain a partial compactification X'_σ of X^0. While X'_σ does not necessarily embed into X, we can equip V'_σ (resp. X'_σ) with a toric (resp. S^1-invariant) Kähler form which agrees with ω_V (resp. ω_ϵ) everywhere outside of an arbitrarily small neighborhood of the compactification divisor.

Denote by $L \subset X^0$ a smooth fiber of π which lies in the region where the Kähler forms agree (so that L is Lagrangian in X'_σ as well).

Lemma 5.7 *The Maslov index 0 holomorphic discs bounded by L inside X'_σ are all contained in X^0 and described by Proposition* 5.1.

Moreover, if L is tautologically unobstructed in X^0 and lies over the chamber U_α, then the points $(L,\nabla) \in U_\alpha^\vee$ define weakly unobstructed objects of $\mathcal{F}(X'_\sigma)$, with

$$W_{X'_\sigma}(L,\nabla) = (1+T^{-\epsilon}w_0)^{\langle \alpha-\alpha_{min},\sigma\rangle} T^{\varpi} \mathbf{v}_\alpha^\sigma \tag{5.7}$$

for some $\varpi \in \mathbb{R}$.

Proof. The Maslov index of a disc in X'_σ with boundary on L is twice its intersection number with the compactification divisor, and Assumption 2.2 is satisfied (in fact X'_σ is affine). Thus all Maslov index 0 holomorphic discs are contained in the open stratum X^0, and Proposition 5.1 holds. (Since L lies away from the compactification divisor, the symplectic area of these discs remains the same as for ω_ϵ.)

Thus, whenever L lies over a chamber U_α it does not bound any holomorphic discs of Maslov index zero or less in X'_σ, and the Maslov index 2 discs can be classified exactly as in the proof of Lemma 5.6. The only difference is that, since we evaluate the symplectic areas of these discs with respect to the Kähler form on X'_σ rather than X, the constant term ϖ in the area formula (5.6) now depends on the choice of the toric Kähler form on V'_σ near the compactification divisor. $\square$

By Remark 2.3 (see also Corollary A.11), the expressions (5.7) determine globally defined regular functions on the mirror of X^0. Thus, we can use Lemma 5.7 to determine the wall-crossing transformations between the affine charts of the mirror.

Consider two adjacent chambers U_α and U_β separated by a wall of potentially obstructed fibers of π, i.e. assume that $\alpha, \beta \in A$ are connected by an edge in the polyhedral decomposition $\mathcal{P}$. Then we have:

Proposition 5.8 *The instanton-corrected gluing between the coordinate charts $U_\alpha^\vee$ and $U_\beta^\vee$ preserves the coordinate w_0, and matches the remaining coordinates via*

$$\mathbf{v}_\alpha^\sigma = (1+T^{-\epsilon}w_0)^{\langle \beta-\alpha,\sigma\rangle}\mathbf{v}_\beta^\sigma \qquad \textit{for all } \sigma \in \mathbb{Z}^n. \tag{5.8}$$

Proof. Let $\{L_t\}_{t\in[0,1]}$ be a path among smooth fibers of π, with L_0 and L_1 tautologically unobstructed and lying over the chambers U_α and U_β respectively. We consider the partial compactifications X^0_+ and X'_σ of X^0 introduced in Lemmas 5.5–5.7; in the case of X'_σ we choose the Kähler form to agree with ω_ϵ over a large open subset which contains the path L_t, so as to be able to apply Lemma 5.7.

Since these partial compactifications satisfy Assumption 2.2, the moduli spaces of Maslov index 0 holomorphic discs bounded by the Lagrangians L_t in X^0_+, X'_σ, and X^0 are the same, and the corresponding wall-crossing transformations are identical (see Appendix A). Noting that the expressions (5.2) and (5.7) are manifestly convergent over the whole completions $(\mathbb{K}^*)^{n+1}$ of $U_\alpha^\vee$ and $U_\beta^\vee$, we appeal to Lemma A.10, and conclude that these expressions for the superpotentials $W_{X^0_+}$ and $W_{X'_\sigma}$ over the chambers $U_\alpha^\vee$ and $U_\beta^\vee$ match under the wall-crossing transformation. Thus w_0 is preserved, and for primitive $\sigma \in \mathbb{Z}^n$ the monomials $\mathbf{v}_\alpha^\sigma$ and $\mathbf{v}_\beta^\sigma$ are related by (5.8). (The case of non-primitive σ follows obviously from the primitive case.) $\square$

This completes the proof of Theorem 1.7. Indeed, the instanton-corrected gluing maps (5.8) coincide with the coordinate change formulas (3.11) between the affine charts for the toric variety Y introduced in §3.3. Therefore, the SYZ mirror of X^0 embeds inside Y, by identifying the completion of the local chart $U_\alpha^\vee$ with the subset of Y_α where w_0 is non-zero. It follows that the SYZ mirror of X^0 is the subset of Y where w_0 is non-zero, namely Y^0.

6 Proof of Theorem 1.5

We now turn to the proof of Theorem 1.5. We begin with an elementary observation:

Lemma 6.1 *If Assumption* 1.4 *holds, then every rational curve $C \simeq \mathbb{P}^1$ in X satisfies $D \cdot C = c_1(X) \cdot C > 0$; so in particular Assumption* 2.2 *holds.*

Proof. $c_1(X) = p_V^* c_1(V) - [E]$, where p_V is the projection to V and $E = p^{-1}(H \times 0)$ is the exceptional divisor. Consider a rational curve C in X (i.e., the image of a nonconstant holomorphic map from $\mathbb{P}^1$ to X), and denote by $C' = p_V(C)$ the rational curve in V obtained by projecting C to V. Applying the maximum principle to the projection to the last coordinate $y \in \mathbb{C}$, we conclude that C is contained either in $p^{-1}(V \times 0) = \tilde{V} \cup E$, or in $p^{-1}(V \times \{y\})$ for some nonzero value of y.

When $C \subset p^{-1}(V \times \{y\})$ for $y \neq 0$, the curve C is disjoint from E and its projection C' is nonconstant, so $c_1(X) \cdot [C] = c_1(V) \cdot [C'] > 0$ by Assumption 1.4.

When C is contained in $\tilde{V}$, the curve C' is again nonconstant, and since the normal bundle of $\tilde{V}$ in X is $\mathcal{O}(-H)$, we have $c_1(X) \cdot [C] = c_1(V) \cdot [C'] - [H] \cdot [C']$, which is positive by Assumption 1.4.

Finally, we consider the case where C is contained in E but not in $\tilde{V}$. Then

$$c_1(X) \cdot [C] = [D] \cdot [C] = [\tilde{V}] \cdot [C] + [p^{-1}(D_V)] \cdot [C] = [\tilde{V}] \cdot [C] + c_1(V) \cdot [C'].$$

The first term is non-negative by positivity of intersection; and by Assumption 1.4 the second one is positive unless C' is a constant curve, and non-negative in any case. However C' is constant only when C is (a cover of) a fiber of the $\mathbb{P}^1$-bundle $p_{|E} : E \to H \times 0$; in that case $[\tilde{V}] \cdot [C] > 0$, so $c_1(X) \cdot [C] > 0$ in all cases. $\square$

As explained in §2.2, this implies that the tautologically unobstructed fibers of $\pi : X^0 \to B$ remain weakly unobstructed in X, and that the SYZ mirror of X is just Y^0 (the SYZ mirror of X^0) equipped with a superpotential W_0 which counts Maslov index 2 holomorphic discs bounded by the fibers of π. Indeed, the conclusion of Lemma 6.1 implies that any component which is a sphere contributes at least 2 to the Maslov index of a stable genus 0 holomorphic curve bounded by a fiber of π. Thus, Maslov index 0 configurations are just discs contained in X^0, and Maslov index 2 configurations are discs intersecting D transversely in a single point.

Observe that each Maslov index 2 holomorphic disc intersects exactly one of the components of the divisor D. Thus, the superpotential W_0 can be expressed as a sum over the components of $D = \tilde{V} \cup p^{-1}(D_V \times \mathbb{C})$, in which each term counts those discs which intersect a particular component. It turns out that the necessary calculations have been carried out in the preceding section: Lemma 5.5 describes the contribution from discs which only hit $\tilde{V}$, and Lemma 5.6 describes the contributions from discs which hit the various components of $p^{-1}(D_V \times \mathbb{C})$. Summing these, and using the notations of §3.3, we obtain that, for any point (L, ∇) of $U_\alpha^\vee$ ($\alpha \in A$),

$$W_0(L, \nabla) = w_{\alpha,0} + \sum_{i=1}^{r} (1 + T^{-\epsilon} w_0)^{\langle \alpha - \alpha_i, \sigma_i \rangle} T^{\varpi_i} \mathbf{v}_\alpha^{\sigma_i} = w_0 + \sum_{i=1}^{r} w_i.$$

Hence W_0 is precisely the leading-order superpotential (3.14). This completes the proof of Theorem 1.5.

Remark 6.2 In the proofs of Lemmas 5.5 and 5.6 we have not discussed in any detail the orientations of moduli spaces of discs, which determine the signs of the various terms appearing in the superpotential. The fact that those are all positive follows from two ingredients.

The first is that, for a standard product torus in a toric variety, equipped with the standard spin structure, the contributions of the various families of Maslov index 2 holomorphic discs to the superpotential are all positive. See [54] for a detailed calculation in the case of the Clifford torus. The fact that all the signs are the same is not surprising, since a monomial change of variables can be used to reduce to a single example, namely the family of discs $D^2 \times \{pt\}$ bounded by a product torus in $\mathbb{C} \times (\mathbb{C}^*)^n$ equipped with the standard spin structure. The same argument also applies to the discs in Lemma 5.5 since those can also be reduced to the toric case.

The second ingredient is a comparison of the orientations of moduli spaces of discs in V and their lifts to X (as in Lemma 5.6). A short calculation shows that, for the standard spin structure, the orientation of the moduli space of lifted discs in X agrees with that induced by the orientation of the moduli space of discs in V and the natural orientation of the orbits of the S^1-action. See the proof of Corollary 8 in [15] for a similar argument. The positivity of the signs in Lemma 5.6 follows.

Remark 6.3 When Assumption 1.4 does not hold, the SYZ mirror of X differs from (Y^0, W_0), since the enumerative geometry of discs is modified by the presence of stable genus 0 configurations of total Maslov index 0 or 2. A borderline case that remains fairly easy is when the strict inequality in Assumption 1.4 is relaxed to

$$c_1(V) \cdot C \geq \max(0, H \cdot C).$$

(This includes the situation where H is a Calabi-Yau hypersurface in a toric Fano variety as an important special case.)

In this case, Assumption 2.2 still holds, so the mirror of X remains Y^0; the only modification is that the superpotential should also count the contributions of configurations consisting of a Maslov index 2 disc together with one or more rational curves satisfying $c_1(X) \cdot C = 0$. Thus, we now have

$$W = (1 + c_0)w_0 + (1 + c_1)w_1 + \cdots + (1 + c_r)w_r,$$

where $c_0, \ldots, c_r \in \Lambda$ are constants (determined by the genus 0 Gromov-Witten theory of X), with $\mathrm{val}(c_i) > 0$.

7 From the blowup X to the hypersurface H

The goal of this section is to prove Theorem 1.6. As a first step, we establish:

Theorem 7.1 *Under Assumption* 1.4*, the B-side Landau-Ginzburg model* (Y, W_0) *is SYZ mirror to the A-side Landau-Ginzburg model* $(X, W^\vee = y)$ *(with the Kähler form ω_ϵ).*

(Recall that y is the coordinate on the second factor of $V \times \mathbb{C}$.)

Proof. [Sketch of proof] This result follows from Theorem 1.5 by the same considerations as in Example 2.4. Specifically, equipping X with the superpotential $W^\vee = y$ enlarges its Fukaya category by adding admissible non-compact Lagrangian submanifolds, i.e., properly embedded Lagrangian submanifolds of X whose image under $W^\vee$ is only allowed to tend to infinity in the direction of the positive real axis; in other terms, the y coordinate is allowed to be unbounded, but only in the positive real direction.

Let $a_0 \subset \mathbb{C}$ be a properly embedded arc which connects $+\infty$ to itself by passing around the origin, encloses an infinite amount of area, and stays away from the projection to $\mathbb{C}$ of the support of the cut-off function χ used to construct ω_ϵ. Then we can supplement the family of Lagrangian tori in X^0 constructed in §4 by considering product Lagrangians of the form $L = p^{-1}(L' \times a_0)$, where L' is an orbit of the T^n-action on V. Indeed, by Proposition 4.5, away from the exceptional divisor the fibers of $\pi : X^0 \to B$ are lifts to X of product tori $L' \times S^1(r) \subset V \times \mathbb{C}$. For large enough r, the circles $S^1(r)$ can be deformed by Hamiltonian isotopies in $\mathbb{C}$ to simple closed curves that approximate a_0 as $r \to \infty$; moreover, the induced isotopies preserve the tautological unobstructedness in X^0 of the fibers of π which do not intersect $p^{-1}(H \times \mathbb{C})$. In this sense, $p^{-1}(L' \times a_0)$ is naturally a limit of the tori $p^{-1}(L' \times S^1(r))$ as $r \to \infty$. The analytic structure near this point is obtained by equation (2.3), which naturally extends as in Example 2.4.

To be more specific, let $L' = \mu_V^{-1}(\zeta_1, \dots, \zeta_n)$ for $(\zeta_1, \dots, \zeta_n)$ a point in the component of $\Delta_V \setminus \mu_V(U_H)$ corresponding to the weight $\alpha \in A$, and equip $L = p^{-1}(L' \times a_0)$ with a local system $\nabla \in \mathcal{H}om(\pi_1(L), U_{\mathbb{K}})$. The maximum principle implies that any holomorphic disc bounded by L in X^0 must be contained inside a fiber of the projection to V (see the proof of Proposition 5.1). Thus L is tautologically unobstructed in X^0, and (L, ∇) defines an object of the Fukaya category $\mathcal{F}(X^0, W^\vee)$, and a point in some partial compactification of the coordinate chart $U_\alpha^\vee$ considered in §5. Denoting by $\gamma_1, \dots, \gamma_n$ the standard basis of $H_1(L) \simeq H_1(L')$ given by the various S^1 factors, in the coordinate chart (5.1) the object (L, ∇) corresponds to

$$(v_{\alpha,1}, \dots, v_{\alpha,n}, w_{\alpha,0}) = \Big(T^{\zeta_1}\nabla(\gamma_1), \dots, T^{\zeta_n}\nabla(\gamma_n), 0\Big).$$

Thus, equipping X^0 with the superpotential $W^\vee$ extends the moduli space of objects under consideration from $Y^0 = Y \setminus w_0^{-1}(0)$ to Y.

Under Assumption 1.4, (L, ∇) remains a weakly unobstructed object of the Fukaya category $\mathcal{F}(X, W^\vee)$. We now study the families of Maslov index 2 holomorphic discs bounded by L in X, in order to determine the corresponding value of the superpotential and show that it agrees with (3.14). Under projection to the y coordinate, any holomorphic disc $u : (D^2, \partial D^2) \to (X, L)$ maps to a holomorphic disc in $\mathbb{C}$ with boundary on the arc a_0, which is necessarily constant; hence the image of u is contained inside $p^{-1}(V \times \{y\})$ for some $y \in a_0$. Moreover, inside the toric variety $p^{-1}(V \times \{y\}) \simeq V$ the holomorphic disc u has boundary on the product torus L'.

Thus, the holomorphic discs bounded by L in X can be determined by reduction to the toric case of (V, L'). For each toric divisor of V there is a family of Maslov index 2 discs which intersect it transversely at a single point and are disjoint from all the other toric divisors; these discs are all regular, and exactly one of them passes through each point of L [56]. The discs which intersect the toric divisor corresponding to a facet of Δ_V with equation $\langle \sigma, \cdot \rangle + \varpi = 0$ have area $\langle \sigma, \zeta \rangle + \varpi$ and weight $T^\varpi \mathbf{v}_\alpha^\sigma$. Summing over all facets of Δ_V, we conclude that

$$W_0(L, \nabla) = \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_\alpha^{\sigma_i}. \tag{7.1}$$

Moreover, because $w_0 = 0$ at the point (L, ∇), the coordinate transformations (3.11) simplify to $\mathbf{v}_{\alpha_i}^{\sigma_i} = \mathbf{v}_{\alpha}^{\sigma_i}$. Thus the expression (7.1) agrees with (3.14). $\square$

Remark 7.2 In order to fill the details of this sketch, we would need a sufficient development of Fukaya categories of A-side Landau-Ginzburg models in order to verify the existence of the analytic charts at infnity. The most straightforward way to do this is to introduce non-compact Lagrangians which are mirror to the powers of an ample line bundle on Y, and check that (i) these Lagrangians generate the Fukaya category and (ii) when r is sufficiently large, the product Lagrangian $L' \times S^1(r) \subset V \times \mathbb{C}$ defines a module over the Floer cochains of this generating family which is equivalent to the one associated to the product of L' with an admissible arc in $\mathbb{C}$ equipped with a bounding cochain which is a multiple of a degree 1 generator coming from a self-intersection at infinity.

Our next observation is that $W^\vee : X \to \mathbb{C}$ has a particularly simple structure. The following statement is a direct consequence of the construction:

Proposition 7.3 *$W^\vee = y : X \to \mathbb{C}$ is a Morse-Bott fibration, with 0 as its only critical value; in fact the singular fiber ${W^\vee}^{-1}(0) = \tilde{V} \cup E \subset X$ has normal crossing singularities along $\mathrm{crit}(W^\vee) = \tilde{V} \cap E \simeq H$.*

Remark 7.4 However, the Kähler form on $\mathrm{crit}(W^\vee) \simeq H$ is not that induced by ω_V, but rather that induced by the restriction of ω_ϵ, which represents the cohomology class $[\omega_V] - \epsilon[H]$. To compensate for this, in the proof of Theorem 1.6 we will actually replace $[\omega_V]$ by $[\omega_V] + \epsilon[H]$.

Proposition 7.3 allows us to relate the Fukaya category of $(X, W^\vee)$ to that of H, using the ideas developed by Seidel in [217], adapted to the Morse-Bott case (see [251]).

Remark 7.5 Strictly speaking, the literature does not include any definition of the Fukaya category of a superpotential without assuming that it is a Lefschetz fibration. The difficulty resides not in defining the morphisms and the compositions, but in defining the higher order products in a coherent way. These technical problems were resolved by Seidel in [219], by introducing a method of defining Fukaya categories of Lefschetz fibration that generalizes in a straightforward way to the Morse-Bott case we are considering. This construction will be revisited in [5]. As the reader will see, in the only example where we shall study such a Fukaya category, the precise nature of the construction of higher products will not enter.

Outside of its critical locus, the Morse-Bott fibration $W^\vee$ carries a natural horizontal distribution given by the ω_ϵ-orthogonal to the fiber. Parallel transport with respect to this distribution induces symplectomorphisms between the smooth fibers; in fact, parallel transport along the real direction is given by (a rescaling of) the Hamiltonian flow generated by $\mathrm{Im}\, W^\vee$, or equivalently, the gradient flow of $\mathrm{Re}\, W^\vee$ (for the Kähler metric).

Given a Lagrangian submanifold $\ell \subset \mathrm{crit}(W^\vee) \simeq H$, parallel transport by the positive gradient flow of $\mathrm{Re}\, W^\vee$ yields an admissible Lagrangian *thimble* $L_\ell \subset X$ (topologically a disc bundle over ℓ). Moreover, any local system ∇ on ℓ induces by pullback a local system $\tilde{\nabla}$ on L_ℓ. However, there is a subtlety related to the nontriviality of the normal bundle to H inside X:

Lemma 7.6 *The thimble L_ℓ is naturally diffeomorphic to the restriction of the complex line bundle $\mathcal{L} = \mathcal{O}(H)$ to $\ell \subset H$.*

Proof. First note that, for the Lefschetz fibration $f(x,y) = xy$ on $\mathbb{C}^2$ equipped with its standard Kähler form, the thimble associated to the critical point at the origin is $\{(x,\bar{x}),\ x \in \mathbb{C}\} \subset \mathbb{C}^2$. Indeed, parallel transport preserves the quantity $|x|^2 - |y|^2$, so that the thimble consists of the points (x,y) where $|x| = |y|$ and $xy \in \mathbb{R}_{\geq 0}$, i.e. $y = \bar{x}$. In particular, the thimble projects diffeomorphically onto either of the two $\mathbb{C}$ factors (the two projections induce opposite orientations).

Now we consider the Morse-Bott fibration $W^\vee : X \to \mathbb{C}$. The normal bundle to the critical locus $\operatorname{crit} W^\vee = \tilde{V} \cap E \simeq H$ is isomorphic to $\mathcal{L} \oplus \mathcal{L}^{-1}$ (where $\mathcal{L}$ is the normal bundle to H inside $\tilde{V}$, while $\mathcal{L}^{-1}$ is its normal bundle inside E). Moreover, $W^\vee$ is locally given by the product of the fiber coordinates on the two line subbundles. The local calculation then shows that, by projecting to either subbundle, a neighborhood of ℓ in L_ℓ can be identified diffeomorphically with a neighborhood of the zero section in either $\mathcal{L}_{|\ell}$ or $\mathcal{L}^{-1}_{|\ell}$. □

Lemma 7.6 implies that, even when $\ell \subset H$ is spin, $L_\ell \subset X$ need not be spin; indeed, $w_2(TL_\ell) = w_2(T\ell) + w_2(\mathcal{L}_{|\ell})$. Rather, L_ℓ is *relatively spin*, i.e. its second Stiefel-Whitney class is the restriction of the *background class* $s \in H^2(X, \mathbb{Z}/2)$ Poincaré dual to $[\tilde{V}]$ (or equivalently to $[E]$). Hence, applying the thimble construction to an object of the Fukaya category $\mathcal{F}(H)$ does not determine an object of $\mathcal{F}(X, W^\vee)$, but rather an object of the s-twisted Fukaya category $\mathcal{F}_s(X, W^\vee)$ (we shall verify in Proposition 7.10 that thimbles are indeed weakly unobstructed objects of this category).

Remark 7.7 While it has not appeared in the literature, the notion of weak unobstructedness of an admissible Lagrangian L is a straightforward generalization of the case of closed Lagrangians. There is a Floer-theoretic A_∞-structure on the ordinary cohomology of L, and a natural A_∞-homomorphism from the ordinary cohomology of L equipped with this A_∞-structure to the endomorphisms of L as an object of the Fukaya category of the potential. This homomorphism is not necessarily an isomorphism, but it is always unital and preserves the curvature $\mathfrak{m}_0$. We say that L is *weakly unobstructed* if the curvature is a multiple of the unit in $H^0(L)$. In the case of thimbles, radial parallel transport allows one to lift Maurer-Cartan elements and bounding cochains from an arbitrarily small neighborhood of the critical fiber to the total space. This implies that an admissible thimble which bounds no holomorphic disc of Maslov index less than 2 in a neighborhood of the critical fiber is weakly unobstructed; and the curvature is then the product of the unit with the count of Maslov index 2 discs passing through a generic point near the critical fiber.

Corollary 7.8 *Under Assumption* 1.4*, there is a fully faithful A_∞-functor from the Fukaya category $\mathcal{F}(H)$ to $\mathcal{F}_s(X, W^\vee)$, which at the level of objects maps (ℓ, ∇) to the thimble $(L_\ell, \tilde{\nabla})$.*

Proof. [Sketch of proof] Let ℓ_1, ℓ_2 be two Lagrangian submanifolds of $\operatorname{crit}(W^\vee) \simeq H$, assumed to intersect transversely (otherwise transversality is achieved by Hamiltonian perturbations, which may be needed to achieve regularity of holomorphic discs in any case), and denote by $L_1, L_2 \subset X$ the corresponding thimbles. (For simplicity we drop the local systems from the notations; we also postpone the discussion of relatively spin structures until further below).

Recall that $\mathcal{H}om_{\mathcal{F}_s(X,W^\vee)}(L_1, L_2)$ is defined by perturbing L_1, L_2 to Lagrangians $\tilde{L}_1, \tilde{L}_2$ whose images under $W^\vee$ are half-lines which intersect transversely and such that the first one lies above the second one near infinity; so for example, fixing a small angle $\theta > 0$, we can take $\tilde{L}_1$ (resp. $\tilde{L}_2$) to be the Lagrangian obtained from ℓ_1 (resp. ℓ_2) by the gradient flow of $\mathrm{Re}(e^{-i\theta}\,W^\vee)$ (resp. $\mathrm{Re}(e^{i\theta}\,W^\vee)$). (A more general approach would be to perturb the holomorphic curve equation by a Hamiltonian vector field generated by a suitable rescaling of the real part of $W^\vee$, instead of perturbing the Lagrangian boundary conditions; in our case the two approaches are equivalent.)

We now observe that $\tilde{L}_1$ and $\tilde{L}_2$ intersect transversely, with all intersections lying in the singular fiber $W^{\vee -1}(0)$, and in fact $\tilde{L}_1 \cap \tilde{L}_2 = \ell_1 \cap \ell_2$. Thus, $\mathcal{H}om_{\mathcal{F}(H)}(\ell_1, \ell_2)$ and $\mathcal{H}om_{\mathcal{F}_s(X,W^\vee)}(L_1, L_2)$ are naturally isomorphic. Moreover, the maximum principle applied to the projection $W^\vee$ implies that all holomorphic discs bounded by the (perturbed) thimbles in X are contained in $(W^\vee)^{-1}(0) = \tilde{V} \cup E$ (and hence their boundary lies on $\ell_1 \cup \ell_2 \subset H \subset \tilde{V} \cup E$).

After quotienting by a suitable reference section, we can view the defining section of H as a meromorphic function on $\tilde{V}$, with $f^{-1}(0) = H$. Since $f = 0$ at the boundary, and since a meromorphic function on the disc which vanishes at the boundary is everywhere zero, any holomorphic disc in $\tilde{V}$ with boundary in $\ell_1 \cup \ell_2$ must lie entirely inside $f^{-1}(0) = H$. By the same argument, any holomorphic disc in E with boundary in $\ell_1 \cup \ell_2$ must stay inside H as well. Finally, Lemma 6.1 implies that stable curves with both disc and sphere components cannot contribute to the Floer differential (since each sphere component contributes at least 2 to the total Maslov index).

This implies that the Floer differentials on $\mathcal{H}om_{\mathcal{F}(H)}(\ell_1, \ell_2)$ and $\mathcal{H}om_{\mathcal{F}_s(X,W^\vee)}(L_1, L_2)$ count the same holomorphic discs. The same argument applies to Floer products and higher structure maps.

To complete the proof it only remains to check that the orientations of the relevant moduli spaces of discs agree. Recall that a relatively spin structure on a Lagrangian submanifold L with background class s is the same thing as a stable trivialization of the tangent bundle of L over its 2-skeleton, i.e. a trivialization of $TL_{|L^{(2)}} \oplus E_{|L^{(2)}}$, where E is a vector bundle over the ambient manifold with $w_2(E) = s$; such a stable trivialization in turn determines orientations of the moduli spaces of holomorphic discs with boundary on L (see [81, Chapter 8], noting that the definition of spin structures in terms of stable trivializations goes back to Milnor [181]).

In our case, we are considering discs in H with boundary on Lagrangian submanifolds $\ell_i \subset H$, and the given spin structures on ℓ_i determine orientations of the moduli spaces for the structure maps in $\mathcal{F}(H)$. If we consider the same holomorphic discs in the context of the thimbles $L_i \subset X$, the spin structure of ℓ_i does not induce a spin structure on $TL_i \simeq T\ell_i \oplus \mathcal{L}_{|\ell_i}$ (what would be needed instead is a relatively spin structure on ℓ_i with background class $w_2(\mathcal{L}_{|H})$). On the other hand, the normal bundle to H inside X, namely $\mathcal{L} \oplus \mathcal{L}^{-1}$, is an $SU(2)$-bundle and hence has a canonical isotopy class of trivialization over the 2-skeleton. Thus, the spin structure on ℓ_i induces a trivialization of $TL_i \oplus \mathcal{L}^{-1}$ over the 2-skeleton of L_i, i.e. a relative spin structure on L_i with background class $w_2(\mathcal{L}^{-1}_{|L_i}) = s_{|L_i}$. Furthermore, because $w_2(\mathcal{L} \oplus \mathcal{L}^{-1}) = 0$, stabilizing by this rank 2 bundle does not affect the orientation of the moduli space of discs [81, Proposition 8.1.16]. Hence the structure maps of $\mathcal{F}(H)$ and $\mathcal{F}_s(X, W^\vee)$ involve the same moduli spaces of holomorphic discs, oriented in the same manner, which completes the proof. $\square$

Remark 7.9 The reason the above is only a sketch of proof is that the construction of the two Fukaya categories requires choices of perturbations, and we have not discussed how to arrange for these choices to yield the same answer. A model for such arguments in a related situation is provided by Seidel in [217, Section (14c)].

Implicit in the statement of Corollary 7.8 is the fact that, if (ℓ, ∇) is weakly unobstructed in $\mathcal{F}(H)$, then $(L_\ell, \tilde{\nabla})$ is weakly unobstructed in $\mathcal{F}_s(X, W^\vee)$. In our setting, the values of the superpotentials for objects of $\mathcal{F}(H)$ and their images in $\mathcal{F}_s(X, W^\vee)$ differ by an additive constant δ. This constant is easiest to determine if we assume that V is affine:

Proposition 7.10 *Under the assumption that V is affine, the functor of Corollary* 7.8 *increases the value of the superpotential by $\delta = T^\epsilon$.*

Proof. [Sketch of proof] Consider a weakly unobstructed object (ℓ, ∇) of $\mathcal{F}(H)$ and the corresponding thimble $L_\ell \subset X$. Holomorphic discs bounded by L_ℓ in X are contained in the level sets of $W^\vee = y$ (by the maximum principle). By Remark 7.7, we only need to study the moduli spaces of such discs for small values of y.

For $y > 0$, the intersection L_ℓ^y of L_ℓ with $(W^\vee)^{-1}(y) \simeq V$ is a circle bundle over ℓ, lying in the boundary of a standard symplectic tubular neighborhood of size ϵ of H in $(W^\vee)^{-1}(y)$ equipped with the restriction of ω_ϵ. Indeed, as $y \to 0$, the fibers of $W^\vee$ degenerate to the normal crossing divisor $\tilde{V} \cup E$. Symplectic parallel transport identifies the standard disc bundle $E \setminus (\tilde{V} \cap E) \simeq H \times D^2(\epsilon)$ inside $(W^\vee)^{-1}(0)$ with a standard symplectic neighborhood U^y of H inside $(W^\vee)^{-1}(y)$ for $y > 0$. The boundary of U^y (a trivial S^1-bundle over H) consists of all points in $(W^\vee)^{-1}(y)$ whose parallel transport converges to $\tilde{V} \cap E \simeq H$ as $y \to 0$, and in particular it contains L_ℓ^y.

However, while the restriction of ω_ϵ to $(W^\vee)^{-1}(y) \simeq V$ is cohomologous to ω_V for all $y > 0$ and agrees with it pointwise for y sufficiently large, the actual forms differ near H for small y. Under the identification $(W^\vee)^{-1}(y) \simeq V$, the neighborhoods U^y are small tubular neighborhoods of H, increasing in size along a suitably normalized gradient flow of $|f|$ as y increases, and agreeing with a standard ω_V-neighborhood of H of size ϵ for $y \gg \epsilon^{1/2}$.

Using that V is affine, H is the vanishing locus of the globally defined holomorphic function f, and the maximum principle applied to f implies that, for small enough y (or for all y if ϵ is small enough), all holomorphic discs bounded by L_ℓ^y in V lie in a neighborhood U'^y of H (possibly larger than U^y).

The complex structure on the neighborhood U'^y of H in V is not biholomorphic to the standard product complex structure on a domain in $H \times \mathbb{C}$, but agrees with it along H. Thus, for small enough y, an arbitrarily C^∞-small perturbation of the almost-complex structure on V (preserving the holomorphicity of f) ensures the existence of a holomorphic projection map $\pi_H : U'^y \to H$, without affecting counts of holomorphic discs; without loss of generality, we can further assume that π_H maps L_ℓ^y to ℓ as an S^1-bundle, with $|f|$ constant in the S^1 fiber over each point of ℓ.

Holomorphic discs with boundary on L_ℓ^y can then be classified by using the projection to H. The Maslov index of a disc $u : D^2 \to (V, L_\ell^y)$ (with image contained in U'^y) is the sum of the Maslov index of $\pi_H \circ u$ and twice the intersection number of u with H. Thus, the weak unobstructedness of ℓ in H implies that of L_ℓ^y, and there are two types of Maslov index 2 discs to consider:

- $\pi_H \circ u$ is a Maslov index 2 disc in H, and u avoids H;
- $\pi_H \circ u$ is constant, and u intersects H transversely once.

In the first case, we observe that, given a point $\hat{p} \in L_\ell^y$, each holomorphic disc $v : D^2 \to (H, \ell)$ through $p = \pi_H(\hat{p})$ has a unique lift u through $\hat{p}$ that avoids H. Indeed, v determines the value of $\log |f|$ along the boundary of the disc u; the (unique) harmonic extension of this function to the entire disc can be expressed as the real part of some holomorphic function g, unique up to a pure imaginary additive constant. We then find that necessarily $f \circ u = \exp(g)$ up to some constant factor which is determined by requiring that the marked point map to $\hat{p}$. This, together with $\pi_H \circ u = v$, determines u. Recalling that L_ℓ^y lives on the boundary of a standard symplectic neighborhood of H, and using that u is disjoint from H, we further observe that the symplectic area of u in $(W^\vee)^{-1}(y)$ is equal to that of v in H, and the holonomy of $\tilde{\nabla}$ along the boundary of u equals that of ∇ along the boundary of v. Moreover, the same argument as in the proof of Corollary 7.8 shows that the orientations of the moduli spaces match. Thus, the total contribution of all these discs corresponds exactly to the superpotential in $\mathcal{F}(H)$.

In the second case, denoting $\pi_H \circ u = p \in \ell$, by construction L_ℓ^y intersects $\pi_H^{-1}(p)$ in a circle which bounds a disc of symplectic area ϵ, and u necessarily maps D^2 biholomorphically onto this disc. These small discs of size ϵ in the normal slices to H are regular, and contribute positively to the superpotential: indeed, their deformation theory splits into that of constant discs in H and that of the standard disc in the complex plane with boundary on a circle with the trivial spin structure (the triviality of the spin structure is due to the twist by the background class s). Thus, these discs are responsible for the additional term T^ϵ in the superpotential for L_ℓ.

For the sake of completeness, we also consider the case $y = 0$, where the intersection of L_ℓ with $(W^\vee)^{-1}(0) = \tilde{V} \cup E$ is simply ℓ. The argument in the proof of Corollary 7.8 then shows that holomorphic discs bounded by ℓ in $\tilde{V} \cup E$ lie entirely within H; however, there is a nontrivial contribution of Maslov index 2 configurations consisting of a constant disc together with a rational curve contained in E, namely the $\mathbb{P}^1$ fiber of the exceptional divisor over a point of $\ell \subset H$. (These exceptional spheres are actually the limits of the area ϵ discs discussed above as $y \to 0$). $\square$

Remark 7.11 The assumption that V is affine can be weakened somewhat: for Proposition 7.10 to hold it is sufficient to assume that the minimal Chern number of a rational curve contained in $\tilde{V}$ is at least 2. When this assumption does not hold, the discrepancy δ between the two superpotentials includes additional contributions from the enumerative geometry of rational curves of Chern number 1 in $\tilde{V}$.

Remark 7.12 The A_∞-functor from $\mathcal{F}(H)$ to $\mathcal{F}_s(X, W^\vee)$ is induced by a Lagrangian correspondence in the product $H \times X$, namely the set of all $(p, q) \in H \times X$ such that parallel transport of q by the gradient flow of $-\mathrm{Re}\, W^\vee$ converges to $p \in \mathrm{crit}\, W^\vee$. This Lagrangian correspondence is admissible with respect to $\mathrm{pr}_2^* W^\vee$, and weakly unobstructed with $\mathfrak{m}_0 = \delta$. While the Ma'u-Wehrheim-Woodward construction of A_∞-functors from Lagrangian correspondences [176] has not yet been developed in the setting considered here, it is certainly the right conceptual framework in which Corollary 7.8 should be understood.

By analogy with the case of Lefschetz fibrations [217], it is expected that the Fukaya category of a Morse-Bott fibration is generated by thimbles, at least under the assumption that the Fukaya category of the critical locus admits a resolution of the diagonal. The argument is expected to be similar to that in [217], except in the Morse-Bott case the key ingredient becomes the long exact sequence for fibered Dehn twists [251]. Thus, it is reasonable to expect that the A_∞-functor of Corollary 7.8 is in fact a quasi-equivalence.

Similar statements are also expected to hold for the *wrapped Fukaya category* of H and the *partially wrapped Fukaya category* of $(X, W^\vee)$ (twisted by s); however, this remains speculative, as the latter category has not been suitably constructed yet.

In any case, Corollary 7.8 and Proposition 7.10 motivate the terminology introduced in Definition 1.2.

Proof. [Proof of Theorem 1.6] While Theorem 7.1 provides an SYZ mirror to the Landau-Ginzburg model $(X, W^\vee)$, in light of the above discussion several adjustments are necessary in order to arrive at a generalized SYZ mirror to H.

1. As noted in Remark 7.4, the restriction of ω_ϵ to $\mathrm{crit}(W^\vee)$ does not agree with the restriction of ω_V to H. To remedy this, in our main construction V should be equipped with a Kähler form in the class $[\omega_V] + \epsilon[H]$ rather than $[\omega_V]$. This ensures that the critical locus of $W^\vee$ is indeed isomorphic to H equipped with the restriction of the Kähler form ω_V.

2. In light of Corollary 7.8, the A-side Landau-Ginzburg model $(X, W^\vee)$ should be twisted by the background class $s = PD([\tilde{V}]) \in H^2(X, \mathbb{Z}/2)$. Namely, the tori we consider in our main argument should be viewed as objects of $\mathcal{F}_s(X, W^\vee)$ rather than $\mathcal{F}(X, W^\vee)$. This modifies the sign conventions for counting discs and hence the mirror superpotential.

3. By Proposition 7.10, the additive constant $\delta = T^\epsilon$ should be subtracted from the superpotential, since the natural A_∞-functor from $\mathcal{F}(H)$ to $\mathcal{F}_s(X, W^\vee)$ increases $\mathfrak{m}_0$ by that amount.

Thus, the mirror space remains the toric variety Y, but the superpotential is no longer

$$W_0 = w_0 + \sum_{i=1}^{r} T^{\varpi_i} \mathbf{v}_{\alpha_i}^{\sigma_i}; \tag{7.2}$$

we now make explicit how each of the above changes affects the potential.

Replacing $[\omega_V]$ by $[\omega_V] + \epsilon[H]$ amounts to changing the equations of the facets of the moment polytope Δ_V from $\langle \sigma_i, \cdot \rangle + \varpi_i = 0$ to $\langle \sigma_i, \cdot \rangle + \varpi_i + \epsilon\lambda(\sigma_i) = 0$ (where $\lambda : \Sigma_V \to \mathbb{R}$ is the piecewise linear function defining $\mathcal{L} = \mathcal{O}(H)$). Accordingly, each exponent ϖ_i in (7.2) should be changed to $\varpi_i + \epsilon\lambda(\sigma_i)$.

Next, we twist by the background class $s = PD([\tilde{V}])$, and view the tori studied in Section 5 as objects of $\mathcal{F}_s(X, W^\vee)$ rather than $\mathcal{F}(X, W^\vee)$. Specifically, s lifts to a class in $H^2(X, L; \mathbb{Z}/2)$ (dual to $[\tilde{V}] \in H_{2n}(X \setminus L)$), and twisting the standard spin structure by this lift of s yields a relatively spin structure on L. By [81, Proposition 8.1.16], this twist affects the signed count of holomorphic discs in a given class $\beta \in \pi_2(X, L)$ by a factor of $(-1)^k$ where $k = \beta \cdot [\tilde{V}]$. Recall from §6 that, of the various families of holomorphic discs that contribute to the superpotential, the only ones that intersect $\tilde{V}$ are those described by Lemma 5.5; thus the only effect of the twisting by the background class s is to change the first term of W_0 from w_0 to $-w_0$.

Finally, we subtract $\delta = T^\epsilon$ from the superpotential, and find that the appropriate superpotential to consider on Y is given by

$$W_0' = -T^\epsilon - w_0 + \sum_{i=1}^{r} T^{\varpi_i + \epsilon\lambda(\sigma_i)} \mathbf{v}_{\alpha_i}^{\sigma_i} = -T^\epsilon v_0 + \sum_{i=1}^{r} T^{\varpi_i} T^{\epsilon\lambda(\sigma_i)} \mathbf{v}_{\alpha_i}^{\sigma_i}.$$

Finally, recall from §3.3 that the weights of the toric monomials v_0 and $\mathbf{v}_{\alpha_i}^{\sigma_i}$ are respectively $(0, 1)$ and $(-\sigma_i, \lambda(\sigma_i)) \in \mathbb{Z}^n \oplus \mathbb{Z}$. Therefore, a rescaling of the last coordinate by a factor of T^ϵ changes v_0 to

$T^\epsilon v_0$ and $\mathbf{v}_{\alpha_i}^{\sigma_i}$ to $T^{\epsilon\lambda(\sigma_i)}\mathbf{v}_{\alpha_i}^{\sigma_i}$. This change of variables eliminates the dependence on ϵ (as one would expect for the mirror to H) and replaces W_0' by the simpler expression

$$-v_0 + \sum_{i=1}^{r} T^{\varpi_i}\mathbf{v}_{\alpha_i}^{\sigma_i},$$

which is exactly W_0^H (see Definition 3.10). □

Remark 7.13 Another way to produce an A_∞-functor from the Fukaya category of H to that of X (more specifically, the idempotent closure of $\mathcal{F}_s(X)$) is the following construction considered by Ivan Smith in [227, Section 4.5].

Given a Lagrangian submanifold $\ell \subset H$, first lift it to the boundary of the ϵ-tubular neighborhood of H inside V, to obtain a Lagrangian submanifold $C_\ell \subset V$ which is a circle bundle over ℓ; then, identifying V with the reduced space $X_{red,\epsilon} = \mu_X^{-1}(\epsilon)/S^1$, lift C_ℓ to $\mu_X^{-1}(\epsilon)$ and "spin" it by the S^1-action, to obtain a Lagrangian submanifold $T_\ell \subset X$ which is a T^2-bundle over ℓ. Then T_ℓ formally splits into a direct sum $T_\ell^+ \oplus T_\ell^-$; the A_∞-functor is constructed by mapping ℓ to either summand.

The two constructions are equivalent: in $\mathcal{F}_s(X, W^\vee)$ the summands $T_\ell^\pm$ are isomorphic to the thimble L_ℓ (up to a shift). One benefit of Smith's construction is that, unlike L_ℓ, the Lagrangian submanifold T_ℓ is entirely contained inside X^0, which makes its further study amenable to T-duality arguments involving X^0 and Y^0.

8 The converse construction

As a consequence of Theorem 1.7, the mirror Y^0 of X^0 can be defined as a variety not only over the Novikov field, but also over the complex numbers. In this section, we impose the maximal degeneration condition (cf. Definition 3.1) which implies that Y^0 is smooth. We then reverse our viewpoint from the preceding discussion: treating T as a numerical parameter and equipping Y^0 with a Kähler form, we shall reconstruct X^0 (as an analytic space that also happens to be defined over complex numbers) as an SYZ mirror. Along the way, we also obtain another perspective on how compactifying Y^0 to the toric variety Y amounts to equipping X^0 with a superpotential. We omit any discussion of Y or Y^0 equipped with A-side Landau-Ginzburg models, which would require a deeper understanding of the corresponding Fukaya categories.

(Note: many of the results in this section were also independently obtained by Chan, Lau and Leung [51].)

To begin our construction, observe that $Y^0 = Y \setminus w_0^{-1}(0)$ carries a natural T^n-action, given in the coordinates introduced in §3.3 by

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot (v_{\alpha,1}, \ldots, v_{\alpha,n}, v_{\alpha,0}) = (e^{i\theta_1}v_{\alpha,1}, \ldots, e^{i\theta_n}v_{\alpha,n}, v_{\alpha,0}).$$

This torus is a subgroup of the $(n+1)$-dimensional torus which acts on the toric variety Y, namely the stabilizer of the regular function $w_0 = -T^\epsilon + T^\epsilon v_0$.

We equip Y^0 with a T^n-invariant Kähler form ω_Y. To make things concrete, take ω_Y to be the restriction of a complete toric Kähler form on Y, with moment polytope

$$\Delta_Y = \{(\xi,\eta) \in \mathbb{R}^n \oplus \mathbb{R} \,|\, \eta \geq \varphi(\xi) = \max_{\alpha\in A}(\langle \alpha, \xi\rangle - \rho(\alpha))\}$$

(cf. (3.8)). We denote by $\tilde{\mu}_Y : Y \to \mathbb{R}^{n+1}$ the moment map for the T^{n+1}-action on Y, and by $\mu_Y : Y^0 \to \mathbb{R}^n$ the moment map for the T^n-action on Y^0. Observing that μ_Y is obtained from $\tilde{\mu}_Y$ by restricting to Y^0 and projecting to the first n components, the critical locus of μ_Y is the union of all codimension 2 toric strata, and the set of critical values of μ_Y is precisely the tropical hypersurface $\Pi_0 \subset \mathbb{R}^n$ defined by φ. Finally, we also equip Y^0 with the T^n-invariant holomorphic $(n+1)$-form given in each chart by

$$\Omega_Y = d\log v_{\alpha,1} \wedge \cdots \wedge d\log v_{\alpha,n} \wedge d\log w_0.$$

Note that this holomorphic volume form scales with ϵ.

Lemma 8.1 *The map $\pi_Y = (\mu_Y, |w_0|) : Y^0 \to B_Y = \mathbb{R}^n \times \mathbb{R}_+$ defines a T^n-invariant special Lagrangian torus fibration on Y^0. Moreover, $\pi_Y^{-1}(\xi, r)$ is singular if and only if $(\xi, r) \in \Pi_0 \times \{T^\epsilon\}$, and obstructed if and only if $r = T^\epsilon$.*

This fibration is analogous to some of the examples considered in [101], [102], [46], [47]; see also Example 3.3.1 in [14].

The statement that $\pi_Y^{-1}(\xi, r)$ is special Lagrangian follows immediately from the observation that Ω_Y descends to the holomorphic 1-form $d\log w_0$ on the reduced space $\mu_Y^{-1}(\xi)/T^n \simeq \mathbb{C}^*$; thus the circle $|w_0| = r$ is special Lagrangian in the reduced space, and its lift to $\mu_Y^{-1}(\xi)$ is special Lagrangian in Y^0.

A useful way to think of these tori is to consider the projection of Y^0 to the coordinate w_0, whose fibers are all isomorphic to $(\mathbb{C}^*)^n$ except for $w_0^{-1}(-T^\epsilon) = v_0^{-1}(0)$ which is the union of all toric strata in Y. In this projection, $\pi_Y^{-1}(\xi, r)$ fibers over the circle of radius r centered at the origin, and intersects each of the fibers $w_0^{-1}(re^{i\theta})$ in a standard product torus (corresponding to the level ξ of the moment map). In particular, $\pi_Y^{-1}(\xi, r)$ is singular precisely when $r = T^\epsilon$ and $\xi \in \Pi_0$.

By the maximum principle, any holomorphic disc in Y^0 bounded by $\pi_Y^{-1}(\xi, r)$ must lie entirely within a fiber of the projection to w_0. Since the regular fibers of w_0 are isomorphic to $(\mathbb{C}^*)^n$, inside which product tori do not bound any nonconstant holomorphic discs, $\pi_Y^{-1}(\xi, r)$ is tautologically unobstructed for $r \neq T^\epsilon$. When $r = T^\epsilon$, $\pi_Y^{-1}(\xi, r)$ intersects one of the components of $w_0^{-1}(-T^\epsilon)$ (i.e. one of the toric divisors of Y) in a product torus, which bounds various families of holomorphic discs as well as configurations consisting of holomorphic discs and rational curves in the toric strata. This completes the proof of Lemma 8.1.

The maximum principle applied to w_0 also implies that every rational curve in Y is contained in $w_0^{-1}(-T^\epsilon)$ (i.e. the union of all toric strata), hence disjoint from the anticanonical divisor $w_0^{-1}(0)$, and thus satisfies $c_1(Y) \cdot C = 0$; in fact Y is a toric Calabi-Yau variety. So Assumption 2.2 holds, and partially compactifying Y^0 to Y does not modify the enumerative geometry of Maslov index 0 discs bounded by the fibers of π_Y. Hence the SYZ mirror of Y is just the mirror of Y^0 equipped with an appropriate superpotential, and we determine both at the same time.

The wall $r = T^\epsilon$ divides the fibration $\pi_Y : Y^0 \to B_Y$ into two chambers; accordingly, the SYZ mirror of Y^0 (and Y) is constructed by gluing together two coordinate charts U' and U'' via a transformation which accounts for the enumerative geometry of discs bounded by the potentially obstructed fibers of π_Y. We now define coordinate systems for both charts and determine the superpotential (for the mirror of Y) in terms of those coordinates. For notational consistency and to avoid confusion, we now denote by τ (rather than T) the Novikov parameter recording areas with respect to ω_Y.

We start with the chamber $r > T^\epsilon$, over which the fibers of π_Y can be deformed into product tori in Y (i.e., orbits of the T^{n+1}-action) by a Hamiltonian isotopy that does not intersect $w_0^{-1}(-T^\epsilon)$ (from the perspective of the projection to w_0, the isotopy amounts simply to deforming the circle of radius r centered at 0 to a circle of the appropriate radius centered at $-T^\epsilon$).

Fix a reference fiber $L^0 = \pi_Y^{-1}(\xi^0, r^0)$, where $\xi^0 \in \mathbb{R}^n$ and $r^0 > T^\epsilon$, and choose a basis $(\gamma_1, \dots, \gamma_n, \gamma_0')$ of $H_1(L^0, \mathbb{Z})$, where $-\gamma_1, \dots, -\gamma_n$ correspond to the factors of the T^n-action on L^0, and $-\gamma_0'$ corresponds to an orbit of the last S^1 factor of T^{n+1} acting on a product torus $\tilde{\mu}_Y^{-1}(\xi^0, \eta^0)$ which is Hamiltonian isotopic to L^0 in Y. (The signs are motivated by consistency with the notations used for X^0.)

A point of the chart U' mirror to the chamber $\{r > T^\epsilon\}$ corresponds to a pair (L, ∇), where $L = \pi_Y^{-1}(\xi, r)$ is a fiber of π_Y (with $r > T^\epsilon$), Hamiltonian isotopic to a product torus $\tilde{\mu}_Y^{-1}(\xi, \eta)$ in Y, and $\nabla \in \mathcal{H}om(\pi_1(L), U_{\mathbb{K}})$. We rescale the coordinates given by (2.3) to eliminate the dependence on the base point (ξ^0, r^0), i.e. we identify U' with a domain in $(\mathbb{K}^*)^{n+1}$ via

$$(L, \nabla) \mapsto (x_1', \dots, x_n', z') = \Big(\tau^{-\xi_1}\nabla(\gamma_1), \dots, \tau^{-\xi_n}\nabla(\gamma_n), \tau^{-\eta}\,\nabla(\gamma_0')\Big). \tag{8.1}$$

(Compare with (2.3), noting that $-\xi_i = -\xi_i^0 + \int_{\Gamma_i} \omega_Y$ and $-\eta = -\eta^0 + \int_{\Gamma_0'} \omega_Y$.)

Lemma 8.2 *In the chart U', the superpotential for the mirror to Y is given by*

$$W^\vee(x_1', \dots, x_n', z') = \sum_{\alpha \in A} (1 + \kappa_\alpha)\tau^{\rho(\alpha)} {x_1'}^{\alpha_1} \dots {x_n'}^{\alpha_n} {z'}^{-1}, \tag{8.2}$$

where $\kappa_\alpha \in \mathbb{K}$ are constants with $\mathrm{val}(\kappa_\alpha) > 0$.

Proof. Consider a point $(L, \nabla) \in U'$, where $L = \pi_Y^{-1}(\xi, r)$ is Hamiltonian isotopic to the product torus $L' = \tilde{\mu}_Y^{-1}(\xi, \eta)$ in Y. As explained above, the isotopy can be performed without intersecting the toric divisors of Y, i.e. without wall-crossing; therefore, the isotopy provides a cobordism between the moduli spaces of Maslov index 2 holomorphic discs bounded by L and L' in Y.

It is well-known that the families of Maslov index 2 holomorphic discs bounded by the standard product torus L' in the toric manifold Y are in one-to-one correspondence with the codimension 1 toric strata of Y. Namely, for each codimension 1 stratum, there is a unique family of holomorphic discs which intersect this stratum transversely at a single point and do not intersect any of the other strata. Moreover, every point of L' lies on the boundary of exactly one disc of each family, and these discs are all regular [56] (see also [13, §4]).

The toric divisors of Y, or equivalently the facets of Δ_Y, are in one-to-one correspondence with the elements of A. The symplectic area of a Maslov index 2 holomorphic disc in (Y, L') which intersects the divisor corresponding to $\alpha \in A$ (and whose class we denote by β_α) is equal to the distance from the point (ξ, η) to that facet of Δ_Y, namely $\eta - \langle \alpha, \xi \rangle + \rho(\alpha)$, whilst the boundary of the disc represents the class $\partial\beta_\alpha = \sum \alpha_i \gamma_i - \gamma_0' \in H_1(L', \mathbb{Z})$. The weight associated to such a disc is therefore

$$z_{\beta_\alpha}(L', \nabla) = \tau^{\eta - \langle \alpha, \xi \rangle + \rho(\alpha)} \nabla(\gamma_1)^{\alpha_1} \dots \nabla(\gamma_n)^{\alpha_n} \nabla(\gamma_0')^{-1} = \tau^{\rho(\alpha)} {x_1'}^{\alpha_1} \dots {x_n'}^{\alpha_n} {z'}^{-1}.$$

Using the isotopy between L and L', we conclude that the contributions of Maslov index 2 holomorphic discs in (Y, L) to the superpotential $W^\vee$ add up to

$$\sum_{\alpha\in A} z_{\beta_\alpha}(L,\nabla) = \sum_{\alpha\in A} \tau^{\rho(\alpha)} {x'_1}^{\alpha_1}\dots {x'_n}^{\alpha_n} z'^{-1}.$$

However, the superpotential $W^\vee$ also includes contributions from (virtual) counts of stable genus 0 configurations of discs and rational curves of total Maslov index 2. These configurations consist of a single Maslov index 2 disc (in one of the above families) together with one or more rational curves contained in the toric divisors of Y (representing a total class $C \in H_2(Y,\mathbb{Z})$). The enumerative invariant $n(L, \beta_\alpha + C)$ giving the (virtual) count of such configurations whose boundary passes through a generic point of L can be understood in terms of genus 0 Gromov-Witten invariants of suitable partial compactifications of Y (see e.g. [51]). However, all that matters to us is the general form of the corresponding terms of the superpotential. Since the rational components contribute a multiplicative factor $\tau^{[\omega_Y]\cdot C}$ to the weight, we obtain that

$$W^\vee = \sum_{\alpha\in A}\Big(1+\sum_{\substack{C\in H_2(Y,\mathbb{Z})\\ [\omega_Y]\cdot C>0}} n(L,\beta_\alpha+C)\,\tau^{[\omega_Y]\cdot C}\Big)\,\tau^{\rho(\alpha)} {x'_1}^{\alpha_1}\dots {x'_n}^{\alpha_n} z'^{-1},$$

which is of the expected form (8.2). ☐

Next we look at the other chart U'', which corresponds to the chamber $r < T^\epsilon$ of the fibration π_Y. Fix again a reference fiber $L^0 = \pi_Y^{-1}(\xi^0, r^0)$, where $\xi^0 \in \mathbb{R}^n$ and $r^0 < T^\epsilon$, and choose a basis $(\gamma_1,\dots,\gamma_n,\gamma''_0)$ of $H_1(L^0,\mathbb{Z})$, where $-\gamma_1,\dots,-\gamma_n$ correspond to the factors of the T^n-action on L^0, and γ''_0 can be represented by a loop in L^0 over which w_0 runs counterclockwise around the circle of radius r^0 while $v_{\alpha,1},\dots,v_{\alpha,n} \in \mathbb{R}_+$ (for some arbitrary choice of α). Note that the fibration $w_0 : Y \to \mathbb{C}$ is trivial over the disc of radius r^0; in fact the coordinates $(w_0, v_{\alpha,1},\dots,v_{\alpha,n})$ (for any α) give a biholomorphism from the subset $\{|w_0| \le r^0\}$ of Y to $D^2(r^0)\times(\mathbb{C}^*)^n$. Then γ''_0 can be characterized as the unique element of $H_1(L^0,\mathbb{Z})$ which arises as the boundary of a section of $w_0 : Y \to \mathbb{C}$ over the disc of radius r^0; we denote by β_0 the relative homotopy class of this section. A point of U'' corresponds to a pair (L,∇) where $L = \pi_Y^{-1}(\xi, r)$ is a fiber of π_Y (with $r < T^\epsilon$), and $\nabla \in \mathcal{H}om(\pi_1(L), U_{\mathbb{K}})$. As before, we rescale the coordinates given by (2.3) to eliminate the dependence on the base point (ξ^0, r^0), i.e. we identify U'' with a domain in $(\mathbb{K}^*)^{n+1}$ via

$$(L,\nabla)\mapsto (x''_1,\dots,x''_n,y'') = \Big(\tau^{-\xi_1}\nabla(\gamma_1),\dots,\tau^{-\xi_n}\nabla(\gamma_n),\tau^{[\omega_Y]\cdot\beta_0}\,\nabla(\gamma''_0)\Big). \tag{8.3}$$

Lemma 8.3 *In the chart U'', the superpotential for the mirror to Y is given by*

$$W^\vee(x''_1,\dots,x''_n,y'') = y''. \tag{8.4}$$

Proof. By the maximum principle applied to the projection to w_0, any holomorphic disc bounded by $L = \pi_Y^{-1}(\xi, r)$ in Y must be contained in the subset $\{|w_0| \le r\} \subset Y$, which is diffeomorphic to $D^2 \times (\mathbb{C}^*)^n$. Thus, for topological reasons, any holomorphic disc bounded by L must represent a multiple of the class β_0. Since the Maslov index is equal to twice the intersection number with $w_0^{-1}(0)$, Maslov index 2 discs are holomorphic sections of $w_0 : Y \to \mathbb{C}$ over the disc of radius r, representing β_0.

The formula (8.4) now follows from the claim that the number of such sections passing through a given point of L is $n(L, \beta_0) = 1$. This can be viewed as an enumerative problem for holomorphic sections of a trivial Lefschetz fibration with a Lagrangian boundary condition, easily answered by applying the powerful methods of [216, §2]. An alternative, more elementary approach is to deform ω_Y among toric Kähler forms in its cohomology class to ensure that, for some $\xi^0 \in \mathbb{R}^n$, $\mu_Y^{-1}(\xi^0)$ is given in one of the coordinate charts Y_α of §3.3 by equations of the form $|v_{\alpha,1}| = \rho_1, \dots, |v_{\alpha,n}| = \rho_n$. (In fact, many natural choices for ω_Y cause this property to hold immediately.) When this property holds, the maximum principle applied to $v_{\alpha,1}, \dots, v_{\alpha,n}$ implies that the holomorphic Maslov index 2 discs bounded by $L^0 = \pi_Y^{-1}(\xi^0, r^0)$ are given by letting w_0 vary in the disc of radius r^0 while the other coordinates $v_{\alpha,1}, \dots, v_{\alpha,n}$ are held constant. All these discs are regular, and there is precisely one disc passing through each point of L^0. It follows that $n(L^0, \beta_0) = 1$. This completes the proof, since the invariant $n(L^0, \beta_0)$ is not affected by the deformation of ω_Y to the special case we have considered, and the value of $n(L, \beta_0)$ is the same for all the fibers of π_Y over the chamber $r < T^\epsilon$. $\square$

We can now formulate and prove the main result of this section:

Theorem 8.4 *The rigid analytic manifold*

$$\mathcal{X}^0 = \{(x_1, \dots, x_n, y, z) \in (\mathbb{K}^*)^n \times \mathbb{K}^2 \mid yz = \tilde{f}(x_1, \dots, x_n)\}, \tag{8.5}$$

where $\tilde{f}(x_1, \dots, x_n) = \sum_{\alpha \in A} (1 + \kappa_\alpha) \tau^{\rho(\alpha)} x_1^{\alpha_1} \dots x_n^{\alpha_n}$, *is SYZ mirror to* (Y^0, ω_Y).

Moreover, the B-side Landau-Ginzburg model $(\mathcal{X}^0, W^\vee = y)$ is SYZ mirror to (Y, ω_Y).

Proof. The two charts U' and U'' are glued to each other by a coordinate transformation which accounts for the Maslov index 0 holomorphic discs bounded by the potentially obstructed fibers of π_Y. There are many families of such discs, all contained in $w_0^{-1}(-T^\epsilon) = v_0^{-1}(0)$. However we claim that the first n coordinates of the charts (8.1) and (8.3) are not affected by these instanton corrections, so that the gluing satisfies $x_1'' = x_1', \dots, x_n'' = x_n'$.

One way to argue is based on the observation that all Maslov index 0 configurations are contained in $w_0^{-1}(-T^\epsilon)$. Consider as in §2.1 a Lagrangian isotopy $\{L_t\}_{t \in [0,1]}$ between fibers of π_Y in the two chambers (with L_{t_0} the only potentially obstructed fiber), and the cycles $C_\alpha = \mathrm{ev}_*[\mathcal{M}_1(\{L_{t_0}\}, \alpha)] \in H_{n-1}(L_{t_0})$ corresponding to the various classes $\alpha \in \pi_2(Y, L_t)$ that may contain Maslov index 0 configurations. The fact that each C_α is supported on $L_{t_0} \cap w_0^{-1}(-T^\epsilon)$ implies readily that $C_\alpha \cdot \gamma_1 = \dots = C_\alpha \cdot \gamma_n = 0$. Since the overall gluing transformation is given by a composition of elementary transformations of the type (2.4), the first n coordinates are not affected.

By Corollary A.11, a more down-to-earth way to see that the gluing preserves $x_i'' = x_i'$ ($i = 1, \dots, n$) is to consider the partial compactification Y_i' of Y^0 given by the moment polytope $\Delta_Y \cap \{\xi_i \leq K\}$ for some constant $K \gg 0$ (still removing $w_0^{-1}(0)$ from the resulting toric variety). From the perspective of the projection $w_0 : Y^0 \to \mathbb{C}^*$, this simply amounts to a toric partial compactification of each fiber, where the generic fiber $(\mathbb{C}^*)^n$ is partially compactified along the i-th factor to $(\mathbb{C}^*)^{n-1} \times \mathbb{C}$. The Maslov index 2 holomorphic discs bounded by $L = \pi_Y^{-1}(\xi, r)$ inside Y_i' are contained in the fibers of w_0 by the maximum principle; requiring that the boundary of the disc pass through a given point $p \in L$ (where we assume $w_0 \neq -T^\epsilon$), we are reduced to the fiber of w_0 containing p, which L intersects in a standard product torus $(S^1)^n \subset (\mathbb{C}^*)^{n-1} \times \mathbb{C}$ (where the radii of the various S^1 factors depend on ξ). Thus, there is exactly one Maslov index 2 holomorphic disc in (Y_i', L) through a generic point

$p \in L$ (namely a disc over which all coordinates except the i-th one are constant). The superpotential is equal to the weight of this disc, i.e. $\tau^{K-\xi_i}\nabla(\gamma_i)$, which can be rewritten as $\tau^K x_i'$ if $r > T^\epsilon$, and $\tau^K x_i''$ if $r < T^\epsilon$. Comparing these two expressions, we see that the gluing between U' and U'' identifies $x_i' = x_i''$.

The gluing transformation between the coordinates y'' and z' is more complicated, but is now determined entirely by a comparison between (8.2) and (8.4): since the two formulas for $W^\vee$ must glue to a regular function on the mirror, y'' must equal the right-hand side of (8.2), hence

$$y''z' = \sum_{\alpha\in A}(1+\kappa_\alpha)\tau^{\rho(\alpha)}{x_1'}^{\alpha_1}\dots{x_n'}^{\alpha_n} = \tilde{f}(x_1',\dots,x_n').$$

This completes the proof of the theorem. $\square$

The first part of Theorem 8.4 is a statement of SYZ mirror symmetry in the opposite direction from Theorem 1.7; the two results taken together relate the symplectic topology and algebraic geometry of the spaces X^0 and Y^0 to each other. More precisely, we would like to treat τ as a fixed complex number and view the mirror of (Y^0, ω_Y) as a complex manifold. The convergence of the function $\tilde{f}$ depends only on that of the constants κ_α, which is unknown in general but holds in practice for a number of examples (see [51] and other work by the same authors). Even when convergence is not an issue, the result reveals the need for care in constructing the mirror map: while our main construction is essentially independent of the coefficients c_α appearing in (3.1) (which do not affect the symplectic geometry of X^0), the direction considered here requires the complex structure of X^0 to be chosen carefully to match with the Kähler class $[\omega_Y]$, specifically we have to take $c_\alpha = 1+\kappa_\alpha$.

The second part of Theorem 8.4 gives a mirror symmetric interpretation of the partial compactification of Y^0 to Y, in terms of equipping X^0 with the superpotential $W^\vee = y$. From the perspective of our main construction (viewing X^0 as a symplectic manifold and Y^0 as its SYZ mirror), we saw the same phenomenon in Section 7.

9 Examples

9.1 Hyperplanes and pairs of pants We consider as our first example the (higher dimensional) pair of pants H defined by the equation

$$x_1 + \dots + x_n + 1 = 0 \tag{9.1}$$

in $V = (\mathbb{C}^*)^n$. (The case $n = 2$ corresponds to the ordinary pair of pants; in general H is the complement of $n+1$ hyperplanes in general position in $\mathbb{CP}^{n-1}$.)

The tropical polynomial corresponding to (9.1) is $\varphi(\xi) = \max(\xi_1,\dots,\xi_n,0)$; the polytope Δ_Y defined by (3.8) is equivalent via $(\xi_1,\dots,\xi_n,\eta)\mapsto(\eta-\xi_1,\dots,\eta-\xi_n,\eta)$ to the orthant $(\mathbb{R}_{\geq 0})^{n+1}\subset\mathbb{R}^{n+1}$. Thus $Y \simeq \mathbb{C}^{n+1}$. In terms of the coordinates $(z_1,\dots,z_{n+1})$ of $\mathbb{C}^{n+1}$, the monomial v_0 is given by $v_0 = z_1\dots z_{n+1}$. Thus, in this example our main results are:

1. the open Calabi-Yau manifold $Y^0 = \mathbb{C}^{n+1} \setminus \{z_1 \dots z_{n+1} = 1\}$ is SYZ mirror to the conic bundle $X^0 = \{(x_1, \dots, x_n, y, z) \in (\mathbb{C}^*)^n \times \mathbb{C}^2 \,|\, yz = x_1 + \dots + x_n + 1\}$;

2. the B-side Landau-Ginzburg model $(Y^0, W_0 = -T^\epsilon + T^\epsilon z_1 \dots z_{n+1})$ is SYZ mirror to the blowup X of $(\mathbb{C}^*)^n \times \mathbb{C}$ along $H \times 0$, where
$$H = \{(x_1, \dots, x_n) \in (\mathbb{C}^*)^n \,|\, x_1 + \dots + x_n + 1 = 0\};$$

3. the B-side Landau-Ginzburg model $(\mathbb{C}^{n+1}, W_0^H = -z_1 \dots z_{n+1})$ is a generalized SYZ mirror of H.

The last statement in particular has been verified in the sense of homological mirror symmetry by Sheridan [221]; see also [3] for a more detailed result in the case $n = 2$ (the usual pair of pants).

If instead we consider the same equation (9.1) to define (in an affine chart) a hyperplane $H \simeq \mathbb{CP}^{n-1}$ inside $V = \mathbb{CP}^n$, with a Kähler form such that $\int_{\mathbb{CP}^1} \omega_V = A$, then our main result becomes that the B-side Landau-Ginzburg model consisting of $Y^0 = \mathbb{C}^{n+1} \setminus \{z_1 \dots z_{n+1} = 1\}$ equipped with the superpotential
$$W_0 = -T^\epsilon + T^\epsilon z_1 \dots z_{n+1} + z_1 + \dots + z_n + T^A z_{n+1}$$
is SYZ mirror to the blowup X of $\mathbb{CP}^n \times \mathbb{C}$ along $H \times 0 \simeq \mathbb{CP}^{n-1} \times 0$.

Even though $\mathbb{CP}^{n-1}$ is not affine, Theorem 1.6 still holds for this example if we assume that $n \geq 2$, by Remark 7.11. In this case, the mirror we obtain for $\mathbb{CP}^{n-1}$ (viewed as a hyperplane in $\mathbb{CP}^n$) is the B-side Landau-Ginzburg model
$$(\mathbb{C}^{n+1}, W_0^H = -z_1 \dots z_{n+1} + z_1 + \dots + z_n + T^A z_{n+1}).$$

Rewriting the superpotential as
$$W_0^H = z_1 + \dots + z_n + z_{n+1}(T^A - z_1 \dots z_n) = \tilde{W}(z_1, \dots, z_n) + z_{n+1}\, g(z_1, \dots, z_n)$$
makes it apparent that this B-side Landau-Ginzburg model is equivalent (e.g. in the sense of Orlov's generalized Knörrer periodicity [196]) to the B-side Landau-Ginzburg model consisting of $g^{-1}(0) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, z_1 \dots z_n = T^A\}$ equipped with the superpotential $\tilde{W} = z_1 + \dots + z_n$, which is the classical toric mirror of $\mathbb{CP}^{n-1}$.

9.2 ALE spaces Let $V = \mathbb{C}$, and let $H = \{x_1, \dots, x_{k+1}\} \subset \mathbb{C}^*$ consist of $k+1$ points, $k \geq 0$, with $|x_1| \ll \dots \ll |x_{k+1}|$ (so that the defining polynomial of H, $f_{k+1}(x) = (x - x_1) \dots (x - x_{k+1}) \in \mathbb{C}[x]$, is near the tropical limit).

The conic bundle $X^0 = \{(x, y, z) \in \mathbb{C}^* \times \mathbb{C}^2 \,|\, yz = f_{k+1}(x)\}$ is the complement of the regular conic $x = 0$ in the A_k-Milnor fiber
$$X' = \{(x, y, z) \in \mathbb{C}^3 \,|\, yz = f_{k+1}(x)\}.$$

In fact, X' is the main space of interest here, rather than its open subset X^0 or its partial compactification X (note that $X' = X \setminus \tilde{V}$). However the mirror of X' differs from that of X simply by excluding the term w_0 (which accounts for those holomorphic discs that intersect $\tilde{V}$) from the mirror superpotential.

The tropical polynomial $\varphi : \mathbb{R} \to \mathbb{R}$ corresponding to f_{k+1} is a piecewise linear function whose slope takes the successive integer values $0, 1, \dots, k+1$. Thus the toric variety Y determined by the polytope $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^2 \,|\, \eta \geq \varphi(\xi)\}$ is the resolution of the A_k singularity $\{st = u^{k+1}\} \subset \mathbb{C}^3$. The $k+2$ edges of Δ_Y correspond to the toric strata of Y, namely the proper transforms of the coordinate axes $s = 0$ and $t = 0$ and the k rational (-2)-curves created by the resolution. Specifically, Y is covered by $k+1$ affine coordinate charts U_α with coordinates $(s_\alpha = v_{\alpha,1}, t_\alpha = v_{\alpha+1,1}^{-1})$, $0 \leq \alpha \leq k$; denoting the toric coordinate $v_{\alpha,0}$ by u, equation (3.9) becomes $s_\alpha t_\alpha = u$, and the regular functions $s = s_0, t = t_k, u \in \mathcal{O}(Y)$ satisfy the relation $st = u^{k+1}$.

Since $w_0 = -T^\epsilon + T^\epsilon v_0 = -T^\epsilon + T^\epsilon u$, the space Y^0 is the complement of the curve $u = 1$ inside Y. With this understood, our main results become:

1. the complement Y^0 of the curve $u = 1$ in the resolution Y of the A_k singularity $\{st = u^{k+1}\} \subset \mathbb{C}^3$ is SYZ mirror to the complement X^0 of the curve $x = 0$ in the Milnor fiber $X' = \{(x, y, z) \in \mathbb{C}^3 \,|\, yz = f_{k+1}(x)\}$ of the A_k singularity;

2. the B-side Landau-Ginzburg model $(Y^0, W_0 = s)$ is SYZ mirror to X';

3. the Landau-Ginzburg models $(Y, W_0 = s)$ and $(X', W^\vee = y)$ are SYZ mirror to each other.

These results show that the oft-stated mirror symmetry relation between the smoothing and the resolution of the A_k singularity (or, specializing to the case $k = 1$, between the affine quadric T^*S^2 and the total space of the line bundle $\mathcal{O}(-2) \to \mathbb{P}^1$) needs to be corrected either by removing smooth curves from each side, or by equipping both sides with superpotentials.

One final comment that may be of interest to symplectic geometers is that $W_0 = s$ vanishes to order $k+1$ along the t coordinate axis, and to orders $1, 2, \dots, k$ along the exceptional curves of the resolution. The higher derivatives of the superpotential encode information about the A_∞-products on the Floer cohomology of the Lagrangian torus fiber of the SYZ fibration, and the high-order vanishing of W_0 along the toric divisors of Y^0 indicates that the A_k Milnor fiber contains Lagrangian tori whose Floer cohomology is isomorphic to the usual cohomology of T^2 as an algebra, but carries non-trivial A_∞-operations. (See also [167] for related considerations.)

Corollary 9.1 *For $\alpha \in \{2, \dots, k+1\}$, let $r \in \mathbb{R}_+$ be such that exactly α of the points $x_1, \dots, x_{k+1}$ satisfy $|x_i| < r$. Then the Floer cohomology of the Lagrangian torus $T_r = \{(x, y, z) \in X' \,|\, |x| = r,\ |y| = |z|\}$ in the A_k Milnor fiber X', equipped with a suitable spin structure, is $\mathrm{HF}^*(T_r, T_r) \simeq H^*(T^2; \Lambda)$, equipped with an A_∞-structure for which the generators a, b of $\mathrm{HF}^1(T_r, T_r)$ satisfy the relations $\mathfrak{m}_2(a, b) + \mathfrak{m}_2(b, a) = 0$; $\mathfrak{m}_i(a, \dots, a) = 0$ for all i; $\mathfrak{m}_i(b, \dots, b) = 0$ for $i \leq \alpha - 1$; and $\mathfrak{m}_\alpha(b, \dots, b) \neq 0$.*

Proof. The condition $|x| = r$ implies that the torus T_r yields a point in the chamber U_α, while the condition that $|y| = |z|$ implies that it lies on the critical locus of W_0: this shows that T_r is a critical point of W_0 of order $\alpha + 1$.

By a construction which is standard in the toric case (see [55]), the restriction of W_0 to a chart of Y modeled after a domain in $H^1(T_r, \Lambda^*)$ (identified with $(\Lambda^*)^2$ by choosing the basis (a, b)) agrees with the map

$$(\exp(\lambda_a), \exp(\lambda_b)) \mapsto \sum_k \mathfrak{m}_k(\lambda_a a + \lambda_b b, \dots, \lambda_a a + \lambda_b b). \tag{9.2}$$

Choosing a to correspond to the generator which vanishes on loops whose projection to $\mathbb{C}$ is constant, the result follows immediately. $\square$ **9.3 Plane curves** For $p, q \geq 2$, consider a smooth Riemann surface H of genus $g = (p-1)(q-1)$ embedded in $V = \mathbb{P}^1 \times \mathbb{P}^1$, defined as the zero set of a suitably chosen polynomial of bidegree (p, q). (The case of a genus 2 curve of bidegree $(3, 2)$ was used in §3 to illustrate the general construction, see Examples 3.2 and 3.12.)

Namely, in affine coordinates f is given by

$$f(x_1, x_2) = \sum_{a=0}^{p} \sum_{b=0}^{q} c_{a,b} \tau^{\rho(a,b)} x_1^a x_2^b,$$

where $c_{a,b} \in \mathbb{C}^*$ are arbitrary, $\rho(a, b) \in \mathbb{R}$ satisfy a suitable convexity condition, and $\tau \ll 1$. The corresponding tropical polynomial

$$\varphi(\xi_1, \xi_2) = \max\{a\xi_1 + b\xi_2 - \rho(a, b) \,|\, 0 \leq a \leq p,\ 0 \leq b \leq q\} \tag{9.3}$$

defines a tropical curve $\Pi_0 \subset \mathbb{R}^2$; see Figure 1. We also denote by H', resp. H^0, the genus g curves with $p + q$ (resp. $2(p + q)$) punctures obtained by intersecting H with the affine subset $V' = \mathbb{C}^2 \subset V$, resp. $V^0 = (\mathbb{C}^*)^2$.

The polytope $\Delta_Y = \{(\xi_1, \xi_2, \eta) \,|\, \eta \geq \varphi(\xi_1, \xi_2)\}$ has $(p + 1)(q + 1)$ facets, corresponding to the regions where a particular term in (9.3) realizes the maximum. Thus the 3-fold Y has $(p + 1)(q + 1)$ irreducible toric divisors $D_{a,b}$ ($0 \leq a \leq p$, $0 \leq b \leq q$) (we label each divisor by the weight of the dominant monomial). The moment polytopes for these divisors are exactly the components of $\mathbb{R}^2 \setminus \Pi_0$, and the tropical curve Π_0 depicts the moment map images of the codimension 2 strata where they intersect (a configuration of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s); see Figure 3 left (and compare with Figure 1 right).

The leading-order superpotential W_0 of Definition 3.10 consists of five terms: $w_0 = -T^\epsilon + T^\epsilon v_0$, where v_0 is the toric monomial of weight $(0, 0, 1)$, which vanishes with multiplicity 1 on each of the toric divisors $D_{a,b}$; and four terms $w_1, \dots, w_4$ corresponding to the facets of Δ_V. Up to constant factors, w_1 is the toric monomial with weight $(-1, 0, 0)$, which vanishes with multiplicity a on $D_{a,b}$; w_2 is the toric monomial with weight $(0, -1, 0)$, vanishing with multiplicity b on $D_{a,b}$; w_3 is the monomial with weight $(1, 0, p)$, with multiplicity $(p - a)$ on $D_{a,b}$; and w_4 is the monomial with weight $(0, 1, q)$, with multiplicity $(q - b)$ on $D_{a,b}$ (compare Example 3.12).

Our main results for the open curve $H^0 \subset V^0 = (\mathbb{C}^*)^2$ are the following:

1. the complement Y^0 of $w_0^{-1}(0) \simeq (\mathbb{C}^*)^2$ in the toric 3-fold Y is SYZ mirror to the conic bundle $X^0 = \{(x_1, x_2, y, z) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2 \,|\, yz = f(x_1, x_2)\}$;
2. the B-side Landau-Ginzburg model (Y^0, w_0) is SYZ mirror to the blowup of $(\mathbb{C}^*)^2 \times \mathbb{C}$ along $H^0 \times 0$;
3. the B-side Landau-Ginzburg model $(Y, -v_0)$ is a generalized SYZ mirror to the open genus g curve H^0.

The B-side Landau-Ginzburg models (Y^0, w_0) and $(Y, -v_0)$ have regular fibers isomorphic to $(\mathbb{C}^*)^2$, while the singular fiber $w_0^{-1}(-T^\epsilon) = v_0^{-1}(0)$ is the union of all the toric divisors $D_{a,b}$. In particular, the singular fiber consists of $(p+1)(q+1)$ toric surfaces intersecting pairwise along a configuration of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s (the 1-dimensional strata of Y), themselves intersecting at triple points (the 0-dimensional strata of Y); the combinatorial structure of the trivalent configuration of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s is exactly given by the tropical curve Π_0. (See Figure 3 left).

If we partially compactify to $V' = \mathbb{C}^2$, then we get:

(2') the B-side Landau-Ginzburg model $(Y^0, w_0 + w_1 + w_2)$ is SYZ mirror to the blowup of $\mathbb{C}^3$ along $H' \times 0$;

(3') the B-side Landau-Ginzburg model $(Y, -v_0 + w_1 + w_2)$ is mirror to H'.

Adding $w_1 + w_2$ to the superpotential results in a partial smoothing of the singular fiber; namely, the singular fiber is now the union of the toric surfaces $D_{a,b}$ where $a > 0$ and $b > 0$ (over which $w_1 + w_2$ vanishes identically) and a single noncompact surface $S' \subset Y$, which can be thought of as a smoothing (or partial smoothing) of $S'_0 = (\bigcup_a D_{a,0}) \cup (\bigcup_b D_{0,b})$.

By an easy calculation in the toric affine charts of Y, the critical locus of $W_{H'} = -v_0 + w_1 + w_2$ (i.e. the pairwise intersections of components of $W_{H'}^{-1}(0)$ and the possible self-intersections of S') is again a union of $\mathbb{P}^1$'s and $\mathbb{A}^1$'s meeting at triple points; the combinatorics of this configuration is obtained from the planar graph Π_0 (which describes the critical locus of $W_{H^0} = -v_0$) by deleting all the unbounded edges in the directions of $(-1, 0)$ and $(0, -1)$, then inductively collapsing the bounded edges that connect to univalent vertices and merging the edges that meet at bivalent vertices (see Figure 3 middle); this construction can be understood as a sequence of "tropical modifications" applied to the tropical curve Π_0.

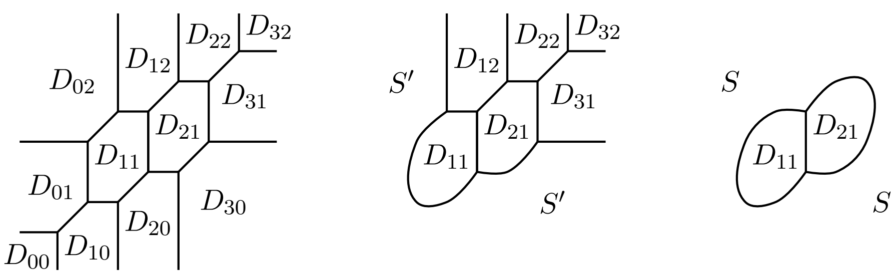


Figure 3: The singular fibers of the mirrors to $H^0 = H \cap (\mathbb{C}^*)^2$ (left) and $H' = H \cap \mathbb{C}^2$ (middle), and of the leading-order terms of the mirror to H (right). Here H is a genus 2 curve of bidegree $(3, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

The closed genus g curve H does not satisfy Assumption 1.4, so our main results do not apply to it. However, it is instructive to consider the leading-order mirrors (Y^0, W_0) to the blowup X of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{C}$ along $H \times 0$ and (Y, W_0^H) to the curve H itself. Indeed, in this case the additional instanton corrections (i.e., virtual counts of configurations that include exceptional rational curves in $\tilde{V}$) are expected to only have a mild effect on the mirror: specifically, they should not affect the topology of the critical locus, but merely deform it in a way that can be accounted for by corrections to the mirror map. We will return to this question in a forthcoming chapter.

The zero set of the leading-order superpotential $W_0^H = -v_0 + w_1 + w_2 + w_3 + w_4$ is the union of the compact toric surfaces $D_{a,b}$, $0 < a < p$, $0 < b < q$, with a single noncompact surface $S \subset Y$, which can be thought of as a smoothing (or partial smoothing) of the union S_0 of the noncompact toric divisors of Y. (There may also be new critical points which do not lie over 0; we shall not discuss them.)

Here again, an easy calculation in the toric affine charts shows that the singular locus of $(W_0^H)^{-1}(0)$ (i.e., the pairwise intersections of components and the possible self-intersections of S) forms a configuration of $3g-3$ $\mathbb{P}^1$'s meeting at triple points. Combinatorially, this configuration is obtained from the planar graph Π_0 by deleting all the unbounded edges, then inductively collapsing the bounded edges that connect to univalent vertices and merging the edges that meet at bivalent vertices (see Figure 3 right); this can be understood as a sequence of tropical modifications turning Π_0 into a closed genus g tropical curve (i.e., a trivalent graph without unbounded edges).

(The situation is slightly different when $p = q = 2$ and $g = 1$: in this case $(W_0^H)^{-1}(0) = D_{1,1} \cup S$, and the critical locus $D_{1,1} \cap S$ is a smooth elliptic curve. In this case, the higher instanton corrections are easy to analyze, and simply amount to rescaling the first term $-v_0$ of the superpotential by a multiplicative factor which encodes certain genus 0 Gromov-Witten invariants of $\mathbb{P}^1 \times \mathbb{P}^1$.)

10 Generalizations

In this section we mention (without details) a couple of straightforward generalizations of our construction.

10.1 Non-maximal degenerations

In our main construction we have assumed that the hypersurface $H \subset V$ is part of a maximally degenerating family $(H_\tau)_{\tau\to 0}$ (see Definition 3.1). This was used for two purposes: (1) to ensure that, for each weight $\alpha \in A$, there exists a connected component of $\mathbb{R}^n \setminus \mathrm{Log}(H)$ over which the corresponding monomial in the defining equation (3.1) dominates all other terms, and (2) to ensure that the toric variety Y associated to the polytope (3.8) is smooth.

(Note that the regularity of $\mathcal{P}$ also ensures the smoothness of H throughout, and of H'_σ in the discussion before Lemma 5.7; without the regularity assumption, smoothness can still be achieved by making generic choices of the coefficients c_α in (3.1).)

In general, removing the assumption of maximal degeneration, some of the terms in the tropical polynomial

$$\varphi(\xi) = \max\{\langle \alpha, \xi\rangle - \rho(\alpha) \,|\, \alpha \in A\}$$

may not achieve the maximum under any circumstances; denote by A_{red} the set of those weights which do achieve the maximum for some value of ξ. Equivalently, those are exactly the vertices of the polyhedral decomposition $\mathcal{P}$ of $\mathrm{Conv}(A)$ induced by the function $\rho : A \to \mathbb{R}$. Then the elements of $A \setminus A_{red}$ do not give rise to connected components of the complement of the tropical curve, nor to facets of Δ_Y, and should be discarded altogether. Thus, the main difference with the maximal degeneration case is that the rays of the fan Σ_Y are the vectors $(-\alpha, 1)$ for $\alpha \in A_{red}$, and the toric variety Y is usually singular.

Indeed, the construction of the Lagrangian torus fibration $\pi : X^0 \to B$ proceeds as in §4, and the arguments in Sections 4 to 6 remain valid, the only difference being that only the weights $\alpha \in A_{red}$ give rise to chambers U_α of tautologically unobstructed fibers of π, and hence to affine coordinate charts $U_\alpha^\vee$ for the SYZ mirror Y^0 of X^0. Replacing A by A_{red} throughout the arguments addresses this issue.

The smooth mirrors obtained from maximal degenerations are crepant resolutions of the singular mirrors obtained from non-maximal ones. Starting from a non-maximal polyhedral decomposition $\mathcal{P}$, the various ways in which it can be refined to a regular decomposition correspond to different choices of resolution. We give a few examples.

Example 10.1 Revisiting the example of the A_k-Milnor fiber considered in §9.2, we now consider the case where the roots of the polynomial f_{k+1} satisfy $|x_1| = \cdots = |x_{k+1}|$, for example $f_{k+1}(x) = x^{k+1} - 1$, which gives

$$X' = \{(x, y, z) \in \mathbb{C}^3 \,|\, yz = x^{k+1} - 1\}.$$

Then the tropical polynomial $\varphi : \mathbb{R} \to \mathbb{R}$ is $\varphi(\xi) = \max(0, (k+1)\xi)$, and the polytope $\Delta_Y = \{(\xi, \eta) \in \mathbb{R}^2 \,|\, \eta \geq \varphi(\xi)\}$ determines the singular toric variety $\{st = u^{k+1}\} \subset \mathbb{K}^3$, i.e. the A_k singularity, rather than its resolution as previously.

Geometrically, the Lagrangian torus fibration π normally consists of $k+2$ chambers, depending on how many of the roots of f_{k+1} lie inside the projection of the fiber to the x coordinate plane. In the case considered here, all the walls lie at $|x| = 1$, and the fibration π only consists of two chambers ($|x| < 1$ and $|x| > 1$).

In fact, $\mathbb{Z}/(k+1)$ acts freely on $X_k^0 = \{(x, y, z) \in \mathbb{C}^* \times \mathbb{C}^2 \,|\, yz = x^{k+1} - 1\}$, making it an unramified cover of $X_0^0 = \{(\hat{x}, y, z) \in \mathbb{C}^* \times \mathbb{C}^2 \,|\, yz = \hat{x} - 1\} \simeq \mathbb{C}^2 \setminus \{yz = -1\}$ via the map $(x, y, z) \mapsto (x^{k+1}, y, z)$. The Lagrangian tori we consider on X_k^0 are simply the preimages of the SYZ fibration on X_0^0, which results in the mirror being the quotient of the mirror of X_0^0 (namely, $\{(\hat{s}, \hat{t}, u) \in \mathbb{K}^3 \,|\, \hat{s}\hat{t} = u,\ u \neq 1\}$) by a $\mathbb{Z}/(k+1)$-action (namely $\zeta \cdot (\hat{s}, \hat{t}, u) = (\zeta\hat{s}, \zeta^{-1}\hat{t}, u)$). As expected, the quotient is nothing other than $Y_k^0 = \{(s, t, u) \in \mathbb{K}^3 \,|\, st = u^{k+1},\ u \neq 1\}$ (via the map $(\hat{s}, \hat{t}, u) \mapsto (\hat{s}^{k+1}, \hat{t}^{k+1}, u)$).

Example 10.2 The higher-dimensional analogue of the previous example is that of Fermat hypersurfaces in $(\mathbb{C}^*)^n$ or in $\mathbb{CP}^n$. Let H be the Fermat hypersurface in $\mathbb{CP}^n$ given by the equation $\sum X_i^d = 0$ in homogeneous coordinates, i.e. $x_1^d + \cdots + x_n^d + 1 = 0$ in affine coordinates, and let X be the blowup of $\mathbb{CP}^n \times \mathbb{C}$ at $H \times 0$. In this case, the open Calabi-Yau manifold X^0 is

$$X^0 = \{(x_1, \ldots, x_n, y, z) \in (\mathbb{C}^*)^n \times \mathbb{C}^2 \,|\, yz = x_1^d + \cdots + x_n^d + 1\}.$$

The tropical polynomial corresponding to H is $\varphi(\xi_1, \ldots, \xi_n) = \max(d\xi_1, \ldots, d\xi_n, 0)$, which is highly degenerate. Thus the toric variety Y associated to the polytope Δ_Y given by (3.8) is singular, in fact it can be described as

$$Y = \{(z_1, \ldots, z_{n+1}, v) \in \mathbb{K}^{n+2} \,|\, z_1 \ldots z_{n+1} = v^d\},$$

which can be viewed as the quotient of $\mathbb{K}^{n+1}$ by the diagonal action of $(\mathbb{Z}/d)^n$ (multiplying all coordinates by roots of unity but preserving their product), via the map $(\tilde{z}_1, \ldots, \tilde{z}_{n+1}) \mapsto (\tilde{z}_1^d, \ldots, \tilde{z}_{n+1}^d, \tilde{z}_1 \ldots \tilde{z}_{n+1})$. As in the previous example, this is consistent with the observation that X^0 is a $(\mathbb{Z}/d)^n$-fold cover of the conic bundle considered in §9.1, where $(\mathbb{Z}/d)^n$ acts diagonally by multiplication on the coordinates $x_1, \ldots, x_n$.

(As usual, considering a maximally degenerating family of hypersurfaces of degree d instead of a Fermat hypersurface would yield a crepant resolution of Y.)

By Theorem 1.6, the affine Fermat hypersurface $H^0 = H \cap (\mathbb{C}^*)^n$ is mirror to the singular B-side Landau-Ginzburg model $(Y, W_0^H = -v)$ or, in other terms, the quotient of $(\mathbb{K}^{n+1}, \tilde{W}_0^H = -\tilde{z}_1 \dots \tilde{z}_{n+1})$ by the action of $(\mathbb{Z}/d)^n$, which is consistent with [221].

Furthermore, by Remark 7.11 the theorem also applies to projective Fermat hypersurfaces of degree $d < n$ in $\mathbb{CP}^n$. Setting $a = \frac{1}{n+1}\int_{\mathbb{CP}^1} \omega_{\mathbb{CP}^n}$, and placing the barycenter of the moment polytope of $\mathbb{CP}^n$ at the origin, we find that

$$\big(Y, W_0^H = -v + T^a(z_1 + \cdots + z_{n+1})\big)$$

is mirror to H (for $d < n$; otherwise this is only the leading-order approximation to the mirror). Equivalently, this can be viewed as the quotient of

$$\Big(\mathbb{K}^{n+1}, \tilde{W}_0^H = -\tilde{z}_1 \dots \tilde{z}_{n+1} + T^a(\tilde{z}_1^d + \cdots + \tilde{z}_{n+1}^d)\Big)$$

by the action of $(\mathbb{Z}/d)^n$, which is again consistent with Sheridan's work.

Example 10.3 We now revisit the example considered in §9.3, where we found the mirrors of nearly tropical plane curves of bidegree (p, q) to be smooth toric 3-folds (equipped with suitable superpotentials) whose topology is determined by the combinatorics of the corresponding tropical plane curve Π_0 (or dually, of the regular triangulation $\mathcal{P}$ of the rectangle $[0, p] \times [0, q]$).

A particularly simple way to modify the combinatorics is to "flip" a pair of adjacent triangles of $\mathcal{P}$ whose union is a unit parallelogram; this affects the toric 3-fold Y by a flip. This operation can be implemented by a continuous deformation of the tropical curve Π_0 in which the length of a bounded edge shrinks to zero, creating a four-valent vertex, which is then resolved by creating a bounded edge in the other direction and increasing its length. The intermediate situation where Π_0 has a 4-valent vertex corresponds to a non-maximal degeneration where $\mathcal{P}$ is no longer a maximal triangulation of $[0, p] \times [0, q]$, instead containing a single parallelogram of unit area; the mirror toric variety Y then acquires an ordinary double point singularity. The two manners in which the four-valent vertex of the tropical curve can be deformed to a pair of trivalent vertices connected by a bounded edge then amount to the two small resolutions of the ordinary double point, and differ by a flip.

10.2 Hypersurfaces in abelian varieties As suggested to us by Paul Seidel, the methods we use to study hypersurfaces in toric varieties can also be applied to the case of hypersurfaces in abelian varieties. For simplicity, we only discuss the case of abelian varieties V which can be viewed as quotients of $(\mathbb{C}^*)^n$ (with its standard Kähler form) by the action of a real lattice $\Gamma_B \subset \mathbb{R}^n$, where $\gamma \in \Gamma_B$ acts by $(x_1, \dots, x_n) \mapsto (e^{\gamma_1}x_1, \dots, e^{\gamma_n}x_n)$. In other terms, the logarithm map identifies V with the product $T_B \times T_F$ of two real Lagrangian tori, the "base" $T_B = \mathbb{R}^n/\Gamma_B$ and the "fiber" $T_F = i\mathbb{R}^n/(2\pi\mathbb{Z})^n$ (which corresponds to the orbit of a T^n-action).

Since the T^n-action on V is not Hamiltonian, there is no globally defined $\mathbb{R}^n$-valued moment map. However, there is an analogous map which takes values in a real torus, namely the quotient of $\mathbb{R}^n$ by the lattice spanned by the periods of ω_V on $H_1(T_B) \times H_1(T_F)$; due to our choice of the standard Kähler form on $(\mathbb{C}^*)^n$, this period lattice is simply Γ_B, and the "moment map" is the logarithm map projecting from V to the real torus $T_B = \mathbb{R}^n/\Gamma_B$.

A tropical hypersurface $\Pi_0 \subset T_B$ can be thought of as the image of a Γ_B-periodic tropical hypersurface $\tilde{\Pi}_0 \subset \mathbb{R}^n$ under the natural projection $\mathbb{R}^n \to \mathbb{R}^n/\Gamma_B = T_B$. Such a tropical hypersurface occurs naturally as the limit of the amoebas (moment map images) of a degenerating family of hypersurfaces H_τ inside the degenerating family of abelian varieties V_τ ($\tau \to 0$) corresponding to rescaling the lattice Γ_B by a factor of $|\log \tau|$. (We keep the Kähler class $[\omega_V]$ and its period lattice Γ_B constant by rescaling the Kähler form of $(\mathbb{C}^*)^n$ by an appropriate factor, so that the moment map is given by the base τ logarithm map, $\mu_V = \mathrm{Log}_\tau : V_\tau \to T_B$.) As in §3 we call $H_\tau \subset V_\tau$ "nearly tropical" if its amoeba $\Pi_\tau = \mathrm{Log}_\tau(H_\tau) \subset T_B$ is contained in a tubular neighborhood of the tropical hypersurface Π_0; we place ourselves in the nearly tropical setting, and elide τ from the notation.

Concretely, the hypersurface H is defined by a section of a line bundle $\mathcal{L} \to V$ whose pullback to $(\mathbb{C}^*)^n$ is trivial; $\mathcal{L}$ can be viewed as the quotient of $(\mathbb{C}^*)^n \times \mathbb{C}$ by Γ_B, where $\gamma \in \Gamma_B$ acts by

$$\gamma_\# : (x_1, \dots, x_n, v) \mapsto (\tau^{-\gamma_1} x_1, \dots, \tau^{-\gamma_n} x_n, \tau^{\kappa(\gamma)} \mathbf{x}^{\lambda(\gamma)} v), \tag{10.1}$$

where $\lambda \in \mathcal{H}om(\Gamma_B, \mathbb{Z}^n)$ is a homomorphism determined by the Chern class $c_1(\mathcal{L})$ (observe that $\mathcal{H}om(\Gamma_B, \mathbb{Z}^n) \simeq H^1(T_B, \mathbb{Z}) \otimes H^1(T_F, \mathbb{Z}) \subset H^2(V, \mathbb{Z})$), and $\kappa : \Gamma_B \to \mathbb{R}$ satisfies a cocycle-type condition in order to make (10.1) a group action. A basis of sections of $\mathcal{L}$ is given by the theta functions

$$\vartheta_\alpha(x_1, \dots, x_n) = \sum_{\gamma \in \Gamma_B} \gamma_\#^*(\mathbf{x}^\alpha), \qquad \alpha \in \mathbb{Z}^n/\lambda(\Gamma_B). \tag{10.2}$$

(Note: for $\gamma \in \Gamma_B$, ϑ_α and $\vartheta_{\alpha+\lambda(\gamma)}$ actually differ by a constant factor.) The defining section f of H is a finite linear combination of these theta functions; equivalently, its lift to $(\mathbb{C}^*)^n$ can be viewed as an infinite Laurent series of the form (3.1), invariant under the action (10.1) (which forces the set of weights A to be $\lambda(\Gamma_B)$-periodic.) We note that the corresponding tropical function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is also Γ_B-equivariant, in the sense that $\varphi(\xi + \gamma) = \varphi(\xi) + \langle \lambda(\gamma), \xi \rangle - \kappa(\gamma)$ for all $\gamma \in \Gamma_B$.

Let X be the blowup of $V \times \mathbb{C}$ along $H \times 0$, equipped with an S^1-invariant Kähler form ω_ϵ such that the fibers of the exceptional divisor have area $\epsilon > 0$ (chosen sufficiently small). Denote by $\tilde{V}$ the proper transform of $V \times 0$, and let $X^0 = X \setminus \tilde{V}$. Then X^0 carries an S^1-invariant Lagrangian torus fibration $\pi : X^0 \to B = T_B \times \mathbb{R}_+$, constructed as in §4 by assembling fibrations on the reduced spaces of the S^1-action. This allows us to determine SYZ mirrors to X^0 and X as in §5 and §6.

The construction can be understood either directly at the level of X and X^0, or by viewing the whole process as a Γ_B-equivariant construction on the cover $\tilde{X}$, namely the blowup of $(\mathbb{C}^*)^n \times \mathbb{C}$ along $\tilde{H} \times 0$, where $\tilde{H}$ is the preimage of H under the covering map $q : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n/\Gamma_B = V$. The latter viewpoint makes it easier to see that the enumerative geometry arguments from the toric case extend to this setting.

As in the toric case, each weight $\bar{\alpha} \in \bar{A} := A/\lambda(\Gamma_B)$ determines a connected component of the complement $T_B \setminus \Pi_0$ of the tropical hypersurface Π_0, and hence a chamber $U_{\bar{\alpha}} \subset B^{reg} \subset B$ over which the fibers of π are tautologically unobstructed. Each of these determines an affine coordinate chart $U_{\bar{\alpha}}^\vee$ for the SYZ mirror of X^0, and these charts are glued to each other via coordinate transformations of the form (3.11).

Alternatively, we can think of the mirror as a quotient by Γ_B of a space built from an infinite collection of charts $U_\alpha^\vee$, $\alpha \in A$, where each chart $U_\alpha^\vee$ has coordinates $(v_{\alpha,1}, \dots, v_{\alpha,n}, w_0)$, glued together by (3.11). Specifically, for each element $\gamma = (\gamma_1, \dots, \gamma_n) \in \Gamma_B$, we identify $U_\alpha^\vee$ with $U_{\alpha+\lambda(\gamma)}^\vee$ via the map

$$\gamma_\#^\vee : (v_{\alpha,1}, \dots, v_{\alpha,n}, w_0) \in U_\alpha^\vee \mapsto (T^{\gamma_1} v_{\alpha,1}, \dots, T^{\gamma_n} v_{\alpha,n}, w_0) \in U_{\alpha+\lambda(\gamma)}^\vee, \tag{10.3}$$

where the multiplicative factors T^{γ_i} account for the amount of symplectic area separating the different lifts to $\tilde{X}$ of a given fiber of π.

Setting $v_0 = 1 + T^{-\epsilon} w_0$, we can again view the SYZ mirror Y^0 of X^0 as the complement of the hypersurface $w_0^{-1}(0) = v_0^{-1}(1)$ in a "locally toric" variety Y covered (outside of codimension 2 strata) by local coordinate charts $Y_\alpha = (\mathbb{K}^*)^n \times \mathbb{K}$ ($\alpha \in A$) glued together by (3.9) and identified under the action of Γ_B. Namely, for all $\alpha, \beta \in A$ and $\gamma \in \Gamma_B$ we make the identifications

$$(v_1, \dots, v_n, v_0) \in Y_\alpha \sim (v_0^{\alpha_1 - \beta_1} v_1, \dots, v_0^{\alpha_n - \beta_n} v_n, v_0) \in Y_\beta, \tag{10.4}$$

$$(v_1, \dots, v_n, v_0) \in Y_\alpha \sim (T^{\gamma_1} v_1, \dots, T^{\gamma_n} v_n, v_0) \in Y_{\alpha + \lambda(\gamma)}. \tag{10.5}$$

Finally, the abelian variety V is aspherical, and any holomorphic disc bounded by $\pi^{-1}(b)$, $b \in B^{reg}$ must be entirely contained in a fiber of the projection to V, so that the only contribution to the superpotential is w_0 (as in the case of hypersurfaces in $(\mathbb{C}^*)^n$). With this understood, our main results become:

Theorem 10.4 *Let H be a nearly tropical hypersurface in an abelian variety V, let X be the blowup of $V \times \mathbb{C}$ along $H \times 0$, and let Y be as above. Then:*

1. *$Y^0 = Y \setminus w_0^{-1}(0)$ is SYZ mirror to $X^0 = X \setminus \tilde{V}$;*
2. *the B-side Landau-Ginzburg model (Y^0, w_0) is SYZ mirror to X;*
3. *the B-side Landau-Ginzburg model $(Y, -v_0)$ is generalised SYZ mirror to H.*

Note that, unlike Theorems 1.5 and 1.6, this result holds without any restrictions: when V is an abelian variety, Assumption 1.4 always holds and there are never any higher-order instanton corrections. On the other hand, the statement of part (3) implicitly uses the properties of Fukaya categories of Landau-Ginzburg models whose proofs are sketched in Section 7 (whereas parts (1) and (2) rely only on familiar versions of the Fukaya category).

The smooth fibers of $-v_0 : Y \to \mathbb{K}$ (or equivalently up to a reparametrization, $w_0 : Y^0 \to \mathbb{K}^*$) are all abelian varieties, in fact quotients of $(\mathbb{K}^*)^n$ (with coordinates $\mathbf{v} = (v_1, \dots, v_n)$) by the identification

$$\mathbf{v}^m \sim v_0^{\langle \lambda(\gamma), m \rangle} T^{\langle \gamma, m \rangle} \mathbf{v}^m \qquad \text{for all } m \in \mathbb{Z}^n \text{ and } \gamma \in \Gamma_B,$$

while the singular fiber is a union of toric varieties

$$v_0^{-1}(0) = \bigcup_{\bar{\alpha} \in \bar{A}} D_{\bar{\alpha}}$$

glued (to each other or to themselves) along toric strata. The moment polytopes for the toric varieties $D_{\bar{\alpha}}$ are exactly the components of $T_B \setminus \Pi_0$, and the tropical hypersurface Π_0 depicts the moment map images of the codimension 2 strata of Y along which they intersect.

Example 10.5 When H is a set of n points on an elliptic curve V, we find that the fibers of $-v_0 : Y \to \mathbb{K}$ are a family of elliptic curves, all smooth except $v_0^{-1}(0)$ which is a union of n $\mathbb{P}^1$'s forming a cycle (in the terminology of elliptic fibrations, this is known as an I_n fiber). In this case the superpotential $-v_0$ has n isolated critical points, all lying in the fiber over zero, as expected.

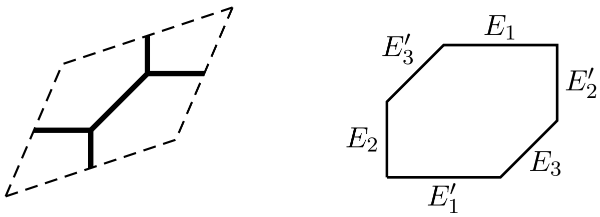


Figure 4: A tropical genus 2 curve on the 2-torus (left); the singular fiber of the mirror Landau-Ginzburg model is the quotient of the toric Del Pezzo surface shown (right) by identifying $E_i \sim E_i'$.

Example 10.6 Now consider the case where H is a genus 2 curve embedded in an abelian surface V (for example its Jacobian torus). The tropical genus 2 curve Π_0 is a trivalent graph on the 2-torus T_B with two vertices and three edges, see Figure 4 left. Since $T_B \setminus \Pi_0$ is connected, the singular fiber $v_0^{-1}(0)$ of the mirror B-side Landau-Ginzburg model is irreducible. Specifically, it is obtained from the toric Del Pezzo surface shown in Figure 4 right, i.e. $\mathbb{P}^2$ blown up in 3 points, by identifying each exceptional curve E_i with the "opposite" exceptional curve E_i' (the proper transform of the line through the two other points). Thus the critical locus of the superpotential is a configuration of three rational curves $E_1 = E_1'$, $E_2 = E_2'$, $E_3 = E_3'$ intersecting at two triple points. (Compare with §9.3: the mirrors are very different, but the critical loci are the same).

11 Complete intersections

In this section we explain (without details) how to extend our main results to the case of complete intersections in toric varieties (under a suitable positivity assumption for rational curves, which always holds in the affine case).

11.1 Notations and statement of the results

Let $H_1, \dots, H_d$ be smooth nearly tropical hypersurfaces in a toric variety V of dimension n, in general position. We denote by f_i the defining equation of H_i, a section of a line bundle $\mathcal{L}_i$ which can be written as a Laurent polynomial (3.1) in affine coordinates $\mathbf{x} = (x_1, \dots, x_n)$; by $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ the corresponding tropical polynomial; and by $\Pi_i \subset \mathbb{R}^n$ the tropical hypersurface defined by φ_i. (To ensure smoothness of the mirror, it is useful to assume that the tropical hypersurfaces $\Pi_1, \dots, \Pi_d$ intersect transversely, though this assumption is actually not necessary).

We denote by X the blowup of $V \times \mathbb{C}^d$ along the d codimension 2 subvarieties $H_i \times \mathbb{C}_i^{d-1}$, where $\mathbb{C}_i^{d-1} = \{y_i = 0\}$ is the i-th coordinate hyperplane in $\mathbb{C}^d$. (The blowup is smooth since the subvarieties $H_i \times \mathbb{C}_i^{d-1}$ intersect transversely). Explicitly, X can be a described as a smooth submanifold of the total space of the $(\mathbb{P}^1)^d$-bundle $\prod_{i=1}^d \mathbb{P}(\mathcal{L}_i \oplus \mathcal{O})$ over $V \times \mathbb{C}^d$,

$$X = \{(\mathbf{x}, y_1, \ldots, y_d, (u_1 : v_1), \ldots, (u_d : v_d)) \,|\, f_i(\mathbf{x})v_i = y_i u_i \ \forall i = 1, \ldots, d\}. \tag{11.1}$$

Outside of the union of the hypersurfaces H_i, the fibers of the projection $p_V : X \to V$ obtained by composing the blowup map $p : X \to V \times \mathbb{C}^d$ with projection to the first factor are isomorphic to $\mathbb{C}^d$; above a point which belongs to k of the H_i, the fiber consists of 2^k components, each of which is a product of $\mathbb{C}$'s and $\mathbb{P}^1$'s.

The action of $T^d = (S^1)^d$ on $V \times \mathbb{C}^d$ by rotation on the last d coordinates lifts to X; we equip X with a T^d-invariant Kähler form for which the exceptional $\mathbb{P}^1$ fibers of the i-th exceptional divisor have area ϵ_i (where $\epsilon_i > 0$ is chosen small enough). As in §3.2, we arrange for the Kähler form on X to coincide with that on $V \times \mathbb{C}^d$ away from the exceptional divisors. We denote by $\mu_X : X \to \mathbb{R}^d$ the moment map.

The dense open subset $X^0 \subset X$ over which we can construct an SYZ fibration is the complement of the proper transforms of the toric strata of $V \times \mathbb{C}^d$; it can be viewed as an iterated conic bundle over the open stratum $V^0 \simeq (\mathbb{C}^*)^n \subset V$, namely

$$X^0 \simeq \{(\mathbf{x}, y_1, \ldots, y_d, z_1, \ldots, z_d) \in V^0 \times \mathbb{C}^{2d} \,|\, y_i z_i = f_i(\mathbf{x}) \ \forall i = 1, \ldots, d\}. \tag{11.2}$$

Consider the polytope $\Delta_Y \subseteq \mathbb{R}^{n+d}$ defined by

$$\Delta_Y = \{(\xi, \eta_1, \ldots, \eta_d) \in \mathbb{R}^n \oplus \mathbb{R}^d \,|\, \eta_i \geq \varphi(\xi_i) \ \forall i = 1, \ldots, d\}, \tag{11.3}$$

and let Y be the corresponding toric variety. For $i = 1, \ldots, d$, denote by $v_{0,i}$ the monomial with weight $(0, \ldots, 0, 1, \ldots, 0)$ (the $(n+i)$-th entry is 1), and set

$$w_{0,i} = -T^{\epsilon_i} + T^{\epsilon_i} v_{0,i}. \tag{11.4}$$

Denote by A the set of connected components of $\mathbb{R}^n \setminus (\Pi_1 \cup \cdots \cup \Pi_d)$, and index each component by the tuple of weights $\vec{\alpha} = (\alpha^1, \ldots, \alpha^d) \in \mathbb{Z}^{n \times d}$ corresponding to the dominant monomials of $\varphi_1, \ldots, \varphi_d$ in that component. Then for each $\vec{\alpha} \in A$ we have a coordinate chart $Y_{\vec{\alpha}} \simeq (\mathbb{K}^*)^n \times \mathbb{K}^d$ with coordinates $\mathbf{v}_{\vec{\alpha}} = (v_{\vec{\alpha},1}, \ldots, v_{\vec{\alpha},n}) \in (\mathbb{K}^*)^n$ and $(v_{0,1}, \ldots, v_{0,d}) \in \mathbb{K}^d$, where the monomial $\mathbf{v}_{\vec{\alpha}}^m = v_{\vec{\alpha},1}^{m_1} \ldots v_{\vec{\alpha},n}^{m_n}$ is the toric monomial with weight $(-m_1, \ldots, -m_n, \langle \alpha^1, m \rangle, \ldots, \langle \alpha^d, m \rangle) \in \mathbb{Z}^{n+d}$. These charts glue via

$$\mathbf{v}_{\vec{\alpha}}^m = \left(\prod_{i=1}^d (1 + T^{-\epsilon_i} w_{0,i})^{\langle \beta^i - \alpha^i, m \rangle} \right) \mathbf{v}_{\vec{\beta}}^m. \tag{11.5}$$

Denoting by $\sigma_1, \ldots, \sigma_r \in \mathbb{Z}^n$ the primitive generators of the rays of the fan Σ_V, and writing the moment polytope of V in the form (3.12), for $j = 1, \ldots, r$ we define

$$w_j = T^{\varpi_j} \mathbf{v}_{\vec{\alpha}_{min}(\sigma_j)}^{\sigma_j}, \tag{11.6}$$

where $\vec{\alpha}_{min}(\sigma_j) \in A$ is chosen so that all $\langle \sigma_j, \alpha^i \rangle$ are minimal. In other terms, $\mathbf{v}_{\vec{\alpha}_{min}(\sigma_j)}^{\sigma_j}$ is the toric monomial with weight $(-\sigma_j, \lambda_1(\sigma_j), \ldots, \lambda_d(\sigma_j)) \in \mathbb{Z}^{n+d}$, where $\lambda_1, \ldots, \lambda_d : \Sigma_V \to \mathbb{R}$ are the piecewise linear functions defining $\mathcal{L}_i = \mathcal{O}(H_i)$.

Finally, define Y^0 to be the subset of Y where $w_{0,1}, \dots, w_{0,d}$ are all non-zero, and define the leading-order superpotentials

$$(11.7) \qquad W_0 = w_{0,1} + \cdots + w_{0,d} + w_1 + \cdots + w_r = \sum_{i=1}^{d}(-T^{\epsilon_i} + T^{\epsilon_i} v_{0,i}) \; + \; \sum_{i=1}^{r} T^{\varpi_j} \mathbf{v}^{\sigma_j}_{\vec{\alpha}_{min}(\sigma_j)},$$

$$(11.8) \qquad W_0^H = -v_{0,1} - \cdots - v_{0,d} + w_1 + \cdots + w_r = -\sum_{i=1}^{d} v_{0,i} \; + \; \sum_{i=1}^{r} T^{\varpi_j} \mathbf{v}^{\sigma_j}_{\vec{\alpha}_{min}(\sigma_j)}.$$

With this understood, the analogue of Theorems 1.5–1.7 is the following

Theorem 11.1 *With the above notations:*

1. *Y^0 is SYZ mirror to the iterated conic bundle X^0;*
2. *assuming that all rational curves in X have positive Chern number (e.g. when V is affine), the B-side Landau-Ginzburg model (Y^0, W_0) is SYZ mirror to X;*
3. *assuming that V is affine, the B-side Landau-Ginzburg model (Y, W_0^H) is a generalized SYZ mirror to the complete intersection $H_1 \cap \cdots \cap H_d \subset V$.*

As in Theorem 10.4, part (3) of this theorem relies on the expected properties of Fukaya categories of Landau-Ginzburg models.

Remark 11.2 Denoting by X_i the blowup of $V \times \mathbb{C}$ at $H_i \times 0$ and by X_i^0 the corresponding conic bundle over V^0, the space X (resp. X^0) is the fiber product of $X_1, \dots, X_d$ (resp. $X_1^0, \dots, X_d^0$) with respect to the natural projections to V. This perspective explains many of the geometric features of the construction.

11.2 Sketch of proof The argument proceeds along the same lines as for the case of hypersurfaces, of which it is really a straightforward adaptation. We outline the key steps for the reader's convenience.

As in §4, a key observation to be made about the T^d-action on X is that the reduced spaces $X_{red,\lambda} = \mu_X^{-1}(\lambda)/T^d$ ($\lambda \in \mathbb{R}^d_{\geq 0}$) are all isomorphic to V via the projection p_V (though the Kähler forms may differ near $H_1 \cup \cdots \cup H_d$). This allows us to build a (singular) Lagrangian torus fibration

$$\pi : X^0 \to B = \mathbb{R}^n \times (\mathbb{R}_+)^d$$

(where the second component is the moment map) by assembling standard Lagrangian torus fibrations on the reduced spaces. The singular fibers of π correspond to the points of X^0 where the T^d-action is not free; therefore

$$B^{sing} = \bigcup_{i=1}^{d} \Pi_i' \times \{(\lambda_1, \dots, \lambda_d) \,|\, \lambda_i = \epsilon_i\},$$

where $\Pi'_i \subset \mathbb{R}^n$ is essentially the amoeba of H_i. The potentially obstructed fibers of $\pi : X^0 \to B$ are precisely those that intersect $p_V^{-1}(H_1 \cup \cdots \cup H_d)$, and for each $\vec{\alpha} \in A$ we have an open subset $U_{\vec{\alpha}} \subset B$ of tautologically unobstructed fibers which project under p to standard product tori in $V^0 \times \mathbb{C}^d$.

Each of the components $U_{\vec{\alpha}} \subset B$ determines an affine coordinate chart $U^\vee_{\vec{\alpha}}$ in the SYZ mirror to X^0. Namely, for $b \in U_{\vec{\alpha}} \subset B$, the Lagrangian torus $L = \pi^{-1}(b) \subset X^0$ is the preimage by p of a standard product torus in $V \times \mathbb{C}^d$. Denoting by $(\zeta_1, \dots, \zeta_n, \lambda_1, \dots, \lambda_d) \in \Delta_V \times \mathbb{R}^d_+$ the corresponding value of the moment map of $V \times \mathbb{C}^d$, and by $(\gamma_1, \dots, \gamma_n, \gamma_{0,1}, \dots, \gamma_{0,d})$ the natural basis of $H_1(L, \mathbb{Z})$, we equip $U^\vee_{\vec{\alpha}}$ with the coordinate system

$$(11.9) \quad (L, \nabla) \mapsto (v_{\vec{\alpha},1}, \dots, v_{\vec{\alpha},n}, w_{0,1}, \dots, w_{0,d}) \\ := \Big(T^{\zeta_1}\nabla(\gamma_1), \dots, T^{\zeta_n}\nabla(\gamma_n), T^{\lambda_1}\nabla(\gamma_{0,1}), \dots, T^{\lambda_d}\nabla(\gamma_{0,d})\Big).$$

For $b \in U_{\vec{\alpha}}$, the Maslov index 2 holomorphic discs bounded by $L = \pi^{-1}(b)$ in X can be determined explicitly as in §5, by projecting to $V \times \mathbb{C}^d$. Specifically, these discs intersect the proper transform of exactly one of the toric divisors transversely in a single point, and there are two cases:

Lemma 11.3 *For any $i = 1, \dots, d$, L bounds a unique family of Maslov index 2 holomorphic discs in X which intersect the proper transform of $V \times \mathbb{C}^{d-1}_i = \{y_i = 0\}$ transversely in a single point; the images of these discs under p are contained in lines parallel to the y_i coordinate axis, and their contribution to the superpotential is $w_{0,i}$.*

Lemma 11.4 *For any $j = 1, \dots, r$, denote by D_{σ_j} the toric divisor in V associated to the ray σ_j of the fan Σ_V, and let $k_i = \langle \alpha^i - \alpha^i_{min}(\sigma_j), \sigma_j \rangle$ ($i = 1, \dots, d$). Then L bounds $2^{k_1+\cdots+k_d}$ families of Maslov index 2 holomorphic discs in X which intersect the proper transform of $D_{\sigma_j} \times \mathbb{C}^d$ transversely in a single point (all of which have the same projections to V), and their total contribution to the superpotential is*

$$\left(\prod_{i=1}^{d} (1 + T^{-\epsilon_i} w_{0,i})^{k_i}\right) T^{\varpi_i} \mathbf{v}_{\vec{\alpha}}^{\sigma_j}.$$

The proofs are essentially identical to those of Lemmas 5.5 and 5.6, and left to the reader. As in §5, the first lemma implies that the coordinates $w_{0,i}$ agree on all charts $U^\vee_{\vec{\alpha}}$, and the second one implies that the coordinates $v_{\vec{\alpha},i}$ transform according to (11.5). The first two statements in Theorem 11.1 follow.

The last statement in the theorem follows from equipping X with the superpotential $W^\vee = y_1 + \cdots + y_d : X \to \mathbb{C}$, which has Morse-Bott singularities along the intersection of the proper transform of $V \times 0$ with the d exceptional divisors, i.e. $\mathrm{crit}(W^\vee) \simeq H_1 \cap \cdots \cap H_d$. As in §7, the nontriviality of the normal bundle forces us to twist the Fukaya category of $(X, W^\vee)$ by a background class $s \in H^2(X, \mathbb{Z}/2)$, in this case Poincaré dual to the sum of the exceptional divisors (or equivalently to the sum of the proper transforms of the toric divisors $V \times \mathbb{C}^{d-1}_i$). The thimble construction then provides a fully faithful A_∞-functor from $\mathcal{F}(H_1 \cap \cdots \cap H_d)$ to $\mathcal{F}_s(X, W^\vee)$. The twisting affects the superpotential by changing the signs of the terms $w_{0,1}, \dots, w_{0,d}$. Moreover, the thimble functor modifies the value of the superpotential by an additive constant, which equals $T^{\epsilon_1} + \cdots + T^{\epsilon_d}$ when V is affine (the i-th term corresponds to a family of small discs of area ϵ_i in the normal direction to H_i). Putting everything together, the result follows by a straightforward adaptation of the arguments in §7.

Appendix A. Moduli of objects in the Fukaya category

A.1 General theory

Let L be an embedded spin Lagrangian of vanishing Maslov class in the Kähler manifold $X^0 = X \setminus D$, where D is an anticanonical divisor which satisfies Assumption 2.2. We begin with a brief overview of the results of [80], which in part implement the constructions of [81] in the setting of de Rham cohomology.

For each positive real number E, Fukaya defines a *curved* A_∞ structure on the de Rham cochains with coefficients in Λ_0/T^E, which we denote by

$$\Omega^*(L; \Lambda_0/T^E\Lambda_0) \equiv \Omega^*(L; \mathbb{R}) \otimes_{\mathbb{R}} \Lambda_0/T^E\Lambda_0.$$

The operations are obtained from the moduli space of holomorphic discs in $X^0 = X \setminus D$ with boundary on L, whose energy is bounded by E. By induction, one obtains an unbounded sequence of real numbers E_i, together with formal diffeomorphisms on $\Omega^*(L; \Lambda_0/T^{E_i}\Lambda_0)$ which pull back the A_∞ structure constructed from discs of energy bounded by E_i to the projection of the A_∞ structure on $\Omega^*(L; \Lambda_0/T^{E_{i+1}}\Lambda_0)$ modulo T^{E_i}. After applying such a formal diffeomorphism, we may therefore assume that the A_∞ map

$$\Omega^*(L; \Lambda_0/T^{E_{i+1}}\Lambda_0) \to \Omega^*(L; \Lambda_0/T^{E_i}\Lambda_0)$$

is defined by projection of coefficient rings. Taking the inverse limit over E_i, we obtain an A_∞ structure on $\Omega^*(L; \Lambda_0)$. By passing to the canonical model (i.e. applying a filtered version of the homological perturbation lemma [133]), we can reduce this A_∞ structure to $H^*(L; \Lambda_0)$.

Fukaya checks that any class $b \in H^1(L; U_\Lambda)$ defines a deformed A_∞ structure on the cohomology. In particular, there is a subset

$$\hat{\mathcal{Y}}_L \subset H^1(L; U_\Lambda)$$

consisting of elements for which this A_∞ structure has vanishing curvature (i.e. solutions to the Maurer-Cartan equation). Gauge transformations [81, Section 4.3] define an equivalence relation on this set; we call the quotient the *moduli space of simple objects supported on L,* which we denote $\mathcal{Y}_L$.

Remark A.1 The original formalism of Fukaya, Oh, Ohta, and Ono [81] considered deformation classes corresponding to $b \in H^1(L; \Lambda_+)$, called bounding cochains, which via exponentiation $\Lambda_+ \to 1 + \Lambda_+$ can also be reinterpreted as local systems. As noted in the discussion following Theorem 1.2 of [80], there are inclusions $1 + \Lambda_+ \subset U_\Lambda \subset \Lambda^*$, and the original construction of Floer cohomology can be generalised to all unitary local systems using an idea of Cho.

The invariance statement of Floer cohomology [81, Theorem 14.1-14.3] asserts that $\mathcal{Y}_L$ does not depend on the choice of auxiliary data (e.g almost-complex structure) in the following sense: let $\mathcal{Y}_L^1$ and $\mathcal{Y}_L^2$ denote the moduli spaces for different choices of auxiliary structures. A homotopy between the auxiliary data induces an isomorphism

$$\mathcal{Y}_L^1 \cong \mathcal{Y}_L^2 \tag{12.1}$$

which is invariant under homotopies of homotopies.

Assumption A.2 *The A_∞ structure on $H^*(L;\Lambda_0)$ is isomorphic to the undeformed structure.*

Remark A.3 For most Lagrangians that we consider, this condition holds automatically because there is a choice of almost complex structure for which the Lagrangian bounds no holomorphic discs which are not constant.

In this setting, the Maurer-Cartan equation vanishes identically, and the gauge equivalence relation is trivial. A choice of isomorphism of the Floer-theoretic A_∞-structure with the undeformed structure (e.g. a choice of almost complex structure for which there are no non-constant holomorphic discs) therefore yields an identification of the moduli space $\mathcal{Y}_L$ of simple objects of the Fukaya category supported on L with its first cohomology with coefficients in U_Λ:

$$\mathcal{Y}_L \equiv H^1(L; U_\Lambda).$$

Let L_t be a Hamiltonian path of Lagrangians in X^0 with vanishing Maslov class, and J_t a family of almost complex structures on X which we assume are fixed at infinity. We describe the isomorphism (12.1) in the special situation which we consider in this chapter. We first identify $H_1(L_0;\mathbb{Z}) \cong H_1(L_t;\mathbb{Z})$ via the given path. A basis for this group yields an identification

$$(z_1, \dots, z_n) : H^1(L_0; U_\Lambda) \to U_\Lambda^n.$$

Assumption A.4 *For the family (L_t, J_t), all stable holomorphic discs represent multiples of a given relative homology class $\beta \in H_2(X, L_0; \mathbb{Z})$.*

The wall-crossing map is then of the form

$$z_i \mapsto h_i(z_\beta) z_i, \tag{12.2}$$

where h_i is a power series with $\mathbb{Q}$ coefficients and leading order term equal to 1, and z_β denotes the monomial $T^{\omega(\beta)} z^{[\partial\beta]}$. Equation (12.2) can be extracted from the construction in Section 11 of [80]. For an explicit derivation, see [241, Lemma 4.4]: for bounding cochains, the transformation corresponds to adding a power series in z_β with vanishing constant term, and Equation (12.2) follows by exponentiation.

By Proposition 5.8, the following assumption holds in the geometric setting of the main theorem:

Assumption A.5 *The power series h_i is the expansion of a rational function in z_β.*

In this case, the transformation in Equation (12.2) converges away from the zeroes and poles of h_i. This is stronger than the general result proved by Fukaya namely that the transformation converges in an analytic neighbourhood of the unitary elements in $H^1(L;\Lambda^*)$.

In order to extend this construction to the non-Hamiltonian setting, we use the main construction of [80] which identifies the moduli space of simple objects supported on Lagrangians near L (but not necessarily Hamiltonian isotopic to it) with an affinoid domain in $H^1(L;\Lambda^*)$ in the sense of Tate.

Given a path $\{L_t\}_{t\in[0,1]}$ between Lagrangians L_0 and L_1 in which there is no wall crossing (e.g. so that no Lagrangian in the family bounds a holomorphic disc), the natural gluing map between these domains is obtained from the flux homomorphism

$$\Phi(\{L_t\}) \in H^1(L_0;\mathbb{R})$$

and the product on cohomology groups

$$H^1(L_0;\mathbb{R}) \times H^1(L_0;\Lambda^*) \to H^1(L_0;\Lambda^*)$$

induced by the map on coefficients $(\lambda, f) \mapsto T^\lambda f$. In the absence of wall crossing we identify $H^1(L_1;\Lambda^*)$ with $H^1(L_0;\Lambda^*)$ via this rescaling map.

Given a general path between Lagrangians L_0 and L_1 (subject to Assumptions A.4 and A.5), this identification is modified by the wall crossing formula given in Equation (12.2), yielding a birational map

$$H^1(L_0;\Lambda^*) \dashrightarrow H^1(L_1;\Lambda^*),$$

defined away from a hypersurface. We glue the moduli spaces of objects supported near L_0 and L_1 using this identification.

Remark A.6 The construction of a map for a Lagrangian path can be reduced to the case of Hamiltonian paths as follows: any path (L_t, J) can be deformed, with fixed endpoints, to a path (L'_t, J_t) which is a concatenation of paths for which the Lagrangian is constant and paths in which there is no wall-crossing. The desired map is then obtained as a composition of the wall-crossing maps for Hamiltonian paths and the rescalings given by the flux homomorphism.

The idea for constructing the deformed path follows the main strategy for proving convergence in [80]. Whenever ϵ is sufficiently small, there is a (compactly supported) diffeomorphism ψ_ϵ taking L_t to $L_{t+\epsilon}$ which preserves the tameness of J. For tautological reasons, there is a path without wall-crossing from (L_t, J) to $(L_{t+\epsilon}, J_{t+\epsilon})$ if $J_{t+\epsilon}$ is the pullback of J by ψ_ϵ. Interpolating between this pullback and $(L_{t+\epsilon}, J)$, via pullbacks of (L_{t+s}, J), we then reach $(L_{t+\epsilon}, J)$ via a path for which the Lagrangian is constant and Assumption A.4 remains satisfied.

Remark A.7 (1) More generally, given a path from L_0 to L_1 that can be decomposed into finitely many sub-paths $\{L_t\}_{t\in[t_j,t_{j+1}]}$, each satisfying Assumption A.4 for some relative class β_j, and for which the wall-crossing transformations are rational functions as in Assumption A.5, we again obtain a wall-crossing map

$$H^1(L_0;\Lambda^*) \dashrightarrow H^1(L_1;\Lambda^*) \tag{12.3}$$

by composing the maps associated to the various sub-paths.

(2) When all the classes β_j have the same boundary in $H_1(L_t,\mathbb{Z})$ and the same symplectic areas, the monomials z_{β_j} are all equal and the birational transformation (12.3) again takes the form of Equation (12.2) up to rescaling of the coefficients.

If we restrict attention to the smooth fibers of a Lagrangian torus fibration, we obtain an embedding of the moduli space $\mathcal{Y}^0_\pi$ of all simple objects supported on such Lagrangians into the rigid analytic space

$$\coprod H^1(L;\Lambda^*)/\sim \tag{12.4}$$

where the equivalence relation identifies points which correspond to each other under the birational wall-crossing transformations of Equation (12.3) induced by all paths among smooth fibres. It does not automatically follow from the above considerations that this quotient is a well-behaved (e.g. separated) analytic space, but in our case this will not be an issue. By the invariance of Floer cohomology [81, Theorem 14.1-14.3], the transformations induced by homotopic paths are equal. The fact that these transformations should in general depend only on the homotopy class of the path in the space of all fibres (i.e. allowing fibres which are not necessarily embedded), is expected to follow as a consequence of forthcoming developments in the study of family Floer cohomology in the presence of singular fibres. In our main example, this independence will be manifest from Proposition 5.8, and the quotient (12.4) can easily be seen to be a smooth analytic space.

Remark A.8 We can think of (12.4) as the natural (analytic) completion of $\mathcal{Y}^0_\pi$. While the points of this completion do not necessarily correspond to unitary local systems on Lagrangians in X^0 with the given Kähler form, in good situations, they can be interpreted as Lagrangians in X^0 equipped with a completed Kähler form. Slightly strengthening Assumption 2.2 by requiring that X^0 be the complement of a nef divisor, we can obtain such a completion by inflation along the divisor at infinity.

It shall be convenient for our purposes to consider a completion which is obtained by gluing only finitely many charts. To this end, assume that $\{L_t\}_{t\in[0,1]}$ is a path of Lagrangians so that the wall-crossing map defines an embedding

$$H^1(L_0; U_\Lambda) \hookrightarrow H^1(L_1; \Lambda^*). \tag{12.5}$$

In this case, the above construction yields that all elements of $\mathcal{Y}_{L_0}$ can be represented in Equation (12.4) by elements of $H^1(L_1; \Lambda^*)$.

More generally, assume that $\{L_\alpha\}_{\alpha\in A}$ is a collection of fibers with the property that for some fixed almost complex structure J, any smooth fiber L can be connected to some fiber L_α in our collection by a path such that the wall-crossing map defines an embedding $H^1(L; U_\Lambda) \to H^1(L_\alpha; \Lambda^*)$. We define

$$\hat{\mathcal{Y}}^0_\pi \equiv \coprod_{\alpha\in A} H^1(L_\alpha; \Lambda^*)/\sim . \tag{12.6}$$

Lemma A.9 *There is a natural analytic embedding of $\mathcal{Y}^0_\pi$ into $\hat{\mathcal{Y}}^0_\pi$.* □

Next, we study the moduli spaces of holomorphic discs in X with boundary on a Lagrangian $L \subset X^0$ of vanishing Maslov class. Since D is an anticanonical divisor, stable holomorphic discs whose intersection number with D is 1 have Maslov index equal to 2. Assumption 2.2 implies that there are no discs of negative Maslov index, and that those of vanishing Maslov index are disjoint from D. For each unitary local system ∇ on L, choice of almost complex structure J, and action cutoff E we obtain a $\Lambda_0/T^E\Lambda_0$-valued de Rham cochain

$$\sum_{\substack{\beta\in\pi_2(X,L)\\ \beta\cdot D=1}} z_\beta(L,\nabla)\,\mathrm{ev}_*[\mathcal{M}_1(L,\beta,J)] \in \Omega^0(L;\Lambda_0/T^E\Lambda_0) \tag{12.7}$$

which is closed with respect to the Floer differential. Passing to the canonical model and to the inverse limit over E we obtain a multiple of the unit in the self-Floer cohomology of (L,∇):

$$\mathfrak{m}_0(L,\nabla,J) = W(L,\nabla,J)\, e_L \in H^0(L;\Lambda). \tag{12.8}$$

Since the moduli spaces of discs of vanishing Maslov index in X and in $X \setminus D$ agree, the invariance of Floer theory and in particular of the potential function [81, Theorem B], as extended to non-unitary local systems in [80], implies that $W(L, \nabla, J)$ gives rise to a well-defined convergent function on $\mathcal{Y}^0_\pi$. Because of this, we shall henceforth drop J from the notation. For non-unitary local systems, $W(L, \nabla)$ may not in general converge, so we have to impose this as an additional assumption. With this in mind, the proof of the following result follows from the unitary case by Remark A.6.

Lemma A.10 *If for each $\alpha \in A$, the map $\nabla \mapsto W(L_\alpha, \nabla)$ converges on $H^1(L_\alpha; \Lambda^*)$, then W defines a regular function on $\hat{\mathcal{Y}}^0_\pi$.* □

We record the following consequence:

Corollary A.11 *If (L_i, ∇_i) and (L_j, ∇_j) are identified by a wall-crossing gluing map, then $W(L_i, \nabla_i) = W(L_j, \nabla_j)$.* □

Remark A.12 Fukaya has announced that rank 1 unitary local systems on immersed Lagrangians which are fibers of π define a rigid analytic space which includes $\hat{\mathcal{Y}}^0_\pi$ as an analytic subset. The general idea is to describe the nearby smooth fibers as the result of Lagrangian surgery, and understand the behaviour of holomorphic discs under such surgeries sufficiently explicitly to produce an analytic structure on this neighbourhood which can be seen to be compatible with the analytic structure on $\hat{\mathcal{Y}}^0_\pi$.

We expect that, in the presence of a potential function, similar ideas can be applied to associate analytic charts to certain admissible non-compact Lagrangians arising as limits of smooth fibers. While we do not develop the general theory in this chapter, Example 2.4 explains how one can use equivalences in the Fukaya category (rather than surgery formulae) to produce the desired charts in the class of examples we encounter.

A.2 Convergence of the wall-crossing

In this section, we verify that the assumptions of Lemma A.9 hold for the smooth fibers of the map $\pi : X^0 \to B$ introduced in Definition 4.4. Recall that the moment map μ_X of the S^1-action descends to a natural map from B to $\mathbb{R}_+$; we write $X^0_\lambda = \mu_X^{-1}(\lambda) \cap X^0$. If ϵ is the blowup parameter in the definition of X, then all fibers of π contained in X^0_λ are smooth whenever $\lambda \neq \epsilon$; and the smooth fibers in X^0_ϵ are exactly those whose image under the blowdown map $p : X^0 \to V^0 \times \mathbb{C}$ is disjoint from $H \times \mathbb{C}$.

Assumption A.2 follows immediately from Proposition 5.1 for all fibers of π whose images under p are disjoint from $H \times \mathbb{C}$, since these bound no holomorphic discs. In general, invariance of Floer cohomology shows that Assumption A.2 is independent of the choice of almost complex structure. Moreover, the identification of the A_∞ structure obtained by deforming by an element in $H^1(L; \Lambda_+)$ with the deformed Floer theory for the associated local system in $H^1(L; 1 + \Lambda_+)$ implies that Assumption A.2 holds for the Floer theory of L equipped with unitary local systems as well, since an analytic function vanishing on $1 + \Lambda_+$ must vanish on all of U_Λ. The same argument shows that the A_∞ structure on L equipped with a non-unitary local system is also undeformed, as long as the valuation is sufficiently small. By Fukaya's work on Family Floer cohomology [80], we conclude that the A_∞ structure on a Lagrangian fibre L' sufficiently close to L is undeformed. Here, sufficiently close means that there is a diffeomorphism preserving the tameness of J and moving L to L'; in compact subsets of the space of smooth fibres, there are uniform bounds on the size of such neighbourhoods, so we conclude that the condition of having undeformed A_∞ structure is open and closed among smooth fibres of π. Therefore, all smooth fibres of π satisfy Assumption A.2.

We next choose Lagrangians $\{L_\alpha\}_{\alpha\in A}$, labelled by the monomials in the equation defining the hypersurface H. We require that L_α be contained in X^0_ϵ, and that its projection to B lie in the chamber $U_\alpha \subset B$ (see Definition 5.3).

Lemma A.13 *Any smooth fiber L of π can be connected to some fiber L_α so that the wall-crossing map defines an embedding*

$$H^1(L; U_\Lambda) \to H^1(L_\alpha; \Lambda^*). \tag{12.9}$$

Proof. There are two cases to consider:

Case 1: Assume that the smooth fiber L lies in X^0_ϵ. Then $\pi_\epsilon(L)$ lies outside of the amoeba of H (cf. Equation (4.4)) and L is tautologically unobstructed (cf. Proposition 5.1). By Remark 5.4, the component of the complement of the amoeba over which L lies determines a chamber U_α, and L can be connected to L_α by a path of tautologically unobstructed fibers. The absence of holomorphic discs in this region implies that there are no non-trivial walls, and hence that the map

$$H^1(L; \Lambda^*) \to H^1(L_\alpha; \Lambda^*) \tag{12.10}$$

is given simply by a rescaling of the coefficients (see the discussion following Equation (12.2)). This completes the argument in this case.

Case 2: Assume that L lies in X^0_λ, with $\lambda \neq \epsilon$. Choose a smooth fiber L^λ_α which is also contained in X^0_λ and whose projection lies in some chamber U_α, and consider the concatenation of a path from L to L^λ_α via Lagrangians contained in X^0_λ with a path from L^λ_α to L_α over the chamber U_α. Since the map associated to the latter path is a simple rescaling as in the previous case, it suffices to show convergence of the wall-crossing map for the path from L to L^λ_α.

To this end, recall from Proposition 5.1 that the simple holomorphic discs bounded by the Lagrangian torus fibers along the path all have the same area $|\lambda - \epsilon|$ and their boundaries all represent the same homology class in $H_1(L_t, \mathbb{Z})$. Thus, the monomials $z_\beta = T^{\omega(\beta)} z^{[\partial\beta]}$ associated to their homology classes are all equal, and by Remark A.7 (2) the wall crossing map is of the form

$$z_i \mapsto h_i(z_\beta) z_i, \tag{12.11}$$

where h_i is a power series with coefficients in $\mathbb{Q}$ and leading order term equal to 1. Whenever we evaluate at a point of $H^1(L; U_\Lambda)$, the valuation of z_β is $|\lambda - \epsilon| > 0$, and so $h_i(z_\beta)$ and its inverse both converge and take values in U_Λ. Thus the leading order term of (12.11) is identity, and the wall-crossing map defines an embedding

$$H^1(L; U_\Lambda) \hookrightarrow H^1(L^\lambda_\alpha; \Lambda^*).$$

Composing this map with the rescaling isomorphism induced by the flux homomorphism of a path over U_α, we arrive at the desired result. □

Appendix B. The geometry of the reduced spaces

In this section we study in more detail the symplectic geometry of the reduced spaces $X_{red,\lambda} = \mu_X^{-1}(\lambda)/S^1$ and prove Lemma 4.1.

Recall from §4.1 that the moment map for the S^1-action on X is given by (4.1), and that the only fixed points apart from $\tilde{V} = \mu_X^{-1}(0)$ occur along $\tilde{H}$, which lies in the level set $\mu_X^{-1}(\epsilon)$. Also recall that, for all $\lambda > 0$, the natural projection to V (obtained by composing $p : X \to V \times \mathbb{C}$ with projection to the first factor) yields a natural identification of $X_{red,\lambda}$ with V.

We will exploit the toric structure of V to construct families of Lagrangian tori in $X_{red,\lambda}$ equipped with the reduced Kähler form $\omega_{red,\lambda}$. The two obstacles are (1) the lack of smoothness along H at $\lambda = \epsilon$, and (2) the lack of T^n-invariance near H.

We start with the first issue, giving a formula for $\omega_{red,\lambda}$ near $\tilde{H}$ and introducing an explicit family of smoothings. Consider a small neighborhood of $\tilde{H}$ where, without loss of generality, we may assume that $\chi \equiv 1$.

Lemma B.1 *Identifying $X_{red,\lambda}$ with V as above, where $\chi \equiv 1$ we have*

$$\omega_{red,\lambda} = \omega_V - \max(0, \epsilon - \lambda)\, c_1(\mathcal{L}) + d\alpha_{0,\lambda}, \tag{13.1}$$

where $c_1(\mathcal{L}) = iF_{\mathcal{L}}/2\pi$ is the Chern form of the chosen Hermitian metric on $\mathcal{L}$, and

$$\alpha_{0,\lambda} = \frac{\min(\lambda, \epsilon)\, d^c(|f(\mathbf{x})|^2)}{2\left(\sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda - \epsilon + \pi|f(\mathbf{x})|^2)^2} + \pi|f(\mathbf{x})|^2 + |\lambda - \epsilon|\right)}. \tag{13.2}$$

Proof. Recall that away from $\tilde{V}$ we can write X as a conic bundle $f(\mathbf{x}) = yz$. Where $f \neq 0$ and $\chi \equiv 1$, the restriction of ω_ϵ to $\mu_X^{-1}(\lambda)$ is equal to

$$p_V^*\omega_V + d\left(\frac{1}{4}|y|^2 d^c(\log|y|^2) + \frac{\epsilon}{4\pi}\frac{|z|^2}{1+|z|^2} d^c(\log|z|^2)\right).$$

Since $d^c \log|y|^2 + d^c \log|z|^2 = d^c \log|f|^2$, using (4.2) we can rewrite the 1-form in this expression as either

$$\frac{1}{4}|y|^2 d^c(\log|f|^2) + \frac{\epsilon - \lambda}{4\pi} d^c(\log|z|^2) \quad \text{or} \quad \frac{\epsilon}{4\pi}\frac{|z|^2}{1+|z|^2} d^c(\log|f|^2) + \frac{\lambda - \epsilon}{4\pi} d^c(\log|y|^2).$$

Now $dd^c \log|y|^2 = 0$, whereas $dd^c \log|z|^2 = -4\pi p_V^* c_1(\mathcal{L})$, so we find that (still where $f \neq 0$ and $\chi \equiv 1$)

$$\begin{aligned}(\omega_\epsilon)_{|\mu_X^{-1}(\lambda)} &= p_V^*\big(\omega_V + (\lambda - \epsilon)c_1(\mathcal{L})\big) + d\left(\frac{d^c(|f(\mathbf{x})|^2)}{4|z|^2}\right) \\ &= p_V^*\omega_V + d\left(\frac{\epsilon}{4\pi}\frac{d^c(|f(\mathbf{x})|^2)}{|y|^2 + |f(\mathbf{x})|^2}\right).\end{aligned} \tag{13.3}$$

The first expression makes sense wherever $z \neq 0$, in particular for $\lambda < \epsilon$; the second one makes sense wherever $y \neq 0$, in particular for $\lambda > \epsilon$. Solving (4.2) for $|y|$, we obtain

$$\begin{aligned} 2\pi|y|^2 &= \sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda - \epsilon + \pi|f(\mathbf{x})|^2)^2} - \pi|f(\mathbf{x})|^2 + (\lambda - \epsilon), \\ \text{and} \quad 2\lambda|z|^2 &= \sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda - \epsilon + \pi|f(\mathbf{x})|^2)^2} + \pi|f(\mathbf{x})|^2 - (\lambda - \epsilon). \end{aligned}$$

Substituting into (13.3) gives the desired expression. $\square$

We can smooth the singularity of $\omega_{red,\lambda}$ by considering the modified Kähler forms given near H by

$$\omega_{sm,\lambda} = \omega_V - \max(0, \epsilon - \lambda)\, c_1(\mathcal{L}) + d\alpha_{\kappa,\lambda}$$

where $\kappa > 0$ is an arbitrarily small constant, and

$$\alpha_{t,\lambda} = \frac{\min(\lambda,\epsilon)\, d^c(|f(\mathbf{x})|^2)}{2\left(\sqrt{4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda - \epsilon + \pi|f(\mathbf{x})|^2)^2 + t^2\tilde{\chi}} + \pi|f(\mathbf{x})|^2 + |\lambda - \epsilon|\right)}, \tag{13.4}$$

where $\tilde{\chi} = \tilde{\chi}(|f(\mathbf{x})|, \lambda)$ is a suitable cut-off function which equals 1 near $\tilde{H}$ and vanishes outside of the region where $\chi \equiv 1$. (We can also assume that $\tilde{\chi}$ vanishes whenever λ is not close to ϵ.) We set $\omega_{sm,\lambda} = \omega_{red,\lambda}$ wherever $\chi \neq 1$. Choosing κ small enough ensures that $\omega_V - \max(0, \epsilon - \lambda)\, c_1(\mathcal{L}) + d\alpha_{t,\lambda}$ is non-degenerate for all $t \in [0, \kappa]$; it is then a Kähler form, because $\alpha_{t,\lambda}$ can be written as d^c of some function of $|f(\mathbf{x})|$.

The Kähler forms $\omega_{sm,\lambda}$ are all smooth, coincide with $\omega_{red,\lambda}$ away from H for all λ, and everywhere when λ is not very close to ϵ. Moreover, $[\omega_{sm,\lambda}] = [\omega_{red,\lambda}]$ by construction, and the dependence of $\omega_{sm,\lambda}$ on λ is piecewise smooth.

Like $\omega_{red,\lambda}$, the Kähler form $\omega_{sm,\lambda}$ is not invariant under the given torus action, but there exist toric Kähler forms in the same cohomology class. Such a Kähler form $\omega'_{V,\lambda}$ can be constructed by averaging $\omega_{sm,\lambda}$ with respect to the standard T^n-action on V:

$$\omega'_{V,\lambda} = \frac{1}{(2\pi)^n} \int_{g \in T^n} g^* \omega_{sm,\lambda}\, dg. \tag{13.5}$$

To see that the T^n-orbits are Lagrangian with respect to $\omega'_{V,\lambda}$, we note that the pullback of $\omega_{sm,\lambda}$ to an orbit represents the trivial cohomology class, since the classes $[\omega_V]$ and $[H]$ are both trivial on a torus fibre. The pullback of $\omega'_{V,\lambda}$ is therefore also trivial in cohomology, but since it is invariant, it must vanish pointwise. This in turn implies that the T^n-action not only preserves $\omega'_{V,\lambda}$ but in fact it is Hamiltonian.

We now state again Lemma 4.1 and give its proof:

Lemma B.2 *There exists a family of homeomorphisms $(\phi_\lambda)_{\lambda \in \mathbb{R}_+}$ of V such that:*

1. *ϕ_λ preserves the toric divisor $D_V \subset V$;*
2. *the restriction of ϕ_λ to V^0 is a diffeomorphism for $\lambda \neq \epsilon$, and a diffeomorphism outside of H for $\lambda = \epsilon$;*
3. *ϕ_λ intertwines the reduced Kähler form $\omega_{red,\lambda}$ and the toric Kähler form $\omega'_{V,\lambda}$;*
4. *$\phi_\lambda = \mathrm{id}$ at every point whose T^n-orbit is disjoint from the support of χ;*
5. *ϕ_λ depends on λ in a continuous manner, and smoothly except at $\lambda = \epsilon$.*

Proof. We proceed in two stages, obtaining ϕ_λ as the composition of two maps $\phi_{sm,\lambda}$, taking $\omega_{red,\lambda}$ to $\omega_{sm,\lambda}$, and $\phi_{avg,\lambda}$ taking $\omega_{sm,\lambda}$ to $\omega'_{V,\lambda}$, each satisfying all the other conditions in the statement. The arguments are quite similar in both cases; we start with the construction of $\phi_{avg,\lambda}$ (Steps 1–2), then proceed with $\phi_{sm,\lambda}$ (Steps 3–4).

Step 1. Let $\beta_\lambda = \omega_{sm,\lambda} - \omega'_{V,\lambda}$. Since $\omega'_{V,\lambda}$ is T^n-invariant, for $\theta \in \mathfrak{t}^n \simeq \mathbb{R}^n$ we have

$$\begin{aligned}\exp(\theta)^*\omega_{sm,\lambda} - \omega_{sm,\lambda} = \exp(\theta)^*\beta_\lambda - \beta_\lambda &= \int_0^1 \frac{d}{dt}\left(\exp(t\theta)^*\beta_\lambda\right)\,dt \\ &= d\left[\int_0^1 \exp(t\theta)^*\left(\iota_{\theta_\#}\beta_\lambda\right)\,dt\right].\end{aligned}$$

Hence, averaging over all elements of T^n, we see that the 1-form

$$a'_\lambda = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n}\int_0^1 \exp(t\theta)^*\left(\iota_{\theta_\#}\beta_\lambda\right)\,dt\,d\theta$$

satisfies $\omega'_{V,\lambda} - \omega_{sm,\lambda} = da'_\lambda$ (i.e., $da'_\lambda = -\beta_\lambda$).

Let $U \subset V$ be the orbit of the support of χ under the standard T^n-action on $X_{red,\lambda} \cong V$. Outside of U, the Kähler forms $\omega_{sm,\lambda} = \omega_{red,\lambda}$ are T^n-invariant, and $\omega_{sm,\lambda}$ and $\omega'_{V,\lambda}$ coincide (in fact they both coincide with ω_V). Therefore, β_λ is supported in U, and consequently so is a'_λ.

Let $\omega'_{t,\lambda} = t\omega'_{V,\lambda} + (1-t)\omega_{sm,\lambda}$ (for $t \in [0,1]$ these are Kähler forms since $\omega'_{V,\lambda}$ and $\omega_{sm,\lambda}$ are Kähler). Denote by v_t the vector field such that $\iota_{v_t}\omega'_{t,\lambda} = -a'_\lambda$ and by ψ_t the flow generated by v_t. Then by Moser's trick,

$$\frac{d}{dt}(\psi_t^*\omega'_{t,\lambda}) = \psi_t^*\left(L_{v_t}\omega'_{t,\lambda} + \frac{d\omega'_{t,\lambda}}{dt}\right) = \psi_t^*(d\iota_{v_t}\omega'_{t,\lambda} + da'_\lambda) = 0,$$

so $\psi_t^*\omega'_{t,\lambda} = \omega_{sm,\lambda}$, and the time 1 flow ψ_1 intertwines $\omega_{sm,\lambda}$ and $\omega'_{V,\lambda}$ as desired. Moreover, because a'_λ is supported in U, outside of U we have $\psi_t = \mathrm{id}$. However, it is not clear that the flow preserves the toric divisors of V.

Step 2. To remedy the issue with the flow not preserving the toric divisors, we modify a'_λ in a neighborhood of D_V. Let $f'_{\lambda,t}$ be a family of C^1 real-valued functions (with locally Lipschitz first derivatives), smooth on V^0, with the following properties:

- the support of $f'_{\lambda,t}$ is contained in the intersection of U with a small tubular neighborhood of D_V;
- at every point $x \in D_V$, belonging to a toric stratum $S \subset V$,

  (13.6) $\qquad$ the 1-form $a'_\lambda + df'_{\lambda,t}$ vanishes on $(T_xS)^\perp$,

  where the orthogonal is with respect to $\omega'_{t,\lambda}$;
- $f'_{\lambda,t}$ depends smoothly on t, and piecewise smoothly on λ.

We construct $f'_{\lambda,t}$ by induction over toric strata of increasing dimension, successively constructing functions $f'_{\lambda,t,\le k} : V \to \mathbb{R}$ which satisfy (13.6) for all strata of dimension at most k and are smooth outside of strata of dimension $< k$. We start by setting $f'_{\lambda,t,\le 0} = 0$, which satisfies (13.6) at the fixed points of the torus action since they lie away from the support of a'_λ.

Assume $f'_{\lambda,t,\leq k}$ constructed, and consider a stratum S of dimension $k+1$. At each point $x \in S$, the restriction of $a'_\lambda + df'_{\lambda,t,\leq k}$ to $(T_xS)^\perp$ is a real-valued linear form, vanishing whenever x belongs to a lower-dimensional stratum, and smooth outside of strata of dimension $< k$. Let $f'^0_{\lambda,t,S}$ be a C^1 function on a neighborhood of S, smooth outside of the strata of dimension $\leq k$, which vanishes on S and whose derivative in the normal directions at each point of S satisfies $(df'^0_{\lambda,t,S})_{|(T_xS)^\perp} = -(a'_\lambda + df'_{\lambda,t,\leq k})_{|(T_xS)^\perp}$. (For instance, identify a neighborhood of S with a subset of its normal bundle in a manner compatible with the toric structure, and take $f'^0_{\lambda,t,S}$ to be linear in the fibers).

Let χ_S be a cut-off function with values in $[0,1]$, defined and smooth outside of the strata of dimension $\leq k$, equal to 1 at all points of a neighborhood of S which are much closer to S than to any other $(k+1)$-dimensional stratum, and with support disjoint from those of the corresponding cut-off functions for all other $(k+1)$-dimensional strata. Specifically, picking an auxiliary metric, we take χ_S to be the product of a standard smooth cut-off function supported in a tubular neighborhood of S with functions $\chi_{S/\Sigma}$ for all strata Σ with $\dim \Sigma \geq k+1$ and $\dim(\Sigma \cap S) \leq k$, chosen so that $\chi_{S/\Sigma}$ equals 1 except near Σ, where it depends on the ratio between distance to S and distance to Σ, equals 1 at all points that lie much closer to S than to Σ, and vanishes at all points that lie closer to Σ than to S.

We note that near a lower-dimensional stratum S', the norm of $d\chi_S$ is bounded by a constant over distance to S'. We then set $f'_{\lambda,t,S} = \chi_S f'^0_{\lambda,t,S}$. By construction, this function is smooth away from strata of dimension $\leq k$. Moreover, near a lower-dimensional stratum S', $f'^0_{\lambda,t,S}$ is bounded by a constant multiple of distance to S times distance to S', so the regularity of $f'_{\lambda,t,S}$ is indeed as desired.

By construction, $f'_{\lambda,t,\leq k+1} = f'_{\lambda,t,\leq k} + \sum_{\dim S = k+1} f'_{\lambda,t,S}$ has the desired properties on all strata of dimension $\leq k+1$. (Note that, since a'_λ vanishes outside of U, so do the various functions we construct.) Finally, we let $f'_{\lambda,t} = f'_{\lambda,t,\leq n-1}$.

We now use Moser's trick again, replacing a'_λ by $\tilde{a}'_{t,\lambda} = a'_\lambda + df'_{\lambda,t}$. Namely, denote by $\tilde{v}_{t,\lambda}$ the vector field such that $\iota_{\tilde{v}_{t,\lambda}} \omega'_{t,\lambda} = -\tilde{a}_{t,\lambda}$. This vector field is locally Lipschitz continuous along D_V, and smooth on V^0; moreover, by construction it is supported in U and, by (13.6), tangent to each stratum of D_V. We thus obtain $\phi_{avg,\lambda}$ with all the desired properties by considering the time 1 flow generated by $\tilde{v}_{t,\lambda}$. (Note: because we have assumed that ω_V defines a complete Kähler metric on V, it is easy to check that even when V is noncompact the time 1 flow is well-defined.)

Step 3. We now turn to the construction of $\phi_{sm,\lambda}$. We interpolate between $\omega_{red,\lambda}$ and $\omega_{sm,\lambda}$ via the family of Kähler forms $\omega_{t,\lambda}$, $t \in [0,\kappa]$, defined by

$$\omega_{t,\lambda} = \omega_V - \max(0, \epsilon - \lambda)\, c_1(\mathcal{L}) + d\alpha_{t,\lambda}$$

where $\chi \equiv 1$ (where $\alpha_{t,\lambda}$ is given by (13.4)) and $\omega_{t,\lambda} = \omega_{red,\lambda}$ wherever $\chi \neq 1$.

These Kähler forms are smooth whenever $t > 0$ or $\lambda \neq \epsilon$. Let $a_{t,\lambda}$ be the 1-form with support contained in the region where $\chi \equiv 1$, and defined by $a_{t,\lambda} = d\alpha_{t,\lambda}/dt$ inside that region. By construction, $d\omega_{t,\lambda}/dt = da_{t,\lambda}$. We use Moser's trick again, and denote by $v_{t,\lambda}$ the vector field such that $\iota_{v_{t,\lambda}}\omega_{t,\lambda} = -a_{t,\lambda}$. This vector field vanishes outside of U, and is smooth except for $t = 0$ and $\lambda = \epsilon$, in which case it is singular along H. We will momentarily check that the flow of $v_{t,\lambda}$ is well-defined even for $\lambda = \epsilon$; the time κ flow then intertwines $\omega_{red,\lambda}$ and $\omega_{sm,\lambda}$ as desired, except it need not preserve the toric divisors of V, an issue which we will address in Step 4 below.

Differentiating (13.4) with respect to t, we have

$$a_{t,\lambda} = \frac{t\,\tilde{\chi}\min(\lambda,\epsilon)\, d^c(|f(\mathbf{x})|^2)}{2\sqrt{\Phi}\big(\sqrt{\Phi} + \pi|f(\mathbf{x})|^2 + |\lambda - \epsilon|\big)^2}, \tag{13.7}$$

where

$$\Phi = 4\pi\epsilon|f(\mathbf{x})|^2 + (\lambda - \epsilon + \pi|f(\mathbf{x})|^2)^2 + t^2\tilde{\chi}. \tag{13.8}$$

Taking the dual vector field, we find that

$$v_{t,\lambda} = \frac{t\,\tilde{\chi}\min(\lambda,\epsilon)\,\nabla^{t,\lambda}(|f(\mathbf{x})|^2)}{2\sqrt{\Phi}\big(\sqrt{\Phi} + \pi|f(\mathbf{x})|^2 + |\lambda-\epsilon|\big)^2}, \tag{13.9}$$

where $\nabla^{t,\lambda}$ is the gradient with respect to the Kähler metric determined by $\omega_{t,\lambda}$.

We restrict our attention to the neighborhood of $\tilde{H}$ where $\tilde{\chi} \equiv 1$, since it is clear that $v_{t,\lambda}$ is well-defined and smooth everywhere else. To estimate the norm of $\nabla^{t,\lambda}(|f(\mathbf{x})|^2)$, we differentiate (13.4) to find that, in this region,

$$d\alpha_{t,\lambda} = \frac{2\min(\lambda,\epsilon)\left(\pi(\epsilon+\lambda)|f|^2 + (\lambda-\epsilon)^2 + t^2 + |\lambda-\epsilon|\sqrt{\Phi}\right) d|f| \wedge d^c|f|}{\sqrt{\Phi}\big(\sqrt{\Phi} + \pi|f|^2 + |\lambda-\epsilon|\big)^2} - \frac{2\pi\min(\lambda,\epsilon)|f|^2\,c_1(\mathcal{L})}{\big(\sqrt{\Phi} + \pi|f|^2 + |\lambda-\epsilon|\big)}. \tag{13.10}$$

(Here we have used the fact that $dd^c|f|^2 = -4\pi|f|^2c_1(\mathcal{L}) + 4d|f| \wedge d^c|f|$.)

When $\lambda - \epsilon$ and $|f(\mathbf{x})|^2$ are much smaller than ϵ, we have $\Phi \sim 4\pi\epsilon|f|^2 + (\lambda-\epsilon)^2 + t^2$. Estimating the various terms in (13.10), we find that the second term tends to zero near H, while the leading-order part of the coefficient of $d|f| \wedge d^c|f|$ is bounded from below by $\epsilon/\sqrt{\Phi}$ (and from above by $4\epsilon/\sqrt{\Phi}$). Hence:

$$d\alpha_{t,\lambda} \gtrsim \frac{\epsilon}{\sqrt{\Phi}}\, d|f| \wedge d^c|f|. \tag{13.11}$$

(where $\gtrsim$ means that the inequality holds up to lower-order terms.) In more geometric terms, the Kähler metrics induced by $\omega_{t,\lambda}$ blow up in the normal direction to H, by an amount of the order of $\epsilon/\sqrt{\Phi}$, while remaining well-behaved in the other directions.

This implies in turn that the norms of $d(|f(\mathbf{x})|^2)$ and $\nabla^{t,\lambda}(|f(\mathbf{x})|^2)$ with respect to the Kähler metric $\omega_{t,\lambda}$ are bounded by $2(\sqrt{\Phi}/\epsilon)^{1/2}|f(\mathbf{x})|$; and, more importantly, the norm of $\nabla^{t,\lambda}(|f(\mathbf{x})|^2)$ with respect to a suitable fixed auxiliary metric is locally bounded by a constant multiple of $(\sqrt{\Phi}/\epsilon)|f(\mathbf{x})|$. Plugging into (13.9), we conclude that the norm of $v_{t,\lambda}$ (again with respect to a smooth auxiliary metric) is bounded by a constant multiple of $t|f(\mathbf{x})|/\Phi \leq t|f(\mathbf{x})|/(t^2 + 4\pi\epsilon|f(\mathbf{x})|^2)$, and hence uniformly bounded. Thus, even though $v_{t,\lambda}$ itself is not continuous at $(t,\lambda,|f(\mathbf{x})|) = (0,\epsilon,0)$, its flow is well-defined and continuous even for $\lambda = \epsilon$, and depends continuously on λ.

Geometrically, for $\lambda - \epsilon$ sufficiently small, near H the leading-order term in $v_{t,\lambda}$ points radially away from H, in the same direction as the gradient of $|f(\mathbf{x})|$ with respect to ω_V, and the time t flow rescales the radial coordinate $r = |f(\mathbf{x})|$ in a suitable manner. A complicated explicit formula for the leading-order term of the rescaling can be obtained by comparing the Kähler areas of small discs in the direction normal to H; for example, for $\lambda = \epsilon$ one finds that the time t flow maps points where $|f(\mathbf{x})| = r_0$ to points where $|f(\mathbf{x})|^2 \approx \frac{1}{2}r_0(r_0 + (r_0^2 + \frac{1}{\pi\epsilon}t^2)^{1/2})$.

Step 4. We now modify the flow constructed in Step 3 in order to arrange for the toric divisors of V to be preserved. We proceed as in Step 2, i.e. we replace the 1-forms $a_{t,\lambda}$ used in Step 3 with $a_{t,\lambda} + df_{t,\lambda}$ for carefully constructed real-valued functions $f_{t,\lambda}$, smooth on V^0 except for $(t, \lambda) = (0, \epsilon)$, such that:

- the support of $f_{t,\lambda}$ is contained in the intersection of U with a small tubular neighborhood of D_V;
- at every point $x \in D_V$, belonging to a toric stratum $S \subset V$,

  (13.12) $$\text{the 1-form } a_{t,\lambda} + df_{t,\lambda} \text{ vanishes on } (T_x S)^{\perp},$$

  where the orthogonal is with respect to $\omega_{t,\lambda}$;
- where it is smooth, $f_{t,\lambda}$ depends smoothly on t, and piecewise smoothly on λ.

We construct $f_{t,\lambda}$ inductively to satisfy (13.12) on toric strata of increasing dimension, by exactly the same method as in Step 2. The main new difficulty is that we need to control the behavior of $f_{t,\lambda}$ near H for (t, λ) close to $(0, \epsilon)$.

We begin with a geometric digression. Fix a collection of smooth foliations $\mathcal{F}_S$ of neighborhoods of $H \cap S$ in V for all toric strata $S \subset V$, with the following properties:

- each leaf of $\mathcal{F}_S$ intersects S transversely at a single point;
- $|f|$ is constant on the leaves; in particular the leaves through $H \cap S$ are contained in H;
- given two strata $S' \subset S$, the leaves of $\mathcal{F}_{S'}$ are unions of leaves of $\mathcal{F}_S$.
- given two strata S and Σ which intersect transversely along a stratum $S' = S \cap \Sigma$, the leaves of $\mathcal{F}_S$ through S' foliate Σ.

The existence of $\mathcal{F}_S$ with these properties follows from the transversality of H to all toric strata. Indeed, near a k-dimensional stratum S' and away from all lower-dimensional strata, consider a toric chart of the form $(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$, and modify the first k coordinates (in a C^∞ manner) so that, near H, $|f|$ only depends on these coordinates, without changing the remaining $n - k$ coordinates. Each stratum $S \supset S'$ is then defined by the vanishing of a certain subset of the last $n - k$ coordinates; we choose the leaves of $\mathcal{F}_S$ to be given by letting these coordinates vary and fixing all others. (More globally, start from a collection of toric charts identifying neighborhoods of strata with toric vector bundles over them, and modify the bundle structures compatibly along H so that $|f|$ is constant in the fibers and the strata containing a given one remain given by distinguished sub-bundles.)

Henceforth, unless stated otherwise, all estimates (on distances, derivatives, etc.) are with respect to a fixed reference metric (independent of t and λ), rather than the metric $g_{t,\lambda}$ determined by $\omega_{t,\lambda}$; and the notation $O(\dots)$ means that an inequality holds up to a constant factor which is uniformly bounded independently of t and λ over any compact subset of V.

Recall that $\omega_{t,\lambda}$ blows up (by a factor of the order of $\epsilon/\sqrt{\Phi}$, cf. (13.11)) in the directions transverse to the complex hyperplane field

$$\Theta = \operatorname{Ker}(d|f|) \cap \operatorname{Ker}(d^c|f|).$$

In what follows, we will often have better estimates on derivatives along Θ than on arbitrary derivatives. We will call derivatives of order (ℓ, m), denoted by $D^{(\ell,m)}(\dots)$, the derivatives of order $\ell + m$ along

ℓ vector fields tangent to Θ and m arbitrary vector fields. Since the hyperplane distribution Θ is not integrable, estimates on higher derivatives in the direction of Θ only make sense up to lower-order derivatives along the level sets of $|f|$; however, the curvature of Θ is $O(|f|^2)$, and the estimates we will obtain below on derivatives of order $(\ell+2, m)$ will generally be no better than $O(|f|^2)$ times the bounds on derivatives of order $(\ell, m+1)$.

Along a stratum S, denote by $\pi^S_{t,\lambda} : TV_{|S} \to TS^\perp$ the orthogonal projection (with respect to $\omega_{t,\lambda}$). Because S is transverse to H, and hence to Θ near H, the behavior of $\omega_{t,\lambda}$ in the directions transverse to Θ implies that, near $H \cap S$, the $\omega_{t,\lambda}$-orthogonal to S becomes nearly tangent to Θ for (t, λ) close to $(0, \epsilon)$. Specifically, near $H \cap S$, the maximum angle (with respect to a fixed reference metric) between a unit vector in $TS^\perp$ and Θ is $O(\epsilon^{-1}\sqrt{\Phi})$. Thus, denoting by $(\pi^S_{t,\lambda})^\parallel$ and $(\pi^S_{t,\lambda})^\perp$ the components of $\pi^S_{t,\lambda}$ along Θ and its orthogonal for the reference metric, pointwise we have $(\pi^S_{t,\lambda})^\perp = O(\epsilon^{-1}\sqrt{\Phi})$. This in turns implies that

$$\left|d^c(|f|^2) \circ \pi^S_{t,\lambda}\right| = O\big(\epsilon^{-1}|f|\sqrt{\Phi}\big).$$

Along the level sets of $|f|$, the coefficient of $d|f| \wedge d^c|f|$ in (13.10) remains constant, and so the geometric behavior of the $\omega_{t,\lambda}$-orthogonals $TS^\perp$ can be controlled uniformly. In particular, the derivatives along Θ of $(\pi^S_{t,\lambda})^\perp$ are bounded by $O(\sqrt{\Phi})$ to all orders. On the other hand, the variation of (13.10) in the directions transverse to the level sets of $|f|$ implies that each derivative in those directions worsens the bounds by a factor of $1/\sqrt{\Phi}$. We conclude that $D^{(\ell,m)}((\pi^S_{t,\lambda})^\perp) = O(\Phi^{(1-m)/2})$. Meanwhile, by a similar reasoning, $D^{(\ell,m)}((\pi^S_{t,\lambda})^\parallel) = O(\Phi^{-m/2})$.

These estimates on $\pi^S_{t,\lambda}$ (and the inequality $|f| \le (\Phi/4\pi\epsilon)^{1/2}$) in turn imply that

$$D^{(\ell,m)}\left(d^c(|f|^2) \circ \pi^S_{t,\lambda}\right) = O\left(\Phi^{(2-m)/2}\right).$$

Thus, the 1-form $a_{t,\lambda}$ from Step 3 satisfies

$$\left|a_{t,\lambda} \circ \pi^S_{t,\lambda}\right| = \frac{t\tilde{\chi}\min(\lambda,\epsilon)\left|d^c(|f|^2) \circ \pi^S_{t,\lambda}\right|}{2\sqrt{\Phi}\big(\sqrt{\Phi} + \pi|f|^2 + |\lambda - \epsilon|\big)^2} = O\left(\frac{t|f|}{\Phi}\right) = O\left(\frac{t}{\sqrt{\Phi}}\right)$$

$$\text{and} \quad D^{(\ell,m)}\left(a_{t,\lambda} \circ \pi^S_{t,\lambda}\right) = O\left(\frac{t}{\Phi^{(m+1)/2}}\right).$$

We now return to our main construction. Starting with $f_{\lambda,t,\le 0} = 0$ as before, assume that we have already constructed $f_{\lambda,t,\le k}$, supported in a neighborhood of the intersection of H with the toric strata of dimension $\le k$, in such a way that (13.12) holds for all strata of dimension $\le k$. We further require that, away from all strata of dimension $\le k-1$, resp. near a stratum S' of dimension $\le k-1$ (and assuming S' is the closest such stratum),

$$D^{(\ell,m)}(f_{t,\lambda,\le k}) = O\left(\frac{t}{\Phi^{(m+1)/2}}\right), \quad \text{resp. } O\left(\frac{t}{\Phi^{(m+1)/2}}\mathrm{dist}_{S'}^{\min(0,2-\ell-m)}\right), \tag{13.13}$$

where $\mathrm{dist}_{S'}$ is the distance to S' with respect to the fixed reference metric.

Let S be a stratum of dimension $k+1$. The above estimates on the derivatives of $\pi^S_{t,\lambda}$, together with (13.13), imply that at any point of S which lies away from the strata of dimension $\le k-1$, resp. near (and closest to) such a stratum S',

(13.14)
$$D^{(\ell,m)}\left((a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^S_{t,\lambda}\right)=O\left(\frac{t}{\Phi^{(m+1)/2}}\right),\ \text{ resp. } O\left(\frac{t}{\Phi^{(m+1)/2}}\operatorname{dist}_{S'}^{\min(0,1-\ell-m)}\right).$$

(Note that, while the quantity in (13.14) involves an additional derivative of $f_{t,\lambda,\le k}$, the extra factor of $\Phi^{-1/2}$ when this derivative is taken in a direction transverse to Θ is offset by the factor of $\Phi^{1/2}$ in the estimates for the transverse component of $\pi^S_{t,\lambda}$.)

Near a stratum $S'\subset S$ with $\dim S'\le k$, condition (13.12) for $f_{t,\lambda,\le k}$ along S' implies that $(a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^S_{t,\lambda}$ vanishes along S'. Since Θ is transverse to S', (13.14) for $(\ell,m)=(1,0)$ in turn implies that, at all points of S which lie near S',

$$\left|(a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^S_{t,\lambda}\right|=O\Big(\frac{t\operatorname{dist}_{S'}}{\sqrt{\Phi}}\Big), \tag{13.15}$$

Meanwhile, since the distance to the nearest k-dimensional stratum is no greater than the distance to the nearest lower-dimensional stratum, the bounds in the second part of (13.14) also hold when $\dim S'=k$. Hence, at any point of S which lies near (and closest) to a stratum $S'\subset S$ of dimension $\le k$,

$$D^{(\ell,m)}\left((a_{t,\lambda}+df_{t,\lambda,\le k})\circ\pi^S_{t,\lambda}\right)=O\left(\frac{t}{\Phi^{(m+1)/2}}\operatorname{dist}_{S'}^{1-\ell-m}\right). \tag{13.16}$$

Define a function $f^0_{\lambda,t,S}$ on a neighborhood of the given $(k+1)$-dimensional stratum S, smooth outside of the leaves of $\mathcal{F}_S$ through strata of dimension $\le k-1$ (and H if $(\lambda,t)=(\epsilon,0)$), which vanishes on S and whose derivative at each point of S satisfies

$$df^0_{\lambda,t,S}=-(a_{t,\lambda}+df_{\lambda,t,\le k})\circ\pi^S_{t,\lambda}. \tag{13.17}$$

Specifically, we identify the leaves of $\mathcal{F}_S$ with open subsets in the fibers of the normal bundle to S, and take $f^0_{\lambda,t,S}$ to be linear in the fibers. We then define $f_{\lambda,t,S}=\chi_S f^0_{\lambda,t,S}$, where χ_S is the same cut-off function as in Step 2.

By construction, $f^0_{t,\lambda,S}=O(t\operatorname{dist}_S/\sqrt{\Phi})$. Moreover, using (13.15), along the leaf of $\mathcal{F}_S$ through a point $x\in S$ which lies near a lower-dimensional stratum S' we have $f^0_{t,\lambda,S}=O(t\operatorname{dist}_{S'}(x)\operatorname{dist}_S/\sqrt{\Phi})$.

The derivative of $f^0_{\lambda,t,S}$ along the leaves of $\mathcal{F}_S$ is the constant extension of (13.17) along $\mathcal{F}_S$; whereas its derivative in the directions transverse to $\mathcal{F}_S$ is a cross-term which grows linearly with distance to S and involves the dependence of (13.17) on the point of S. Moreover, the leaves of $\mathcal{F}_S$ are tangent to the level sets of $|f|$ near H, and hence nearly tangent to Θ: the maximum angle between vectors in $T\mathcal{F}_S$ and Θ is $O(|f|)$. It then follows from (13.14) that, away from $(k-1)$-dimensional strata,

$$D^{(\ell,m)}(f^0_{\lambda,t,S})=O\left(\frac{t}{\Phi^{(m+1)/2}}\right). \tag{13.18}$$

Meanwhile, along the leaf of $\mathcal{F}_S$ through a point $x\in S$ which lies near (and closest to) a stratum $S'\subset S$ with $\dim S'\le k$, (13.16) implies that

$$D^{(\ell,m)}(f^0_{\lambda,t,S})=O\left(\frac{t}{\Phi^{(m+1)/2}}\left(\operatorname{dist}_{S'}(x)^{2-\ell-m}+\operatorname{dist}_{S'}(x)^{1-\ell-m}\operatorname{dist}_S(\cdot)\right)\right).$$

The leaf of $\mathcal{F}_S$ through x locally stays close to a leaf through S', which by construction is contained in some other stratum of D_V. In particular, as soon as the distance to S is sufficiently large compared to $\mathrm{dist}_{S'}(x)$, points on the leaf through x lie closer to some other stratum Σ of dimension $\geq k+1$ (and not containing S) than to S, and so the cut-off function χ_S vanishes identically. Thus, over the support of χ_S, $\mathrm{dist}_{S'}(\cdot)$ and $\mathrm{dist}_{S'}(x)$ are within bounded factors of each other. Since $\mathrm{dist}_S \leq \mathrm{dist}_{S'}$, we conclude that, at all points of the support of χ_S which lie near (and closest to) S',

$$D^{(\ell,m)}(f^0_{\lambda,t,S}) = O\left(\frac{t}{\Phi^{(m+1)/2}}\mathrm{dist}_{S'}^{2-\ell-m}\right). \tag{13.19}$$

Now we observe that the derivatives of the cut-off function χ_S are $O(1)$ away from strata of dimension $\leq k$, and near a stratum $S' \subset S$ of dimension $\leq k$ the derivatives of order r are $O(1/\mathrm{dist}^r_{S'})$. Thus, (13.18) and (13.19) imply that away from k-dimensional strata, resp. near (and closest to) $S' \subset S$ with $\dim S' \leq k$,

$$D^{(\ell,m)}(f_{\lambda,t,S}) = O\left(\frac{t}{\Phi^{(m+1)/2}}\right), \quad \text{resp. } O\left(\frac{t}{\Phi^{(m+1)/2}}\mathrm{dist}_{S'}^{2-\ell-m}\right). \tag{13.20}$$

We now set

$$f_{t,\lambda,\leq k+1} = f_{t,\lambda,\leq k} + \sum_{\dim S = k+1} f_{t,\lambda,S}.$$

By construction, $f_{t,\lambda,\leq k+1}$ is supported in a neighborhood of the intersection of H with the strata of dimension at most $k+1$, and satisfies (13.12) for all strata of dimension $\leq k+1$. Indeed, by (13.20), $df_{t,\lambda,S}$ vanishes along strata of dimension $\leq k$, so (13.12) continues to hold for those; whereas, over the interior of the stratum S, $df_{t,\lambda,S} = df^0_{t,\lambda,S}$, and the contributions from other $(k+1)$-dimensional strata vanish.

Moreover, $f_{t,\lambda,\leq k+1}$ satisfies the estimate (13.13) (with $k+1$ instead of k), as needed for the induction to proceed. Indeed, this follows immediately from the estimates (13.13) for $f_{t,\lambda,\leq k}$ (note that the second estimate also holds near k-dimensional strata, since the distance to the nearest k-dimensional stratum is no greater than that to the nearest lower-dimensional stratum), and (13.20) for $f_{t,\lambda,S}$.

Thus, we can indeed carry out the construction of $f_{t,\lambda,\leq k}$ with the desired properties by induction on k. Finally, we let $f_{t,\lambda} = f_{t,\lambda,\leq n-1}$.

As a consequence of the estimates (13.20) on individual terms, $f_{t,\lambda}$ is C^1 with locally Lipschitz first derivatives, and smooth on V^0, except along H for $(t,\lambda) = (0,\epsilon)$. By construction, it is supported in the intersection of U with a neighborhood of D_V, and satisfies (13.12) for all toric strata.

By (13.13), $|df_{t,\lambda}| = O(t/\Phi)$, while $|df_{t,\lambda|\Theta}| = O(t/\sqrt{\Phi})$.

Because the Kähler form $\omega_{t,\lambda}$ blows up like $\epsilon/\sqrt{\Phi}$ in the directions transverse to Θ, we conclude that the Hamiltonian vector field of $f_{t,\lambda}$ with respect to $\omega_{t,\lambda}$ is bounded by $O(t/\sqrt{\Phi})$ (again with respect to the fixed reference metric), hence locally uniformly bounded. (Recall that $\sqrt{\Phi} \geq t$ wherever $\tilde{\chi} \equiv 1$, while the other terms are bounded below wherever $\tilde{\chi} < 1$.) Moreover, the regularity of $f_{t,\lambda}$ implies that this vector field is locally Lipschitz continuous, and smooth on V^0, except along H for $(t,\lambda) = (0,\epsilon)$.

Combining this with the outcome of Step 3, we find that the vector field $\tilde{v}_{t,\lambda}$ defined by $\iota_{\tilde{v}_{t,\lambda}}\omega'_{t,\lambda} = -a_{t,\lambda} - df_{t,\lambda}$ is smooth on V^0 (and locally Lipschitz continuous along D_V), except along H for $(t,\lambda) = (0,\epsilon)$, and its norm (again with respect to a smooth reference metric) is bounded by $O(t/\sqrt{\Phi})$, hence locally uniformly bounded. Thus, even though $\tilde{v}_{t,\lambda}$ is not continuous along H for $(t,\lambda) = (0,\epsilon)$, its flow is well-defined and continuous even for $\lambda = \epsilon$. We then obtain $\phi_{sm,\lambda}$ with all the desired properties by considering the time κ flow generated by $\tilde{v}_{t,\lambda}$. $\square$

Part II

Noncommutative motivic invariants

Hodge theoretic aspects of mirror symmetry.

1 Hodge theory and Mirror Symmetry

This chapter is a first in a series aiming to develop a general procedure associating a 2-dimensional cohomological field theory in the sense [155] (CohFT in short) to a certain structure in derived algebraic geometry. More precisely, for any Calabi-Yau A_∞-category satisfying appropriate finiteness conditions (smoothness and compactness), and such that a noncommutative analog of the Hodge $\Rightarrow$ de Rham spectral sequence collapses, we associate an infinite-dimensional family of CohFTs. The additional parameters needed to specify the CohFT are of a purely cohomological nature. Conjecturally, our procedure applied to the Fukaya category should give (higher genus) Gromov-Witten invariants of the underlying symplectic manifold.

This program was first outlined by the second author in several talks given in 2003-2004, and some aspects of it were later developed in depth by K.Costello [62], [60], [63], [61]. The whole picture turns out to be very intricate, and in the process of writing we realized that we have to address a large variety of problems. In this installment we do not discuss the general plan of our approach but rather focus on those features of A_∞ or dg categories that can be captured by Hodge theoretic constructions. We propose a formalism that starts with Homological Mirror Symmetry and extrapolates a geometric picture for the requisite categories that makes them amenable to study via old and new Hodge theory. Our hope is that this geometric treatment will provide new invariants and will expand the scope of possible applications in symplectic geometry and algebraic geometry.

Mirror symmetry was introduced in physics as a special duality between two $N = 2$ super conformal field theories. Traditionally a $N = 2$ super conformal field theory is constructed as a quantization of a non-linear σ-model with target a compact Calabi-Yau manifold equipped with a Ricci flat Kähler metric and a closed 2-form - the so called B-field. Two Calabi-Yau manifolds X and Y form a *mirror pair* $X|Y$ if the associated $N = 2$ super conformal field theories are mirror dual to each other [65].

Homological Mirror Symmetry was introduced in 1994 by the second author for the case of Calabi-

Yau manifolds but today the realm of its applicability is much broader. In particular many of our considerations in the present work are governed by an analogue of Homological Mirror Symmetry for geometries with potentials. We study the effect of such mirror symmetry on the associated categories of D-branes and especially on the associated non-commutative Hodge structures on homological invariants, i.e. on the Hochschild and cyclic homology and cohomology of such categories. We study mirror pairs consisting of a compact manifold on one side, and of a Landau-Ginzburg model with a proper potential on a non-compact manifold having $c_1 = 0$ on the other. We formulate the mirror symmetry conjecture on the Hodge theoretic level in both directions. That is, we compare the non-commutative Hodge structures associated with a compact complex manifold and a mirror holomorphic Landau-Ginzburg model, and also the non-commutative Hodge structures associated with a compact complex manifold with a chosen smooth anticanonical divisor and with the mirror symplectic Landau-Ginzburg model. This picture is clearly non-symmetric and has to be generalized. In order to completely understand the Hodge theoretic aspect of mirror symmetry, one will have to allow for non-trivial potentials on both sides of the duality and include the novel toric dualities between formal Landau-Ginzburg models of Clarke [57] and the new smooth variations of non-commutative Hodge structures of Calabi-Yau type that we attach to anticanonical $\mathbb{Q}$-divisors in section 4.3. We plan to return to such a generalization in a future work.

Due to its foundational nature the chapter comes out somewhat long winded and technical for which we apologize. It is organized in three major parts:

The first part introduces and develops the abstract theory of non-commutative (**nc**)Hodge structures. This theory is a variant of the formalism of semi-infinite Hodge structures that was introduced by Barannikov [21], [22], [23]. We discuss the general theory of **nc**-Hodge structures in the abstract and analyze the various ways in which the Betti, de Rham and Hodge filtration data can be specified. In particular we compare **nc** and ordinary Hodge theory and explain how **nc**-Hodge theory fits within the setup of categorical non-commutative geometry. We also pay special attention to the **nc**-aspect of Hodge theory and its interaction with the classification of irregular connections on the line via topological data. One of the most useful technical results in this part is the gluing Theorem 2.35 which allows us to assemble **nc**-Hodge structures out of some simple geometric ingredients. This theorem is used later in the chapter for constructing **nc**-Hodge structures attached to geometries with a potential.

The second part explains how symplectic and complex geometry give rise to **nc**-Hodge structures and how these structures can be viewed as interesting invariants of Gromov-Witten theory, projective geometry and the theory of algebraic cycles. In particular we analyze the Betti part of the **nc**-Hodge theory of a projective space (viewed as a symplectic manifold) and use this analysis to propose a general conjecture for the integral structure on the cohomology of the Fukaya category of a general compact symplectic manifold. The formula for the integral structure uses only genus zero Gromov-Witten invariants and characteristic classes of the tangent bundle. Our conjecture is in complete agreement with the recent work of Iritani [123] who made a similar proposal based on mirror symmetry for toric Fano orbifolds. We also discuss in detail the origin of the Stokes data for holomorphic geometries with potentials and investigate the possible categorical incarnations of this data.

In the third part we study **nc**-Hodge structures and their variations under the Calabi-Yau condition. We extend and generalize the standard treatment of the deformation theory of Calabi-Yau spaces in order to get a theory which works equally well in the **nc**-context and to be able to properly define the canonical coordinates in Homological Mirror Symmetry. We approach the deformation-obstruction problem both algebraically and by Hodge theoretic means and we obtain unobstructedness results, generalized pre Frobenius structures and some interesting geometric properties of period domains for **nc**-Hodge structure. We also study global and infinitesimal deformations and describe different constructions of Betti and de Rham **nc**-Hodge data for ordinary geometry, relative geometry, geometry with potentials and abstract **nc**-geometry.

2 Non-commutative Hodge structures

In this section we will discuss the notion of a pure non-commutative (**nc**) Hodge structure. The **nc**-Hodge structures are analogues of the classical notion of a pure Hodge structure on a complex vector space. Both the **nc**-Hodge structures discussed presently and Simpson's non-abelian Hodge structures [224] generalize classical Hodge theory. In Simpson's theory, one allows for non-linearity in the substrate of the Hodge structure: the non-abelian Hodge structures of [224] are given by imposing Hodge and weight filtrations on non-linear topological invariants of a Kähler space, e.g. on cohomology with non-abelian coefficients, or on the homotopy type. In contrast the **nc**-Hodge structures discussed in this chapter consist of a novel filtration-type data (the twistor structure of [225], [113], [207]) which are still specified on a vector space, e.g. on the periodic cyclic homology of an algebra.

Similarly to ordinary Hodge theory **nc**-Hodge structures arise naturally on the de Rham cohomology of non-commutative spaces of categorical origin.

2.1 Hodge structures

We will give several different descriptions of a **nc**-Hodge structure in terms of local data. We begin with the notion of a rational and unpolarized **nc**-Hodge structures, ignoring for the time being the existence of polarizations and integral lattices.

2.1.1 Notation The **nc**-Hodge structures will be described in terms of geometric data on the punctured complex line, so we fix once and for all a coordinate u on $\mathbb{C}$ and the compactification $\mathbb{C} \subset \mathbb{P}^1$. We will write $\mathbb{C}[[u]]$ for the algebra of formal power series in u, and $\mathbb{C}((u))$ for the field of formal Laurent series in u. Similarly, we will write $\mathbb{C}\{u\}$ for the algebra of power series in u having positive radius of convergence, and $\mathbb{C}\{u\}[u^{-1}]$ for the field of meromorphic Laurent series in u with a pole at most at $u = 0$.

2.1.2 Meromorphic connections on the complex line We will need some standard notions and facts related to meromorphic connections on Riemann surfaces. We briefly recall those next. More details can be found in e.g. [205, chapter II].

Let $\mathscr{M}$ be a finite dimensional vector space over $\mathbb{C}\{u\}[u^{-1}]$, and let ∇ be a ***meromorphic connection*** on $\mathscr{M}$. Explicitly ∇ is given by a $\mathbb{C}$-linear map $\nabla_{\frac{d}{du}} : \mathscr{M} \to \mathscr{M}$ which satisfies the Leibniz rule for multiplication by elements in $\mathbb{C}\{u\}[u^{-1}]$. A ***holomorphic extension of*** $\mathscr{M}$ is a free finitely generated $\mathbb{C}\{u\}$-submodule $\mathscr{H} \subset \mathscr{M}$, such that

$$\mathscr{H} \otimes_{\mathbb{C}\{u\}} \mathbb{C}\{u\}[u^{-1}] = \mathscr{M}.$$

Traditionally (see e.g. [205, section 0.8]) a holomorphic extension is called a *lattice*. We will avoid this classical terminology since it may lead to confusion with the integral lattice structures that we need.

As usual the data $(\mathscr{M}, \nabla)$ or $(\mathscr{H}, \nabla)$ should be viewed as local models for geometric data on a Riemann surface: $(\mathscr{M}, \nabla)$ is the local model of a germ of a meromorphic bundle with connection on a Riemann surface, and $(\mathscr{H}, \nabla)$ is the local model of a holomorphic bundle with meromorphic connection on a Riemann surface.

Suppose $(\mathscr{M}, \nabla)$ is a meromorphic bundle with connection over $\mathbb{C}\{u\}[u^{-1}]$ and let $\mathscr{H} \subset \mathscr{M}$ be a holomorphic extension. We say that $\mathscr{H}$ is **logarithmic** with respect to ∇ if $\nabla(\mathscr{H}) \subset \mathscr{H}\frac{du}{u}$. We say that $(\mathscr{M}, \nabla)$ has at most a **regular singularity at** 0 if we can find a holomorphic extension $\mathscr{H} \subset \mathscr{M}$ which is logarithmic with respect to ∇.

Remark 2.1 The Riemann-Hilbert correspondence implies (see e.g. [205, II.3.7]) that the functor of taking the germ at $0 \in \mathbb{P}^1$:

$$\begin{pmatrix} \text{finite rank algebraic vector bundles with} \\ \text{connections on } \mathbb{A}^1 - \{0\} \text{ with a regular} \\ \text{singularity at } \infty \end{pmatrix} \xrightarrow{\mathfrak{G}_0} \begin{pmatrix} \text{finite dimensional } \mathbb{C}\{u\}[u^{-1}]\text{-} \\ \text{vector spaces with meromorphic} \\ \text{connections} \end{pmatrix}$$

is an equivalence of categories. For future reference we choose once and for all an inverse $\mathfrak{B}_0$ of $\mathfrak{G}_0$. We will call $\mathfrak{B}_0$ the **algebraization functor** or the **Birkhoff extension functor**.

Suppose $\mathscr{H}$ is a *holomorphic bundle* over $\mathbb{C}\{u\}$ equipped with a meromorphic connection ∇. Let $\mathscr{M} = \mathscr{H} \otimes_{\mathbb{C}\{u\}} \mathbb{C}\{u\}[u^{-1}]$ and let $(M, \nabla) = \mathfrak{B}_0(\mathscr{M}, \nabla)$ be the corresponding Birkhoff extension. The algebraic bundle M on $\mathbb{A}^1 - \{0\}$ admits a natural extension to a holomorphic bundle H on $\mathbb{A}^1$: on a small punctured disc centered at $0 \in \mathbb{A}^1$, the bundle M is analytically isomorphic to $\mathscr{M}$, and so $\mathscr{H}$ gives us an extension to the full disc. In particular $\mathfrak{G}_0$ and $\mathfrak{B}_0$ can be promoted to a pair of inverse equivalences

$$\begin{pmatrix} \text{finite rank algebraic vector bundles on } \mathbb{A}^1 \\ \text{equipped with a meromorphic connection} \\ \text{with poles at most at } 0 \text{ and } \infty\text{, and a reg-} \\ \text{ular singularity at } \infty \end{pmatrix} \underset{\mathfrak{B}^0}{\overset{\mathfrak{G}_0}{\rightleftarrows}} \begin{pmatrix} \text{finite rank free } \mathbb{C}\{u\}\text{-modules} \\ \text{equipped with a meromorphic} \\ \text{connection} \end{pmatrix}$$

which we will denote again by $\mathfrak{G}_0$ and $\mathfrak{B}_0$.

2.1.3 Stokes data Let $(\mathscr{H}, \nabla)$ be a holomorphic bundle with meromorphic connection over $\mathbb{C}\{u\}$. We will need the Deligne-Malgrange description of the associated meromorphic connection $(\mathscr{M}, \nabla)$ via a filtered sheaf on the circle. We briefly recall this description next. More details can be found in [171] and [20]. Let $(M, \nabla) := \mathfrak{B}_0((\mathscr{M}, \nabla))$ be the Birkhoff extension of $(\mathscr{M}, \nabla)$ to $\mathbb{P}^1$. Consider the circle $\boldsymbol{S}^1 := \mathbb{C}^\times/\mathbb{R}_+^\times$. The sheaf of local ∇-horizontal sections of M^{an} on $\mathbb{C}^\times$ is a locally constant sheaf on $\mathbb{C}^\times$, which by contractability of $\mathbb{R}_+^\times$ induces a locally constant sheaf S of $\mathbb{C}$-vector spaces on $\boldsymbol{S}^1$.

The sheaf S is equipped with a natural local filtration by subsheaves $\{\mathsf{S}_{\leq \omega}\}_{\omega \in \mathsf{Del}}$, where

(i) Del is the complex local system on $\boldsymbol{S}^1$ for which for every open $U \subset \boldsymbol{S}^1$ the space of sections $\mathsf{Del}(U)$ is defined to be the space of all holomorphic one forms ω on the sector

$$\mathrm{Sec}(U) := \left\{ re^{i\varphi} \middle| \, r > 0, \varphi \in U \right\}$$

which are of the form

$$\omega = \left(\sum_{\substack{a \in \mathbb{Q} \\ a < -1}} c_a u^a \right) du,$$

where at most finitely many $c_a \neq 0$ and the branches u^a are chosen arbitrarily.

Note that the germs of sections of Del are naturally ordered: if $\omega', \omega'' \in \mathsf{Del}(U)$, $\varphi \in U$, and if

$$\omega' - \omega'' = c_a u^a + \begin{pmatrix} \text{higher} \\ \text{order terms} \end{pmatrix},$$

then one sets

$$\omega' <_\varphi \omega'' \quad \Leftrightarrow \quad \mathrm{Re}\left(\frac{c_a e^{i\varphi(a+1)}}{a+1} \right) < 0.$$

(ii) For every $\varphi \in \boldsymbol{S}^1$ and every $\omega \in \mathsf{Del}_\varphi$ the stalk

$$\left(\mathsf{S}_{\leq \omega}\right)_\varphi \subset \mathsf{S}_\varphi$$

is defined to be the subspace

$$\left(\mathsf{S}_{\leq \omega}\right)_\varphi := \left\{ s \in \mathsf{S}_\varphi = \Gamma\left(\mathbb{R}_+^\times e^{i\varphi}, M^{\mathrm{an}}\right)^\nabla \;\middle|\; \begin{array}{l} e^{-\int \omega} s \text{ has moderate growth in} \\ \text{the direction } \varphi \text{, i.e.} \\ \quad \left\| e^{-\int \omega} s \right\|_{|\mathbb{R}_+^\times e^{i\varphi}} = \boldsymbol{O}\left(r^{-N}\right), \\ \text{when } r \to 0,\, N \gg 0. \end{array} \right\}$$

Here $\| \bullet \|$ is the Hermitian norm of a section of M computed in some (any) meromorphic trivialization of M^{an} near $u = 0$.

Definition 2.2 *The filtration we just defined is the* ***Deligne-Malgrange-Stokes filtration****, and the* Del*-filtered sheaf* S *is called the* ***Stokes structure*** *associated to* $(\mathscr{M}, \nabla)$.

Remark 2.3 The Deligne-Malgrange-Stokes filtration satisfies the following property. First of all, there exists a covariantly local system of finite sets $\mathsf{Del}_{(\mathscr{M},\nabla)} \subset \mathsf{Del}$ canonically associated with $(\mathscr{M}, \nabla)$ such that the filtration by all of Del is determined by a filtration by all $\mathsf{Del}_{(\mathscr{M},\nabla)}(U)$ and all consecutive factors are non-trivial at all points of S^1 except finitely many (called the directions of the Stokes rays). Outside the Stokes rays the set $\mathsf{Del}_{\mathscr{M},\nabla}(\phi)$ is totally ordered. It is easy to see that when we cross a Stokes ray then the order changes by flipping the order on several disjoint intervals (e.g. $\{1,2,3,4,5,6\} \to \{2,1,3,6,5,4\}$). Moreover, on the subquotients corresponding to these intervals, two filtrations coming from the left and from the right of the anti Stokes ray are opposed to each other. This implies that the graded pieces with respect to the Deligne-Malgrange-Stokes filtration are locally constant sytems of vector spaces on the total space of stalks of the sheaf $\mathsf{Del}_{(\mathscr{M},\nabla)}$ (which is a disjoint union of finite coverings of S^1).

Remark 2.4 A fundamental theorem of Deligne and Malgrange [171, Theorem 4.2], [20, Theorem 4.7.3] asserts that the functor $(\mathscr{M}, \nabla) \mapsto \left(\mathsf{S}, \{\mathsf{S}_{\leq \omega}\}_{\omega \in \mathsf{Del}}\right)$ is an equivalence between the category of meromorphic connections over $\mathbb{C}\{u\}[u^{-1}]$ and the category of Del-filtered local systems on S^1 satisfying the conditions described in Remark 2.3. We will use this equivalence to define the Betti part of a nc-Hodge structure.

2.1.4 The definition of a nc-Hodge structure

After these preliminaries we are now ready to define nc-Hodge structures.

Definition 2.5 *A* ***rational pure*** nc***-Hodge structure*** *consists of the data* $(H, \mathscr{E}_B, \overset{\sim}{\longrightarrow})$, *where*

- H *is a* $\mathbb{Z}/2$*-graded algebraic vector bundle on* $\mathbb{A}^1$.
- $\mathscr{E}_B$ *is a local system of finite dimensional* $\mathbb{Z}/2$*-graded* $\mathbb{Q}$*-vector spaces on* $\mathbb{A}^1 - \{0\}$.
- $\overset{\sim}{\longrightarrow}$ *is an* analytic *isomorphism of holomorphic vector bundles on* $\mathbb{A}^1 - \{0\}$*:*

$$\overset{\sim}{\longrightarrow}: \mathscr{E}_B \otimes \mathscr{O}_{\mathbb{A}^1-\{0\}} \xrightarrow{\cong} H_{|\mathbb{A}^1-\{0\}}.$$

Note: *The isomorphism* $\overset{\sim}{\longrightarrow}$ *induces a natural flat holomorphic connection* ∇ *on* $H_{|\mathbb{A}^1-\{0\}}$.

These data have to satisfy the following axioms:

(nc-filtration axiom) *∇ is a meromorphic connection on H with a pole of order ≤ 2 at $u = 0$ and a regular singularity at ∞. More precisely, there exist:*

- *a holomorphic frame of H near $u = 0$ in which*

$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du$$

with $A_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$, $r = \mathrm{rank}\ H$.

- *a meromorphic frame of H near $u = \infty$ in which*

$$\nabla = d + \left(\sum_{k \geq -1} B_k u^{-k} \right) d(u^{-1})$$

and $B_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$.

($\mathbb{Q}$-structure axiom) *The $\mathbb{Q}$-structure $\mathscr{E}_B$ on (H, ∇) is compatible with Stokes data. More precisely, let $\left(\mathbf{S}, \{\mathbf{S}_{\leq \omega}\}_{\omega \in \mathbf{Del}}\right)$ be the Stokes structure corresponding to the germ $(\mathscr{H}, \nabla) := \mathfrak{G}_0(H, \nabla)$, and let $\mathbf{S}_B \subset \mathbf{S}$ be the $\mathbb{Q}$-structure on $\mathbf{S}$ induced from $\mathscr{E}_B$ via the isomorphism $\xrightarrow{\sim}$. We require that the Deligne-Malgrange-Stokes filtration on $\mathbf{S}$ is defined over $\mathbb{Q}$, i.e.*

$$(\mathbf{S}_{\leq \omega} \cap \mathbf{S}_B) \otimes_{\mathbb{Q}} \mathbb{C} = \mathbf{S}_{\leq \omega}$$

for all local sections $\omega \in \mathbf{Del}$.

(opposedness axiom) *The $\mathbb{Q}$-structure $\mathbf{S}_B$ induces a real structure on $\mathbf{S}$ and hence a complex conjugation $\tau : \mathbf{S} \to \mathbf{S}$. Let $\widehat{H}$ be the holomorphic bundle on $\mathbb{P}^1$ obtained as the gluing of $H^{\mathrm{alg}}_{|\{|u| \leq 1\}}$ and $\left(\gamma^* \overline{H^{\mathrm{alg}}}\right)_{|\{|u| \geq 1\}}$ via τ, where where $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ is the real structure on $\mathbb{P}^1$ which fixes the unit circle, i.e. $\gamma(u) := 1/\bar{u}$. Then we require that $\widehat{H}$ be holomorphically trivial, i.e. $\widehat{H} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$.*

A morphism $\boldsymbol{f} : (H_1, \mathscr{E}_{B,1}, \xrightarrow{\sim}_1) \to (H_2, \mathscr{E}_{B,2}, \xrightarrow{\sim}_2)$ of $\mathbf{nc}$-Hodge structures is a pair $\boldsymbol{f} = (f, f_B)$, where $f : H_1 \to H_2$, is an algebraic map of vector bundles which intertwines the connections, and $f_B : \mathscr{E}_{B,1} \to \mathscr{E}_{B,2}$ is a map of $\mathbb{Q}$-local systems, such that $f \circ \xrightarrow{\sim}_1 = \xrightarrow{\sim}_2 \circ (f_B \otimes \mathrm{id}_{\mathcal{O}})$. We will write $(\oplus\mathbb{Q} - \mathbf{nc}\mathsf{HS})$ for the category of pure $\mathbf{nc}$-Hodge structures.

Remark 2.6 The meromorphic connection (M, ∇) where $M = H \otimes_{\mathbb{C}[u]} \mathbb{C}[u, u^{-1}]$ can be thought of as the de Rham data of the $\mathbf{nc}$-Hodge structure, the local system $\mathbf{S}_B$ of rational vector spaces over $\boldsymbol{S}^1$ endowed with the rational Stokes filtration (see $\mathbb{Q}$-structure axiom) can be thought of as the Betti data, and the holomorphic extension H of M can be thought of as the analogue of the Hodge filtration.

2.1.5 Variations of nc-Hodge structures One can also define variations of **nc**-Hodge structures:

Definition 2.7 *Let S be a complex manifold. A **variation of pure nc-Hodge structures** over S is a triple $(H, \mathscr{E}_B, \overset{\sim}{\to})$, where*

- *H is a holomorphic $\mathbb{Z}/2$-graded vector bundle on $\mathbb{A}^1 \times S$ which is algebraic in the $\mathbb{A}^1$-direction.*
- *$\mathscr{E}_B$ is a local system of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on $(\mathbb{A}^1 - \{0\}) \times S$.*
- *$\overset{\sim}{\to}$ is an analytic isomorphism of holomorphic vector bundles*

$$\overset{\sim}{\to} : \mathscr{E}_B \otimes \mathcal{O}_{(\mathbb{A}^1-\{0\})\times S} \overset{\cong}{\to} H_{|(\mathbb{A}^1-\{0\})\times S}.$$

Let ∇ be the induced meromorphic connection on H. The data $(H, \mathscr{E}_B, \overset{\sim}{\to})$ should satisfy:

(nc-filtration axiom) *The connection ∇ has a regular singularity along $\{\infty\} \times S$ and Poincaré rank ≤ 1 along $\{0\} \times S$, i.e.*

$$u^2 \cdot \nabla_{\frac{\partial}{\partial u}} : H \to H$$

is a holomorphic differential operator on H of order ≤ 1.

(Griffiths transversality axiom) *For every locally defined vector field $\xi \in T_S$ we have that*

$$u \cdot \nabla_{\xi} : H \to H,$$

is a holomorphic differential operator on H of order ≤ 1.

($\mathbb{Q}$-structure axiom) *The Stokes structure on the local system **S** on $\boldsymbol{S}^1 \times S$ is well defined, i.e. the steps in the Deligne-Malgrange-Stokes filtration on **S** are sheaves on $\boldsymbol{S}^1 \times S$. Furthermore the $\mathbb{Q}$-structure $\mathscr{E}_B$ is compatible with the Stokes data as in Definition 2.5.*

(opposedness axiom) *The relative version of the gluing construction for **nc**-Hodge structures gives a globally defined complex vector bundle $\widehat{H}$ on $\mathbb{P}^1 \times S$, which is holomorphically trivial in the $\mathbb{P}^1$ direction. Moreover, with respect to the extension $\widehat{H}$ the connection ∇ is meromorphic with Poincaré rank one along $(\{0\} \times S) \cup (\{\infty\} \times S)$.*

2.1.6 Relation to other definitions Various special cases and partial versions of our notion of a **nc**-Hodge structure have been studied before in slightly different but related setups. We list a few of the relevant notions and references without going into detailed comparisons:

- A version of ($\mathbb{Z}$-graded) **nc**-Hodge structures appears in the fundamental work of K.Saito (see [208], [210], [209] and references therein) on the Hodge theoretic invariants of quasi-homogeneous hypersurface singularities under the name ***weight system***.

- A version of the notion of a variation of (complex) **nc**-Hodge structure appears in the work of Cecotti-Vafa in Conformal Field Theory [48], [49], [50], [29] under the name ***tt^*-geometry***.
- Various versions of the notion of a (complex,polarized) **nc**-Hodge structure appear in algebraic geometry and non-abelian Hodge theory in the works of Simpson [224], [225] and T.Mochizuki [183], [184], [185], [186] under the names of ***(tame or wild) harmonic bundle*** or ***pure twistor structure***, and in the work of Sabbah [207] under the name ***integrable pure twistor D-module***.
- The analytic germ of a (complex) variation of **nc**-Hodge structures appears in mirror symmetry in the work of Barannikov [21], [22], [23] and Barannikov and the second author [24] under the name ***semi-infinite Hodge structure***. The integral structures on semi-infinite Hodge structure were recently introduced and studied in the work of Iritani [123].
- A version of the notion of a (real) **nc**-Hodge structure appears in singularity theory in the work of Hertling [113], [114] and Hertling-Sevenchek [115] under the name ***TER structure***. Hertling and Sevenchek also consider polarized and mixed **nc**-Hodge structures. Those appear under the names ***TERP structure*** and ***mixed TERP structure*** respectively. In particular in [115] Hertling and Sevenchek study the degenerations of of TERP structures and prove a version of Schmid's nilpotent orbit theorem which gives rise to the notion of a limiting mixed TERP structure. Degenerations of variants of **nc**-Hodge structures, as well as limiting mixed **nc**-Hodge structures appear also in the works of Sabbah [206] and S.Szabo [229].

2.1.7 Relation to usual Hodge theory Recall (see e.g. [67]) that a ***pure rational Hodge structure*** of weight w is a triple $(V, F^\bullet V, V_{\mathbb{Q}})$ where:

- V is a complex vector space,
- $V_{\mathbb{Q}} \subset V$ is a $\mathbb{Q}$-subspace such that $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, and
- $F^\bullet V$ is a ***Hodge filtration*** of weight w on V, i.e $F^\bullet V$ is a decreasing finite exhaustive filtration by complex subspaces which satisfies $F^p V \oplus \overline{F^{w+1-p} V} = V$, where the complex conjugation on V is the one given by the real structure $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes \mathbb{R} \subset V$.

A ***pure Hodge structure*** is a direct sum of pure Hodge structures of various weights, and a morphism of pure Hodge structures is a linear map of complex vector spaces, which maps the rational structures into each other and is strictly compatible with the filtrations. We will write $(\oplus\mathbb{Q} - HS)$ for the category of pure rational Hodge structures. It is well known [67] that $(\oplus\mathbb{Q} - HS)$ is an abelian $\mathbb{Q}$-linear tensor category. For every $w \in \mathbb{Z}$ we have a $\otimes$-invertible object in $(\oplus\mathbb{Q} - HS)$ of pure weight $2w$: the ***Tate Hodge structure*** $\mathbb{Q}(w)$ given by $\mathbb{Q}(w) := (\mathbb{C}, F^\bullet, \mathbb{Q})$, where $F^i = \mathbb{C}$ for $i \leq w$ and $F^i = \{0\}$ for $i > w$.

It turns out that pure Hodge structures can be viewed as **nc**-Hodge structures. This is achieved through a version of the Rees module construction (see e.g. [224]) which converts a filtered vector space into a bundle over the affine line $\mathbb{A}^1$. Specifically, given a pure Hodge structure $(V, F^\bullet V, V_\mathbb{Q})$ of weight w we consider the rank one meromorphic bundle with connection

$$\boldsymbol{\mathcal{T}}_{\frac{w}{2}} := \left(\mathbb{C}\{u\}[u^{-1}], d - \frac{w}{2} \cdot \frac{du}{u}\right)$$

and we set

- $\mathscr{H} := \mathscr{H}^{w \bmod 2} := \sum_i u^{-i} F^i V\{u\}$ viewed as a $\mathbb{C}\{u\}$-submodule in $\mathbb{C}\{u\}[u^{-1}] \otimes_\mathbb{C} V$. Clearly, this submodule is preserved by the operator $\nabla_{u\frac{d}{du}}$ for the connection $\nabla := \left(d - \frac{w}{2} \cdot \frac{du}{u}\right) \otimes \mathrm{id}_V$, i.e. $(\mathscr{H}, \nabla)$ is a logarithmic holomorphic extension of the meromorphic bundle with connection $\boldsymbol{\mathcal{T}}_{\frac{w}{2}} \otimes_\mathbb{C} V$.

  Note: Consider the algebraization $(H, \nabla) = \mathfrak{B}_0(\mathscr{H}, \nabla)$ of $(\mathscr{H}, \nabla)$. The fiber $H_1 := H/(u-1)H$ of H at $1 \in \mathbb{A}^1$ is canonically identified with V. By definition the connection ∇ on H has monodromy $(-1)^w \,\mathrm{id}_V$ and so preserves any rational subspace in V.

- $\mathscr{E}_B := \mathscr{E}_B^{w \bmod 2}$ - the $\mathbb{Q}$-local system on $\mathbb{A}^1 - \{0\}$ defined as the subsheaf $\mathscr{E}_B \subset H$ consisting of sections whose value at 1 is in $V_\mathbb{Q} \subset V = H/(u-1)H$. In other words $\mathscr{E}_B$ is the locally constant sheaf on $\mathbb{A}^1 - \{0\}$ with fiber $V_\mathbb{Q}$ and monodromy $(-1)^w \,\mathrm{id}_{V_\mathbb{Q}}$.

- $\xrightarrow{\sim}$ is the isomorphism of complex local systems, corresponding to the embedding $\mathscr{E}_B \subset H$.

Remark 2.8 On every simply connected open (in the analytic topology) subset $U \subset \mathbb{A}^1 - \{0\}$ the bundle with connection $\boldsymbol{\mathcal{T}}_{\frac{w}{2}}$ has a horizontal section $u^{w/2}$. In particular on such opens we have $H_{|U} = \sum_i u^{w/2} u^{-i} F^i[u]$.

The data $\left(H, \mathscr{E}_B, \xrightarrow{\sim}\right)$ satisfy tautologically the **($\mathbb{Q}$-structure axiom)** and the **(opposedness axiom)** from Definition 2.5. Indeed, the **($\mathbb{Q}$-structure axiom)** is satisfied since by definition ∇ has a regular singularity at 0 and so $\mathsf{S}_{\leq \omega} = \mathsf{S}$ or 0 for all ω. The **(opposedness axiom)** is satisfied as it is equivalent in the case of regular singularities to the oposedness property in the definition of the usual Hodge structures.

Thus, the assignment $(V, F^\bullet V, V_\mathbb{Q}) \to \left(H, \mathscr{E}_B, \xrightarrow{\sim}\right)$ gives a functor

$$\mathfrak{n} : (\oplus\mathbb{Q} - HS) \to (\oplus\mathbb{Q} - \textbf{nc}\mathsf{HS})$$

which by definition factors through the orbit category (see e.g. [149] for the definition of an orbit category)

$$\pi : (\oplus\mathbb{Q} - HS) \to (\oplus\mathbb{Q} - HS)/(\bullet \otimes \mathbb{Q}(1)),$$

i.e we have $\mathfrak{N} = \mathfrak{n} \circ \pi$ for a functor

$$\mathfrak{N} : (\oplus\mathbb{Q} - HS)/(\bullet \otimes \mathbb{Q}(1)) \to (\oplus\mathbb{Q} - \textbf{nc}\mathsf{HS}).$$

The proof of the following statement is an immediate consequence from the definition.

Lemma 2.9 *The functor* $\mathfrak{N}$ *is fully faithful and its essential image consists of all* **nc***-Hodge structures that have regular singularities and monodromy* $= \mathrm{id}$ *on* H^0 *and* $= -\mathrm{id}$ *on* H^1.

Remark 2.10 It is straightforward to check that the functor $\mathfrak{N}$ can also be defined in families and embeds the category of variations of Hodge structures (modulo the Tate twist) into the category of variations of **nc**-Hodge structures.

2.1.8 nc-Hodge structures of exponential type As we saw in section 2.1.7 the usual Hodge structures give rise to special **nc**-Hodge structures with regular singularities. The **nc**-Hodge structures with regular singularities are also important because they can serve as building blocks of general **nc**-Hodge structures. Let $(H, \mathscr{E}_B, \overset{\sim}{\to})$ be a **nc**-Hodge structure, let $(\mathscr{H}, \nabla) = \mathfrak{G}_0((H, \nabla))$ be the germ of (H, ∇) at zero, and assume that $A_{-2} \neq 0$, i.e. ∇ has a second order pole. According to Turrittin-Levelt formal decomposition theorem (see e.g. [170], [20], [205, II.5.7 and II.5.9]) we can find a finite base change $p_N : \mathbb{C} \to \mathbb{C}$, $p_N(t) := t^N = u$, so that $p_N^*(\mathscr{H}, \nabla)[t^{-1}]$ is formally isomorphic to a direct sum of regular singular connections on meromorphic bundles multiplied by exponents of Laurent polynomials. More precisely we can find polynomial tails $g_i(t) \in \mathbb{C}[t^{-1}]$, $\mathbb{C}\{t\}[t^{-1}]$-vector spaces $\mathscr{R}_i$ and meromorphic connections

$$(\nabla_i)_{\frac{d}{dt}} : \mathscr{R}_i \to \mathscr{R}_i,$$

each with at most regular singularity at 0, so that we have an isomorphism of formal meromorphic connections over $\mathbb{C}((t))$:

$$\Psi : p_N^*(\mathscr{H}, \nabla) \bigotimes_{\mathbb{C}\{t\}[t^{-1}]} \mathbb{C}((t)) \xrightarrow{\cong} \left(\bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{g_i} \bigotimes_{\mathbb{C}\{t\}[t^{-1}]} (\mathscr{R}_i, \nabla_i) \right) \bigotimes_{\mathbb{C}\{t\}[t^{-1}]} \mathbb{C}((t)).$$

Here $\boldsymbol{\mathcal{E}}^f$ denotes the rank one holomorphic bundle with meromorphic connection $(\mathbb{C}\{t\}, d - df)$, and $(\mathscr{R}_i, \nabla_i)$ denote meromorphic bundles with connections having regular singularities.

Remark 2.11 The bundle $\boldsymbol{\mathcal{E}}^f$ has a non-vanishing horizontal section, namely e^f. In particular the multivalued flat sections of $\boldsymbol{\mathcal{E}}^{g_i} \otimes (\mathscr{R}_i, \nabla_i)$ are given by multiplying multivalued flat sections of $(\mathscr{R}_i, \nabla_i)$ by e^{g_i}.

In the examples coming from Mirror Symmetry that we are interested in, the base change p_N is not needed for the decomposition to work. In this case we can take $g_i(u) = \boldsymbol{c}_i/u$ where $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ denote the distinct eigenvalues of A_{-2}. Because of this we introduce the following definition (see also [115, Definition 8.1]):

Definition 2.12 *We say that a* **nc***-Hodge structure* $(H, \mathscr{E}_B, \overset{\sim}{\to})$ *is of* ***exponential type*** *if there exists a formal isomorphism*

$$\Psi : (\mathscr{H} \otimes_{\mathbb{C}\{u\}} \mathbb{C}[[u]], \nabla) \overset{\cong}{\to} \bigoplus_{i=1}^{m} \left(\boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i) \right) \otimes_{\mathbb{C}\{u\}} \mathbb{C}[[u]]$$

where $(\mathscr{R}_i, \nabla_i)$ *are meromorphic bundles with connections with regular singularities and* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ *denote the distinct eigenvalues of* A_{-2}.

Remark 2.13 • There are various sufficient conditions that will guarantee that a given **nc**-Hodge structure is decomposable without base change. For instance, this will be the case if A_{-2} has distinct eigenvalues, or if $A_{-1} = 0$. More generally, it suffices to require that we can find holomorphic functions $\ell_i(u) \in \mathbb{C}\{u\}$ so that $\ell_i(0) = \boldsymbol{c}_i$ for $i = 1, \ldots, m$ and the characteristic polynomial of $u^2 A(u)$ is $\det\left(\boldsymbol{c} \cdot \mathrm{id} - u^2 A(u)\right) = \prod_{i=1}^{m} (\boldsymbol{c} - \ell_i)^{\nu_i}$.

• Not every irregular connection with a pole of order two is of exponential type. Indeed the rank two connection

$$\nabla = d - \begin{pmatrix} 0 & u^{-2} \\ u^{-1} & \frac{u^{-1}}{2} \end{pmatrix}$$

has a horizontal section

$$\begin{pmatrix} e^{-2u^{-\frac{1}{2}}} \\ u^{\frac{1}{2}} e^{-2u^{-\frac{1}{2}}} \end{pmatrix},$$

and so one needs a quadratic base change for the formal decomposition to work for this connection.

• If a **nc**-Hodge structure $(H, \mathscr{E}_B, \overset{\sim}{\to})$ is of exponential type, then one can check (see [115, Lemma 8.2]) that for each $i = 1, \ldots, m$ we can find a unique holomorphic extension $\mathscr{H}_{\boldsymbol{c}_i} \subset \mathcal{R}_i$ in which the connection has a second order pole and so that Ψ induces a formal isomorphism of holomorphic bundles with meromorphic connections

$$\Psi : (\mathscr{H}, \nabla) \otimes \mathbb{C}[[u]] \xrightarrow{\cong} \left(\bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \bigotimes_{\mathbb{C}\{u\}} (\mathscr{H}_{\boldsymbol{c}_i}, \nabla_i) \right) \otimes \mathbb{C}[[u]],$$

over $\mathbb{C}[[u]]$.

The **nc**-Hodge structures with regular singularities or the **nc**-Hodge structures of exponential type comprise full subcategories

$$(\oplus\mathbb{Q} - \mathbf{nc}\mathsf{HS})^{\mathrm{reg}} \subset (\oplus\mathbb{Q} - \mathbf{nc}\mathsf{HS})^{\mathrm{exp}} \subset (\oplus\mathbb{Q} - \mathbf{nc}\mathsf{HS})$$

in $(\oplus\mathbb{Q} - \mathbf{nc}\mathsf{HS})$. In fact, in the exponential type case one can state the **nc**-Hodge structure axioms in an easier way. The simplification comes from the fact that in this case the Deligne-Malgrange-Stokes filtration is given by subsheaves $\mathbf{S}_{\leq\lambda}$ of $\mathbf{S}$ that are labeled by $\lambda \in \mathbb{R}$ and consisting of solutions decaying faster than $\boldsymbol{O}\left(\exp\left(\frac{\lambda+o(1)}{r}\right)\right)$, $r = |u|$. Indeed, tracing through the definition one sees that in the exponential case for a ray defined by φ the jumps of the steps of the Deligne-Malgrange-Stokes filtration occur exactly at the numbers $\mathrm{Re}(\boldsymbol{c}_i e^{-i\varphi})$. Furthermore, the associated graded pieces for the filtration are local systems on the circle and in fact coincide with the regular pieces $(\mathscr{R}_i, \nabla_i)$ that appear in the formal decomposition of the connection. Hence one arrives at the following

Definition 2.14 *A* ***rational pure*** **nc*****-Hodge structure of exponential type*** *consists of the data* $(H, \mathscr{E}_B, \overset{\sim}{\to})$*, where*

- H *is a* $\mathbb{Z}/2$*-graded algebraic vector bundle on* $\mathbb{A}^1$.
- $\mathscr{E}_B$ *is a local system of finite dimensional* $\mathbb{Z}/2$*-graded* $\mathbb{Q}$*-vector spaces on* $\mathbb{A}^1 - \{0\}$.
- $\overset{\sim}{\to}$ *is an* analytic *isomorphism of holomorphic vector bundles on* $\mathbb{A}^1 - \{0\}$*:*

$$\overset{\sim}{\to}: \mathscr{E}_B \otimes \mathcal{O}_{\mathbb{A}^1-\{0\}} \xrightarrow{\cong} H_{|\mathbb{A}^1-\{0\}}.$$

These data have to satisfy the following axioms:

(nc-filtration axiom)$^{\mathbf{exp}}$ *The connection* ∇ *induced from* $\overset{\sim}{\to}$ *is a meromorphic connection of exponential type on* H *with a pole of order* ≤ 2 *at* $u = 0$ *and a regular singularity at* ∞*. More precisely, there exist:*

- *a holomorphic frame of* $\mathscr{H}$ *near* $u = 0$ *in which*

$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k\right) du$$

with $A_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$*,* $r = \mathrm{rank}_{\mathbb{C}\{u\}} \mathscr{H}$.

- *a holomorphic frame of* $\mathscr{H}$ *near* $u = \infty$ *in which*

$$\nabla = d + \left(\sum_{k \geq -1} B_k u^{-k}\right) d(u^{-1})$$

and $B_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$.

- *a formal isomorphism over* $\mathbb{C}((u))$*:*

$$(\mathscr{H}[u^{-1}], \nabla) \overset{\cong}{\to} \bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i)$$

where $(\mathscr{R}_i, \nabla_i)$ *are meromorphic bundles with connections with regular singularities and* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ *denote the distinct eigenvalues of* A_{-2}.

($\mathbb{Q}$-structure axiom)$^{\mathbf{exp}}$ *The $\mathbb{Q}$-structure $\mathscr{E}_B$ on (H, ∇) is compatible with Stokes data in the following sense. The filtration $\{\mathsf{S}_{\leq\lambda}\}_{\lambda\in\mathbb{R}}$ of S by the subsheaves $\mathsf{S}_{\leq\lambda}$, whose stalk at $\varphi \in \boldsymbol{S}^1$ is given by*

$$(\mathsf{S}_{\leq\lambda})_{\varphi} := \left\{ s \in \mathsf{S}_{\varphi} = \Gamma\left(\mathbb{R}_+^{\times} e^{i\varphi}, H\right) \;\middle|\; \begin{array}{l} s \text{ is a } \nabla\text{-horizontal section of } H \text{ over the} \\ \text{ray } \mathbb{R}_+^{\times} e^{i\varphi}, \text{ for which} \\ \\ \left\| s\left(re^{i\varphi}\right)\right\| = \boldsymbol{O}\left(\exp\left(\dfrac{\lambda + \boldsymbol{o}(1)}{r}\right)\right). \\ \\ \text{when } r \to 0. \end{array} \right\}$$

is defined over $\mathbb{Q}$, i.e.

$$(\mathsf{S}_{\leq\lambda} \cap \mathsf{S}_B) \otimes_{\mathbb{Q}} \mathbb{C} = \mathsf{S}_{\leq\lambda}$$

for all $\lambda \in \mathbb{R}$.

(opposedness axiom)$^{\mathbf{exp}}$ = (opposedness axiom)

Remark 2.15 It is instructive to understand more explicitly the behavior of the Deligne-Malgrange-Stokes filtration for **nc**-Hodge structures (or more generally irregular connections) of exponential type. As before we denote by S the complex local system on the circle $\boldsymbol{S}^1$ corresponding to a **nc**-Hodge structure for which A_{-2} has distinct eigenvalues $\boldsymbol{c}_1, \ldots \boldsymbol{c}_m$.

By definition, for every φ, the steps in the Deligne-Malgrange-Stokes filtration $(\mathsf{S}_{\leq\lambda})_{\varphi}$ jump exactly when λ crosses one of the numbers $\mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})$. More invariantly, the assignment $\varphi \in \boldsymbol{S}^1 \mapsto \{\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})\} \subset \mathbb{R}$ is a sheaf Λ of finite sets of real numbers (possibly with repetitions) on $\boldsymbol{S}^1$. For a general value of φ, the real numbers $\{\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})\}$ are all distinct but for finitely many special values of φ some of $\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})$ will coalesce. More precisely we have the Stokes rays $\mathbb{R}_{>0} \cdot i(\boldsymbol{c}_b - \boldsymbol{c}_a)$ and the associated set $\mathsf{SD} \subset [0, 2\pi)$ of Stokes directions: i.e. $\varphi \in \mathsf{SD}$, if and only if there is some pair $a \neq b$ s.t. $\boldsymbol{c}_a - \boldsymbol{c}_b = re^{i\left(\frac{\pi}{2}+\varphi\right)}$ for some $r > 0$. Clearly for every open arc $U \subset \boldsymbol{S}^1$, which does not intersect SD the restriction $\Lambda_{|U}$ is a local system of finite sets of cardinality m. Moreover the values $\varphi \in \mathsf{SD}$ are precisely the ones for which some of $\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})$ become equal to each other.

Now recall that for any given $\varphi \in \boldsymbol{S}^1$, the subspaces $(\mathsf{S}_{\leq\lambda})_{\varphi} \subset \mathsf{S}_{\varphi}$ do not change if we move $\lambda \in \mathbb{R}$ continuously without passing through some element of Λ_{φ}. In other words, we can label the steps of the Deligne-Malgrange-Stokes filtration by local sections of Λ, and so that at each $\varphi \in \boldsymbol{S}^1$ the steps are ordered according to the order on Λ_{φ} induced from the embedding $\Lambda_{\varphi} \subset \mathbb{R}$. The finite set $\mathsf{SD} \subset \boldsymbol{S}^1$ of Stokes directions breaks the circle into disjoint arcs. Over each such arc U we have that $\Lambda_{|U}$ is a local system of finite sets of real numbers with m linearly ordered flat sections and the steps Deligne-Malgrange-Stokes filtration of $\mathsf{S}_{|U}$ are labeled naturally by these sections. If we move from U to an adjacent arc U' by passing across a Stokes direction $\phi \in \mathsf{SD}$, then some of the elements in the labelling set get identified at ϕ and get reordered when we cross over to U' (see Figure 1).

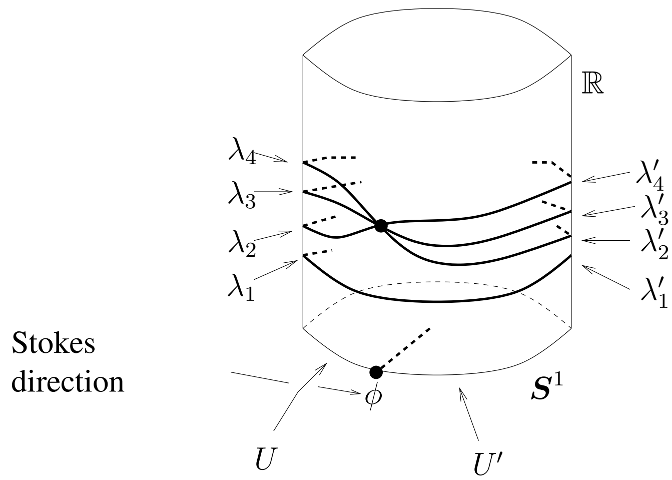


Figure 1: The system of labels for the Deligne-Malgrange-Stokes filtration.

In fact, if $\lambda_1 < \ldots < \lambda_m$ are the ordered flat sections of $\Lambda_{|U}$, and $\lambda'_1 < \ldots < \lambda'_m$ are the ordered flat sections of $\Lambda_{|U'}$, then the transition from the λ's to the λ''s is always such that certain groups of consecutive λ's are totally reordered into groups of consecutive λ''s. For instance in Figure 1 the passage from $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ to $\{\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4\}$ across the Stokes point $\phi \in \mathsf{SD}$ has the effect of relabelling: $\lambda_1 \mapsto \lambda'_1$, $\lambda_2 \mapsto \lambda'_4$, $\lambda_3 \mapsto \lambda'_3$, and $\lambda_4 \mapsto \lambda'_2$.

This behavior of the labelling set and the behavior of the associated filtration can be systematized in the following:

Definition 2.16 *Let S be a finite dimensional local system of $\mathbb{Z}/2$-graded complex vector spaces over $\boldsymbol{S}^1$. Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m$ be distinct complex numbers, let Λ be the sheaf of finite sets of real numbers on $\boldsymbol{S}^1$ given by $\varphi \mapsto \{\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi})\}$, and let $\mathsf{SD} \subset \boldsymbol{S}^1$ be the associated set of Stokes directions.*

*An **abstract Deligne-Malgrange-Stokes filtration of S of exponential type and exponents** $(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$ is a filtration by subsheaves $\mathsf{S}_{\leq\lambda}$ such that:*

- *$\mathsf{S}_{\leq\lambda}$ is labeled by local continuous sections λ of Λ and is locally constant on any arc which does not intersect SD.*

- *Suppose $\varphi \in \mathsf{SD}$, and let $U, U' \subset \boldsymbol{S}^1 - \mathsf{SD}$ be the two arcs adjacent at φ. Let $\lambda_1 < \cdots < \lambda_m$ and $\lambda'_1 < \cdots < \lambda'_m$ be the ordered flat sections of $\Lambda_{|U}$ and $\Lambda_{|U'}$ respectively. Trivialize S on $U \cup U' \cup \{\varphi\}$ by identifying the flat sections with the elements of the fiber S_φ and let*

$$0 \subset F_{\leq\lambda_1} \subset \ldots \subset F_{\leq\lambda_m} \subset \mathsf{S}_\varphi, \qquad \textit{and} \qquad 0 \subset F'_{\leq\lambda'_1} \subset \ldots \subset F'_{\leq\lambda'_m} \subset \mathsf{S}_\varphi$$

be the filtrations corresponding to this trivialization and the filtrations $\mathsf{S}_{\leq\lambda}$ on U and U' respectively, i.e.

$$F_{\leq\lambda_i} := \lim_{\substack{\psi\in U\\ \psi\to\varphi}} (\mathsf{S}_{\leq\lambda_i})_\psi \qquad \textit{and} \qquad F'_{\leq\lambda'_i} := \lim_{\substack{\psi\in U'\\ \psi\to\varphi}} \left(\mathsf{S}_{\leq\lambda'_i}\right)_\psi$$

Let $1\leq i_1 < j_1 \leq i_2 < j_2 \leq \cdots \leq i_k < j_k \leq m$ be the sequence of integers such that $\lambda_a = \lambda'_a$ for $a \notin [i_1, j_1]\cup[i_2, j_2]\cup\cdots\cup[i_k, j_k]$, and for each interval $[i_s, j_s]$ we have that $\lambda'_{j_s} = \lambda_{i_s}$, $\lambda_{j_s-1} = \lambda_{i_s+1}, \ldots \lambda'_{i_s} = \lambda_{j_s}$. Then we require that:

- *for each $a \notin [i_1, j_1]\cup[i_2, j_2]\cup\cdots\cup[i_k, j_k]$ we have $F_{\leq\lambda_a} = F'_{\lambda'_a}$;*
- *for each $s = 1, \ldots, k$, $F_{\leq\lambda_{j_s}} = F_{\leq\lambda'_{j_s}}$ and the filtrations*

$$\begin{array}{ccccccc}
F_{\leq\lambda_{i_s}}/F_{\leq\lambda_{i_s-1}} & \subset & F_{\leq\lambda_{i_s+1}}/F_{\leq\lambda_{i_s-1}} & \subset & \cdots & \subset & F_{\leq\lambda_{j_s}}/F_{\leq\lambda_{i_s-1}} \\
 & & & & & & \| \\
F'_{\leq\lambda'_{i_s}}/F'_{\leq\lambda'_{i_s-1}} & \subset & F'_{\leq\lambda'_{i_s+1}}/F'_{\leq\lambda'_{i_s-1}} & \subset & \cdots & \subset & F'_{\leq\lambda'_{j_s}}/F'_{\leq\lambda'_{i_s-1}}
\end{array}$$

are $(j_s - i_s)$-opposed.

Remark 2.17 The above discussion generalizes immediately from connections of exponential type to arbitrary meromorphic connections (see remark 2.3). One gets a collection of curves drawn on the boundary of the cylinder which can be interpreted as a projection to 0-jets of a Legendrian link in the contact manifold of 1-jets of functions on S^1.

The categories of nc-Hodge structures, of nc-Hodge structures of exponential type, or of nc-Hodge structures with regular singularities all behave similarly to ordinary Hodge structures. For instance one can introduce the notion of polarization on nc-Hodge structures, which specializes to the usual notion in the case of ordinary Hodge structures. (This will not be needed for our discussion so we will not spell it out here. The interested reader may wish to consult [114], [115], [154] for the details of the definition.) In fact we have the following

Lemma 2.18 *The categories $(\oplus\mathbb{Q} - \mathsf{nc}\mathsf{HS})^{\mathrm{reg}} \subset (\oplus\mathbb{Q} - \mathsf{nc}\mathsf{HS})^{\mathrm{exp}} \subset (\oplus\mathbb{Q} - \mathsf{nc}\mathsf{HS})$ are $\mathbb{Q}$-linear abelian categories. The respective categories of polarizable nc-Hodge structures are semi-simple.*

Proof: The statement is a manifestation of Simpson's **Meta-Theorem** from [225]. The opposedness axiom implies that the respective categories are abelian and the existence of polarizations implies the semi-simplicity. The proofs follow verbatim the argument in usual Hodge theory or the argument in [225]. Alternatively one can use the comparison statement [115, Lemma 3.9] identifying the nc-Hodge structures with pure twistor structures and then invoke [225, Lemma 1.3 and Lemma 3.1]. □

The bundles with connections $(\mathscr{H}_{\boldsymbol{c}_i}, \nabla_i)$ can be thought of as the regular singular constituents of the nc-Hodge structure $(H, \mathscr{E}_B, \xrightarrow{\sim})$. The $(\mathscr{H}_{\boldsymbol{c}_i}, \nabla_i)$'s are invariants of the nc-Hodge structure but of course they do not give a complete set of invariants (see the third point in 2.13). As usual we need additional

Stokes data (see e.g. [205]) in order to reconstruct the pair $(\mathscr{H}, \nabla)$ from its regular constituents. To understand how the rest of **nc**-Hodge structure arises from the constituents we need to understand how the rational structure $\mathscr{E}_B$ interacts with the Stokes data. This process is very similar to the interaction between Betti, de Rham and Dolbeault cohomology in ordinary Hodge theory and we will describe it in detail in section 2.3.

The **nc**-Hodge structures one finds in geometric examples are very often regular (e.g. in the case of ordinary Hodge structures) or at worst have exponential type. It is also expected that the **nc**-Hodge structures arising in mirror symmetry will always be of exponential type but at the moment this is only supported by experimental evidence.

We will discuss in detail some of this evidence in the subsequent sections. Before we get to the examples however, it will be instructive to comment on the reason for introducing the **nc**-Hodge structures at the first place. The geometric significance of these structures stems from the fact that they appear naturally on the cohomology of non-commutative spaces of categorical nature.

2.2 Hodge structures in nc geometry The version of non-commutative geometry that is most relevant to **nc**-Hodge structures is the one in which a proxy for the notion of a non-commutative space (**nc**-space) is a category, thought of as the (unbounded) derived category of quasi-coherent sheaves on that space.

2.2.1 Categorical nc-geometry The basic notion here is:

Definition 2.19 *A **graded complex nc-space** (respectively a **complex nc-space**) is a $\mathbb{C}$-linear differential graded (respectively $\mathbb{Z}/2$-graded) category C which is homotopy complete and cocomplete.*

Notation: We will often write C_X for the category to signify that it describes the sheaf theory of some **nc**-space X, even when we do not have a geometric construction of X.

The categorical point of view to non-commutative geometry goes back to the works of Bondal [33], Bondal-Orlov [38, 39] with many non-trivial examples computed in the later works of Orlov [194, 196, 195], Caldero-Keller [42, 43], Aroux, Orlov, and the first author, [16, 17], Kuznetsov [162, 161, 163], etc. More recently this approach to **nc**-geometry became the central part of a long term research program initiated by the second author and was studied systematically in the works of the second author and Soibelman [159, 154], Toën [234], and Toën-Vaquie [237].

Remark 2.20 **(i)** We do not spell out here the notions of homotopy completeness and cocompleteness in dg categories since on one hand they are quite technical and on the other hand will not be used later in the chapter. It is worth mentioning though that some effort is required to define these notions. In the original approach of the second author described in his 2005 IAS lectures and in his 2007 course at the University of Miami the homotopy completeness and cocompletness in C was defined by a universal property for homotopy coherent diagrams of objects in the dg category labeled by simplicial sets. Alternatively [236] one may use the model category $(C^{\mathrm{op}} - \mathsf{mod})$ of C^{op}-dg modules, whose equivalences are the quasi-isomorphisms, and whose fibrations are the epimorphisms. In these terms one says that C is homotopy complete if the full subcategory of $(C^{\mathrm{op}} - \mathsf{mod})$ consisting of quasi-representable objects is preserved by all small homotopy limits (defined via the given model structure). Similarly we say that C is homotopy cocomplete if C^{op} is homotopy complete.

(ii) Note that in the above definition the category C is automatically triangulated as follows already from the existence of *finite* homotopy limits, and Karoubi closed by the standard mapping telescope construction [32].

Example 2.21 The two main types of **nc**-spaces are the following:

usual schemes: Usual complex schemes can be viewed as (graded) **nc**-spaces. Given a scheme X over $\mathbb{C}$, the corresponding category C_X is the derived category $D(\mathsf{Qcoh}(X)w)$ of quasi-coherent sheaves on X taken with an appropriate dg enhancement (see [37]). In particular, the closed point $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ corresponds to the category C_{pt} of complexes of $\mathbb{C}$-vector spaces.

modules over an algebra: If A is a differential graded (or $\mathbb{Z}/2$-graded) unital associative algebra over $\mathbb{C}$, then we get a **nc**-space $\oplus\mathbf{ncSpec}(A)$ such that $C_{\oplus\mathbf{ncSpec}(A)} = (A - \mathsf{mod})$ is the category of dg modules over A which admit an exhaustive increasing filtration whose associated graded are sums of shifts of A.

To illustrate how the above notion of a **nc**-space fits with the **nc**Hodge structures we will concentrate on the case of **nc**-affine spaces, i.e. **nc**-spaces equivalent to $\oplus\mathbf{ncSpec}(A)$ for some differential $\mathbb{Z}/2$-graded algebra A over $\mathbb{C}$. Note that because of derived Morita equivalences an affine **nc**-space X does not determine an algebra A uniquely, i.e. different algebras can give rise to the same **nc**-space.

Remark 2.22 The condition is not as restrictive as it appears at a first glance. In fact almost all **nc**-spaces that one encounters in practice are affine. For instance usual quasi-compact quasi-separated schemes of finite type over $\mathbb{C}$ are affine when viewed as **nc**-spaces. This follows from a deep theorem of Bondal and van den Bergh [41] which asserts that for such a scheme X the category $C_X = D(\mathsf{Qcoh}(X))$ has a compact generator $\mathcal{E}$. That is, we can find an object $\mathcal{E} \in C_X$ so that

$$\mathrm{Hom}(\mathcal{E}, \bullet) : C_X \to C_{\mathrm{pt}}$$

commutes with homotopy colimits and has a zero kernel. In particular the dg algebra computing the category C_X is given in terms of the generator $\mathcal{E}$, i.e.

$$C_X \cong (Hom(\mathcal{E}, \mathcal{E})^{\mathrm{op}} - \mathsf{mod}).$$

Suppose now that $X = \oplus\mathbf{ncSpec}(A)$. Recall that an object $\mathcal{E} \in C_X = (A - \mathsf{mod})$ is ***perfect*** if $\mathrm{Hom}(\mathcal{E}, \bullet)$ preserves small homotopy colimits. We will write $\oplus\mathsf{Perf}_X$ for the full subcategory of perfect objects in C_X. We now have the following definition (see e.g. [159], [154] or [237]):

Definition 2.23 *A complex differential $\mathbb{Z}/2$-graded algebra is called*

smooth: *if $A \in \oplus\mathsf{Perf}_{\oplus\mathbf{ncSpec}(A\otimes A^{\mathrm{op}})}$;*

compact: *if $\dim_{\mathbb{C}} H^{\bullet}(A, d_A) < +\infty$ or equivalently if $A \in \oplus\mathsf{Perf}_{\mathrm{pt}}$.*

Note: One can check (see e.g. [159] or [237]) that the properties of X being smooth ad compact do not depend on the choice of the algebra A which computes C_X. Also, for a usual scheme X of finite type over $\mathbb{C}$, smoothness and compactness in the scheme-theoretic sense are equivalent to smoothness and compactness in the **nc**-sense.

2.2.2 The main conjecture The analogy with commutative geometry suggests that one should look for pure **nc**-Hodge structures on the cohomology of smooth an proper **nc**-spaces. More precisely we have the following basic conjecture

Conjecture 2.24 *Let X be a smooth and compact **nc**-space over $\mathbb{C}$. Then the periodic cyclic homology $HP_{\bullet}(C_X)$ of C_X carries a natural functorial pure $\mathbb{Q}$-**nc**-Hodge structure with regular singularities.*

*Furthermore if the $\mathbb{Z}/2$-grading on X can be refined to a $\mathbb{Z}$-grading, then the **nc**-Hodge structure on $HP_{\bullet}(C_X)$ is an ordinary pure Hodge structure, i.e. belongs to the essential image of the functor $\mathfrak{N}$.*

2.2.3 Cyclic homology There are some natural candidates for the various ingredients of the conjectural **nc**-Hodge structure on $HP_{\bullet}(C_X)$. Assuming that $X \cong \oplus\mathbf{ncSpec}(A)$ is **nc**-affine, we can compute $HP_{\bullet}(C_X)$ in terms of A. Namely

$$HP_{\bullet}(C_X) = HP_{\bullet}(A) = HP_{\bullet}\left(C_{\bullet}^{\mathrm{red}}(A, A)((u)), \partial + u \cdot B\right),$$

where

- u is an even formal variable (of degree 2 in the $\mathbb{Z}$-graded case);
- $C_{-k+1}^{\mathrm{red}}(A, A)((u)) := A \otimes (A/\mathbb{C} \cdot 1_A)^{\otimes k} \otimes \mathbb{C}((u))$, for all $k \geq 0$;

- $\partial = b + \delta$, where

$$b(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^{\deg(a_0 \otimes \cdots \otimes a_i)} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^{\deg(a_0 \otimes \cdots \otimes a_n)(\deg(a_n)+1)+1} a_n a_0 \otimes \cdots \otimes a_{n-1},$$

is the Hochschild differential, and

$$\delta(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n} (-1)^{\deg(a_0 \otimes \cdots \otimes a_{i-1})} a_0 \otimes \cdots \otimes d_A a_i \otimes \cdots \otimes a_n$$

is the differential induced from d_A via the Leibniz rule;

-
$$B(a_0 \otimes \cdots \otimes a_n) :=$$
$$\sum_{i=0}^{n} (-1)^{(\deg(a_0 \otimes \cdots \otimes a_i)-1)(\deg(a_{i+1} \otimes \cdots \otimes a_n)-1)} 1_A \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i,$$

is Connes' cyclic differential.

2.2.4 The degeneration conjecture and the vector bundle part of the nc-Hodge structure Note that by construction $HP_\bullet(C_X)$ is a module over $\mathbb{C}((u))$. We can also look at the negative cyclic homology $HC_\bullet^-(C_X)$ of C_X. By definition $HC_\bullet^-(C_X)$ is the cohomology of the complex

$$\left(C_\bullet^{\mathrm{red}}(A, A)[[u]], \partial + u \cdot B \right),$$

and so is a module over $\mathbb{C}[[u]]$. The specialization

$$HC_\bullet^-(C_X)/uHC_\bullet^-(C_X)$$

of this module at $u = 0$ maps to the cohomology of the complex

$$(C_\bullet^{\mathrm{red}}(A, A), \partial)$$

of reduced Hochschild chains for A which by definition is the Hochschild homology $HH_\bullet(A)$ of A. The Hochschild-to-cyclic spectral sequence implies that

$$\dim_{\mathbb{C}((u))} HP_\bullet(A) \leq \dim_{\mathbb{C}} HH_\bullet(A) \tag{2.1}$$

If X is a smooth and compact **nc**-space, the Hochschild chain complex of C_X is the derived tensor product over $A \otimes A^{\mathrm{op}}$ of a perfect complex with finite dimensional cohomology with itself. In particular $HH_\bullet(C_X) := HH_\bullet(A)$ is a finite dimensional $\mathbb{C}$-vector space, and so by (2.1) we have that $HP_\bullet(C_X)$

is finite dimensional over $\mathbb{C}((u))$. Thus the $\mathbb{C}[[u]]$-module $HC^{-}_{\bullet}(C_X)$ is finitely generated and so corresponds to the formal germ at $u = 0$ of an algebraic $\mathbb{Z}/2$-graded coherent sheaf on $\mathbb{A}^1_{\mathbb{C}}$. The fiber of this sheaf at $u = 0$ is $HH_{\bullet}(C_X)$ and the generic fiber is $HP_{\bullet}(C_X)$. In [159], [154] the second author proposed the so called ***degeneration conjecture*** asserting that for a smooth and compact **nc**-space $X = \oplus\mathbf{ncSpec}(A)$ we have an equality of dimensions in (2.1). In other words the degeneration conjecture assert that for a smooth and compact **nc**-space the $\mathbb{C}[[u]]$-module $HC^{-}_{\bullet}(C_X)$ is free of finite rank and thus corresponds to an algebraic vector bundle on the one dimensional formal disc $\mathbb{D} := \mathrm{Spf}(\mathbb{C}[[u]])$.

Remark 2.25 There is a lot of evidence supporting the validity of this conjecture. The work of Weibel [250] shows that if X is a usual quasi-compact and quasi-separated complex scheme the Hochschild and periodic cyclic homology of X viewed as a **nc**-space can be identified with the algebraic de Rham and Dolbeault cohomology of X respectively. Combined with the degeneration of the Hodge-to-de-Rham spectral sequence in the smooth proper case this shows that the degeneration conjecture holds true for usual schemes. Also recently in a very exciting sequence of papers [130], [129] Kaledin proved the degeneration conjecture for graded **nc**-spaces $X = \oplus\mathbf{ncSpec}(A)$ for which A is concentrated in non-negative degrees. The case of graded **nc**-spaces $X = \oplus\mathbf{ncSpec}(A)$ for which A is concentrated in non-positive degrees was also settled by Shklyarov [222]. The general graded case and the $\mathbb{Z}/2$-graded case are still wide open.

2.2.5 The meromorphic connection in the u-direction The next observation is that the $\mathbb{C}\{u\}[u^{-1}]$-module $HP_{\bullet}(C_X)$ comes equipped with a natural meromorphic connection. Indeed, recall that by the work of Getzler [91] there is a version of the Gauss-Manin connection which exists on the periodic cyclic homology of any flat family of differential graded algebras (see also [239], [131]). An analogous statement holds in the $\mathbb{Z}/2$-graded case as explained e.g. in [159, Section 11.5]. The Gauss-Manin connection for any family of dg algebras $\mathcal{A}_x$ over the formal disc $\mathrm{Spf}\,\mathbb{C}[[x]]$ with a formal parameter x, is an operator

$$\nabla^{\mathrm{GM}}_{u\frac{\partial}{\partial x}} : H^{\bullet}(C^{\mathrm{red}}(\mathcal{A}_x, \mathcal{A}_x)[[u, x]], \partial_{\mathcal{A}_x} + u \cdot B_{\mathcal{A}_x}) \to H^{\bullet}(C^{\mathrm{red}}(\mathcal{A}_x, \mathcal{A}_x)[[u, x]], \partial_{\mathcal{A}_x} + u \cdot B_{\mathcal{A}_x})$$

satisfying the Leibniz rule with respect to the multiplications by u and x (compare this with the **(Griffiths transversality axiom)** in Definition 2.7 from Section 2.1.5).

Suppose now A is a differential $\mathbb{Z}/2$-graded algebra with product m_A, differential d_A, and a strict unit 1_A. Then we can form a flat family $\mathcal{A} \to \mathbb{A}^1 - \{0\}$ of differential $\mathbb{Z}/2$-graded algebras parameterized by the punctured affine line $\mathbb{A}^1 - \{0\}$. The fiber $\mathcal{A}_t$ of $\mathcal{A}$ over a point $t \in \mathbb{A}^1_{\mathbb{C}} - \{0\}$ is the d($\mathbb{Z}/2$)g algebra for which the underlying $\mathbb{Z}/2$-graded vector space is A and

$$\begin{aligned} m_{\mathcal{A}_t} &:= t \cdot m_A, \\ d_{\mathcal{A}_t} &:= t \cdot d_A, \\ 1_{\mathcal{A}_t} &:= t^{-1} \cdot 1_A. \end{aligned}$$

Looking at the scaling properties of ∂ and B we see that the identity morphism on the level of cochains induces a natural isomorphism

$$H^{\bullet}\left(C_{\bullet}^{\mathrm{red}}(\mathcal{A}_t,\mathcal{A}_t)[[u]],\partial_{\mathcal{A}_t}+u\cdot B_{\mathcal{A}_t}\right)\xrightarrow{\cong}H^{\bullet}(C_{\bullet}^{\mathrm{red}}(A,A)[[u]],\partial+ut^{-2}\cdot B). \tag{2.2}$$

This isomorphism does not come from a quasi-isomorphism of complexes, as the identity map is not a morphims of complexes: the differentials do not coincide but differ by the factor t. If A is smooth and compact, then the negative cyclic homology of the family of algebras $\mathcal{A}_t$ gives rise to an algebraic vector bundle $\widetilde{HC}^{-}$ on the product $(\mathbb{A}^1-\{0\})\times\mathbb{D}$. Here $\mathbb{D}:=\operatorname{Spf}\mathbb{C}[[u]]$ denotes the one dimensional formal disc. We will write (t,u) for the coordinates on $(\mathbb{A}^1-\{0\})\times\mathbb{D}$. We will be interested in fact only in the formal neigborhood of point $t=1$ where we can choose as a local coordinate $x:=\log(t)$. The Getzler-Gauss-Manin connection then can be viewed as a relative holomorphic connection ∇^{GM} on $\widetilde{HC}^{-}$ which differentiates only along $\mathbb{A}^1_{\mathbb{C}}-\{0\}$. On the other hand the formal completion of the group $\mathbb{C}^{\times}$ at 1 acts on $(\mathbb{A}^1_{\mathbb{C}}-\{0\})\times\mathbb{D}$ by $(t,u)\mapsto(\mu t,\mu^2u)$ for $\mu\in\mathbb{C}^{\times}$. The isomorphism (2.2) gives rise to a $\mathbb{C}^{\times}$-equivariant structure on the vector bundle $\widetilde{HC}^{-}$ and the infinitesimal action of $d/d\mu$ associated with this equivariant structure gives a holomorphic differential operator $\mathbf{\Lambda}\in\operatorname{Diff}^{\leq 1}(\widetilde{HC}^{-},\widetilde{HC}^{-})$ with symbol equal to

$$\left(t\frac{\partial}{\partial t}+2u\frac{\partial}{\partial u}\right)\cdot\mathrm{id}_{\widetilde{HC}^{-}}.$$

Hence

$$\nabla_{u^2\frac{\partial}{\partial u}}:=\frac{u}{2}\cdot\mathbf{\Lambda}-\nabla^{\mathrm{GM}}_{\frac{ut}{2}\frac{\partial}{\partial t}}$$

is a first order differential operator on $\widetilde{HC}^{-}$ with symbol

$$u^2\frac{\partial}{\partial u}\cdot\mathrm{id}_{\widetilde{HC}^{-}}$$

and so after restricting $\widetilde{HC}^{-}$ to $\{1\}\times\mathbb{D}$ this operator gives a meromorphic connection ∇ on the $\mathbb{C}[[u]]$-module $HC_{\bullet}^{-}(C_X)$ with at most a second order pole at $u=0$. Note also that if the algebra A is $\mathbb{Z}$-graded, then the family $\mathcal{A}_t$ t is easily seen to be trivial and the connection ∇ has the first order pole at $u=0$ with monodromy equal to $(-1)^{\mathrm{parity}}$.

2.2.6 The $\mathbb{Q}$-structure The categorical origin of the rational (or integral) structure of the conjectural **nc**-Hodge structure is more mysterious. Conceptually the correct rational structure should come from the Betti cohomology or, say, the topological K-theory of the **nc**-space. There are two natural approaches to constructing the rational structure $\mathscr{E}_B\subset HP_{\bullet}(C_X)$:

(a) The soft algebra approach ([154]). Let again $X=\oplus\mathbf{ncSpec}(A)$ be an affine **nc**-space, and assume X is compact. By analogy with the classical geometric case one expects that there should exist a nuclear Frechét d$(\mathbb{Z}/2)$g algebra $A_{C^{\infty}}$ so that

- The K-theory of A_{C^∞} satisfies Bott periodicity, i.e. $K_i(A_{C^\infty}) = K_{i+2}(A_{C^\infty})$ for all $i \geq 0$.
- There is a homomorphism $\varphi : A \to A_{C^\infty}$ of d($\mathbb{Z}/2$)g algebras for which $\varphi_* : HP_\bullet(A) \to HP_\bullet(A_{C^\infty})$ is an isomorphism, and the image of the Chern character map
$$ch : K_\bullet(A_{C_\infty}) \to HP_\bullet(A_{C^\infty})$$
is an integral lattice, and hence gives a rational structure $\mathscr{E}_B \subset HP_\bullet(A)$.

Note: If X is a smooth and compact complex variety and if $\mathcal{E} \in \oplus\mathsf{Perf}(X)$ is a vector bundle generating C_X, then one may take

$$A := A^{0,\bullet}(X, \mathcal{E}^\vee \otimes \mathcal{E}), \bar{\partial})$$
$$A_{C^\infty} := A^{0,0}(X, \mathcal{E}^\vee \otimes \mathcal{E}).$$

Note that the algebra A_{C^∞} is Morita equivalent to $C^\infty(X)$.

(b) The semi-topological K-theory approach (Bondal, Toën, [235]). Assume again that $X = \oplus\mathbf{ncSpec}(A)$ is a smooth and compact graded $\mathbf{nc}$-affine $\mathbf{nc}$-space. Consider the moduli stack $\mathscr{M}_X$ of all objects in $\oplus\mathsf{Perf}_X$. This is an ∞-stack which by a theorem of Toën and Vaquie [237] is locally geometric and locally of finite presentation. Moreover for any $a, b \in \mathbb{N}$ the substack $\mathscr{M}_X^{[a,b]} \subset \mathscr{M}_X$ consisting of objects of amplitude in the interval $[a, b]$ is a geometric $b - a + 1$-stack. The functor sending a complex scheme to the underlying topological space in the analytic topology gives rise by a left Kan extension to a topological realization functor

$$| \bullet | : \mathrm{Ho}\,(\mathsf{Stacks}/\mathbb{C}) \to \mathrm{Ho}(\mathsf{Top})$$

from the homotopy category of stacks to the homotopy category of complex spaces. Following Friedlander-Walker [78] we define the semi-topological K-group of the $\mathbf{nc}$-space X to be

$$K_0^{st}(X) := \pi_0(|\mathscr{M}_X|).$$

The group structure here is induced by the direct sum $\oplus$ of A-modules: the monoid $(\pi_0(|\mathscr{M}_X|), \oplus)$ is actually a group. To see this note that for any A-module E we have that $[E \oplus E[1]] = 0$ in $\pi_0(|\mathscr{M}_X|)$. Indeed we have distinguished triangles

$$E \longrightarrow 0 \longrightarrow E[1] \longrightarrow E[1]$$

$$E \longrightarrow E \oplus E[1] \longrightarrow E[1] \longrightarrow E[1]$$

the first of which corresponds to $\mathrm{id} \in \mathrm{Ext}^1(E[1], E) = \mathrm{Hom}(E, E)$, and the second corresponds to $0 \in \mathrm{Ext}^1(E[1], E) = \mathrm{Hom}(E, E)$. Since $\mathrm{Ext}^1(E[1], E) = \mathrm{Hom}(E, E)$ is a vector space, it follows that id deforms to 0 continuously and so the second terms in the above triangles represent the same point in $\pi_0(|\mathscr{M}_X|)$.

More generally $\oplus$ makes $|\mathscr{M}_X|$ into an H-space $\mathbb{K}^{st}(X)$ which is the degree zero part of a natural spectrum. Using this one can define $K_i^{st}(X)$ for all $i \geq 0$.

Next note that since C_X is triangulated it is a module over the category $\oplus\mathsf{Perf}_{\mathrm{pt}}$ of complexes of $\mathbb{C}$-vector spaces with finite dimensional total cohomology. In particular $K_\bullet^{st}(X)$ is a graded module over $K_\bullet^{st}(\mathrm{pt})$. It can be checked that

$$\mathbb{K}^{st}(\mathrm{pt}) = BU = \mathbb{K}^{\mathrm{top}}(\mathrm{pt}),$$

and so $K_\bullet^{st}(X)$ is a graded $\mathbb{Z}[u]$-module ($\deg u = 2$).

Now we can define

$$K_\bullet^{\mathrm{top}}(X) := K_\bullet^{st}(X)[u^{-1}] = K_\bullet^{st}(X) \otimes_{\mathbb{Z}[u]} \mathbb{Z}[u, u^{-1}].$$

Again one expects that there is a Chern character map

$$ch : K_\bullet^{\mathrm{top}}(X) \to HP_\bullet(C_X)$$

whose image gives a rational structure $\mathscr{E}_B$ on $HP_\bullet(C_X)$.

Note: If X is a smooth and compact complex variety, then the Friedlander-Walker comparison theorem [77] implies that $K^{\mathrm{top}}(D(\mathsf{QCoh}(X))) \cong K^{\mathrm{top}}(X^{\mathrm{top}})$, where X^{top} is the topological space underlying X.

2.2.7 Questions Even though we have some good candidates for the ingredients H, ∇, $\mathscr{E}_B$ of the conjectural **nc**-Hodge structure associated with a **nc**-space, there are several important problems that need to be addressed before one can prove Conjecture 2.24:

- show that the connection ∇ has regular singularities (this is automatically true in the $\mathbb{Z}$-graded case);
- show that ∇ preserves the rational structure;
- show that the opposedness axiom hold.

One can show that for a smooth compact nc-space the coefficient A_{-2} in the u-connection is a nilpotent operator, which gives an evidence in favor of the regular singularity question.

In fact Conjecture 2.24 and the above questions are special cases of a general conjecture which predicts the existence of a general **nc**-Hodge structure on the periodic cyclic homology of a curved d$(\mathbb{Z}/2)$g category which is formally smooth and compact. We will not discuss the general conjecture or the relevant constructions here but we will revisit these questions in some interesting geometric examples in Section 3.

2.3 Gluing data

In this section we discuss how general **nc**-Hodge structures of exponential type can be glued together out of **nc**-Hodge structures with regular singularities and additional gluing data.

2.3.1 nc-De Rham data The de Rham part of a **nc**-Hodge structure of exponential type can be prescribed in three equivalent ways:

ncdR(i) A pair $(\mathscr{M}, \nabla)$, where $\mathscr{M}$ is a finite dimensional vector space over $\mathbb{C}\{u\}[u^{-1}]$ and ∇ is a meromorphic connection. These data should satisfy the following

Property ncdR(i): There exist:

- a frame $\underline{e} = (e_1, \dots, e_r)$ of $\mathscr{M}$ over $\mathbb{C}\{u\}[u^{-1}]$ in which
$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du$$
with $A_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$, $r = \mathrm{rank}_{\mathbb{C}\{u\}[u^{-1}]} \mathscr{M}$. In other words, there is a holomorphic extension $\mathscr{H} = \mathbb{C}\{u\}e_1 \oplus \dots \oplus \mathbb{C}\{u\}e_r$ in which ∇ has at most a second order pole.

- a formal isomorphism over $\mathbb{C}((u))$:
$$(\mathscr{M}, \nabla) \otimes_{\mathbb{C}\{u\}[u^{-1}]} \mathbb{C}((u)) \overset{\cong}{\to} \bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i)$$
where $(\mathscr{R}_i, \nabla_i)$ are meromorphic bundles with connections with regular singularities and $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m \in \mathbb{C}$ denote the distinct eigenvalues of A_{-2}.

ncdR(ii) A pair (M, ∇), where M is an algebraic vector bundle on $\mathbb{A}^1 - \{0\}$ and ∇ is a connection on M. These data should satisfy the following

Property ncdR(ii): M can be extended to an algebraic vector bundle $\widetilde{M}$ on $\mathbb{P}^1$, and

- with respect to this extension and appropriate local trivializations at zero and infinity we must have
$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du \qquad \text{near } 0,$$
$$\nabla = d + \left(\sum_{k \geq -1} B_k u^{-k} \right) d(u^{-1}) \qquad \text{near } \infty.$$

In other words $\nabla : \widetilde{M} \to \widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}(2 \cdot \{0\} + \{\infty\})$.

• There is a formal isomorphism over $\mathbb{C}((u))$:

$$(M, \nabla) \otimes_{\mathbb{C}[u,u^{-1}]} \mathbb{C}((u)) \overset{\cong}{\to} \bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i)$$

where $(\mathscr{R}_i, \nabla_i)$ are meromorphic bundles with connections with regular singularities and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ denote the distinct eigenvalues of A_{-2}.

ncdR(iii) An algebraic holonomic $\mathcal{D}$-module $\boldsymbol{M}$ on $\mathbb{A}^1$. The $\mathcal{D}$-module $\boldsymbol{M}$ should also satisfy the following

Property ncdR(iii): $\boldsymbol{M}$ has regular singularities and $H^{\bullet}_{DR}(\mathbb{A}^1, \boldsymbol{M}) = 0$.

The **nc**-de Rham data of types **ncdR(i)**, **ncdR(ii)**, and **ncdR(iii)** form obvious full subcategories in the categories of meromorphic connections over $\mathbb{C}\{u\}[u^{-1}]$, algebraic vector bundles with connections on $\mathbb{A}^1 - \{0\}$, and coherent algebraic $\mathcal{D}$-modules on $\mathbb{A}^1$ respectively. We have the following

Lemma 2.26 *The categories of* **nc***-de Rham data of types* **nc*dR(i)***, **nc*dR(ii)***, *and* **nc*dR(iii)*** *are all equivalent.*

Proof. In essence we have already discussed the equivalence **ncdR(i)** $\Leftrightarrow$ **ncdR(ii)** in Remark 2.1. Explicitly $(\mathscr{M}, \nabla) = \mathfrak{G}_0((M, \nabla)) = (M \otimes_{\mathbb{C}[u,u^{-1}]} \mathbb{C}\{u\}[u^{-1}], \nabla)$.

We define a functor $\mathfrak{F}$: (data **(iii)**) $\to$ (data **(ii)**) as follows. Let $\boldsymbol{M}$ be a regular holonomic algebraic $\mathcal{D}$-module on $\mathbb{A}^1$ with trivial de Rham cohomology. Denote the coordinate on $\mathbb{A}^1$ by v. The vanishing of de Rham cohomology means that the action $\frac{d}{dv} : \boldsymbol{M} \to \boldsymbol{M}$ is an invertible operator. Consider the algebraic Fourier transform $\boldsymbol{\Phi M}$ which is the same vector space as $\boldsymbol{M}$ endowed with action of the Weyl algebra defined by

$$\tilde{v} := \frac{d}{dv}$$

$$\frac{d}{d\tilde{v}} := -v$$

where $\tilde{v}$ is the coordinate on the dual line. By our assumptions $\boldsymbol{\Phi M}$ is a holonomic $\mathcal{D}$-module on which $\tilde{v}$ acts invertibly. Hence $\boldsymbol{\Phi M}$ is the direct image of a holonomic $\mathcal{D}$-module $\boldsymbol{\Phi M}'$ on $\mathbb{A}^1 - \{0\}$ under the embedding

$$\left(\mathbb{A}^1 - \{0\}\right) \hookrightarrow \mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[\tilde{v}])$$

Finally, making the change of coordinates $u = 1/\tilde{v}$ we obtain a $\mathcal{D}$-module M on $\mathbb{A}^1 - \{0\}$ with coordinate u.

We claim that $\mathfrak{F}(\boldsymbol{M}) := M$ obtained in this way satisfies the property **ncdR(ii)**, and that by this construction one obtains all such modules. It follows from the well-known properties of the Fourier transform that $\mathbf{\Phi}\boldsymbol{M}$ has no singularities in $\mathbb{A}^1 - \{0\}$ and the its singularity at $\tilde{v} = 0$ is regular. Hence M is a vector bundle on $\mathbb{A}^1 - \{0\}$ endowed with connection with regular singularity at ∞. It only remains to to use the well-known fact (see e.g. [172, Chapters IX-XI] or [138, Theorem 2.10.16]) that the exponential type property for M is equivalent to the property of $\boldsymbol{M}$ to have only regular singularities. □

Remark 2.27 The characterization of the exponential type property in terms of the Fourier transform can be stated more precisely (see [172, Chapters IX-XI] or [138, Theorem 2.10.16]): For an algebraic holonomic $\mathcal{D}$-module M on the complex affine line, the following two conditions are equivalent:

1) M has regular singularities;
2) the Fourier transform $\mathbf{\Phi}M$ of M has no singularities outside 0, its singularity at 0 is regular, and its singularity at infinity is of exponential type.

Explicitly $\mathbf{\Phi}M$ being of exponential type at infinity means that if x is a coordinate on $\mathbb{A}^1$ centered at 0, then after passing to the formal completion $(\mathbf{\Phi}M) \otimes_{\mathbb{C}[x]} \mathbb{C}((x^{-1}))$ the resulting module will be isomorphic to a finite sum

$$\bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i x} \otimes (\mathscr{R}_i, \nabla_i)$$

where $(\mathscr{R}_i, \nabla_i)$ are $\mathcal{D}$-modules with a regular singularity at infinity.

Remark 2.28 Note that the de Rham data **ncdR(i)** is analytic in nature, whereas **ncdR(ii)** and **ncdR(iii)** are algebraic. In fact from the proof it is clear that **ncdR(ii)** and **ncdR(iii)** and their equivalence still make sense if we replace $\mathbb{C}$ with any field of characteristic zero.

2.3.2 nc-Betti data The (rational) Betti part of a **nc**-Hodge structure of exponential type can be prescribed in four ways:

ncB(i) A (middle perversity) perverse sheaf $\mathscr{G}^\bullet$ of $\mathbb{Q}$-vector spaces on the Riemann surface $\mathbb{C}$ (taken with the analytic topology) satisfying the following

Property ncB(i): $R\Gamma(\mathbb{C}, \mathscr{G}^\bullet) = 0$.

ncB(ii) A constructible sheaf $\mathscr{F}$ of $\mathbb{Q}$-vector spaces on the Riemann surface $\mathbb{C}$ (taken with the analytic topology) satisfying the following

Property ncB(ii): $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$.

ncB(iii) A finite collection of distinct points $S = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\} \subset \mathbb{C}$, and

- a collection $U_1, U_2, \dots, U_n$ of finite dimensional non-zero $\mathbb{Q}$-vector spaces,
- a collection of linear maps $T_{ij} : U_j \to U_i$, for all $i, j = 1, \dots, n$,

satisfying the following

Property ncB(iii): $T_{ii} \in GL(U_i)$.

ncB(iv) A local system S of $\mathbb{Q}$-vector spaces on $\boldsymbol{S}^1$ equipped with a filtration $\{\mathsf{S}_{\leq\lambda}\}_{\lambda\in\mathbb{R}}$ by subsheaves of $\mathbb{Q}$-vector spaces, satisfying the following

Property ncB(iv): The filtration $\{\mathsf{S}_{\leq\lambda} \otimes \mathbb{C}\}_{\lambda\in\mathbb{R}}$ of $\mathsf{S} \otimes \mathbb{C}$ is a Deligne-Malgrange-Stokes filtration of exponential type. In other words, there exist complex numbers $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n \in \mathbb{C}$ so that:

- For every $\varphi \in \boldsymbol{S}^1$, the filtration $\{(\mathsf{S}_{\leq\lambda} \otimes \mathbb{C})_\varphi\}_{\lambda\in\mathbb{R}}$ of the stalk $(\mathsf{S} \otimes \mathbb{C})_\varphi$ jumps exactly at the real numbers $\{\mathrm{Re}\left(\boldsymbol{c}_k e^{-i\varphi}\right)\}_{k=1}^n$.
- The associated graded sheaves of $\mathsf{S} \otimes \mathbb{C}$ with respect to $\{\mathsf{S}_{\leq\lambda} \otimes \mathbb{C}\}_{\lambda\in\mathbb{R}}$ are local systems on $\boldsymbol{S}^1$.

Again there are natural equivalences of the different types of Betti data (for ncB(iii) the equivalence depends on certain choices of paths as one can see from the proof of Theorem 2.29 and the statement of Lemma 2.30.). Consider the full subcategories (ncB(i)) and (ncB(ii)) of nc-Betti data of types ncB(i) and ncB(ii) in the category of perverse sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{C}$ and in the category of constructible sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{C}$ respectively. We have the following

Theorem 2.29 *The categories of nc-Betti data of types ncB(i) and ncB(ii) are naturally equivalent. More precisely, the natural functors*

$$\mathcal{H}^{-1} : D^b_{\mathrm{constr}}(\mathbb{C}, \mathbb{Q}) \to \mathsf{Constr}(\mathbb{C}, \mathbb{Q}) \qquad \textit{and} \qquad [1] : \mathsf{Constr}(\mathbb{C}, \mathbb{Q}) \to D^b_{\mathrm{constr}}(\mathbb{C}, \mathbb{Q})$$

induce mutually inverse equivalences of the full subcategories (ncB(i)) $\subset D^b_{\mathrm{constr}}(\mathbb{C}, \mathbb{Q})$ *and* (ncB(ii)) $\subset \mathsf{Constr}(\mathbb{C}, \mathbb{Q})$.

Proof. First we look at the data ncB(i) more closely. Suppose X is a complex analytic space underlying a complex quasi-projective variety. Recall (see e.g. [28], [137], [69]) that a bounded complex $\mathscr{G}^\bullet$ of sheaves of $\mathbb{C}$-vector spaces on X is called a (middle perversity) perverse sheaf if it has constructible cohomology sheaves $\mathcal{H}^k(\mathscr{G}^\bullet)$ and if

- for all $k \in \mathbb{Z}$, we have $\dim_{\mathbb{R}}\{x \in X | \ \mathcal{H}^{-k}(\boldsymbol{i}_x^*\mathscr{G}^\bullet) \neq 0\} \leq 2k$,

- for all $k \in \mathbb{Z}$, we have $\dim_{\mathbb{R}}\{x \in X \mid \mathcal{H}^k(\boldsymbol{i}_x^! \mathscr{G}^\bullet) \neq 0\} \leq 2k$.

Here $\boldsymbol{i}_x : x \hookrightarrow X$ denotes the inclusion of the point x in X.

For future reference we will write $D^b_{\mathrm{constr}}(X, \mathbb{Q})$ for the derived category of complexes of $\mathbb{Q}$-vector spaces on X with constructible cohomology, $\mathsf{Perv}(X, \mathbb{Q}) \subset D^b_{\mathrm{constr}}(X, \mathbb{Q})$ for the full subcategory of middle perversity perverse sheaves, and $\mathsf{Constr}(X, \mathbb{Q}) \subset D^b_{\mathrm{constr}}(X, \mathbb{Q})$ for the full subcategory of constructible sheaves.

From the definition it is clear that if $\mathscr{G}^\bullet$ is a perverse sheaf on $\mathbb{C}$, then $\mathscr{G}^\bullet$ has at most two non-trivial cohomology sheaves $\mathcal{H}^{-1}(\mathscr{G}^\bullet)$ and $\mathcal{H}^0(\mathscr{G}^\bullet)$. Moreover the support of $\mathcal{H}^0(\mathscr{G}^\bullet)$ has dimension ≤ 0. Now the cohomology $R\Gamma^\bullet(\mathscr{G}^\bullet) = \mathbb{H}^\bullet(\mathbb{C}, \mathscr{G}^\bullet)$ can be computed via the hypercohomology spectral sequence

$$E_2^{pq} = H^p(\mathbb{C}, \mathcal{H}^q(\mathscr{G}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(\mathbb{C}, \mathscr{G}^\bullet).$$

Since $\mathscr{G}^\bullet$ has only two cohomology sheaves, the E_2 sheet of this spectral sequence is

$$
\begin{tikzcd}
0 & H^0(\mathbb{C},\mathcal{H}^0(\mathscr{G}^\bullet)) \arrow[drr] & H^1(\mathbb{C},\mathcal{H}^0(\mathscr{G}^\bullet)) \arrow[drr] & H^2(\mathbb{C},\mathcal{H}^0(\mathscr{G}^\bullet)) & \cdots \\
-1 & H^0(\mathbb{C},\mathcal{H}^{-1}(\mathscr{G}^\bullet)) & H^1(\mathbb{C},\mathcal{H}^{-1}(\mathscr{G}^\bullet)) & H^2(\mathbb{C},\mathcal{H}^{-1}(\mathscr{G}^\bullet)) & \cdots \\
 & 0 & 1 & 2 &
\end{tikzcd}
$$

By Artin's vanishing theorem for constructible sheaves [8, Corollary 3.2] he have $H^p(\mathbb{C}, \mathcal{H}^q(\mathscr{G}^\bullet)) = 0$ for all q and all $p > 1$. Furthermore since $\mathcal{H}^0(\mathscr{G}^\bullet)$ has at most zero dimensional support we have $H^1(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) = 0$. In particular the spectral sequence degenerates at E_2 and the only potentially non-trivial cohomology groups of $\mathscr{G}^\bullet$ are

$$
\begin{aligned}
\mathbb{H}^{-1}(\mathbb{C}, \mathscr{G}^\bullet) &= H^0(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)), \text{ and} \\
\mathbb{H}^{0}(\mathbb{C}, \mathscr{G}^\bullet) &= H^1(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)) \oplus H^0(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)).
\end{aligned}
$$

Thus under the assumption that $R\Gamma(\mathscr{G}^\bullet) = 0$ we get that $H^0(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) = 0$, i.e. that $\mathcal{H}^0(\mathscr{G}^\bullet) = 0$. In other words $\mathscr{G}^\bullet = \mathscr{F}[1]$ for some constructible sheaf $\mathscr{F}$ with $R\Gamma(\mathscr{F}) = 0$.

To finish the proof of the theorem we need to show that for every constructible sheaf $\mathscr{F}$ with $R\Gamma(\mathscr{F}) = 0$, the object $\mathscr{F}[1]$ will be perverse (for the middle perversity). For this we will have to look more closely at constructible sheaves on the complex line.

Suppose $\mathscr{F}$ is a constructible sheaf of $\mathbb{Q}$ vector spaces on $\mathbb{C}$. Then there is a finite set $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ of points in $\mathbb{C}$ so that $\mathbb{C} - S$ is the maximal open set on which $\mathscr{F}$ restricts to a local system. Let $\mathbb{F} := \mathscr{F}_{|\mathbb{C}-S}$ denote this local system. Let $\mathbb{C} - S \overset{\boldsymbol{j}}{\hookrightarrow} \mathbb{C} \overset{\boldsymbol{i}}{\hookleftarrow} S$ be the natural inclusions and let $\varphi : \mathscr{F} \to j_* j^* \mathscr{F} = j_* \mathbb{F}$ be the adjunction homomorphism.

Before we can describe $\mathbb{F}$ and $\mathscr{F}$ via the quiver-like data of type **ncB(iii)** we will need to make some rigidifying choices. First we fix a base point $\boldsymbol{c}_0 \in \mathbb{C} - S$. For $i = 1, \ldots, n$ we choose a collection of a small disjoint discs $\boldsymbol{D}_i \subset \mathbb{C}$, each $\boldsymbol{D}_i$ centered at $\boldsymbol{c}_i$. For each disc we fix a point $\boldsymbol{o}_i \in \partial \boldsymbol{D}_i$ and denote by $\mathfrak{l}_i$ the loop starting and ending at $\boldsymbol{o}_i$ and tracing $\partial \boldsymbol{D}_i$ once in the counterclockwise direction. We fix an ordered system of non-intersecting paths $\{a_i\}_{i=1}^n \subset \mathbb{C} - (\cup_{i=1}^n \boldsymbol{D}_i)$ which connect the base point $\boldsymbol{c}_0$ with the each of the $\boldsymbol{o}_i$ as in Figure 2.

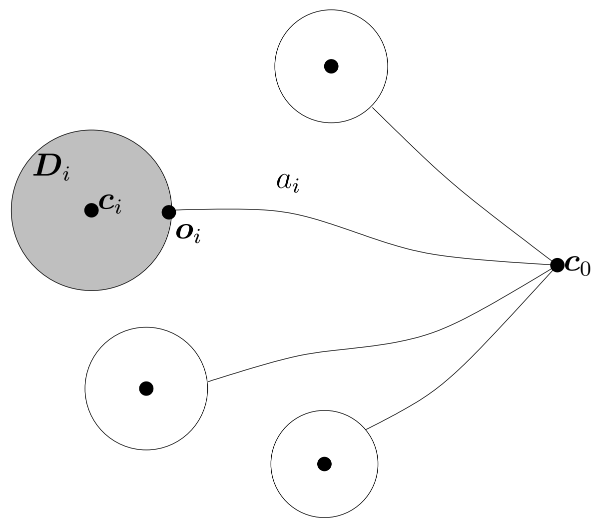


Figure 2: A system of paths for $S \subset \mathbb{C}$.

Let $\mathrm{mon}_{\mathfrak{l}_i} : \mathbb{F}_{\boldsymbol{o}_i} \to \mathbb{F}_{\boldsymbol{o}_i}$ be the monodromy operator associated with the local system $\mathbb{F}$ and the loop $\mathfrak{l}_i$. The stalk $(\boldsymbol{j}_*\mathbb{F})_{\boldsymbol{c}_i}$ of the constructible sheaf $\boldsymbol{j}_*\mathbb{F}$ at $\boldsymbol{c}_i$ can be identified naturally with the subspace $\mathbb{F}_{\boldsymbol{o}_i}^{\mathrm{mon}_{\mathfrak{l}_i}}$ of invariants for the local monodromy. Taking stalks at each $\boldsymbol{c}_i \in S$ we get $\mathbb{Q}$-vector spaces $\mathscr{F}_{\boldsymbol{c}_i}$ and the adjunction map $\varphi : \mathscr{F} \to \boldsymbol{j}_*\mathbb{F}$ induces linear maps

$$\varphi_{\boldsymbol{c}_i} : \mathscr{F}_{\boldsymbol{c}_i} \to \mathbb{F}_{\boldsymbol{o}_i}^{\mathrm{mon}_{\boldsymbol{c}_i}} \subset \mathbb{F}_{\boldsymbol{o}_i}.$$

Note that, by descent, specifying the constructible sheaf $\mathscr{F}$ is equivalent to specifying the collection of points $S \subset \mathbb{C}$, the local system $\mathbb{F}$ on $\mathbb{C} - S$, the collection of vector spaces $\{\mathscr{F}_{\boldsymbol{c}_i}\}_{i=1}^n$ and the collection of linear maps $\{\varphi_{\boldsymbol{c}_i}\}_{i=1}^n$. In particular, the compactly supported pullback of $\mathscr{F}[1]$ via the inclusion $\boldsymbol{i}_{\boldsymbol{c}_i} : \{\boldsymbol{c}_i\} \hookrightarrow \mathbb{C}$ can be computed in terms of these linear algebraic data and is given explicitly by the complex

$$\boldsymbol{i}^!_{\boldsymbol{c}_i}(\mathscr{F}[1]) = \Big[\underset{-1}{\mathscr{F}_{\boldsymbol{c}_i}} \xrightarrow{\varphi_{\boldsymbol{c}_i}} \underset{0}{\mathbb{F}_{\boldsymbol{o}_i}} \xrightarrow{1-\mathrm{mon}_{\mathfrak{l}_i}} \underset{1}{\mathbb{F}_{\boldsymbol{o}_i}} \Big].$$

By definition $\mathscr{F}[1]$ is a perverse sheaf iff for all $\boldsymbol{c}_i \in S$ the complex of vector spaces $\boldsymbol{i}^!_{\boldsymbol{c}_i}(\mathscr{F}[1])$ has no cohomology in strictly negative degrees, i.e. iff $\varphi_{\boldsymbol{c}_i}$ is injective for all $i = 1, \ldots, n$.

Next we rewrite the condition $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$ in terms of the descent data $(\mathbb{F}, \{\mathscr{F}_{\boldsymbol{c}_i}\}, \{\varphi_{\boldsymbol{c}_i}\})$. To simplify notation let $U := \mathscr{F}_{\boldsymbol{c}_0}$, $V_i = \mathscr{F}_{\boldsymbol{c}_i}$ for $i = 1, \ldots, n$. Let $T_i : U \to U$ be the monodromy operator for the local system $\mathbb{F}$ and the $\boldsymbol{c}_0$-based loop γ_i obtained by first tracing the path a_i from $\boldsymbol{c}_0$ to $\boldsymbol{o}_i$, then tracing the loop $\mathfrak{l}_i$, and then tracing back a_i in the opposite direction. Similarly we have linear maps $\psi_i : V_i \to U^{T_i} \subset U$ obtained by conjugating $\varphi_{\boldsymbol{c}_i} : V_i \to \mathbb{F}_{\boldsymbol{o}_i}$ with the operator of parallel transport in $\mathbb{F}$ along the path a_i.

The descent data for $\mathscr{F}$ with respect to the open cover $\mathbb{C} = (\mathbb{C}-S)\cup(\cup_{i=1}^n \boldsymbol{D}_i)$ are now completely encoded in the linear algebraic data $(U, \{V_i\}_{i=1}^n, \{T_i\}_{i=1}^n, \{\psi_i\}_{i=1}^n)$. Cover $\mathbb{C}$ by the two opens $\mathbb{C}-S$ and $\cup_{i=1}^n \boldsymbol{D}_i$. The intersection of these two opens is the disjoint union of punctured discs $\coprod_{i=1}^n (\boldsymbol{D}_i - \boldsymbol{c}_i)$. The Mayer-Vietoris sequence for $\mathscr{F}$ and this cover identifies $R\Gamma(\mathbb{C}, \mathscr{F})$ with the complex:

$$
\begin{tikzcd}
\bigoplus_{i=1}^n V_i \arrow[r, "\oplus_{i=1}^n \psi_i"] & U^{\oplus n} \arrow[r, "\oplus_{i=1}^n (1-T_i)"] & U^{\oplus n} \\
\bigoplus & \bigoplus & \\
U \arrow[uur, "\mathrm{id}_U^{\oplus n}"] \arrow[r, "\oplus_{i=1}^n (1-T_i)"'] & U^{\oplus n} \arrow[uur, "-\mathrm{id}_U^{\oplus n}"'] & \\
0 & 1 & 2
\end{tikzcd}
$$

In other words we have a quasi-isomorphism of complexes of $\mathbb{Q}$-vector spaces:

$$
R\Gamma(\mathbb{C}, \mathscr{F}) \cong
\begin{tikzcd}
\bigoplus_{i=1}^n V_i \arrow[dr, "\oplus_{i=1}^n \psi_i"] & \\
\bigoplus & U^{\oplus n} \\
U \arrow[ur, "\mathrm{id}_U^{\oplus n}"'] & \\
0 & 1
\end{tikzcd}
$$

The acyclicity of this complex is equivalent to the conditions

(a) the maps $\psi_i : V_i \to U$ are injective for all $i = 1, \ldots, n$, and

(b) the map $U \to \oplus_{i=1}^n U/V_i$ is an isomorphism.

Thus the acyclicity of $R\Gamma(\mathbb{C}, \mathscr{F})$ implies the perversity of $\mathscr{F}[1]$. The theorem is proven.

□

The conditions (a) and (b) from the proof of Theorem 2.29 suggest a better way of recording the linear algebraic content of $\mathscr{F}$. Namely, if we set $U_i := U/V_i$, then we can use (b) to identify U with $\oplus_{i=1}^n U_i$, V_i with $\oplus_{j\neq i} U_j$ and the map $\psi_i : V_i \hookrightarrow U$ with the natural inclusion $\oplus_{j\neq i} U_j \subset \oplus_{i=1}^n U_i$. The only thing left is the data of the monodromy operators $T_i \in GL(U)$, $i = 1, \ldots, n$. However for each i we have embedding

$$V_i \xhookrightarrow{\psi_i} \mathrm{Ker}\left[U \overset{(1-T_i)}{\longrightarrow} U \right]$$

and so under the decomposition $U = \oplus_{i=1}^n U_i$ the automorphism T_i has a block form

$$
T_i = \begin{pmatrix}
1 & 0 & \cdots & 0 & T_{1i} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & T_{2i} & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & T_{ii} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & T_{i+1,i} & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & T_{ni} & 0 & \cdots & 1
\end{pmatrix}
$$

where $T_{i|U_i} = \sum_{j=1}^{n} T_{ji}$, and $T_{ji} : U_i \to U_j$. The linear maps T_{ji} are unconstrained except for the obvious condition that for all i the map T_i should be invertible, which is equivalent to $T_{ii} : U_i \to U_i$ being invertible for all $i = 1, \ldots, n$. Also since S was chosen to be such that $\mathbb{C} - S$ is the maximal open on which $\mathscr{F}$ is a local system, it follows that $U_i \neq \{0\}$ for all $i = 1, \ldots, n$.

In other words we have proven the following

Lemma 2.30 *Fix the set of points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ and choose the discs $\{\boldsymbol{D}_i\}_{i=n}$ and the system of paths $\{a_i\}_{i=1}^n$. The functor assigning to a constructible sheaf $\mathscr{F}$ with singularities at S the data $(\{U_i\}_{i=1}^n, \{T_{ij}\})$ establishes an equivalence between the groupoid of all data of type* **ncB(ii)** *with singularities exactly at S and all data of type* **ncB(iii)** *with the given S.*

The bridge between the **nc**- de Rham and Betti data is provided as usual by the Riemann-Hilbert correspondence. This is tautological but we record it for future reference:

Lemma 2.31 *The de Rham functor:*

$$\boldsymbol{M} \to \operatorname{cone}\left(\boldsymbol{M} \otimes_{\mathbb{C}[u]} \mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1} \xrightarrow{\frac{\partial}{\partial u}} \boldsymbol{M} \otimes_{\mathbb{C}[u]} \mathcal{O}^{\mathrm{an}}_{\mathbb{A}^1} \right)$$

establishes an equivalence between the categories (**ncdR(iii)**) *and* (**ncB(i)**)$\otimes\mathbb{C}$.

Finally, note that Theorem 2.29, together with Lemma 2.31, and Deligne's classification [20, Theorem 4.7.3] of germs of irregular connections give immediately:

Lemma 2.32 *The data data* (**ncB(ii)**) *and* (**ncB(iv)**) *are equivalent.*

Proof. Let $\mathscr{F}$ be a constructible sheaf of $\mathbb{Q}$-vector spaces on $\mathbb{C}$. Define a local system S of $\mathbb{Q}$-vector spaces on $\boldsymbol{S}^1$ as the restriction of $\mathscr{F}$ to the circle "at infinity", i.e. define the stalk of S at $\varphi \in \boldsymbol{S}^1$ to be

$$\mathsf{S}_\varphi := \lim_{r \to +\infty} \mathscr{F}_{re^{i\varphi}}.$$

Next, for any $\lambda \in \mathbb{R}$ and any $\varphi \in \boldsymbol{S}^1$ consider the half-plane

$$\mathfrak{H}_{\varphi,\lambda} := (\lambda + \{u \in \mathbb{C} | \operatorname{Re}(u) \geq 0\}) \cdot e^{i\varphi},$$

as shown on Figure 3.

Now suppose that $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$. By the long exact sequence for the cohomology of the pair $\mathfrak{H}_{\varphi,\lambda} \subset \mathbb{C}$ we get that $H^i(\mathbb{C}, \mathfrak{H}_{\varphi,\lambda}; \mathscr{F}) = 0$ unless $i = 1$. The Deligne-Malgrange-Stokes filtration on S is then given explicitly by

$$\mathsf{S}_{\varphi,\leq\lambda} := H^1(\mathbb{C}, \mathfrak{H}_{\varphi,\lambda}; \mathscr{F}) \subset \mathsf{S}_\varphi.$$

$\square$

For the purposes of **nc**-Hodge theory all these statements can be summarized in the following

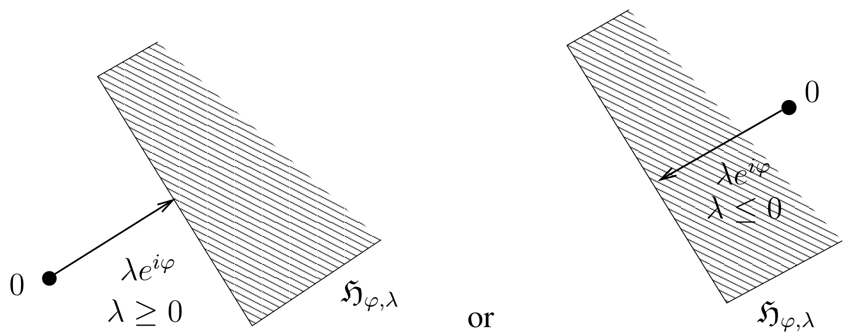


Figure 3: The half-plane $\mathfrak{H}_{\varphi,\lambda}$.

Theorem 2.33 *There is natural equivalence of categories*

$$\left(\begin{array}{l}\text{triples } (H, \mathscr{E}_B, \overset{\sim}{\to}) \text{ satisfying} \\ \text{the } \textbf{(nc-filtration axiom)}^{\mathbf{exp}} \text{ and} \\ \text{the } \textbf{(}\mathbb{Q}\textbf{-structure axiom)}^{\mathbf{exp}}\end{array}\right) \leftrightarrow \left(\begin{array}{l}\text{quadruples } ((H,\nabla), \mathscr{F}_B, \boldsymbol{f}), \text{ where} \\ \bullet\ H \text{ is an algebraic } \mathbb{Z}/2\text{-graded vector} \\ \quad \text{bundle on } \mathbb{C} \text{ and } \nabla \text{ is a meromor-} \\ \quad \text{phic connection on } H \text{ satisfying the} \\ \quad \textbf{(nc-filtration axiom)}^{\mathbf{exp}}; \\ \bullet\ \mathscr{F}_B \in \mathsf{Constr}(\mathbb{C},\mathbb{Q}), \text{ satisfying } R\Gamma(\mathbb{C},\mathscr{F}_B) = \\ \quad 0; \\ \bullet\ \boldsymbol{f} \text{ is an isomorphism} \\ \quad \boldsymbol{f} : \mathscr{F}_B \otimes \mathbb{C} \to DR\left(\boldsymbol{\Phi}\left[\iota_*\left((H,\nabla)_{|\mathbb{A}^1-\{0\}}\right)\right]\right) \\ \quad \text{in } D^b_{\text{constr}}(\mathbb{C},\mathbb{C})\end{array}\right)$$

Here as before

DR *is the de Rham complex functor from the derived category of regular holonomic $\mathbb{D}$-modules to the derived category of constructible sheaves,*

ι *is the inclusion map* $\iota : \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{A}^1$ *given by* $\iota(v) = v^{-1}$*, and*

$\boldsymbol{\Phi}(\bullet)$ *is the Fourier-Laplace transform for $\mathcal{D}$-modules on $\mathbb{A}^1$.*

Proof. Follows immediately from previous equivalences. □

2.4 Structure results

In this section we collect a few results clarifying the structure properties of the **nc**-Hodge structures of exponential type.

2.4.1 A quiver description of nc-Betti data Since the gluing data **ncB(iii)** are of essentially combinatorial nature, it is natural to look for a quiver interpretation of this data. To that end consider the algebra

$$\mathscr{A}_n := \left\langle \begin{array}{l} \boldsymbol{p}_1,\ldots,\boldsymbol{p}_n \\ T, T_{11}^{-1},\ldots,T_{nn}^{-1} \end{array} \middle| \begin{array}{l} \boldsymbol{p}_1+\boldsymbol{p}_2+\ldots+\boldsymbol{p}_n = 1 \\ \boldsymbol{p}_i\boldsymbol{p}_j = \boldsymbol{p}_j\boldsymbol{p}_i \text{ for } i \neq j,\ \boldsymbol{p}_i^2 = \boldsymbol{p}_i \\ T_{ii}^{-1}\boldsymbol{p}_i T \boldsymbol{p}_i = \boldsymbol{p}_i T \boldsymbol{p}_i T_{ii}^{-1} = \boldsymbol{p}_i \end{array} \right\rangle \tag{2.3}$$

This is the path algebra of the complete quiver having n ordered vertices, $n^2 - n$ arrows connecting all pairs of distinct vertices, and $2n$-loops - two at each vertex, with the only relations being that the two loops at every given vertex are inverses to each other.

Note that our description of the gluing data **ncB(iii)** now immediately gives the following

Lemma 2.34 *For a given set of points $S = \{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n\} \subset \mathbb{C}$, the category of gluing data **ncB(iii)** with singularities at S is equivalent to the category of finite dimensional representations of $\mathscr{A}_n$.*

In particular since the braid group $\boldsymbol{B}_n$ on n-strands acts naturally on the data **ncB(iii)** we get a homomorphism $\boldsymbol{B}_n \to \mathrm{Aut}(\mathscr{A}_n)$ from the braid group to the group of algebra automorphisms of $\mathscr{A}_n$.

2.4.2 Gluing of nc-Hodge structures It is natural to expect that the usual classification of connections with second order poles in terms of formal regular type and Stokes multipliers can be promoted to a similar classification of **nc**-Hodge structures. The search for such a classification leads naturally to the following theorem:

Theorem 2.35 *Let $\left\{\left(H, \mathscr{E}_B, \overset{\sim}{\to}\right)\right\}$ be a **nc**-Hodge structure of exponential type. Then specifying $\left\{\left(H, \mathscr{E}_B, \overset{\sim}{\to}\right)\right\}$ is equivalent to specifying the following data:*

(regular type) *A finite set $S = \{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_n\} \subset \mathbb{C}$ and a collection $\left\{\left((\mathscr{R}_i, \nabla_i), \mathscr{E}_{B,i}, \overset{\sim}{\to}_i\right)\right\}_{i=1}^n$ of **nc**-Hodge structures with regular singularities.*

(gluing data) *A base point $\boldsymbol{c}_0 \in \mathbb{C} - S$, a collection of discs $\{\boldsymbol{D}_i\}_{i=1}^n$ and paths $\{a_i\}_{i=1}^n$, chosen as in the proof of Theorem 2.29, and for every $i \neq j$, $i, j \in \{1,\ldots,n\}$ a map of rational vector spaces*

$$T_{ij} : \left(\mathscr{E}_{B,j}\right)_{\boldsymbol{c}_0} \longrightarrow \left(\mathscr{E}_{B,i}\right)_{\boldsymbol{c}_0}$$

Proof. It will be convenient to introduce formal counterparts to the de Rham parts of the **nc**-Hodge structures appearing in the statement of the theorem. We consider the following:

formal(a) A pair $(\mathscr{M}^{\text{for}}, \nabla^{\text{for}})$, where $\mathscr{M}^{\text{for}}$ is a finite dimensional vector space over $\mathbb{C}((u))$ and ∇^{for} is a meromorphic connection on $\mathscr{M}^{\text{for}}$ of exponential type.

formal(b) A finite set of points $S = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\} \subset \mathbb{C}$ and a collection $\{(\mathscr{R}_i^{\mathrm{for}}, \nabla_i^{\mathrm{for}})\}_{i=1}^n$ where each $\mathscr{R}_i^{\mathrm{for}}$ is a non-zero finite dimensional vector space over $\mathbb{C}((u))$ and each ∇_i^{for} is a meromorphic connection on $\mathscr{R}_i^{\mathrm{for}}$ with a regular singularity.

formal(c) A finite collection of points $S = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\} \subset \mathbb{C}$, and

- a collection $U_1, U_2, \dots, U_n$ of finite dimensional non-zero $\mathbb{Q}$-vector spaces,
- a collection of linear maps $T_{ii} \in GL(U_i)$, for all $i = 1, \dots, n$,

By Remark 2.13 the natural functor from the category of data **formal(b)** to the category of data **formal(a)**, which is given by

$$(\textbf{formal(b)}) \longrightarrow (\textbf{formal(a)})$$

$$\left(S; \{(\mathscr{R}_i^{\mathrm{for}}, \nabla_i^{\mathrm{for}})\}_{i=1}^n\right) \longrightarrow \bigoplus_{i=1}^n \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i^{\mathrm{for}}, \nabla_i^{\mathrm{for}}) =: (\mathscr{M}^{\mathrm{for}}, \nabla^{\mathrm{for}})$$

is an equivalence of categories.

Also we have the following

Lemma 2.36 *The categories of data **formal(b)** and **formal(c)** are naturally equivalent.*

Proof. Indeed, consider the category $\mathcal{C}$ of all data consisting of a finite set of points $S = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\} \subset \mathbb{C}$ and a collection $\{(\mathscr{R}_i, \nabla_i)\}_{i=1}^n$ where each $\mathscr{R}_i$ is a non-zero finite dimensional vector space over $\mathbb{C}\{u\}[u^{-1}]$ and each ∇_i is a meromorphic connection on $\mathscr{R}_i$ with a regular singularity and non-trivial monodromy. Then we have natural functors

$$\begin{tikzcd} & (\textbf{formal(b)}) \\ \mathcal{C} \arrow[ur, "(\bullet)\otimes\mathbb{C}((u))"] \arrow[dr, "\mathsf{mon}"'] & \\ & (\textbf{formal(c)}) \end{tikzcd}$$

where $(\bullet)\otimes\mathbb{C}((u))$ is the passage to a formal completion and mon is given by assigning to each $(\mathscr{R}_i, \nabla_i)$ the pair (U_i, T_i), where U_i is the fiber of the Birkhoff extension $\mathfrak{B}_0(\mathscr{R}_i, \nabla_i)$ of $(\mathscr{R}_i, \nabla_i)$ at $1 \in \mathbb{A}^1$, and T_i is the monodromy of $\mathfrak{B}_0(\mathscr{R}_i, \nabla_i)$ around the unit circle traced in the positive direction.

This proves the lemma since mon is an equivalence by the Riemann-Hilbert correspondence and $(\bullet) \otimes \mathbb{C}((u))$ is an equivalence by the formal decomposition theorem [205, II.5.7]).

$\square$

Note that these equivalences are compatible with the corresponding equivalence of analytic de Rham data and Betti data. More precisely we have a commutative diagram of functors

$$
\begin{tikzcd}
(\textsf{\textbf{nc}}\mathbf{dR(i)}) \arrow[r] \arrow[d, dotted, leftrightarrow] & (\mathbf{formal(a)}) \arrow[d, dotted, leftrightarrow] \\
(\textsf{\textbf{nc}}\mathbf{B(iii)}) \arrow[r] & (\mathbf{formal(c)})
\end{tikzcd}
\tag{2.4}
$$

Here the right vertical equivalence is the composition of the equivalences $(\mathbf{formal(a)}) \cong (\mathbf{formal(b)}) \cong (\mathbf{formal(c)})$ that we just discussed. The left vertical equivalence is the composition of the equivalence $(\textsf{\textbf{nc}}\mathrm{dR(i)}) \cong (\textsf{\textbf{nc}}\mathrm{dR(iii)})$ given in Lemma 2.26, the equivalence $(\textsf{\textbf{nc}}\mathrm{dR(iii)}) \cong (\textsf{\textbf{nc}}\mathrm{B(i)})$ from Lemma 2.31, the equivalence $(\textsf{\textbf{nc}}\mathrm{B(i)}) \cong (\textsf{\textbf{nc}}\mathrm{B(ii)})$ given in Theorem 2.29, and the equivalence $(\textsf{\textbf{nc}}\mathrm{B(ii)}) \cong (\textsf{\textbf{nc}}\mathrm{B(iii)})$ from Lemma 2.30.

Horizontally we have the forgetful functors

$$
\begin{aligned}
(\textsf{\textbf{nc}}\mathbf{dR(i)}) &\longrightarrow (\mathbf{formal(a)}) \\
(\mathscr{M}, \nabla) &\longmapsto (\mathscr{M}, \nabla) \otimes_{\mathbb{C}\{u\}[u^{-1}]} \mathbb{C}((u)),
\end{aligned}
$$

and

$$
\begin{aligned}
(\textsf{\textbf{nc}}\mathbf{B(iii)}) &\longrightarrow (\mathbf{formal(c)}) \\
\left(S; \{U_i\}_{i=1}^n, \{T_{ij}\}_{i,j=1}^n\right) &\longmapsto \left(S; \{U_i\}_{i=1}^n, \{T_{ii}\}_{i=1}^n\right).
\end{aligned}
$$

Next we need the following

Lemma 2.37 *Suppose that $(\mathscr{M}, \nabla)$ is some de Rham data of type* $\textsf{\textbf{nc}}\mathbf{dR(i)}$ *and let*

$$
(\mathscr{M}^{\mathrm{for}}, \nabla^{\mathrm{for}}) = (\mathscr{M}, \nabla) \otimes_{\mathbb{C}\{u\}[u^{-1}]} \mathbb{C}((u))
$$

be the corresponding formal data. Then:

(a) *the map*

$$
\begin{pmatrix} \mathbb{C}\{u\}\text{-submodules } \mathscr{H} \subset \mathscr{M}, \text{ on} \\ \text{which } \nabla \text{ has a pole of order} \leq 2 \end{pmatrix} \xrightarrow{(\bullet)\otimes\mathbb{C}[[u]]} \begin{pmatrix} \mathbb{C}[[u]]\text{-submodules } \mathscr{H}^{\mathrm{for}} \subset \mathscr{M}^{\mathrm{for}}, \text{ on} \\ \text{which } \nabla^{\mathrm{for}} \text{ has a pole of order} \leq 2 \end{pmatrix},
$$

is bijective.

(b) *If $\Psi : (\mathscr{M}^{\mathrm{for}}, \nabla^{\mathrm{for}}) \to \bigoplus_{i=1}^n \boldsymbol{\mathcal{E}}^{c_i/u} \otimes (\mathscr{R}_i^{\mathrm{for}}, \nabla_i^{\mathrm{for}})$ is a formal isomorphism, then the map*

$$
\begin{pmatrix} \mathbb{C}[[u]]\text{-submodules } \mathscr{H}^{\mathrm{for}} \subset \mathscr{M}^{\mathrm{for}}, \text{ on} \\ \text{which } \nabla^{\mathrm{for}} \text{ has a pole of order} \leq 2 \end{pmatrix} \xleftarrow[\Psi]{} \begin{pmatrix} \mathbb{C}[[u]]\text{-submodules } \mathscr{H}_i^{\mathrm{for}} \subset \mathscr{R}_i^{\mathrm{for}}, \text{ for} \\ \text{all } i = 1, \ldots, n, \text{ on which } \nabla_i^{\mathrm{for}} \text{ has a} \\ \text{pole of order} \leq 2 \end{pmatrix},
$$

is bijective.

Proof. (a) Pick some frame $\underline{e}$ of $\mathscr{M}$ over $\mathbb{C}\{u\}[u^{-1}]$ and let $\mathscr{H}^0 := \mathbb{C}\{u\} \cdot \underline{e} \subset \mathscr{M}$ be the submodule of all sections in $\mathscr{M}$ that are holomorphic in this frame. Now any $\mathbb{C}\{u\}$-submodule $\mathscr{H} \subset \mathscr{M}$, on which ∇ has a pole of order ≤ 2 will be a $\mathbb{C}\{u\}$-submodule of $\mathscr{M}$ which is commensurable with $\mathscr{H}^0$, i.e. we will have $u^N \mathscr{H}^0 \subset \mathscr{H} \subset u^{-N}\mathscr{H}^0$ for $N \gg 1$. However the formal completion functor $(\bullet) \otimes_{\mathbb{C}\{u\}} \mathbb{C}[[u]]$ establishes an isomorphism between the Grassmanian $GL_r(\mathbb{C}\{u\}[u^{-1}])/GL_r(\mathbb{C}\{u\})$ and the affine Grassmanian $GL_r(\mathbb{C}((u))/GL_r(\mathbb{C}[[u]])$. But this map preserves the condition that a submodule $\mathscr{H}$ is invariant under $\nabla_{u^2 d/du}$ which proves **(a)**.

(b) As already mentioned in Remark 2.13 this is proven in [115, Lemma 8.2]. Alternatively we can reason as in the proof of part **(a)**. Let $\mathscr{H}$ be a $\mathbb{C}[[u]]$-submodule in $\mathscr{M}^{\text{for}}$ which is commensurable with $\mathscr{H}^{0,\text{for}}$ and preserved by $\nabla_{u^2 \frac{d}{du}}$. The operator $\nabla_{u^2 \frac{d}{du}}$ acts on the infinite-dimensional topological complex vector space $\mathscr{M}^{\text{for}}$ with finitely many infinite Jordan blocks with eigenvalues $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$. The corresponding generalized eigenspaces are exactly modules $\boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u}\mathscr{R}_i^{\text{for}}$. Hence

$$\mathscr{H}^{\text{for}} = \oplus_i \left(\mathscr{H}^{\text{for}} \cap \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u}\mathscr{R}_i^{\text{for}} \right)$$

Therefore we obtain extensions $\mathscr{R}_i^{\text{for}}$ with second order poles and regular singularity.

Combining the previous lemma with the equivalences in diagram (2.4) and the description of **nc**-Hodge structures from Section 2.1.8 gives the theorem. □

2.5 Deformations of nc-spaces and gluing

In this section we will briefly examine how the gluing construction for **nc**-Hodge structures varies with parameters. In particular, we will look at deformations of **nc**-spaces and the way the gluing data for the **nc**-Hodge structures on the cohomology of these spaces interacts with the appearance of a curvature in the d($\mathbb{Z}/2$g algebra computing the sheaf theory of the space.

2.5.1 The cohomological Hochschild complex

Suppose $X = \oplus\textbf{ncSpec}A$ is a **nc**-affine **nc**-space. Recall that the cohomological Hochschild complex is defined as

$$C^\bullet(A,A) := \prod_{n\geq 0} \operatorname{Hom}_{\mathbb{C}-\mathsf{Vect}}\left((\mathbf{\Pi}A)^{\otimes n}, A\right),$$

Its shift $\mathbf{\Pi}C^\bullet(A,A)$ is a Lie superalgebra with respect to the Gerstenhaber bracket [90], and can be interpreted as the Lie algebra of continuous derivations of the free topological algebra $\prod_{n\geq 0}((\mathbf{\Pi}A)^{\otimes n})^\vee$. The multiplication m_A and differential d_A of A combine into a cochain $\gamma_A := m_A + d_A \in C^\bullet(A,A)$ satisfying $[\gamma_A, \gamma_A] = 0$.

The formal deformation theory of X is controlled by a d($\mathbb{Z}/2$)g Lie algebra structure $\mathbf{\Pi}C^\bullet(A,A)$ endowed with the differential $[\gamma_A, \bullet]$. It is convenient to consider also the reduced Hochschild complex

$$C^\bullet_{\text{red}}(A,A) := \prod_{n\geq 0} \operatorname{Hom}_{\mathbb{C}-\mathsf{Vect}}\left((\mathbf{\Pi}\left(A/\mathbb{C}\cdot 1_A\right))^{\otimes n}, A\right),$$

which is naturally a subspace of $C^\bullet(A,A)$. The reduced complex is (after the parity change) a dg Lie subalgebra in $\mathbf{\Pi}C^\bullet(A,A)$. Moreover it is quasi-isomorphic to $\mathbf{\Pi}C^\bullet(A,A)$. Hence, for deformation theory purposes one can replace $\mathbf{\Pi}C^\bullet(A,A)$ by $\mathbf{\Pi}C^\bullet_{\mathrm{red}}(A,A)$.

Let $\boldsymbol{\gamma} = \sum_{i\geq 1}\gamma_i t^i \in tC^{\mathrm{even}}_{\mathrm{red}}(A,A)[[t]]$ be a formal path consisting of solutions of the Maurer-Cartan equation, i.e.

$$d\boldsymbol{\gamma} + \frac{1}{2}[\boldsymbol{\gamma},\boldsymbol{\gamma}] = 0 \quad (\Leftrightarrow\ [\boldsymbol{\gamma}+\gamma_A, \boldsymbol{\gamma}+\gamma_A] = 0)\,.$$

Such a solution defines so called formal deformation of the d$(\mathbb{Z}/2)$g algebra A as a weak (or curved) A_∞-algebra (see e.g. [165] for the definition and [213] for a more detailed analysis). We can use the cochain $\boldsymbol{\gamma}+\gamma_A \in C^{\mathrm{even}}(A,A)[[t]]$ to twist the notion of an A-module. We will write $A_{\boldsymbol{\gamma}}$ for the (weak) A_∞-algebra over $\mathbb{C}[[t]]$ corresponding to A and $\boldsymbol{\gamma}+\gamma_A$ and $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ for the $\mathbb{C}[[t]]$-linear dg category of all modules over $A_{\boldsymbol{\gamma}}$. By definition $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ is the category of dg modules over a bar-type resolution of $A_{\boldsymbol{\gamma}}$ [159]. As an algebra the relevant bar dg algebra is the completed tensor product

$$\prod_{n\geq 0}\left((\mathbf{\Pi}A)^{\otimes n}\right)^\vee \widehat{\otimes}\mathbb{C}[[t]] \tag{2.5}$$

where the algebra structure comes from the usual algebra structure on $\mathbb{C}[[t]]$ and the tensor algebra structure on $\prod_{n\geq 0}\left((\mathbf{\Pi}A)^{\otimes n}\right)^\vee$. Thus for every $\boldsymbol{\gamma} \in tC^{\mathrm{even}}_{\mathrm{red}}(A,A)[[t]]$ which solves the Mauer-Cartan equation we get a differential $\boldsymbol{\gamma}+\gamma_A$ on the graded algebra (2.5). The bar dg algebra of $A_{\boldsymbol{\gamma}}$ is now defined as the dg algebra

$$B_{\boldsymbol{\gamma}} := \left(\prod_{n\geq 0}\left((\mathbf{\Pi}A)^{\otimes n}\right)^\vee \widehat{\otimes}\mathbb{C}[[t]], \boldsymbol{\gamma}+\gamma_A\right).$$

The dg category $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ is by definition the category of dg modules over $B_{\boldsymbol{\gamma}}$ which are topologically free as modules of the underlying algebra, i.e. after forgetting the differential, and also satisfying the condition of unitality at $t=0$.

As before this category can be viewed as the category $C_{\mathbb{X}_{\boldsymbol{\gamma}}} := (A_{\boldsymbol{\gamma}} - \mathsf{mod})$ of quasi-coherent sheaves on a **nc**-affine **nc**-space $\mathbb{X}_{\boldsymbol{\gamma}} \to \mathbb{D}$ defined over the formal disc $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[t]])$. More generally we will get a **nc**-space $\mathbb{X}$ over the formal scheme of solutions to the Maurer-Cartan equation and $\mathbb{X}_{\boldsymbol{\gamma}} \to \mathbb{D}$ is the restriction of $\mathbb{X}$ to the formal path $\boldsymbol{\gamma}+\gamma_A$ sitting inside that formal scheme.

Similarly we can use $\boldsymbol{\gamma}$ to twist the notion of a Hochschild cohomology class for A. Namely we can consider the Hochschild cohomology of the A_∞-algebra $A_{\boldsymbol{\gamma}}$. It is given explicitly as the cohomology

$$HH^\bullet_{\boldsymbol{\gamma}}(A) := H^\bullet\left(C^\bullet(A,A)[[t]], [\boldsymbol{\gamma}+\gamma_A, \bullet]\right),$$

and is a commutative algebra with respect to the cup product. Note also that the algebra $HH^\bullet_{\boldsymbol{\gamma}}(A)$ comes equipped with a unit $[1_A]$ and a distinguished even element $[\boldsymbol{\gamma}+\gamma_A]$, i.e. a structure similar to the one discussed in Section 2.2.5.

Remark 2.38 • If $\boldsymbol{\gamma}$ has no component of degree zero, i.e. if

$$\boldsymbol{\gamma} \in tC^{\mathrm{even}}_{\mathrm{red},+}(A,A)[[t]],\ \text{where}\ C^\bullet_{\mathrm{red},+}(A,A) = \prod_{n\geq 1}\mathrm{Hom}_{\mathbb{C}-\mathsf{Vect}}\left((\mathbf{\Pi}\,(A/\mathbb{C}\cdot 1_A))^{\otimes n}, A\right),$$

then A_γ is an honest (strong) A_∞-algebra, and the category $(A_\gamma - \mathsf{mod})$ will typically have many interesting objects. Furthermore, in this case smoothness and compactness are stable under deformations. That is, if A is smooth (respectively compact) over $\mathbb{C}$, then A_γ is smooth (respectively compact) over $\mathbb{C}[[t]]$.

- If the $n = 0$ component of γ is non-trivial, i.e. if the corresponding A_∞ structure has a non-trivial m_0, then the category $(A_\gamma - \mathsf{mod})$ may have no non-zero objects. The basic example of this is when $A = \mathbb{C}$ and $\gamma = t \cdot 1_A$.

If the original algebra A has the degeneration property, then it is easy to see that the Hodge-to-de Rham spectral sequence will degenerate for the periodic cyclic homology of A_γ. In other words the formal **nc**-space $\mathbb{X}$ will give rise to a variation of **nc**-Hodge structures over the formal scheme of solutions of the Maurer-Cartan equation for A. When we have a non-trivial $n = 0$ component in γ this may lead to a paradoxical situation in which we have a family of **nc**-spaces over $\mathbb{D}$ which has no sheaves over the generic point but has non-trivial de Rham cohomology (i.e. periodic cyclic homology) generically. This suggests the following important

Question 2.39 What is the geometrical meaning of $HH^\bullet_\gamma(A)$, $HH_\bullet(A_\gamma)$, $HH^-_\bullet(A_\gamma)$, and $HP_\bullet(A_\gamma)$, when γ has non-trivial $n = 0$ component and the objects of $(A_\gamma - \mathsf{mod})$ dissapear over $\mathbb{D}^\times$?

Remark 2.40 Note that if γ solves the Maurer-Cartan equation, then for any $c \in t\mathbb{C}[[t]]$, the cochain $\gamma + c \cdot 1_A$ will also solve the Maurer-Cartan equation[1]. So we have a natural mechanism for modifying formal paths of solutions of the Maurer-Cartan equation. We will exploit this mechanism in the next section.

2.5.2 Corrections by constants

The unpleasant phenomenon of having **nc**-spaces with no sheaves and non-trivial cohomology at the generic point is related to the gluing description for **nc**-Hodge structures. The idea is that the A-modules that dissapear at the generic point of $\mathbb{D}$ may reappear again if we modify the weak A_∞ algebra A_γ appropriately. The periodic cyclic homologies of the different admissible modifications of A_γ then correspond to the regular pieces in the gluing description of the **nc**-de Rham data given by $HP_\bullet(A_\gamma)$. More precisely we have the following

Conjecture 2.41 *Suppose that A is a smooth and compact $d(\mathbb{Z}/2)g$ algebra. Let $\gamma \in tC^{\text{even}}_{\text{red}}(A, A)[[t]]$ be a formal even path of solutions of the Maurer-Cartan equation for A. Then the periodic cyclic homology $HP_\bullet(A_\gamma)$ carries a canonical functorial structure of a variation of $\mathbb{Q}$-**nc**-Hodge structures*

[1] In fact this is the main reason for all the hassle with the unit and the reduced complex in this section.

of exponential type over $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[t]])$. Furthermore there exists a positive integer N and a finite collection of pairwise distinct Puiseux series

$$\boldsymbol{c}_i = \sum_{j \geq 1} c_{i,j} t^{\frac{j}{N}}, \quad c_{i,j} \in \mathbb{C}$$

such that:

- *The series $\boldsymbol{c}_i$ are the distinct eigenvalues of the operator of mutiplication by the class $[\boldsymbol{\gamma} + \gamma_A]$ in the supercommutative algebra $HH^{\bullet}_{\boldsymbol{\gamma}}(A) \widehat{\otimes}_{\mathbb{C}[[t]]} \mathbb{C}((t))$.*
- *For each i the category $(A_{\boldsymbol{\gamma}+\boldsymbol{c}_i \cdot 1_A} - \mathsf{mod})$ is a non-trivial $\mathbb{C}\left[\left[t^{1/N}\right]\right]$-linear $d(\mathbb{Z}/2)g$ category which are smooth and compact over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$ and is computed by a $d(\mathbb{Z}/2)g$ algebra B_i defined over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$ and quasi-isomorphic to the (weak) A_∞-algebra $A_{\boldsymbol{\gamma}+\boldsymbol{c}_i \cdot 1_A}$.*
- *The Hochschild homologies $HH_\bullet(B_i)$ are flat $\mathbb{C}\left[\left[t^{1/N}\right]\right]$-modules and we have*

$$\sum_i \mathrm{rk}_{\mathbb{C}[[t^{1/N}]]}(HH_\bullet(B_i)) = \mathrm{rk}_{\mathbb{C}[[t^{1/N}]]}\, HH_\bullet\left(A_{\boldsymbol{\gamma}}\right) = \dim_{\mathbb{C}} HH_\bullet(A).$$

- *The variation of* **nc***-Hodge structures $HP_\bullet\left(A_{\boldsymbol{\gamma}}\right)$ viewed as a variation over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$ has as regular constituents the variations of* **nc***-Hodge structures on $HP_\bullet(B_i)$ whose existence is predicted by Conjecture* 2.24.

In particular Conjecture 2.41 says that the categorical and Hodge theoretic content of the algebra $A_{\boldsymbol{\gamma}}$ consists of the following data:

(categories) A finite collection of smooth and compact $\mathbb{C}\left[\left[t^{1/N}\right]\right]$-linear d$(\mathbb{Z}/2)$g categories $(B_i - \mathsf{mod})$.

(gluing) A finite collection of distinct Piuseux series $\boldsymbol{c}_i \in \mathbb{C}\left[\left[t^{1/N}\right]\right]$, and formal **nc**-gluing data which glues the variations of regular **nc**-Hodge structures on $HP_\bullet(B_i)$ into a variation of **nc**-Hodge structure of exponential type over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$.

In the above discussion we have tacitly replaces the analytic setting from Section 2.3 by a formal setting. One can check that both the de Rham and Betti data make sense here, e.g. one can speak about homotopy classes of non-intersecting paths to points $\boldsymbol{c}_i$ thinking about t as a small real positive parameter.

Remark 2.42 This situation is analogous to a well known setup in singularity theory. Namely, if we have a germ of an isolated hypersurface singularity given by an equation $f = 0$, and if we have a deformation of f which has several critical values, then the Milnor number of the original singularity is equal to the sum of the Milnor numbers of the simpler critical points of the deformed function. In fact, as we will see in section 3.2 the singularity setup is a rigorous manifestation of the above conjectural picture.

2.5.3 Singular deformations Suppose next that A is compact but not smooth (or smooth but non-compact) d$(\mathbb{Z}/2)$g algebra and let again $\boldsymbol{\gamma} \in tC^{\text{even}}_{\text{red}}(A, A)[[t]]$ be a formal path of solutions of the Maurer-Cartan equation. We expect that the usual definition of smoothness and compactness can be modified to give a notion of smoothness together with compactness of $A_{\boldsymbol{\gamma}}$ at the generic point, i.e. over $\mathbb{C}((t))$, even when the objects in $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ dissapear over $\mathbb{C}((t))$.

In the case when $A_{\boldsymbol{\gamma}}$ is smooth and compact over $\mathbb{C}((t))$, i.e. when the deformation given by $\boldsymbol{\gamma}$ is a smoothing deformation, we also expect Conjecture 2.41 to hold at the generic point. More precisely, we expect to have Puiseux series $\boldsymbol{c}_i$ as above for which the associated categories $(A_{\boldsymbol{\gamma}+\boldsymbol{c}_i 1_A} - \mathsf{mod})$ are non-trivial and smooth and compact over $\mathbb{C}\left(\left(t^{1/N}\right)\right)$. We also expect that the periodic cyclic homology $HP_\bullet(A_{\boldsymbol{\gamma}})$ is equipped with a variation of **nc**-Hodge structures of exponential type over $\mathbb{C}((t))$ so that the periodic cyclic homologies of the categories $(A_{\boldsymbol{\gamma}+\boldsymbol{c}_i 1_A} - \mathsf{mod})$ are the regular pieces of this variation after we base change to $\mathbb{C}\left(\left(t^{1/N}\right)\right)$. Finally, the Puiseux series $\{\boldsymbol{c}_i\}$ should be the eigenvalues of the operator of multiplication by $[\boldsymbol{\gamma} + \gamma_A] \in HH^\bullet(A_{\boldsymbol{\gamma}}) \widehat{\otimes}_{\mathbb{C}[[t]]} \mathbb{C}((t))$.

3 Examples and relation to mirror symmetry

In this section we discuss examples of **nc**-Hodge structures arising from smooth and compact Calabi-Yau geometries and we study how these structures are affected by mirror symmetry. Specifically we look at a generalization of Homological Mirror Symmetry which relates categories of boundary topological field theories (or D-branes) associated with the following two types of geometric backgrounds:

A**-model backgrounds:** Pairs (X, ω), where X is a compact C^∞-manifold, and ω is a symplectic form on X satisfying a convergence property (see below).

B**-model backgrounds:** Pairs $\mathsf{w} : Y \to \text{disc} \subset \mathbb{C}$, where Y is a complex manifold with trivial canonical class, and w is a proper holomorphic map.

We will explain how each such background (both in the A and the B model) gives rise to the geometric and Hodge theoretic data described in Section 2.5.2. Namely we get:

- A finite collection $\{Z_i^{A/B}\}$ of smooth compact **nc**-spaces. In fact $\{Z_i^{A/B}\}$ will be (see Section 4.4.1 for the definition) odd/even Calabi-Yau **nc**-spaces of dimension $\left(\frac{\dim_{\mathbb{R}} X}{2} \mod 2\right) \Big/ (\dim_{\mathbb{C}} Y \mod 2)$.
- Complex numbers $\boldsymbol{c}_i^{A/B}$ and Betti gluing data $\left\{T_{ij}^{A/B}\right\}$ for the regular **nc**-Hodge structures on the periodic cyclic homology of $Z_i^{A/B}$.

In particular the data $\left(HC_\bullet^-\left(Z_i^A\right), \left\{\boldsymbol{c}_i^A\right\}, \left\{T_{ij}^A\right\}\right)$ and $\left(HC_\bullet^-\left(Z_i^B\right), \left\{\boldsymbol{c}_i^B\right\}, \left\{T_{ij}^B\right\}\right)$ each glue into a **nc**-Hodge structure of exponential type. The **generalized Homological Mirror Symmetry Conjecture** now asserts that if two A/B-model backgrounds $(X, \omega)/(Y, \mathsf{w})$ are mirror to each other, then the associated **nc**-geometry and **nc**-Hodge structure packages are isomorphic:

$$\left(Z_i^A, \left\{\boldsymbol{c}_i^A\right\}, \left\{T_{ij}^A\right\}\right) \cong \left(Z_i^B, \left\{\boldsymbol{c}_i^B\right\}, \left\{T_{ij}^B\right\}\right).$$

3.1 A-model Hodge structures: symplectic manifolds

Suppose (X, ω) is a compact symplectic manifold of dimension $\dim_{\mathbb{R}} X = 2d$. In the case when X is a Calabi-Yau variety (in particular $c_1(X) = 0$) one has a family of superconformal field theories attached to X in the large volume limit (i.e. after the rescaling $\omega \to \omega/\hbar$ where $0 < \hbar \ll 1$), and the A-twist gives a topological quantum field theory (see [121]). In mathematical terms it means that we have Gromov-Witten invariants and a $\mathbb{Z}$-graded Fukaya category associated to $(X, \omega/\hbar)$. One the other side, Gromov-Witten invariants can be defined for an arbitrary compact symplectic manifold, not necessarily the one with $c_1(X) = 0$. Our goal in this section is to describe what is an analog of the Fukaya category for general (X, ω).

Namely, it is expected that for (X, ω) of large volume the Fukaya category of (X, ω) is a weak $\mathbb{Z}/2$-graded A_∞-category which will satisfy the generalized smoothness and compactness properties conjectured in Section 2.5.3. Briefly this should work as follows. Following Fukaya-Oh-Ohta-Ono [81] consider a finite collection $\mathfrak{L} = \{L_i\}$ of transversal oriented spin Lagrangian submanifolds in X and form a "degenerate" version $\mathrm{Fuk}_{\mathfrak{L}}$ of Fukaya's category which only involves the L_i. More precisely we take $\mathsf{Ob}\,(\mathrm{Fuk}_{\mathfrak{L}}) = \{L_i\}$, and define

$$\mathrm{Hom}_{\mathrm{Fuk}_{\mathfrak{L}}}(L_i, L_j) = \begin{cases} \mathbb{C}^{L_i \cap L_j}, & i \neq j, \\ A^\bullet(L_i, \mathbb{C}), & i = j. \end{cases}$$

Here $\mathbb{C}^{L_i \cap L_j}$ is taken with the ordinary algebra structure but is put in degree equal to the Maslov grading $\mathrm{mod}\ 2$, and $A^\bullet(L_i, \mathbb{C})$ is the dg algebra of C^∞ differential forms on L_i.

We consider a 1-parameter family of symplectic manifolds

$$\left(X, \frac{\omega}{\hbar}\right), \quad \hbar \in \mathbb{R}_{>0}, \quad \hbar \to 0. \tag{3.1}$$

It will be convenient to introduce a new parameter $q := \exp(-1/\hbar)$ (note that $q \to 0$ when $\hbar \to 0$). Denote by $\mathbb{C}_q$ the usual Novikov ring:

$$\mathbb{C}_q := \left\{ \sum_{i=0}^{\infty} a_i q^{E_i} \,\middle|\, \begin{array}{l} \text{formal series where } a_i \in \mathbb{C} \text{ and } E_i \in \mathbb{R} \\ \text{with } \lim_{i\to\infty} E_i = +\infty. \end{array} \right\}$$

In the case $[\omega] \in H^2(X, \mathbb{Z})$ one can replace the Novikov ring $\mathbb{C}_q$ by more familiar algebra $\mathbb{C}((q))$ of Laurent series. The three-point genus zero Gromov-Witten invariants of the symplectic family (3.1) give rise (see e.g. [155],[168],[223], [65],[85]) to a $\mathbb{C}_q$-valued (small) quantum deformation of the cup product on $H^\bullet(X, \mathbb{C})$:

$$*_q : H^\bullet(X, \mathbb{C})^{\otimes 2} \to H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}_q$$

Conjecturally the series for the quantum product is absolutely convergent for sufficiently small q.

What is constructed in [81] is a solution $\boldsymbol{\gamma}$ of the Maurer-Cartan equation in the cohomological Hochschild complex of $\mathrm{Fuk}_{\mathfrak{L}}$ with coefficients in the series in $\mathbb{C}_q$ with strictly positive exponents (equal to the areas of non-trivial pseudo-holomorphic discs). The meaning of the quantum product is the cup-product in the Hochschild cohomology of the deformed weak category.

The $\mathrm{d}\mathbb{Z}/2\mathrm{g}$ category $\mathrm{Fuk}_{\mathfrak{L}}$ over $\mathbb{C}_q$ is compact but not smooth. If the collection $\mathfrak{L}$ is chosen to be big enough, i.e. if it generates the full Fukaya category, then $\mathrm{Fuk}_{\mathfrak{L}}$ is the large volume limit of $\mathrm{Fuk}(X, \omega)$, i.e. the limit in which all disc instantons for ω are supressed.

Now the formalism of Section 2.5.3 should associate with $\mathrm{Fuk}_{\mathfrak{L}} = (A - \textsf{mod})$ and $\boldsymbol{\gamma}$ a finite collection $\{\boldsymbol{c}_i\}$ of formal series in positive powers of q and a collection $\{\mathrm{Fuk}_i\}$ of non-trivial smooth and compact modifications of the Fukaya category whose Hochschild homologies are the regular singularity constitutents of the Hochschild homology of the q-family of Fukaya categories near the large volume limit. In this geometric context, we expect that the $\{\boldsymbol{c}_i\}$ are the eigenvalues of the quantum multiplication operator $c_1(T_X) *_q (\bullet)$ acting on $H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}[[u]]$. Some evidence for this comes from the observation that when $c_1(T_X)$ vanishes in $H^2(X, \mathbb{Z})$, then the Fukaya category is $\mathbb{Z}$-graded thus is a fixed point of the renormalization group. There is also a more explicit direct argument identifying the class $c_1(T_X)$ with the infnitesimal generator of the renormalization group, but we will not discuss it here.

The formalism of Section 2.5.3 now predicts that the periodic cyclic homology of the Fukaya category, which additively should be the same as the de Rham cohomology of X, should carry a natural **nc**-Hodge structure satisfying the degeneration conjecture from Section 2.2.4. This expectation is supported by ample evidence coming from mirror symmetry for Calabi-Yau complete intersections. Here we present further evidence by describing a natural **nc**-Hodge structure on the de Rham cohomology of a symplectic manifold and by showing that as ω approaches the large volume limit this structure fits in a natural variation of **nc**-Hodge structures.

Using the quantum product $*_q$ we will attach to (X, ω) a variation $((\mathscr{H}, \nabla), \mathscr{E}_B, \overset{\sim}{\longrightarrow})$ of **nc**-Hodge structures over a small disc $\{q \in \mathbb{C} | \, |q| < r\}$ in the q-plane. First we describe the **nc**-Hodge filtration $(\mathscr{H}, \nabla)$ and its variation in the q-direction:

- $\mathscr{H} := H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}\{u, q\}$ and

$$\mathscr{H}^0 := \left(\bigoplus_{k = d \bmod 2} H^k(X, \mathbb{C}) \right) \otimes \mathbb{C}\{u, q\}$$

$$\mathscr{H}^1 := \left(\bigoplus_{k = d+1 \bmod 2} H^k(X, \mathbb{C}) \right) \otimes \mathbb{C}\{u, q\}$$

- ∇ is a meromorphic connection on $\mathscr{H}$ with poles along the coordinate axes $u = 0$ and $q = 0$, given by

$$\nabla_{\frac{\partial}{\partial u}} := \frac{\partial}{\partial u} + u^{-2} \left(\kappa_X *_q \bullet \right) + u^{-1} \textsf{Gr},$$

$$\nabla_{\frac{\partial}{\partial q}} := \frac{\partial}{\partial q} - q^{-1} u^{-1} \left([\omega] *_q \bullet \right),$$

where:

$\kappa_X \in H^2(X, \mathbb{Z})$ denotes the first Chern class of the cotangent bundle of X computed w.r.t. any ω-compatible almost complex structure, and

$\textsf{Gr} : \mathscr{H} \to \mathscr{H}$ is the grading operator defined to be $\textsf{Gr}_{|H^k(X,\mathbb{C})} := \frac{k-d}{2} \, \mathrm{id}_{H^k(X,\mathbb{C})}$.

The data $(\mathscr{H}, \nabla)$ define a q-variation of (the de Rham part of) **nc**-Hodge structures. Defining the $\mathbb{Q}$-structure is much more delicate. To gain some insight into the shape of the rational local system $\mathscr{E}_B$ one can look at the monodromy in the q direction of the algebraic bundle with connection

$$(H, \nabla)_{|(\mathbb{A}^1 - \{0\}) \times \{q \in \mathbb{C}|\ |q| < R\}}, \qquad (H, \nabla) = \mathfrak{B}_{\text{along } u}((\mathscr{H}, \nabla)).$$

In some cases the fact that $\mathscr{E}_B$ should be preserved by ∇ and the Stokes filtration is rational with respect to $\mathscr{E}_B$ is enough to determine $\mathscr{E}_B$ completely:

Proposition 3.1 *Let $X = \mathbb{CP}^{n-1}$ and let ω be the Fubini-Studi form. Let (H, ∇) be the holomorphic bundle with meromorphic connection on $(\mathbb{A}^1 - \{0\}) \times \{q \in \mathbb{C}|\ |q| < R\}$ defined above. Let $\psi \in H$ be a holomorphic section which is covariantly constant with respect to ∇. Then*

(a) For every $u \neq 0$, $\psi \neq 0$ the limit (in a sector of the q-plane)

$$\psi_{\text{cl}}(u) = \lim_{q \to 0} \left(\exp \left(-\frac{\log(q)}{u} ([\omega] \wedge (\bullet)) \right) \right) \psi$$

exists. Furthermore, ψ_{cl} satisfies the differential equation

$$\left(\frac{d}{du} + u^{-2} \kappa_X \wedge + u^{-1} \mathbf{Gr} \right) \psi_{\text{cl}} = 0.$$

(b) The vector

$$\psi_{\text{const}}(u) := \exp(\log(u) \mathbf{Gr}) \exp \left(\frac{\log(u)}{u} \kappa_X \wedge (\bullet) \right) \psi_{\text{cl}} \in H^\bullet(X, \mathbb{C})$$

is independent of u.

(c) Define the rational structure $\mathscr{E}_B \subset H^\nabla$ as the subsheaf of all covariantly constant sections ψ for which the vector $\psi_{\text{const}} \in H^\bullet(X, \mathbb{C})$ belongs to the image of the map

$$H^\bullet(X, \mathbb{Q}) \xrightarrow{\mathfrak{d}} H^\bullet(X, \mathbb{C}) \xrightarrow{\widehat{\Gamma}(X) \wedge (\bullet)} H^\bullet(X, \mathbb{C}),$$

where $\mathfrak{d} \in GL(H^\bullet(X, \mathbb{C}))$ is the operator of multiplication by $(2\pi i)^{k/2}$ on $H^k(X, \mathbb{C})$, and $\widehat{\Gamma}(X)$ is a new characteristic class of X defined as

$$\widehat{\Gamma}(X) := \exp \left(\boldsymbol{C} ch_1(T_X) + \sum_{n \geq 2} \frac{\zeta(n)}{n} ch_n(T_X) \right),$$

where

$$\boldsymbol{C} = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n) \right)$$

is Euler's constant, and $\zeta(s)$ is Riemann's zeta function.

Then the inclusion $\mathscr{E}_B \subset H^{\nabla}$ is compatible with Stokes data, i.e. the rational structure $\mathscr{E}_B$ satisfies **($\mathbb{Q}$-structure axiom)^{exp}**.

The calculation presented below was known already to B.Dubrovin [70, Section 4.2.1], where he also obtained a Taylor expansions of a power of a Gamma function in quantum cohomology, although he did not identify it with a characteristic class.

Proof of Proposition 3.1**.** In the standard basis $\{1, h, h^2, \ldots, h^{n-1}\}$ of $H^\bullet(\mathbb{P}^{n-1}, \mathbb{C})$ the connection ∇ on H is given by

$$\nabla_{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u} + u^{-2}\begin{pmatrix} 0 & & & nq \\ n & 0 & & \\ & \ddots & \ddots & \\ & & n & 0 \end{pmatrix} + u^{-1}\begin{pmatrix} \frac{1-n}{2} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \frac{n-1}{2} \end{pmatrix}$$

$$\nabla_{\frac{\partial}{\partial q}} = \frac{\partial}{\partial q} - q^{-1}u^{-1}\begin{pmatrix} 0 & & & q \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix},$$

If $\psi = \sum_{i=1}^{n} \psi_i h^{i-1}$ is a local section of H, a straightforward check shows that the condition on ψ to be ∇-horizontal is solved by the following ansatz:

$$\psi_n = u^{\frac{1-n}{2}} \int\limits_{\Gamma_{u,q}} \exp(\mathcal{F}) \prod_{i=1}^{n-1} \frac{dz_i}{z_i}$$

$$\psi_{n-1} = \left(uq\frac{\partial}{\partial q}\right)\psi_n$$

$$\psi_{n-2} = \left(uq\frac{\partial}{\partial q}\right)^2\psi_n$$

$$\cdots$$

$$\psi_1 = \left(uq\frac{\partial}{\partial q}\right)^{n-1}\psi_n.$$

Here $\mathcal{F}$ is the function on $(\mathbb{C}^\times)^{n-1}$ with coordinates $z_1, \ldots, z_{n-1}$ depending on parameters $u, q \neq 0$ and given by

$$\mathcal{F}(z_1, z_2, \ldots, z_{n-1}; u, q) := u^{-1}\left(z_1 + z_2 + \ldots z_{n-1} + \frac{q}{z_1 z_2 \ldots z_{n-1}}\right).$$

The integral is taken over some fixed $(n-1)$-dimensional semi-algebraic non-compact cycle $\Gamma_{u,q}$ in $(\mathbb{C}^\times)^{n-1}$ (depending on the parameters u,q) which is going to infinity in directions where $\mathrm{Re}(\mathcal{F}) \to -\infty$.

More generally, the domain of integration $\Gamma_{u,q}$ used for defining ψ_n can be taken to be a $(n-1)$-dimensional rapid decay homology chain in $(\mathbb{C}^\times)^{n-1}$. The rapid decay homology cycles on smooth complex algebraic varieties are the natural domains of integration for periods of cohomology classes of irregular connections. The rapid decay homology was introduced and studied by Hien [116], [117], following previous works of Sabbah [204] and Bloch-Esnault [31]. In particular by a recent work of Mochizuki [187], [188] and Hien [117] it follows that (after a birational base change) taking periods induces a perfect pairing between the de Rham cohomology of an irregular connection and the rapid decay homology. This powerful general theory is not really needed in our case where the manifold is the affine algebraic torus $(\mathbb{C}^\times)^{n-1}$ but it does provide a useful perspective.

Explicitly the non-compact cycles that we will use to generate horizontal sections of (H,∇) will be the $(n-1)$-dimensional relative cycles for a pair $(\boldsymbol{\mathcal{X}}, Z)$ constructed as follows. Start with a smooth projective compactification $\mathfrak{X}$ of $(\mathbb{C}^\times)^{n-1}$ with a normal crossing boundary divisor D which is adapted to $\mathcal{F}$ in the sense that if u and q are nonzero, the divisors of zeroes and poles of $\mathcal{F}$ in $\mathfrak{X}$ do not intersect with each other, and locally at points of D the function $\mathcal{F}$ can be written as a product of an invertible holomorphic function and a monomial in the local coordinates. Let $\boldsymbol{\mathcal{X}}$ be the real oriented blow-up of $\mathfrak{X}$ along the divisor D. Now consider the real boundary $\partial\boldsymbol{\mathcal{X}}$ of $\boldsymbol{\mathcal{X}}$, i.e. the union of all the boundary divisors of the real oriented blow-up. The boundary $\partial\boldsymbol{\mathcal{X}}$ contains a natural open real semi-algebraic subset $Z \subset \partial\boldsymbol{\mathcal{X}}$ consisting of all points $b \in \partial\boldsymbol{\mathcal{X}}$, such that $|\mathcal{F}(z;u,q)| \to \infty$ when $z \to b$, and for points $z \in t(\mathbb{C}^\times)^{n-1}$ near b the argument of $\mathcal{F}(z;u,q)$ lies strictly in the left half-plane of $\mathbb{C}$. Note that the real blow-up X has the same homotopy type as $\mathfrak{X} - D = (\mathbb{C}^\times)^{(n-1)}$ and so relative cycles on $(\boldsymbol{\mathcal{X}}, Z)$ can be thought of as non-compact cycles on $(\mathbb{C}^\times)^{(n-1)}$. Moreover since Z is defined by our condition on the argument of $\mathcal{F}$, it follows that relative cycles with boundaries in Z give rise to well defined integrals of $\exp(\mathcal{F})\prod z_i^{-1}dz_i$.

Next observe the integrals over relative cycles with integral coefficients, i.e. elements in $H_{n-1}(\boldsymbol{\mathcal{X}}, Z; \mathbb{Z})$ give rise to a covariantly constant integral lattice in the bundle (H,∇). Furthermore the Deligne-Malgrange-Stokes filtration is integral with respect to this lattice. Indeed if we fix a real number λ, then whenever $\mathrm{Re}\,(\mathcal{F}) < \lambda \cdot |u|^{-1}$, it follows that $|\exp(\mathcal{F})| < \exp(\lambda \cdot |u|^{-1})$ when $u \to 0$. Hence the steps of the Deligne-Malgrange-Stokes filtration of (H,∇) are easy to describe in this language: they correspond to periods of $\exp(\mathcal{F})\prod z_i^{-1}dz_i$ on relative cycles on $(\boldsymbol{\mathcal{X}}, Z)$ whose boundary is contained in half-planes of the form $\mathrm{Re}(\mathcal{F}) < \mathrm{const}$. The periods over cycles with integral coefficients and the same boundary property then give a full integral lattice in each such step.

Now to finish the proof of the proposition we have just to calculate the limiting lattice (which is independent of u and q) consisting of vectors $\psi_{\mathrm{const}} \in H^\bullet(X,\mathbb{C})$ defined in terms of ψ by the formula in part (b) of the statement of the proposition.

For a general ∇-horizontal local section $\psi = \sum_{i=1}^n \psi_i h^{i-1}$ in a sector at 0 in the q-plane (for given $u \neq 0$) one has an asymptotic expansion of ψ at $q \to 0$ given by:

$$\psi_n = \sum_{i=0}^{n-1} a_i(u)(\log q)^i + \boldsymbol{O}(q(\log q)^n) + \dots, \tag{3.2}$$

Then we have that the "classical limit" (at $q \to 0$ where the quantum multiplication becomes classical) is given by

$$\psi_{\rm cl}(u) = \begin{pmatrix} (n-1)!u^{n-1}a_{n-1}(u) \\ (n-2)!u^{n-2}a_{n-2}(u) \\ \vdots \\ 0!u^0 a_0(u) \end{pmatrix}.$$

Now we restrict to the case where all variables are real, $u < 0$, $q > 0$ and the contour of integration being the positive octant $\{(z_1, \ldots, z_n) \in \mathbb{C}^n | \, z_i > 0 \; \forall i\}$.

Function $\psi_n = \psi_n(u, q)$ decays exponentially fast at $q \to +\infty$ for a given $u < 0$, hence one can extract its asymptotic expansion at $q \to 0$ through the Mellin transform:

$$\int\limits_0^{+\infty} \psi_n q^s \frac{dq}{q} = \sum_{i=0}^{\infty} a_i(u) \frac{i!(-1)^i}{s^{i+1}} + \boldsymbol{O}(1), \;\; s \to 0.$$

This integral can be calculated explicitly

$$\int\limits_0^{+\infty} \psi_n q^s \frac{dq}{q} = u^{\frac{1-n}{2}} \underbrace{\int\limits_0^{+\infty} \cdots \int\limits_0^{+\infty}}_{n \text{ times}} \frac{dq}{q} \prod_{i=1}^{n-1} \frac{dz_i}{z_i} \exp\left(u^{-1}\left(z_1 + z_2 + \ldots z_{n-1} + \frac{q}{z_1 z_2 \ldots z_{n-1}}\right)\right) q^s$$

$$= u^{\frac{1-n}{2}} \underbrace{\int\limits_0^{+\infty} \cdots \int\limits_0^{+\infty}}_{n-1 \text{ times}} \prod_{i=1}^{n-1} \frac{dz_i}{z_i} \exp\left(u^{-1} \sum_i^{n-1} z_i\right)$$

$$\cdot \underbrace{\int_0^{+\infty} \exp\left(\frac{q}{u z_1 z_2 \ldots z_{n-1}}\right) q^s \frac{dq}{q}}_{\overset{||}{\Gamma(s)(-u z_1 z_2 \ldots z_{n-1})^s}}$$

$$= u^{\frac{1-n}{2}} (-u)^s \Gamma(s) \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{i=1}^{n-1} \left(\frac{dz_i}{z_i} z_i^s \exp \frac{z_i}{u}\right)$$

$$= u^{\frac{1-n}{2}} (-u)^s \Gamma(s) \left((-u)^s \Gamma(s)\right)^{n-1}$$

$$= u^{\frac{1-n}{2}} (-u)^{ns} \Gamma(s)^n.$$

The conclusion is that the chosen branch $\psi_{\rm cl}(u)$ is completely defined by the expansion

$$u^{\frac{1-n}{2}} (-u)^{ns} \Gamma(s)^n = \frac{\psi_{{\rm cl},n}(u)}{(-u)^0 s} + \frac{\psi_{{\rm cl},n-1}(u)}{(-u)^1 s^2} + \cdots + \frac{\psi_{{\rm cl},1}(u)}{(-u)^{n-1} s^n} + \boldsymbol{O}(1), \;\; s \to 0$$

Furthermore, all the other branches can be obtained by acting on the branch we know by the monodromy transformations (around $q = 0$)

$$\frac{(2\pi\sqrt{-1})^i}{u^i}\begin{pmatrix}0 & & & 0\\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{pmatrix}^i,$$

for $i = 0, \ldots, n-1$.

Section ψ_{cl} satisfies the differential equation

$$\left(\frac{d}{du} + u^{-2}\kappa_X \wedge + u^{-1}\textbf{Gr}\right)\psi_{\mathrm{cl}} = 0.$$

which is the classical limit (at $q \to 0$) of the equation

$$\nabla_{\frac{\partial}{\partial u}}(\psi) = 0$$

One can check that the operator $\frac{d}{du} + u^{-2}\kappa_X \wedge + u^{-1}\textbf{Gr}$ can be written as

$$\exp\left(-\frac{\log(u)}{u}\kappa_X \wedge (\bullet)\right)\exp(-\log(u)\textbf{Gr}) \circ \frac{d}{du} \circ \exp(\log(u)\textbf{Gr})\exp\left(\frac{\log(u)}{u}\kappa_X \wedge (\bullet)\right)$$

This follows from the commutation relation

$$[\kappa_X \wedge (\bullet), \textbf{Gr}] = -\kappa_X \wedge (\bullet)$$

Finally, in the above formulas one can replace $\log(u)$ by $\log(-u)$ (and also $u^{\frac{1-n}{2}}$ by $(-u)^{\frac{1-n}{2}}$ with principal values at the domain $u < 0$. Having this modification in mind, we conclude that the vector

$$\psi_{\mathrm{const}} = \psi_{\mathrm{const}}(u) := \exp(\log(-u)\textbf{Gr})\exp\left(\frac{\log(-u)}{u}\kappa_X \wedge (\bullet)\right)\psi_{\mathrm{cl}} \in H^\bullet(X, \mathbb{C})$$

is independent of u, and in particular it coincides with $\psi_{\mathrm{cl}}(-1)$, as for $u = -1$ the correction matrices relating $\psi_{\mathrm{const}}(u)$ and $\psi_{\mathrm{cl}}(u)$ are identity matrices. Therefore the vector ψ_{const} is given by Taylor coefficients

$$\psi_{\mathrm{const},1}s^0 + \cdots + \psi_{\mathrm{const},n}s^{n-1} = s^n\Gamma(s)^n + \boldsymbol{O}(s^n) = \Gamma(1+s)^n + \boldsymbol{O}(s^n)$$

We see that $\psi_{\mathrm{const}} \in H^\bullet(X, \mathbb{C})$ (after rescaling by operator $\mathfrak{d}$ from the Proposition) with the value of the multiplicative characteristic class associated with the series $\Gamma(1+s) = 1 + \boldsymbol{O}(s) \in \mathbb{C}[[s]]$ and the tangent bundle T_X, because $[T_X] = n[\mathcal{O}(1)] - [\mathcal{O}]$ for $X = \mathbb{CP}^n$, and by the classical expansion

$$\log(\Gamma(1+s)) = \boldsymbol{C}s + \sum_{k \geq 2}\frac{\zeta(k)}{k}s^k$$

The action of the monodromy corresponds (up to torsion) to the multiplication by $\kappa_X \in H^\bullet(X, \mathbb{Z})$. $\square$

The previous proposition suggests the following general definition:

Definition 3.2 *The rational structure on (H, ∇) is the local subsystem $\mathscr{E}_B \subset H_{|\mathbb{A}^1 - \{0\}}$ of multivalued ∇-horizontal sections whose values at 1 belong to the image of*

$$H^\bullet(X, \mathbb{Q}) \xrightarrow{\mathfrak{d}} H^\bullet(X, \mathbb{C}) \xrightarrow{\widehat{\Gamma}(TX) \wedge (\bullet)} H^\bullet(X, \mathbb{C}),$$

where $\mathfrak{d} \in GL(H^\bullet(X, \mathbb{C}))$ is the operator of multiplication by $(2\pi i)^{k/2}$ on $H^k(X, \mathbb{C})$, and $\widehat{\Gamma}(TX)$ is a new characteristic class of X defined as

$$\widehat{\Gamma}(TX) := \prod_{i=1}^{d} \Gamma(1 + \lambda_i),$$

where $\Gamma(s)$ is the classical gamma function and λ_i are the Chern roots of T_X computed in any ω-admissible almost complex structure.

Remark 3.3 Apart from the calculation in Proposition 3.1 there are a few other (loose) motivations for this definition:

- The class $\widehat{\Gamma}$ appears in the context of deformation quantization in the work of the second author [153, Section 4.6].
- The number $\chi(X)\zeta(3)$ appears in the mirror formula for the quintic threefold.
- Golyshev's description [93], [94] of the **nc**-motives associated with the Landau-Ginzburg mirror of a toric Fano involves similar hypergeometric series.
- The same class $\widehat{\Gamma}$ was derived and a definition similar to Definition 3.2 was proposed in the recent work of Iritani [123] for the case of toric orbifolds by tracing out the mirror image of rational structure of the mirror Landau-Ginzburg model.

Conjecture 3.4 *The triple $(H, \mathscr{E}_B, \xrightarrow{\sim})$ associated above with a symplectic manifold (X, ω) is a variation of* **nc***-Hodge structures of exponential type.*

Remark 3.5 **(i)** In general it is not clear if the **($\mathbb{Q}$-structure axiom)$^{\mathbf{exp}}$** holds in this case. It does hold trivially in the graded case, i.e. when X is a Calabi-Yau.

(ii) At the moment the "exponential type" part of the conjecture is not supported by any evidence beyond the graded case in which the **nc**-Hodge structure is regular. It is possible that for non-Kähler symplectic manifolds the **nc**-Hodge structure on the de Rham cohomology is not of exponential type.

3.2 B-model Hodge structures: holomorphic Landau-Ginzburg models

Suppose we have an algebraic map $\mathsf{w} : Y \to \mathbb{C}$, where Y is a smooth quasi-projective manifold and w has a compact critical locus $\mathrm{crit}(\mathsf{w}) \subset Y$. Let $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m\} \subset \mathbb{C}$ denote the critical values of w.

A pair (Y, w) like that is called a ***holomorphic Landau-Ginzburg model*** and often arises (see e.g. [122, 121]) as the mirror of a symplectic manifold underlying a hypersurface, or a complete intersection in a toric variety. Remarkably the pair (Y, w) give rise to a natural nc-space $\mathsf{nc}(Y, \mathsf{w})$. The category $C_{\mathsf{nc}(Y,\mathsf{w})}$ can be described in two equivalent ways (in fact these descriptions are valid even if the critical locus of w is not compact). First note that it is enough to define $\oplus\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$ since that the category $C_{\mathsf{nc}(Y,\mathsf{w})}$ can be thought of as the homotopy colimit completion of $\oplus\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$. For the latter we have two models:

$\oplus\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$ **as a category of matrix factorizations:** This model was proposed originally by the second author as a mathematical description of the category of D-branes and was subsequently studied extensively in the physics and mathematics literature, see [134, 135] and [194, 196, 195].

A matrix factorization on (Y, w) is a pair $\left(E = E^0 \oplus E^1, d_E \in \mathrm{End}(E)^{\mathrm{opp}}\right)$, where

E is a $\mathbb{Z}/2$-graded algebraic vector bundle on Y, and

d_E is an odd endomorphism satisfying $d_E^2 = \mathsf{w} \cdot \mathrm{id}_E$.

In the case when Y is affine the $\mathbb{Z}/2$-graded complex $\underline{\mathrm{Hom}}((E, d_E), (F, d_F)$ of homomorphisms between two matrix factorizations is defined as $\underline{\mathrm{Hom}}((E, d_E), (F, d_F) := (\mathrm{Hom}(E, F), d)$ where for a $\varphi : E \to F$ we have $d\varphi := \varphi \circ d_E - d_F \circ \varphi$. For general Y the same definition works if we replace $\mathrm{Hom}(E, F)$ by some acyclic model, e.g. if we use the Dolbeault resolution. The resulting category $\mathsf{MF}(Y, \mathsf{w})$ of matrix factorizations is a $\mathbb{C}$-linear d$(\mathbb{Z}/2)$g category. We define $\oplus\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$ to be the derived category $D^b(\mathsf{MF}(Y, \mathsf{w}))$ of the category of matrix factorizations.

To construct $D^b(\mathsf{MF}(Y, \mathsf{w}))$ one notes that in addition to being a d$(\mathbb{Z}/2)$g category $\mathsf{MF}(Y, \mathsf{w})$ can also be viewed as a curved d$(\mathbb{Z}/2)$g category with central curvature w (see e.g. [199] for the definition) or as a $\mathbb{Z}/2$-graded weak A_∞-category, i.e. an A_∞ category with an m_0-operation given by w (see e.g. [213], [165] for the definition). In particular we can form the associated homotopy category (in the A_∞-sense) which by definition will be the derived category of matrix factorizations.

Alternatively, one can use the following two step construction proposed by Orlov. First we pass to the homotopy category of $\mathsf{MF}(Y, \mathsf{w})$, i.e. we consider the category whose objects are matrix factorizations and whose morphisms are given by the quotient of $\underline{\mathrm{Hom}}((E, d_E), (F, d_F))$ by homotopy equivalences. Next (following the standard wisdom) we need to quotient $\mathrm{Ho}(\mathsf{MF}(Y, \mathsf{w}))$ by the subcategory of acyclic factorizations. Since the matrix factorizations are not complexes, they do not have cohomology and so we can not define acyclicity in the usual way. But there is another point of view on acyclicity. If we have a short exact sequence of usual complexes, then the total complex of this diagram will be an acyclic complex. So we define ***acyclic*** matrix factorizations as the total matrix factorization of an exact sequence of factorizations. With this definition we get a thick subcategory in the homotopy category $\mathrm{Ho}(\mathsf{MF}(Y, \mathsf{w}))$ matrix factorizations and then we can pass to the Serre quotient of $\mathrm{Ho}(\mathsf{MF}(Y, \mathsf{w}))$ by this thick subcategory. We set $D^b(\mathsf{MF}(Y, \mathsf{w}))$ to be this Serre quotient.

$\oplus\mathsf{Perf}_{C_{\mathbf{nc}(Y,\mathsf{w})}}$ **as a category of singularities:** This model was proposed originally by D. Orlov as an alternative to the matrix factorization description which is localized near the critical set of w. Orlov proved the equivalence of the two models, various versions of the localization theorem, and proved several duality statements relating derived categories of singularities to other familiar categories [194, 196, 195].

Suppose Z is a quasi-projective complex scheme. The ***derived category*** $D^b_{\mathrm{Sing}}(Z)$ ***of singularities*** of Z is defined as the quotient

$$D^b_{\mathrm{Sing}}(Z) := D^b(\mathsf{Coh}(Z))/\oplus\mathsf{Perf}_Z$$

of the (dg enhancement of the) bounded derived category $D^b(\mathsf{Coh}(Z))$ of coherent sheaves on Z by the thick subcategory of perfect complexes on Z. The syzygy theorem implies that $D^b_{\mathrm{Sing}}(Z) = 0$ whenever Z is smooth and so $D^b(\mathsf{Coh}(Z))$ can be thought of as an invariant of the singularities of Z.

If now $\mathsf{w} : Y \to \mathbb{C}$ is a holomorphic Landau-Ginzburg model we write Y_c for the fiber $\mathsf{w}^{-1}(c)$ and set

$$\oplus\mathsf{Perf}_{C_{\mathbf{nc}(Y,\mathsf{w})}} := D^b_{\mathrm{Sing}}(Y_0).$$

Note that if $0 \in \mathbb{A}^1$ is not a critical value of w, then with this definition we will get $\oplus\mathsf{Perf}_{C_{\mathbf{nc}(Y,\mathsf{w})}} = 0$. In order to get non-trivial categories we will use the critical values $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ to shift the potentail $\mathsf{w} \rightsquigarrow \mathsf{w} - \boldsymbol{c}_i$ and associate with $\mathbf{nc}(Y, \mathsf{w})$ honest categories $\oplus\mathsf{Perf}_i := \oplus\mathsf{Perf}_{C_{\mathbf{nc}(Y,\mathsf{w}-\boldsymbol{c}_i)}} = D^b_{\mathrm{Sing}}(Y_{\boldsymbol{c}_i})$. Conjecturally, these categories are smooth and compact.

Mirror symmetry suggests that the $\mathbf{nc}$-space $\mathbf{nc}(Y, \mathsf{w})$ gives rise to the B-model geometric and Hodge theoretic data described in Section 2.5.2, and in particular that the periodic cyclic homology of $C_{\mathbf{nc}(Y,\mathsf{w})}$ carries a canonical $\mathbf{nc}$-Hodge structure. In fact we have already described the geometric part of the data, namely the numbers $\{\boldsymbol{c}_i\}$ and the categories $\{\oplus\mathsf{Perf}_i\}$. These data of course fix the regular type (in the sense of Theorem 2.35) of the $\mathbf{nc}$-Hodge structure but we are still missing the gluing data. Here we propose a construction of the Hodge structure on the periodic cyclic homology of $C_{\mathbf{nc}(Y,\mathsf{w})}$ but similarly to the A-model we have to rely on the actual geometry of (Y, w) in order to produce the gluing data. At present it is not clear if the gluing data can be reconstructed from the category $C_{\mathbf{nc}(Y,\mathsf{w})}$ or more generally from its one parameter deformation.

First we discuss the appropriate cohomologies of the Landau-Ginzburg model. Let

$$\begin{aligned}\mathscr{H}^{\bullet}_{\mathrm{for}} &:= H^{\bullet}_{DR}((Y, \mathsf{w}); \mathbb{C}) \\ &= \mathbb{H}^{\bullet \bmod 2}_{\mathrm{Zar}}\left(Y, (\Omega^{\bullet}_Y[[u]], ud_{DR} + d\mathsf{w}\wedge)\right)\end{aligned}$$

be the $\mathbb{Z}/2$-graded $\mathbb{C}[[u]]$-module of algebraic de Rham cohomology of the potential w. In the case when $\mathrm{crit}(\mathsf{w})$ is compact, the $\mathbb{C}[[u]]$-module $\mathscr{H}^{\bullet}_{\mathrm{for}}$ is known to be free by the work of Barannikov and the second author (unpublished), Sabbah [203], or Ogus-Vologodsky [191]. This implies the following

Lemma 3.6 *Assume that Y is quasi-projective and the critical locus of* w *is compact. Then we have:*

(i) *The fiber of $\mathscr{H}^{\bullet}_{\mathrm{for}}$ at $u = 0$ is the algebraic Dolbeault cohomology*

$$\mathbb{H}^{\bullet}_{Zar}\left(Y, (\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)\right) \cong \mathbb{H}^{\bullet}_{an}\left(Y, (\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)\right)$$

of the potential w.

(ii) *There is a canonical isomorphism*

$$\mathbb{H}^{\bullet}_{Zar}\left(Y, (\Omega^{\bullet}_{Y}[[u]], ud_{DR} + d\mathsf{w}\wedge)\right) \cong \mathbb{H}^{\bullet}_{an}\left(Y, (\Omega^{\bullet}_{Y}[[u]], ud_{DR} + d\mathsf{w}\wedge)\right)$$

(iii) *If the map* w *is proper then $\mathscr{H}^{\bullet}_{\mathrm{for}}$ is the formal germ at $u = 0$ of an algebraic vector bundle on the affine line*

$$\mathscr{H}^{\bullet}_{\mathrm{alg}} := \mathbb{H}^{\bullet\, mod\, 2}_{Zar}\left(Y, (\Omega^{\bullet}_{Y}[u], ud_{DR} + d\mathsf{w}\wedge)\right)$$

Proof. The cohomology sheaves of the complex $(\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)$ are supported on the critical locus of w and so, by our compactness assumption, must be coherent sheaves on Y both in the analytic and in the Zariski topology. The hypercohomology spectral sequence then implies that the hypercohomology of the complex $(\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)$ is finite dimensional and the spectral sequence associated with the filtration induced by multiplication by u implies that $\mathbb{H}^{\bullet}_{\mathrm{Zar/an}}\left(Y, (\Omega^{\bullet}_{Y}[[u]], ud_{DR} + d\mathsf{w}\wedge)\right)$ is a finite rank $\mathbb{C}[[u]]$-module. Furthermore, the same spectral sequence implies that

$$\dim_{\mathbb{C}((u))} \mathbb{H}^{\bullet}_{\mathrm{Zar/an}}\left(Y, (\Omega^{\bullet}_{Y}((u)), ud_{DR} + d\mathsf{w}\wedge)\right) \leq \dim_{\mathbb{C}} \mathbb{H}^{\bullet}_{\mathrm{Zar/an}}\left(Y, (\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)\right).$$

The freeness statement of Barannikov and the second author (see e.g. [203]) now gives that these two dimensions are equal and so $\mathbb{H}^{\bullet}_{\mathrm{Zar}}\left(Y, (\Omega^{\bullet}_{Y}[[u]], ud_{DR} + d\mathsf{w}\wedge)\right)$ is a free finite rank module over $\mathbb{C}[[u]]$. This proves part **(i)** of the lemma.

For part **(ii)** we only need to notice that the two spaces in question are computed by spectral sequences associated with the filtrations by the powers of u and that these spectral sequences have E_2-sheets whose entries are finite sums of copies of $\mathbb{H}^{\bullet}_{\mathrm{Zar}}\left(Y, (\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)\right)$ and $\mathbb{H}^{\bullet}_{\mathrm{an}}\left(Y, (\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)\right)$ respectively. Each of these can in turn be computed from the hypercohomology spectral sequence for the complex $(\Omega^{\bullet}_{Y}, d\mathsf{w}\wedge)$ of (Zariski or analytic) coherent sheaves. But the cohomology sheaves of this complex are supported on the zero locus of $d\mathsf{w}$ which by assumption is projective. Hence by GAGA the Zariski and analytic cohomologies of this complex are naturally isomorphic. This gives isomorphisms of the hypercohomology and filtration spectral sequences in the Zariski and the analytic setup respectively and so the two types of hypercohomologies are isomorphic.

Finally, part **(iii)** was also proven by Barannikov and the second author, and by Sabbah [203]. $\square$

Remark 3.7 The isomorphism in part **(ii)** of the previous lemma is not convergent for $u \to 0$ in general. Indeed if $u \neq 0$ is a complex number, then the complex vector space $\mathbb{H}^{\bullet}_{\mathrm{an}}\left(Y, (\Omega^{\bullet}_{Y}, ud_{DR} + d\mathsf{w}\wedge)\right)$ is the same as the usual de Rham cohomology $H^{\bullet}_{DR}(Y, \mathbb{C})$ of Y. Indeed, for such a fixed $u \neq 0$, the complex $(\Omega^{\bullet}_{Y}, ud_{DR} + d\mathsf{w}\wedge)) \cong \left(\Omega^{\bullet}_{Y}, d_{DR} + u^{-1}d\mathsf{w}\wedge\right))$ is the holomorphic de Rham complex of the local system $(\mathcal{O}_{Y}, d_{DR}+u^{-1}d\mathsf{w})$. But the multiplication by $\exp(-u^{-1}\mathsf{w})$ is an analytic automorphism of the line bundle $\mathcal{O}_Y$ which gauge transforms the connection $d_{DR}+u^{-1}d\mathsf{w}$ into the trivial connection d_{DR}. Hence $\exp(-u^{-1}\mathsf{w})$ identifies $(\Omega_Y^\bullet, ud_{DR}+d\mathsf{w}\wedge)$ with the holomorphic de Rham complex $(\Omega_Y^\bullet, d_{DR})$ and $\mathbb{H}^\bullet_{\text{an}}(Y,(\Omega_Y^\bullet, ud_{DR}+d\mathsf{w}\wedge))$ with $H^\bullet_{DR}(Y,\mathbb{C})$. On the other hand, the space $\mathbb{H}^\bullet_{\text{Zar}}(Y,(\Omega_Y^\bullet, ud_{DR}+d\mathsf{w}\wedge))$ depends on the potential in an essential way. For instance, if $\mathsf{w}: Y\to\mathbb{A}^1$ is a Lefschetz fibration, then the complex $(\Omega_Y^\bullet, d\mathsf{w}\wedge)$ is just the Koszul complex associated with the regular section $d\mathsf{w}\in\Omega^1_Y$. In particular the space $\mathbb{H}^\bullet_{\text{Zar}}(Y,(\Omega_Y^\bullet, d_{DR}+d\mathsf{w}\wedge))\cong\mathbb{H}^\bullet_{\text{Zar}}(Y,(\Omega_Y^\bullet, d\mathsf{w}\wedge))$ has dimension equal to the number of critical points of w. More generally $\mathbb{H}^\bullet_{\text{Zar}}(Y,(\Omega_Y^\bullet, d_{DR}+d\mathsf{w}\wedge))$ can be identified (see e.g. [132]) with the cohomology of the perverse sheaf of vanishing cycles of w.

Remark 3.8 Under our assumptions, the algebraic de Rham and Dolbeault cohomologies $H^\bullet_{DR}((Y,\mathsf{w});\mathbb{C})$ and $H^\bullet_{Dol}((Y,\mathsf{w});\mathbb{C})$ of the potential w can be identified respectively with the periodic cyclic and Hochschild homologies $HP_\bullet(C_{\mathbf{nc}(Y,\mathsf{w})})$ and $HH_\bullet(C_{\mathbf{nc}(Y,\mathsf{w})})$ of the **nc**-space $C_{\mathbf{nc}(Y,\mathsf{w})}$ (more precisely, of the collection of categories $\oplus\mathsf{Perf}_i$ labeled by numbers $\{\boldsymbol{c}_i\}$). This can be done, e.g. by choosing strong generators $\mathcal{E}_i$ of $\oplus\mathsf{Perf}_i$, and then identifying $HP_\bullet(C_{\mathbf{nc}(Y,\mathsf{w})})$ and $HH_\bullet(C_{\mathbf{nc}(Y,\mathsf{w})})$ with the periodic cyclic and Hochschild homologies of the curved d($\mathbb{Z}/2$)g algebra, which consists of the d($\mathbb{Z}/2$)g algebra $R\operatorname{Hom}(\mathcal{E},\mathcal{E})$ and a central curvature given by w. A detailed proof of the comparison theorem giving the identifications $H^\bullet_{DR}((Y,\mathsf{w});\mathbb{C})\cong HP_\bullet(C_{\mathbf{nc}(Y,\mathsf{w})})$ and $H^\bullet_{Dol}((Y,\mathsf{w});\mathbb{C})\cong HH_\bullet(C_{\mathbf{nc}(Y,\mathsf{w})})$ can be found in the recent work of Junwu Tu [240].

We will construct a **nc**-Hodge structure on $H^\bullet_{DR}((Y,\mathsf{w});\mathbb{C})$ by using the dual description of **nc**-Hodge structures given in Theorem 2.35. Here we will assume that we choose an open subset (in the analytic topology) $Y'\subset Y$ such that

- $\operatorname{crit}(\mathsf{w})\subset Y'$,
- $\mathsf{w}(Y')$ is an open disc in $\mathbb{C}$,
- the closure $\overline{Y}'$ of Y' is a manifold with corners,
- the restriction of w to the part of the boundary of $\overline{Y}'$ lying over $\mathsf{w}(Y')$ is a smooth fibration.

In the case when w is already proper one can choose Y' to be the pre-image under w of an open disc in $\mathbb{C}$ containing all the critical values $\boldsymbol{c}_i$.

Label the critical values of w: $S=\{\boldsymbol{c}_1,\dots,\boldsymbol{c}_n\}$, and let $\boldsymbol{c}_0\in\mathsf{w}(Y')-S$. Choose a system of paths $\{a_i\}_{i=1}^n$ and discs $\boldsymbol{D}_i$ as in the proof of Theorem 2.35. Choose $\boldsymbol{c}_0$-based loops $\gamma_1,\dots,\gamma_n$, so that γ_i goes once around c_i in the counterclockwise direction, all γ_i intersect only at $\boldsymbol{c}_0$, and each γ_i encloses the path a_i and the disc $\boldsymbol{D}_i$ (see Figure 4). Let Γ_i denote the closed region in $\mathbb{C}$ enclosed by γ_i. Adjusting if necessary the choice of the γ_i we can ensure also the each Γ_i is convex. From now on we will always assume that this is the case.

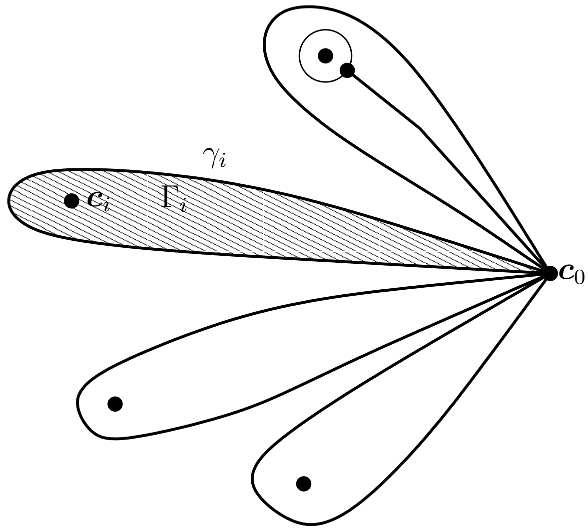


Figure 4: A system of thickened loops for $S \subset \mathbb{C}$.

For $i = 1, \ldots, n$ set $Y_i := \mathsf{w}^{-1}(\Gamma_i) \cap Y'$ and consider the $\mathbb{Q}$-vector spaces of relative cohomology

$$U_i := H^\bullet(Y_i, Y_{\boldsymbol{c}_0}; \mathbb{Q}),$$

and

$$\begin{aligned} U &:= \oplus_{i=1}^n U_i \\ &= H^\bullet(\mathsf{w}^{-1}\left(\cup_{i=1}^n \Gamma_i\right), Y_{\boldsymbol{c}_0}; \mathbb{Q}) \\ &= H^\bullet(Y, Y_{\boldsymbol{c}_0}; \mathbb{Q}). \end{aligned}$$

Let $T_i : U \to U$ be the monodromy along γ_i. By definition T_i satisfies

$$(T_i - 1)_{|\oplus_{j \neq i} U_j} = 0$$

and so we get operators $T_{ji} : U_i \to U_j$, such that $T_{i|U_i} = \sum_{j=1}^n T_{ji}$. By construction the operator T_{ii} is the monodromy along γ_i of the local system on Γ_i of local relative cohomology, i.e. the local system of $\mathbb{Q}$-vector spaces whose fiber at $\boldsymbol{c} \in \Gamma_i$ is $H^\bullet(Y_i, Y_{\boldsymbol{c}}; \mathbb{Q})$. Hence T_{ii} is an isomorphism, and so the data $(S, \{U_i\}_{i=1}^n, \{T_{ij}\})$ are **nc**-Betti data of type **ncB(iii)**.

Remark 3.9 (a) By Lemma 2.30 the data $(S, \{U_i\}_{i=1}^n, \{T_{ij}\})$ are the same thing as a constructible sheaf $\mathscr{F}$ of $\mathbb{Q}$-vector spaces on $\mathbb{C}$, satisfying $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$. The sheaf $\mathscr{F}$ can be described directly in terms of the geometry of (Y, w): for a $\boldsymbol{c} \in \mathbb{C}$ the stalk $\mathscr{F}_{\boldsymbol{c}}$ of $\mathscr{F}$ at $\boldsymbol{c}$ is the relative cohomology $H^\bullet(Y, Y_{\boldsymbol{c}}; \mathbb{Q})$.

(b) The geometric construction of $\mathscr{F}$ makes sense for every cohomology theory K. Indeed for every such K we can form a constructible sheaf of abelian groups ${}^{\mathsf{K}}\mathscr{F}$ whose stalk at $\boldsymbol{c} \in \mathbb{C}$ is $\mathsf{K}(Y, Y_{\boldsymbol{c}})$ and which again satisfies $R\Gamma(\mathbb{C},{}^{\mathsf{K}}\mathscr{F}) = 0$. The vanishing of cohomology here is not obvious but can be proven as follows. Given a disk $\boldsymbol{D} \subset \mathsf{w}(Y') \subset \mathbb{C}$ s.t. $\partial\boldsymbol{D} \cap S = \varnothing$, and given any point $\boldsymbol{c} \in \partial\boldsymbol{D}$ consider the abelian group $A(\boldsymbol{D},\boldsymbol{c}) := \mathsf{K}(\mathsf{w}^{-1}(\boldsymbol{D}), Y_{\boldsymbol{c}})$. The collection of abelian groups $A(\boldsymbol{D},\boldsymbol{c})$ satisfies:

- $A(\boldsymbol{D},\boldsymbol{c})$ are locally constant under small perturbations of $(\boldsymbol{D},\boldsymbol{c})$, and
- for every decomposition $(\boldsymbol{D},\boldsymbol{c}) = (\boldsymbol{D}_1,\boldsymbol{c}) \cup (\boldsymbol{D}_2,\boldsymbol{c})$ of $\boldsymbol{D}$ obtained by cutting D along a chord starting at c, we have $A(\boldsymbol{D},\boldsymbol{c}) = A(\boldsymbol{D}_1,\boldsymbol{c}) \oplus A(\boldsymbol{D}_2,\boldsymbol{c})$.

This immediately gives us an equivalent description of ${}^{\mathsf{K}}\mathscr{F}$ via data of type **ncB(iii)**, which in turn yields the vanishing of cohomology of ${}^{\mathsf{K}}\mathscr{F}$.

Next, in order to complete the data **ncB(iii)** to a full-fledged **nc**-Hodge structure of exponential type, we need to construct:

- a collection $\{(\mathscr{R}_i,\nabla_i)\}_{i=1}^m$ of holomorphic bundles $\mathscr{R}_i$ over $\mathbb{C}\{u\}$ equipped with meromorphic connections ∇_i with at most second order pole and regular singularities, and
- for each $i = 1,\ldots,m$, an isomorphism $\mathfrak{f}_i$ between the local system on $\boldsymbol{S}^1$ induced from $(\mathscr{R}_i,\nabla_i)$ and the local system on $\boldsymbol{S}^1$ corresponding to the vector space $U_i\otimes\mathbb{C}$ and the monodromy operator T_{ii}.

As explained above the local system on the circle corresponding to the vector space $U_i \otimes \mathbb{C}$ and the monodromy operator T_{ii} can be described geometrically as the sheaf of complex vector spaces on the loop γ_i, whose stalk at $\boldsymbol{c} \in \gamma_i$ is $H^\bullet((Y_i, Y_{\boldsymbol{c}});\mathbb{C})$. We will exploit this geometric picture to produce $(\mathscr{R}_i,\nabla_i)$ and the isomorphism $\mathfrak{f}_i$. The most convenient way to define the ∇_i is by using a Betti-to-de Rham cohomology isomorphism given by oscillating integrals.

Fix $i \in \{1,\ldots,m\}$ and let $Z := Y_i$, $\boldsymbol{\Delta} := \Gamma_i - \boldsymbol{c}_i \subset \mathbb{C}$, $\boldsymbol{f} := \mathsf{w} - \boldsymbol{c}_i$. By construction we have:

Z is a C^∞-manifold with boundary which is the closure of an open (in the classical topology) subset in the quasi projective complex manifold Y.

$\boldsymbol{\Delta} \subset \mathbb{C}$ is a closed disc containing zero.

$\boldsymbol{f} : Z \to \boldsymbol{\Delta}$ is an analytic surjective map whose only critical value is zero and whose critical locus $\mathrm{crit}(\boldsymbol{f}) \subset Z$ is compact.

Consider now the $\mathbb{Z}/2$-graded $\mathbb{C}[[u]]$-module $H^\bullet_{DR}((Z,\boldsymbol{f});\mathbb{C})$ of de Rham cohomology of $(Z,\boldsymbol{f})$. By lemma 3.6 we know that $H^\bullet_{DR}((Z,\boldsymbol{f});\mathbb{C})$ is a free $\mathbb{C}[[u]]$-module which can be computed as the cohomology of the complex $(\mathcal{A}^\bullet(Z)[[u]], d_{\mathrm{tot}})$, where $\mathcal{A}^\bullet(Z)[[u]]$ are the global C^∞ complex valued differential forms on Z, and $d_{\mathrm{tot}} := \bar\partial + u\partial + d\boldsymbol{f}\wedge$. The $\mathbb{C}[[u]]$-module $H^\bullet_{DR}((Z,\boldsymbol{f});\mathbb{C})$ carries a natural meromorphic connection ∇ differentiating in the u-direction and having a second order pole at

$u = 0$. This connection is induced from a connection $\boldsymbol{\nabla}$ on the $\mathbb{C}[[u]]$-module $\mathcal{A}^\bullet(Z)[[u]]$ which also has a second order pole and is defined by the formula

$$\boldsymbol{\nabla}_{u^2\frac{d}{du}} := u^2\frac{d}{du} - \boldsymbol{f}\cdot(\bullet) + u\mathsf{Gr} : \mathcal{A}^\bullet(Z)[[u]] \longrightarrow \mathcal{A}^\bullet(Z)[[u]],$$

where

$$\mathsf{Gr}_{|\mathcal{A}^{p,q}(Z)[[u]]} := \frac{q-p}{2}\cdot \mathrm{id}_{\mathcal{A}^{p,q}(Z)[[u]]}$$

is the grading operator coming from **nc**-geometry (compare with 2.1.7).

With this definition we have

Lemma 3.10 *The operator* $\boldsymbol{\nabla}_{u^2\frac{d}{du}}$ *satisfies:*

(a) $\left[\boldsymbol{\nabla}_{u^2\frac{d}{du}}, d_{\mathrm{tot}}\right] = \frac{u}{2}\cdot d_{\mathrm{tot}}$.

(b) $\boldsymbol{\nabla}_{u^2\frac{d}{du}}$ *preserves* $\ker(d_{\mathrm{tot}})$ *and* $\mathrm{im}(d_{\mathrm{tot}})$ *and so induces a meromorphic connection* ∇ *with a second order pole on the* $\mathbb{C}[[u]]$*-module* $H^\bullet_{DR}((Z,\boldsymbol{f});\mathbb{C})$.

Proof. We compute

$$\begin{aligned}\left[\boldsymbol{\nabla}_{u^2\frac{d}{du}}, d_{\mathrm{tot}}\right] &= \left[u^2\frac{d}{du} - \boldsymbol{f} + u\mathsf{Gr}, \bar{\partial} + u\partial + d\boldsymbol{f}\wedge\right]\\ &= \left[u^2\frac{d}{du}, u\partial\right] - [\boldsymbol{f}, u\partial] + \left[u\mathsf{Gr}, \bar{\partial} + u\partial + d\boldsymbol{f}\wedge\right]\\ &= u^2\partial + ud\boldsymbol{f}\wedge + \frac{u\bar{\partial}}{2} - \frac{ud\boldsymbol{f}\wedge}{2} - \frac{u^2\partial}{2}\\ &= \frac{u}{2}\cdot d_{\mathrm{tot}}.\end{aligned}$$

Part **(b)** follows immediately from **(a)** ∎

Suppose now that $\alpha = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + \cdots \in \mathcal{A}^\bullet(Z)[[u]]$, $\alpha_i = \sum \alpha_i^{p,q}$, $\alpha_i^{p,q} \in \mathcal{A}^{p,q}(Z)$ is a d_{tot}-cocycle. Then the differential $d + u^{-1}d\boldsymbol{f}\wedge = \bar{\partial} + \partial + u^{-1}d\boldsymbol{f}\wedge = u^{-1/2}u^{\mathsf{Gr}}d_{\mathrm{tot}}u^{-\mathsf{Gr}}$ will kill the element

$$u^{\mathsf{Gr}}\alpha := \sum_{\substack{i\geq 0\\ 0\leq p,q\leq \dim Z}} \alpha_i^{p,q}u^{i+\frac{q-p}{2}} \in \mathcal{A}^\bullet(Z)((u^{1/2})).$$

Therefore the expression $e^{\frac{\boldsymbol{f}}{u}}u^{\mathsf{Gr}}\alpha$ satisfies formally

$$d\left(e^{\frac{\boldsymbol{f}}{u}}u^{\mathsf{Gr}}\alpha\right) = 0,$$

i.e. is d-closed. Moreover, the action of the operator $\boldsymbol{\nabla}_{u^2\frac{d}{du}}$ on α translates to the action of $u^2\frac{d}{du}$ on the above expression modulo formally exact forms.

Consider now a closed connected arc $\delta \subset \partial\mathbf{\Delta} = \gamma_i$ and let $\mathrm{Sec}(\delta) \subset \mathbf{\Delta}$ be the corresponding open sector (see Figure 5) with vertex at $0 \in \mathbf{\Delta}$, and boundary made out of the arc δ and the segments connecting 0 with the end points of δ. Note that the convexity of $\mathbf{\Delta}$ assures that $\mathrm{Sec}(\delta) \subset \mathbf{\Delta}$. Denote by $\mathrm{Sec}(\delta)^\vee \subset \mathbb{C}$ the dual angle sector consisting of $u \in \mathbb{C}$ such that $\mathrm{Re}(w/u) < 0$ for all $w \in \mathrm{Sec}(\delta)$.

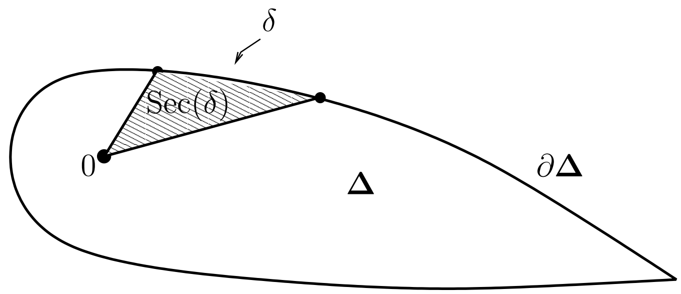


Figure 5: A sector in $\mathbf{\Delta}$.

Clearly, for each class in the relative integral homology $H_\bullet(Z, \boldsymbol{f}^{-1}(\delta); \mathbb{Z})$ we can choose a relative chain $\mathfrak{c}$ representing it, so that $\mathfrak{c}$ satisfies:

(†)

- $\mathfrak{c}$ is piece-wise real analytic;
- $\boldsymbol{f}(\mathrm{supp}(\mathfrak{c})) \subset \mathrm{Sec}(\delta)$;
- $\boldsymbol{f}(\mathrm{supp}(\partial\mathfrak{c})) \subset \delta$.

For every such relative chain $\mathfrak{c}$ we now have:

Lemma 3.11 *For every d_{tot}-closed formal power series of forms $\alpha \in \mathcal{A}^\bullet_Z(Z)[[u]]$ and every relative chain $\mathfrak{c} \in C_\bullet(Z, \boldsymbol{f}^{-1}(\delta); \mathbb{Z})$ satisfying (†) the oscillating integral*

$$\int_{\mathfrak{c}} e^{\frac{\boldsymbol{f}}{u}} u^{\mathsf{Gr}} \alpha$$

is well defined as an asymptotic series in $u^{\mathbb{Q}}(\log u)^{\mathbb{N}}$ in the sector $\mathrm{Sec}(\delta)^\vee$.

Proof. Let $N \geq 0$ be a non-negative integer. Clearly the expression

$$e^{\boldsymbol{f}/u} u^{\mathsf{Gr}} \left(\sum_{0 \leq i \leq N} \alpha_i u^i \right)$$

is a well defined analytic function on $Z \times \mathrm{Sec}(\delta)^\vee$. Using the fact that $(d + u^{-1} d\boldsymbol{f} \wedge) u^{\mathsf{Gr}} \alpha = 0$ and the Malgrange-Sibuya theory of asymptotic sectorial solutions to analytic differential equations, we get that

$$\int_{\mathfrak{c}} e^{\boldsymbol{f}/u} u^{\mathsf{Gr}} \left(\sum_{0 \leq i \leq N} \alpha_i u^i \right) \simeq \sum_{j \in \mathbb{Q}, k \in \mathbb{N}} c_{j,k} u^j (\log(u))^k \tag{3.3}$$

is asymptotic to a series in $u^{\mathbb{Q}}(\log u)^{\mathbb{N}}$ in which the logarithms enter with bounded powers. Thus the limit of (3.3) as $N \to \infty$ is asymptotic to a series in $u^{\mathbb{Q}}(\log u)^{\mathbb{N}}$ on $\mathrm{Sec}(\delta)^{\vee}$. $\square$

The previous lemma shows that the $\mathbb{C}[[u]]$-module with connection $(H^{\bullet}_{DR}((Z, \boldsymbol{f}); \mathbb{C}), \nabla)$ is formally isomorphic to a meromorphic local system of the form $\boldsymbol{\mathcal{E}}^{\boldsymbol{f}/u} \otimes (\mathscr{R}_i, \nabla_i)$, where $\mathscr{R}_i$ is a free $\mathbb{C}[[u]]$-module, and ∇_i has regular singularities. Furthermore the lemma shows that the oscillating integrals above identify the local system on γ_i given by $(\boldsymbol{c} \in \gamma_i) \mapsto H^{\bullet}((Y_i, Y_{\boldsymbol{c}}), \mathbb{Q})$ with a rational structure on $(\mathscr{R}_i \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u)), \nabla_i)$. In particular the data $\{(\mathscr{R}_i, \nabla_i)\}_{i=1}^{m}$ and $(S, \{U_i\}, \{T_{ij}\})$ constitute the regular type and gluing data (in the sense of Theorem 2.35) of a **nc**-Hodge structure of exponential type.

Usually if one tries to make a Landau-Ginzburg model with proper map w from non-proper examples above, one gets new parasitic critical points. Choosing an appropriate domain $Y' \subset Y$ one can define the gluing data for the relevant critical points.

3.3 Mirror symmetry examples

Finally, in order to give a general idea of the mirror correspondence, we briefly discuss three examples of Landau-Ginzburg models mirror dual to symplectic manifolds of positive, vanishing, and negative anti-canonical class respectively.

- For $X = \mathbb{CP}^n$ one of the possible mirror dual Landau-Ginzburg models is given by $Y = (\mathbb{C}^{\times})^n$ endowed with potential

$$\mathsf{w}(z_1, \dots, z_n) = z_1 + \dots + z_n + \frac{q}{z_1 \dots z_n}$$

  where $q \in \mathbb{C}^{\times}$ is a parameter. In this model the map w is not proper. This can be repaired by compactifying the fibers of w to $(n-1)$-dimensional projective Calabi-Yau varieties. The compactification is not unique, it depends on combinatorial data, but the compactified space has the same critical points as Y. In general, for symplectic manifolds (X, ω) with ω representing the anticanonical class, one can combine equations for the connection in q and u directions and get a beautiful variation of Hodge structures with strong arithmetic properties as predicted by our considerations in section 3.1 (see also Golyshev's work [93], [94]).

- For a smooth projective Calabi-Yau variety X one can take for Y the product $(X^{\vee} \times \mathbb{A}^{2N}, \mathsf{w})$ where $X^{\vee}$ is a Calabi-Yau variety mirror dual to X, $N \geq 1$ is arbitrary integer and w is the pullback from $\mathbb{A}^{2N}$ of a non-degenerate quadratic form. In general, the complex dimension of the Landau-Ginzburg model is equal to half of the real dimension of X modulo 2.

- For X being a complex curve of genus $g \geq 2$ (considered as a symplectic manifold), the first author proposed several years ago a mirror Landau-Ginzburg model (Y, w) which is a complex algebraic 3-dimensional manifold with non-vanishing algebraic volume element, such that locally (in the analytic topology) near each point the pair (Y, w) is isomorphic to

$$\mathsf{w} : \mathbb{C}^3 \to \mathbb{C}, \qquad (x, y, z) \mapsto xyz$$

The set of critical point of w is the union of $3g-3$ copies of $\mathbb{CP}^1$ glued along points $0, \infty$ meeting 3 curves at a point. The graph obtained by contracting each copy of $\mathbb{C}^\times$ to an edge is a connected 3-valent graph with g loops, representing a maximal degeneration point in the Deligne-Mumford moduli stack of stable genus g curves.

4 Generalized Tian-Todorov theorems and canonical coordinates

In this section we will examine more closely the other direction of the mirror symmetry correspondence, i.e. the situation in which symplectic Landau-Ginzburg models appear as mirrors of complex manifolds with a fixed anti-canonical section. In order to understand the Hodge theoretic implications of this process we first revisit a classical concept in the subject: the notion of canonical coordinates.

4.1 Canonical coordinates for Calabi-Yau variations of nc-Hodge structures

4.1.1 Variations over supermanifolds We begin with a reformulation of the definition of variations of **nc**-Hodge structures (Definition 2.7) to allow for bases that are supermanifolds:

Definition 4.1 *For a complex analytic supermanifold S, a **variation of nc-Hodge structures over** S (respectively a **variation of nc-Hodge structures over** S **of exponential type**) is a triple $(H, \mathscr{E}_B, \overset{\sim}{\to})$, where*

- *H is a holomorphic $\mathbb{Z}/2$-graded vector bundle on $\mathbb{A}^1 \times S$ which is algebraic in the $\mathbb{A}^1$-direction;*
- *$\mathscr{E}_B$ is a local system of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on $(\mathbb{A}^1 - \{0\}) \times S$;*
- *$\overset{\sim}{\to}$ is an analytic isomorphism of holomorphic vector bundles*

$$\overset{\sim}{\to} : \mathscr{E}_B \otimes \mathcal{O}_{(\mathbb{A}^1-\{0\})\times S} \overset{\cong}{\to} H_{|(\mathbb{A}^1-\{0\})\times S};$$

so that:

◊ *the induced meromorphic connection ∇ on $H_{|(\mathbb{A}^1-\{0\})\times S}$ satisfies: locally on S, for every section ξ of T_S, the operators $\nabla_{u^2\frac{\partial}{\partial u}}$, $\nabla_{u\xi}$ extend to operators on $\mathbb{A}^1 \times S$, and*

◊ *the triple $(H, \mathscr{E}_B, \overset{\sim}{\to})$ satisfies the* **($\mathbb{Q}$-structure axiom)** *and the* **(Opposedness axiom)** *(respectively (H, ∇) is of exponential type and $(H, \mathscr{E}_B, \overset{\sim}{\to})$ satisfies the* **($\mathbb{Q}$-structure axiom)$^{\mathbf{exp}}$** *and the* **(Opposedness axiom)$^{\mathbf{exp}}$**).

Remark 4.2 From now on we will suppress the $\mathbb{Q}$-structure and the opposedness axioms since they will not play any special role in our analysis. At any given stage of the discussion they can be added without any harm or alteration to the arguments.

4.1.2 Calabi-Yau variations Suppose now $(H, \mathscr{E}_B, \stackrel{\sim}{\rightarrow})$ is a variation of **nc**-Hodge structures over a supermanifold S. For any point $x \in S$ let $H_{0,x}$ denote the fiber of H at $(0, x) \in \mathbb{A}^1 \times S$. We get a canonical map

$$\mu_x : T_x S \to \mathrm{End}\,(H_{0,x})\,,$$

defined as follows: Extend the tangent vector $v \in T_x S$ to some analytic vector field ξ defined in a neighborhood of x. Consider the holomorphic first order differential operator $\nabla_{u\xi} : H \to H$. By construction this operator has symbol $(u\xi) \otimes \mathrm{id}_H$. In particular, the restriction of $\nabla_{u\xi}$ to the slice $\{0\} \times S \subset \mathbb{A}^1 \times S$ will have zero symbol, and so will be an $\mathcal{O}$-linear endomorphism of $H_{|\{0\}\times S}$. We define $\mu_x(v)$ to be the action of this $\mathcal{O}$-linear map on the fiber $H_{(0,x)}$. It is straightforward to check that this action is independent of the extension ξ and depends on v only.

Definition 4.3 *Let S be a complex analytic supermanifold. We say that a variation $(H, \mathscr{E}_B, \stackrel{\sim}{\rightarrow})$ of **nc**-Hodge structures on S is of **Calabi-Yau type at a point** $x \in S$ if there exists an (even or odd) vector $h \in H_{(0,x)}$, so that the linear map*

$$\begin{aligned} T_x S &\longrightarrow H_{(0,x)} \\ v &\longmapsto (\mu_x(v))(h) \end{aligned}$$

*is an isomorphism. Such a vector h will be called a **generating vector for** H **at** x.*

It follows from the definition that if S is the base of a variation of **nc**-Hodge structures which is of Calabi-Yau type at a point $x \in S$, then the tangent space $T_x S$ is a unital commutative associative algebra acting on $H_{0,x}$ via the map μ_x and such that $H_{0,x}$ is a free module of rank one. The condition on a variation to have a Calabi-Yau type (even or odd) is an open condition on $x \in S$. Variations of **nc**-Hodge structures of Calabi-Yau type should arise naturally on the periodic cyclic homology of smooth and compact d$(\mathbb{Z}/2)$g categories which are Calabi-Yau in the sense of [159]. The basic geometric example of a Calabi-Yau variation is an extension of the setup we discussed in section 3.1:

Example 4.4 Let (X, ω) be a compact symplectic manifold with $\dim_{\mathbb{R}} X = 2d$. Conjecturally there exists a non-empty open subset $S \subset H^\bullet(X, \mathbb{C})$ so that the big quantum product $*_x$ is absolutely convergent for all $x \in S$ (the product is given by a formula similar to one on page 276). The manifold S has a natural structure of a supermanifold being an open subset in the affine super space $H^\bullet(X, \mathbb{C})$. As in section 3.1 we define a variation of **nc**-Hodge structures $(H, \mathscr{E}_B, \stackrel{\sim}{\rightarrow})$ on S by taking H to be the trivial vector bundle on $\mathbb{A}^1 \times S$ with fiber $H^\bullet(X, \mathbb{C})$, and defining the connection ∇ on H by the formulas:

$$\nabla_{\frac{\partial}{\partial u}} := \frac{\partial}{\partial u} + u^{-2}\,(\kappa_X *_x \bullet) + u^{-1}\mathbf{Gr},$$

$$\nabla_{\frac{\partial}{\partial t^i}} := \frac{\partial}{\partial t^i} - q^{-1}u^{-1}\,(t_i *_x \bullet)\,,$$

where the (t_i) form a basis on $H^\bullet(X,\mathbb{C})$, and (t^i) are the dual linear coordinates.

Clearly, if we restrict (H,∇) to $S\cap H^2(X,\mathbb{C})$ we will get back the bundle with connection we defined in section 3.1. We now define the integral lattice $\mathscr{E}_B$ and isomorphism $\xrightarrow{\sim}$ on S as the ∇-horizontal extensions of the integral lattice and isomorphism we had defined on $S\cap H^2(X,\mathbb{C})$. Finally, in order to match the framework of **nc**-geometry, we should change the parity of the bundle H in the case $d = 1 \bmod 2$.

4.1.3 Decorated Calabi-Yau variations The variations of **nc**-Hodge structures of Calabi-Yau type need to be decorated by a few additional pieces of data before we can extract canonical coordinates from them. To motivate our choice of such data we first recall the Deligne-Malgrange classification of logarithmic holomorphic extension of regular connections.

Let S be a complex analytic supermanifold, let $\boldsymbol{D}$ be a one dimensional complex disc, and let $\mathscr{E}$ be a complex local system on $(\boldsymbol{D}-\{\mathrm{pt}\})\times S$ and let $(\mathscr{E},\nabla)$ be the associated holomorphic bundle $E := \mathscr{E}\otimes\mathcal{O}_{(D-\{\mathrm{pt}\})\times S}$ on $(D-\{\mathrm{pt}\})\times S$ with the induced flat connection ∇. Suppose $\widetilde{E}$ is a holomorphic bundle on $\boldsymbol{D}\times S$ which extends E and on which ∇ has a logarithmic pole. The restriction $\widetilde{E}_{|\{\mathrm{pt}\}\times S}$ is a holomorphic bundle on S and ∇ induces: a holomorphic connection ${}^{\widetilde{E}}\nabla$ and an $\mathcal{O}_S$-linear residue endomorphism $\mathrm{Res}_{\widetilde{E}}(\nabla)$ on $\widetilde{E}_{|\{\mathrm{pt}\}\times S}$. Furthermore the integrability of ∇ on $(\boldsymbol{D}-\{\mathrm{pt}\})\times S$ implies that ${}^{\widetilde{E}}\nabla$ is also integrable and that the endomorphism $\mathrm{Res}_{\widetilde{E}}(\nabla)$ is covariantly constant with respect to ${}^{\widetilde{E}}\nabla$ [205, Section 0.14b].

Recall next that by Deligne's extension theorem (see e.g. [66, Chapter II.5] or [205, Corollary II.2.21]) meromorphic bundles with connections with regular singularities always admit functorial holomorphic extensions across the pole divisor. Deligne's extension procedure is not unique and depends on the choice of a set-theoretic section of the quotient map $\mathbb{C}\to\mathbb{C}/\mathbb{Z}$. We fix $\mathcal{V}$ to be the unique Deligne extension of E for which ∇ has a logarithmic pole at $\{\mathrm{pt}\}\times S$ and a residue with eigenvalues whose real parts are in the interval $(-1,0]$. Now the classification theorem of Deligne-Malgrange [205, Theorem III.1.1] asserts that there is a natural equivalence of categories

$$\begin{pmatrix}\text{Holomorphic extensions of } E \text{ to}\\ \boldsymbol{D}\times S \text{ for which } \nabla \text{ has a loga-}\\ \text{rithmic singularity along } \{\mathrm{pt}\}\times\\ S\end{pmatrix}\longleftrightarrow\begin{pmatrix}\text{Decreasing filtrations of } \mathscr{E} \text{ by}\\ \mathbb{C}\text{-local subsystems on } (\boldsymbol{D}-\\ \{\mathrm{pt}\})\times S\end{pmatrix}.$$

The equivalence depends on the chosen Deligne extension and is explicitly given as follows. Let t be a complex coordinate on $\boldsymbol{D}$ which vanishes at $\mathrm{pt}\in\boldsymbol{D}$. Consider the restriction $\mathcal{V}/t\mathcal{V}$ of $\mathcal{V}$ to $\{\mathrm{pt}\}\times S$. This is a holomorphic bundle on S equipped as above with the holomorphic connection ${}^{\mathcal{V}}\nabla$ and the covariantly constant residue endomorphism $\mathrm{Res}_{\mathcal{V}}(\nabla)$. Suppose now that $\widetilde{E}$ is another holomorphic bundle on $\boldsymbol{D}\times S$ which extends E and on which ∇ has a logarithmic pole. For any $k\in\mathbb{Z}$ we define a subbundle $(\mathcal{V}/t\mathcal{V})^k\subset\mathcal{V}/t\mathcal{V}$ by setting

$$(\mathcal{V}/t\mathcal{V})^k := \frac{\mathcal{V}\cap t^k\widetilde{E}}{t\mathcal{V}\cap t^k\widetilde{E}}$$

where $\mathcal{V}$ and $\widetilde{E}$ are viewed as subsheaves in the meromorphic bundle E.

By construction the sub-bundles $(\mathcal{V}/t\mathcal{V})^k$ are preserved both by ${}^{\mathcal{V}}\nabla$ and by the residue endomorphism $\mathrm{Res}_{\mathcal{V}}(\nabla)$ and so give rise to ∇-covariantly constant meromorphic subbundles of E, or equivalently to $\mathbb{C}$-local subsystems of $\mathscr{E}$.

Alternatively we can use a more intrinsic description of holomorphic extensions of $(\mathscr{E}, \nabla)$ which is beter adapted to our examples and in particular to Example 4.8. Namely, instead of relying on the Deligne extension and the induced filtration we can use decreasing filtrations $\mathscr{E}_{\leq \lambda}$ of $\mathscr{E}$ labeled by real numbers $\lambda \in \mathbb{R}$ and such that on the associated graded pieces the monodromy on $\boldsymbol{D} - \{\mathrm{pt}\}$ has eigenvalues in $\mathbb{R}_+ \times \exp(2\pi i \lambda)$.

We can now introduce the additional data that one needs for the canonical coordinates

Definition 4.5 *Let S be a complex supermanifold and let $(H, \mathscr{E}_B, \overset{\sim}{\to})$ be a variation of* **nc**-*Hodge structures of Calabi-Yau type on S. A* ***decoration*** *on $(H, \mathscr{E}_B, \overset{\sim}{\to})$ is a pair $(\widetilde{H}, \psi)$ where:*

$\widetilde{H}$ *is an extension of H to $(\mathbb{Z}/2)$-graded vector bundle on $\mathbb{P}^1 \times S$ so that ∇ has a regular singularity at $\{\infty\} \times S$.*

ψ *is a ${}^{\widetilde{H}}\nabla$-covariantly constant section of $\widetilde{H}_{\{\infty\}\times S}$.*

A decoration is called ***rational*** *iff the $\mathbb{R}$-filtration on the local system $\mathscr{E}_B \otimes \mathbb{C}$ is compatible with the rational structure, and if the vector $\psi(x) \in \widetilde{H}_{\{\infty\}\times\{x\}} = \mathrm{gr}(\mathscr{E}_B \otimes \mathbb{C})_x$ is rational, i.e. if $\psi(x) \in \mathrm{gr}(\mathscr{E}_B)_x$.*

The previous discussion applied to the local system $\mathscr{E}_B \otimes \mathbb{C}$, the disc $\boldsymbol{D} = \{|u| > 1\} \cup \{\infty\}$ and the point $\mathrm{pt} = \infty$ shows that the data of a decoration are equivalent to the data $((\mathscr{E}_B \otimes \mathbb{C})_{\leq \bullet}, \psi)$, where $(\mathscr{E}_B \otimes \mathbb{C})_{\leq \bullet}$ is a decreasing filtration of $\mathscr{E}_B \otimes \mathbb{C}$ (labeled by real numbers) and ψ is a covariantly constant section (along S) of the corresponding logarithmic holomorphic extension of H. We will freely go back and forth between these two points of view.

Any decorated variation $(H, \mathscr{E}, \overset{\sim}{\to}; \widetilde{H}, \psi)$ of **nc**-Hodge structures of Calabi-Yau type gives rise to a natural open domain $U \subset S$ defined by

$$U := \left\{ x \in S \;\middle|\; \begin{array}{l} \widetilde{H}_{\mathbb{P}^1\times\{x\}} \text{ is holomorphically trivial and if} \\ s \in \Gamma\left(\mathbb{P}^1 \times \{x\}\right) \text{ is such that } s_x(\infty) = \psi(\infty, x), \\ \text{then } s_x(0) \text{ is a generating vector for } (H, \mathscr{E}, \overset{\sim}{\to}). \end{array} \right\}$$

Furthermore for every $x \in U$ we get a natural map $\mathsf{can}_x : T_x S \to \widetilde{H}_{\infty,x}$ defined as the composition

$$\begin{tikzcd} T_x S \arrow[r, "\mu_x(\bullet)(s_x(0))"] \arrow[rrr, bend right, "\mathsf{can}_x"'] & H_{0,x} \arrow[r, "\mathrm{ev}_{(0,x)}^{-1}"] & \Gamma\left(\widetilde{H}_{|\mathbb{P}^1\times\{x\}}\right) \arrow[r, "\mathrm{ev}_{(\infty,x)}"] & \widetilde{H}_{\infty,x}. \end{tikzcd}$$

where $\mathrm{ev}_{(t,x)} : \Gamma\left(\mathbb{P}^1, \widetilde{H}_{|\mathbb{P}^1\times\{x\}}\right) \to \widetilde{H}_{t,x}$ denotes the natural evaluation of sections, which is invertible by the triviality assumption on $\widetilde{H}_{|\mathbb{P}^1\times\{x\}}$.

The pullback of the flat connection ${}^{\widetilde{H}}\nabla$ by the map can induces a flat connection on $TS_{|U}$. The canonical coordinates on S come from the following easy claim whose proof we omit

Claim 4.6 *The flat connection* $\mathsf{can}^*\left({}^{\widetilde{H}}\nabla\right)$ *on* $TS_{|U}$ *is torsion free and so gives rise to a natural affine structure and affine coordinates on* U*. If the decoration is rational then the tangent bundle* $TS_{|U}$ *carries a natural rational structure.*

Remark 4.7 **(i)** The canonical coordinates on U corresponding to a decorated **nc**-variation of Hodge structures are only affine coordinates and are defined only up to a translation.

(ii) For any $u \in \mathbb{A}^1 - \{0\}$ we can introduce another affine structure which is a **vector structure**. In fact, we get an analytic isomorphism between U and a domain in $H_{u,\bullet} = (\mathscr{E}_B)_{u,\bullet} \otimes \mathbb{C}$:

$$x \in U \mapsto \mathrm{ev}_{(u,x)} \, \mathrm{ev}^{-1}_{(\infty,x)}(\psi(x)) \in H_{(u,x)}.$$

One can use this to show that the local Torelli theorem holds for decorated Calabi-Yau variations of **nc**-Hodge structures.

Example 4.8 The setup of Example 4.4 gives not only a variation of **nc**-Hodge structures but in fact gives a rationally decorated **nc**-Hodge structure of Calabi-Yau type. Indeed by definition the fibers of H are identified with $\mathbf{\Pi}^d H^\bullet(X,\mathbb{C})$. The monodromy of the connection around $\infty \in \mathbb{P}^1$ is the operator acting by $(-1)^{i+d}\exp(\kappa_X \wedge (\bullet))$ on $H^i(X,\mathbb{C})$. Consider the monodromy invariant filtration on $H^\bullet(X,\mathbb{C})$ whose step in degree $\frac{d-i}{2}$ is $H^{\geq i}(X,\mathbb{C})$. Let $\widetilde{H}$ be the corresponding logarithmic extension of H and let ψ be the section of $\widetilde{H}$ corresponding to the image of $1 \in H^0(X,\mathbb{C}) \subset H^\bullet(X,\mathbb{C})$. The bundle $\widetilde{H}_{|\{\infty\}\times S}$ is trivialized and $\nabla_{\frac{\partial}{\partial t^i}} = \frac{\partial}{\partial t^i}$ in this trivialization. This gives the desired decoration $(\widetilde{H},\psi)$ and the associated canonical coordinates are the standard canonical coordinates in Gromov-Witten theory.

4.1.4 Generalized decorations The notion of a decorated Calabi-Yau variation of **nc**-Hodge structures can be generalized in various ways. For instance, instead of specifying a covariantly constant filtration on H giving the extension $\widetilde{H}$ we can start with any holomorphic bundle H' defined on $\{u \in \mathbb{P}^1 \,|\, |u| \geq R\}$, and an identification of C^∞-bundles

$$p_1^*\left(H'_{|\{|u|=R\}}\right) \cong (\mathscr{E}_B \otimes \mathbb{C})_{\{|u|=R\}\times S},$$

where $p_1 : \{|u| = R\} \times S \to \{|u| = R\}$ is the projection on the first factor.

Furthermore (locally in S) the holomorphic bundle $p_1^* H'$ on $\{u \in \mathbb{P}^1 \,|\, |u| \geq R\} \times S$ carries a flat connection defined along S only. We can use the above identification to glue this together with H along

$\{|u| = R\} \times S$ to get a bundle $\widetilde{H}$ on $\mathbb{P}^1 \times S$ equipped with a flat connection $\nabla_{/S}$ along S. This generalizes the first part of the decoration. For the second part we will take a $\nabla_{/S}$-covariantly constant section ψ of $\widetilde{H}_{|\{\infty\}\times S}$. Now the same definition of the set U and the canonical map can make sense in this context. The resulting connection on $TS_{|U}$ is again torsion free.

4.1.5 Formal variations of Calabi-Yau type The notion of a Calabi-Yau variation extends readily to the formal context. Suppose $S = \operatorname{Spf} \mathbb{C}[[x_1, \ldots, x_N, \xi_1, \ldots, \xi_M]]$ be a formal algebraic supermanifold, where x_i are even and ξ_j are odd formal variables. The de Rham part of formal variation of **nc**-Hodge structures on S is a pair (H, ∇) where H is a $(\mathbb{Z}/2)$-graded algebraic vector bundle over $\mathbb{D} \times S$, where $\mathbb{D}$ is the one dimensional formal disc $\mathbb{D} := \operatorname{Spf}(\mathbb{C}[[u]])$. Here ∇ is a meromorphic connection on H such that $\nabla_{u^2 \frac{\partial}{\partial u}}$, $\nabla_{u \frac{\partial}{\partial x_i}}$, $\nabla_{u \frac{\partial}{\partial \xi_j}}$ are regular differential operators on H.

We say that such a pair (H, ∇) has the Calabi-Yau property if we can find a vector $h \in H_{0,0}$, so that the natural linear map $T_0 S \to H_{0,0}$, $v \mapsto \mu_0(v)(h)$ is an isomorphism.

Finally a decoration of a formal Calabi-Yau de Rham data (H, ∇) is a pair $(\boldsymbol{e}, h)$, where $\boldsymbol{e}$ is a trivialization $\boldsymbol{e} : H_{|\mathbb{D}\times\{0\}} \to H_{0,0} \otimes \mathcal{O}_{\mathbb{D}\times\{0\}}$, and $h \in H_{0,0}$ is a generating vector for the Calabi-Yau property.

Again a decorated de Rham data of Calabi-Yau type gives an affine structure and canonical formal coordinates on S.

4.2 Algebraic framework: dg Batalin-Vilkovisky algebras

In this section we discuss the aspects of algebraic deformation theory relevant to the study of **nc**-Hodge structures. We will work over $\mathbb{C}$ but all algebraic considerations in this section make sense over any field of characteristic zero.

4.2.1 Preliminaries on L_∞ algebras Our main objects of interest here will be differential $\mathbb{Z}/2$-graded algebras over $\mathbb{C}$ or more generally $\mathbb{Z}/2$-graded L_∞-algebras over $\mathbb{C}$. We begin with a definition:

Definition 4.9 *A complex differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ (or a $\mathbb{Z}/2$-graded L_∞-algebra) is called **homotopy abelian** if it is L_∞ quasi-isomorphic to an abelian $d(\mathbb{Z}/2)g$ Lie algebra.*

Remark 4.10 Homotopy abelian differential $\mathbb{Z}/2$-graded Lie algebras can be characterized in a variety of ways. In particular we have the following statements that follow readily from the definition:

- A differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is homotopy abelian if and only if all the higher operations m_n vanish on its L_∞ minimal model $\mathfrak{g}^{\min} = H^\bullet(\mathfrak{g}, d_{\mathfrak{g}})$, i.e. $m_n = 0$ for $n \geq 1$.
- A differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is homotopy abelian if and only if there exist $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$

Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and morphisms of d$(\mathbb{Z}/2)$g Lie algebras:

$$\begin{tikzcd} & \mathfrak{g}_1 \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathfrak{g} & & \mathfrak{g}_2 \end{tikzcd}$$

so that $\mathfrak{g}_2$ is an abelian d$(\mathbb{Z}/2)$g Lie algebra, and the morphisms $\mathfrak{g}_1 \to \mathfrak{g}$ and $\mathfrak{g}_1 \to \mathfrak{g}_2$ are quasi-isomorphisms.

- A differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is homotopy abelian if and only if the Lie algebra cohomology algebra $H^\bullet(\mathfrak{g}, \mathbb{C})$ is free, i.e. is isomorphic to the algebra of formal power series on some (possibly infinitely many) supervariables. Here the Lie algebra cohomology is defined as

$$H^\bullet(\mathfrak{g}, \mathbb{C}) := H^\bullet\left(\prod_{n\geq 0} \mathrm{Hom}_{(\mathbb{C}-\mathsf{Vect})}(\mathrm{Sym}^n\, \mathbf{\Pi}\mathfrak{g}, \mathbb{C})^\bullet, d\right)$$

where d is the cochain Cartan-Eilenberg differential.

After the prioneering work of Deligne and Drinfeld in the 80's, it is by now a common wisdom (see e.g. [173, Chapter III.9]) that dg Lie algebras give rise to solutions of moduli problems. In particular a homotopy abelian d$(\mathbb{Z}/2)$g Lie algebra $\mathfrak{g}$ gives rise to a moduli space - the formal supermanifold $\oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g}, d_\mathfrak{g})} := \mathrm{Spf}(H^\bullet(\mathfrak{g}, \mathbb{C}))$.

The property of being homotopy abelian is preserved by suitably non-degenerate deformations and various other natural operations:

Proposition 4.11 **(i)** *Let $\mathfrak{g}$ be a flat family of d$(\mathbb{Z}/2)$g Lie algebras (or $(\mathbb{Z}/2)$-graded L_∞ algebras) over $\mathbb{C}[[u]]$. That is $\mathfrak{g}$ is a flat $(\mathbb{Z}/2)$-graded $\mathbb{C}[[u]]$-module, and the Lie bracket and differential on $\mathfrak{g}$ are $\mathbb{C}[[u]]$-linear. Assume further that*

(A) $\mathfrak{g}_{\mathrm{gen}} := \mathfrak{g} \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u))$ *is homotopy abelian over* $\mathbb{C}((u))$*, and*

(B) $H^\bullet(\mathfrak{g}, d_\mathfrak{g})$ *is a flat* $\mathbb{C}[[u]]$*-module.*

Then the special fiber $\mathfrak{g}_0 := \mathfrak{g}\widehat{\otimes}_{\mathbb{C}[[u]]}\mathbb{C}$ is also a homotopy abelian d$(\mathbb{Z}/2)$g Lie algebra over $\mathbb{C}$.

(ii) *If $\mathfrak{g}$ is a homotopy abelian d$(\mathbb{Z}/2)$g Lie algebra over $\mathbb{C}$, and $\mathfrak{g}_1 \to \mathfrak{g}$ is a morphism of L_∞-algebras inducing a monomorphism $H^\bullet(\mathfrak{g}_1, d_{\mathfrak{g}_1}) \hookrightarrow H^\bullet(\mathfrak{g}, d_\mathfrak{g})$, then $\mathfrak{g}_1$ is homotopy abelian as well.*

(iii) *If $\mathfrak{g}$ is a homotopy abelian d$(\mathbb{Z}/2)$g Lie algebra over $\mathbb{C}$, and $\mathfrak{g} \to \mathfrak{g}_2$ is a morphism of L_∞-algebras inducing an epimorphism $H^\bullet(\mathfrak{g}, d_\mathfrak{g}) \twoheadrightarrow H^\bullet(\mathfrak{g}_2, d_{\mathfrak{g}_2})$, then $\mathfrak{g}_2$ is homotopy abelian as well.*

Proof. The proof is standard so we only mention some of the highlights of the argument. First note that parts **(ii)** and **(iii)** follow immediately by passing to minimal models. For part **(i)** we note first that the assumption *(B)* implies (and is in fact equivalent to) the existence of $\mathbb{C}[[u]]$-linear quasi-isomorphisms p_1, p_2 of complexes:

$$(H^\bullet(\mathfrak{g}_0, d_{\mathfrak{g}_0})[[u]], 0) \;\cong\; (H^\bullet(\mathfrak{g}, d_{\mathfrak{g}}), 0) \underset{p_2}{\overset{p_1}{\rightleftarrows}} (\mathfrak{g}, d_{\mathfrak{g}}),$$

and a $\mathbb{C}[[u]]$-linear homotopy h so that

$$\begin{aligned} p_2 \circ p_1 &= \mathrm{id} \\ p_1 \circ p_2 &= \mathrm{id} + [d_{\mathfrak{g}}, h]. \end{aligned}$$

Next note that the homological perturbation theory of [157] carries over verbatim to the L_∞-context and gives explicit expressions for the higher products m_n on $(H^\bullet(\mathfrak{g}_0, d_{\mathfrak{g}_0})[[u]], 0)$ as a polynomial expression in p_1, p_2 and h. In particular the operations m_n are all $\mathbb{C}[[u]]$-linear and are given by universal expressions. But by assumption *(A)* we know that the higher operations are zero after tensoring with $\otimes_{\mathbb{C}[[u]]}\mathbb{C}((u))$ and so $m_n = 0$ as formal power series in u for all $n \geq 1$. This implies that $m_{n|u=0} = 0$ for all $n \geq 1$ and so the proposition is proven. $\square$

4.2.2 DG Batalin-Vilkovisky algebras Recall [173, Chapter III.10] the notion of a dg BV algebra:

Definition 4.12 *A differential $\mathbb{Z}/2$-graded Batalin-Vilkovisky algebra over $\mathbb{C}$ is the data (A, d, Δ), where A is a $\mathbb{Z}/2$-graded suppercommutative associative unital algebra, and $d : A \to A$, $\Delta : A \to A$ are odd $\mathbb{C}$-linear maps satisfying:*

- *$d(1) = \Delta(1) = 0$,*
- *d is a differential operator of order ≤ 1 on A,*
- *Δ is a differential operator of order ≤ 2 on A,*
- *$d^2 = \Delta^2 = d\Delta + \Delta d = 0$.*

Note that the first two properties in the definition imply that d is a derivation of A. Also $\mathfrak{g} := \mathbf{\Pi}A$ together with $[a, b] := \Delta(ab) - \Delta(a)b - (-1)^{\deg(a)} a\Delta(b)$ is a Lie superalgebra with two anti-commuting differentials d and Δ.

Definition 4.13 *We will say that a d$(\mathbb{Z}/2)$g Batalin-Vilkovisky algebra (A, d, Δ) has the* ***degeneration property*** *if for every $N \geq 1$ we have that $H^\bullet(A[u]/(u^N), d + u\Delta)$ is a free $\mathbb{C}[u]/(u^N)$-module.*

Equivalently (A, d, Δ) has the degeneration property iff $H^\bullet(A[[u]], d + u\Delta)$ is a topologically free (flat) $\mathbb{C}[[u]]$-module. This in turn is equivalent to the existence of a (non-unique) isomorphism of topological $\mathbb{C}[[u]]$-modules:

$$T: \; H^\bullet(A[[u]], d + u\Delta) \xrightarrow{\cong} H^\bullet(A, d)[[u]]. \tag{4.1}$$

In this situation we will always normalize T so that $T_{|u=0} = \mathrm{id}_{H^\bullet(A,d)}$.

The degeneration property for dg Batalin-Vilkovisky algebras defined above is weaker than the $\partial\bar{\partial}$-lemma used Barannikov and the second author in [24] and by Manin in [173, 174]. In particular it has potentially a wider scope of applications - a feature that we will exploit next. We begin with a general smoothness result which was also proven by J.Terilla [231].

Theorem 4.14 *Suppose (A, d, Δ) is a d$(\mathbb{Z}/2)$g Batalin-Vilkovisky algebra which has the degeneration property. Let $\mathfrak{g} := \mathbf{\Pi}A$ be the associated super Lie algebra with anti-commuting differentials d and Δ. Then:*

- **(1)** *The d$(\mathbb{Z}/2)$g Lie algebra $(\mathfrak{g}, d)$ is homotopy abelian, i.e. is quasi-isomorphic to $H^\bullet(\mathfrak{g}, d)$ endowed with the trivial bracket and the zero differential. In particular the associated moduli space $\oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g},d)}$ is (non-canonically) isomorphic to a formal neighborhood of 0 in the super affine space $\mathbf{\Pi}H^\bullet(\mathfrak{g}, d)$.*
- **(2)** *Every choice of a normalized degeneration isomorphism T as in equation (4.1) gives an identification of formal manifolds*

$$\Phi_T : \ \oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g},d)} \xrightarrow{\cong} \begin{pmatrix} \textit{formal neighborhood of} \\ 0 \textit{ in } \mathbf{\Pi}H^\bullet(\mathfrak{g}, d) \end{pmatrix}$$

Proof. Part **(1)** of the theorem follows immediately from

Lemma 4.15 *The d$(\mathbb{Z}/2)$g Lie algebra $(\mathfrak{g}((u)), d + u\Delta)$ is homotopy abelian over $\mathbb{C}((u))$.*

Proof. Consider the formal completion at zero $\widehat{A}$ of the vector superspace underlying $A = \mathbf{\Pi}\mathfrak{g}$ as an algebraic supermanifold, and let as before $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[u]])$ be the formal one dimensional disc. The d$(\mathbb{Z}/2)$g Lie algebra structure on $\mathfrak{g}[[u]]$ is encoded in an odd vector field $\boldsymbol{\xi} \in \Gamma(\widehat{A} \times \mathbb{D}, T)$ on the supermanifold $\widehat{A} \times \mathbb{D}$, defined by

$$\dot{a} := \boldsymbol{\xi}(a) = da + u\Delta a + \frac{1}{2}[a, a].$$

There is a natural automorphism (i.e. a formal change of coordinates) $F : \widehat{A} \times \mathbb{D}^\times \to \widehat{A} \times \mathbb{D}^\times$ on the formal supermanifold $\widehat{A} \times \mathbb{D}^\times$ given by

$$F(a) := u\left(\exp\left(\frac{a}{u}\right) - 1\right) = a + \frac{1}{u}\frac{1}{2!}a^2 + \frac{1}{u^2}\frac{1}{3!}a^3 + \cdots ,$$

and in the new coordinates $b = F(a)$ the vector field $\boldsymbol{\xi}$ is linear:

$$\begin{aligned} \dot{b} = \dot{a} \cdot \exp\left(\frac{a}{u}\right) &= \left(da + u\Delta a + \frac{1}{2}[a, a]\right) \cdot \exp\left(\frac{a}{u}\right) \\ &= u \cdot \left(\frac{da}{u} + u\Delta\left(\frac{a}{u}\right) + u\frac{1}{2}\left[\frac{a}{u}, \frac{a}{u}\right]\right) \cdot \exp\left(\frac{a}{u}\right) \\ &= u \cdot (d + u\Delta)\exp\left(\frac{a}{u}\right) = (d + u\Delta)b. \end{aligned}$$

So in the b-coordinates, the vector field $\boldsymbol{\xi}$ depends only on the differential $d + u\Delta$ and does not depend on any higher operations. Passing to the minimal model we see that $(\mathfrak{g}((u)), d + u\Delta)$ is homotopy abelian, which proves the lemma. $\square$

The lemma implies that the hypothesis *(A)* of Proposition 4.11 **(i)** holds. On the other hand the hypothesis *(B)* holds by the degeneration assumption. Therefore by Proposition 4.11 **(i)** we conclude that $(\mathfrak{g}, d)$ is homotopy abelian. This proves part **(1)** of the theorem.

Next we construct the identification Φ_T. Given a formal path in $\oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g},d)}$, i.e. a family of solutions (up to guage equivalence)

$$
\begin{aligned}
a(\varepsilon) &= a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots \in \varepsilon A[[\varepsilon]] \\
d(a(\varepsilon)) &+ \frac{1}{2}[a(\varepsilon), a(\varepsilon)] = 0
\end{aligned}
$$

of the Maurer-Cartan equation in $(\mathfrak{g}, d)$, we have to construct the corresponding formal path through the origin in $H^\bullet(\mathfrak{g}, d)$.

As a first step choose a lift of the formal arc $a(\varepsilon)$ to a formal series in two variables $\tilde{a}(\varepsilon, u) \in \varepsilon A[[\varepsilon, u]]$ such that

$$
\begin{aligned}
&(d{+}u\Delta)\tilde{a} + \frac{1}{2}[\tilde{a}, \tilde{a}] = 0, \\
&a(\varepsilon, 0) = a(\varepsilon).
\end{aligned}
$$

Consider the reparameterization

$$
\tilde{b} = F(\tilde{a}) = u\left(\exp\left(\frac{\tilde{a}}{u}\right) - 1\right) \in \varepsilon A((u))[[\varepsilon]].
$$

Arguing as before we see that $\tilde{b}$ satisfies $(d + u\Delta)\tilde{b} = 0$. So if we expand

$$
\tilde{b} = \tilde{b}_1\varepsilon + \tilde{b}_2\varepsilon^2 + \cdots, \qquad \text{where } \tilde{b}_n \in A((u)), \text{ satisfy } (d + u\Delta)\tilde{b}_n = 0,
$$

we can define cohomology classes $\left[\tilde{b}_n\right] \in H^\bullet(A((u)), d + u\Delta)$. We can now apply the isomorphism $T \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u))$ to the series

$$
\sum_{n\geq 1} \left[\tilde{b}_n\right] \varepsilon^n \in \varepsilon H^\bullet(A((u)), d + u\Delta)[[\varepsilon]],
$$

to obtain an element

$$
T\left(\sum_{n\geq 1} \left[\tilde{b}_n\right] \varepsilon^n\right) \in \varepsilon H^\bullet(A, d)((u))[[\varepsilon]].
$$

In fact one has the following lemma whose proof we will skip since it is a somewhat tedious application of homological perturbation theory:

Lemma 4.16 *There exists a lift $\tilde{a}(\varepsilon, u)$ of $a(\varepsilon)$ such that the associated class $T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\right)$ belongs to $\varepsilon H^\bullet(A,d)[[\varepsilon]] \subset \varepsilon H^\bullet(A,d)((u))[[\varepsilon]]$. Any such lift $\widetilde{a}$ produces the same class $T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\right)$ and this class depends only on the gauge equivalence class of the original arc a, i.e. on the image $\underline{a}(\varepsilon)$ of $a(\varepsilon)$ in $\oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g},d)}$.*

Now by definition the map Φ_T assigns the class $T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\right) \subset \varepsilon H^\bullet(A,d)[[\varepsilon]]$ to the formal arc $\underline{a}(\varepsilon)$. □

4.2.3 Geometric interpretation The previous discussion can be repackaged geometrically as follows. A $(\mathbb{Z}/2)$-graded Batalin-Vilkovisky algebra (A,d,Δ), gives rise to a family $\mathscr{M} \to \mathbb{D} = \mathrm{Spf}(\mathbb{C}[[u]])$ of formal manifolds over the one dimensional formal disc. The family $\mathscr{M}$ is the total space of the relative moduli space $\oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g},d+u\Delta)}$ over $\mathbb{C}[[u]]$. If (A,d,Δ) has the degeneration property, then by Lemma 4.15 we have an affine structure on the generic fiber $\mathscr{M}^{\mathrm{gen}} := \mathscr{M} \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u))$ of the family (see Figure 6) given by the map F.

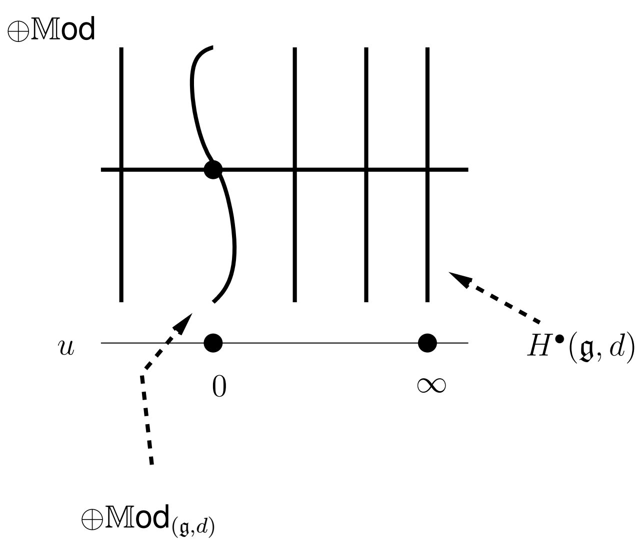


Figure 6: The relative moduli $\oplus\mathbb{M}\mathsf{od} \to \mathbb{P}^1$.

Furthermore the map T can be viewed as an extension of the affine bundle $\mathscr{M}^{\mathrm{gen}} \to \mathbb{D}^\times$ to a trivial bundle on $\mathbb{P}^1 - \{0\}$ of formal super affine spaces, where the fiber at ∞ is the super affine space $H^\bullet(\mathfrak{g},d)$. This results into a family $\oplus\mathbb{M}\mathsf{od} \to \mathbb{P}^1$ of formal super manifolds, which is a trivial vector bundle outside of zero but has a non-linear fiber at $0 \in \mathbb{P}^1$. Moreover by picking the closed point in each fiber we get a section of $\oplus\mathbb{M}\mathsf{od} \to \mathbb{P}^1$, which is just the zero section of the vector bundle $\oplus\mathbb{M}\mathsf{od}_{|\mathbb{P}^1-\{0\}} \to \mathbb{P}^1 - \{0\}$. The normal bundle to this section in $\oplus\mathbb{M}\mathsf{od}$ is trivial (hence $\oplus\mathbb{M}\mathsf{od}$ is trival as a non-linaer bundle over $\mathbb{P}^1$), and the map Φ_T gives a (non-linear) trivialization of $\oplus\mathbb{M}\mathsf{od}$ over $\mathbb{P}^1$. This type of geometry was already discussed in [52].

4.2.4 Relation to Calabi-Yau variations of nc-Hodge structures Suppose (A, d, Δ) is a d$\mathbb{Z}/2$g Batalin-Vilkovisky algebra which has the degeneration property. In this generality one does not expect to find a natural connection on $H^\bullet(A, d+u\Delta)$ along u, i.e. one does not expect to have a general formal analogue of a **nc**-Hodge structure.

However, a natural connection along the u-line may exist if we specify some additional data on (A, d, Δ). Following the analogy with the **nc**-Hodge structure associated with a symplectic manifold and the Gromov-Witten invariants, it is sufficient to specify:

- an even element $\kappa \in A$, with $d\kappa = 0$, and
- a grading operator $\mathbf{Gr} : A \to A$,

so that if we consider $\Gamma_{-1} := \mathbf{Gr} : A \to A$, and $\Gamma_{-2} : A \to A$ - the operator of multiplication by κ, then we have the commutation relations:

$$\begin{aligned}
[\Gamma_{-1}, \Delta] &= -\frac{1}{2}\Delta \\
[\Gamma_{-2}, d] &= 0 \\
d &= [\Gamma_{-1}, d] + [\Gamma_{-2}, \Delta] .
\end{aligned}$$

These commutation relations imply the identity

$$\left[u\frac{\partial}{\partial u} + u^{-1}\Gamma_{-2} + \Gamma_{-1}, d + u\Delta\right] = \frac{1}{2}(d + u\Delta),$$

which is consistent with the general formulas from Section 2.2.5. In particular, we can define a connection on $H^\bullet(A, d + u\Delta)$ along the u-line by setting

$$\nabla_{\frac{\partial}{\partial u}} := \frac{\partial}{\partial u} + u^{-2}\Gamma_{-2} + u^{-1}\Gamma_{-1}.$$

Example 4.17 Let Y be a (possibly non-compact) d-dimensional Calabi-Yau manifold with a fixed holomorphic volume form $\mathbf{\Omega}_Y$. Let $\mathsf{w} : Y \to \mathbb{C}$ be a proper holomorphic function. This geometry gives rise to a natural dg Batalin-Vilkovisky algebra:

$$\begin{aligned}
A &:= \Gamma_{C^\infty}\left(Y, \wedge^\bullet T_Y^{1,0} \otimes \wedge^\bullet A_Y^{0,1}\right), \\
d &:= \bar{\partial} + \iota_{d\mathsf{w}}, \\
\Delta &:= \oplus\mathsf{div}_{\mathbf{\Omega}_Y} = \iota_{\mathbf{\Omega}_Y}^{-1} \circ \partial \circ \iota_{\mathbf{\Omega}_Y},
\end{aligned}$$

where $\iota_{\mathbf{\Omega}_Y} : \wedge^\bullet T_Y^{1,0} \to \wedge^{d-\bullet}\Omega_Y^{1,0}$ denotes the contraction with $\mathbf{\Omega}_Y$.

As discussed in section 3.2 in this situation we get a connection along u which conjecturally defines a nc-Hodge structure. The connection is defined the above formula with Γ_{-2} = the operator of multiplication by $-\mathrm{w}$, and $\Gamma_{-1} = \mathsf{Gr} : A \to A$, the grading operator which is equal to $\frac{q+p-d}{2} \cdot \mathrm{id}$ on $\Gamma_{C^\infty}\left(Y, \wedge^p T_Y^{1,0} \otimes \wedge^q A_Y^{0,1}\right)$.

We will elaborate on this geometric picture in the next section.

4.3 B-model framework: manifolds with anticanonical sections

4.3.1 The classical Tian-Todorov theorem. Let X be a compact Kähler manifold. By Kodaira-Spencer theory we know that the deformations of X are controlled by the dg Lie algebra

$$\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right) := \left(\Gamma_{C^\infty}\left(X, T_X^{1,0} \otimes_{\mathcal{C}_X^\infty} A_X^{0,\bullet}\right), \bar{\partial}\right).$$

The classical Tian-Todorov theorem [232], [233] can be formulated as follows:

Theorem 4.18 *If X is a compact Kähler manifold with $c_1(X) = 0 \in \mathrm{Pic}(X)$, then $\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right)$ is homotopy abelian. In particular the formal moduli space of X is smooth.*

Proof. Since $c_1(X) = 0 \in \mathrm{Pic}(X)$ we can find a unique up to scale holomorphic volume form $\mathbf{\Omega}_X$ on X. As in example 4.17 the pair $(X, \mathbf{\Omega}_X)$ gives rise to a dg Batalin-Vilkovisky algebra (A, d, Δ):

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_X^{1,0} \otimes \wedge^\bullet A_X^{0,1}\right) \\ d &:= \bar{\partial} \\ \Delta &:= \oplus \mathsf{div}_{\mathbf{\Omega}_X} = \iota_{\mathbf{\Omega}_X}^{-1} \circ \partial \circ \iota_{\mathbf{\Omega}_X}. \end{aligned}$$

Consider the associated dg Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}}) := (\mathbf{\Pi} A, d)$. We have a natural inclusion of dg Lie algebras

$$\begin{array}{ccc} \left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right) & \hookrightarrow & (\mathfrak{g}, d_{\mathfrak{g}}) \\ \| & & \| \\ \left(\Gamma_{C^\infty}\left(X, T_X^{1,0} \otimes_{\mathcal{C}_X^\infty} A_X^{0,\bullet}\right), \bar{\partial}\right) & \hookrightarrow & \Gamma_{C^\infty}\left(X, \wedge^\bullet T_X^{1,0} \otimes \wedge^\bullet A_X^{0,1}\right) \end{array}$$

which embeds $\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right)$ as a direct summand in $(\mathfrak{g}, d_{\mathfrak{g}})$, and so induces and embedding $H^\bullet\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right) \subset H^\bullet(\mathfrak{g}, d_{\mathfrak{g}})$ in cohomology. So by Proposition 4.11 it suffices to check that $(\mathfrak{g}, d_{\mathfrak{g}})$ is homotopy abelian.

On the other hand the contraction map $\iota_{\mathbf{\Omega}_X}$ gives an isomorphism of bicomplexes between the dg Batalin-Vilkovisky algebra (A, d, Δ) and the Dolbeault bicomplex $(A^\bullet(X), \bar{\partial}, \partial)$. Since X is assumed compact and Kähler, the Hodge-to-de Rham spectral sequence degenerates on X which is equivalent to the equality $\dim H_{dR}^k(X, \mathbb{C}) = \dim(\oplus_{p+q=k} H^p(X, \Omega_X^q))$ which implies that the Dolbeault bicomplex $(A^\bullet(X), \bar{\partial}, \partial)$ has the degeneration property. Thus by Theorem 4.14 **(1)** it follows that $(\mathfrak{g}, d_{\mathfrak{g}})$ is homotopy abelian. The theorem is proven. □

4.3.2 Canonical coordinates on the moduli of Calabi-Yau manifolds. Let X be a Calabi-Yau manifold, i.e. a d-dimensional compact Kähler manifold with $c_1(X) = 0$ in $\mathrm{Pic}(X)$. Let (A, d, Δ) be the dg Batalin-Vilkovisky algebra defined in section 4.3.1. The contraction map $\iota_{\boldsymbol{\Omega}_X}$ identifies the $\mathbb{C}[[u]]$-module $H^\bullet\left(\mathfrak{g}[[u]], d + u\Delta\right)$ with the Rees module of the **nc**-Hodge filtration on $H^\bullet_{dR}(X, \mathbb{C})$ for which $H^{p,q}(X) \subset F^{\frac{p-q}{2}}$. Now choose one of the following equivalent pieces of data:

- a filtration $G^\bullet$ on $H^\bullet_{dR}(X, \mathbb{C})$ which is opposed to the **nc**-Hodge filtration,
- a splitting of the **nc**-hodge filtration,
- an extension of the associated **nc**-Hodge structure to a trivial bundle on $\mathbb{P}^1$ such that the connection has at most a first order pole at infinity.

Each such choice gives rise to an affine structure on $\oplus\mathbb{M}\mathsf{od}_{(\mathfrak{g}, d_{\mathfrak{g}})}$. This affine structure is the same as the one described in section 4.1.3 corresponding to the **nc**-Hodge structure above and the decoration ψ given by the class $[\boldsymbol{\Omega}_X]$ in the associated graded $\mathsf{gr}_{G^\bullet} H^\bullet_{dR}(X, \mathbb{C})$.

In mirror symmetry considerations a choice of this type arises naturally when X is a Calabi-Yau manifold near a large complex structure limit point. Concretely, suppose $X = X_{\mathfrak{z}}$ is member in a holomorphic family $\{X_z\}$ of compact d-dimensional Calabi-Yau manifolds parameterized by z in a polydisc $\prod_{i=1}^M \{z_i \in \mathbb{C} \,|\, 0 < |z_i| \ll 1\}$, and such that:

- $M = \dim_{\mathbb{C}} H^1\left(X_z, T_{X_z}\right)$;
- for each $i = 1, \ldots, M$ the monodromy operator $t_i \in GL\left(H^1\left(X_{\mathfrak{z}}, T_{X_{\mathfrak{z}}}\right)\right)$ assigned to the circle (traced counterclockwise)
$$\gamma_i = \left\{ z \,\middle|\, \begin{array}{l} z_j = \mathfrak{z}_j,\ j \neq i, \\ |z_i| = |\mathfrak{z}_i| \end{array} \right\}$$
is unipotent of order d.

In this setup, the filtration $G^\bullet$ of $H^\bullet\left(X_{\mathfrak{z}}, \mathbb{C}\right)$ invariant under all unipotent operators $\prod_{i=1}^M t_i^{a_i}$, $a_i \in \mathbb{Z}_{>0}$ will be opposed to the Hodge filtration and will thus give us canonical coordinates on the polydisc. This affine structure corresponds to a rational decoration of a Calabi-Yau variation of **nc**-Hodge structures.

4.3.3 Generalizations. Here we generalize the previous discussion to the case of varieties with divisors.

(i) Let X be a d-dimensional smooth projective variety over $\mathbb{C}$, and let $D \subset X$ be a normal crossings anti-canonical divisor, i.e $\mathcal{O}_X(D) = K_X^{-1} \in \mathrm{Pic}(X)$. Typically such an X will be a Fano or a quasi-Fano. If D is smooth, then by adjunction D will be a Calabi-Yau. Specifying such a divisor is equivalent to specifying a logarithmic volume form on X. This is a unique up to scale n-form $\boldsymbol{\Omega}_{X \log D} \in \Gamma\left(X, \Omega_X^d(\log D)\right)$ on X which has a first order pole along D and does not vanish anywhere on $X - D$.

Let $T_{X,D}$ be the subsheaf of T_X of holomorphic vector fields on X which at the points of D are tangent to D. This is a locally free subsheaf of T_X of rank d which controls the deformation theory of the pair (X, D). The relevant dg Batalin-Vilkovisky algebra (A, d, Δ) is an obvious generalization of the one in the absolute case:

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_{X,D} \otimes_{\mathcal{C}^\infty_X} \wedge^\bullet A^{0,1}_X\right) \\ d &:= \bar{\partial} \\ \Delta &:= \oplus \mathsf{div}_{\mathbf{\Omega}_{X \log D}} = \iota^{-1}_{\mathbf{\Omega}_{X \log D}} \circ \partial \circ \iota_{\mathbf{\Omega}_{X \log D}}, \end{aligned}$$

where $\iota_{\mathbf{\Omega}_{X \log D}} : \wedge^\bullet T_{X,D} \to \Omega^{d-\bullet}_X(\log D)$ is the isomorphism given by contraction with $\mathbf{\Omega}_{X \log D}$.

Again the map $\iota_{\mathbf{\Omega}_{X \log D}}$ identifies (A, d, Δ) with the logarithmic Dolbeault bicomplex $\left(A^{\bullet,\bullet}(\log D), \bar{\partial}, \partial\right)$. In particular, for all $u \neq 0$ we get an identification of the cohomology of the complex $(A, d + u\Delta)$ with the cohomology of the total complex of the double complex $\left(\Omega^{\bullet,\bullet}_X(\log D), \bar{\partial}, \partial\right)$, which is equal [247, Section 6.1] to the cohomology of the open variety $X - D$. In other words for all $u \neq 0$ we have an isomorphism

$$H^\bullet(A, d + u\Delta) \cong H^\bullet_{dR}(X - D, \mathbb{C}). \tag{4.2}$$

Now mixed Hodge theory implies the following

Lemma 4.19 *The logarithmic dg Batalin-Vilkovisky algebra (A, d, Δ) has the degeneration property. In particular the formal moduli of the pair (X, D) is smooth.*

We will return to the proof of this lemma in section 4.3.4 but first we will discuss a couple of variants of this geometric setup.

(ii) Suppose X is a smooth projective d-dimensional Calabi-Yau manifold. Let as before $\mathbf{\Omega}_X$ be the holomorphic volume form on X. Let $D \subset X$ be a normal crossings divisor. Typically if D is smooth, it will be a variety of general type.

Consider the dg Batalin-Vilkovisky algebra (A, d, Δ) given by

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_{X,D} \otimes_{\mathcal{C}^\infty_X} \wedge^\bullet A^{0,1}_X\right) \\ d &:= \bar{\partial} \\ \Delta &:= \oplus \mathsf{div}_{\mathbf{\Omega}_X} = \iota^{-1}_{\mathbf{\Omega}_X} \circ \partial \circ \iota_{\mathbf{\Omega}_X}, \end{aligned}$$

The contraction $\iota_{\mathbf{\Omega}_X}$ identifies this algebra with the dg Batalin-Vilkovisky algebra

$$\left(\Gamma_{C^\infty}\left(X, \Omega^\bullet_X(\mathrm{rel}\, D) \otimes_{\mathcal{C}^\infty_X} \wedge^\bullet A^{0,1}_X\right), \bar{\partial}, \partial\right),$$

where $\Omega^k_X(\mathrm{rel}\, D) \subset \Omega^k_X$ denotes the subsheaf of all holomorphic k-forms that restrict to $0 \in \Omega^k_{D-\mathrm{sing}(D)}$. The cohomology of the total complex associated with this double complex is the de Rham cohomology of the pair (X, D), and so again we get an identification

$$H^\bullet(A, d + u\Delta) \cong H^\bullet_{dR}(X, D; \mathbb{C}) \tag{4.3}$$

valid for all fixed $u \neq 0$. Again using this identification and mixed Hodge theory one deduces the following

Lemma 4.20 *The dg Batalin-Vilkovisky algebra (A, d, Δ) has the degeneration property and hence the formal moduli space of the pair (X, D) is smooth.*

(iii) The setups **(i)** and **(ii)** have a natural common generalization. Fix a smooth projective complex variety of dimension d, a normal crossings divisor $D = \cup_{i\in I} D_i \subset X$, and a collection of weights $\{a_i\}_{i\in I} \subset [0,1] \cap \mathbb{Q}$, so that

$$\sum_{i\in I} a_i[D_i] = -K_X \in \mathrm{Pic}(X) \otimes \mathbb{Q}.$$

Represent the a_i's by reduced fractions, take $N \geq 1$ to be the least common multiple of the denominators of these fractions and such that

$$\sum_{i\in I} (Na_i)[D_i] = -NK_X \in \mathrm{Pic}(X),$$

and set $n_i := a_i N$. In particular we have a unique up to scale section $\widetilde{\mathbf{\Omega}}_X \in \Gamma\left(X, K_X^{\otimes(-N)}\right)$ whose divisor is $\sum_{i\in I} n_i D_i$. In this situation we can again promote the Dolbeault dg Lie algebra which computes the deformation theory of (X, D) to a dg Batalin-Vilkovisky algebra (A, d, Δ), where

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_{X,D} \otimes_{\mathcal{C}^\infty_X} \wedge^\bullet A_X^{0,1}\right) \\ d &:= \bar\partial \\ \Delta &:= \oplus\mathsf{div}_{\widetilde{\mathbf{\Omega}}_X}. \end{aligned}$$

The divergence operator $\oplus\mathsf{div}_{\widetilde{\mathbf{\Omega}}_X}$ is defined as follows. Restricting the section $\widetilde{\mathbf{\Omega}}_X$ to $X - D$ we get a nowhere vanishing section of $K_{X-D}^{\otimes(-N)}$, i.e. a flat holomorphic connection on K_{X-D}. If $U \subset X - D$ is a simply connected open, then we can choose $\mathbf{\Omega}_U$ a holomorphic volume form on U which is covariantly constant for this flat connection, and define the associated divergence operator $\oplus\mathsf{div}_{\mathbf{\Omega}_U} := \iota_{\mathbf{\Omega}_U}^{-1} \circ \partial \circ \iota_{\mathbf{\Omega}_U}$. But by the flatness of the connection it follows that any other covariantly constant volume form on U will be proportional to $\mathbf{\Omega}_U$ with a constant proportionality coefficient. Since by definition $\oplus\mathsf{div}_{c\mathbf{\Omega}_U} = \oplus\mathsf{div}_{\mathbf{\Omega}_U}$ for any constant c we get a well defined divergence operator on $X - U$. Furthermore locally this divergence operator is a given by a holomorphic volume form which is a branch of $\left(\widetilde{\mathbf{\Omega}}_X\right)^{-1/N}$ and so by continuity it gives a well defined map of locally free sheaves $\oplus\mathsf{div}_{\widetilde{\mathbf{\Omega}}_X} : \wedge^i T_{X,D} \to \wedge^{i-1} T_{X,D}$.

Again we claim that

Lemma 4.21 *The dg Batalin-Vilkovisky algebra (A, d, Δ) has the degeneration property and the formal moduli space of the pair (X, D) is smooth.*

Proof. The proof of this lemma again reduces to mixed Hodge theory via a map similar to the isomorphisms (4.2) and (4.3). However constructing this map is a bit more involved than the arguments we used to construct (4.2) and (4.3).

Consider the root stack $Z = X\left\langle \left\{\frac{D_i}{N}\right\}_{i\in I}\right\rangle$ as defined in e.g. [175], [126]. By construction Z is a smooth proper Deligne-Mumford stack, equipped with a finite and flat morphism $\pi : Z \to X$.

Conceptually the best way to define the stack Z is as a moduli stack classifying (special) log structures associated with X, the divisor D and the number N (see [175] for the details). Etale locally on X the stack Z can be described easily as a quotient stack. Indeed choose etale locally an identification of X with a neighborhood of zero in $\mathbb{A}^d$ with coordinates $z_1, \ldots, z_d$, so that $D = D_1 \cup \cdots \cup D_r$ and D_i is identified with the hyperplane $z_i = 0$. Then the corresponding etale local patch in Z is canonically isomorphic to the stack quotient

$$\big[\mathbb{A}^d / \underbrace{\boldsymbol{\mu}_N \times \cdots \times \boldsymbol{\mu}_N}_{r\text{-times}}\big],$$

where $\boldsymbol{\mu}_N \subset \mathbb{C}^\times$ is the group of N-th roots of unity, and $(\zeta_1, \ldots, \zeta_r) \in \boldsymbol{\mu}_N^{\times r}$ acts as $(z_1, \ldots, z_r, z_{r+1}, \ldots, z_d) \mapsto (\zeta_1 z_1, \ldots, \zeta_r z_r, z_{r+1}, \ldots, z_d)$.

In particular, this description shows (see [175, Theorem 4.1]) that:

- The map π is an isomorphism over $X - D$ and in general identifies X with the coarse moduli space of Z;
- There is a strict normal crossings divisor $\widetilde{D} = \cup_{i\in I} \widetilde{D}_i \subset Z$, such that

  $$\mathcal{O}_Z\left(-N\widetilde{D}_i\right) = \pi^* \mathcal{O}_X\left(-D_i\right)$$

  as ideal subsheaves of $\mathcal{O}_Z$;
- For all j we have the Hurwitz formula $\Omega^j_Z(\log \widetilde{D}) = \pi^* \Omega^j_X(\log D)$.

In particular we have canonical isomorphisms

$$\pi^* K_X \cong \mathcal{O}_Z\left(-\sum_{i\in I} n_i \widetilde{D}_i\right)$$

$$\pi^* K_X \cong K_Z \otimes \mathcal{O}_Z\left((1-N)\sum_{i\in I} \widetilde{D}_i\right)$$

the first given by the section $\pi^* \widetilde{\boldsymbol{\Omega}}_X$ and the second coming from the Hurwitz formula.

There is a natural complex local system of rank one on $X - D$ with monodromy in $\boldsymbol{\mu}_N$ associated with the choices of N-th root of the section $\widetilde{\boldsymbol{\Omega}}_X$. It is easy to see that the pullback of this local system admits a canonical extension (as a local system) to Z, which we denote by $\boldsymbol{\Xi}$. Moreover, we have a canonical meromorphic section $\boldsymbol{\Omega}_Z$ of $K_Z \otimes_{\mathbb{C}} \boldsymbol{\Xi}$ with divisor $\sum_{i\in I}(N - 1 - n_i)\widetilde{D}_i$. It is easy to check locally by using the etale local description of Z as a quotient stack the contraction $\iota_{\boldsymbol{\Omega}_Z}$ gives a well defined isomorphism of locally free sheaves:

$$\iota_{\boldsymbol{\Omega}_Z} : \ \wedge^j T_{Z,\widetilde{D}} \xrightarrow{\cong} \Omega_Z^{d-j}\left(\log \widetilde{D}_{(1)}, \mathrm{rel}\, \widetilde{D}_{(0)}\right) \otimes_{\mathbb{C}} \boldsymbol{\Xi}.$$

Here

$$\widetilde{D}_{(0)} := \cup_{i\in I_0} \widetilde{D}_i \qquad I_0 = \{i \in I | a_i = 0\}$$
$$\widetilde{D}_{(1)} := \cup_{i\in I_1} \widetilde{D}_i \qquad I_1 = \{i \in I | a_i = 1\}.$$

Now taking into account the Hurwitz isomorphism $\wedge^j T_{Z,\widetilde{D}} \cong \pi^* \wedge^j T_{X,D}$ and using adjunction, we can view $\iota_{\boldsymbol{\Omega}_Z}$ as an isomorphism

$$\wedge^j T_{X,D} \xrightarrow{\cong} \left(\pi_* \Omega_Z^{d-j}\left(\log \widetilde{D}_{(1)}, \mathrm{rel}\, \widetilde{D}_{(0)}\right) \otimes_{\mathbb{C}} \boldsymbol{\Xi}\right) \tag{4.4}$$

It is immediate from the definition that the isomorphism (4.4) (taken for all j) identifies the dg Batalin-Vilkovisky algebra (A, d, Δ) with the Dolbeault bicomplex

$$\left(\Gamma_{C^\infty}\left(X, \left(\pi_* \Omega_Z^{\bullet}\left(\log \widetilde{D}_{(1)}, \mathrm{rel}\, \widetilde{D}_{(0)}\right) \otimes_{\mathbb{C}} \boldsymbol{\Xi}\right) \otimes_{\mathcal{C}^\infty_X} A_X^{0,\bullet}\right), \bar{\partial}, \partial\right).$$

But the above complex equipped with the differential $\partial + \bar{\partial}$ is the Dolbeault resolution of the complex of sheaves $\pi_*\left(\Omega_Z^{\bullet}\left(\log \widetilde{D}_{(1)}, \mathrm{rel}\, \widetilde{D}_{(0)}\right) \otimes_{\mathbb{C}} \boldsymbol{\Xi}, \partial\right)$ which is equal to the derived direct image $R\pi_*\left(\Omega_Z^{\bullet}\left(\log \widetilde{D}_{(1)}, \mathrm{rel}\, \widetilde{D}_{(0)}\right) \otimes_{\mathbb{C}}\right.$ since π is finite. Now combined with the Leray spectral sequence for π this gives, for all $u \neq 0$ an isomorphism

$$H^\bullet(A, d + u\Delta) \cong H^\bullet_{dR}\left(Z - \widetilde{D}_{(1)}, \widetilde{D}_{(0)} - \widetilde{D}_{(1)}; \boldsymbol{\Xi}\right), \tag{4.5}$$

which specializes to both isomorphisms (4.2) and (4.3).

Now the fact that Z is a smooth and proper Deligne-Mumford stack and mixed Hodge theory (see 4.3.4) for $\left(Z - \widetilde{D}_{(1)}, \widetilde{D}_{(0)} - \widetilde{D}_{(1)}\right)$ endowed with local system $\boldsymbol{\Xi}$ imply that (A, d, Δ) has the degeneration property. □

Remark 4.22 The fact that the root stack in the previous proof can be viewed as the moduli stack of special log structures is very interesting. It suggests that the setup we just discussed may fit naturally in the recent approach of Gross-Siebert [104, 105] to mirror symmetry and instanton corrections via log degenerations of toric Fano manifolds (see also [158, 157]). The relationship between these two setups is certainly worth studying and we plan to return to it in the future.

(iv) Yet another generalization of the previous picture arises when we take the variety X to be a normal-crossings Calabi-Yau. More precisely assume that X is a strict normal crossings variety with irreducible components $X = \cup_{i \in I} X_i$ equipped with a holomorphic volume form $\boldsymbol{\Omega}_X$ on $X - X^{\mathrm{sing}}$ such that the restriction of $\boldsymbol{\Omega}_X$ on each X_i has a logarithmic pole along $X_i \cap (\cup_{j \neq i} X_j)$ and the residues of these restricted forms cancel along each $X_i \cup X_j$. Taking a colimit along the projective system of all finite intersections of components of X we get again a dg Batalin-Vilkovisky algebra $A_{\mathrm{tot}}(X) = \mathsf{colim}_{J \subset I}\, A\left(\cap_{i \in J} X_i\right)$ and again by using mixed Hodge theory we can check that this algebra has the degeneration property.

4.3.4 Mixed Hodge theory in a nutshell.

In this section we briefly recall the basic arguments from Deligne's mixed Hodge theory [68] that are necessary for proving the degeneration property of the dg Batalign-Vilkovisky algebras in section 4.3.3 **(i)-(iv)**.

Suppose we are given:

- a finite ordered collection (X_α) of smooth complex projective varieties;
- for every α a choice of a $\mathbb{Z} \times \mathbb{Z}$-graded complex of sheaves of differential forms which are either C^∞ or are $C^{-\infty}$ (i.e. currents) and constrained so that their wave front (singular support) is contained in a given conical Lagrangian in $T^\vee X_\alpha$ which is the conormal bundle to a normal crossings divisor in X_α;
- a collection of integers $n_\alpha \in \mathbb{Z}$.

Consider the complex $C^{\mathrm{tot}} = \oplus_\alpha C^\bullet_\alpha[n_\alpha]$ equipped with three differentials $\partial, \bar\partial, \delta$, where $\delta = \sum_{\alpha<\beta} \delta_{\alpha\beta}$, and the $\delta_{\alpha\beta}$ come from pullbacks and pushforwards for some maps $X_\beta \hookrightarrow X_\alpha$ or $X_\alpha \hookrightarrow X_\beta$. The statement we need now can be formulated as follows:

Claim 4.23 *For every $k \geq 1$ the cohomology*

$$H^\bullet\left(C^{\mathrm{tot}}[u]/(u^k), \bar\partial + \delta + u\partial\right)$$

is a free $\mathbb{C}[u]/(u^k)$-module.

Proof. If X is smooth projective over $\mathbb{C}$ and if $\left(A^\bullet(X), \bar\partial\right)$ is the $\bar\partial$-complex of (either C^∞ or $C^{-\infty}$) differential forms on X, then the inclusion

$$\left(\ker\partial, \bar\partial\right) \hookrightarrow \left(A^\bullet(X), \bar\partial\right)$$

is a quasi-isomorphism.

This implies that the horizontal arrows in the diagram of complexes

$$\begin{array}{ccc}
\left(\ker\partial[u]/(u^k), \bar\partial + \delta + u\partial\right) & \longrightarrow & \left(C^{\mathrm{tot}}[u]/(u^k), \bar\partial + \delta + u\partial\right) \\
\| & & \\
\left(\ker\partial[u]/(u^k), \bar\partial + \delta\right) & \longrightarrow & \left(C^{\mathrm{tot}}, \bar\partial + \delta\right)[u]/(u^k),
\end{array}$$

are quasi-isomorphisms. Indeed, this follows by noticing that there are natural filtrations on both sides (by the powers of u and the index α) which give rise to convergent spectral sequences and induce the quasi-isomorphic inclusion $\left(\ker\partial, \bar\partial\right) \hookrightarrow \left(C^{\mathrm{tot}}, \bar\partial\right)$ on the associated graded. This proves the claim. $\square$

Remark 4.24 • Note that the same reasoning implies that the natural map

$$\left(\ker\partial, \bar\partial + \delta\right) \twoheadrightarrow \left(\ker\partial/\operatorname{im}\partial, \bar\partial + \delta\right) = \left(H^\bullet(X_\alpha), \delta\right),$$

is also a quasi-isomorphism, which reduces the problem of computing $H^\bullet\left(C^{\mathrm{tot}}[u]/(u^k), \bar\partial + \delta + u\partial\right)$ to a homological algebra question on a complex of finite dimensional vector spaces.

• There is useful variant of the theory, also discussed in [68]: the previous discussion immediately generalizes to the case of cochain complexes of a collection of projective manifolds with coefficients in some unitary local systems.

Next we discuss a few examples and applications of the geometric setup from section 4.3.3.

4.3.5 The moduli stack of Fano varieties. As a consequence of section 4.3.3 **(iii)** we get a new proof and a refinement of the following result of Ran [200], [146]:

Theorem 4.25 *Let X be a complex Fano manifold, that is let X be a smooth proper $\mathbb{C}$-variety for which K_X^{-1} is ample. Then the versal deformations of X are unobstructed.*

Proof: Choose $N > 1$ so that $K_X^{\otimes(-N)}$ is very ample and all the higher cohomology groups $H^k\left(X, K_X^{\otimes(-N)}\right)$ vanish for $k \geq 1$. Choose a generic section $\widetilde{\mathbf{\Omega}}_X \in H^0\left(X, K_X^{\otimes(-N)}\right) = 0$ whose zero locus is a smooth and connected divisor $D \subset X$.

Consider now $\mathfrak{g} = \mathbf{\Pi} R\Gamma\left(X, \wedge^{\bullet} T_{X,D}\right)$ with the Schouten bracket. By Lemma 4.21 this d($\mathbb{Z}/2$)g Lie algebra is homotopy abelian and so as in the proof of Theorem 4.18 we conclude that $\mathfrak{g}^{(1)} = R\Gamma\left(X, T_{X,D}\right)$ is homotopy abelian. Since this d($\mathbb{Z}/2$)g Lie algebra governs the deformation theory of (X, D) as a variety with a divisor, it follows that the formal germ of the deformation space of the pair (X, D) is smooth. Next we will need the following simple

Lemma 4.26 *Suppose (X', D') is a small deformation of (X, D) as a variety with divisor. Then X' is still a Fano with $K_{X'}^{\otimes(-N)}$ is very ample and $D' \in \left|K_{X'}^{\otimes(-N)}\right|$.*

Proof: The condition of $K_X^{\otimes(-N)}$ being very ample is open in the moduli of X. Furthermore by definition $K_X^{\otimes(-N)} \otimes \mathcal{O}_X(-D) = \mathcal{O}_X$ and so by the small deformation hypothesis it follows that $K_{X'}^{\otimes(-N)} \otimes \mathcal{O}_{X'}(-D)$ is in the connected component of the identity of $\mathrm{Pic}(X')$. But X' is a Fano and so $\mathrm{Lie}(\mathrm{Pic}^0(X')) = H^1(X', \mathcal{O}_{X'}) = 0$. Hence $K_{X'}^{\otimes(-N)} \otimes \mathcal{O}_{X'}(-D) = \mathcal{O}_X$ as well. $\square$

The theorem now follows easily. The versal deformation space of smooth connected D's for a given X is smooth and isomorphic to a domain in $\mathbb{P}^{h^0\left(X, K_{X'}^{\otimes(-N)}\right)-1}$. Since the dimension of these projective spaces is locally constant in X by Riemann-Roch and vanishing of the higher cohomologies, it follows that the map from the versal deformation space of the pairs (X, D) to the versal deformation stack of X is smooth. In other words the versal deformation stack of X has a presentation in the smooth topology with a smooth atlas - the versal deformation space for (X, D). Hence the versal deformations of X are a smooth stack. $\square$

4.3.6 Algebras for the Landau-Ginzburg model. Consider again the setup of a holomorphic Landau-Ginzburg model. Suppose Y is smooth and quasi-projective over $\mathbb{C}$ and of dimension $\dim Y = d$. Suppose there exists a nowhere vanishing algebraic volume form $\mathbf{\Omega}_Y \in \Gamma(Y, K_Y)$, and let $\mathsf{w} : Y \to \mathbb{A}^1$ be a regular function with compact critical locus.

This data gives a dg Batalin-Vilkovisky algebra (A, d, Δ) where

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(Y, \wedge^\bullet T_Y^{1,0} \otimes_{\mathcal{C}_Y^\infty} \wedge^\bullet A_Y^{0,1}\right) \\ d &:= \bar{\partial} + \iota_{d\mathsf{w}} \\ \Delta &:= \oplus \mathsf{div}_{\widetilde{\mathbf{\Omega}}_Y}. \end{aligned}$$

Again the contraction $\iota_{\mathbf{\Omega}_Y}$ identifies (A, d, Δ) with the twisted Dolbeault bicomplex $\left(A^\bullet(Y), \bar{\partial} + d\mathsf{w}\wedge, \partial\right)$. The latter satisfies the degeneration property by the work of Barannikov and the second author, Sabbah [203], or Ogus-Vologodsky [191]

Remark 4.27 It will be interesting to combine the previous discussion with the discussion in section 4.3.3 **(iii)** or with the broken Calabi-Yau geometry from section 4.3.3 **(iv)**. Suppose we have a quasi-projective smooth complex Y, a regular function $\mathsf{w} : Y \to \mathbb{A}^1$ with compact critical locus, and suppose we are given a normal crossings divisor $D = \cup_{i\in I} D_i$ and a system of weights $\{a_i\}_{i\in I}$ as in section 4.3.3 **(iii)**. Then we can write the w-twisted version of the dg Batalin-Vilkovisky algebra for (Y, D) which by general nonsense will compute the deformation theory of the data (Y, D, w). Similarly we can add a potential to a Y which itself is a normal-crossings Calabi-Yau, as in section 4.3.3 **(iv)**. We expect that the resulting algebras will again have the degeneration property but we have not investigated this question.

4.4 Categorical framework: spherical functors

In this section we briefly discuss some algebraic aspects of the deformation theory of **nc**-spaces (see section 2.2.1). For simplicity we will discuss the $\mathbb{Z}$-graded case but in fact all definitions and statements readily generalize to the $\mathbb{Z}/2$ case.

4.4.1 Calabi-Yau nc-spaces. Suppose $X = \oplus\mathbf{ncSpec}(A)$ is a graded **nc**-affine **nc**-space over $\mathbb{C}$. If X is smooth, then $A \in \oplus\mathsf{Perf}_{X\times X^{\mathrm{op}}} = \oplus\mathsf{Perf}\,(A \otimes A^{\mathrm{op}} - \mathsf{mod})$ and we define the ***smooth dual*** of A to be $A^! := \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A \otimes A)$. Similarly if X is compact, then $A \in \oplus\mathsf{Perf}_{\mathrm{pt}}$ and we define the ***compact dual*** of A to be $A^* := \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) \in (A \otimes A^{\mathrm{op}} - \mathsf{mod})$.

If X is both a smooth and compact **nc**-space, then we have isomorphisms

$$A^! \otimes_A A^* \cong A^* \otimes_A A^! \cong A$$

in the category $(A \otimes A^{\mathrm{op}} - \mathsf{mod})$. The endofunctor $S_X : C_X \to C_X$ given by the A-bimodule A^* is called ***the Serre functor*** of X. It is an autoequivalence of C_X which is central (i.e. commutes with all autoequivalences). Moreover for any two objects $\mathcal{E}, \mathcal{F} \in \oplus\mathsf{Perf}_X$ there is a functorial identification

$$\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})^\vee \cong \mathrm{Hom}_X(\mathcal{F}, S_X\mathcal{E}).$$

With this notation we have the following definition (see also [159]):

Definition 4.28 *We say that a smooth graded* $\mathbf{nc}$*-affine* $\mathbf{nc}$*-space* $X = \oplus\mathbf{ncSpec}(A)$ *is a* ***Calabi-Yau of dimension*** $d \in \mathbb{Z}$ *if* $A^! \cong A[-d]$ *in* $(A \otimes A^{\mathrm{op}} - \mathsf{mod})$*. We say that a compact* $\mathbf{nc}$*-affine* $\mathbf{nc}$*-space* $X = \oplus\mathbf{ncSpec}(A)$ *is a* ***Calabi-Yau of dimension*** $d \in \mathbb{Z}$ *if* $A^* \cong A[d]$ *in* $(A \otimes A^{\mathrm{op}} - \mathsf{mod})$.

The definition works also in the $\mathbb{Z}/2$-graded case, where the dimension d is understood as an element of $\mathbb{Z}/2$.

For a $\mathbf{nc}$-space which is both smooth and compact the two conditions are equivalent and are equivalent to having an isomorphism of endofunctors $S_X \cong [d]$.

Remark 4.29 This definition of a Calabi-Yau structure on a smooth compact $\mathbf{nc}$-space is somewhat simplistic and should be taken with a grain of salt. The true definition (see [159]) implies the isomorphism of functors $S_X \cong [d]$ but also involves higher homotopical data which is encoded in a cyclic category structure on C_X. We will suppress the cyclic structure here in order to simplify the discussion.

We are interested in $\mathbf{nc}$-space analogues of the Tian-Todorov theorem. The unobstructedness of graded smooth and compact $\mathbf{nc}$-Calabi-Yau spaces was recently analyzed by Pandit [198] via the T^1-lifting property of Ran [200] and Kawamata [146]. Here we formulate the following general

Theorem 4.30 *Suppose that* X *is a smooth and compact* $\mathbf{nc}$*-Calabi-Yau space of dimension* $d \in \mathbb{Z}$ *(or of dimension* $d \in \mathbb{Z}/2$ *in the* $\mathbb{Z}/2$*-graded case). Assume that* X *satisfies the degeneration conjecture (see section* 2.2.4*). Then:*

- *the Hochschild cochain algebra* $C^\bullet(X)$ *of* X *is a homotopy abelian* L_∞ *algebra;*
- *the formal moduli space* $\oplus\mathbb{M}\mathsf{od}_X$ *of* X *is a formal supermanifold, i.e.*

$$\oplus\mathbb{M}\mathsf{od}_X := \oplus\mathbb{M}\mathsf{od}_{C^\bullet(A,A)} \cong \operatorname{Spf}\mathbb{C}[[x_1,\ldots,x_N,\xi_1,\ldots,\xi_M]];$$

- *the negative cyclic homology of the universal family over* $\oplus\mathbb{M}\mathsf{od}_X$ *gives a vector bundle* $H \to \oplus\mathbb{M}\mathsf{od}_X \times \mathbb{D}$ *which is equipped with a flat meromorphic connection* ∇ *so that* $\nabla_{u\partial/\partial x_i}$*,* $\nabla_{u\partial/\partial \xi_j}$*, and* $\nabla_{u^2\partial/\partial u}$ *are regular;*
- (H, ∇) *is the de Rham part of a Calabi-Yau variation of* $\mathbf{nc}$*-Hodge structures.*

We will only sketch some of the highlights of the proof of this theorem here since going into full details will take us too far afield. The proof is based on a mildly generalized version of Deligne's conjecture (see e.g. [156], [230]) which states that the Hochschild cochain complex of an affine $\mathbf{nc}$-space is also an algebra over the operad of chains of the little discs operad. The first step is to show that under the Calabi-Yau assumption the Hochschild cochain complex $C^\bullet(X)$ is also naturally an algebra over the cyclic operad of chains of the framed little discs operad (i.e. the operad of little discs with a marked point point on the boundary). Next one shows that the validity of the degeneration conjecture for X implies that the induced S^1-action on the cochain complex, is homotopically trivial. Finally by a topological argument one deduces from this the fact that all the higher L_∞ operations on $C^\bullet(X)$ must vanish.

Remark 4.31 It seems certain[2] that from deformation quantization it follows that if X is a smooth and projective Calabi-Yau variety, then the data described in the above theorem is canonically isomorphic to the formal completion of the variation of **nc**-Hodge structures described in section 4.3.2.

For a general smooth and compact **nc**-Calabi-Yau space we expect that the formal variation of **nc**-de Rham data in theorem 4.30 converges to give an analytic de Rham data which contains a compatible **nc**-Betti data $\mathscr{E}_B$ and so extends to an honest variation of **nc**-Hodge structures.

4.4.2 Spherical functors. In this section we introduce a special version of the general notion of a spherical functor [6] which is tailored to the Calabi-Yau condition. We begin with a definition:

Definition 4.32 *Let X and Y be two graded **nc**-spaces. A **morphism** $f : X \to Y$ is a triple of functors*

$$\begin{tikzcd} C_X \arrow[d, "f_*"] \\ C_Y \arrow[u, bend left, "f^*"] \arrow[u, bend right, "f^!"'] \end{tikzcd}$$

so that (f^*, f_*) and $(f_*, f^!)$ are $(\text{left}, \text{right})$ pairs of adjoint functors.

Suppose now X, Y are smooth and compact graded **nc**-spaces and let Y be a **nc**-Calabi-Yau of dimension d.

Definition 4.33 *A morphism $f : X \to Y$ is **spherical** if:*

- **(a)** *the cone of the natural adjunction morphism $\mathrm{id}_{C_X} \to f^! \circ f_*$ is isomorphic to the shifted Serre functor of X: $\operatorname{cone}\left(\mathrm{id}_{C_X} \to f^! \circ f_*\right) \cong S_X[1-d]$,*
- **(b)** *the natural map $f^! \to S_X[1-d] \circ f^*$, induced from the isomorphism in* **(a)** *and the adjunction $f^! \to f^! \circ f_* \circ f^*$ is an isomorphism of functors.*

Remark 4.34 **(a)** If f is spherical, then the associated ***reflection functor*** $\mathcal{R}_f := \operatorname{cone}\left(f_* \circ f^! \to \mathrm{id}_{C_Y}\right)$ is an auto-equivalence of C_Y [6].
(b) Similarly to the definition of a Calabi-Yau structure the above notion of a spherical functor should be viewed as a weak preliminary version of a stronger more refined notion which has to involve higher homotopical data and has yet to be defined carefully.

[2] We borrowed this delightful expression from [132].

Example 4.35 **(i)** Let $X = \mathrm{pt}$, and let Y be a d-dimensional smooth and compact **nc**-Calabi-Yau and let $\mathcal{E} \in C_Y$ be a spherical object, i.e. an object for which the complex of $\mathbb{C}$-vector spaces $\mathrm{Hom}_Y(\mathcal{E}, \mathcal{E})$ is quasi isomorphic to $(H^\bullet(S^d, \mathbb{C}), 0)$. The morphism of **nc**-spaces $f : \mathrm{pt} \to Y$ given by $f_*(V) = \mathcal{E} \otimes V$, for any $V \in C_{\mathrm{pt}} = (\mathsf{Vect}_{\mathbb{C}})$ is spherical.

(ii) Let X be smooth and projective of dimension $d+1$, and let $i : Y \hookrightarrow X$ be a smooth anti-canonical divisor in X. The Y is a d-dimensional Calabi-Yau and we have a natural spherical **nc**-morphism $f : X \to Y$ given by $f_* := i^*$, $f^! := i_*$, etc.

(iii) Let Y be a smooth projective d-dimensional Calabi-Yau. Let $i : X \hookrightarrow Y$ be a smooth hypersurface. Then we have a natural spherical **nc**-morphism $f : X \to Y$ given by $f_* = i_*$, $f^! = i^!$, and $f^* = i^*$.

Remark 4.36 The geometry of Example 4.35 **(ii)**, where X is taken to be a smooth projective Fano, and $i : Y \hookrightarrow X$ is a smooth anti-canonical divisor, can be encoded algebraically in the categories $C_X = D(\mathsf{Qcoh}(X))$, $C_Y = D(\mathsf{Qcoh}(Y))$, the functor $f_* = i^*$, and another natural triple of categories:

- the compact category $D_{\substack{\text{compact}\\ \text{support}}}(\mathsf{Qcoh}(X - Y)) = \ker(f_*)$,
- the compact category $D_{\mathrm{supp}\, Y}(\mathsf{Qcoh}(X))$ = the subcategory in $D(\mathsf{Qcoh}(X))$ generated by $i_* D(\mathsf{Qcoh}(Y))$,
- the smooth category $D(\mathsf{Qcoh}(X - Y))$ = the quotient $D(\mathsf{Qcoh}(X))/D_{\mathrm{supp}\, Y}(\mathsf{Qcoh}(X))$.

There is a similar triple of categories for the setup in Example 4.35 **(iii)**. It will be very interesting to describe the categorical data that encodes anti-canonical divisors with normal crossings or more generally the fractional anti-canonical divisor setup from section 4.3.3 **(iii)**. It seems likely that in this situation one gets a system of nested categories and functors with a "spherical" condition imposed on the whole system rather than on individual functors. This is a very interesting question that we plan to investigate in the future.

Remark 4.37 It is clear from the examples above that spherical functors give a unifying framework for handling different type of geometric pairs.

Suppose that X and Y are smooth and compact **nc**-spaces, Y is a **nc**-Calabi-Yau, $f : X \to Y$ is a spherical map, and the degeneration conjecture holds for both X and Y. In this situation we expect that the deformation theory of $f : X \to Y$ is controlled by a homotopy abelian d$(\mathbb{Z}/2)$g Lie algebra which is L_∞-quasi-isomorphic to

$$\mathrm{cone}\left(C_\bullet(Y) \xrightarrow{f^!} C_\bullet(X) \right) [1 - d]. \tag{4.6}$$

Moreover, using f_* (or $f^!$) we can build a new **nc**-space Z by taking C_Z to be the semi-orthogonal extension $C_Z = \langle C_X, C_Y \rangle$, where we set

$$\mathrm{Hom}_Z\left(C_Y, C_X\right) := 0$$
$$\mathrm{Hom}_Z\left(\mathcal{E}, \mathcal{F}\right) := \mathrm{Hom}_Y\left(f_*\mathcal{E}, \mathcal{F}\right) \quad \text{for all } \mathcal{E} \in C_X, \mathcal{F} \in C_Y.$$

We expect that the deformation theory of $f : X \to Y$ is equivalent to the deformation theory of Z and in particular that the L_∞ algebra $C^\bullet(Z)$ is quasi-isomorphic to the algebra (4.6).

Remark 4.38 We should point out that even though deformation quantization provides a conceptual bridge between the categorical framework and the geometric framework of the previous section, the actual connection between the two frameworks is tenuous at best. The source of the problem lies in the fact that the deformation quantization of general Poisson maps can be obstructed [252].

4.5 A-model framework: symplectic Landau-Ginzburg models

We already noted in Examples 4.4 and 4.8 that there are natural canonical coordinates and a Calabi-Yau variation of **nc**-Hodge structures that one can attach to the A-model on a compact symplectic manifold. An interesting open problem is to find an algebraic description of these coordinates and variation in terms of some $d(\mathbb{Z}/2)g$ Batalin-Vilkovisky algebra that is naturally attached to the Fukaya category. This question is hard and we will not study it directly here. Instead we will look at the question of finding canonical coordinates and variation in another symplectic context, i.e. for symplectic Landau-Ginzburg models, and try to get an insight into a possible algebraic formulation in that case. It will be interesting to compare our formalism with the recent work of Fan-Jarvis-Ruan [75] on the symplectic geometry of quasi-homogeneous Landau-Ginzburg potentials with isolated singularities but at the moment we do not see a direct relationship.

4.5.1 Symplectic geometry with potentials.

The objects we would like to understand are triples (Y, w, ω), where

- Y is a C^∞-manifold and ω is a C^∞-symplectic form on Y.
- $\mathsf{w} : Y \to \mathbb{C}$ is a proper C^∞-map such that there exists an $R > 0$ so that over $\{z \in \mathbb{C} | \, |z| \geq R\}$ the map w is a smooth fibration with fibers symplectic submanifolds in (Y, ω).

Similarly to the case of compact symplectic manifolds one can associate Gromov-Witten invariants to such a geometry. Specifically, if we fix $n \geq 1$, $g \geq 0$, and $\beta \in H_2(Y, \mathbb{Z})$, then we can use stable pseudo-holomorphic pointed curves in Y to define a natural linear (correlator) map

$$I^{(1)}_{g,\beta,n-1} : \quad H^\bullet(Y, \mathbb{Q})^{\otimes(n-1)} \otimes H^\bullet\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right) \longrightarrow H^\bullet(Y, \mathbb{Q}).$$

Indeed, note that Poincaré duality gives an identification

$$H^\bullet(Y, \mathbb{Q}) \cong H_\bullet(Y, Y_R; \mathbb{Q})[-\dim Y],$$

where $R > 0$ is as above and $Y_R = \mathsf{w}^{-1}(\{z \in \mathbb{C} | |z| \geq R\}) \subset Y$. Combining this identification with the isomorphism $(H^\bullet)^\vee = H_\bullet$ we see that $I^{(1)}_{g,\beta,n-1}$ will be given by a class in $H_\bullet(Y,\mathbb{Q})^{\otimes(n-1)} \otimes H_\bullet(Y, Y_R; \mathbb{Q}) \otimes H_\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Next consider the usual moduli stack $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ of stable pseudo-holomorphic maps. Here it will be convenient to assume that an almost-complex structure on Y tamed by ω is chosen in such a way that $\mathsf{w}_{|Y_R}$ is holomorphic. The stack $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ is non compact but near infinity it parameterizes only pseudo-holomorphic maps $\varphi : C \to Y$ such that $\mathsf{w} \circ \varphi : C \to \mathbb{C}$ is constant and $\mathsf{w} \circ \varphi(C) \in \mathbb{C}$ is close to infinity. Thus the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ is well defined as a class in the relative homology

$$\begin{aligned}\left[\overline{\mathcal{M}}_{g,n}(Y,\beta)\right]_{\mathrm{vir}} &\in H_\bullet\left(Y^n \times \overline{\mathcal{M}}_{g,n}, Y^{n-1} \times Y_R \times \overline{\mathcal{M}}_{g,n}; \mathbb{Q}\right) \\ &= H_\bullet(Y,\mathbb{Q})^{\otimes(n-1)} \otimes H_\bullet(Y, Y_R; \mathbb{Q}) \otimes H_\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).\end{aligned}$$

We define $I^{(1)}_{g,\beta,n-1}$ to be the map given by the relative virtual fundamental class $\left[\overline{\mathcal{M}}_{g,n}(Y,\beta)\right]_{\mathrm{vir}}$.

This collection of correlates satisfies analogues of the usual axioms of a cohomological field theory [155] but we will not discuss them here. Consider now a cohomology class

$$x = (x_2, x_{\neq 2}) \in H^\bullet(Y,\mathbb{C}) = H^2(Y,\mathbb{C}) \oplus H^{\neq 2}(Y,\mathbb{C}),$$

where $H^\bullet(Y,\mathbb{C})$ is viewed as a supermanifold over $\mathbb{C}$.

Now for every such x we define a quantum product

$$\bullet *_x \bullet : \quad H^\bullet(Y,\mathbb{C}) \otimes H^\bullet(Y,\mathbb{C}) \longrightarrow H^\bullet(Y,\mathbb{C})$$

by the formula

$$\alpha_1 *_x \alpha_2 := \sum_{m \geq 0} \sum_{\beta \in H_2(Y,\mathbb{Z})} \exp\left(\langle \beta, x_2 \rangle\right) \cdot \frac{1}{m!} I^{(1)}_{g,\beta,m+1}\big((\alpha_1 \otimes \alpha_2 \otimes \underbrace{x_{\neq 2} \otimes \cdots \otimes x_{\neq 2}}_{m \text{ times}}) \otimes 1_{\overline{\mathcal{M}}_{0,m+1}}\big).$$

Now this quantum multiplication together with the usual formulas (see Examples 4.4 and 4.8) can be used to define a decorated variation of **nc**-Hodge structures over the (conjecturally non-empty) domain in $H^\bullet(Y,\mathbb{C})$ where the series defining $*_x$ is absolutely convergent.

Remark 4.39 There are some interesting variants of this construction. For instance we can take a symplectic manifold (Y, ω) with no potential and a pseudo-convex boundary. In this situation $\overline{\mathscr{M}}_{g,n}(Y, \beta)$ is already compact, as long as $\beta \neq 0$. Also in a symplectic Landau-Ginzburg model (Y, ω, w) we can allow for w to be non-proper and instead require that its fibers have pseudo-convex boundary. Finally one can consider a symplectic Y equipped with a proper map $Y \to \mathbb{C}^k$, holomorphic at infinity and with $k \geq 2$.

4.5.2 Categories of branes Let (Y, ω, w) be a symplectic geometry with a proper potential. There are two natural categories that we can attach to this geometry: the Fukaya category of the general fiber of w, and the Fukaya-Seidel category of w. Understanding the structure properties of these categories or even defining them properly is a difficult task which requires a lot of effort and hard work. We will not explain any of these intricate details but will rather use the Fukaya and Fukaya-Seidel categories as conceptual entities. For details of the definitions and a rigorous development of the theory we refer the reader to the main sources [81, 85, 217, Sei07a]. The categories that we are interested in are:

(1) The Fukaya-Seidel category $\oplus\mathsf{FS}(Y,\omega,\mathsf{w})$ of the potential w has objects which are unitary local systems $\mathbb{V}$ on (graded) ω-Lagrangian submanifolds $L \subset Y$ such that:

- $\mathsf{w}(L) \subset (\text{compact}) \cup \mathbb{R}_{\leq 0}$;
- The restriction of L over the ray $\mathbb{R}_{\leq 0}$ is a fibration on $\mathbb{R}_{\leq -R}$ and when $z \in \mathbb{R}_{\leq 0}$, and $z \to -\infty$, we have that the fiber $L_z \subset Y_z$ is a Lagrangian submanifold in the symplectic manifold $\left(Y_z, \omega_{|Y_z}\right)$.

The morphisms between two objects $(L_1, \mathbb{V}_1)$ and $(L_2, \mathbb{V}_2)$ are defined as homomorphisms between the fibers of the local systems at the intersection points of the two Lagrangians. As usual to make this work one has to perturb one of the Lagrangians, say L_2 by a Hamiltonian isotopy to ensure transversality of the intersection. A new feature of this setup (compared to the situation of symplectic manifolds with no potential) is that the allowable isotopies are tightly controlled - they correspond to small wiggling, see Figure 7, of the tail of the tadpole-like image of the Lagrangian in $\mathbb{C}$ and a lift of this wiggling to Y given by a non-linear symplectic connection identifying the fibers of Y.

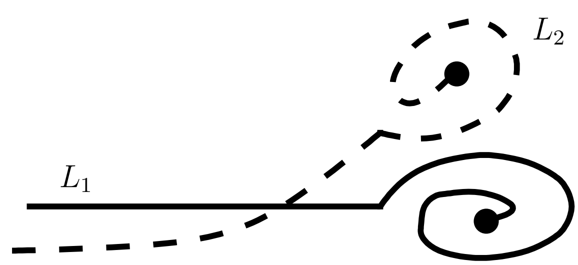


Figure 7: Tadpole-like w-images of two Lagrangian submanifolds.

The compositions of morphisms are given by correlators counting pseudo-holomorphic discs whose boundary is contained in the given Lagrangian submanifolds.

(2) The Fukaya category $\mathrm{Fuk}(Y_z)$ of a fiber $\left(Y_z, \omega_{|Y_z}\right)$ over a point $z \in \mathbb{C}$ which is not a critical value for w. The objects in this category are again pairs consisting of (graded) Lagrangian submanifolds in Y_z equipped with unitary local systems, and morphisms and compositions are defined again by maps between the fibers of the local systems at the intersection points and by counting discs. The parallel transport w.r.t. a non-linear symplectic connection on $\mathsf{w} : Y \to \mathbb{C}$ identifies symplectically all fibers $\left(Y_z, \omega_{|Y_z}\right)$ over points $z \in \mathbb{R}_{\leq 0}$ when $z \to -\infty$. We will denote any one such fiber as $(Y_{-\infty}, \omega_{-\infty})$.

Now observe that by intersecting a Lagrangian $L \subset Y$ with the fiber $Y_{-\infty}$ we get an assignment $L \mapsto L_{-\infty} := L \cap Y_{-\infty}$. We expect that this assignment can be promoted to a spherical functor (see also [Sei07a] for a similar discussion)

$$F: \ \oplus \mathsf{FS}(Y, \mathsf{w}, \omega) \longrightarrow \mathrm{Fuk}(Y_{-\infty}, \omega_{-\infty})$$

so that the associated spherical twist $\mathcal{R}_F : \mathrm{Fuk}(Y_{-\infty}, \omega_{-\infty}) \to \mathrm{Fuk}(Y_{-\infty}, \omega_{-\infty})$ categorifies the monodromy around the circle $\{z \in \mathbb{C} | \, |z| = R\}$.

In this situation one can also define relative Gromov-Witten invariants

$$J^{(1)}_{g,\beta,n-2}: \quad H^{\bullet}(Y, Y_{-\infty}; \mathbb{Q}) \otimes H^{\bullet}(Y, \mathbb{Q})^{\otimes (n-2)} \otimes H^{\bullet}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right) \longrightarrow H^{\bullet}(Y, Y_{-\infty}; \mathbb{Q}).$$

For we again use the duality $(H^{\bullet})^{\vee} \cong H_{\bullet}$ and the Poincaré duality $H^{\bullet}(Y, Y_{-\infty}; \mathbb{Q}) \cong H_{\bullet}(Y, Y_{+\infty}; \mathbb{Q})$ to rewrite $J^{(1)}_{g,\beta,n-2}$ as a class in

$$H_{\bullet}(Y, Y_{-\infty}; \mathbb{Q}) \otimes H_{\bullet}(Y, Y_{+\infty}; \mathbb{Q}) \otimes H_{\bullet}(Y, \mathbb{Q})^{\otimes (n-2)} \otimes H_{\bullet}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right).$$

This class can again be defined as a virtual fundamental class space $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ of stable pseudo-holomorphic maps. Again we can interpret the virtual class as a relative homology class:

$$\begin{aligned}
\left[\overline{\mathcal{M}}_{g,n}(Y, \beta)\right]_{\mathrm{vir}} &\in H_{\bullet}\left(Y^{n} \times \overline{\mathcal{M}}_{g,n},\, Y^{n-2} \times \left((Y^{-}_{R,\varepsilon} \times Y) \cup (Y \times Y^{+}_{R,\varepsilon})\right) \times \overline{\mathcal{M}}_{g,n}; \mathbb{Q}\right) \\
&= H_{\bullet}(Y, Y^{-}_{R,\varepsilon}; \mathbb{Q}) \otimes H_{\bullet}(Y, Y^{+}_{R,\varepsilon}; \mathbb{Q}) \otimes H_{\bullet}(Y, \mathbb{Q})^{\otimes (n-2)} \otimes H_{\bullet}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right) \\
&= H_{\bullet}(Y, Y_{-\infty}; \mathbb{Q}) \otimes H_{\bullet}(Y, Y_{+\infty}; \mathbb{Q}) \otimes H_{\bullet}(Y, \mathbb{Q})^{\otimes (n-2)} \otimes H_{\bullet}\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right),
\end{aligned}$$

and so it gives the desired map $J^{(1)}_{g,\beta,n-2}$.

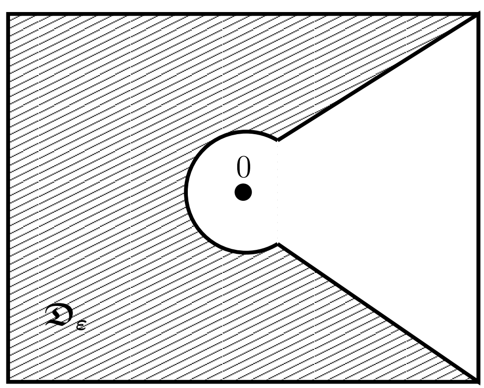


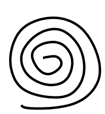
Figure 8: The domain $\mathfrak{D}_{\varepsilon}$.

Here $1 \gg \varepsilon > 0$, and $Y^{\pm}_{R,\varepsilon} = \mathsf{w}^{-1}(\pm \mathfrak{D}_{\varepsilon})$, where $\mathfrak{D}_{\varepsilon} \subset \mathbb{C}$ is the domain given by (see Figure 8)

$$\mathfrak{D}_{\varepsilon} := \left\{ z \in \mathbb{C} \;\middle|\; |z| \geq R \text{ and } \operatorname{Arg} z \in \left(\frac{\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon \right) \right\}.$$

Again the relative invariants $J^{(1)}_{g,\beta,n-2}$ give rise to a quantum multiplication and through the usual formulas from Examples 4.4 and 4.8 we again get a decorated variation of **nc**-Hodge structures over a (conjecturally non-empty) domain in $H^\bullet(Y,\mathbb{C})$ with fiber $H^\bullet(Y,Y_{-\infty})$.

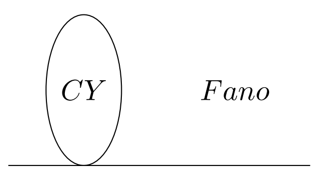
4.5.3 Mirror symmetry. In conclusion we systematize all the objects introduced above in a mirror table (see also [13]) describing the corresponding A and B-model entities in parallel:

<table>
<tr><th>invariants</th><th>A-model</th><th>B-model</th></tr>
<tr><td>geometry</td><td>a triple (Y,w,ω) where:
$\mathsf{w} : Y \to \mathbb{C}$ is a proper C^∞-map (Y,ω) is symplectic with $c_1(T_Y)=0$</td><td>a pair $Z\subset X$ where:
X is smooth projective, and $Z \subset X$ is a smooth anticanonical divisor</td></tr>
<tr><td>cohomology</td><td>$\left.\begin{matrix} H^\bullet(Y,\mathbb{C}) \\ H^\bullet(Y,Y_{-\infty};\mathbb{C}) \\ H^\bullet(Y_{-\infty},\mathbb{C}) \end{matrix}\right\}$ variations of **nc**HS over a domain in $H^\bullet(Y,\mathbb{C})$</td><td>$\left.\begin{matrix} H^\bullet(X-Z,\mathbb{C}) \\ H^\bullet(X,\mathbb{C}) \\ H^\bullet(Z,\mathbb{C}) \end{matrix}\right\}$ variations of **nc**HS over a domain in $H^\bullet(X-Z,\mathbb{C})$</td></tr>
<tr><td rowspan="4">categories</td><td>$\begin{matrix} \mathrm{Fuk}(Y_{-\infty}) \\ \Big\uparrow{\scriptstyle F} \\ \oplus\mathsf{FS}(Y) \end{matrix}$: $\mathrm{Fuk}(Y_{-\infty})$ is a CY category and F is a spherical functor</td><td>$\begin{matrix} D(Z) \\ \Big\uparrow{\scriptstyle F} \\ D(X) \end{matrix}$: $D(Z)$ is a CY category and F is a spherical functor</td></tr>
<tr><td>The part of $\oplus\mathsf{FS}(Y)$ consisting of Lagrangians fibered over where the circle is of radius $R \gg 0$.</td><td>$D_{\mathrm{supp}\, Z}(X)$: a full compact (non smooth) subcategory in $D(X)$</td></tr>
<tr><td>The part of $\oplus\mathsf{FS}(Y)$ consisting of compact Lagrangian sumanifolds in Y</td><td>$D_{\substack{\text{compact}\\ \text{support}}}(X-Z)$: a full compact (non smooth) subcategory in $D(X)$</td></tr>
<tr><td>The wrapped $\oplus\mathsf{FS}$ category: the Hom space between $(L_1,\mathbb{V}_1)$ and $(L_2,\mathbb{V}_2)$ is the sum of $\mathrm{Hom}(\mathbb{V}_1,\mathbb{V}_2)_x$, $x\in L_1\cap L_2$, and L_2 is deformed so that $\mathsf{w}(L_2)$ becomes a spiral:

wrapped infinitely many times</td><td>$D(X-Z)$: a smooth (non compact) category</td></tr>
</table>

Interpretations of spectra.

1 Categorical linear systems

The studies of homological mirror symmetry (HMS) as correspondence of Lefshetz pencils was initiated in [144] as part of the general theory of categorical linear systems. In this chapter, we look at the monodromy of these linear systems. We utilise these monodromies by introducing a new notion of noncomutative spectrum. We will use the setup and the notations from [144]. We start with a pencil where the fibers are CY varieties and the global pencils constitute mirrors of Fano manifolds. We have the following category diagram:



$$F(CY) \to F(Fano)$$

Here $F(CY), F(Fano)$ are the corresponding Fukaya-Seidel categories. $\operatorname{Im}\Phi(F(CY)) = \mathcal{A}$ is a localization category $F(CY)/\sim$. (Using HMS we can use $D^b(X)$ - the category of coherent sheaves on algebraic varieties X.)

This localization category has a filtration:

$$\mathcal{A} \supset \mathcal{F}_{\lambda_1} \supset \cdots \supset \mathcal{F}_{\lambda_n}$$

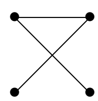
where:

- λ_i are the asymptotics of limiting stability conditions.

- $Z = z^{\lambda_i}(\cdots)$
- $\mathcal{F}_{\lambda_i} = \{F \text{ s.t. } Z(F) = z^{\lambda_i}(\cdots)\}$
- λ_i are also the asymptotics of the PDE

$$\left(\frac{\partial}{\partial u} + u^{-2}K + u^{-1}G\right)$$

The above filtration can also be seen as the monodromy of the perverse sheaf of categories over the skeleton. Following [144] we think of the category as a perverse sheaf of categories over lagrangian skeleton. In the diagram bellow we describe our findings in [144].



Homotopy of Skeleta $\updownarrow$ dim of CAT $\longleftrightarrow$ Monodromy of Perverse Sheaves $\updownarrow$ Asymptotics of Stab Conditions $\updownarrow$ Orlov Spectra

The main idea in current chapter is to give an interpretation of the above λ_i filtration as a noncommutative spetrum and a spectrum of Landau-Ginzburg (LG) models. We use the theory of LG models as generalized theory of singularity.

The above considerations lead to birational invariants, which will appear in more details in [142], [145]. (For definitions and general theory of LG models and HMS we refer to [143].)

We will base our birational considerations on the following major notions and ideas:

1. **Quantum spectrum**. The quantum spectrum is defined in [142]. Let $K\cdot$ be the quantum multiplication by canonical class. It defines the following splitting of cohomology:

$$\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}.$$

   Here λ_i are the eigenvalues of $K\cdot$. We call these eigenvalues *quantum spectrum*. The main theorem proven in [142] is:

   MAIN THEOREM: The splitting $\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}$ is a birational invariant.

2. **Noncommutative spectrum**. The noncommutative spectrum is defined in [142].

   In the current chapter we extend these ideas and give some examples.

A) We build analogues with low dimensional topology and give several new directions for research.

B) We extend the definition of a noncommutative spectrum to multispectra. Possible applications are discussed.

Our considerations are only the tip of the iceberg. We propose a correspondence between nonrationality over algebraically nonclosed fields and complexity of the discriminant loci of the moduli space of LG models. We will consider some arithmetics applications in Section 3. In fact one can define several different spectra.

In addition to the **quantum spectrum** mentioned above, one can define several other spectra:

- **Noncommutative spectrum;**
  defined by the asymptotics of the quantum equation.
- **Givental spectrum;**
  defined by the solutions of the Givental's equation.
- **Spectrum of LG model – multiplier ideal sheaf;**
  defined as the Steenbrink spectrum of a new singularity theory of the LG model.
- **Asymptotics of stability conditions – stability spectrum;**
  defined as asymptotics of limiting stability conditions.
- **Serre dimension of the Kuznetsov's component;**
  defined as a categorical dimension.
- **Arnold-Varchenko-Steenbrink spectrum of the affine cone.**
  defined as the classical spectrum of the affine cone singularity over X.
- **R-charges** – the assymptotics of RG flow – the same as asymptotics of Kähler-Ricci flow – see Section 6.

We will discuss relations among some of them. Understanding the complete scope of relations is an intriguing problem. We initiate the study of these connections in this chapter. We will develop this connections in upcoming papers [140], [108].

C) We also propose a parallel between the existence of Kähler-Einstein metrics and the top number of the noncommutative spectra. Recall that

$$lct(X,G) = \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ l.c.s. } \forall G \text{ inv. } D\}$$

We note the following parallel:

nonrationality of (X,G) orbifold	$\exists$ of K.E. metric on (X,G)
$\delta > \dim X - 2$ X is not rational δ is lct for sing	$lct(X,G)$ $> \dfrac{\dim X}{\dim X + 1}$

In the above table lct is the log canonical treshold.

We take this parallel further:

D) We connect the noncommutative spectra with elliptic genus and conformal field theory. We connect orbifoldization of elliptic genus with spectra of singular varieties. This leads to a categorical interpretation of Birkar's boundness theorem. We propose the idea of categorical resolution and "boundness" of conformal field theories – the central charges correspond to the noncommutative spectra.

As a consequence we propose a parallel between Zamolodchikov's c-theorem and uppersemicontinuity condition of noncommutative spectra.

We will call the monotonicity of the highest number of the spectrum *uppersemicontinuity*. In other words, the highest number of the spectrum is decreasing monotonically when moving from the boundary of Frobenius manifold to its general point.

The chapter is organized as follows. We explain the general theory in Section 2. The Fano applications are considered in Section 2. The arithmetics applications are considered in Section 3. The parallel with 3-dimensional topology are discussed in Section 4. The extension to multispectra is discussed in Section 5. In Section 6, we consider the connection of spectra with elliptic genus. We make a connection between Birkar's theory and the conformal field theories.

2 Noncommutative spectra

In this section we introduce the idea of noncommutative spectra – an idea which belongs to M. Kontsevich. We describe new birational invariants and describe some easy applications.

2.1 Definitions of quantum and nc spectra

Let X be a projective algebraic variety over $\mathbb{C}$, with a given ample line bundle. The Gromov-Witten invariants in genus zero define a potential $\mathcal{F}_0$: formal series on $H^\bullet(X)$ with coefficients in $\mathbb{Q}[[T]]$ – see e.g. [143]. We briefly recall two conjectures (see e.g. [142]).

1. First we have:

Conjecture 2.1 *$\mathcal{F}_0$ is convergent for a point $\gamma \in H^\bullet(X)$ and for $T \in \mathbb{C}$, both close to 0.*

2. Assuming Γ-conjecture (see e.g. [143]) we get that **nc** Hodge structures are parametrized by a domain

$$M \subset H^\bullet(X, \mathbb{C})/H^2(X, 2\pi i\mathbb{Z}),$$

which is a meromorphic connection on the trivial bundle over u-plane $\mathbb{C}_u$ with fiber $H^\bullet(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G$$

(Recall that the Γ-conjecture gives a lattice, hypothetically compatible with Stokes filtrations along rays at $u \to 0$. For more details see [143].)

We define the operator $K = K(\gamma)$ as the quantum product with $c_1(T_X)+\sum_{i\neq 2}(2-i)\gamma_i$. It depends on the point $\gamma = (\gamma_i \in H^i(X))_{i=0,\dots,2\dim_{\mathbb{C}} X}$ in Frobenius manifold $\mathcal{M}$. We also define the operator G as a constant operator given by $G_{|H^i(X)} = \frac{i-\dim_{\mathbb{C}} X}{2} \cdot id_{H^i(X)}$.

We use the example bellow to introduce and demonstrate two important definitions. Let X be a smooth 3-dimensional cubic in $\mathbb{P}^4$. Operators K, G on 4-dimensional space $H^{\text{even}}(X) = \oplus_{i=0}^{3} H^{2i}(X)$ with the basis being powers of the hyperplane section, at point $\gamma = 0 \in \mathcal{M}$, are:

$$K = 2 \cdot \begin{pmatrix} 0 & 6 & 0 & 36 \\ 1 & 0 & 15 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

Solutions of the quantum equation

$$\left(\frac{d}{du} + \frac{1}{u^2} K + \frac{1}{u} G \right) \psi(u) = 0 \tag{2.1}$$

grow at $u \to 0$ as

$$\sim u^{-\frac{5}{6}}, \sim u^{-\frac{1}{6}}.$$

Definition 2.2 ***Quantum spectrum*** *is the spectrum of K, a finite subset $\{z_a\} = Spec_X \subset \mathbb{C}$ (depends on the point γ in $\mathcal{M}$).*

Definition 2.3 ***Noncommutative spectrum****: The asymptotics of the sub-exponential growth solutions of the equation* 2.1 *above form the* ***noncommutative spectrum*** *or* ***nc spectrum****.*

In what follows we will denote by δ minus two times the lowest number of **noncommutative spectrum**. In the above example

$$\delta = \frac{5}{3}.$$

Consider a purely even affine submanifold $\mathcal{M}^{\text{alg}} \subset \mathcal{M}$, given by deformations of quantum product by linear combinations of algebraic classes $H^{\text{alg}}_{\mathbb{Q}}(X) \subset H^{\text{even}}(X, \mathbb{Q})$.

Conjecture 2.4 *For any point in $\mathcal{M}^{alg}$ and a choice of disjoint paths from ∞ to points of the corresponding quantum spectrum (see Figure* 1*), we obtain a semi-orthogonal decomposition $\boldsymbol{D}^b(Coh(X)) = \langle \mathcal{C}_1, \dots, \mathcal{C}_r \rangle$ where r is the number of elements of the spectrum.*

All categories $\mathcal{C}_1, \dots, \mathcal{C}_r$ are saturated (i.e. smooth and proper), equal to local Fukaya-Seidel categories for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists.

Example 2.5 *1. $X = \mathbb{P}^n$, the* ***quantum spectrum*** *is $\mu_{n+1} = \{z \in \mathbb{C} \mid z^{n+1} = 1\}$ (for some point in $\mathcal{M}$)*

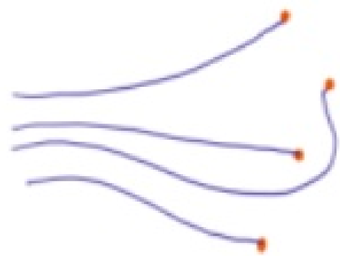
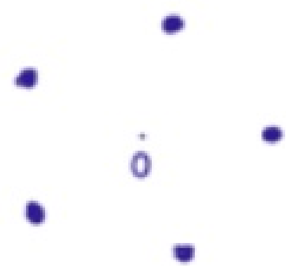


Figure 1: Gabrielov paths *(Red dots correspond to eigenvalues of quantum multiplication.)*



This gives $\boldsymbol{D}^b(CohX) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$.

2. *Conjectural blow-up formula: If* $\widetilde{X} = Bl_Y(X)$ *where* $Y \subset X$ *is a smooth closed subvariety of codimension* $m \geq 2$*, then the* ***quantum spectrum*** $Spec_{\widetilde{X}}$ *looks like*

with $(m-1)$ shifted copies of $Spec_Y$ around one copy of $Spec_X$. (Here the blue dots correspond to eigenvalues of quantum multiplication added after blow ups.)

3. *If X is a Calabi-Yau manifold or a manifold of general type the **quantum spectrum** is just a point.*

4. *The above considerations lead to the following theorem proven in [142]:*

   MAIN THEOREM*: The splitting $\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}$ is a birational invariant.*

2.2 Dimension theory

In this section, we introduce Serre dimension which (with some exceptions) is equal to the number δ from the noncommutative spectrum. We see that sometimes elementary pieces $\mathcal{C}_a = \mathcal{C}_{z_a}$, $z_a \in Spec_X$ (could be combined as some points of the spectrum collide), are themselves equivalent to derived categories of coherent sheaves on some varieties, of certain dimensions $\leq \dim X$.

In general, for a saturated category $\mathcal{C}$ one can define its **Serre dimension** [74]

$$\dim_{\text{Serre}} \mathcal{C} := \lim_{|k| \to +\infty} \left\{ \frac{i}{k} \mid Ext^i(Id_{\mathcal{C}}, S^k_{\mathcal{C}}) \neq 0 \right\} \subset \mathbb{R}.$$

Here $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the Serre functor [35]:

$$Hom_{\mathcal{C}}(E, F)^{\star} = Hom_{\mathcal{C}}(F, S_{\mathcal{C}}E), \quad \forall E, F \in Ob(\mathcal{C}).$$

In general, Serre dimension could be an empty set, or an interval.

For categories $\boldsymbol{D}^b(Coh(X))$, it is exactly the dimension $\dim X \in \mathbb{Z}_{\geq 0}$. For a fractional Calabi-Yau category $S^k_{\mathcal{C}} \sim [n]$, the Serre dimension is equal to Calabi-Yau dimension $\frac{n}{k}$, hence fractional.

Example 2.6 *Fukaya-Seidel category of $Y = \mathbb{C}_x$, $W = x^d$, $d \geq 2$: $\dim_{\mathit{Serre}} = 1 - \frac{2}{d}$.*

Let us assume that (H, ∇) is a connection with second order pole and regular singularity (i.e. all solutions have polynomial growth). Then the order of growth defines a filtration by subbundles, preserved by connection ∇, the indices form the **subexponential growth spectrum = nc spectrum**.

ESSENTIAL EXAMPLE.

Consider the hypersurface $X \subset \mathbb{P}^n$ of Calabi-Yau/general type. The connection on the image of $H^\bullet(\mathbb{P}^n)$ in $H^\bullet(X)$ under restriction map, i.e. the span of powers of $c_1(\mathcal{O}(1)) \in H^2(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G, K = \text{ classical product with } c_1(T_X)$$

The **nc spectrum** is

$$(-\dim X/2, -\dim X/2, \ldots)$$

for X a manifold of general type and so

$$\delta = \dim X.$$

For X a Calabi-Yau manifold **nc spectrum** is

$$(-\dim X/2, 1-\dim X/2, \ldots, +\dim X/2)$$

and $\delta = \dim X$. Similar behavior happens for Calabi-Yau when we replace the multiplication by $c_1(T_X) = 0$, by the multiplication by an inhomogeneous class $c_1(T_X) + \sum_{i\neq 2}(2-i)\gamma_i$, $\gamma_i \in H^i(X)$, $i \in 2\mathbb{Z}$.

2.2.1 More general example

Let us consider a weighted projective space $\mathbb{P}^{\omega_0,\ldots,\omega_n}$ and generic complete intersection X of hypersurfaces of degrees $d_1, \ldots, d_m$. In what follows we investigate the connection between **nc spectrum**, **Givental spectrum** and **Steenbrink spectrum** in this example.

Recall that such a complete intersection is called **well-formed** iff (here unions are understood **with multiplicities**)

$$(\star) \qquad \bigcup_i \{\frac{1}{\omega_i}, \ldots, \frac{\omega_i - 1}{\omega_i}\} \subset \bigcup_j \{\frac{1}{d_j}, \ldots, \frac{d_j - 1}{d_j}\}$$

We call the numbers from $\star$ **Givental spectrum**.

Well formed X is smooth, and does not meet singularities of $\mathbb{P}^{\omega_0,\ldots,\omega_n}$. Let us assume that X is a Fano variety, i.e. $\sum_i \omega_i > \sum_j d_j$.

We define the Givental's hypergeometric operator:

$$\prod_i \omega_i^{\omega_i} \cdot \partial^{\dim X} - \prod_j d_j^{d_j} \cdot q \cdot \frac{\prod_j (\partial + \frac{1}{d_j}) \cdots (\partial + \frac{d_j - 1}{d_j})}{\prod_i (\partial + \frac{1}{\omega_i}) \cdots (\partial + \frac{\omega_i - 1}{\omega_i})}, \quad \partial := q\frac{d}{dq}, u = c\cdot q^{-\frac{1}{\sum_i \omega_i - \sum_j d_j}}$$

The **nc spectrum** of the Laplace operator of the Givental's hypergeometric operator is:
$-\frac{\dim X}{2}$ + {complement in $(\star)$} $\cdot (\sum_i \omega_i - \sum_j d_j) \to$ numbers $s_0 \leq s_1 \leq \cdots$.
The **adjusted Steenbrink spectrum** is:
$(s_0, s_1 + 1, s_2 + 2, \ldots)$.
The adjusted Steenbrink spectrum is symmetric with center at 0.

Example 2.7 *Let use consider complete intersection of two hypersurfaces of degree $d_1 = 2$, $d_2 = 4$ in $\mathbb{P}^6 = \mathbb{P}^6(1,1,1,1,1,1,1)$.*

*The **growth spectrum** is*

$$(-\frac{7}{4}, -\frac{6}{4}, -\frac{6}{4}, -\frac{5}{4})$$

In other words the solutions of the quantum equation grow as

$$u^{-\frac{7}{4}}, \log(u)u^{-\frac{6}{4}}, u^{-\frac{6}{4}}, u^{-\frac{5}{4}}$$

.

*Adding $(0,1,2,3)$ to **nc spectrum** we obtain **adjusted Steenbrink spectrum**:*

$$(-\frac{7}{4}, -\frac{1}{2}, +\frac{1}{2}, +\frac{7}{4})$$

.

2.3 Some computational tools

We briefly discuss some methods for calculations. We start with:

Theorem 2.8 (Saito's Theorem) *([226]) $P_f(t) = Sp_f(t)$*

Here $P_f(t) = \sum_\alpha (\dim J_\alpha) t^\alpha$ is the Poincare series and $Sp_f(t) = \sum_i (n_i.t^i)$ - is the spectrum polynomial and n_i - are the multiplicity of spectral number.

Recall that for $f(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = \lambda f(x_1, \dots, x_n)$ we define weight
$wt.(x_1^{a_1}, \dots, x_n^{a_n}) = \sum_{i=1}^n (1 + a_i)w_i$

Example 2.9 *Let us look at the example of three dimensional cubic from a new point of view:*
$f(x_1, \dots, x_5) = x_1^3 + \dots + x_5^3$
$P_f(t) = t^{\frac{5}{3}} + 5t^2 + 10t^{\frac{7}{3}} + 5t^3 + t^{\frac{10}{3}}$
$\delta = \frac{10}{3} - \frac{5}{3} = \frac{5}{3}$

Let us denote by $Cone(X)$ the cone over a hypersurface X and C is the Fukaya-Seidel category associated with the most singular fiber of the LG model of X. By Orlov's theorem we have $D^b(Cone(X/G)) = C$.

Denote by S_l the lowest number of the Steenbrink spectrum and by S_h the highest number of the Steenbrink spectrum for $Cone(X/G)$. An A-side conjectural version of Orlov's theorem suggests:

Conjecture 2.10 *The Steenbrink spectra of $Cone(X)$ determines noncommutative spectrum associated with X. The following identity holds*

$$\delta = S_h - S_l.$$

Let $\mathcal{C}$ be a Calabi-Yau category s.t. Serre functor satisfies $S^a = [b]$.
$HH_\bullet(\mathcal{C}) = \oplus HH^i(\mathcal{C})[\delta]$

Definition 2.11 *The homomorphism*

$$\epsilon : (Q \times \mathbb{Z}_2) \to Aut(\mathcal{C})$$

defines a categorical covering. The covering structure is recorded by multiplication in the A_∞.

In the example 2.8 we get $t^{\frac{10}{3}}$, $t^{\frac{5}{3}}$ define $\frac{10}{3} - \frac{5}{3}$, which produces degree of a covering.

Example 2.12 $x_1^4 + \dots + x_5^4$*. We consider this hypersurface as an affine cone. We compute the Poincare polynomial and obtain:*
$P_f = t^{\frac{5}{4}} + \dots + t^{\frac{15}{4}} \Rightarrow \delta = \frac{15}{4} - \frac{5}{4}$

Example 2.13 $x_1^3 + \cdots + x_5^3$. *We consider this hypersurface as an affine cone. Here we can compute the Bernstein polynomial*

$b_f(t) = (t+1)(t+2)(t+3)(t+\frac{5}{3})(t+\frac{7}{3})(t+\frac{8}{3})(t+\frac{10}{3})$

and obtain:

$\delta = \frac{10}{3} - \frac{5}{3}$

2.4 New nonrationality results

In this section we record the results of our method and compare them with already known results. We use the simplest of invariants – δ. We hope that other numbers of the noncommutative spectrum can be used as well. In fact it seems that these numbers mirror classical theory of multiplier ideal sheaves and characterize the stratification of the base loci of the anticanonical system for Fano's.

We have defined

$$\delta = dim(X) - 2(N-d)/d$$

As an immediate consequence we get in [142].

Theorem 2.14 1. *Let X be a Fano smooth hypersurface of degree d in $\mathbb{P}^{5-1}$ such that*

$$d > 5/2.$$

Then X is not rational.

2. *Let X be a Fano smooth hypersurface of degree d in $\mathbb{P}^{6-1}$ such that*

$$d \geq 6/2$$

and $H^{2,2}(X,\mathbb{Z}) = \mathbb{Z}$. Then X is not rational.

3. *Let us assume uppersemicontinuity condition. Let X be a Fano smooth hypersurface of odd dimension and of degree d in $\mathbb{P}^{N-1}$ such that*

$$d > N/2$$

Then X is not rational.

4. *Let X be a Fano smooth hypersurface of even dimension $k = (N-2)/2$ and of degree d such that*

$$d > N/2$$

and $H^{k,k}(X,\mathbb{Z}) = \mathbb{Z}$. Then X is not rational.

We briefly describe the idea of the proof.

Proof.

The above formulae is equivalent to $\delta > dim(X) - 2$.

1. $dim(X) = 3$ Assume that X is rational so it is obtained via sequence of blow ups and blow downs with centers curves.

   According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to 1.

   Our **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2 = 1.$$

   - a contradiction.

2. $dim(X) = 4$. Assume $\delta > 2$. The fact that $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ ensures that $\delta > 2$ stays unchanged under deformations. Assume that X is rational so it is obtained via sequence of blow ups and blow downs with centers points, surfaces, curves.

   According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to 2.

   The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2 = 2.$$

   - a contradiction.

   The case $d = 3$, $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ will be treated in [142]. Let us briefly mention the idea. We have a splitting

$$\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}.$$

   With the exception of one all of these H_{λ_i} are one dimensional. The high dimensional one has a symmetric noncommutative Hodge structure. With 20 dimensional space of deformation this noncommutative Hodge structure cannot come from a commutative surface.

3. $dim(X) = N - 2$, $N - 2$ is odd. In this case $\delta > dim(X) - 2$.

   Assume that X is rational so it is obtained via sequence of blow ups and blow downs.

   According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to $dim(X) - 2$. According to uppersemicontinuity these asymptotics can only go down. The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2.$$

   - a contradiction.

4. $dim(X) = N - 2 = 2k$, $N - 2$ is even $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$. In this case $\delta > dim(X) - 2$. The fact that $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$ ensures that $\delta > dim(X) - 2$ does not go down.

   Assume that X is rational so it is obtained via sequence of blow ups and blow downs.

   According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to $dim(X) - 2$. According to uppersemicontinuity these asymptotics can go only down. The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2.$$

   - a contradiction.

□

Similarly we have [142].

Theorem 2.15 *Let X be a smooth Fano complete intersection of hypersurfaces of degrees $d_1, \ldots, d_m$ in $\mathbb{P}^N$. Denote by d_t the sum $d_1 + \cdots + d_n$ and by d_m the minimal degree.*

In this case the Arnold number (the largest number of the noncommutative spectrum) is equal to:

$$\delta = dim(X) - 2((d_t - d_m)/d_m)$$

.

1. *Let X be 3 dimensional and $\delta > 1$. Then X is not rational.*

2. *Let X be 4 dimensional, $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ and $\delta > 2$. Then X is not rational.*

Let us assume uppersemicontinuity condition.

3. *Let X be of odd dimension and $\delta > dim(X) - 2$. Then X is not rational.*

4. *Let X be of even dimension $2k$, $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$ and $\delta > dim(X) - 2$. Then X is not rational.*

The same result works for well formed complete intersection in weighted projective spaces. The formulae for δ is similar:

$$\delta = \dim X - 2\frac{\omega_{\text{sum}} - d_{\text{sum}}}{d_{\max}}, \quad \omega_{\text{sum}} := \sum_j \omega_j \text{ for } \mathbb{P}^{\omega_0,\ldots,\omega_n}$$

3 Application to Arithmetics

The GW invariants can be defined over algebraically nonclosed fields L. Therefore the techniques of noncommutative spectrum can be used to investigate nonrationality over algebraically nonclosed fields

L. Of course changing the fields does not change the GW invariants but it changes algebraic cycles. Changing algebraic cycles affects deformations of LG models and as a result the spectrum of quantum multiplication by the canonical class. In this case we do not need an uppersemicontinuity - the restriction on deformation comes from algebraic cycles.

Recall the example from the introduction - the two dimensional cubic: $X : X_0^3 + \cdots + X_3^3 = 0$. Consider X over algebraically nonclosed field L s.t. $\operatorname{Pic} X_L = 1$. After analyzing the Sarkisov links we conclude that X is not rational.

We will look at this example from the point of view of the spectrum. We begin with:

Theorem 3.1 *Let X be a Fano stack of dimension at most 4 over a field L such that image of $CH(X)$ in $\sum_i H^i(X,\mathbb{Z})$ is generated by powers of anticanonical class. Assume that Arnold constant (the highest number in the spectrum) is bigger than $dim(X) - 2$. Then X is not rational.*

The same theorem works in the case when dimension of X is greater than four but with the assumption of uppersemicontinuity condition.

Proof. We give a proof under assumption of an isomorphism between the quantum cohomologies and Jacobian ring proven in many cases. The quantum multiplication by the canonical class K corresponds to multiplication of the class of W.

$$\begin{array}{ccc} QH(H^r) & \cong & \text{Jac}(W) \\ \text{multi } K & & \text{mult by } W \end{array}$$

$$\begin{array}{ccc} QH & \cong & \text{Jac}(W) \\ \cup & & \cup \\ \text{subring} & & \text{subring} \\ \text{generated by } K & & \text{generated by } W \end{array}$$

$$\begin{array}{ccc} W \text{ the LG} & \longrightarrow & P\text{polynomial of} W \\ \text{counterpart of } K & & W + P(W) \\ & & \text{all deformations} \\ & & \text{have the same critical values} \\ & & \text{as } W \end{array}$$

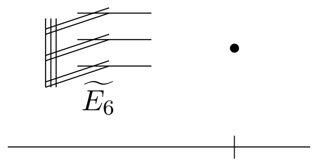
It follows that the spectrum of the most singular fiber of W does not go down since this most singular fiber does not split further under deformations. So we have $\delta > \dim X_L - 2 = 2$.

From another point the main assumption and the fact that we blow up points, curves and surfaces implies that $\delta = 2$ - a contradiction. In the case of dimension higher than 4 the proof is the same.

□

We return to the case of cubic surface. We assume existence of a point in X_L over L. Its Landau-Ginzburg models is:

$$w = \frac{(x+y+1)^3}{xy} \text{ for cubic}$$



the deformation
does not change

If the $PicX_L = \mathbb{Z}$ then W have only two singular fibers.
We compute:

$$\delta = 2 - 2\frac{4-3}{3} = \frac{4}{3}$$
$$\Rightarrow X \text{ is not rational}$$

Since the $\operatorname{Pic} X_L = \mathbb{Z}$ the deformation of W is restricted so we cannot morsify and δ does not go down to 0. So X_L is not rational.

We move to considering a cubic with $\operatorname{Pic} X_L = \mathbb{Z} + \mathbb{Z}$:

1. In the case $\operatorname{Pic} X_L = \mathbb{Z}+\mathbb{Z} \Rightarrow$ we get a conic bundle with 5 singular fibers. By Noether formulae:

   $$8 - S = k^2 = 3,$$

   so we have 5 singular fibers. (The classical Iskovskikh criteria $|2K_{\mathbb{P}^1} + S| = |-4p + 5p| \neq \emptyset$ gives nonrationality.)

   We will use spectrum in order to compute nonrationality. We compute the Bernstein polynomial for a cubic as an affine cone with a singularity at zero.

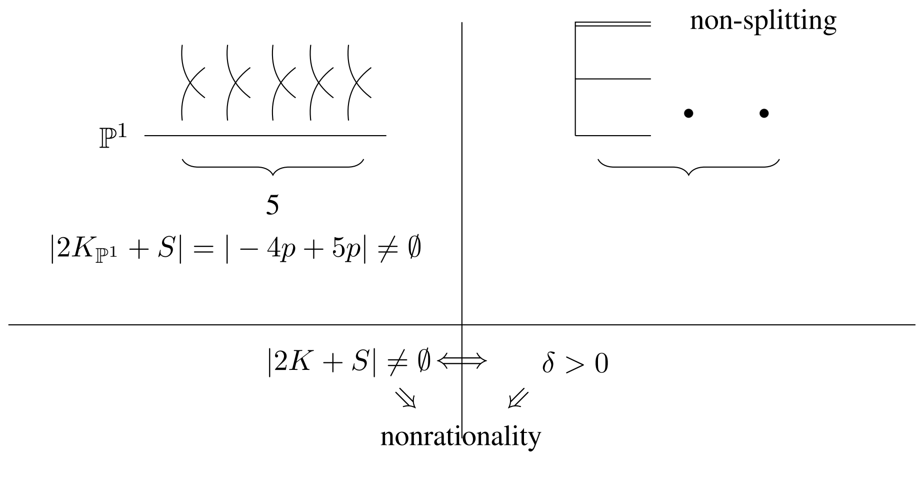
   We have $8 - C = 3$. $C = 5$ pts.

   $|2K + C| = |-4 + 5| = \mathcal{O}_{\mathbb{P}^1}(1) \neq \emptyset$

   $f = a^5x^2 + b^5y^2 + c^5z^2$

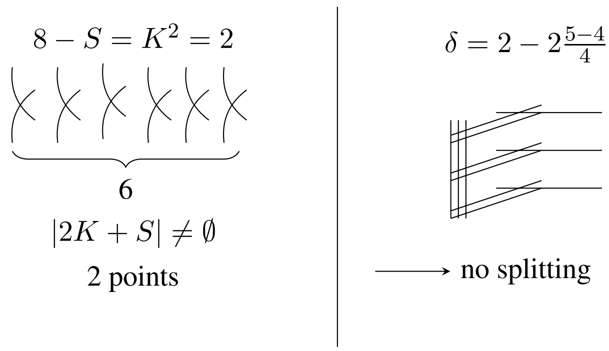
   $f = (s+1)^2(s+2)^2(s+\frac{3}{2})^2 \cdots (s+\frac{3}{10})$

   So $\delta = \frac{3}{2} - \frac{3}{10} \neq 0$ and X_L is nonrational.

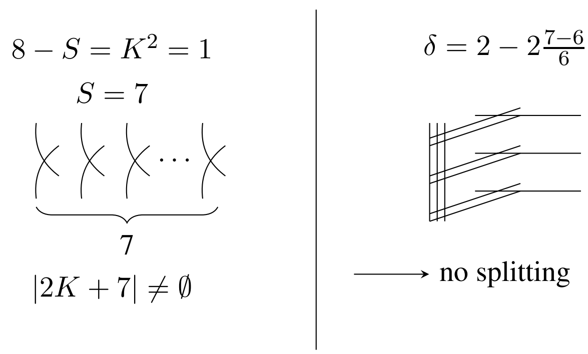


2. We consider del Pezzo surface $X_L =$ of degree 4 in $\mathbb{P}^3(1,1,1,2)$ with $\operatorname{Pic} X_L = \mathbb{Z} + \mathbb{Z}$ It is a conic bundle with 6 singular fibers. (The classical Iskovskikh criteria $|2K_{\mathbb{P}^1} + S| = |-4p+6p| \neq \emptyset$ gives nonrationality.)

   As before we use the Bernstein polynomial to show that $\delta > 0$ and X_L are not rational.



3. Consider del Pezzo surface X_L of degree 6 in $\mathbb{P}(1,1,2,3)$.



   As before we use the Bernstein polynomial to show that $\delta > 0$ and X_L are not rational.

The above observations suggest the following conjecture.

Conjecture 3.2 *Let X_L be a conic bundle over $\mathbb{P}^2$ (or another rational surface). Assume that the following holds:*

$$\begin{tikzcd} |2K+S| \neq \emptyset \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \textit{nonsplitting} \arrow[dl, leftrightarrow] \\ & \delta > dim(X_L) - 2 & \end{tikzcd}$$

Then X_L is not rational.

Let us consider a stack X/G. In this case the GW invariant of X are different from the ones of X/G. From another point the new contributions to cohomologies do form as twisted sectors which do not interact with the quantum span of the anticanonical divisor.

We denote the cohomologies associated to twisted sectors by $H_{\gamma_1,...} + \cdots, H_{\gamma_k}$. We have the following splitting of quantum cohomologies.

$$QH(X)^G = H + H_{\gamma_1} + \cdots H_{\gamma_k}$$

It leads to the following conjecture.

Conjecture 3.3 *Let X/G be a stack defined over a field L such that the image of $CH(X)$ in $\sum_i H^i(X,\mathbb{Z})$ is generated by powers of anticanonical class.*

Assume that $\delta > dim(X/G) - 2$. Then X/G is not rational.

The proof is very similar to the proof of the previous theorem. As before we have:

$$\begin{array}{ccc} QH = H + H_{\gamma_1} + \cdots H_{\gamma_k} & \longrightarrow & \text{Jac}(W_m) + J_{\gamma_1} + \cdots J_{\gamma_k} \\ \langle 1, K(1)_1 \rangle \text{ deformed} & \cong & \langle W_m \rangle + P(W_m) \\ & & = \text{no new eigenvalues} \end{array}$$

Here we denote by W_m the potential modified by the contributions of the age factors. As before we do not have further splitting of the cohomology and the inequality $\delta > dim(X/G) - 2$ implies nonrationality.

We will look at some examples of del Pezzo stacks.

Using this theorem we consider several examples of del Pezzo stacks - all hypersurfaces in weighted projective $\mathbb{P}^3$. Consider the case of weights: $3, 3, 5, 5$ and a hypersurface of degree 15. In this case $\delta = 2 - 2(16-15)/15 = 28/15 > 0$ so we have nonrationality. We can conpute the spectrum applying theorem 5.5. Using Singular we compute the Steenbrink spectrum of $Cone(X)$ - $(0,1), \ldots, (28/15, 1)$. So $\delta = 48/15$. We obtain nonrationality.

Remark 3.4 *Observe that choice of the field L and the condition $Im(CH \to H) = \langle 1, K(1), K^2(1), \cdots \rangle$ are essential. Without these assumptions the most singular fiber of W_m splits to singularities A_4, A_2, A_2 and further which makes $\delta = 0$.*

Similarly consider the weights: $3, 5, 7, 11$ and a hypersurface of degree 25. The Steenbrink spectrum of $Cone(X)$ is $(0,1), \ldots, (48/25, 1)$. So $\delta = 48/25$. We obtain nonrationality.

This methods work in all Johnson-Kollár examples as well as in higher dimension - for more see [127].

4 Low dimensional topology invariants

We explain a parallel between quantum spectrum and classical 3-dim, 4-dim invariants. First we recall the classical theory. We start with theory of knots and Alexander polynomials. Consider the singular curve:

$$f(z,w) = z^p + w^q, (z,w) \in \mathbb{C}^2$$
$$S_\epsilon = \{|z|^2 + |w|^2 = \epsilon^2\} \subset \mathbb{C}^2, 0 < \epsilon << 1$$
$$K_{p,q} = f^{-1}(0) \cap S_\epsilon \text{ a knot}$$

Alexander polynomial of this torus knot is:

$$\Delta_{p,q} = t^{-\frac{(p-1)(q-1)}{2}} \cdot \frac{(t-1)(t^{pq}-1)}{(t^p-1)(t^q-1)}$$

We define $Sp(f) := \sum_{\alpha\in\mathbb{Q}} n_{f,\alpha} t^\alpha$ the *Steenbrink spectrum*

$$Steen = \{\alpha_1, \alpha_2, \ldots, \alpha_\mu\}, \mu = (p-1)(q-1)$$

Fact 1 $\Delta_{K_{p,q}} = t^{-\frac{\mu}{2}} \prod_{i=1}^{\mu} \Phi_{\alpha_i}(t)$, $\Phi_{\alpha_i}(t) = (t - e^{2\pi i\alpha_i})$

Example 4.1 (($(p,q) = (2,3)$))

$$\Delta_{K_{2,3}} = t^{-\frac{\mu}{2}} \frac{(t^6-1)(t-1)}{(t^2-1)(t^3-1)} = t^{-\frac{\mu}{2}}(t - e^{2\pi i\frac{5}{6}})(t - e^{2\pi i\frac{7}{6}})$$

$Steen = \{\frac{5}{6}, \frac{7}{6}\}$. *Also using Thom-Sebastiani theorem we get:*

$$Steen = \{Steen(z^2)\} + \{Steen(w^3)\} = \{\frac{1}{2}\} + \{\frac{1}{3}, \frac{2}{3}\} = \{\frac{5}{6}, \frac{7}{6}\}$$

Example 4.2 (($(p,q) = (2,5)$))

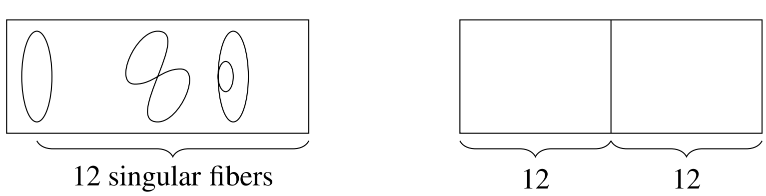
$$\Delta_{K_{2,5}} = t^{-\frac{\mu}{2}} \frac{(t^{10}-1)(t-1)}{(t^2-1)(t^5-1)}$$

$Steen = \{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\}$. *Using Thom-Sebastiani we get:*

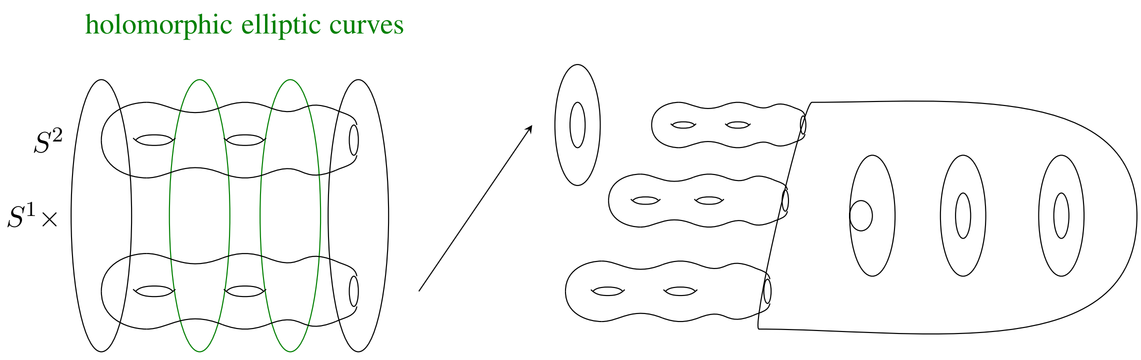
$$Steen(z^2 + w^5) = \{Steen(z^2)\} + \{Steen(w^5)\} = \{\frac{1}{2}\} + \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$$

We move 1 dimension higher. Consider an elliptic surface $E(n)$: an elliptic fibration.

$E(1) = \mathbb{P}^2_{p_1,\ldots,p_9}$ $\qquad$ $E(2) = K3$



We describe fibered knot surgery and its connections with Seiberg Witten invariants SW.



$S^1 \times (\sum_g \times S^1)$ into $\qquad\qquad$ $E(n)$

Under surgery:

$$SW_{E_K(n)} = \sum_{K\in\mathbb{Z}} SW(K[F])t^K = SW_{E(n)}(t)\Delta_K(t), SW_{E(n)} = (t-t^{-1})^{n-2}$$

where F is the fiber of $E_K(n)$.

Theorem 4.3 **($Gr = SW$)** *Coefficients of Δ_K count holomorphic curves $g = 1$ in the class $K[F]$ in $E_K(n)$.*

We explore the connection with spectra. Recall that:

$$\begin{array}{ccc} \sum_g & \to & S^3 - K \\ & & \downarrow \\ & & S_1 \end{array}$$

Φ the monodromy of the surgery (char polynomial of $\Delta_k(t)$) produces an endofunctor on $Fuk(\sum_g)$ and $Fuk(Sym^k \sum_g)$ (or $FS(\sum_g)$?).

Conjecture 4.4 *Φ defines filtration on $HH(Fuk(\sum_g))$ which corresponds to Steen.*

Conjecture 4.5 *$\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f)$ has a filtration*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f) \supset \mathcal{F}_{\alpha_1} \supset \mathcal{F}_{\alpha_2} \cdots$$

given by the spectra.

Let $\mathcal{F}$ be mirror of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f)$. Consider the quantum differential equation 2.1

$$\{\text{asymptotics of 2.1}\} \leftrightarrow \{\text{Spectrum of } f\}$$

Conjecture 4.6 *Entropy of Φ: $\eta(\Phi)$ is the first coefficient of $\Delta_K(t)$.*

These simple observations suggest the following questions:

Question 4.7 *Does the spectrum define canonical filtration on Floer homology?*

Question 4.8 *What is the symplectic meaning of this filtration? We expect it is connected with the structures of the Lagrangian skeleta.*

We discuss further applications. We define modular spectrum of a link M - link of singularity $X_f \leftarrow Y_{1,q}$ as the Steenbrink $Steen(Y_{1,q})$. We give a brief example to fix notations.

Example 4.9 $M = \Sigma(2,3,5)$

$$Y_{1,q} - E_8$$

$$WRT(M) \leftrightarrow (1, 7, 11, 13, 17, 19, 23, 29)$$

Here $WRT(M)$ is the Witten-Reshetikhin-Turaev (WRT) invariant of the 3-manifold M.
We pose the following:

Question 4.10 *Is there a categorical meaning of WRT?*

We will discuss some of these questions in the next section.

4.1 Spectra and WRT

Let M be a smooth 3-manifold which is a link of an isolated normal surface singularity in $\mathbb{C}^3$. In the following sections, we study topological invariants of M and their relation to spectra. GPPV invariants[1] $\hat{Z}_b(q)$ [110], [53] are q-series that refine the WRT invariants.

Series $Z_b(q)$ can be expressed as a linear combination of false theta functions in the case of Seifert manifolds with 3 singular fibres. Corresponding theta functions can be conjecturally written as components of a vector-valued modular form, which is know for some examples, including links of ADE singularities [53]. Induced representation of $SL(2,\mathbb{Z})$ is a subrepresentation of $2m$-dimensional Weil representation for some integer m and θ functions are labelled by residue classes modulo $2m$. We are interested in these residue classes for all components of the modular form, not just those that correspond to $\hat{Z}_b$. We call this set *Modular spectrum* for convenience. A precise definition depends on the conjectural existence of a natural vector-valued modular form. It was posed as a question in [53] what is a deeper meaning of these residue classes.

Example 4.11 *The relation with the spectrum started with an observation about E_8 singularity, defined by the equation $x^2 + y^3 + z^5 = 0$. Its link is a Poincaré homology sphere, Seifert manifold*

[1] also called BPS q-series or homological blocks

$M(-2, 1/2, 2/3, 4/5)$. *WRT invariants of this manifold have been studied in [166]. Lawrence and Zagier defined two functions holomorphic inside the unit circle:*

$$\theta_+(\tau) = q^{1/120}(1 + 11q + 19q^3 + 29q^7 - 31q^8 - 41q^{14} - \dots$$
$$\theta_-(\tau) = q^{49/120}(7 + 13q + 17q^2 + 23q^4 - 37q^{11} - 43q^{15} - \dots$$

The first function gives WRT as the radial limits at the roots of unity. Both functions together form a vector-valued modular form for $SL(2, \mathbb{Z})$.

Those functions can be written as a linear combination of theta functions assigned to residue classes modulo 60 (see Section 2):

$$\theta_+(\tau) = \theta^1_{30,1}(\tau) + \theta^1_{30,11}(\tau) + \theta^1_{30,19}(\tau) + \theta^1_{30,29}(\tau) + \dots$$
$$\theta_-(\tau) = \theta^1_{30,7}(\tau) + \theta^1_{30,13}(\tau) + \theta^1_{30,17}(\tau) + \theta^1_{30,23}(\tau) + \dots$$

The spectrum of E_8 *singularity is*

$$\{1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30\}$$

and we can see that the numerators of the elements of spectrum correspond to residue classes of the theta functions while the denominator corresponds to the modulus.

This example can be generalized in two ways. One is the class of Brieskorn homology spheres $x^{p_1} + y^{p_2} + z^{p_3} = 0$ for a_0, a_1, a_2 pairwise coprime. An analogical relation of theta functions and spectrum is true for them as described in Section 3. It is remarkable since the spectrum contains negative numbers and this is reflected in topology.

Theorem 4.1 *Let* M *be a Brieskorn homology sphere, i.e. the link of the singularity* X *given by the equation* $x^{p_1} + y^{p_2} + z^{p_3} = 0$ *Then*

Modular spectrum of M *= Steenbrink spectrum of* X.

Another generalization is the class of ADE singularities. Here we need to take a spectrum of a different but related singularity - universal Abelian cover.

Theorem 4.2 *Let* M *be a link of ADE singularity* X *and* Y *be the corresponding maximal Abelian cover. Then*

Modular spectrum of M *= Steenbrink spectrum of* Y.

This phenomenon can be certainly generalized to Seifert manifolds, where $\hat{Z}_b$ have been explicitly computed recently. For more general plumbed 3-manifolds, the singularities to consider are splice-quotients and their universal covers, where the spectrum is difficult to compute, however much can be said about the topology itself using ideas from singularity theory and simpler invariants than spectrum. For these generalizations, see [108]. On the topology side, since the description of $\hat{Z}_b$ using false theta functions is limited to 3 singular fibres of Seifert fibration on M, we need to replace theta function labels by something more general. The poles of Borel plane [111] seem to be a good candidate.

4.1.1 Theta functions

We will follow the notation in [53]. In particular we denote $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$.

Definition 4.12 *Let m be a positive integer and r a residue class $\bmod 2m$. We define weight 1/2 theta function and weight 3/2 unary theta function as (respectively)*

$$\theta_{m,r}(\tau, z) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} q^{\ell^2/4m} y^\ell; \theta^1_{m,r}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} \ell\, q^{\ell^2/4m}, \tag{4.1}$$

Unary theta functions form a (rank $2m$) vector-valued modular form of weight 3/2. Its matrices S and T define *Weil representation* of $\widetilde{SL}(2,\mathbb{Z})$, the double cover of $SL(2,\mathbb{Z})$.

Definition 4.13 False theta function (or Eichler integral) *of $\theta_{m,r}$ is*

$$\Psi_{m,r}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} \operatorname{sgn}(\ell)\, q^{\ell^2/4m}. \tag{4.2}$$

False theta functions keep a weaker modular property - quantum modularity [256].

Note also the obvious relations:

$$\Psi_{m,r}(\tau) = \Psi_{m,-r}(\tau) \tag{4.3}$$

$$\Psi_{m,r+2m}(\tau) = \Psi_{m,r}(\tau) \tag{4.4}$$

The basic idea is the correspondence $\frac{r}{m}$ as an element of the spectrum of certain singularity related to the 3-manifold and $\Psi_{m,r}(\tau)$ as an Eichler integral of a certain theta function assigned to a 3-manifold.

4.1.2 GPPV invariants

A plumbed 3-manifold M admits GPPV invariants [110], which are q-series $\hat{Z}_b(q)$ defined using plumbing graph of M and labeled by elements of $H_1(M)$ or $spin^c$ structures. These invariants can be computed by an explicit integral formula [53]. It is an intriguing question whether the series Z_b can be written as components of (quantum) modular forms.

The vector-valued modular forms described in [53] have usually more components than is the number of $Z_b(q)$ (as in the example E_8 in the introduction). It is not clear what is the meaning of these components for the 3-manifold and how to get an intrinsic definition of them.

4.1.3 Example of Brieskorn homology sphere $\Sigma(3, 4, 5)$

Here we give an example of theorem 4.1. Homology sphere $\Sigma(3, 4, 5)$ is the link of $x^3+y^4+z^5 = 0$. This case has been studied in [53], p. 67. They describe a representation of $\widetilde{SL}(2,\mathbb{Z})$ given by theta functions $\theta^1_{m,r}$ and corresponding false theta functions $\Psi_{m,r}$. The number m is $3 \cdot 4 \cdot 5 = 60$.

False theta functions:

$$\begin{aligned}
\Psi_{60,1} - \Psi_{60,31} - \Psi_{60,41} - \Psi_{60,49} \\
\Psi_{60,2} + \Psi_{60,22} + \Psi_{60,38} + \Psi_{60,58} \\
\Psi_{60,7} + \Psi_{60,17} + \Psi_{60,23} - \Psi_{60,47} \\
\Psi_{60,11} + \Psi_{60,19} + \Psi_{60,29} - \Psi_{60,59} \\
\Psi_{60,13} - \Psi_{60,37} - \Psi_{60,43} - \Psi_{60,53} \\
\Psi_{60,14} + \Psi_{60,26} + \Psi_{60,34} - \Psi_{60,46}
\end{aligned}$$

If we use the relation $\Psi_{m,2m+r} = \Psi_{m,r}$ and multiply first and fifth row by -1 (change of the basis of the representation) we obtain

$$\begin{aligned}
\Psi_{60,-1} + \Psi_{60,31} + \Psi_{60,41} + \Psi_{60,49} \\
\Psi_{60,2} + \Psi_{60,22} + \Psi_{60,38} + \Psi_{60,58} \\
\Psi_{60,7} + \Psi_{60,17} + \Psi_{60,23} + \Psi_{60,73} \\
\Psi_{60,11} + \Psi_{60,19} + \Psi_{60,29} + \Psi_{60,61} \\
\Psi_{60,-13} + \Psi_{60,37} + \Psi_{60,43} + \Psi_{60,53} \\
\Psi_{60,14} + \Psi_{60,26} + \Psi_{60,34} + \Psi_{60,46}
\end{aligned}$$

Now the labels r of $\Psi_{m,r}$ are exactly the numerators of the elements of Steenbrink spectrum of $x^3+y^4+z^5=0$. The terms in each sum correspond to the orbits of a natural action of $\mathbb{Z}_2^2$ on the spectrum. Note that since the theta functions only depend on $r \pmod{2m}$ the relevant spectrum is spectrum modulo 2 (we cannot hope to recover the full Hodge-theoretic information from topology).

The series $Z_0(q)$ is at the fifth row. It contains the term labelled by the smallest number in the spectrum: -13/60.

Remark 4.14 *As conjectured in [53], components of the representation should correspond to non-abelian $SL(2,\mathbb{C})$ connections (it is true for Brieskorn spheres). If we use this identification and restrict it to real connections, we recover the classical relation of the signature of Milnor fiber of the Brieskorn singularity and Casson invariant of M [76].*

4.1.4 ADE singularities

Before we get to the relation of GPPV and the spectrum, we need to recall the notion of universal Abelian cover of an isolated singularity (see, for example, [189]). Recall that a closed oriented 3-manifold M is a $\mathbb{Q}$-homology sphere if $H_*(M,\mathbb{Q}) = H_*(S^3,\mathbb{Q})$.

Definition 4.15 *Let X be a germ of an isolated normal surface singularity whose link M is a $\mathbb{Q}$-homology sphere. The universal Abelian cover Y of X is a maximal Abelian cover of the germ ramified at the singular point.*[2]

[2] The covering group is then $H_1(M,\mathbb{Z})$

$\hat{Z}_b$ and modular forms of the links of ADE singularities were computed in [53], see also [118]. Using their results, we obtain theorem 4.2. All ADE singularities, their Abelian covers and invariants are summarized in table 4.1.4.

manifold M	X	Y	false thetas of M	spectrum of Y
lens space	A_n	$\mathbb{C}^2$	no thetas	empty
$M(-2; \frac{1}{2}, \frac{1}{2}, \frac{n-3}{n-2})$	D_n	A_{n-3}	$\Psi_{1,n-2}, \Psi_{2,n-2}, \ldots, \Psi_{n-3,n-2}$	$(1, 2, \ldots, n-3)/(n-2)$
$M(-2; \frac{1}{2}, \frac{2}{3}, \frac{2}{3})$	E_6	D_4	$\Psi_{6,1} + \Psi_{6,5}, 2\Psi_{6,3}$	$(1, 3, 3, 5)/6$
$M(-1; \frac{1}{2}, \frac{2}{3}, \frac{3}{4})$	E_7	E_6	$\Psi_{12,1} + \Psi_{12,7}, \Psi_{12,4} + \Psi_{12,8}, \Psi_{12,5} + \Psi_{12,11}$	$(1, 4, 5, 7, 8, 11)/12$
$\Sigma(2, 3, 5)$	E_8	E_8	4.11,[166]	$(1, 7, 11, 13, 17, 19, 23, 29)/30$

Table 1: Labels of false theta functions for M, the link of singularity X, correspond to the spectrum of the universal Ab. cover Y of X.

4.2 Topological invariants of plane curve singularity

We give some ideas of the categorical origin of these topological invariants. Let $C = \{f(x, y) = 0\}$ be a germ of a plane curve having an isolated singularity at the origin p and $L_{C,p}$ be an algebraic link of the plane curve singularity. There have been lots of works studying relations between algebraic geometry of C and topology of $L_{C,p}$. For example, the Alexander polynomial of $L_{C,p}$ can be computed via the ring of functions $\mathcal{O}_C$ thanks to the works of Campillo-Delgado-Gusein-Zade (cf. [44]) and the HOMFLY-PT polynomial of $L_{C,p}$ can be expressed in terms of Hilbert schemes of the plane curve singularity thanks to the works of Oblomkov-Shende (cf. [190]) and Maulik (cf. [177]). On the other hand, there have been lots of interests in mirror symmetry of hypersurface singularities these days (see [72] and references therein for more details) and plane curve singularities again have provided natural testing grounds for mirror symmetry conjecture. Takahashi conjectured that for an invertible polymial f, the category of graded matrix factorization $\mathrm{HMS}^{L_f}(f)$ will be equivalent to the Fukaya-Seidel category $\mathrm{Fuk}^{\to}(f^T)$ of the Berglund-Hübsch mirror polynomial f^T and recently there have been lots of works in this direction and both categories have been intensively studied. For example, it turns out that $\mathrm{HMS}^{L_f}(f)$ has a full exceptional collection and admits a Gepner type stability condition when f is of ADE type. Here, we will discuss the relation between Hilbert schemes of plane curve singularities, certain topological data of some algebraic links, and matrix factorizations. To be more precise, we will consider the images of ideals which belong to certain Hilbert scheme $C_p^{[*]}$ in the category $\mathrm{HMF}^{L_f}(f)$ when $f = x^2 + y^3$. Then we can check that the images have interesting properties. For example, a natural stratification on (some parts of) the Hilbert scheme $C_p^{[*]}$ corresponds to an indecomposable object in $\mathrm{HMS}^{L_f}(f)$. We can also verify that the difference between the Alexander polynomial and the HOMFLY-PT polyonomial of $L_{C,p}$ can be expressed in terms of $\mathrm{HMF}^{L_f}(f)$.

4.2.1 Hilbert schemes

Let $C = \{f(x, y) = 0\}$ be the germ of a plane curve with an isolated singularity at the origin at $p = (0, 0)$.

Definition 4.16 *Let $C_p^{[l]}$ be the Hilbert scheme of length l zero dimensional subschemes of C which are set-theoretically supported at p. And let $C_p^{[*]} := \bigcup_l C_p^{[l]}$.*

The normalization induces an embedding $\mathcal{O}_C \to \mathbb{C}[[t]]$. And the natural valuation induces a valuation $\mathcal{O}_C \to \mathbb{N}$. Let $\Gamma = \nu(\mathcal{O})$ be the semigroup. Let $I \subset \mathcal{O}_C$ be a L_f-graded ideal. Then $\mathcal{O}_C/I$ gives an element in $D_{\mathrm{sg}}^{L_f}(R_f)$.

Proposition 4.17 *Let f be a weighted homogeneous polynomial. Then there is a $\mathbb{C}^*$-action on $C_p^{[*]}$. A $\mathbb{C}^*$-invariant ideal gives an $\mathbb{Z}$-graded ideal.*

Proof. The obvious $\mathbb{C}^*$-action on f induces an action on $C_p^{[*]}$ and having a $\mathbb{C}^*$-action is equivalent to having a $\mathbb{Z}$-grading. □

The following remark tells us that not all ideals of $\mathcal{O}_C$ give nontrivial elements in $\mathrm{HMF}^{L_f}(f)$.

Remark 4.18 *Let g be a nonzero divisor in $\mathcal{O}_C$. Then $\mathcal{O}/(g)$ is a perfect complex.*

Proof. We have the following short exact sequence.

$$0 \to \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_C/(g) \to 0$$

Therefore $\mathcal{O}/(g)$ is a perfect complex. □

4.2.2 Example $f = x^2 + y^3$

We can compute L_f as follows.

$$L_f = \mathbb{Z}\vec{x} \oplus \mathbb{Z}\vec{y} \oplus \mathbb{Z}\vec{f}/(\vec{f} - 2\vec{x} - 3\vec{y}) \cong \mathbb{Z}$$

$$R_f = \mathcal{O}_C = \mathbb{C}[[x, y]]/(x^2 + y^3) = \mathbb{C}[[t^2, t^3]]$$

There is a stratification on the Hilbert scheme as follows.

$$(1)$$

$$(t^i + ut^{i+1}), \;\; i \geq 2, u \in \mathbb{C}$$

$$(t^i, t^{i+1}), \;\; i \geq 2$$

The $\mathbb{C}^*$-invariant parts of the Hilbert scheme are as follows.

$$(1)$$

$$(t^i), \;\; i \geq 2$$

$$(t^i, t^{i+1}), \;\; i \geq 2$$

The semigroup Γ is $\{0, 2, 3, 4, 5, 6, 7, \cdots\}$.

The Koszul resolution of $\mathbb{C}[[x, y]]/(x, y)$ induces an L_f-graded matrix factorization $F = (F_0, F_1, f_0, f_1)$ of f where $P(\vec{f}) := S(-\vec{x}) \oplus S(-\vec{y})$ and

$$F_0 := S \oplus \wedge^2 P(\vec{f}), \;\; F_1 := P(\vec{f}).$$

Proposition 4.19 *The matrix factorizations correspond to the ideal* (t^i, t^{i+1}) *is the image of the above matrix factorization under the autoequivalence* $(\overrightarrow{l})$ *for some* $\overrightarrow{l} \in L_f$.

Proof. Let $M = \mathbb{C}[[x,y]]/(x,y)$. Let M^{stab} be the above matrix factorization. Note that (t^i, t^{i+1}) is isomorphic to (t^2, t^3) as an R_f-modules. The only difference between them is grading and hence we obtain the desired conclusion. □

Proposition 4.20 *The ideal* (t^i, t^{i+1}) *is an exceptional object in* $\mathrm{HMF}^{L_f}(f)$.

Proof. Because $\mathbb{C}[[x,y]]/(x,y)$ is an exceptional object (cf. [89]), we see that (t^i, t^{i+1}) is also exceptional. □

Then we have the following.

Corollary 4.21 *The ideal* (t^i, t^{i+1}) *is an indecomposable object in* $\mathrm{HMF}^{L_f}(f)$.

It is well-known that there are only finitely many indecomposable objects in $\mathrm{HMF}^{L_f}(f)$ up to autoequivalences.

Theorem 4.22 *The difference between the Alexander polynomial and the HOMFLY-PT polynomial is a categorical invariant.*

Proof. The difference between the Alexander polynomial and the HOMFLY-PT polynomial of $L_{C,p}$ is the integration over ideals of type (t^i, t^{i+1}). And every element of the form (t^i, t^{i+1}) can be obtained from (t^2, t^3) by applying translations. From the above discussion, we see that these ideals give non-trivial elements in $\mathrm{HMF}^{L_f}(f)$. Therefore, one can see that the difference can be written in terms of $\mathrm{HMF}^{L_f}(f)$. □

5 Generalization of Spectra

We extend the connection of spectra with Alexander polynomial initiated in the previous section. We extend the correspondence:

$$\boxed{\text{Multivariable Alexander Polynomials}} \longleftrightarrow \boxed{\text{multispectra}}$$

Theorem of Libgober [169] says that we can associate to spectrum of $f_1, f_2, \ldots \leftrightarrow$ faces of quasi-adjunction. We will give a categorical version of this process:

5.1 Splitting of a potential

Consider a Landau-Ginzburg model with a potential $W = W_1 + W_2$ We consider the associated Fukaya-Seidel categories $FS(W_1)$, $FS(W_2)$, $FS(W)$.

We start with the tower:

$$\begin{tikzcd}
FS(W_1+W_2) \arrow[r] \arrow[d] & FS(W_1) \arrow[d] \\
FS(W_1) \arrow[r] & FS(W_1\cap W_2)
\end{tikzcd}$$

Example 5.1 **($X_3^5 \subset \mathbb{P}^6$ 5-dim cubic)**

$$\begin{aligned}
\boldsymbol{D}^b(X_3^5) &\cong FS(W_1+W_2)\\
\boldsymbol{D}^b(X_6^4) &\cong FS(W_1)\\
\boldsymbol{D}^b(X_6^4) &\cong FS(W_2)
\end{aligned}$$

Conjecture 5.2 *The NC spectra of X_3^5 is a superposition of X_6^4 and X_6^4.*

We have the P.D.E.

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G$$

Conjecture 5.3 *The P.D.E. of X_6^4 and P.D.E. of X_6^4 produce the P.D.E. of X_3^5 via convolution.*

$$PDE(X_6^4) *_A PDE(X_6^4) \cong PDE(X_3^5)$$

We see that asymptotics are superposition of asymptotics.

Corollary 5.4 *Let $\widetilde{\mathbb{P}^N_X}$ is a blow-up of $\mathbb{P}^N$ along X. Then the faces of quasiadjuction contain*

$$(-(\dim X)/2, \ldots, -(\dim X)/2)$$

In general, we have

$$Spec(\{A_i\}) \twoheadrightarrow Spec(\{K\})$$

Here the algebra $\{K\}$ is the algebra generated by canonical bundle. $\{A_i\}$ is the algebra generated by algebraic cycles. The above epimorphism defines a deeper filtration.

Question 5.5 *Is this new filtration a birational invariant?*

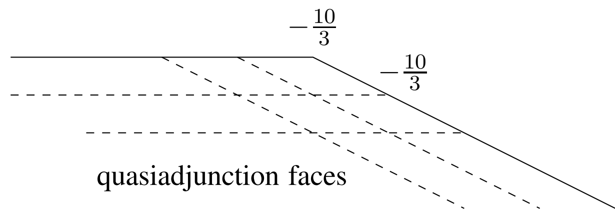
Question 5.6 *Does the algebra defined by splitting produce birational invariants?*

We consider the example of 5-dim cubics.

$$\begin{tikzcd}
& \boldsymbol{D}^b(X_3^5) & \\
\boldsymbol{D}^b(X_{3,2}^4) \arrow[ur, hook] & & \boldsymbol{D}^b(X_{3,2}^4) \arrow[ul, hook']
\end{tikzcd}$$

$$\delta_1(X_3^5) = \frac{7}{3}$$
$$\delta_1(X_{3,2}^5) = 4 - 2\frac{6-3-2}{3} = \frac{10}{3}$$

We compute the quasiadjunction of the above splitting.



Observation

We notice that in the above spliting $-(\dim X)/2, \dots, (\dim X)/2$ do not belong to quasiadjunction faces of the polygon. This suggest a different proof of the nonrationality 3-dimensional cubic.

5.2 Category filtrations

For a category $\mathcal{C}$ and A, B and a noncommutative Hodge structure $\mathcal{H}, \nabla, Herm > 0$, we define a sequence of stability conditions $\mathcal{J}_1, \dots, \mathcal{J}_k$ corresponding to asymptotics of stability spectrum.

We consider the asymptotics of integral $\int_{\Gamma'(0)} \alpha_{(0)} \sim$ Asymptotics at $z = 0$. These asymptotics define **stability spectrum.**

Example 5.7 *Consider the category A_n - 1 dimensional Fukaya-Seidel categories. So we have $x^j e^{\frac{p}{u}} dx$ is a stability condition. Here p is a polynomial of degree $< (n-1)$.*

<u>Step 1</u> *We have $\alpha = dx$.*

<u>Step 2</u> *We move to define Kähler metric on moduli space of stability conditions. We begin with $K_{ij}(u, \bar{u}) = \iint_{\mathbb{C}} x^i x^j e^{\frac{p}{u} - \frac{\bar{p}}{\bar{u}}} dx d\bar{x}$*

$$\Phi : |u| \leq 1 \to GL(n+1, \mathbb{C})$$
$$\forall |u| = 1, \Phi(u)\Phi^t(u) = K_{ij}$$

We define Hermitian form

$$H(u) = \Phi(u)\Phi^t(u)$$
$$\textit{Asymptotics} \int x^i e^{\frac{p}{u}} dx$$

define asymptotics and the noncommutative spectrum.

As we saw the asymptotics of the integral $\lim_{n \to 0} Z_n = \sum u^{\alpha_i}$ define stability and nc spectra. We move in to investigate the connection with analysis.

We have the following:

Theorem 5.8 *The stability conditions $\mathcal{J}_1, \ldots, \mathcal{J}_k$ define a filtration on $\mathcal{C}$:*

$$\mathcal{F}_{\leq i}(\mathcal{C}) = \textit{semistable Obj}(\mathcal{E})$$

such that

$$Z_{\mathcal{J}_i}(\mathcal{E}) \leq \mathcal{O}(|\mathcal{J}|^j)$$

This theorem will be discussed in detail in [142]. We will make some use of this filtration in what follows. We consider a Fano X and a splitting of a canonical divisor $K_X = D_1 + D_2$.

$$X - \text{Fano}$$
$$K_X = D_1 + D_2$$

On the mirror side we have spitting of the potential $W = W_1 + W_2$.

$$\begin{tikzcd}
FS(W_1) \arrow[r] & FS(W) \\
Fuk(CY) \arrow[u] \arrow[r] & FS(W_2) \arrow[u]
\end{tikzcd}$$

Monodromy of W_1 gives a filtration:

$$FS(W_1) \supset \mathcal{F}_{\lambda_1} \supset \cdots \supset \mathcal{F}_{\lambda_n}$$

Monodromy of W_2 gives a filtration:

$$FS(W_2) \supset \mathcal{F}_{\mu_1} \supset \cdots \supset \mathcal{F}_{\mu_n}$$

giving a double filtration

$$FS(W) \supset \mathcal{F}_{\mu_1,\lambda_1} \supset \cdots$$
$$FS(W) \supset \mathcal{F}_{\nu_1} \supset \cdots$$

The behavior of λ_i, μ_j is of Thom Sebastiani type generalized

$$\nu_i \overset{ThomSebastiani}{=} (\lambda_i, \mu_i)$$

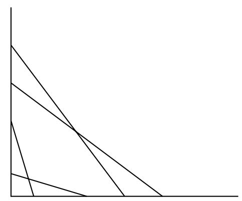
In fact, we have a correspondence:

$$\left\{ \begin{array}{c} \text{Choices} \\ \text{of} \\ W_1, W_2, \ldots \end{array} \right\} \to \left\{ \begin{array}{c} \text{generalized} \\ \text{Thom Sebastiani} \\ \lambda_i \;\; \mu_i \;\; \nu_i \\ \vdots \;\; \vdots \;\; \vdots \end{array} \right\}$$

Question 5.9 *Can one produce out of λ_i, μ_i, ν_i new birational invariants?*

We discuss briefly a couple of examples.

Example 5.10 (Polytope of quasiadjunction $(x^2+y^3)(x^3+y^2)$)



$(x^2+y^3)(x^3+y^2)$

$$2u+3v=\tfrac{1}{2}\,\tfrac{3}{2}\,\tfrac{5}{2}$$

$$3u+2v=\tfrac{1}{2}\,\tfrac{3}{2}\,\tfrac{5}{2}$$

The Alexander polynomial is:

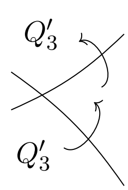
$$(t_1^2t_2^3+1)(t_1^3t_2^2+1)$$

Example 5.11 (3-dim cubic)

$$-K_X = 2H$$

$$f = Q_3'Q_3' \qquad \textit{two cubics}$$

$$\begin{matrix} \lambda_1 = \dfrac{5}{3} \\ \lambda_2 = \dfrac{5}{3} \end{matrix} \quad \to \quad \delta = \dfrac{5}{3}$$



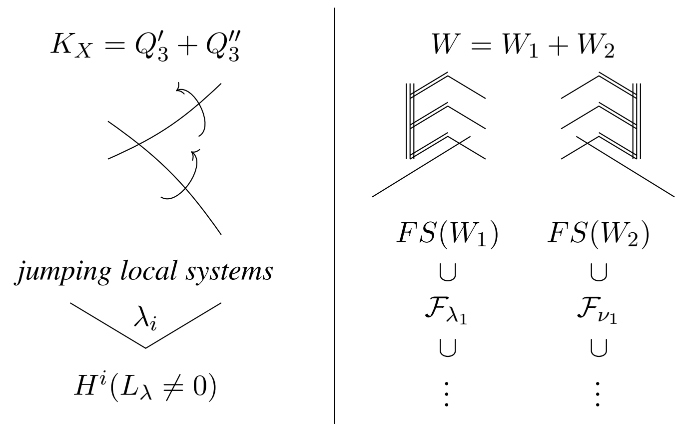
local Alexander polynomials

$$\Downarrow$$

$$\frac{5}{3}$$

Mirror

$$\begin{matrix} W = & W_1 & + & W_2 \\ & | & & | \\ & Q_3'' & & Q_3'' \\ & \frac{5}{3} & & \frac{5}{3} \end{matrix} \qquad \textit{no deformations}$$



6 Spectrum, orbifoldization and conformal field theory

In this section we propose a new point of view of noncommutative spectra. Details will appear elsewhere see e.g. [140], [108].

Our approach is based on the parallel between:

- Birkar's proof [30] of boundness of Fano's.
- Zamolodchikov's [97] c-theorem.

We combine these two directions with categorical resolution of singularities. The final outcome is creating theory of noncommutative spectra similar to Arnold-Varchenko-Steenbrink spectrum.

We will describe a procedure of computing noncommutative spectrum as equivariant part of Steenbrink spectrum of the corresponding affine cone.

Steenbrink Spectrum $\xrightarrow[Equivariant]{Elliptic}$ Noncommutative Spectrum.

We consider the following examples.

1. Let X be a hypersurface (Fermat) of degree d in $\mathbb{P}^N$

$$x_0^d + \cdots x_N^d$$

by Steenbrink $(y^{\frac{1}{d}} + \cdots + y^{\frac{d-1}{d}})^{N+1}$.

This is the fixed part of the Elliptic genus when applied to 5-dim. cubic.

Recall that

$$x_0^3 + \cdots x_6^3 = 0$$

has Steenbrink Spectrum

$$(y^{\frac{1}{3}} + y^{\frac{2}{3}})^7$$

We orbitalize using action of $\mathbb{Z}_3$

$$\frac{1}{3}y^{-\frac{7}{2}}\left(\sum_{0\leq a\leq 3}\left(\frac{y^{\frac{1}{3}}-y\omega^{-a}}{y^{\frac{1}{3}}-\omega^{-a}}\right)^7+\sum\left(y^{\frac{6}{3}}\right)^7\right)$$

So after that, we get

$$-21(y^{-\frac{7}{2}}+y^{\frac{1}{2}})+y^{-\frac{7}{6}}+y^{\frac{7}{6}}$$

$$\Rightarrow\left(-\frac{7}{6},\frac{7}{6}\right)\text{ - noncommutative spectrum}$$

2. Similarly for 2-dim. cubic $y^{-\frac{2}{3}}+2+y^{\frac{2}{3}}$.
For K3 ($x_0^4+\cdots+x_3^4=0$), we have $2y^{-1}+20+2y$.

Proposition 6.1 *For CY, the procedure gives* $-\frac{\dim X}{2},\ldots,\frac{\dim X}{2}$.

Proposition 6.2 *For general type, the procedure gives* $-\frac{\dim X}{2},\ldots,\frac{\dim X}{2}$.

Proposition 6.3 *The uppersemicontinuity for Steenbrink spectrum brings uppersemicontinuity for noncommutative spectrum.*

We consider the Berglund-Hübsch Mirror Symmetry.

$$X^\vee=\mathbb{C}^{n+1}/\Gamma\xrightarrow{f}\mathbb{C}$$

where $X^\vee$ is the mirror of $X\subset\mathbb{P}^N$. So we have:

Conjecture 6.4 $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(X^\vee,f)^{eq}=Fuk^0(X)$.

Now we present a program which not only explains Conjecture 6.1 but suggests a far going program of categorical resolutions. We begin by:

Conjecture 6.5 *Let $r:X\to X_{sing}$ be a resolution of singularity. There exists a category $\mathcal{C}_0$ which does not depend on r.*

In the case of orbifold we can be more precise:

Conjecture 6.6 *There exists a piece $\mathcal{H}_0\subset H^i(X)$ which does not depend on r. Then $\mathcal{H}_0\cong IH(X_{sing})$.*

We have:

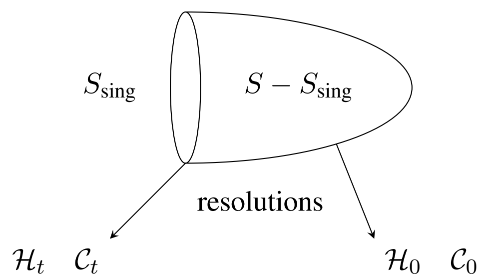
$$H^{\mathrm{String}}(X_{\mathrm{sing}})=IH(X_{\mathrm{sing}})+T_{S_1}+\cdots T_{S_w}$$

Here IH are the intersection cohomologies of X. The noncommutative spectrum is defined over $IH(X_{\mathrm{sing}})$. We can combine above conjecture with our orbifoldization procedure. We observe that the twisted sectors we need to take are precisely the ones on which the group acts with determinant equal to one. The above considerations can be lifted to categorical level.

Conjecture 6.7 *Consider a resolution $S' \overset{res}{\longleftarrow} S$ of terminal singularities. Assume $S - S_{sing}$ has a volume form. Then*

1. *$\mathcal{H}_0$ is independent of r;*
2. *$\mathcal{C}_0$ is a CY-category, subcategory of $Perf(X)$ is independent of r.*

We would like to make a parallel between Birkar's theory and category theory.



In the above setting $S - S_{\text{sing}}$ determines $\mathcal{H}_0$ and S_{sing} the rest of semi-orthogonal decomposition. We have a correspondence between classical and categorical notions:

$$K_X, B \longleftrightarrow S_{\text{sing}}$$

$$B'_{\text{complement}} \longleftrightarrow S/S_{\text{sing}}$$

$$\text{volumes} \longleftrightarrow \text{Categorical Entropy } h$$

Let $\mathcal{C}^d_{\mathcal{E}}$ be a log Calabi-Yau category. (We fix the biggest number in the spectra and d is the categorical dimension.)

Question 6.8 Φ *is a functor of $\mathcal{C}^d_{\mathcal{E}}$. Are $h(\Phi)$ bounded?*

Question 6.9 *Is Aut$(\mathcal{C}^d_{\mathcal{E}})$ of Jordan type? (Here Aut$(\mathcal{C}^d_{\mathcal{E}})$ is the group of autoequivalences).*

Question 6.10 *Is $F(\mathcal{C}^d_{\mathcal{E}})$ a bounding family? (Here $F(\mathcal{C}^d_{\mathcal{E}})$ is the family parametrizing the categories with dimension d and bounded the biggest number of the spectra from below. Proper definition will take effort.)*

Question 6.11 *Consider the splitting*

$$\mathcal{C} = \bigcup_{i \geq 0}^{\lambda(\mathcal{E},d)} \mathcal{C}_i$$

$$\mathcal{H} = \bigcup_{i \geq 0}^{\lambda(\mathcal{E},d)} \mathcal{H}_i$$

Show that $\lambda(\mathcal{E}, d)$ is finite.

Question 6.12 *Are categorical dimensions of $\mathcal{C}^{\lambda_i}_{\mathcal{E},d}$ bounded?*

The above considerations suggest the following parallels.

Fano	Category	CFT
Birkar's Theory $\mathcal{E}, d$ Boundness	σ, d Boundness of log CY theory	Behavior of σ, d theory
Jordan Property of Birational Aut	Jordan Property of Aut $\mathrm{D^b}$	
	uppersemicontinuity of Spectra	Zamolodchikov Theorem

The Zamolodchikov's c theorem suggests semicontinuity of the noncommutative spectra - see [48], [107]. This correspondence will be discussed elsewhere.

Our findings in the previous sections suggest that in the case of X, an algebraic surface, we have the following correspondence.

$$\left\{ \begin{array}{c} \text{Additional basic} \\ \text{for } H^2(X) \text{ classes} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{Phantoms} \\ \text{of } \boldsymbol{D}^b(X) \text{ classes} \end{array} \right\} \qquad\qquad \left\{ \ \delta > 0 \ \right\}$$

The above findings suggest that new (A, B) structures can be used to define new invariants, A side invariants for the B side.

We have the following parallel:

Resolution of singularity	Surgery
Creation of Spectra	Creation of Spectra

Conjecture 6.13 *Log transform (rational blow down) creates nontrivial $\delta > 0$.*

This suggests the following questions.

Question 6.14 *Can we have symplectic 4-fold with the same basic classes but different spectra?*

We have a connection with k-spectra of CFT. This observations lead to: symplectic Poincare conjectures.

- Find a 4-dim symplectic manifold s.t. $X \overset{\text{homeo}}{\cong} \mathbb{P}^2$ and $\delta(X) > 0$.
- Find a 4-dim symplectic manifold s.t. $X \overset{\text{homeo}}{\cong} \mathbb{P}^1 \times \mathbb{P}^1$ and $\delta(X) > 0$.
- Find a $2n$-dim symplectic manifold s.t. $X \cong \mathbb{P}^n$ and $\delta(X) > 0$.

The parallel between RG flow and Kaehler Ricci flow suggests that the other R-charges can also lead to birational invariants.

Renormalisation group flow and defects lines in the LG model could lead to higher invariants. We investigate these phenomena further in [109].

Scientific Contributions

What have we achieved in this dissertation? We have facilitated the use of Homological Mirror Symmetry to solve deep problems in Birational Geometry, e.g. nonrationality questions. The only algebraic geometry applications of Homological Mirror Symmetry before that were counting curves.

Birational Geometry is a central part of Algebraic Geometry - we still need to admit we do not understand the non rationality of cubics. The first steps in that direction were done by Riemann - the theory of elliptic integral proves nonrationality of a one dimensional smooth cubic. In dimension two rationality questions was done by Enriques, Castelnuovo, Zariski. Some spectacular results were obtained in dimension 3 and higher by Clemens, Griffiths, Voisin, Kollár, Tschinkel.

This dissertation offers a completely different method based on Homological Mirror Symmetry .

Homological Mirror Symmetry is a subject with many faces from different subjects from logic to arithmetics. In this dissertation we concentrate on the connection with birational geometry and Hodge theory. The reason for that is the application to nonrationality questions we have in mind.

We start with simple example related to rational surfaces where it is easy to investigate mirror site of the birational geometry.

In first part of the thesis we prove Homological Mirror Symmetry for the projective plane and for Del Pezzo surfaces - see Theorem 1.2 and Theorem 1.4. In these two examples it becomes clear that birational transformations lead to theory of singularities on the mirror side.

In the third part we expand this observation in any dimension. It becomes clear that birational transformations are nothing but new singular fibers in he Landau Ginzburg models - see Theorem 1.7.

In order to go deeper in birational transformations we need to expand Hodge theoretic invariants - see section Noncommuative Hodge Structure. We do this in the second part of the thesis. We introduce noncommutative Hodge theory - theory of quantum D - modules. These leads to two spectra - see the section Interpretation of spectra:

1. The eigenvalues of quantum multiplication by cannonical class - quantum spectrum.
2. The assymptotics of the solutions of the quantum differential equation.

The last part of the thesis suggests how these two spectra lead to spectacular birational applications. Indeed we can use these spectra to show nonrationality of generic four dimetional cubic - more than sixty years old problems in birational geometry. Many other Fano varieties are considered. Of course this is only the tip of the iceberg. The dynamics of the other numbers in the spectra will bring new obstruction to rationality.

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Indices

List of Names

List of Terms