Autor's note on the scientific contributions of the thesis: SYMPLECTIC TOPOLOGY, NON - COMMUTATIVE GEOMETRY, AND MIRROR SYMMETRY

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Contents

1 Introduction

The main purpose of this dissertation is developing Homological Mirror Symmetry for the benefit of classical birational geometry.

In 1994 International Congress of Mathematicians in Zürich, Kontsevich speculated that mirror symmetry for a pair of Calabi-Yau manifolds X and Y could be explained as an equivalence of a triangulated category constructed from the algebraic geometry of X (the derived category of coherent sheaves on X) and another triangulated category constructed from the symplectic geometry of Y (the derived Fukaya category).

2 Physics background

Edward Witten originally described the topological twisting of the $N = (2, 2)$ super-symmetric field theory into what he called the A and B model topological string theories. These models concern maps from Riemann surfaces into a fixed target – usually a Calabi-Yau manifold. Most of the mathematical predictions of mirror symmetry are embedded in the physical equivalence of the A-model on Y with the B-model on its mirror X. When the Riemann surfaces have empty boundary, they represent the worldsheets of closed strings. To cover the case of open strings, one must introduce boundary conditions to preserve the super-symmetry. In the A-model, these boundary conditions come in the form of Lagrangian submanifolds of Y with some additional structure (often called a brane structure). In the B-model, the boundary conditions come in the form of holomorphic (or algebraic) submanifolds of X with holomorphic (or algebraic) vector bundles on them. These are the objects one uses to build the relevant categories. They are often called A and B branes, respectively. Morphisms in the categories are given by the massless spectrum of open strings stretching between two branes.

The closed string A and B models only capture the so-called topological sector – a small portion of the full string theory. Similarly, the branes in these models are only topological approximations to the full dynamical objects that are D-branes.

3 Contributions

The mathematics resulting from this small piece of string theory has been both deep and difficult.

The main outcomes of the theses are:

- Building a solid mathematical categorical correspondense.

  A major part of this dissertation is to give well defined mathematical theory of Homological Mirror Symmetry in the case of Fano manifolds.

  The main body of the dissertation is developing the Homological Mirror Symmetry. The mirror Symmetry started as a theory allowing counting curves - done by physicists. For us that counting of curves – Gromov Witten theory is a tool which leads to higher structures and as a result to higher order applications.

The point we take is that Birational geometry (derived categories) is mirror to theory of singularities (the category of vanishing cycles).

We start with the very simple case – rational surfaces, where birational geometry is rather easy. First we develop the categorical foundations of Homological Mirror Symmetry. We establish the fact that the birational transformations lead to creation of new singularities on the mirror site. We extend this correspondence in general – this is the main content of the first part of the dissertation.

- Hodge theory, singularity theory, birational geometry.

  In the second part, we develop the theory of Non-commutative Hodge structures. We also show that the quantum differential equation and its asymptotics corresponds to the spectrum of singularities of the Landau Ginzburg model. This is the second part of the dissertation.

  The development of conformal field theory begins with the two-dimensional case. It is consolidated with the 1983 article by Belavin, Polyakov and Zamolodchikov.

  In the two-dimensional quantum theory we have the Witt algebra of infinitesimal conformal transformations which is centrally extended, with a central charge and other renormalization charges – spectra of dimensions.

  Alexander Zamolodchikov has proven the Zamolodchikov C-theorem, and tells us that renormalization group flow in two dimensions is irreversible.

  Computing the charges of conformal field theories is a challenging exercise in general. In the case of massive theories one can use geometry in order to compute them.

  The theory of spectra of singularities was developed in a parallel way to the theory of central charges. In fact, it was developed in the same city – in Moscow by Arnold and Varchenko. The spectra of singularity corresponds to the charges of conformal field theories and the Zamolodchikov C-theorem is the semi-continuity theorem in the theory of spectra of singularities.

  In the second part of the theses we develop the theory of Non-commutative Hodge structures - a mathematical approach to the physics theory mentioned above.

  We also show that the quantum differential equation and its asymptotics corresponds to the spectrum of singularities of the Landau Ginzburg model.

  All invariants of the Noncommutative Hodge theory we build serve as a Birational Invariants.

- We build new birational invariants.

  The new Non-commutative Hodge structures invariants lead to new birational invariants. These invariants lead to spectacular birational applications - proof of nonrationalities of generic four dimensional cubic - a long standing question in Algebraic Geometry.

4 Dessimination

The results of this theses were reported in the following talks:

- Homological Mirror symmetry, SUNY Stony Brook. 2004
- Homological Mirror symmetry, University of New Mexico, 2004
- Homological Mirror symmetry, Caltech, 2004.
- Homological Mirror symmetry, WAGS 2004.
- Homological Mirror symmetry, UCSD 2004.
- Homological Mirror symmetry, UPenn 2004.
- Stability conditions for categories in Math Biology, pres. Hebrew University 2004.
- Homological Mirror Symmetry, University of Kyoto, 2005.
- Homological Mirror Symmetry, UCR, 2005.
- Homological Mirror Symmetry, U de Nice, 2005.
- Homological Mirror Symmetry, KSU, 2005.
- Homological Mirror Symmetry, Cambridge, 2005.
- Homological Mirror Symmetry, Arrowhead, 2005.
- Homological Mirror Symmetry, UCDavis, 2005.
- Homological Mirror Symmetry, SIAG, Seattle, 2005.
- Homological Mirror Symmetry,, Canbera, 2005.
- Derived Categories in Topology, Sydney, 2005.
- Birational Geometry and HMS, UPenn 2005.
- Birational Geometry and HMS, Vienna 2005.
- Birational Geometry and HMS, Moscow, 2005.
- Birational Geometry and HMS, Nice 2005.
- Birational Geometry and HMS, Stanford 2005.
- Birational Geometry and HMS, NYU 2005.

- Birational Geometry and HMS, Gatech 2005.
- Birational Geometry and HMS, UMich 2005.
- Algebraic Cycles and HMS, NYU 2006.
- Birational Geometry and HMS, Caltech, 2006.
- Birational Geometry and HMS, Moscow, 2006.
- Birational Geometry and HMS, Nice 2006.
- Birational Geometry and HMS, Glasgow 2006.
- Birational Geometry and HMS, Leeds 2006.
- Birational Geometry and HMS, OSU 2006.
- Birational Geometry and HMS, UMich 2006.
- Birational Geometry and HMS, KSU 2006.
- Birational Geometry and HMS, MPI Bonn, 2006.
- Homological Mirror Symmetry, Melbourne, 2006.
- Derived Categories in Topology, KIS, Seoul, 2006.
- Birational Geometry and HMS, UPenn 2006.
- Algebraic Cycles and HMS, ETH 2007.
- Algebraic Cycles and HMS, Toronto 2007.
- Algebraic Cycles and HMS, UPenn 2007.
- Algebraic Cycles and HMS, U of M 2007.
- Algebraic Cycles and HMS, Augsburg 2007.
- Algebraic Cycles and HMS, Florence 2007.
- Algebraic Cycles and HMS, Lausanne 2007.
- Algebraic Cycles and HMS, Split 2007.
- Algebraic Cycles and HMS, Warwick, 2007.
- Noncommutative Hodge Theory 1,2,3, TUW, 2007.

- Noncommutative Hodge Theory and Homological Mirror Symmetry Salamanca, 2008.
- Nonrationality of Conic Bundles , Bilbao 2008.
- Nonrationality of Conic Bundles , Bonn 2008.
- Nonrationality of Conic Bundles I, II, Pisa 2008.
- Nonrationality of Conic Bundles , Triese 2008.
- Noncommutative Hodge Theory and Homological Mirror Symmetry, MSRI, 2008.
- Noncommutative Hodge Theory and Homological Mirror Symmetry, Edingburgh , 2008.
- Nonrationality of Conic Bundles , Clay 2009.
- Nonrationality of Conic Bundles Budapest 2009.
- Nonrationality of Conic Bundles , Edingburgh , 2009.
- Nonrationality of Conic Bundles , MIT , 2009.
- Nonrationality of Conic Bundles , Moscow , 2009.
- DG schemes and HMS Vienna, 2009.
- DG schemes and HMS Goekova, 2009.
- DG schemes and HMS Salamanca, 2009.
- Spectra and Fukaya category Marecias 2009
- Spectra and Fukaya category Warwick 2009
- Spectra and Fukaya category Bonn 2009
- Spectra and Fukaya category U of Miami 2009
- Spectra and Fukaya category U of Michigan 2009
- Spectra and Fukaya category U of Minnesota 2009