Cent. Eur. J. Math. • 7(4) • 2009 • 571-605
DOI: 10.2478/s11533-009-0056-x

Central European Journal of **Mathematics**

Homological Mirror Symmetry for manifolds of general type

Research Article

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Received 22 August 2009; accepted 1 September 2009

Abstract: In this paper we outline the foundations of Homological Mirror Symmetry for manifolds of general type. Both Physics and Categorical prospectives are considered.

MSC: 57D37, 57R17, 14J33

Keywords: Homological mirror Symmetry • K theory • Categories

To Fedya, our teacher and friend, with admiration

1. Introduction

This paper discusses Homological Mirror Symmetry for manifolds of general type and the way this symmetry interacts with interesting conjectural dualities in complex and symplectic geometry. Traditionally, [7, 8], the mirror symmetry phenomenon is studied in the most relevant for the Physics case of Calabi-Yau manifolds. It was observed that some pairs of such manifolds ("mirror partners"), which naturally appear in mathematical models for physics string theory, exhibit properties (have numerical invariants) that are symmetic ("mirror") to each other. There were several attempts to mathematically formalize this mirror phenomenon, one of which was the celebrated Homological Mirror Symmetry Conjecture stated by Kontsevich in 1994. Given two Calabi-Yau mirror partners, they are naturally complex manifolds

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and symplectic manifolds. Accordingly, there are two categories defined on each of them: the bounded derived category of coherent sheaves (reflecting the complex structure of the manifolds and "physically" related to D-branes of type B), and the derived Fukaya category (a symplectic construct and related to D-branes of type A). What the Conjecture states is that, for any pair of mirror partners, the category of B-branes on each of the manifold is equivalent to the category of A-branes on the mirror partner.
Kontsevich's conjecture was verified for large number of Calabi-Yau manifolds: elliptic curves (Polischuk-Zaslow) [24], abelian varieties (Fukaya) [10], K3 surfaces (Seidel) [28].
A basic ingredient in these considerations is the construction of (physics) mirror partners. An approach by Hori and Vafa, interpreting the latter as a holomorphic fibration (Landau-Ginzburg model), turned out to be working well even beyond the case of Calabi-Yau manifolds. Specifically, this approach provides "mirror partners" for all complete intersections in toric manifolds. When the pairs of manifolds constructed this way are not Calabi-Yau, there is no physical ground to call them mirror symmetric. Moreover, the Landau-Ginzburg partner in this case, as given by the Hori-Vafa approach, is not compact. So, it was expected that appropriate modifications had to be made in the construction of this partner as well as in the definition of the relevant categories in order to make it possible to extract valuable geometric information out of this fibration. Furthermore, the formulation of the conjecture itself needed to be changed in order to make it a true statement. Suppose the A- and B-brane categories can be defined (with modifications on the Landau-Ginzburg partner) and compared. Then by definition, when these are cross-equivalent as in the HMSC, the manifolds are called (homologically) mirror symmetric. The task now would be to show that any complete intersection in a toric manifold and its Landau-Ginzburg partner provide a pair of (homologically) mirror partners. Indeed, an of idea how to change the categories exists, and it was proved that when the starting manifold is a del Pezzo surface (a Fano manifold), then the Landau-Ginzburg partner is a (homologically) mirror partner of the surface. The same result holds true for weighted projective planes and Hirzebruch surfaces as well.
In this paper we discuss how the HMSC should be modified so that it holds true in the case of manifolds of general type (with negative first Chern class). The conjectures we make are based on numerous examples some of which we present here. In particular, we study in considerable detail the Landau-Ginzburg partner of the hyperelliptic curves or of the associated (see [5, Theorem 2.7]) intersections of two quadrics.
Our main objective is to study how the HMSC for non-Calabi-Yau manifolds will interact with interesting geometric dualities on the A and the B side of the mirror correspondence. In this direction we propose a conjecture relating the complex or symplectic geometry of a pair of manifolds, one of which is a Fano, and the other is of general type:

Conjecture 3.3.

Let F and G be projective manifolds. Suppose F is a Fano manifold, and G is a manifold of general type or a Calabi-Yau manifold. Suppose further that there exists a fully faithful functor

$$\Phi : D^b(G) \hookrightarrow D^b(F)$$

between the derived categories of F and G. Write $\underline{F}$ and $\underline{G}$ for the C^∞ manifolds underlyng F and G. Then:

(i) For every complexified Kähler class α_F on F there exists a complexified Kähler class α_G on G and a fully faithful functor

$$\Psi : \overline{D\mathrm{Fuk}}(\underline{G}, \alpha_G) \hookrightarrow \overline{D\mathrm{Fuk}}(\underline{F}, \alpha_F)$$

between the corresponding Karoubi completed Fukaya categories.

(ii) If $K_\Phi \in D^b(G \times F)$ is a kernel object for Φ (such an object exists by [20]), then Ψ is given by a kernel object K_Ψ in $\overline{D\mathrm{Fuk}}(\underline{G} \times \underline{F}, -p_G^* \alpha_G + p_F^* \alpha_F)$ which is uniquely determined by K.

In particular, this conjecture predicts the existence of a functor (of geometric nature) between the Fukaya categories of a curve of genus two and the intersection of two quadrics in the projective 5-space. Using Seidel's recent proof [29] of HMS for the genus two curve, we can recast the above conjecture as a statement about B-branes in the mirror Landau-Ginzburg models. Our main result, Theorem 6.1 is that in a large complex structure limit a fully faithful functor between these Landau-Ginzburg models does exist. This proves the large volume case of Conjecture 3.3 when G is a curve of genus two and F is the associated degree four Fano variety in $\mathbb{P}^5$.

The analogue of Theorem 6.1 is expected to be true for G a hyper-elliptic curve of genus g and F the associated intersections of two quadrics in $\mathbb{P}^{2g+1}$. The recent proof [9] of mirror symmetry for higher genus curves by Efimov again allows us to reformulate the conjecture as a statement about holomorphic Landau-Ginzburg models and we expect that our method of proof will work in that case as well.

The paper is organized as follows. In section 2 we collect all the changes needed in the classical (regarding Calabi-Yau manifolds) definitions for the case of Landau-Ginzburg partners (fibrations over the complex line). In particular, we discuss the issue of the non-uniqueness of (partial) compactification and further desingularization of the originally given Landau-Ginzburg partner. We conclude the section by giving a warm-up example (of a degree four del Pezzo surface) where our procedures work perfectly in agreement with the expectations. In section 3, based on physics considerations, we state the main conjectures about how the HMS conjecture should look like in the case of manifolds of general type. In section 4 we study the Landau-Ginzburg partners of hyper-elliptic curves. The emphasis here is on the case of genus two curves. In the following section 5, we investigate the Landau-Ginzburg partner of the complete intersection of two quadrics in the 5-dmiensional projective space, and compare it with the results for genus 2 curve in section 4. In section 6 we compare the two mirrors and prove a large volume version of Conjecture 3.3 modulo the Homological Mirror Symmetry Conjecture for the intersection of two quadrics. In section 7 we show that the derived categories of three dimensional Landau-Ginzburg models are preserved by simple flops, a fact that is used repeatedly in our constructions of good models for Landau-Ginzburg mirrors. In the last section 8 we try to make the mirror correspondence for genus two curves more explicit by matching explicitly representattive objects in the Fukaya category with objects in the mirror Landau-Ginzburg model and comparing the corresponding Floer homologies and Ext groups.

Before we proceed, let us state the conjectures explaining how the HMS should behave in the case of manifolds of general type. To explain the conjectures we will need to introduce some notations. Consider

$\boxed{(\underline{X}, \omega)}$: a symplectic manifold associated with a projective manifold X of general type, i.e. $\underline{X}$ is the C^∞-manifold underlying X and ω is a symplectic form such that $c_1(X) = [-\omega]$. For instance take ω to be the curvature of a Hermitian connection on the canonical line bundle K_X.

$\boxed{\mathrm{w} : Y \to \mathbb{C}}$: a holomorphic Landau-Ginzburg mirror of the symplectic manifold $(\underline{X}, \omega)$. For instance, in the case when X is a complete intersection in a toric variety, the mirror $\mathrm{w} : Y \to \mathbb{C}$ can be constructed by the Hori-Vafa prescription [13].

With this notation we have the following:

Conjecture 3.1.

If $(\underline{X}, \omega)|(Y, \mathrm{w})$ is such a mirror pair, then:

- *The derived Fukaya category $D\mathrm{Fuk}(\underline{X}, \omega)$ embeds as a direct summand into the category $D^b(Y, \mathrm{w})$ of B-branes of $\mathrm{w} : Y \to \mathbb{C}$.*
- *The orthogonal complement of $D\mathrm{Fuk}(\underline{X}, \omega)$ in $D^b(Y, \mathrm{w})$ is very simple: it is a direct sum of several copies of the category of graded modules over a Clifford algebra of a symmetric bilinear form on a complex vector space of dimension $n = \dim_{\mathbb{C}} Y$.*

There is also an analogous mirror conjecture in which the A and B sides of the theory are switched. More precisely if denote the C^∞-manifold underlying Y by $\underline{Y}$, then the complex manifold X of general type should determine a symplectic structure η on $\underline{Y}$ with respect to which $\mathrm{w} : \underline{Y} \to \mathbb{C}$ becomes a symplectic Lefschetz fibration.

With this notation we have the following:

Conjecture 3.2.

If $X|(\underline{Y}, \mathrm{w}, \eta)$ is such a mirror pair, then the category $D^b(X)$ of B-branes on X is equivalent to the A-brane category $D\mathrm{FS}(\underline{Y}_D, \mathrm{w}, \eta)$ of a potential $\mathrm{w} : \underline{Y}_D \to D$, where $0 \in D \subset \mathbb{C}$ is a suitably chosen disk, and $\underline{Y}_D = \mathrm{w}^{-1}(D)$.

Remark.

While this work has been written two related works have appeared. The first is Seidel's paper [29] in which he proves HMS of genus 2 curves by a different approach. The second is the recent work of A.Efimov [9] who had proved the HMS conjecture for curves of arbitrary genus.

2. Some definitions

We will use the setup of HMS in situations when one of the partners is a Landau-Ginzburg fibration. For projective manifolds the categories of topological branes are well understood [12, 15]. In the simplest setup the category of branes in the B model is identified with the bounded derived category of coherent sheaves $D^b(X)$ on a complex manifold X, and the category of A-branes is the derived Fukaya category $D\mathrm{Fuk}(\underline{Y}, \omega)$ of a compact symplectic manifold $(\underline{Y}, \omega)$ [11]. It is a triangulated category which is the homotopy category of the category of twisted complexes over the geometrically defined Fukaya category of Lagrangian submanifolds equipped with complex local systems. More generally one has to enhance the categories $D^b(X)$ and $D\mathrm{Fuk}(\underline{Y}, \omega)$ with natural dg or A_∞-structures.

In the Landau-Ginzburg setting the categories of braness have to be modified in an appropriate manner [12, 15]. We begin by recalling the definition of the categories of D-branes in topological Landau-Ginzburg theories. First we deal with the symplectic (A-brane) side of the picture. We will follow Seidel's treatment of the A-twist of the Landau-Ginzburg model [27]. Historically the idea was introduced first by M.Kontsevich and later by K.Hori. The first non-trivial case was worked out by Seidel [25] who constructed a Fukaya-type A_∞-category associated to a symplectic Lefschetz pencil. Since this construction is a model for the general definition we briefly review it next.

Let $(\underline{Y}, \omega)$ be an open symplectic manifold, and let $\mathrm{w} : \underline{Y} \to \mathbb{C}$ be a symplectic Lefschetz fibration, i.e. a C^∞ complex-valued function with isolated non-degenerate critical points $p_1, \dots, p_r$. This means that the smooth parts of the fibers of w are symplectic submanifolds of $(\underline{Y}, \omega)$, and that near each p_i we can find a ω-adapted almost complex structure on $\underline{Y}$ so that in almost-complex local coordinates w is given by $\mathrm{w}(z_1, \dots, z_n) = \mathrm{w}(p_i) + z_1^2 + \cdots + z_n^2$. Fix a regular value λ_0 of w, and consider an arc $\gamma \subset \mathbb{C}$ joining λ_0 to a critical value $\lambda_i = \mathrm{w}(p_i)$. Using the horizontal distribution which is symplectic orthogonal to the fibers of f, we can transport a cycle vanishing at p_i along the arc γ to obtain a Lagrangian disc $D_\gamma \subset \underline{Y}$ fibered above γ, whose boundary is an embedded Lagrangian sphere L_γ in the fiber $\underline{Y}_0 = \mathrm{w}^{-1}(\lambda_0)$. The Lagrangian disc D_γ is called the *Lefschetz thimble* over γ, and its boundary L_γ is the Lagrangian vanishing cycle associated to the critical point p_i and to the arc γ.

Let $\gamma_1, \dots, \gamma_r$ be a collection of arcs in $\mathbb{C}$ joining the reference point λ_0 to the various critical values of w, intersecting each other only at λ_0, and ordered in the clockwise direction around p_0. Each arc γ_i gives rise to a Lefschetz thimble $D_i \subset \underline{Y}$, whose boundary is a Lagrangian sphere $L_i \subset \underline{Y}_0$. After a small perturbation we can always assume that these spheres intersect each other transversely inside $\underline{Y}_0$. Following Seidel [25] we have

Definition 2.1.

Fix a commutative unital coefficient ring R. The *directed Fukaya-Seidel category* $\mathrm{FS}((\underline{Y}, \omega, \mathrm{w}); \{\gamma_i\})$ *over* R is the following A_∞-category:

- the objects of $D\mathrm{FS}((\underline{Y}, \omega, \mathrm{w}); \{\gamma_i\})$ are the Lagrangian vanishing cycles $L_1, \dots, L_r$;
- the morphisms between the objects are given by

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} CF^*(L_i, L_j; R) = R^{|L_i \cap L_j|} & \text{if } i < j \\ R \cdot id & \text{if } i = j \\ 0 & \text{if } i > j; \end{cases}$$

  and the differential m_1, composition m_2 and higher order products m_k are defined in terms of Lagrangian-Floer homology inside $\underline{Y}_0$.

More precisely,

$$m_k : \mathrm{Hom}(L_{i_0}, L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}(L_{i_{k-1}}, L_{i_k}) \to \mathrm{Hom}(L_{i_0}, L_{i_k})[2-k]$$

is trivial when the inequality $i_0 < i_1 < \cdots < i_k$ fails to hold (i.e. it is always zero in this case, except for m_2 where composition with an identity morphism is given by the obvious formula).

When $i_0 < \cdots < i_k$, m_k is defined by fixing a generic ω-compatible almost-complex structure on $\underline{Y}_0$ and counting pseudo-holomorphic maps from a disc with $k+1$ cyclically ordered marked points on its boundary to $\underline{Y}_0$, mapping the marked points to the given intersection points between vanishing cycles, and the portions of boundary between them to $L_{i_0}, \dots, L_{i_k}$ respectively.

The derived Fukaya-Seidel category $D\mathrm{FS}((\underline{Y}, \omega, \mathrm{w}); \{\gamma_i\})$ is defined as the homotopy category of the category of twisted complexes (see [4] for the definition) over $\mathrm{FS}((\underline{Y}, \omega, \mathrm{w}); \{\gamma_i\})$.

Next we turn to the holomorphic (B-brane) side of the picture and discuss the B-twisted Landau-Ginzburg model. Following [21] we define

Definition 2.2.

Let Y be a complex algebraic variety and let $\mathrm{w} : Y \to \mathbb{C}$ be a regular function. The *derived category* $D^b(Y, \mathrm{w})$ *of the potential* w is defined as the disjoint union of the quotient categories $Q(Y_c)$ of all singular fibers $Y_c := \mathrm{w}^{-1}(c)$, $c \in \mathrm{crit}(\mathrm{w})$. The category $Q(Y_c)$ is the quotient category of derived category of coherent sheaves $D^b(Y_c)$ modulo the full triangulated subcategory of perfect complexes on Y_c.

The categories $D^b(Y, \mathrm{w})$ and $DFS(\underline{Y}, \omega, \mathrm{w})$ together with the derived category $D^b(X)$ of coherent sheaves of a complex variety X and the Fukaya category $D\mathrm{Fuk}(\underline{X}, \omega)$ play a central role in the HMS conjecture for non-Calabi-Yau manifolds. In particular, when the symplectic manifold $(\underline{X}, \omega)$ underlies a canonically polarized variety of general type and $(\underline{X}, \omega)|(Y, \mathrm{w})$ or $X|(\underline{Y}, \omega, \mathrm{w})$ are mirror pairs, the HMS conjecture predicts a precise relationship between $D\mathrm{Fuk}(\underline{X}, \omega)$ and $D^b(Y, \mathrm{w})$, as well as between $D^b(X)$ and $DFS(\underline{Y}, \omega, \mathrm{w})$. Next we investigate this relationship in more detail.

Suppose $(\underline{X}, \omega)$ is a compact symplectic manifold which underlies some complete intersection in a toric variety. The standard procedure for constructing the Landau-Ginzburg partner of $(\underline{X}, \omega)$ is based on the Hori-Vafa ansatz [13]. For every such $(\underline{X}, \omega)$ the Hori-Vafa prescription produces an affine complex Landau-Ginzburg mirror $(Y^{\mathrm{aff}}, \mathrm{w}^{\mathrm{aff}})$ where Y^{aff} is a closed algebraic submanifold in an affine torus $(\mathbb{C}^*)^n$, and the superpotential $\mathrm{w}^{\mathrm{aff}}$ is the restriction of a regular function on $(\mathbb{C}^*)^n$. Physically the properties of $(\underline{X}, \omega)$ are encoded in the critical level sets of the mirror superpotential $\mathrm{w}^{\mathrm{aff}}$. Often the affine mirror data $(Y^{\mathrm{aff}}, \mathrm{w}^{\mathrm{aff}})$ is too crude as an invariant and does not capture all the information in the A-model on $(\underline{X}, \omega)$. This inadequacy of $(Y^{\mathrm{aff}}, \mathrm{w}^{\mathrm{aff}})$ usually manifests itself geometrically in the fact that the critical loci of $\mathrm{w}^{\mathrm{aff}}$ are not compact and to obtain a viable Landau-Ginzburg theory one has to partially compactify Y^{aff} so that $\mathrm{w}^{\mathrm{aff}}$ becomes a proper map [13, Section 7.3]. Suppose $\mathrm{w}^{\mathrm{prop}} : Y^{\mathrm{prop}} \to \mathbb{C}$ is such partial compactification. Usually Y^{prop} is a variety with complicated singularities. The next step is to find a crepant resolution Y of Y^{prop}. After doing this and extending appropriately the superpotential, one gets a projective morphism $\mathrm{w} : Y \to \mathbb{C}$ with total space Y which is a smooth non-compact Calabi-Yau.

The pair (Y, w) is the Landau-Ginzburg partner of $(\underline{X}, \omega)$. By definition the category $D^b(Y, \mathrm{w})$ of B-branes for (Y, w) depends only on the structure of the singular fibers of w. The Landau-Ginzburg partner we construct is not uniquely defined by any means - but this was not to be expected either. Even in the case of Calabi-Yau manifolds the mirror partners are not uniquely defined. In our setup, there is also the additional issue of the different ways of building Y. Our numerous computations show that no matter how one (partially) compactifies and desingularizes the affine Hori-Vafa mirror, at least combinatorially the structure of the critical levels is the same. We expect that this will be the case on the level of categories as well. That is, we expect that $D^b(Y, \mathrm{w})$ will depend on $(\underline{X}, \omega)$ only. When X is 3-dimensional this is indeed true: the possible desingularizations of Y^{prop} differ by flops, and the category we get is the same (see Section 7).

In the following example we illustrate our approach for constructing a Landau-Ginzburg partner, and show that the produced categories have the expected by HMSC behaviour. The example deals with a del Pezzo surface in which case HMSC is a theorem [2]. We have chosen this example since the degree four del Pezzo is an intersection of two quadrics in $\mathbb{P}^4$ and the analysis of its mirror illustrates well how mirror constructions for intersection of quadrics should be approached.

Example 2.1.

Following the procedure from [13] (see also the begininning of our section 3) we compute the mirror of the manifold $(\underline{X}, \omega)$ which underlies a degree four Del Pezzo surface X realized as the intersection of two quadrics in $\mathbb{P}^4$. We get the following affine Hori-Vafa miror:

$$Y^{\mathrm{aff}} = \begin{cases} x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = e^{-t} =: A \\ x_1 + x_2 = -1 \\ x_3 + x_4 = -1 \end{cases}$$

and a potential

$$\mathrm{w}^{\mathrm{aff}} = x_5.$$

If we solve for x_2 and x_4 from the second and third equation and for x_5 from the first equation, we get

$$Y^{\mathrm{aff}} = \mathbb{C}^* \times \mathbb{C}^*, \text{ with coordinates } x_1 \text{ and } x_3$$

and

$$\mathrm{w}^{\mathrm{aff}} = \frac{A}{x_1(x_1+1)x_3(x_3+1)}.$$

In order to partially compactify and resolve the total space of the potential we do the following. Since the compatification is defined modulo birational transformations, we can start with any compactification of $\mathbb{C}^* \times \mathbb{C}^*$. We will use the four-point blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ where the points of the blow ups are $(x_1, x_3) = (\infty, 0), (\infty, -1), (0, \infty), (-1, \infty)$. On this blow-up the potential becomes a morphism to $\mathbb{P}^1$ and so we will get a fiberwise compactification $\mathrm{w} : Y \to \mathbb{C}$ if we delete the fiber at infinity, i.e. the proper transforms of the four lines $x_1 = 0$, $x_3 = 0$, $x_1 = -1$, and $x_3 = -1$ where the morphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ has poles. This gives us a superpotential whose critical locus consists of one isolated point $(x_1, x_3) = (-1/2, -1/2)$, and a degenerate critical locus consisting of the proper transforms of the lines $x_1 = \infty$ and $x_3 = \infty$.
We make a change of coordinates $t = 1/x_1$ and $u = 1/x_3$, so that $\mathrm{w} = t^2u^2/(t+1)(u+1)$ and the critical points are $(-2, -2)$ and the proper transforms of the two coordinate axes $t = 0$ and $u = 0$, while the poles are at t or $u = -1$ or ∞. We blow up $(0, -1)$, $(0, \infty)$, $(-1, 0)$, $(\infty, 0)$. We can deform w to a new superpotential $\mathrm{w}' = t(t-e)u(u-e)/(t+1)(u+1)$, where e is a small constant. This deformation preserves the compactness property - namely, w′ is well-defined over the blown-up space where we were defining w, and its critical points lie inside a compact subset, and do not escape to infinity when e tends to 0. In fact, one can also check that the topology of the generic fiber isn't changed (in all cases it is a four-times punctured elliptic curve). The potential w′ has only isolated critical points – 8 of them. Therefore we can compute the vanishing cycles for w′. As usual we do not count critical points over the fiber over ∞ in $\mathbb{P}^1$.
The 8 critical points of w′ are:

1) in the 4-component fiber $\mathrm{w}' = 0$, the four points $(0, 0)$, $(0, e)$, $(e, 0)$, (e, e).

2) for non-zero values of w′, the critical points correspond to solving:

$$\begin{cases} d\log \mathrm{w}'/dt = 1/t + 1/(t-e) - 1/(t+1) = 0, \\ d\log \mathrm{w}'/du = 1/u + 1/(u-e) - 1/(u+1) = 0, \end{cases} \quad \text{i.e.} \quad \begin{cases} t^2 + 2t - e = 0 \\ u^2 + 2u - e = 0 \end{cases}$$

which gives four points lying close to $(e/2, e/2)$, $(e/2, -2)$, $(-2, e/2)$, $(-2, -2)$ (if e is small). Note that $\mathrm{w}'(e/2, e/2) \simeq e^4/16$, $\mathrm{w}'(e/2, -2) = \mathrm{w}'(-2, e/2) \simeq e^2$, $\mathrm{w}'(-2, -2) \simeq 16$.

So, we have w′ with 4 critical values ($\lambda_1 = \cdots = \lambda_4 = 0$, $\lambda_5 \simeq e^4/16$, $\lambda_6 = \lambda_7 \simeq e^2$, $\lambda_8 \simeq 16$). We fix a reference fiber $\mathrm{w}' = \lambda_0 = e^4/32$, and arcs γ_i joining λ_0 to the critical values, to define vanishing cycles:

1. For $L_1, \ldots, L_4$ we choose a straight line from λ_0 to 0.
2. For L_5 we choose a straight line from λ_0 to λ_5.
3. For L_6, L_7 we choose an arc from λ_0 to $\lambda_6 = \lambda_7$ passing below the real axis.
4. For L_8 we choose an arc from λ_0 to λ_8 passing below the real axis.

We can find the vanishing cycles by viewing each fiber (except for $\mathrm{w}' = 0$ which is too singular) as a double cover of the t-plane branched at 4 points. The branch points of the fiber $\mathrm{w} = \lambda$ are given by roots of: $\lambda^2(t+1)^2 + e^2t^2(t-e)^2 + (4+2e)\lambda t(t-e)(t+1) = 0$, which we can plot for various values of λ to find the vanishing cycles. We will just summarize the answer:

- $L_1, \ldots, L_4$ are all disjoint (they occur in the same fiber $\mathrm{w}' = 0$).
- $L_1, \ldots, L_4$ all intersect each of $L_5, \ldots, L_8$ in exactly one point.
- L_5 intersects L_6 and L_7 in 2 points, L_8 in 4 points.
- L_6 and L_7 don't intersect; L_8 intersects L_6 and L_7 in 2 points.

The matrix of the corresponding quiver is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & 1 & 1 \\ & & & & 1 & 2 & 2 & 4 \\ & & & & & 1 & 0 & 2 \\ & & & & & & 1 & 2 \\ & & & & & & & 1 \end{pmatrix}$$

All entries correspond to actual numbers of morphisms, and the answer perfectly matches the derived category of a 5-point blowup of $\mathbb{P}^2$ (or equivalently a 4-point blowup of $\mathbb{P}^1 \times \mathbb{P}^1$), which is indeed a Del Pezzo surface of degree 4.
According to definition 2.1 the Fukaya-Seidel category is the A_∞-category of modules over the A_∞-version of the path algebra of this quiver where the higher products are given by disk instantons. With additional work, one can check that $m_1 = 0$ and $m_k = 0\ \forall k \geq 3$, and compute the composition m_2. So we have the following [2]:

Theorem 2.1.
The categories $D^b(X)$ and $\mathrm{DFS}(\underline{Y}, \mathrm{w}, \eta)$ are equivalent. The same statement holds true for the derived categories of noncommutative deformations of X and the categories associated with deformed symplectic structures on $\mathrm{w} : \underline{Y} \to \mathbb{C}$.

3. Main conjectures

Let X be a manifold of general type (i.e. X has a positive-definite canonical class). In this section we will formulate and motivate an analogue of the HMS conjecture for such manifolds.
A quantum sigma-model with target X is free in the infrared limit, while in the ultraviolet limit it is strongly coupled. In order to make sense of this theory at arbitrarily high energy scales, one has to embed it into some asymptotically free $N = 2$ field theory, for example into a gauged linear sigma-model (GLSM). Here "embedding" means finding a GLSM such that the low-energy physics of one of its vacua is described by the sigma-model with target X. In mathematical terms, this means that X has to be realized as a complete intersection in a toric variety.
The GLSM usually has other vacua as well, whose physics is not related to X. Typically, these extra vacua have a mass gap. To learn about X by studying the GLSM, it is important to be able to recognize the extra vacua. Let Z be a toric variety defined as a symplectic quotient of $\mathbb{C}^N$ by a linear action of the gauge group $G \simeq U(1)^k$. The weights of this action will be denoted Q_{ia}, where $i = 1, \dots, N$ and $a = 1, \dots, k$. Let X be a complete intersection in X given by homogeneous equations $G_\alpha(X) = 0$, $\alpha = 1, \dots, m$. The weights of G_α under the G-action will be denoted $d_{\alpha a}$. The GLSM corresponding to X involves chiral fields Φ_i, $i = 1, \dots, N$ and Ψ_α, $\alpha = 1, \dots, m$. Their charges under the gauge group G are given by matrices Q_{ia} and $d_{\alpha a}$, respectively. The Lagrangian of the GLSM depends also on complex parameters t_a, $a = 1, \dots, k$. On the classical level, the vector t_a is the level of the symplectic quotient, and thus parametrizes the complexified Kähler form on Z. The Kähler form on X is the induced one. On the quantum level the parameters t_a are renormalized and satisfy linear RG equations:

$$\mu \frac{\partial t_a}{\partial \mu} = \beta_a = \sum_i Q_{ia}.$$

In the Calabi-Yau case all β_a vanish, and the parameters t_a are not renormalized.
The mirror Landau-Ginzburg model has (twisted) chiral fields Λ_a, $a = 1, \dots, k$, Y_i, $i = 1, \dots, N$ and Υ_α, $\alpha = 1, \dots, m$. The superpotential is given by

$$\mathrm{w} = \sum_a \Lambda_a \left(\sum_i Q_{ia} Y_i - \sum_\alpha d_{\alpha a} \Upsilon_\alpha - t_a \right) + \sum_i e^{-Y_i} + \sum_\alpha e^{-\Upsilon_\alpha}.$$

The vacua are in one-to-one correspondence with the critical points of w. By definition, massive vacua are those corresponding to non-degenerate critical points. An additional complication is that before computing the critical points one has to partially compactify the target space of the Landau-Ginzburg model.

One can determine which vacua are "extra" (i.e. unrelated to X) as follows. The infrared limit is the limit $\mu \to 0$. Since t_a depend on μ, so do the critical points of w. A critical point is relevant for X (i.e. is not an extra vacuum) if and only if the critical values of e^{-Y_i} all go to zero as μ goes to zero. In terms of the original variables Φ_i, this means that vacuum expectation values of $|\Phi_i|^2$ go to $+\infty$ in the infrared limit. This is precisely the condition which justifies the classical treatment of vacua in the GLSM. It is instructive here to recall that that the classical space of vacua in the GLSM is precisely X.
Now let us state the analogue of the HMS for complete intersections X which are of general type.

Conjecture 3.1.

If $(\underline{X}, \omega)|(Y, \mathrm{w})$ is a Hori-Vafa mirror pair, then:

- *The Fukaya category $D\mathrm{Fuk}(\underline{X}, \omega)$ embeds as a direct summand into the category $D^b(Y, \mathrm{w})$ of B-branes of $\mathrm{w} : Y \to \mathbb{C}$.*
- *The orthogonal complement of $D\mathrm{Fuk}(\underline{X}, \omega)$ in $D^b(Y, \mathrm{w})$ is very simple: it is a direct sum of several copies of the category of graded modules over a Clifford algebra of a symmetric bilinear form on a complex vector space of dimension $n = \dim_{\mathbb{C}} Y$.*

There is also an analogous mirror conjecture in which the A and B sides of the theory are switched. More precisely if denote the C^∞-manifold underlying Y by $\underline{Y}$, then the complex manifold X of general type should determine a symplectic structure η on $\underline{Y}$ with respect to which $\mathrm{w} : \underline{Y} \to \mathbb{C}$ becomes a symplectic Lefschetz fibration.
With this notation we have the following:

Conjecture 3.2.

If $X|(\underline{Y}, \mathrm{w}, \eta)$ is such a mirror pair, then the category $D^b(X)$ of B-branes on X is equivalent to the A-brane category $D\mathrm{FS}(\underline{Y}_D, \mathrm{w}, \eta)$ of a potential $\mathrm{w} : \underline{Y}_D \to D$, where $0 \in D \subset \mathbb{C}$ is a suitably chosen disk, and $\underline{Y}_D = \mathrm{w}^{-1}(D)$.

Let us explain the physical reasoning behind these conjectures. To any $N = (2, 2)$ quantum field theory in two dimensions one can assign the categories of A-branes and B-branes. These categories are "topological", in the sense that they do not depend on the flat two-dimensional metric necessary to defined the quantum field theory. Since rescaling the metric is equivalent to rescaling the renormalization scale μ, this means that the categories of topological D-branes are μ-independent. By construction, the GLSM and its mirror Landau-Ginzburg model have identical physics in the infrared limit $\mu \to 0$, up to a mirror involution, so their categories of topological D-branes must be equivalent, up to exchange of A and B-branes.
It is important to note at this point that categories of topological D-branes are well-defined even when the topological twisting is not. For example, if $c_1(X) \neq 0$, then the B-type twist for the sigma-model with target X does not exist, but the category of B-branes on X is well-defined. The reason for this is that the structure of the category is specified by the boundary chiral ring, whose definition uses only the properties of the field theory on a flat world-sheet (with boundaries). If the topological twist exists, it provides additional structures on the category of branes (such as a Serre functor).
The neighborhood of each vacuum can be regarded as a separate physical theory, with its own categories of topological D-branes, which are full sub-categories in the respective categories of D-branes in the "parent" theory. This implies that the category of A-branes on X is a full sub-category of the category of B-branes for the mirror Landau-Ginzburg model, and vice versa.
The category of B-branes for a Landau-Ginzburg model has very strong localization properties [23]: it is a direct sum of categories determined by "critical-level schemes" of w. In a generic situation the values of w at all extra vacua are different and also different from the values of w at relevant vacua. This implies that each extra vacuum contributes independently, and that the total category of B-branes for the Landau-Ginzburg model is a sum of a category equivalent to the category of A-branes on X and the contributions of extra vacua.
Finally, if a critical point of w is non-degenerate, then the corresponding category of B-branes is equivalent to the category of graded modules over the Clifford algebra with n generators, where n is the number of variables on which w depends. We summarize the discussion above in the Table 1 below.
In this formalism we need to take a Karoubi closure on both sides of HMSC.

A side	B side
$\underline{X}$ — compact manifold, ω — symplectic form on $\underline{X}$. $D\mathrm{Fuk}(X,\omega) = \begin{cases} \text{Obj: } (L_i, \mathcal{E}) \\ \text{Mor: } HF(L_i, L_j) \end{cases}$ L_i — Lagrangian submanifold of X, $\mathcal{E}$ — flat $U(1)$-bundle on L_i	X — smooth projective variety over $\mathbb{C}$ $D^b(X) = \begin{cases} \text{Obj: } C_i^\bullet \\ \text{Mor: } Ext(C_j^\bullet, C_j^\bullet) \end{cases}$ $C_i^\bullet$ — complex of coherent sheaves on X
$(\underline{Y}, \eta)$ — open symplectic manifold, $\mathrm{w} : \underline{Y} \to \mathbb{C}$ — a proper C^∞ map with symplectic fibers. $DFS\,(\underline{Y}, \mathrm{w}, \eta) = \begin{cases} \text{Obj: } (L_i, \mathcal{E}) \\ \text{Mor: } HF(L_i, L_j) \end{cases}$ L_i — Lagrangian submanifold of $\underline{Y}_{\lambda_0}$, $\mathcal{E}$ — flat $U(1)$-bundle on L_i. $D\mathrm{FS}_{\lambda_i, r_i}(\underline{Y}, \mathrm{w}, \eta)$ – the Fukaya-Seidel category of $(\underline{Y}_{\lvert t-\lambda_i \rvert < r_i}, \mathrm{w}, \eta)$.	Y — smooth quasi-projective variety over $\mathbb{C}$, $\mathrm{w} : Y \to \mathbb{C}$ — proper algebraic map. $D^b(Y, \mathrm{w}) = \bigsqcup_t D^b(Y_t) \Big/ \mathrm{Perf}(Y_t)$ $D^b_{\lambda_i, r_i}(Y, \mathrm{w}) = \bigsqcup_{\lvert t-\lambda_i \rvert \leq r_i} D^b(Y_t) \Big/ \mathrm{Perf}(Y_t)$

Table 1. Kontsevich's HMS conjecture

The above statement of HMS interacts in a subtle way with various other physical and geometric operations, such as orientifold projections, phase flows in gauged linear sigma models, or large N-duality. In this direction we focus on a conjecture relating the complex or symplectic geometry of a pair of manifolds, one of which is a Fano, and the other is of general type:

Conjecture 3.3.

Let F and G be projective manifolds. Suppose F is a Fano manifold, and G is a manifold of general type or a Calabi-Yau manifold. Suppose further that there exists a fully faithful functor

$$\Phi : D^b(G) \hookrightarrow D^b(F)$$

between the derived categories of F and G. Write $\underline{F}$ and $\underline{G}$ for the C^∞ manifolds underlyng F and G. Then:

(i) For every complexified Kähler class α_F on F there exists a complexified Kähler class α_G on G and a fully faithful functor

$$\Psi : \overline{D\mathrm{Fuk}}(\underline{G}, \alpha_G) \hookrightarrow \overline{D\mathrm{Fuk}}(\underline{F}, \alpha_F)$$

between the corresponding Karoubi completed Fukaya categories.

(ii) If $K_\Phi \in D^b(G \times F)$ is a kernel object for Φ (such an object exists by [20]), then Ψ is given by a kernel object K_Ψ in $\overline{D\mathrm{Fuk}}(\underline{G} \times \underline{F}, -p_G^*\alpha_G + p_F^*\alpha_F)$ which is uniquely determined by K.

It is hard to make this conjecture more precise mainly because at the moment we do not have a good enough grasp on all objects in the Fukaya category of a symplectic manifold. Heuristically, the kernel object K_Ψ should be a coisotropic A-brane in the sense of [14], and one should be able to write the connection data for this coisotropic brane in terms of a (super) connection data on K_Φ which satisfies an appropriate form of the Hermite-Yang-Mills equations and is uniquely determined by the holomorphic structure on K_Φ and the Kähler structure $-p_G^*\alpha_G + p_F^*\alpha_F)$ on $\underline{G} \times \underline{F}$.

4. Hyperelliptic curves

In this section we study various aspects of Conjectures 3.1 and 3.2 in the case of hyperelliptic curves. In particular we give two explicit models for the Landau-Ginzburg mirror of a genus two curve.

4.1. Curves of genus two

Let $\underline{C}$ be a differentiable compact Riemann surface of genus two. One presentation of such a surface is as the C^∞ manifold underlying a divisor of bi-degree $(2,3)$ in the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In other words $\underline{C}$ is defined as the zeros of a generic form $p(z_1 : z_2; w_1 : w_2) \in \Gamma(\mathbb{F}_0, \mathcal{O}(2,3))$. Here z_1, z_2, w_1, and w_2 are the homogeneous coordinates of the rulings of $\mathbb{F}_0$. If we realize the latter as a GIT quotient $\mathbb{C}^4 /\!\!/ \mathbb{C}^{\times 2}$, then the weights of the torus action are given in the table:

z_1	z_2	w_1	w_2	p
1	1	0	0	-3
0	0	1	1	-2

Let $B + i\omega$ be a complexified Kähler form on the toric surface $\mathbb{F}_0$, which is homogeneous under the natural torus action. In other words B is a torus invariant closed real two form on $\mathbb{F}_0$, and ω is a torus invariant Kähler form. The Hori-Vafa mirror of $(\underline{C}, B + i\omega)$ is constructed in stages:

(i) Build the affine Hori-Vafa mirror $\mathrm{w}^{\mathrm{aff}} : Y^{\mathrm{aff}} \to \mathbb{C}$ of $(\underline{C}, B + i\omega)$ corresponding to the embedding in $\mathbb{F}_0$;

(ii) Compactify partially $\left(Y^{\mathrm{aff}}, \mathrm{w}^{\mathrm{aff}}\right)$ to a family $\mathrm{w}^{\mathrm{prop}} : Y^{\mathrm{prop}} \to \mathbb{C}$, so that Y^{prop} is a quasi-projective Gorenstein variety with trivial canonical class, and $\mathrm{w}^{\mathrm{prop}}$ is proper;

(iii) Construct a crepant (possibly stacky) resolution $\mathrm{w} : Y \to \mathbb{C}$ of the compactified family $\mathrm{w}^{\mathrm{prop}} : Y^{\mathrm{prop}} \to \mathbb{C}$;

In fact from the point of view of Conjectures 3.1 and 3.2 these construction steps should be supplemented by

(iv) Select a disk $0 \in D \subset \mathbb{C}$ so that the restricted family $(Y_D := \mathrm{w}^{-1}(D), \mathrm{w})$ is the mirror of the genus two curve.

Explicitly this means that:

(A to B) There is an equivalence of Karoubi closures

$$\overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega) \cong \overline{D^b}(Y_D, \mathrm{w}) = \bigsqcup_{t \in D} \overline{D^b(Y_t)/\operatorname{Perf}(Y_t)};$$

(B to A) The differentiable manifolds and functions underlying the holomorphic Landau-Ginzburg mirrors for the various $B + i\omega$ are all naturally diffeomorphic to a fixed C^∞ fibration $\mathrm{w} : \underline{Y} \to \mathbb{C}$. Moreover fpr any choice of a complex structure on $\underline{C}$, i.e. for any choice of a section $p(z_1 : z_2; w_1 : w_2) \in \Gamma(\mathbb{F}_0, \mathcal{O}(2,3))$, the complex curve $C_p : \{p = 0\}$

has an associated complexified symplectic form η_p on $\underline{Y}_D := \mathrm{w}^{-1}(D)$ so that $\operatorname{Im}\eta_p$ restricts to a symplectic form on the fibers of w and we have an equivalence

$$D^b(C_p) \cong \overline{\mathrm{DFS}}(\underline{Y}_D, \mathrm{w}, \eta_p).$$

where the bar on the right hand side denotes the Karoubi closure.

The first step in the constuction is just an implementation of the Hori-Vafa recipe described in the previous section. Following [13] we define the affine mirror of $(\underline{C}, B + i\omega)$ to be the pencil $\mathrm{w}^{\mathrm{aff}} : Y^{\mathrm{aff}} \to \mathbb{C}$ given by the function $\mathrm{w} = x_1 + \cdots + x_5$ over the threefold $Y^{\mathrm{aff}} \subset (\mathbb{C}^*)^5$ defined by the equations:

$$Y^{\mathrm{aff}} \ : \ \begin{cases} x_1 \cdot x_2 = a_1 \cdot x_3^3 \\ x_4 \cdot x_5 = a_2 \cdot x_3^2. \end{cases}$$

Here x_1, x_2, x_3, x_4, x_5 are the natural coordinates on $(\mathbb{C}^*)^5$, and a_1 and a_2 are non-zero complex constants corresponding to the $(B + i\omega)$-volume of the two rulings of $\mathbb{F}_0$. Concretely we have

$$a_1 := \exp\left(-\int_{\{\mathrm{pt}\}\times\mathbb{P}^1}(B + i\omega)\right), \text{ and } a_2 := \exp\left(-\int_{\mathbb{P}^1\times\{\mathrm{pt}\}}(B + i\omega)\right).$$

The most delicate and involved part of the above prescription are the steps (ii) and (iii), and carrying those out will occupy the rest of the section.

For step (ii) we note that equivalently we can think of Y^{aff} as the subvariety in $(\mathbb{C}^*)^4 \times \mathbb{C}$ with coordinates $(x_1, x_2, x_3, x_4; w)$, defined by the equations

$$Y^{\mathrm{aff}} \ : \ \begin{cases} x_4^2 - x_4(w - x_1 - x_2 - x_3) + a_2 x_3^2 = 0, \\ \qquad\qquad\qquad\qquad -x_1 x_2 + a_1 x_3^3 = 0. \end{cases}$$

In these terms the superpotential w becomes simply the projection on the factor $\mathbb{C}$ in the product $(\mathbb{C}^*)^4 \times \mathbb{C}$. To obtain the partial compactification $(Y^{\mathrm{prop}}, \mathrm{w}^{\mathrm{prop}})$ we embed $(\mathbb{C}^*)^4$ as the standard torus inside $\mathbb{P}^4$, and take Y^{prop} to be the closure of Y^{aff} in $\mathbb{P}^4 \times \mathbb{C}$, while $\mathrm{w}^{\mathrm{prop}}$ is again the projection on the factor $\mathbb{C}$.

Proceeding with step (iii) we have to resolve the singularities of Y^{prop}. The structure of the resolution is summarized in the following:

Theorem 4.1.

There exists a crepant resolution Y of Y^{prop} so that the the fibers of the function $\mathrm{w} : Y \to \mathbb{C}$ induced from $\mathrm{w}^{\mathrm{prop}}$ can be described as follows.

- *For $w \neq 0, \frac{1}{a_2}\left(\frac{1 \pm 2\sqrt{a_1}}{3}\right)^3$, the fiber $Y_w := \mathrm{w}^{-1}(w)$ is an elliptic K3 surface with two singular fibers of type I_9 and six singular fibers of type I_1. So, Y_w is a K3 surface of Picard rank 18.*
- *For $w = \frac{1}{a_2}\left(\frac{1 \pm 2\sqrt{a_1}}{3}\right)^3$, the fiber Y_w is singular and has a single isolated A_1-singularity. Its minimal resolution is an elliptic K3 with a $2 \cdot I_9 + 4 \cdot I_1 + I_2$ configuration of singular fibers.*
- *The central fiber Y_0 of the fibration $\mathrm{w} : Y \to \mathbb{C}$ is a union of three rational surfaces: $Y_0 = \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{S}_3$. These surfaces intersect in three copies of $\mathbb{P}^1$:*

$$q_1 = \widetilde{S}_1 \cap \widetilde{S}_3, \qquad q_2 = \widetilde{S}_2 \cap \widetilde{S}_3, \quad \textit{and} \quad q_3 = \widetilde{S}_1 \cap \widetilde{S}_2.$$

The three curves q_i all meet in two points: $q_1 \cap q_2 \cap q_3 = \{M, N\}$. In particular Y_0 is a type III degeneration of a K3 surface. The geometry of the components of Y_0 and the position of the curves q_i on them is described as follows.

(i) The component $\widetilde{S}_3$ is a blow up of a Hirzebruch surface $\mathbb{F}_2$ at 4 points of depth 2 on two generic sections of $\mathbb{F}_2$ and such that two pairs of these 4 points belong to two (generic) fibers of $\mathbb{F}_2$. The proper transforms of the two sections are the curves q_1 and q_2. Their intersection points are M and N.

(ii) To describe the union $\widetilde{S}_1 \cup \widetilde{S}_2$ we start as in (i) by blowing up $\mathbb{F}_2$ at the same points but only once this time. Consider the proper transforms of the two sections as in (i) but call them here q' and q''. Take two copies of thus blown up $\mathbb{F}_2$ and identify them along q'. The curves q'' will be identified with q_1 and q_2 on F. To get the union $\widetilde{S}_1 \cup \widetilde{S}_2$ we have to blow up two more times either the first or the second $\mathbb{F}_2$ at points infinitesimally near the already blown up ones on q'.[1]

The rank of the Picard group of Y_0 is 21.

Proof. As explained above we can obtain the partial compactification $\mathrm{w}^{\mathrm{prop}} : Y^{\mathrm{prop}} \to \mathbb{C}$ by closing Y^{aff} in $\mathbb{P}^4 \times \mathbb{C}$. By slightly abusing notation we will write $(x_0 : x_1 : x_2 : x_3 : x_4)$ for the the homogeneous coordinates on $\mathbb{P}^4$ and w for the coordinate on $\mathbb{C}$. In these terms we have

$$Y^{\mathrm{prop}} \ : \ \begin{cases} x_4^2 - x_4(\mathrm{w}x_0 - x_1 - x_2 - x_3) + a_1 x_3^2 = 0 \\ \qquad\qquad\qquad\qquad -x_0x_1x_2 + a_2x_3^3 = 0 \end{cases}$$

and $\mathrm{w}^{\mathrm{prop}}$ is induced by the projection to the second factor of $(\mathbb{C}^*)^4 \times \mathbb{C}$.
First we need to understand the general fiber of the desingularization of Y^{prop}:

Claim 4.1.
There is a crepant resolution

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "\mathrm{w}"'] & Y^{\mathrm{prop}} \arrow[d, "\mathrm{w}^{\mathrm{prop}}"] \\ & \mathbb{C} \end{tikzcd}$$

of the singularities of Y^{prop} so that:

- *For $w \neq 0$, $\frac{1}{a_2}\left(\frac{1\pm 2\sqrt{a_1}}{3}\right)^3$, the fiber $Y_w = \mathrm{w}^{-1}(w)$ is an elliptic K3 surface with two singular fibers of type I_9 and 6 singular fibers of type I_1.*
- *For $w = \frac{1}{a_2}\left(\frac{1\pm 2\sqrt{a_1}}{3}\right)^3$, the fiber Y_w has an isolated singularity o type A_1 and its minimal resolution is an elliptic K3 with a $2 \cdot I_9 + 4 \cdot I_1 + I_2$ configuration ofsingular fibers.*

Proof. The singular locus of Y^{prop} consists of nine sections of $\mathrm{w}^{\mathrm{prop}}$: three of transversal singularity type A_1

$$l_1 : (1 : w : 0 : 0 : 0; w) \qquad l_2 : (1 : 0 : w : 0 : 0; w) \qquad l_3 : (0 : 1 : -1 : 0 : 0; w),$$

and six of transversal type A_2

$$\begin{array}{lll} m_1 : (1 : 0 : 0 : 0 : w; w) & m_2 : (1 : 0 : 0 : 0 : 0; w) & m_3 : (0 : 0 : 1 : 0 : 0; w) \\ m_4 : (0 : 1 : 0 : 0 : 0; w) & m_5 : (0 : 0 : 1 : 0 : -1; w) & m_6 : (0 : 1 : 0 : 0 : -1; w). \end{array}$$

[1] *The fact that this construction of $\widetilde{S}_1 \cup \widetilde{S}_2$ is legitimate follows from the fact that (before blowing the two last points up) there is an obvious $\mathbb{Z}_2$ symmetry in the preimage of Y_0^{prop} which exchanges $\widetilde{S}_1$ and $\widetilde{S}_2$ and leaves the component $\widetilde{S}_3$ invariant. This in turn shows that the choices of the sections on the $\mathbb{F}_2$ we are making have to be in certain agreement with each other.*

The singularities of Y^{prop} can be resolved by blowing up these lines in the ambient space $\mathbb{P}^4 \times \mathbb{C}$.
The fiber Y_w^{prop} is transversal to the components of the singular locus of Y^{prop} for $w \neq 0$, and hense inherits $3A_1$ and $6A_2$ singular points which form the singular locus of the generic fiber of φ. These points get desingularized when we blow up the lines. From this it folows that the critical points of $\mathrm{w} : (Y - Y_0) \to (\mathbb{C} - 0)$ come from the ones of $\mathrm{w}^{\mathrm{prop}} : Y^{\mathrm{prop}} \to \mathbb{C}$. A direct computation shows that the latter map has only two critical points, at $w = \frac{1}{a_2}\left(\frac{1\pm 2\sqrt{a_1}}{3}\right)^3$.
Before turning to the structure of the central fiber Y_0, let us note one more feature of the manifold Y. If we denote by S the surface $\{x_0x_1x_2 = a_2x_3^3\} \subset \mathbb{P}^3$, then Y is a double cover of $S \times \mathbb{C}$ via the map which forgets the variable x_4 (that is, via projection with center $(0:0:0:1) \in \mathbb{P}^4$ fiberwise). Accordingly, every fiber Y_w is a double cover of S. This additional structure of Y will help us understand the geometry of the fibers Y_w.
Consider first $w \neq 0$. The cubic surface S has three A_2 singularities which are the pairwise intersection points of the three lines,

$$t_0 : x_0 = x_3 = 0, \qquad t_1 : x_1 = x_3 = 0, \qquad t_2 : x_2 = x_3. = 0.$$

Note that these lines are all contained S and that the triangle formed by this lines is the intersection of S with the plane $x_3 = 0$. The fiber Y_w^{prop} is a double cover of S branched at the union of two cubic curves on S:

$$D_1 = \{w \cdot x_0 - x_1 - x_2 - x_3 - 2\sqrt{a_1}x_3 = 0\} \cap S, \qquad D_2 = \{w \cdot x_0 - x_1 - x_2 - x_3 + 2\sqrt{a_1}x_3 = 0\} \cap S$$

which intersect transversally at the three points

$$P_0 = (0:1:-1:0), \qquad P_1(1:0:w:0), \qquad P_2(1:w:0:0)$$

belonging to t_0, t_1, and t_2 respectively. In particular, we see that there are two sources of singularities for Y_w^{prop}: the singular locus of S, and the points of $D_1 \cup D_2$. Hence Y_w is a double cover of the surface S' obtained by first desingularizing S and then blowing up additionally the points P_0, P_1, P_2. The surface S' is a rational elliptic surface. It has one fiber of type I_9 and three fibers of type I_1. For $w = \frac{1}{a_2}\left(\frac{1+2\sqrt{a_1}}{3}\right)^3$, one of these I_1 fibers is D_1, while for $w = \frac{1}{a_2}\left(\frac{1-2\sqrt{a_1}}{3}\right)^3$ one of them is D_2. The surface Y_w is a double cover of S' branched at the fibers of S' corresponding to D_1 and D_2. As a result we get that Y_w is an elliptic fibration with two I_9 and six I_1 fibers for $w \neq \frac{1}{a_2}\left(\frac{1\pm 2\sqrt{a_1}}{3}\right)^3$, while for $w = \frac{1}{a_2}\left(\frac{1\pm 2\sqrt{a_1}}{3}\right)^3$ the double cover has a singular point of type A_1 at the preimage of the node of D_1 and D_2 respectively. The double cover interpretation also shows that the generic Y_w is a smooth K3 surface of Picard number 18. This proves the claim. $\square$

We turn now to the case $w = 0$ and the description of the central fiber. The fiber Y_0^{prop} has six singular points: $4A_2$ and $1A_1$ singular points at infinity ($x_0 = 0$), and a deeper singular point at $(1:0:0:0:0;0)$ which corresponds to the intersection point of the components l_1, l_2, m_1, and m_2 of the singular locus of Y^{prop}. In terms of the double cover $Y_0^{\mathrm{prop}} \to S$ we see that the branch divisor has two singular points: P_0 (at infinity) and the intersection $t_1 \cap t_2$. The latter point corresponds to a collision of P_1 and P_2 as the generic branch divisor specializes to the the branch divisor of $Y_0^{\mathrm{prop}} \to S$. As before we see that the points at infinity get resolved with the blowing up the lines l_3, m_3, m_4, m_5, and m_6. The new feature here is the resolution at the point $(1:0:0:0:0;0)$.
Consider Y^{prop} away from $x_0 = 0$. We will denote this threefold by $V^a := Y^{\mathrm{prop}} - \{x_0 = 0\}$. It is a toric variety which in the $\mathbb{C}^5$ with coordinates x_1, x_2, x_3, x_4, x_5 is given by the equations

$$V^a \ : \ \begin{cases} x_1x_2 = a_1x_3^3 \\ x_4x_5 = a_2x_3^2. \end{cases}$$

Note also that the lines l_1, l_2, m_1, and m_2 (which all meet at the origin) are toric subvarieties of V^a. The desingularization of Y^{prop} at the origin can be easily understood in terms of the toric geometry of V^a. Indeed, the Newton polytope of V^a is shown in the picture below.
Desingularization of V^a amounts to subdividing this polytope. There are different ways to subdivide the rectangle. One can get from one subdivision to another geometrically by perfoming flops on the desingularized varieties. We choose to

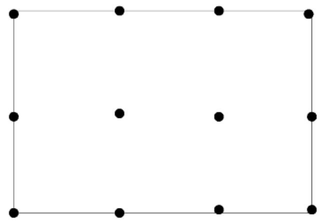

Figure 1. The Newton polytope of V^a.

subdivide the rectangle by first blowing up m_2. Then we blow up (in any order) the proper preimages of the lines m_1, l_1, and l_2. The result after these blow ups can be explained as follows.
(i) Denote by V^b the proper transform of V^a after we blow up m_2 in $\mathbb{C}^5$. The Newton polytope of V^b is depicted in Figure 2.

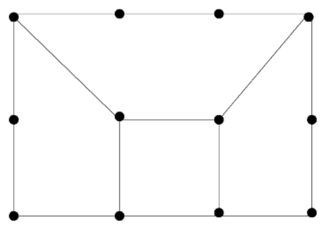

Figure 2. The Newton polytope of V^b.

The central fiber V_0^b is a union of three components: two exceptional $\mathbb{P}^2$'s, call them H_1 and H_2, and the proper transform F^a of V_0^a. Going back to the double cover structure on Y_0^{prop}, we see that V_0^a is a double cover of $S^a = S - \{x_0 = x_3 = 0\} \subset \mathbb{C}^3$ branched at two curves which intersect at the origin of $\mathbb{C}^3$. These curves intersect in such a way that the blow up of m_2 takes them appart, and F^a is actually a double cover of a smooth surface $\widetilde{S}^a$ (the only singularity of S^a gets resolved) branched in a smooth divisor. We see then that the components of V_0^b are all smooth surfaces.
We are now ready to describe the proper transform F of Y_0^{prop} that we get after we resolve the whole Y^{prop}. As mentioned above, Y_0^{prop} is a double cover of S branched in two curves, and the singularities of Y_0^{prop} arise from the singularities of S and the singularities of the branch curve. On the surface S the efect of blowing up the lines m_1, m_3, m_4, m_5, and m_6 in the ambient space, can be seen as a resolution of the singularities of the branch curve. Call the resulting surface $\widetilde{S}'$. The proper transform of Y_0^{prop} under these modifications is a double cover of $\widetilde{S}'$ branched at two curves which intersect at a point which belongs to the line l_3. After resolving along l_3, we obtain the ultimate proper transform F as a double cover of the blow up of $\widetilde{S}'$ at the point where l_3 intersects it, branched in a smooth divisor. So F is itself smooth. The surface $\widetilde{S}'$ can also be obtained from $\mathbb{P}^2$ by blowing up two of the vertices of a triangle three times each so that the side these vertices define gets blown up only twice, and the other two get blown up three times each. The branch curves are then the preimages of two lines, in general position w.r.t. the triangle, and with a commom point on the side of the triangle blown up only twice. Accordingly, after we blow up along l_3, we can identify $\widetilde{S}'$ with the Hirzebruch surface $\mathbb{F}_1$ blown up six times at points belonging to a fiber and two sections, in general position, so that the sections get blown up three times each. From this point of view the branch divisors will consist of two general fibers of the blown up $\mathbb{F}_1$. As a result we get a description of Fas the blowing up of two general sections on a Hirzebruch surface $\mathbb{F}_2$ at points belonging to two general fibers (four points in all) and their infinitesimally near ones so that the sections get blown up three times in each of the ponts. We still need to resolve along m_1, l_1, and l_2. As before, the toric picture will help us understand how this resolution will affect the picture we already have. We will complete this in (ii) and (iii) below.
(ii) Next we study how F, H_1 and H_2 intersect each other. Consider the blow up of $\mathbb{C}^5$ at m_2 in the affine chart where

$$x_1 = u_1/u_4, \qquad x_2 = u_2/u_4, \qquad x_3 = u_3/u_4.$$

The equations of the proper transform of V^a are

$$V^b \ : \ \begin{cases} (u_1/u_4)\cdot(u_2/u_4) = a_1\cdot x_4\cdot(u_3/u_4)^3 \\ w - x_4\cdot(1+u_1/u_4+u_2/u_4+u_3/u_4) = a_2\cdot x_4\cdot(u_3/u_4)^2 \end{cases}$$

In Y the exceptional divisor of this blow up is given by $x_4 = w = (u_1/u_4)\cdot(u_2/u_4) = 0$, so it is a union of two planes which we called H_1 and H_2 respectively. In the $\mathbb{P}^3$ with homogeneous coordinates $(u_1:\dots:u_4)$. The proper transform V^b has a fiber above $w=0$ consisting of three components: the exceptional planes H_1, H_2, and the proper transform of V_0^a.

The planes H_1 and H_2 intersect in the line $q_3:\ u_1 = u_2 = 0 \subset \mathbb{P}^3$. The intersection of H_1 (respectively H_2) with F is a curve q_1 (respectively q_2) which is the preimage of one of the exceptional curves of the desingularization $\widetilde{S}$ of S. These (-2) exceptional curves, as obvious from the explanations above, meet the branch divisor of the double cover $F\to\widetilde{S}$ at two points each. This is why, on F, they will lift to $\mathbb{P}^1$'s of self-intersection -4. The curve q_i is a conic on H_i, $i=1,2$. All three curves intersect in two triple points.

For future references let us point out that H_i considered as toric subvarieties of V^b have three fixed points and three fixed lines each under the torus action. Two of the points are common (belong to q_3, which is one of the fixed lines for each of the planes), and the respective third points belong to the individul planes. As it follows by direct inspection, the conics q_1 and q_2 intersect q_3 away from the fixed points on the latter, and the former lines pass through the third fixed point on the corresponding plane. Finally, the lines $u_4=0$ on each of H_1 and H_2 are tangent at the torus-fixed points to q_1 and q_2 while the analogous lines with $u_3=0$ intersect the conics in two points (one of them is a torus-fixed one). Also, as it can be computed in the coordinates above, the (-1)-curves on $\widetilde{S}$ that result from the blow-up of $\mathbb{F}_1$ lift to two pairs of (-1)-curves on F in such a way that they intersect also the lines with $u_3=0$ in each H_i.

(iii) Finally we explain the desingularization of V^b. The proper transforms of m_1, l_1 and l_2 intersect $V_0^b = H_1\cup H_2\cup F$ in singular points of V_0^b. Hence, they intersect it in points of the curves q_i, $i=1,2,3$. These proper transforms are toric subvarieties of V^b, so they intersect H_i in torus-fixed points. By the toric picture of V^b we see that H_1 and H_2 intersect only one of the lines l_1 and l_2, and both intersect the line m_1. Hence, l_1 and l_2 intersect the exceptional planes at the torus-fixed points on the conics q_i, while m_1 intesects q_3 at the fixed point with $u_4\neq 0$.

Blowing up the three lines (and thus resolving the generic fiber of Y^{prop}) blows up H_1 and H_2 in two points each (they become del Pezzo surfaces, $\widetilde{H}_1$ and $\widetilde{H}_2$, of degree 7 each). The common line of these del Pezzos, the proper transform of q_3, has self-intersection 0 on each of them. In this process F gets blown up in two points (to the surface F^b), and the curves of intersection of this surface with $\widetilde{H}_1$ and $\widetilde{H}_2$ have self-intersection 3 on the del Pezzos and -5 on F^b. The subdivision of the Newton polytope we get at this point are shown on Figure 3.

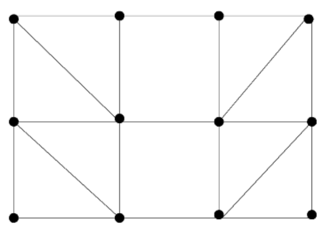

Figure 3. The Newton polytope of the resolution of V^b.

There are still two singular points on V^b to be resolved. According to the toric picture these are the common points of $\widetilde{H}_1$ and $\widetilde{H}_2$ only. These actually are the fixed points under the torus action on the intersection of the two degree 7 del Pezzos (and are far from F^b). We see also that the two points can be resolved by using small resolutions as Figure 4 illustrates.

The resulting intersection line of the two exceptional components, call the components S_1 and S_2, will be either $0,-2$ or $-1,-1$ or $-2,0$ depending on the choice of the small resolution. Do now two flops in the exceptional curves on S_3 from the last blow up. The proper preimage of Y_0^{prop} will then be the surface F, and the exceptional surfaces S_1 and S_2 get blown up one more time. The central fiber Y_0 of the desingularization Y will consist of three components: $\widetilde{S}_3$, $\widetilde{S}_1$

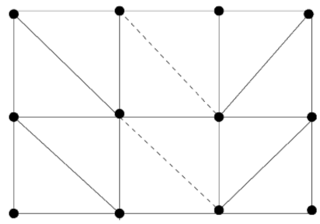

Figure 4. The Newton polytope of the small resolution.

and $\widetilde{S}_2$. These intersect in three curves ($\mathbb{P}^1$s) with self-intersections $(-4, 2)$ for $(F, \widetilde{S}_i)$, and $(-2, 0)$, $(-1, -1)$, or $(0, -2)$ on $(\widetilde{S}_1, \widetilde{S}_2)$.

The component $\widetilde{S}_3$ corresponding F, is then a blow up of the Hirzebruch surface $\mathbb{F}_2$ in the following sequence of 14 points. Choose two generic sections of $\mathbb{F}_2$ and mark on them four points defined by the intersection with two generic fibers. The proper transform $\widetilde{S}_3$ of Y_0^{prop} is the blow up of $\mathbb{F}_2$ at each of the marked points, then at infinitesimally near points of order one and two to the marked ones where, at each step, the infinitesimally near points belong to the proper transforms of the chosen sections. The proper transforms of the chosen sections after these 12 blow ups are the curves of intersection q_1 and q_2 of F with the exceptional components H_1 and H_2. The component $\widetilde{S}_3$ is the blow up of F at two more points (the effect on F of blowing up the lines l_1 and l_2 on V). These are points on q_1 and q_2 which belong to exceptional curves of the blown up $\mathbb{F}_2$ as explained above.

For the sake of later considerations, we prefer to work with a different desingularization of Y^{prop}. As noticed above, different desingularizations differ by flops from the one we just constructed. Flops can be performed along $(-1, -1)$-lines, and these are exactly the exceptional curves in Y_0 which meet only two components of Y_0. In particular, we can do flops 4 times on F in such a way that this component gets obtained from $\mathbb{F}_2$ by blowing up the two sections in points of depth two instead of three. This will result in blowing up $\widetilde{S}_i$ two more times (at points on q_i). For the sake of symmetry we also prefer to do the small resolutions discussed above in both components S_1 and S_2. We will be considering this desingularization and be using the same notation for the central fiber and its components. □

As a corollary, we get the combinatorial structure of the singularities of $\mathrm{w} : Y \to \mathbb{C}$.

Corollary 4.1.

The set of critical points of w *consists of:*

- *two isolated points* $y_\pm$ *with* $\mathrm{w}(y_\pm) = \frac{1}{a_2}\left(\frac{1 \pm 2\sqrt{a_1}}{3}\right)^3$, *and*
- *three* $\mathbb{P}^1$*'s:* q_1, q_2, q_3, *passing through two other points* M *and* N. *(See figure 5)).*

Note that the positive dimensional part of the critical locus, i.e. the union $q_1 \cup q_2 \cup q_3$, *is a singular curve of genus 2.*

Remark 4.1.

• From the proof of Theorem 4.1 we see that $y_\pm$ are ordinary double points in their respective fibers and so the categories $D^b_{\mathrm{sing}}(Y_{\mathrm{w}(y_\pm)})$ are just categories of modules over a Clifford algebra. Therefore for step (iv) above it is natural to take the disk $D \subset \mathbb{C}$ to be any disk which contains $0 \in \mathbb{C}$ bud does not contain the critical values $\frac{1}{a_2}\left(\frac{1 \pm 2\sqrt{a_1}}{3}\right)^3$

• In view of the completion theorem [23, Theorem 2.10] the Karoubi closure $\overline{D^b}(Y, \mathrm{w})$ of the category $D^b(Y, \mathrm{w})$ is determined by the points $y_\pm$ and a formal neighborhood of the singular curve $q_1 \cup q_2 \cup q_3$ inside the fiber Y_0.

Our next task is to analyze the the structure of the category $D^b(Y_D, \mathrm{w}) = D^b_{\mathrm{sing}}(Y_0)$ in more detail and in particular to understand how this category depends on the moduli of the Landau-Ginzburg model (Y, w). To do that we will need a better model for (Y, w) in which the components of the singular fiber Y_0 appear in a more symmetric way.

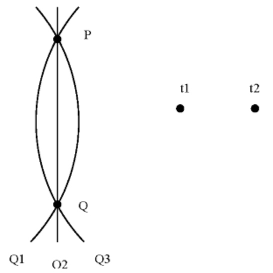

Figure 5. The critical points of w.

Since Y is three dimensional, the category of a superpotential $\mathrm{w} : Y \to \mathbb{C}$ does not change if we modify Y by a simple flop (see Section 7) we can try to simplify the zero fiber of our superpotential by performing flops on the compactification constructed in Theorem 4.1. Before we can explain the outcome of these modifications we will need the following

Construction. Fix a complex number λ. Consider the ruled surface $\mathbb{F}_2$ with its natural projection $\pi : \mathbb{F}_2 := \mathbb{P}(\mathcal{O}(2) \oplus \mathcal{O}) \to \mathbb{P}^1$. Let $\delta \subset \mathbb{F}_2$ be the unique (-2)-section of π. Fix four distinct points p_0, p_∞, p_1, p_2 of cross ration $(p_0, p_\infty, p_1, p_2) = \lambda$, and let P_1, P_2 be points in $\mathbb{F}_2 - \delta$ with $\pi(P_i) = p_i$, $i = 1, 2$. Consider the pencil of curves in the linear system $|\mathcal{O}(\delta) \otimes \pi^*\mathcal{O}(2)|$ passing through P_1 and P_2. Let Q' and Q'' be two irreducible curves that generate this pencil. Note that Q' and Q'' are sections of π that do not intersect δ and so they each have self intersection 2. Note that if we fix the points p_0, p_∞, p_1, p_2 the automorphisms of $\mathbb{F}_2$ that act along the fibers of π and preserving δ will act transitively on the data (P_1, P_2, Q', Q'').

Indeed these automorphisms are just affine automorphims of the total space of $\mathcal{O}(2)$ and so are generated by scaling along the fibers or translations by sections of $\mathcal{O}(2)$. Since we can always find a quadratic polynomial to take two prescribed values at p_1 and p_2, we can use a translation to bring P_1 and P_2 to points on the zero section of $\mathcal{O}(2)$. After this translation Q' and Q'' will be just two distinct sections of $\mathcal{O}(2)$ which vanishes at p_1, p_2. Since up to scale there is a unique non-zero section ξ of $\mathcal{O}(2)$ which vanishes at p_1, p_2 we can find constants $c' \neq c''$ so that $Q' = c' \cdot \xi$ and $Q'' = c'' \cdot \xi$. Translating by $(-c' \cdot \xi)$ and scaling by $(c'' - c')^{-1}$ will keep the points P_1, P_2 on the zero section but will move the sections Q' and Q'' to the zero section and the section ξ respectively.

In particular, up to isomorphism the data (P_1, P_2, Q', Q'') is determined by the four points p_0, p_∞, p_1, p_2. Consider the rational surface obtained by blowing up $\mathbb{F}_2$ at the four points on Q' and Q'' sitting over the points p_0 and p_∞. By what we just explained up to isomorphism this surface is uniquely determined by the ordered qudruple of points p_0, p_∞, p_1, p_2 in $\mathbb{P}^1$ Since the quadruple of points has only one modulus - the cross ratio λ - it follows that up to isomorphism the rational surface is determined by λ. For future reference we will denote this surface by $\mathfrak{S}^\lambda$. The proper transforms of Q' and Q'' are two rational curves on $\mathfrak{S}^\lambda$ intersecting at two points. We will denote these curves by $\mathfrak{q}'$ and $\mathfrak{q}''$. Notice that the surfaces $\mathfrak{S}^\lambda$ and $\mathfrak{S}^{1/\lambda}$ are naturally isomorphic since we can pass from one to the other by switching the order of the points p_1 and p_2.

We are now ready to describe the better model for the Landau-Ginzburg mirror of $(\underline{C}, B + i\omega)$:

Theorem 4.2.

There exists a crepant resolution Y *of* Y^{prop} *so that the the fibers of the function* $\mathrm{w} : \mathsf{Y} \to \mathbb{C}$ *induced from* $\mathrm{w}^{\mathrm{prop}}$ *can be described as follows.*

- *For* $\mathrm{w} \neq 0, \frac{1}{a_2}\left(\frac{1 \pm 2\sqrt{a_1}}{3}\right)^3$, *the fiber* $\mathsf{Y}_w := \mathrm{w}^{-1}(w)$ *is an elliptic K3 surface with two singular fibers of type* I_9 *and six singular fibers of type* I_1, *i.e.* Y_w *is a K3 surface of Picard rank 18.*

- *For* $w = \frac{1}{a_2}\left(\frac{1 \pm 2\sqrt{a_1}}{3}\right)^3$, *the fiber* Y_w *is singular and has a single isolated* A_1*-singularity. Its minimal resolution is an elliptic K3 with a* $2 \cdot I_9 + 4 \cdot I_1 + I_2$ *configuration of singular fibers.*

- *The central fiber* Y_0 *of the fibration* $\mathrm{w} : \mathsf{Y} \to \mathbb{C}$ *is a union of three rational surfaces:* $\mathsf{Y}_0 = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3$. *These surfaces intersect in three copies of* $\mathbb{P}^1$:

$$\mathsf{q}_1 = \mathsf{S}_1 \cap \mathsf{S}_3, \qquad \mathsf{q}_2 = \mathsf{S}_2 \cap \mathsf{S}_3, \quad \textit{and} \quad \mathsf{q}_3 = \mathsf{S}_1 \cup \mathsf{S}_2.$$

The three curves $\mathfrak{q}_i$ *all meet in two points:* $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3 = \{M, N\}$. *Explicitly* Y_0 *and the position of the curves* $\mathfrak{q}_i$ *on it can be described as follows. Fix once and for all an ordering* λ_1, λ_2 *of the roots of the polynomial* $x^2 + x + a_2 = 0$ *and let* $\lambda = \lambda_1/\lambda_2$ *be their ratio. Consider three copies* S_i, $i = 1, 2, 3$ *of the surface* $\mathfrak{S}^\lambda$, *each with two marked rational curves* $\mathfrak{q}'_i$ *and* $\mathfrak{q}''_i$. *The fiber* Y_0 *is the normal crossings surface obtained by gluing the components* $\{\mathsf{S}_i\}_{i=1,2,3}$ *along the curves* $\mathfrak{q}'_i$ *and* $\mathfrak{q}''_i$, *where we identify* $\mathfrak{q}''_i$ *with* $\mathfrak{q}'_{i+1}$ *(for all* $i \mod 3$*) and the identification are such that the ordered quadruples of marked points on each curve are matched.*

The rank of the Picard group of Y_0 *is 21.*

Proof. The proof of the theorem follows immediately from the proof of Theorem 4.1. We will not repeat any of the arguments and will just indicate the flops that one needs in order to construct Y from Y.
After blowing up the line m_2 in Y^{prop} two of the components we had in the central fiber were the planes H_1 and H_2 intersecting along the line q_3. Furthermore each H_i contained a conic q_i as depicted on Figure 6.

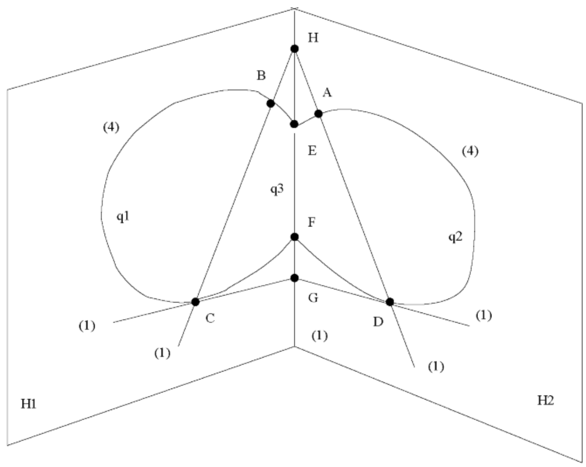

Figure 6. The planes H_1 and H_2.

The self-intersection numbers of the various curves in Figure 6 are indicated in parentheses. The points H, G, C, and D appearing in Figure 6 are fixed by the torus action. The lines GC and GD are tangent lines to the conics q_1 and q_2 at C and D respectively. The cross ratios (D, F, E, A), (G, F, E, H), and (C, F, E, B) are all equal to λ. This observation will be essential in construction of Y.
Next we proceed with the flops on page 585. The picture for H_1 and H_2 after doing these flops but before performing the small resolutions (at G and H) is sketeched in Figure 7.
It is clear from Figure 7 that the two components of Y_0 corresponding to H_1 and H_2 can be obtained by blowing up $\mathbb{F}_2$ at two points of the curve q_3. Therefore, these two components look similar to the F component. To get to the final picture in Theorem 4.1, we have to perform small resolution at H and G in different components. In particular the curve q_3 on the two $\mathbb{F}_2$ will be blown up at 3 points total. To get the picture for Y_0, we have to blow up points on the conics q_1 and q_2 as well. Specifically we have to blow up q_1 and q_2 three times each at A, C and B, D respectively. The whole process is summarized in Figure 8. These blow-ups correspond to flops on the total space Y: they can be achieved by flopping suitable (-1) curves on the component F. The important fact here is that the intersection curves q_1, q_2, and q_3, after all resolutions and flops, meet at E and F, and also have two more marked points with (-1) curves at each one on the corresponding surface.

□

Remark 4.2.

Note that the description of the fiber Y_0 in the previous theorem shows that if we vary the symplectic curve $(\underline{C}, B + i\omega)$ the complex algebraic surface Y_0 varies but has one dimensional moduli parameterized by the cross-ratio parameter λ,

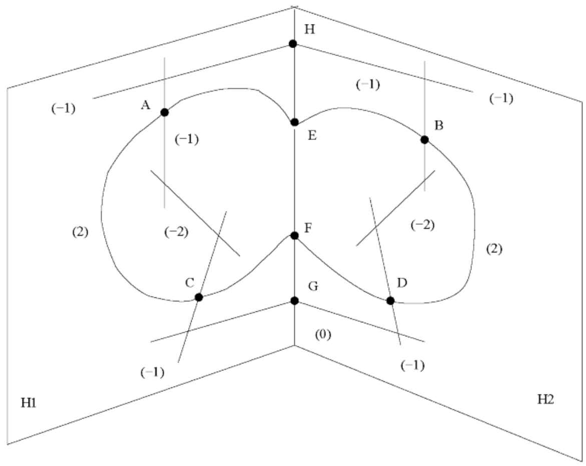

Figure 7. The planes H_1 and H_2 after flops in the exceptional curves on $\widetilde{S}_3$.

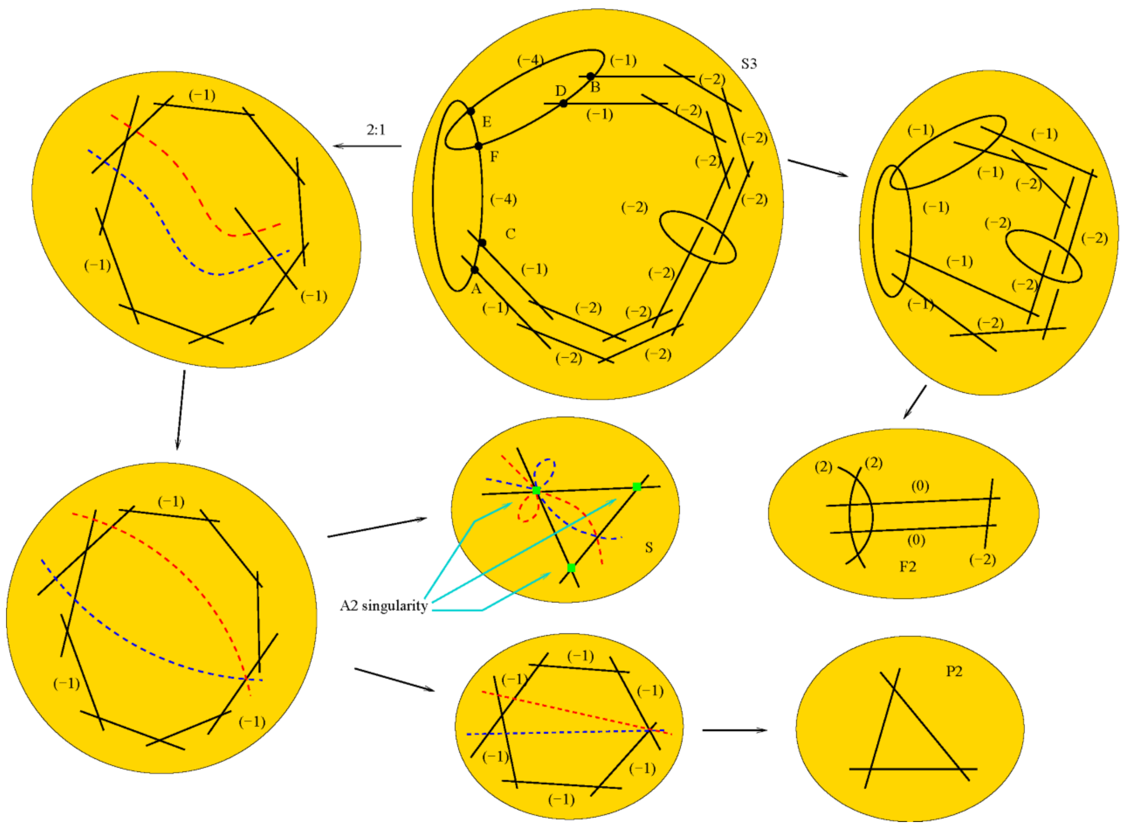

Figure 8. The effect of the flops on $\widetilde{S}_3$. The dashed curves depict the branch divisors of the double cover structure on $\widetilde{S}_3$.

which in turn depends only on the symplectic volume parameter $a_2 = \exp\left(-\int_{\mathbb{P}^1\times \mathrm{pt}}(B+i\omega)\right)$. This is not surprising since the map from torus equivariant Kähler classes on $\mathbb{P}^1\times\mathbb{P}^1$ has a one dimensional kernel and so there is a non-trivial relation between a_1 and a_2 when we view them as Kähler parameters on C.

The formal neighborhood of the singular locus of Y_0 likely depends of the gluing parameter λ as well. Through the localization theorem [23, Theorem 2.10] this gives and apparent dependence on λ for the Karoubi closure

$$\overline{D_{\text{sing}}}(Y_0) = \overline{D^b}(Y_D, w).$$

This observation suggests that the conjectural (A to B) equivalence

$$\overline{D\text{Fuk}}(\underline{C}, B + i\omega) \cong \overline{D^b}(Y_D, w) \tag{A to B}$$

predicted by HMS is really a family of different equivalences labeled by λ.
One can try to detect the λ-dependence in (A to B) on the level of K-groups. From the work of Abouzaid [1] it is known that the K-group of the Karoubi-closed derived Fukaya category $\overline{D\text{Fuk}}(\underline{C}, B+i\omega)$ is not finitely generated. More precisely from [1] it follows that the K-group of the Fukaya category is given by

$$\begin{aligned} K_0(\overline{D\text{Fuk}}(\underline{C}, B + i\omega)) &= H_1(\underline{C}, \mathbb{Z}) \oplus \mathbb{Z}/\chi(\underline{C}) \oplus \mathbb{C}^\times \\ &= \mathbb{Z}^4 \oplus \mathbb{Z}/2 \oplus \mathbb{C}^\times. \end{aligned}$$

Combining this with the recent proof by Seidel [29] of the mirror equivalence (A to B) we conclude that

$$K_0(\overline{D^b}(Y_D, w)) \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2 \oplus \mathbb{C}^\times,$$

but perhaps there is a whole family of such isomorphisms which depends non-trivially on λ.
In fact in the last formula can be verified by a direct calculation [22] in $\overline{D^b}(Y_D, w)$ but we will skip this verification here. Some evidence for that is provided by the fact that the K-group of the uncompleted category $D^b(Y_D, w) = D^b_{\text{sing}}(Y_0)$ depends [22] non-trivially on the moduli of Y_0.
Secifically suppose that λ is a primitive n-th root of unity and that $a_2 \in \mathbb{C}$ is such that λ is the ratio of the roots $x^2 + x + a_2 = 0$ in some order. (In other words we have $a_2 = \lambda/(\lambda + 1)^2$.) If Y_0 is the singular surface of modulus λ constructed in theorem 4.2, then one can check [22] that

$$K_0(D^b_{\text{sing}}(Y_0)) \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2n.$$

4.2. General hyperelliptic curve

Let us consider a hyperelliptic curve C of genus $k-1$. We can think of it as a difisor of bi-degree $(2, k)$ in the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})$. Similarly to the the genus two case $k = 3$ we get the weights of the torus action in the table:

z_1	z_2	w_1	w_2	p
1	1	0	0	$-k$
0	0	1	1	-2

The Hori-Vafa procedure [13] gives an affine mirror $(Y^{\text{aff}}, w^{\text{aff}})$ of $(\underline{C}, B+i\omega)$ where Y^{aff} is the threefold in $\mathbb{C}^{\text{f5}}$ defined by the equations

$$Y^{\text{aff}} \ : \ \begin{cases} x_1 \cdot x_2 = a_1 \cdot x_3^k \\ x_4 \cdot x_5 = a_2 \cdot x_3^2. \end{cases} \tag{1}$$

and the pencil $w^{\text{aff}} : Y^{\text{aff}} \to \mathbb{C}$ is defined by the formula

$$w^{\text{aff}} = x_1 + \cdots + x_5. \tag{2}$$

The corresponding partial compactification and resolution can be studied in a way similar to the genus two case. We omit the detailed description of the singular fibers of the resulting $w : Y \to \mathbb{C}$. For completeness we give the combinatorial description of the critical locus of w.

Theorem 4.3.

Let (Y, w) be a Landau-Ginzburg mirror of $(\underline{C}, B + i\omega)$ which compactifies and resolves the affine mirror given by (1) and (2). Then:

- w *has exactly k critical values;*
- *The zero dimensional part of the critical set of* w *consists of $k-1$ isolated points of Y, the corresponding fibers of* w *have simple A_1 singularities at those points;*
- *The positive-dimensional part of the critical locus of* w *sits over the critical value* 0 *and consists of $3k-6$ rational curves forming a degenerate curve of genus $k-1$ whose dual graph is shown in figure 9.*

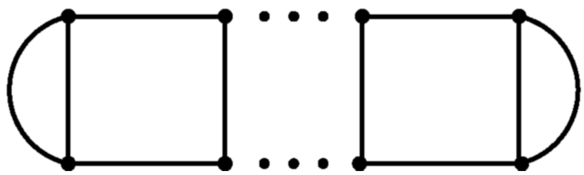

Figure 9. Hyperelliptic curves

Recently Seidel [29] and [9] Efimov proposed an alternative construction of the Hori-Vafa mirror of the hyper-elliptic curve C. The idea is not to treat C as a curve emebedded in a toric surface but rather as an etale $\mathbb{Z}/(2k-1)$-Galois cover of a genus zero orbifold. Understanding the mirror of such an orbifold and tracing the effect of the Galois action leads to an affine Landau-Ginzburg model whos total space has quotient singularities. These in turn can be resolved by introducing appropriate orbifold structures - a procedure which is much more tracktable than the toric resolution procedure we employed in the previous section. Not surprisingly the critical locus of the superpotential on the orbifold Landau-Ginzburg mirror has the same combinatorial structure as the one we obtained in Theorem 4.3.
By analyzing this geometry and combining it with a version of the McKay correspondence with potentials Efimov proves [9] the (A to B) equivalence

$$\overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega) \cong \overline{D^b}(Y_D, \mathrm{w}) \qquad (A \text{ to } B)$$

predicted by HMS. Here as before D is a sufficiently small disk in $\mathbb{C}$ which only contains the critical value 0 of w.
The above orbifold procedure is not directly compatible with our construction of the mirror. However, as it was pointed out to us by P.Seidel, the trick with orbifold resolutions can be easily modified to fit with our birational model. We will sketch this modification here since the resulting geometry ties very nicely with the approach of Seidel and Efimov. Start again with the affine Hori-Vafa mirror $(Y^{\mathrm{aff}}, \mathrm{w}^{\mathrm{aff}})$ given by (1) and (2). Before we compactify the fibers of $\mathrm{w}^{\mathrm{aff}}$ and start resolving the singularities of the resulting total space we can simplify the problem by first resolving the singularities of Y^{aff} and then choosing a partial compactification compatible with this resolution. This can be done in two steps - first pass to an orbifold cover of Y^{aff} with mild singularities, and then construct a crepant resolution of that cover. Concretely, consider the quadric cone $Z^{\mathrm{aff}} : z_1z_2 = z_4z_5$ sitting in the $\mathbb{C}^4$ with coordinates z_1, z_2, z_4, z_5. Consider also the natural map $Z^{\mathrm{aff}} \to Y^{\mathrm{aff}}$ given by $(z_1, z_2, z_4, z_5) \mapsto (z_1^k, z_2^k, z_1z_2, z_4^2, z_5^2)$. The map $Z^{\mathrm{aff}} \to Y^{\mathrm{aff}}$ is a ramified $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$-Galois cover, where $(\lambda, \mu) \in (\mathbb{Z}/k) \times (\mathbb{Z}/2)$ acts on Z by $(\lambda, \mu) \cdot (z_1, z_2, z_4, z_5) := (\lambda z_1, \lambda^{-1} z_2, \mu z_4, \mu^{-1} z_5)$. Choose now a small resolution $\widehat{Z} \to Z^{\mathrm{aff}}$ of this quadratic cone which is compatible with the group action. For instance we can take $\widehat{Z}$ to be the blow-up of Z along the plane $z_1 = z_4 = 0$. Note that by construction $\widehat{Z}$ embeds in $\mathbb{C}^5 \times \mathbb{P}^1$ and this embedding is equivariant for the natural $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$-actions. Proceeding as before we can partially compactify $\widehat{Z}$ by closing it inside $\mathbb{P}^4 \times \mathbb{C} \times \mathbb{P}^1$. The resulting space Z is smooth and the quotient variety $Z/((\mathbb{Z}/k) \times (\mathbb{Z}/2))$ is a partial compactification and a small modification of Z^{aff}. By Hartogs' theorem the potential $\mathrm{w}^{\mathrm{aff}}$ induces a proper holomorphic function $\boldsymbol{f} : Z \to \mathbb{C}$ and the group $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$ acts along the fibers of $\boldsymbol{f}$. In particular $\boldsymbol{f}$ will descend to a function (which we denote again by $\boldsymbol{f}$) on the stack quotient $\mathcal{Z} := [Z/((\mathbb{Z}/k) \times (\mathbb{Z}/2))]$ and so the orbifold Landau-Ginzburg model $(\mathcal{Z}, \boldsymbol{f})$ can be viewed as a stacky crepant resolution of $(Y^{\mathrm{prop}}, \mathrm{w}^{\mathrm{prop}})$. The categorical McKay correspondence of [6] and [16] applies to this situation and allows us to identify the derived categories of $\mathcal{Z}$ and the crepant resolution Y. Combined with our argument from section 7 this implies that we have an equivalence of categories $D^b(Y, \mathrm{w}) \cong D^b(\mathcal{Z}, \boldsymbol{f})$.

This interpretation of the B-model category has some definite computational advantages. First, since $\mathcal{Z}$ is a quotient stack, we can compute $D^b(\mathcal{Z}, \boldsymbol{f})$ as the $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$-equivariant derived category of the potential $\boldsymbol{f} : Z \to \mathbb{C}$. Also, since the critical locus of $\boldsymbol{f}$ is disjoint from the boundary $Z - \widehat{Z}$ we can use the localization theorem from [21] to argue that $D^b(\mathcal{Z}, \boldsymbol{f})$ is equivalent to the $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$-equivariant derived category of the potential $\boldsymbol{f} : \widehat{Z} \to \mathbb{C}$. The later category should be easy to compute since $\widehat{Z}$ is just the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$ and so its derived category is computed by a quiver. The superpotential $\boldsymbol{f}$ gives an A_∞-deformation of this algebra and smashing this deformation with group $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$ we get an explicit A_∞ algebra that computes the B-model category $D^b(\mathcal{Z}, \boldsymbol{f})$.
We also have a similar orbifold picture for the symplectic hyperellyptic curve $(\underline{C}, B + i\omega)$ on the A-side. Indeed, we can view $\underline{C}$ as an etale $(\mathbb{Z}/k) \times (\mathbb{Z}/2)$-Galois cover of an orbifold S^2 with four orbifold points with stabilizers $\mathbb{Z}/k$, $\mathbb{Z}/k$, $\mathbb{Z}/2$, and $\mathbb{Z}/2$ respectively. Using the techniques of Seidel and Efimov we can describe the Fukaya category of such a $2k$-sheeted cover in terms of curves on the four punctured sphere. By repeating the analysis in [9] one should be able to match this description of the Fukaya category directly with the B-side deformed smash algebra that we constructed above. Note that the orbifold $\mathbb{P}^1$ with four orbifold points of indices $(k, k, 2, 2)$ appears naturally on the B-side as well. Specifically the orbifold $\left[\widehat{Z}/((\mathbb{Z}/k) \times (\mathbb{Z}/2))\right]$ can be identified with the total space of a vector bundle $L \oplus L$ on this orbifold $\mathbb{P}^1$, where L is the unique theta characteristic on the orbifold $\mathbb{P}^1$.
The above orbifold picture provides yet another description of the mirror of the hyperelliptic curve and gives an alternative setting for proving Efimov's theorem.

Remark 4.3.
The shape of the critical locus of w suggests that both $D^b(Y_D, \mathrm{w})$ and $D\mathrm{Fuk}(\underline{C}, B + i\omega)$ may be amenable to cutting and pasting computations. In particular it will be interesting to try and compute $D\mathrm{Fuk}(\underline{C}, B + i\omega)$ by gluing $\underline{C}$ small pieces - e.g. by using a pair of pants decomposition. It will also be interesting to see how the mapping class group interacts with the mirror functor.

We will leave this question for the future. For now we proceed with the analysis of the mirror of the intersection of two quadrics which is the other half of our conjecture.

5. Intersection of two quadrics in $\mathbb{P}^5$

In this section, we give a detailed description of the Landau-Gizburg mirror $\mathrm{w} : Y \to \mathbb{C}$ of the Fano three-fold X_4 of degree 4 in $\mathbb{P}^5$. We will show that the topological structure of the critical set of w is identical with the one corresponding to the genus two curve C. Assumimg the HMS conjecture, we also give evidence supporting our Conjecture 3.3, which in this case predicts that under an appropriate identification of complexified Kähler classes the Fukaya category of $\underline{C}$ maps fully faithfully to the Fukaya category of the three-fold $\underline{X}_4$.
The Fano manifold X_4 is a complete intersection of two quadrics in $\mathbb{P}^5$. If we equip the underlying C^∞-manifold $\underline{X}_4$ with the unique up-to-scale complexified Kähler form which is a restriction of a torus uinvariant form on $\mathbb{P}^5$, then according to [13] its affine Landau-Ginzburg mirror is given by the subvariety in $\mathbb{C}^{6*}$ cut out by the system of equations:

$$Y^{\mathrm{aff}} \ : \ \begin{cases} x_1x_2x_3x_4x_5x_6 = 1 \\ x_1 + x_2 = -1 \\ x_3 + x_4 = -1 \end{cases}$$

and equippped with the superpotential $\mathrm{w}^{\mathrm{aff}} = x_5 + x_6$.
To construct the partial coompactification Y^{prop} of Y^{aff} we consider the closure of this variety in $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \times \mathbb{P}^1_{(z_0:z_1)} \times \mathbb{C}_w$ given by the equation

$$Y^{\mathrm{prop}} \ : \ x_1(x_0 - x_1)y_1(y_0 - y_1)z_1(wz_0 - z_1) = x_0^2y_0^2z_0^2.$$

Here the coordinates of each $\mathbb{P}^1$-factor are indicated in the subscripts. As before the extension $\mathrm{w}^{\mathrm{prop}}$ of the potential $\mathrm{w}^{\mathrm{aff}}$ is given by projecting Y^{prop} onto the w-line.
The desingurization Y of this variety and the structure of the singular fibers of the resulting potential w are summarized in the following theorem.

Theorem 5.1.

There exists a crepant resolution Y of Y^{prop} so that the fibers of the function $\mathrm{w} : Y \to \mathbb{C}$ induced from w^{prop} can be described as follows:

- *For $w \neq -8, 0, 8$, the fiber $Y_w := \mathrm{w}^{-1}(w)$ is smooth and has a structure of an elliptic fibration over $\mathbb{P}^1$ with five singular fibers: one of type I_8, two of type I_1^*, and two of type I_1. Hence, Y_w is an elliptic $K3$ surface with Picard rank 19.*
- *For $w = \pm 8$ the fiber Y_w is singular with an isolated A_1 singularity. Its minimal resolution is an elliptic $K3$ with an $I_8 + 2 \cdot I_1^* + I_2$ configuration of singular fibers.*
- *The central fiber Y_0 of the fibration $w : Y \to \mathbb{P}^1$ is a union of three rational surfaces $Y = \widetilde{S}_1 \cup \widetilde{S}_2 \cup \widetilde{S}_3$. These three surfaces intersect in three copies of $\mathbb{P}^1$:*

$$q_1 = \widetilde{S}_1 \cap \widetilde{S}_3, \quad q_2 = \widetilde{S}_2 \cap \widetilde{S}_3, \quad \textit{and} \quad q_3 = \widetilde{S}_1 \cap \widetilde{S}_2.$$

The curves q_1, q_2, and q_3 intersect in two points, R_1 and R_2. Hence, Y_0 is a type III deformation of a K3 surface. The geometry of the surfaces and the position of the curves on them is explained as follows.

(i) To construct $\widetilde{S}_3$, choose four generic rulings on $\mathbb{F}_0$ which intersect at points P_1, P_2, Q_1, and Q_2 so that P_1 and Q_1 are "diagonal" points w.r.t. the chosen rulings. Choose further two generic $(1,1)$ divisors, d_1 and d_2 such that d_1 passes through P_1 and Q_1, and d_2 passes through P_2 and Q_2. Let $d_1 \cap d_2 = \{R_1, R_2\}$. The component $\widetilde{S}_3$ is the blow up of $\mathbb{F}_0$ in points of depth two on $d_1 \cup d_2$ and support at P_1, P_2, Q_1, Q_2. The proper transforms of the divisors d_1 and d_2 are the curves q_1 and q_2 on $\widetilde{S}_3$.

(ii) The set up for constructing $\widetilde{S}_1 \cup \widetilde{S}_2$ is the same as in (i). Only now we blow up $\mathbb{F}_0$ in the vertices of the quadrilateral only once; consider two copies of this modified surface and identify them along the proper transform of d_1 so that the points R_1 and R_2 from the two copies of the surface match; then we blow up either the first or the second component at two points on d_1 but different from R_1 and R_2. The resulting variety is $\widetilde{S}_1 \cup \widetilde{S}_2$. The two divisors d_2 correspond to q_1 and q_2 on $\widetilde{S}_1$ and $\widetilde{S}_2$ respectively. The proper transform of d_1, on either of $\widetilde{S}_i$, is the curve q_3 on the corresponding component of Y_0.
This way of constructing $\widetilde{S}_1 \cup \widetilde{S}_2$ is clarified by the fact that, before we blow up the last two points on d_1, there is an obvious $\mathbb{Z}_2$ symmetry of the union of the three components which identifies the two exceptional ones as we do here. According to this symmetry, for the case of the third component, we have to choose on the corresponding $\mathbb{F}_0$ the configuration of diagonal divisors symmetric with respect to the $\mathbb{Z}_2$ symmetry which exchanges the members of one of the rulings.

The rank of the Picard group of Y_0 is 21.

Proof. Consider the intersection $X_4 = Q_1 \cap Q_2 \subset \mathbb{P}^5$ of two generic quadrics in $\mathbb{P}^5$. The space X_4 has a unique up to scale complexified symplectic form which is a restriction of a torus invariant complexified symplectic form on $\mathbb{P}^5$. For concretensess we normalize this form to be $2\pi i \omega_{FS}$ where ω_{FS} is the Fubini-Studi form on $\mathbb{P}^5$.
The Hori-Vafa recipe [13] identifies the affine Landau-Ginzburg mirror of $(\underline{X}_4, 2\pi i\omega_{FS})$ with the family $\mathrm{w}^{\text{aff}} : Y^{\text{aff}} \to \mathbb{C}$ where

$$Y^{\text{aff}} \ : \ \begin{cases} x_1x_2x_3x_4x_5x_6 = 1 \\ \quad x_1 + x_2 = -1 \\ \quad x_3 + x_4 = -1 \end{cases}$$

and the superpotential is given by $\mathrm{w}^{\text{aff}} = x_5 + x_6$. Note that any rescaling of the Fubini-Studi form will only affect the affine mirror by introducing some non-zero complex constant r instead of the 1 in the right-hand-side of the first equation. The resulting affine varieties with potentials are clearly isomorphic (say by scaling x_5 and x_6 by the square root of r). Thus the scaling parameter for the symplectic volume of X_4 is not a modulus of the mirror Landau-Ginzburg model and we can work with the normalized mirror above without any loss if generality.
We consider the closure of this variety in $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \times \mathbb{P}^1_{(z_0:z_1)} \times \mathbb{C}_w$ given by the equation

$$Y^{\text{prop}} \ : \ x_1(x_0 - x_1)y_1(y_0 - y_1)z_1(wz_0 - z_1) = x_0^2y_0^2z_0^2$$

A crepant resolution Y of this variety together with the induced superpotential will be Landau-Ginzburg mirror fibration of the Fano manifold $(\underline{X}_4, 2\pi i\omega_{FS})$. The singular locus of Y^{prop} consists of the following one dimensional components:

$$\begin{array}{ll}
l_1 : [(0:1); (1:0); (1:0); w] & m_1 : [(1:0); (0:1); (1:0); w] \\
l_2 : [(0:1); (1:0); (1:w); w] & m_2 : [(1:0); (0:1); (1:w); w] \\
l_3 : [(0:1); (1:1); (1:0); w] & m_3 : [(1:1); (0:1); (1:0); w] \\
l_4 : [(0:1); (1:1); (1:w); w] & m_4 : [(1:1); (0:1); (1:w); w] \\
n_1 : [(1:0); (1:0); (0:1); w] & p_1 : [(0:1); (y_0:y_1); (1:0); 0] \\
n_2 : [(1:0); (1:1); (0:1); w] & p_2 : [(x_0:x_1); (0:1); (1:0); 0] \\
n_3 : [(1:1); (1:0); (0:1); w] & \\
n_4 : [(1:1); (1:1); (0:1); w] &
\end{array}$$

All these components have transversal singularity of type A_1, and, at generic points, can be resolved by blowing up the components in the ambient space. The fiber Y_w^{prop}, for $w \neq 0$, intersects transversally 12 of the lines above. The fiber Y_w of the desingularization Y is then the resolution of the $12(A_1)$ singularities of Y_w^{prop}.
Again we begin with a description of the general fiber.

Claim 5.1.

For $w \neq -8, 0, 8$, Y_w is a smooth $K3$ surface and has a structure of an elliptic fibration over $\mathbb{P}^1$ with five singular fibers: one of type I_8, two of type I_1^*, and two of type I_1. The rank of the Picard group of Y_w is 19. The fibers $Y_{\pm 8}$ are singular with an isolated A_1 singularity. The minimal resolution of either of these two fibers is an elliptic $K3$ with an $I_8 + 2 \cdot I_1^* + I_2$ configuration of singular fibers.

Proof. Observe that Y^{prop} is a double cover of $S' \times \mathbb{C}_w$ where

$$S' : \{x_1(x_0 - x_1)y_1(y_0 - y_1)T_1 = x_0^2\, y_0^2\, T_0\} \subset \mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)} \times \mathbb{P}^1_{(T_0:T_1)}$$

and the cover map $\varphi' : Y^{\text{prop}} \to S' \times \mathbb{C}_w$ is induced by the double cover $\psi : \mathbb{P}^1_{(z_0:z_1)} \to \mathbb{P}^1_{(T_0:T_1)}$ where $\psi((z_0 : z_1)) = (z_0^2 : z_1(w\,z_0 - z_1))$. The surface S' has four singular points of type A_1 and with respect to the projection to $\mathbb{P}^1_{(T_0:T_1)}$, is a rational elliptic fibration with singular fibers of type I_4, at $T_0 = 0$, type I_1 at $T_1 = 16T_0$, and a double fiber of nodal type, at $T_1 = 0$. The four A_1 singularities of S' belong to this latter fiber (neither of these is a nodal point there). The restriction of φ' to Y_w^{prop} is a double cover map $\varphi'_w : Y_w^{\text{prop}} \to S'$ The branching for ψ occurs at $T_0 = 0$ and $T_1 = (w/2)^2\,T_0$. Hence, the branch divisor of φ'_w consists of two fibers of S': at $T_0 = 0$ and at $T_1 = (w/2)^2\,T_0$. This implies that, for $w \neq -8, 0, 8$, Y_w^{prop} has a structure of an elliptic fibration over $\mathbb{P}^1_{(T_0:T_1)}$ with singularities induced by those of S', 8 total, and of the self-intersection of the branch divisor, 4 total. These are resolved by blowing up the lines l_i, m_i, n_i, $(i = 1, \ldots, 4)$, and the result is an elliptic fibration $Y_w \to \mathbb{P}^1$ with an $I_8 + 2 \cdot I_1^* + 2I_1$ configuration of singular fibers.
Denote by S the resolved S' blown up additionally in the nodes of its I_4 fiber. Then S is a rational elliptic fibration with three singular fibers. One of them looks like an I_8 fiber but has has four disjoint components of self-intersection -4 and other four of self-intersection -1. The second singular fiber is of type I_1^*. The third is a type I_1 fiber. The surface Y_w is a double cover of S branched in a union of a smooth fiber and of the four -4 lines of the I_8-like fiber It is easy to see now that there are only 24 (-2)-lines on Y_w. The fibration structure on Y_w gives two relations between these (-2)-lines in $\mathrm{Pic}(Y_w)$. If we repeat, word for word, our considerations for the projections of Y_w^{prop} to the rest of the $\mathbb{P}^1$s (with coordinates $(x_0 : x_1)$ and $(y_0 : y_1)$), we will get two more ways to express Y_w as an elliptic fibration with the same types and numbers of singular fibers. This way we get 4 more relations between the -2-lines in $\mathrm{Pic}(Y_w)$. A direct check shows that among the six relations we get this way only five are linearly independent. Since the Picard group on such a K3 surface is generated by the (-2)-components of the fibers, we get that the rank of this group for Y_w is no more than 19. By a direct chase one can find 19 linearly independent such lines in our case.
The fibers $Y_{\pm 8}$ are branched over S at the four -4-lines as before, and at the I_1 fiber of S. This way we immediately get the structure of $Y_{\pm 8}$. This completes the proof of the claim. □

Next we turn to the central fiber Y_0^{prop} of Y^{prop}. This fiber will be affected also by the resolution of the lines p_1 and p_2, which are contained in this fiber. Along these two lines, Y_0^{prop} has a self-intersection of nodal type and resolving them will produce two extra components in Y_0. The double cover $Y^{\text{prop}} \to S'$ will help us understand the structure of Y_0. This cover is branched along the fiber I_4 as well as in the double nodal fiber (for $T_1 = 0$). The double portion of the branch divisor is responsible for the non-normal singularities of Y_0^{prop}, and to resolve these is equivalent to (partially) "deleting" the doble-lines part from the branch divisor.
To properly carry out this "deletion", we start by blowing up the line p_1 given by $x_0 = z_1 = w = 0$. By setting

$$(w : x_0/x_1 : z_1/z_0) = (u_0 : u_1 : u_2)$$

we get the blown up Y^{prop} in the form

$$(x_0/x_1 - 1) \cdot y_1 \cdot (y_0 - y_1) \cdot u_2 \cdot (u_0 - u_2) = u_1^2 \cdot y_0^2$$

where $x_0/x_1 = 0$ gives the exceptional divisor, and the proper transform of Y_0^{prop} is given by $u_0 = 0$.
It is easy to check (see Figure 10) that the exceptional divisor, D_1, has five singular points of type A_1:

$$\begin{aligned}
&u_1 = y_1 = u_2 = 0;\\
&u_1 = y_1 = u_0 - u_2 = 0;\\
&u_1 = y_0 - y_1 = u_2 = 0;\\
&u_1 = y_0 - y_1 = u_0 - u_1 = 0;\\
&y_0 = u_2 = u_0 = 0.
\end{aligned}$$

The first four singularities get resolved by blowing up the lines l_i, $i = 1, \cdots, 4$ while the last one gets resolved by blowing up the line p_2.
Before desingularizing D_1, we can realise it as a double cover as follows. Consider the plane $\mathbb{P}^2$ with coordinates $(u_0 : u_1 : u_2)$, and the pencil of conics in it given by the equation $\lambda \cdot u_2(u_0 - u_1) = \mu \cdot u_0^2$. This pencil defines a fibration of conics over $\mathbb{P}^1$ (the line has coordinates $(\lambda : \mu)$). Then the equations $\lambda = y_1(y_0 - y_1)$ and $\mu = y_0^2$ define D_1 as a double cover over that fibration. It is straightforward to see that the desingularization of D_1 together with the curves of intersection with the rest of the components of the central fiber of Y can be obtained from a Hirzebruch surface $\mathbb{F}_0$ by blowing it up eight times in the following centers. Choose a quadrilateral of rulings on $\mathbb{F}_0$ and two $(1, 1)$- divisors which intersect the quadrilateral diagonally. Blow up $\mathbb{F}_2$ in the vertices of that quadrilateral first, and then in the two points where the proper transform of one of the diaginal divisors intersect two of the exceptional curves. This is how the resolved D_1 looks like. The proper transform of the diagonal blown up twice, q_1, is the intersection of D_1 with F while the proper transform of the diagonal blown up four times, i.e. q_3 , is the curve of intersection of D_1 with D_2.
The second exceptional component of Y_0, due to the blow up in p_2 , can be treated the same way (here we have to blow up the lines m_i, $i = 1, \cdots, 4$). The only difference is that here we have to blow up only once the vertices of the quadrilateral. To make the picture more symmetric, we can flop D_1 in a -1-curve intersecting q_3.
Finally let us look at the geometry of the third component of Y_0, namely the proper transform of Y_0^{prop} under the desingularization we perform (see Figure 11).
As we saw Y_0^{prop} is a double cover of S' branched in the I_4 and the double nodal fibers. There are several singular points, besides the non-normal ones.
The blow up of p_1 and p_2 amounts to deleting the components of the double nodal fiber from the branch locus of $Y_0^{\text{prop}} \to S'$. But the four singular A_1 points on that fiber remain parts of the branch locus.
The blow up of $n_1 \cup n_2 \cup n_3 \cup n_4$ amounts in producing the I_8-type fiber on S' (with four disjoint components of self-intersection -4 and four of self-intersection -2) and resolving four of the singular points on Y_0^{prop}. Only the -4-components of this fiber belong to the branch locus of the cover. The completion of this process produces the surface S with one I_8-like fiber and one nodal fiber with four singular points on it.
So, the proper preimage $\widetilde{S}_3$ of Y_0^{prop} (the third components of Y_0) is a double cover of S branched in four lines: the four (-4)-lines from the I_8-like fiber, and in four singular points, in the nodal fiber. It is easy to realize this surface as a blow up of $\mathbb{F}_0$ in eight points as follows.

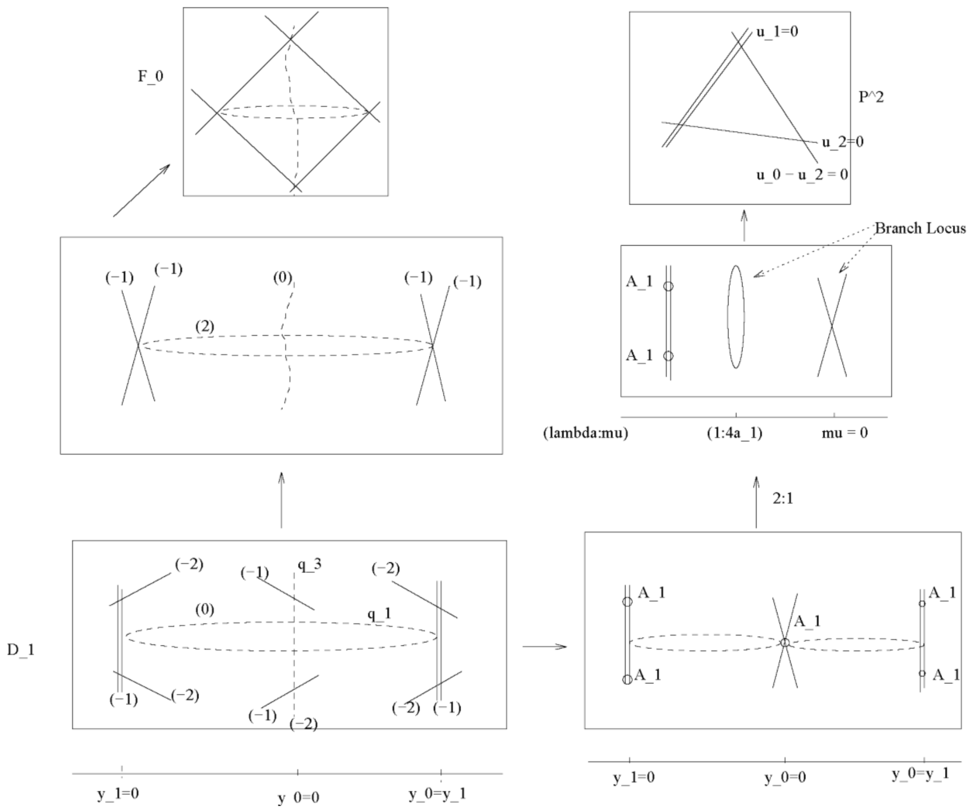

Figure 10. The desingularization of D_1.

Choose, as it was in the case of D_1 and D_2, two $(1,1)$-divisors intersecting diagonally a quadrilateral made by four general rulings on $\mathbb{F}_0$. Blow up the surface in the four verices of the quadrilateral considered as schemes of depth two on the diagonal divisors. Each divisor gets blown up four times in infinitesimally near points to the vertices of the quadrilateral. This is how the third component $\widetilde{S}_3$ of Y_0 looks like. The diagonal divisors will be the curves of intersection of F with D_1 and D_2 , q_1 and q_2 respectively.
It is easy to see that the points on q_i where -1-curves intersect them on one component match with the analogous points on the second component to which q_i belongs. This fact shows that the choices of the rulings and diagonal divisors for each individual component of Y_0 have to be compatible with each other. □

From this analysis one immediately gets

Corollary 5.1.
The zero-dimentional part of the critical set of $\mathrm{w} : Y \to \mathbb{C}$ *consists of two points. The rest of the critical set consists of three rational curves which intersect in two points as shown in Figure 12 below.*

6. The large volume comparison of Fukaya categories

With Theorem 4.2 and Theorem 5.1 at our disposal we can now use Homological Mirror Symmetry to attack Conjecture 3.3 when $G = C$ is a genus two curve and $F = X_4$ is a Fano threefold of degree four.

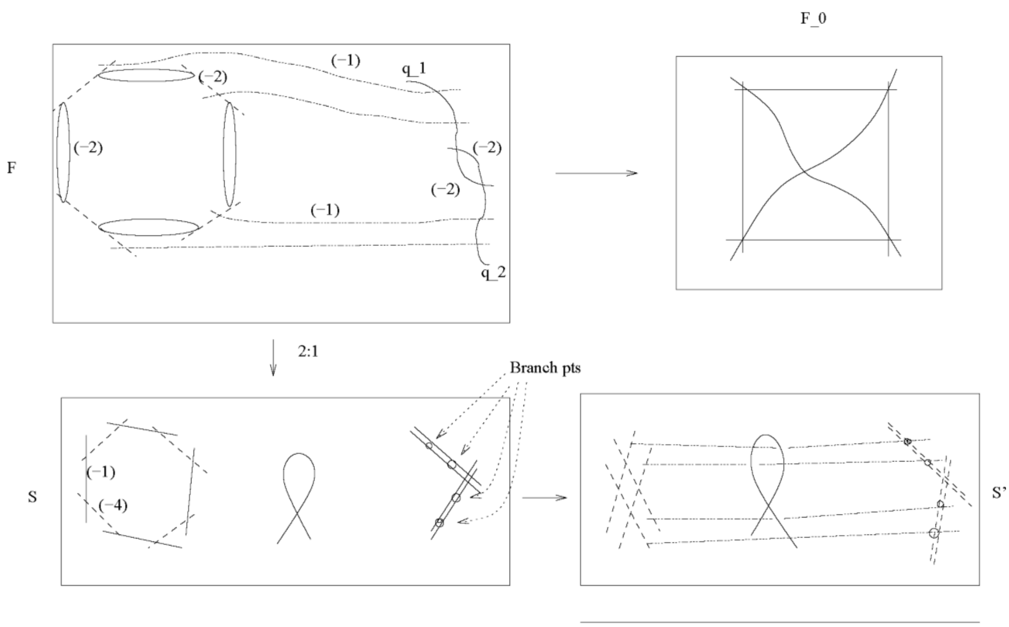

Figure 11. Obtaining the component $\widetilde{S}_3$.

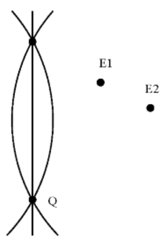

Figure 12. The critical points of w for the intersection of two quadrics in $\mathbb{P}^5$.

Recall that if C is a smooth complex curve of genus two, then C gives rise to a pencil of quadrics in $\mathbb{P}^5$ whose base locus is a smooth Fano threefold X_4 of degree four. Explicitly since C is hyperelliptic we have a double cover $C \to \mathbb{P}^1$ whose branch points in the appropriate affine coordinate on $\mathbb{P}^1$ are given by some complex numbers $c_1, ...c_6$. If $(x_1 : x_2 : \cdots : x_6)$ are homogeneous coordinates on $\mathbb{P}^5$, then we can associate with C the pencil of quadrics in $\mathbb{P}^5$ spanned by $\sum_{i=1}^6 x_i^2$ and $\sum_{i=1}^6 c_i x_i^2$. The degree four Fano threefold corresponding to C is the complete intersection

$$X_4 \; : \; \sum_{i=1}^{6} x_i^2 = 0, \qquad \sum_{i=1}^{6} c_i x_i^2 = 0.$$

By [5, Theorem 2.7] the derived category of coherent sheaves of C has a natural fully faithful embedding in the derived category of coherent sheaves on X_4. More precisely $D^b(C)$ embeds in $D^b(X_4)$ as the orthogonal of two line bundles:

$$D^b(X_4) = \left\langle D^b(C), \mathcal{O}_{X_4}, \mathcal{O}_{X_4}(1) \right\rangle .$$

In fact [5, Theorem 2.7] gives an explicit description of the kernel object $K_\Phi \in D^b(C \times X_4)$ defining the inclusion $\Phi : D^b(C) \hookrightarrow D^b(X_4)$. Therefore C and X_4 constitute a geometric pair that satisfies the hypotheses of Conjecture 3.3. The conjecture then predicts that for any choice of a complexified Kähler structure α_F on $\underline{X}_4$ there exists an associated choice of a complexified Kähler structure α_G on $\underline{C}$, and a way to interpret K_Φ as a coisotropic brane on $\left(\underline{C} \times \underline{X}_4, -p_C^* \alpha_G + p_{X_4}^* \alpha_F\right)$ which induces a fully faithful functor

$$\Psi \ : \ \overline{D\mathrm{Fuk}}(C, \alpha_G) \hookrightarrow \overline{D\mathrm{Fuk}}(X_4, \alpha_F)$$

of Fukaya categories.

Before we can address the question of existence of Ψ we need to properly understand teh matching of the complexified Kähler classes α_F and α_G. The idea is to use the mirror maps which identify certain Kähler moduli of C and X_4 with the complex moduli of the mirror Landau-Ginzburg models constructed in theorems 4.2 and 5.1. This presents a problem since the Kähler moduli relevant to the mirror maps are the ones coming from equivariant Kähler classes on the ambient toric varieties. Since $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ and $X_4 \subset \mathbb{P}^5$ we have two ambient classes that show up as parameters for the Landau-Ginzburg mirror of C, and only one parameter for the Landau-Ginzburg mirror of X_4. In Section 4.1 we saw that only one of the two ambient classes for $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ shows up as a modulus of the complex Landau-Ginzburg mirror of C. Namely we showed that:

- If $B + i\omega$ is a complexified Kähler class on $\mathbb{P}^1 \times \mathbb{P}^1$, then the $B + i\omega$-symplectic volumes a_1 and a_2 of the two rulings appear as deformation parameters for the Landau-Ginzburg mirror of $(\underline{C}, B + i\omega)$ but only a_2 is a modulus of the formal neighborhood of the singularities of the zero fiber of the superpotetial.

- If α_F is a complex multiple of the Fubini-Studi form on $\mathbb{P}^5$, then the α_F-symplectic volume of the hyperplane in $\mathbb{P}^5$ appears as a deformation parameter for the Landau-Ginzburg mirror of $(\underline{X}_4, \alpha_F)$ but this parameter gives an iso-trivial deformation of the central fiber and so is not a modulus of the Landau-Ginzburg model.

This shows that we have a one parameter ambiguity for what α_G can be, whereas α_F is unique up to an isomorphism. To remove this ambiguity we will consider the large volume limit of the curve of genus 2. Specifically we will look at the situation where $|a_2| \to +\infty$. We have the following

Theorem 6.1.

Suppose that the Homological Mirror Symmetry conjecture holds for the Fano variety X_4, i.e. suppose that the Karoubi closure of the derived Fukaya category of $(\underline{X}_4, 2\pi i \omega_{FS})$ is equivalent to the Karoubi closure of the derived category of the mirror Landau-Ginzburg model. Then Conjecture 3.3 holds for C and X_4 in the large volume limit. Specifically there is a fully faithful functor

$$\Psi \ : \ \lim_{\left|\int_{\mathbb{P}^1 \times \{\mathrm{pt}\}} B\right| \to -\infty} \overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega) \hookrightarrow \overline{D\mathrm{Fuk}}(\underline{X}_4, 2\pi i \omega_{FS}).$$

Proof. Part of the statement of the theorem is to make sense of limiting Fukaya category on the left hand side. We will use mirror symmetry to show the existence of the limit and to construct the embedding Ψ in terms of the mirror Landau-Ginzburg models.

Using Seidel's proof [29, Theorem 1.1] of the Homological Mirror Conjecture for C we can identify the category $\overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega)$ with the Karoubi closure $\overline{D^b_{\mathrm{sing}}}(\mathrm{Y}_0)$ of the derived category of singularities of the zero fiber of the mirror Landau-Ginzburg model. By [23, Theorem 2.10] $\overline{D^b_{\mathrm{sing}}}(\mathrm{Y}_0)$ depends only on the formal neighborhood of the singularities in Y_0, while by our Theorem 4.2 it follows that this formal neighborhood depends on a single moduli parameter λ. In the proof of Theorem 4.2 we identified λ on one hand with the ratio of the roots of the quadratic polynomial $x^2 + x + a_2$ and on the other hand with the cross-ratio of the four points on $\mathbb{P}^1$ necessary for constructing and gluing together the three components of $\mathbf{Y_0}$. Since Seidel's theorem identifies $\overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega)$ with the Karoubi closure of the derived category of singularities of the Y_0 of modulus λ and the limit $\lambda \to -1$ corresponds to $|a_2| \to +\infty$ we see that $\lim_{|a_2| \to +\infty} \overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega)$ is naturally identified with the Karoubi closure of the derived category of singularities of the Y_0 corresponding to the value $\lambda = -1$ of the modulus parameter. This gives a rigorous meaning of the limiting category $\lim_{|a_2| \to +\infty} \overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega)$.

Similarly in the proof of Theorem 5.1 we argued that the zero fiber Y_0 of the Landau-Ginzburg mirror of $(\underline{X}_4, 2\pi i \omega_{FS})$ is built out of three rational surfaces which are constructed and glued together out of four points on $\mathbb{P}^1$ with cross-ratio (-1). Furthermore in Corollary 5.1 and Corollary 4.1 we saw that the critical loci of the Landau-Ginzburg mirrors of a genus two curve and an intersection of two quadrics in $\mathbb{P}^5$ are scheme-theoretically isomorphic and that the combinatorial structures of the full formal neighborhoods of these critical loci in the fibers of the respective superpotentials are also the same. In particular it folllows that the formal neighborhood of the singularities in the Y_0 corresponding to the value $\lambda = -1$ of the modulus parameter is isomorphic to the formal neighborhood of the singularities in the zero fiber Y_0 of the Landau-Ginzburg mirror (Y, w) of $(\underline{X}_4, 2\pi i\omega_{FS})$. By Corollary 5.1 and [23, Theorem 2.10] the category $\overline{D^b_{\text{sing}}}(Y_0)$ embeds as an admissible subcategory in $\overline{D^b}(Y, \mathrm{w})$ which by our assumption is mirror equivalent to the Karoubi closed Fukaya category $\overline{D\mathrm{Fuk}}(\underline{X}_4, 2\pi i\omega_{FS})$). This completes the proof of the theorem. □

The fact that the Fukaya category of X_4 has no moduli suggests that the previous theorem should hold without taking the large volume limit of the Fukaya categories of C. In other words, we expect that for every $B + i\omega$ on C we have a fully faithful functor

$$\Psi_{a_2} \ : \ \overline{D\mathrm{Fuk}}(\underline{C}, B + i\omega) \hookrightarrow \overline{D\mathrm{Fuk}}(\underline{X}_4, 2\pi i\omega_{FS})$$

given by an explicit kernel which depends only on K_Φ and the a_2 period of $B + i\omega$.

This statement again has a mirror incarnation relating the complex Landau-Ginzburg mirrors of $(\underline{C}, B + i\omega)$ and $(\underline{X}_4, 2\pi i\omega_{FS})$. More precisely consider the zero fiber Y_0 of the Landau-Ginzburg mirror of $(\underline{X}_4, 2\pi i\omega_{FS})$ constructed in Theorem 5.1 and the zero fiber Y_0^λ of the Landau-Ginzburg mirror of $(\underline{C}, B + i\omega)$ constructed in Theorem 4.2. Here the superscript λ indicate that the zero fiber of the mirror of $(\underline{C}, B + i\omega)$ corresponds to the moduli parameter λ which is the ratio of the roots of $x^2 + x + a_2 = 0$ with a_2 being the second period of $B + i\omega$. With this notation, the existence of Ψ_{a_2} can be recast in mirror terms as the existence for all λ of an equivalence

$$\phi_\lambda \ : \ \overline{D^b_{\text{sing}}}(\mathrm{Y}_0^\lambda) \xrightarrow{\cong} \overline{D^b_{\text{sing}}}(Y_0).$$

In the remainder of this section we discuss a possible construction of the functor Ψ_{a_2} which does not use mirror symmetry and is performed directly on the level of Fukaya categories.

For this construction we will use yet another geometric interpretation of X_4. If C is a smooth curve of genus two, the Narasimhan-Ramanan theorem [19] identifies the corresponding Fano X_4 with the moduli space of stable rank 2 bundle on C of degree one. The underlying C^∞ manifold of $\underline{X}_4$ can then be identified with the moduli space of flat $SO(3)$ connections on $\underline{C}$. Now the observation is that for every Lagrangian $s \subset C$ we can construct a Lagrangian $L_s \subset X_4$. By definition L_s is the moduli space of flat $SO(3)$ connections on C that have trivial monodromy on the loop s. The universal $SO(3)$ connection on $\underline{C} \times \underline{X}_4$ allows us to transform connections on loops s to connections on L_s and it is natural to expect that this assignment extends to a fully faithful functor between the Fukaya categories.

As a simple check on this expectation, we will show that the Floer homology on both sides match in the following two very simple cases:

1) if two loops are disjoint then the corresponding moduli spaces should be disjoint;

2) if two loops intersect once then the corresponding moduli spaces should intersect once.

To be specific let $\boldsymbol{a}_1$, $\boldsymbol{b}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}_2$ be four loops in C which constitute a standard system of generators of $\pi_1(C)$, subject to the relation $[\boldsymbol{a}_1, \boldsymbol{b}_1] \cdot [\boldsymbol{a}_2, \boldsymbol{b}_2] = 1$. Fix a symplectic form ω on $\underline{C}$ of unit volume. Then by [19] we can identify $\underline{X}_4$ with the moduli of pairs (E, ∇) where E complex rank 2 bundles on $\underline{C}$, and ∇ is an irreducible connection with curvature $-i\pi\omega\,\mathrm{id}_E$. Every such connection is characterized by its holonomy along the four generating loops $\boldsymbol{a}_1$, $\boldsymbol{b}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}_2$. Moreover, the 1-chain $[\boldsymbol{a}_1, \boldsymbol{b}_1] \cdot [\boldsymbol{a}_2, \boldsymbol{b}_2]$ (an 8-sided polygon) bounds a disc in C, and the area of that disc is exactly the area of C, i.e. is equal to 1. Therefore by Stokes formula and by the formula for the curvature of the connection, the holonomy along this chain is given by $\exp(-i\pi\,\mathrm{id}_E) = -\mathrm{id}_E$. Hence, the moduli space $\underline{X}_4$ can be identified with the space of 4-tuples (A_1, A_2, B_1, B_2) in $SU(2)$ such that $[A_1, B_1][A_2, B_2] = -\mathrm{I}$, modulo conjugation.

Note that since the connection is not flat but only projectively flat, the holonomy varies when we move the loop in its homotopy class; however, if we move the loop by a Hamiltonian isotopy then the holonomy does not change, because the area swept by the loop is 0. This is consistent with the fact we are interested in Lagrangians in C up to Hamiltonian isotopy only.

Now let us consider connections that are trivial both on $\boldsymbol{a}_1$ and on $\boldsymbol{a}_2$. Since we need to have $A_1 = A_2 = \mathrm{I}$, it follows that $[A_1, B_1] \cdot [A_2, B_2] = [\mathrm{I}, B_1] \cdot [\mathrm{I}, B_2] = \mathrm{I}$ is never equal to $-\mathrm{I}$. Hence there are no solutions and the two Lagrangians $L_{\boldsymbol{a}_1} = \{A_1 = Id\}$ and $L_{\boldsymbol{a}_2} = \{A_2 = Id\}$ are disjoint. On this example we have $HF(L_{\boldsymbol{a}_1}, L_{\boldsymbol{a}_2}) = HF(\boldsymbol{a}_1, \boldsymbol{a}_2) = 0$.

Similarly consider the Lagrangians in X_4 defined by say $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$. We need quadruples with $A_1 = B_1 = \mathrm{I}$, so the intersection of $L_{\boldsymbol{a}_1}$ with $L_{\boldsymbol{b}_1}$ consists of the conjugacy classes of pairs (A_2, B_2) in $SU(2)$ such that $[A_2, B_2] = -\mathrm{I}$. Up to conjugation we can diagonalize A_2 in the form $diag(\exp(it), \exp(-it))$. Here t is not multiple of π since otherwise A_2 would be central. Then, writing the equation $[A_2, B_2] = -Id$ explicitly in terms of coefficients we get that

- B_2 is anti-diagonal and
- $A_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ or $A_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

Since all such pairs (A_2, B_2) are conjugate the intersection of $L_{\boldsymbol{a}_1}$ with $L_{\boldsymbol{b}_1}$ consists of a single point, the same as the intersection of $\boldsymbol{a}_1$ with $\boldsymbol{b}_1$. In other words Floer homologies have rank 1 in both cases.

Remark 6.1.

It is an interesting question to understand the geometric nature of additional Lagrangian objects in the Fukaya category of X_4. A natural candidate for one of them is the Lagrangian of all connections that come as pullbacks from the quotient of C by an (orientation-reversing) involution.

Remark 6.2.

The geometric relation between C and X_4 is not special to the genus two case. Mukai [18] showed that there are many other Fano threefolds that can be realized as Brill-Noether loci in the moduli space of bundles over some curve of high genus. We expect that an anlogue of Conjecture 3.3 statement should hold in these cases as well. In paticular we expect that Conjecture 3.3 should hold when F is the Fano threefold X_{16} and G is a curve of genus 3, or when F is a Fano threefold X_{12} and G is a curve of genus 7. Some calculations for the mirrors of reductive Fano varieties give strong evidence for that.

Remark 6.3.

One can also describe the connection between Fukaya category of S - a generic intersection of quadric and cubic in $\mathbb{P}^5$ and the hypersurface N of degree six in weigthed $\mathbb{P}^4$ with weights $(1, 1, 1, 2, 3)$. The affine mirror of S is threefold:

$$Y_S^{\mathrm{aff}} \ : \ \begin{cases} x_1 + x_2 = -1 \\ x_3 + x_4 + x_5 = -1 \\ x_1 \cdot x_2 \cdot \ldots \cdot x_6 = A \end{cases}$$

equipped with the superpotential $\mathrm{w}_S^{\mathrm{aff}} = x_6$.
Similarly we can describe the mirror of N. Consider the potential given by $\mathrm{w}_N^{\mathrm{aff}} = u_1 + \cdots + u_4 + u_5 + v$ over the fivefold Y_N^{aff} in $\mathbb{C}^{*5} \times \mathbb{C}$ given by the equation:

$$Y_N^{\mathrm{aff}} \ : \ u_1 \cdot u_2 \cdot u_3 \cdot u_4^2 \cdot u_5^3 = a \cdot v^6.$$

Even though the fibers of $\mathrm{w}_S^{\mathrm{aff}}$ are two dimensional and the fibers of $\mathrm{w}_N^{\mathrm{aff}}$ are four dimensional we expect that the critical sets of $\mathrm{w}_S^{\mathrm{aff}}$ and $\mathrm{w}_N^{\mathrm{aff}}$ have the same topology. Moreover, following an algebro geometric conjecture made by A.Kuznetsov [17] and relating $D^b(S)$ and $D^b(N)$, we expect that Conjecture 3.3 holds with $F = N$ and $G = S$.

7. Derived categories of singularities and flops

In this section we fill in the details of a statement that we frequently used in our analysis: the fact that the derived category of a potential on a three-dimensional smooth variety does not change under a simple flop. This statement follows from a more general derived equivalence of Landau-Ginzburg models that we procced to describe.
Let $Y \to S$ and $Y' \to S$ be two quasi-projective flat schemes over a quasi-projective scheme S. Let $\mathcal{E} \in D^b(Y \times_S Y')$ be an object such that: $\mathcal{E}$ has finite Tor-amplitude over Y, finite Ext-amplitude over Y', and $\mathrm{supp}(\mathcal{E})$ is projective over Y and Y'. Under these assumptions the integral transform $\Phi_{\mathcal{E}} := Rp_{Y'*}(Lp_Y^*(\bullet) \otimes \mathcal{E})$ sends $D^b(Y)$ to $D^b(Y')$. The right adjoint to

$\Phi_{\mathcal{E}}$ is defined by the formula $Rp_{Y*}R\,\mathrm{hom}(\mathcal{E}, p^{!}_{Y'}(\bullet))$, where $p^{!}_{Y'}$ is the right adjoint to $p_{Y'*}$. Since $\mathcal{E}$ has finite Ext-dimension over Y' the right adjoint sends $D^b(Y')$ to $D^b(Y)$. Moreover, in this case the functor $\Phi_{\mathcal{E}}$ sends the subcategory of perfect complexes $\mathrm{perf}(Y)$ to the subcategory of perfect complexe $\mathrm{perf}(Y')$.

Let $i : s \hookrightarrow S$ be a point. Consider the fibers $i_Y : Y_s \hookrightarrow Y$ and $i'_Y : Y'_s \hookrightarrow Y'$ and denote by $\mathcal{E}_s$ the object of $D^b(Y_s \times Y'_s)$ which is the inverse image $Li^*_{Y\times_S Y'}\mathcal{E}$ with respect to embedding $i_{Y\times_S Y'} : Y_s \times Y'_s \hookrightarrow Y \times_S Y'$. It can be easily checked that the object $\mathcal{E}_s$ also has finite Tor-amplitude over Y_s, finite Ext-amplitude over Y'_s and its support is projective over Y_s and Y'_s as well. Hence we obtain the functor $\Phi_{\mathcal{E}_s} : D^b(Y_s) \to D^b(Y'_s)$, which sends $\mathrm{perf}(Y_s)$ to $\mathrm{perf}(Y'_s)$ and has a right adjoint.

A direct calculation gives $Ri_{Y'*}\Phi_{\mathcal{E}_s} \cong \Phi_{\mathcal{E}}Ri_{Y*}$. Using this relation one checks immediately that if the functor $\Phi_{\mathcal{E}}$ is an equivalence then the functor $\Phi_{\mathcal{E}_s}$ is an equivalence as well.

Note also that if the functor $\Phi_{\mathcal{E}_s} : D^b(Y_s) \xrightarrow{\sim} D^b(Y'_s)$ is an equivalence, then it induces equivalence between categories of perfect complexes, because the perfect complexes are characterized as the homologically finite objects in the derived category of coherent sheaves, i.e. objects A for which $R\,\mathrm{Hom}(A, B)$ has only finitely many nontrivial cohomologies for any B. Thus any equivalence $\Phi_{\mathcal{E}_s}$ induces an equivalence between triangulated categories of singularities $D^b_{\mathrm{sing}}(Y_s)$ and $D^b_{\mathrm{sing}}(Y'_s)$.

We apply this observation to our situation. Let $\mathrm{w} : Y \to \mathbb{C}$ be a 3-dimensional Landau-Ginzburg-model. And let $C \in Y_0$ be a rational curve in the fiber over 0 such that the normal bundle to C in Y is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Let us consider the flop of Y in C. More precisely, in this situation we can consider the blow up $\widetilde{Y}$ of $C \subset Y$ and then blow down the exceptional divisor $C \times \mathbb{P}^1$ to $\mathbb{P}^1$, i.e. blow down along the ruling C. We obtain another Landau-Ginzburg-model $\mathrm{w}' : Y' \to \mathbb{C}$. Let us take as $\mathcal{E}$ the structure sheaf $\mathcal{O}_{\widetilde{Y}}$ as an object of $D^b(Y \times_{\mathbb{C}} Y')$. It is shown in [5] that the functor $\Phi_{\mathcal{E}}$ is an equivalence between derived categories of coherent sheaves on Y and Y'. Since Y and Y' are smooth, any object on the product has finite Tor-amplitude over Y, and finite Ext-amplitude over Y'. Moreover, the support of $\mathcal{E}$ is projective over Y and Y'. Thus the previous discussion above applies and we get that the fibers Y_0 and Y'_0 have equivalent derived categories of coherent sheaves and equivalent triangulated categories of singularities. In other words we have just proven the following:

Proposition 7.1.
Let $\mathrm{w} : Y \to \mathbb{C}$ and $\mathrm{w}' : Y' \to \mathbb{C}$ be two Landau-Ginzburg-models which are related to each other via flop in a rational curve C from the fiber Y_0 over 0 whose normal bundle is isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then the triangulated categories of singularities $D^b_{\mathrm{sing}}(Y_0)$ and $D^b_{\mathrm{sing}}(Y'_0)$ are equivalent.

8. Computations with mirror branes

In this section we return to the HMS question and identify objects in the derived category of the Landau-Ginzburg mirror of a genus two curve which are natural candidates for mirrors of the standard basis of loops on the curve.

Let now Y_0 be the fiber of a Landau-Ginzburg model $\mathrm{w} : Y \to \mathbb{C}$ which is a union of three rational surfaces S_i where $i = 1, 2, 3$. We denote by F_i the complement to S_i in Y_0 which is the union of S_j and S_k, where $j \neq k \neq i$. Since the relative canonical sheaf $\omega_{Y/\mathbb{C}}$ is trivial. We have that the restriction of the line bundle $\mathcal{O}_Y(S_i)$ on S_i gives us the canonical sheaf ω_{S_i}. Note that the canonical sheaf ω_{S_i} also coincides with the restriction of $\mathcal{O}_Y(-F_i)$. Let C_i be the intersection $S_i \cap F_i$. It is a curve on S_i, which consists of two rational components meeting in two points and is a divisor in the anticanonical system on S_i.

Consider the short exact sequence on Y

$$0 \to \mathcal{O}_Y(-S_i) \to \mathcal{O}_Y \to \mathcal{O}_{S_i} \to 0.$$

It induces $\mathbb{Z}/2$-periodic resolution of $\mathcal{O}_{S_i}$ as the sheaf on Y_0

$$\{\cdots \to \mathcal{O}_{Y_0} \to \mathcal{O}_{Y_0}(-S_i) \to \mathcal{O}_{Y_0}\} \to \mathcal{O}_{S_i} \to 0 \tag{3}$$

Let L and M be two line bundles on Y_0. Denote by L_i and M_i, $i = 1, 2, 3$, the coherent sheaves on Y_0 which are obtained by restrictions of L and M on S_i, respectively. Tensoring the resolution (3) with the line bundle L we obtain a

$\mathbb{Z}/2$-periodic resolution for the coherent sheaf L_i

$$\{\cdots \to L \to L(-S_i) \to L\} \to L_i \to 0 \tag{4}$$

Using this resolution and the fact that the restriction of $O(S_i)$ on S_i coincides with the canonical sheaf ω_{S_i} we can calculated $R\operatorname{Hom}(L_i, M_j)$ on Y_0. When $j = i$ we get the following complex

$$0 \to L_i^* \otimes M_i \xrightarrow{0} L_i^* \otimes M_i \otimes \omega_{S_i} \xrightarrow{u} L_i^* \otimes M_i \xrightarrow{0} \cdots$$

here u is obtained from the injection $\mathcal{O}_{S_i}(-F_i) \to \mathcal{O}_{S_i}$ by tensoring with $L_i^* \otimes M_i$.
When $j \neq i$ we get the complex

$$0 \to L_j^* \otimes M_j \xrightarrow{v} L_j^* \otimes M_j(Q_{ij}) \xrightarrow{0} L_j^* \otimes M_j \xrightarrow{v} \cdots$$

where v is obtained from the injection $\mathcal{O}_{S_j} \to \mathcal{O}_{S_j}(Q_{ij})$ by tensoring with $L_j^* \otimes M_j$.
Summarizing we get

Proposition 8.1.

Let L and M be two line bundles on Y_0. And let L_i and M_i, $i = 1, 2, 3$, be the coherent sheaves on Y_0 which are obtained by restrictions of L and M on S_i, respectively. Then

1) *when $j = i$ we have $\operatorname{Ext}^{2p+1}_{Y_0}(L_i, M_i) \cong 0$ and $\operatorname{Ext}^{2p+2}_{Y_0}(L_i, M_i) \cong (L^* \otimes M)|_{C_i}$,*

2) *when $j \neq i$ we have $\operatorname{Ext}^{2p}_{Y_0}(L_i, M_j) \cong 0$ and $\operatorname{Ext}^{2p+1}_{Y_0}(L_i, M_j) \cong (L_j^* \otimes M_j)(Q_{ij})|_{Q_{ij}}$.*

Consider now the triangulated category of singularities $D^b_{\text{sing}} Y_0$. By Proposition 1.21 of [21] we know that Ext's between objects in $D^b_{\text{sing}} Y_0$ can be calculated as Ext's in $\operatorname{Coh}(Y_0)$. More precisely, in our case we have that

$$\operatorname{Hom}_{D^b_{\text{sing}} Y_0}(L_i, M_j[N]) \cong \operatorname{Ext}^N_{Y_0}(L_i, M_j)$$

for $N > 2$. But since shift by $[2]$ in $D^b_{\text{sing}} Y_0$ is identity, we can calculated all Hom's in triangulated categories of singularities of Y_0 taking Ext's in category of coherent sheaves on it. Considering spectral sequence from local to global Ext's, which is degenerated in our case we immediately obtain the following

Proposition 8.2.

Let L and M be two line bundles on Y_0. And let L_i and M_i, $i = 1, 2, 3$, be the coherent sheaves on Y_0 which are obtained by restrictions of L and M on S_i, respectively. Then

1) *when $j = i$ we have*

$$\operatorname{Hom}_{D^b_{\text{sing}} Y_0}(L_i, M_i) \cong H^0(C_i, L^* \otimes M)|_{C_i})$$
$$\operatorname{Hom}_{D^b_{\text{sing}} Y_0}(L_i, M_i[1]) \cong H^1(C_i, L^* \otimes M)|_{C_i}),$$

2) *when $j \neq i$ we have*

$$\operatorname{Hom}_{D^b_{\text{sing}} Y_0}(L_i, M_j) \cong H^1(Q_{ij}, (L_j^* \otimes M_j)(Q_{ij})|_{Q_{ij}})$$
$$\operatorname{Hom}_{D^b_{\text{sing}} Y_0}(L_i, M_j[1]) \cong H^0(Q_{ij}, (L_j^* \otimes M_j)(Q_{ij})|_{Q_{ij}}).$$

Corollary 8.1.

For any L_i as above we have

$$\operatorname{Hom}_{D^b_{\text{sing}} Y_0}(L_i, L_i) = \mathbb{C} \quad \textit{and} \quad \operatorname{Hom}(L_i, L_i[1]) = \mathbb{C}$$

This follows from the fact that $H^0(C_i, \mathcal{O}_{C_i}) = \mathbb{C}$ and $H^1(C_i, \mathcal{O}_{C_i}) = \mathbb{C}$.
Let us take L_i and M_j, when $i \neq j$. Consider the restriction of these sheaves on $Q_{ij} \cong \mathbb{P}^1$. Let $L_i|_{Q_{ij}} \cong O(l_i)$ and $M_j|_{Q_{ij}} \cong O(m_j)$. Then Proposition 8.2 and the property that the normal bundle to Q_{ij} in S_j is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$ imply that

$$\mathrm{Hom}_{D^b_{\mathrm{sing}} Y_0}(L_i, M_j) = H^1(\mathbb{P}^1, O(m_j - l_i - 1)) \quad \text{and} \quad \mathrm{Hom}_{D^b_{\mathrm{sing}} Y_0}(L_i, M_j[1]) = H^0(\mathbb{P}^1, O(m_j - l_i - 1)).$$

On each surface S_i there are two (-1)-curves each of which meet the curve Q_{ij} in one point. Denote them as E'_{ij} and E''_{ij}. Consider a set of four sheaves $\mathcal{O}_{S_1}(E'_{12}), \mathcal{O}_{S_1}(E'_{13}), \mathcal{O}_{S_2}, \mathcal{O}_{S_3}$ on Y_0. Then nontrivial Hom's in the triangulated category of singularities between them are only the following

$$\mathrm{Hom}_{D^b_{\mathrm{sing}} Y_0}(\mathcal{O}_{S_1}(E'_{12}), \mathcal{O}_{S_2}) = \mathbb{C}, \quad \mathrm{Hom}_{D^b_{\mathrm{sing}} Y_0}(\mathcal{O}_{S_2}, \mathcal{O}_{S_1}(E_{12})'[1]) = \mathbb{C},$$

$$\mathrm{Hom}_{D^b_{\mathrm{sing}} Y_0}(\mathcal{O}_{S_1}(E'_{13}), \mathcal{O}_{S_3}) = \mathbb{C}, \quad \mathrm{Hom}_{D^b_{\mathrm{sing}} Y_0}(\mathcal{O}_{S_3}, \mathcal{O}_{S_1}(E_{13})'[1]) = \mathbb{C}.$$

Corollary 8.2.
Hom's between objects $\mathcal{O}_{S_1}(E'_{12}), \mathcal{O}_{S_1}(E'_{13}), \mathcal{O}_{S_2}, \mathcal{O}_{S_3}$ in triangulated category of singularities of Y_0 coincide with Floer cohomologies between Lagrangian circles on a curve of genus two which represent the standard basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{b}_1, \boldsymbol{b}_2$ in the first integral homologies of the curve.

9. Appendix

In this appendix we collect some important technical facts on the structure of the singular fibers of the Landau-Ginzburg mirrors of a curve of genus two or the intersection of two quadrics in $\mathbb{P}^5$.

9.1. Local analytic structure at the curves of intersection

Our description of the component $\widetilde{S}_3$ of the zero fiber of either the mirror of C or the mirror of X_4 shows that this surface has an elliptic fibration in which one I_2 fiber is given by the union of the curves $q_1 \cup q_2$. This is why the formal or analytic neighborhoods of $q_1 \cup q_2$ can be identified in the two mirrors. The global elliptic fibrations on the $\widetilde{S}_3$ components are different though: in the case of the genus two curve the fibration has two fibers of type I_1, and one of type I_2^* while in the case of the intersection of two quadrics in $\mathbb{P}^5$ there are two fibers of type I_1 and one of type I_8.
The same applies to the curve q_3. In our birational models for the mirrors they are singular (nodal) fibers in conic fibrations, and hense have the same analytic neighbourhoods. In order to identify the full analytic neighborhods of the singularities of the central fibers for the two mirrors we have to identify the respective curves q_3 so that **four** points on them match. Since the cross ratios of the four tuple of points in the two mirrors are different (λ and (-1) respectively) it is not a priori clear if this can be done geometrically. We still expect that the corresponding categories are equivalent though.

9.2. Symmetries in the central fibers

Here is one more feature the fibrations we are working with have.
As mentioned in the previous sections, the pairs diagonal $(1,1)$-divisors in $\mathbb{F}_0$ and the pairs of positive sections in $\mathbb{F}_2$ have to be chosen so that certain agreements are met. This follows from the symmetry (coming basically from the initial equations of the varieties Y^{aff}) according to which the exceptional components of the central fibers are isomorphic (this is before we blow up the last two points as explained before, and via an involution of the central fibers which leave the third component invariant).
In the case of $\mathbb{F}_0$ corresponding to the component $\widetilde{S}_3$, this implies that the pair of diagonals have to get exchanged by an involution of $\mathbb{F}_0$ acting so that maps one of the members of the one pair of rulings to the other one and leaves the

members of the other pair invariant. This actually forces to have the same property on both other components as well. This in turn gives us that the three components form an S_3 symmetric configuration up to some extra blow-ups.
The same can be said about the sections of $\mathbb{F}_2$. In a sense, this reflects the symmetry we had on the general fiber related to the three involutions acting there.

Acknowledgements

First and second authors are partially supported by NSF Grant DMS0600800 and by Clay Math Institute.

We have benefited greatly from discussions M.Kontsevich, P.Seidel, K.Hori, C.Vafa and E.Witten. We are very thankful to D.Auroux and T.Pantev, without whom this paper would have never been written. We would like to thank IPAM, UCLA and especially M.Green and H.D.Cao for the wonderful working conditions of the program Symplectic Geometry and Physics, during which the work on this paper was initiated. The first two authors are grateful to KITP, Santa Barbara and especially D.Gross, R.Dijgraaf, D.Freed and T.Pantev, for the wonderful athmosphere during the program Geometry, Duality and Strings, when most of this work was done.

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