Interpretations of Spectra

L. Katzarkov, K. S. Lee, J. Svoboda, and A. Petkov

Abstract The studies of homological mirror symmetry as correspondence of Lefshetz pencils was initiated as part of the general theory of categorical linear systems. In this paper, we look at the monodromy of these linear systems via a new notion of noncommutative spectrum.

Keywords Mirror symmetry · Landau-Ginzburg models · Spectra

1 Introduction

The studies of homological mirror symmetry (HMS) as correspondence of Lefshetz pencils was initiated in [31] as part of the general theory of categorical linear systems. In this paper, we look at the monodromy of these linear systems. We utilise these monodromies by introducing a new notion of noncommutative spectrum. We will use the setup and the notations from [31]. We start with a pencil where the fibers are CY varieties and the global pencils constitute mirrors of Fano manifolds. We have the following category diagram:

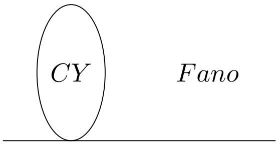
L. Katzarkov (✉) · K. S. Lee · J. Svoboda
University of Miami, Miami, FL, USA
e-mail: lkatzarkov@gmail.com

L. Katzarkov
HSE Moscow, Moscow, Russia

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str. bl. 8, 1113 Sofia, Bulgaria

A. Petkov
FMI, Sofia, Bulgaria

I. Cheltsov et al. (eds.), *Birational Geometry, Kähler–Einstein Metrics and Degenerations*, Springer Proceedings in Mathematics & Statistics 409,
https://doi.org/10.1007/978-3-031-17859-7_20



$$F(CY) \to F(Fano)$$

Here $F(CY)$, $F(Fano)$ are the corresponding Fukaya–Seidel categories. Im $\Phi(F(CY)) = \mathcal{A}$ is a localization category $F(CY)/\sim$. (Using HMS we can use $D^b(X)$—the category of coherent sheaves on algebraic varieties X.)

This localization category has a filtration:

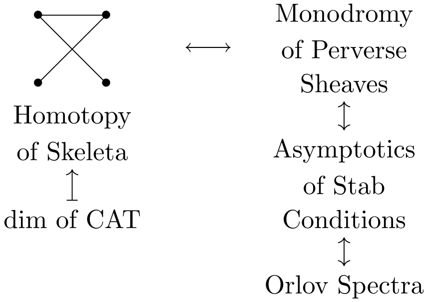
$$\mathcal{A} \supset \mathcal{F}_{\lambda_1} \supset \cdots \supset \mathcal{F}_{\lambda_n}$$

where:

- λ_i are the asymptotics of limiting stability conditions.
- $Z = z^{\lambda_i}(\cdots)$
- $\mathcal{F}_{\lambda_i} = \{F \text{ s.t. } Z(F) = z^{\lambda_i}(\cdots)\}$
- λ_i are also the asymptotics of the PDE

$$\left(\frac{\partial}{\partial u} + u^{-2}K + u^{-1}G\right)$$

The above filtration can also be seen as the monodromy of the perverse sheaf of categories over the skeleton. Following [31] we think of the category as a perverse sheaf of categories over lagrangian skeleton. In the diagram bellow we describe our findings in [31].



The main idea in current paper is to give an interpretation of the above λ_i filtration as a noncommutative spectrum and a spectrum of Landau-Ginzburg (LG) models. We use the theory of LG models as generalized theory of singularity.

The above considerations lead to birational invariants, which will appear in more details in [29, 34]. (For definitions and general theory of LG models and HMS we refer to [30].)

We will base our birational considerations on the following major notions and ideas:

(1) **Quantum spectrum**. The quantum spectrum is defined in [29]. Let $K\cdot$ be the quantum multiplication by canonical class. It defines the following splitting of cohomology:

$$\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}.$$

Here λ_i are the eigenvalues of $K\cdot$. We call these eigenvalues *quantum spectrum*. The main theorem proven in [29] is:

MAIN THEOREM: **The splitting $\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}$ is a birational invariant.**

(2) **Noncommutative spectrum**. The noncommutative spectrum is defined in [29]. In the current paper we extend these ideas and give some examples.

(A) We build analogues with low dimensional topology and give several new directions for research.

(B) We extend the definition of a noncommutative spectrum to multispectra. Possible applications are discussed.
Our considerations are only the tip of the iceberg. We propose a correspondence between nonrationality over algebraically nonclosed fields and complexity of the discriminant loci of the moduli space of LG models. We will consider some arithmetics applications in Sect. 3. In fact one can define several different spectra.
In addition to the **quantum spectrum** mentioned above, one can define several other spectra:

- **Noncommutative spectrum;**
defined by the asymptotics of the quantum equation.
- **Givental spectrum;**
defined by the solutions of the Givental's equation.
- **Spectrum of LG model—multiplier ideal sheaf;**
defined as the Steenbrink spectrum of a new singularity theory of the LG model.
- **Asymptotics of stability conditions—stability spectrum;**
defined as asymptotics of limiting stability conditions.
- **Serre dimension of the Kuznetsov's component;**
defined as a categorical dimension.
- **Arnold-Varchenko-Steenbrink spectrum of the affine cone.**
defined as the classical spectrum of the affine cone singularity over X.
- **R-charges**—the assymptotics of RG flow—the same as asymptotics of Kähler-Ricci flow—see Sect. 6.

We will discuss relations among some of them. Understanding the complete scope of relations is an intriguing problem. We initiate the study of these connections in this paper. We will develop this connections in upcoming papers [27, 32].

(C) We also propose a parallel between the existence of Kähler-Einstein metrics and the top number of the noncommutative spectra. Recall that

$$lct(X, G) = \sup \{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ l.c.s. } \forall G \text{ inv. } D\}$$

We note the following parallel:

nonrationality of (X, G) orbifold	$\exists$of K.E. metric on (X, G)
$\delta > \dim X - 2$ $X$is not rational $\delta$is$lct$for sing	$lct(X, G) > \frac{\dim X}{\dim X + 1}$

In the above table lct is the log canonical threshold.
We take this parallel further:

(D) We connect the noncommutative spectra with elliptic genus and conformal field theory. We connect orbifoldization of elliptic genus with spectra of singular varieties. This leads to a categorical interpretation of Birkar's boundness theorem. We propose the idea of categorical resolution and "boundness" of conformal field theories—the central charges correspond to the noncommutative spectra.
As a consequence we propose a parallel between Zamolodchikov's c-theorem and uppersemicontinuity condition of noncommutative spectra.
We will call the monotonicity of the highest number of the spectrum uppersemicontinuity. In other words, the highest number of the spectrum is decreasing monotonically when moving from the boundary of Frobenius manifold to its general point.

The paper is organized as follows. We explain the general theory in Sect. 2. The Fano applications are considered in Sect. 2. The arithmetics applications are considered in Sect. 3. The parallel with 3-dimensional topology are discussed in Sect. 4. The extension to multispectra is discussed in Sect. 5. In Sect. 6, we consider the connection of spectra with elliptic genus. We make a connection between Birkar's theory and the conformal field theories.

2 Noncommutative Spectra

In this section we introduce the idea of noncommutative spectra—an idea which belongs to M. Kontsevich. We describe new birational invariants and describe some easy applications.

2.1 Definitions of Quantum and Nc Spectra

Let X be a projective algebraic variety over $\mathbb{C}$, with a given ample line bundle. The Gromov-Witten invariants in genus zero define a potential $\mathcal{F}_0$: formal series on $H^\bullet(X)$ with coefficients in $\mathbb{Q}[[T]]$—see e.g. [30]. We briefly recall two conjectures (see e.g. [29]).

1. First we have:

Conjecture 2.1 $\mathcal{F}_0$ is convergent for a point $\gamma \in H^\bullet(X)$ and for $T \in \mathbb{C}$, both close to 0.

2. Assuming Γ-conjecture (see e.g. [30]) we get that **nc** Hodge structures are parametrized by a domain

$$M \subset H^\bullet(X, \mathbb{C})/H^2(X, 2\pi i\mathbb{Z}),$$

which is a meromorphic connection on the trivial bundle over u-plane $\mathbb{C}_u$ with fiber $H^\bullet(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G$$

(Recall that the Γ-conjecture gives a lattice, hypothetically compatible with Stokes filtrations along rays at $u \to 0$. For more details see [30].)

We define the operator $K = K(\gamma)$ as the quantum product with $c_1(T_X) + \sum_{i\neq 2}(2-i)\gamma_i$. It depends on the point $\gamma = (\gamma_i \in H^i(X))_{i=0,\dots,2\dim_{\mathbb{C}} X}$ in Frobenius manifold $\mathcal{M}$. We also define the operator G as a constant operator given by $G_{|H^i(X)} = \frac{i-\dim_{\mathbb{C}} X}{2} \cdot id_{H^i(X)}$.

We use the example bellow to introduce and demonstrate two important definitions. Let X be a smooth 3-dimensional cubic in $\mathbb{P}^4$. Operators K, G on 4-dimensional space $H^{\text{even}}(X) = \oplus_{i=0}^3 H^{2i}(X)$ with the basis being powers of the hyperplane section, at point $\gamma = 0 \in \mathcal{M}$, are:

$$K = 2 \cdot \begin{pmatrix} 0 & 6 & 0 & 36 \\ 1 & 0 & 15 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

Solutions of the quantum equation

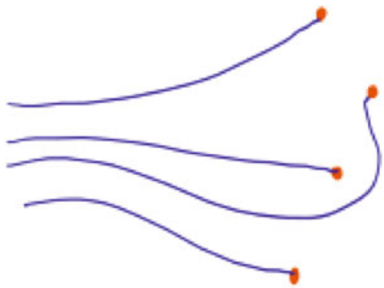


Fig. 1 Gabrielov paths *(Red dots correspond to eigenvalues of quantum multiplication)*

$$\left(\frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G\right)\psi(u) = 0 \tag{1}$$

grow at $u \to 0$ as

$$\sim u^{-\frac{5}{6}}, \sim u^{-\frac{1}{6}}.$$

Definition 2.2 Quantum spectrum is the spectrum of K, a finite subset $\{z_a\} = Spec_X \subset \mathbb{C}$ (depends on the point γ in $\mathcal{M}$).

Definition 2.3 Noncommutative spectrum: The asymptotics of the sub-exponential growth solutions of the Eq. 1 above form the **noncommutative spectrum** or **nc spectrum**.

In what follows we will denote by δ minus two times the lowest number of **noncommutative spectrum**. In the above example

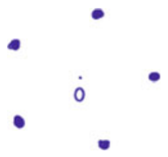
$$\delta = \frac{5}{3}.$$

Consider a purely even affine submanifold $\mathcal{M}^{\mathrm{alg}} \subset \mathcal{M}$, given by deformations of quantum product by linear combinations of algebraic classes $H^{\mathrm{alg}}_{\mathbb{Q}}(X) \subset H^{\mathrm{even}}(X, \mathbb{Q})$.

Conjecture 2.4 For any point in $\mathcal{M}^{\mathrm{alg}}$ and a choice of disjoint paths from ∞ to points of the corresponding quantum spectrum (see Fig. 1), we obtain a semi-orthogonal decomposition $\mathrm{D}^{\mathrm{b}}(Coh(X)) = \langle \mathcal{C}_1, \ldots, \mathcal{C}_r \rangle$ where r is the number of elements of the spectrum.

All categories $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are saturated (i.e. smooth and proper), equal to local Fukaya-Seidel categories for the mirror LG dual $(Y, W : Y \to \mathbb{C})$, if it exists.

Example 1 (1) $X = \mathbb{P}^n$, the **quantum spectrum** is $\mu_{n+1} = \{z \in \mathbb{C} \mid z^{n+1} = 1\}$ (for some point in $\mathcal{M}$)



This gives $\mathrm{D}^{\mathrm{b}}(Coh X) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$.

(2) Conjectural blow-up formula: If $\widetilde{X} = Bl_Y(X)$ where $Y \subset X$ is a smooth closed subvariety of codimension $m \geq 2$, then the **quantum spectrum** $Spec_{\widetilde{X}}$ looks like with $(m-1)$ shifted copies of $Spec_Y$ around one copy of $Spec_X$. (Here the blue dots correspond to eigenvalues of quantum multiplication added after blow ups.)

(3) If X is a Calabi-Yau manifold or a manifold of general type the **quantum spectrum** is just a point.

(4) The above considerations lead to the following theorem proven in [29]: **MAIN THEOREM**: The splitting $\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}$ is a birational invariant.

2.2 *Dimension Theory*

In this section, we introduce Serre dimension which (with some exceptions) is equal to the number δ from the noncommutative spectrum. We see that sometimes elementary pieces $\mathcal{C}_a = \mathcal{C}_{z_a}$, $z_a \in Spec_X$ (could be combined as some points of the spectrum collide), are themselves equivalent to derived categories of coherent sheaves on some varieties, of certain dimensions $\leq \dim X$.

In general, for a saturated category $\mathcal{C}$ one can define its **Serre dimension** [49]

$$\dim_{\mathrm{Serre}} \mathcal{C} := \lim_{|k| \to +\infty} \left\{ \frac{i}{k} \mid Ext^i(Id_{\mathcal{C}}, S_{\mathcal{C}}^k) \neq 0 \right\} \subset \mathbb{R}.$$

Here $S_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is the Serre functor [48]:

$$Hom_{\mathcal{C}}(E, F)^{\star} = Hom_{\mathcal{C}}(F, S_{\mathcal{C}} E), \quad \forall E, F \in Ob(\mathcal{C}).$$

In general, Serre dimension could be an empty set, or an interval.

For categories $\mathrm{D}^{\mathrm{b}}(Coh(X))$, it is exactly the dimension $\dim X \in \mathbb{Z}_{\geq 0}$. For a fractional Calabi-Yau category $S_{\mathcal{C}}^k \sim [n]$, the Serre dimension is equal to Calabi-Yau dimension $\frac{n}{k}$, hence fractional.

Example 2 Fukaya-Seidel category of $Y = \mathbb{C}_x$, $W = x^d$, $d \geq 2$: $\dim_{\text{Serre}} = 1 - \frac{2}{d}$.

Let us assume that (H, ∇) is a connection with second order pole and regular singularity (i.e. all solutions have polynomial growth). Then the order of growth defines a filtration by subbundles, preserved by connection ∇, the indices form the **subexponential growth spectrum = nc spectrum**.

Essential Example
Consider the hypersurface $X \subset \mathbb{P}^n$ of Calabi-Yau/general type. The connection on the image of $H^\bullet(\mathbb{P}^n)$ in $H^\bullet(X)$ under restriction map, i.e. the span of powers of $c_1(\mathcal{O}(1)) \in H^2(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G,\ K = \text{ classical product with } c_1(T_X)$$

The **nc spectrum** is

$$(-\dim X/2, -\dim X/2, \ldots)$$

for X a manifold of general type and so

$$\delta = \dim X.$$

For X a Calabi-Yau manifold **nc spectrum** is

$$(-\dim X/2, 1 - \dim X/2, \ldots, +\dim X/2)$$

and $\delta = \dim X$. Similar behavior happens for Calabi-Yau when we replace the multiplication by $c_1(T_X) = 0$, by the multiplication by an inhomogeneous class $c_1(T_X) + \sum_{i \neq 2}(2 - i)\gamma_i$, $\gamma_i \in H^i(X)$, $i \in 2\mathbb{Z}$.

2.2.1 More General Example

Let us consider a weighted projective space $\mathbb{P}^{\omega_0,\ldots,\omega_n}$ and generic complete intersection X of hypersurfaces of degrees $d_1, \ldots, d_m$. In what follows we investigate the connection between **nc spectrum**, **Givental spectrum** and **Steenbrink spectrum** in this example.

Recall that such a complete intersection is called **well-formed** iff (here unions are understood **with multiplicities**)

$$\bigcup_i \left\{\frac{1}{\omega_i}, \ldots, \frac{\omega_i - 1}{\omega_i}\right\} \subset \bigcup_j \left\{\frac{1}{d_j}, \ldots, \frac{d_j - 1}{d_j}\right\} \qquad \star$$

We call the numbers from $\star$ **Givental spectrum**.

Well formed X is smooth, and does not meet singularities of $\mathbb{P}^{\omega_0,\ldots,\omega_n}$. Let us assume that X is a Fano variety, i.e. $\sum_i \omega_i > \sum_j d_j$.

We define the Givental's hypergeometric operator:

$$\prod_i \omega_i^{\omega_i} \cdot \partial^{\dim X} - \prod_j d_j^{d_j} \cdot q \cdot \frac{\prod_j (\partial + \frac{1}{d_j}) \cdots (\partial + \frac{d_j - 1}{d_j})}{\prod_i (\partial + \frac{1}{\omega_i}) \cdots (\partial + \frac{\omega_i - 1}{\omega_i})}, \quad \partial := q\frac{d}{dq}, u = c \cdot q^{-\frac{1}{\sum_i \omega_i - \sum_j d_j}}$$

The **nc spectrum** of the Laplace operator of the Givental's hypergeometric operator is:

$-\frac{\dim X}{2}$ + {complement in $(\star)$} $\cdot(\sum_i \omega_i - \sum_j d_j) \rightarrow$ numbers $s_0 \leq s_1 \leq \cdots$.

The **adjusted Steenbrink spectrum** is:

$(s_0, s_1 + 1, s_2 + 2, \ldots)$.

The adjusted Steenbrink spectrum is symmetric with center at 0.

Example 3 Let use consider complete intersection of two hypersurfaces of degree $d_1 = 2, d_2 = 4$ in $\mathbb{P}^6 = \mathbb{P}^6(1, 1, 1, 1, 1, 1, 1)$.

The **growth spectrum** is

$$\left(-\frac{7}{4}, -\frac{6}{4}, -\frac{6}{4}, -\frac{5}{4}\right)$$

In other words the solutions of the quantum equation grow as

$$u^{-\frac{7}{4}}, \log(u)u^{-\frac{6}{4}}, u^{-\frac{6}{4}}, u^{-\frac{5}{4}}$$

.

Adding $(0, 1, 2, 3)$ to **nc spectrum** we obtain **adjusted Steenbrink spectrum**:

$$\left(-\frac{7}{4}, -\frac{1}{2}, +\frac{1}{2}, +\frac{7}{4}\right)$$

.

2.3 Some Computational Tools

We briefly discuss some methods for calculations. We start with:

Theorem 2.5 (Saito's Theorem) ([46]) $P_f(t) = Sp_f(t)$.

Here $P_f(t) = \sum_\alpha (\dim J_\alpha) t^\alpha$ is the Poincare series and $Sp_f(t) = \sum_i (n_i . t^i)$—is the spectrum polynomial and n_i—are the multiplicity of spectral number.

Recall that for $f(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda f(x_1, \ldots, x_n)$ we define weight $wt.(x_1^{a_1}, \ldots, x_n^{a_n}) = \sum_{i=1}^n (1 + a_i) w_i$.

Example 4 Let us look at the example of three dimensional cubic from a new point of view:
$f(x_1, \ldots, x_5) = x_1^3 + \cdots + x_5^3$
$P_f(t) = t^{\frac{5}{3}} + 5t^2 + 10t^{\frac{7}{3}} + 5t^3 + t^{\frac{10}{3}}$
$\delta = \frac{10}{3} - \frac{5}{3} = \frac{5}{3}$.

Let us denote by $Cone(X)$ the cone over a hypersurface X and C is the Fukaya-Seidel category associated with the most singular fiber of the LG model of X. By Orlov's theorem we have $D^b(Cone(X/G)) = C$.

Denote by S_l the lowest number of the Steenbrink spectrum and by S_h the highest number of the Steenbrink spectrum for $Cone(X/G)$. An A-side conjectural version of Orlov's theorem suggests:

Conjecture 2.6 The Steenbrink spectra of $Cone(X)$ determines noncommutative spectrum associated with X. The following identity holds

$$\delta = S_h - S_l.$$

Let $\mathcal{C}$ be a Calabi-Yau category s.t. Serre functor satisfies $S^a = [b]$.
$HH_\bullet(\mathcal{C}) = \oplus HH^i(\mathcal{C})[\delta]$

Definition 2.7 The homomorphism

$$\epsilon : (Q \times \mathbb{Z}_2) \to Aut(\mathcal{C})$$

defines a categorical covering. The covering structure is recorded by multiplication in the A_∞.

In the example 2.8 we get $t^{\frac{10}{3}}$, $t^{\frac{5}{3}}$ define $\frac{10}{3} - \frac{5}{3}$, which produces degree of a covering.

Example 5 $x_1^4 + \cdots + x_5^4$. We consider this hypersurface as an affine cone. We compute the Poincare polynomial and obtain:
$P_f = t^{\frac{5}{4}} + \cdots + t^{\frac{15}{4}} \Rightarrow \delta = \frac{15}{4} - \frac{5}{4}$.

Example 6 $x_1^3 + \cdots + x_5^3$. We consider this hypersurface as an affine cone. Here we can compute the Bernstein polynomial
$b_f(t) = (t+1)(t+2)(t+3)(t+\frac{5}{3})(t+\frac{7}{3})(t+\frac{8}{3})(t+\frac{10}{3})$
and obtain:
$\delta = \frac{10}{3} - \frac{5}{3}$.

2.4 New Nonrationality Results

In this section we record the results of our method and compare them with already known results. We use the simplest of invariants—δ. We hope that other numbers of the noncommutative spectrum can be used as well. In fact it seems that these numbers mirror classical theory of multiplier ideal sheaves and characterize the stratification of the base loci of the anticanonical system for Fano's.

We have defined

$$\delta = dim(X) - 2(N - d)/d$$

As an immediate consequence we get in [29].

Theorem 2.8 *(1) Let X be a Fano smooth hypersurface of degree d in* $\mathbb{P}^{5-1}$ *such that*

$$d > 5/2.$$

Then X is not rational.

(2) Let X be a Fano smooth hypersurface of degree d in $\mathbb{P}^{6-1}$ *such that*

$$d \geq 6/2$$

and $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$. *Then X is not rational.*

(3) Let us assume uppersemicontinuity condition. Let X be a Fano smooth hypersurface of odd dimension and of degree d in $\mathbb{P}^{N-1}$ *such that*

$$d > N/2$$

Then X is not rational.

(4) Let X be a Fano smooth hypersurface of even dimension $k = (N - 2)/2$ *and of degree d such that*

$$d > N/2$$

and $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$. *Then X is not rational.*

We briefly describe the idea of the proof.

Proof The above formulae is equivalent to $\delta > dim(X) - 2$.

(1) $dim(X) = 3$ Assume that X is rational so it is obtained via sequence of blow ups and blow downs with centers curves.
According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to 1.
Our **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2 = 1.$$

- a contradiction.

(2) $dim(X) = 4$. Assume $\delta > 2$. The fact that $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ ensures that $\delta > 2$ stays unchanged under deformations. Assume that X is rational so it is obtained via sequence of blow ups and blow downs with centers points, surfaces, curves. According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to 2.
The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2 = 2.$$

- a contradiction.
The case $d = 3$, $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ will be treated in [29]. Let us briefly mention the idea. We have a splitting

$$\mathcal{H} = \oplus_{\lambda_i} H_{\lambda_i}.$$

With the exception of one all of these H_{λ_i} are one dimensional. The high dimensional one has a symmetric noncommutative Hodge structure. With 20 dimensional space of deformation this noncommutative Hodge structure cannot come from a commutative surface.

(3) $dim(X) = N - 2$, $N - 2$ is odd. In this case $\delta > dim(X) - 2$.
Assume that X is rational so it is obtained via sequence of blow ups and blow downs.
According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to $dim(X) - 2$. According to uppersemicontinuity these asymptotics can only go down. The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2.$$

- a contradiction.

(4) $dim(X) = N - 2 = 2k$, $N - 2$ is even $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$. In this case $\delta > dim(X) - 2$. The fact that $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$ ensures that $\delta > dim(X) - 2$ does not go down.
Assume that X is rational so it is obtained via sequence of blow ups and blow downs.
According to the **ESSENTIAL EXAMPLE** the maximal asymptotics we get under blow ups are integers less or equal to $dim(X) - 2$. According to uppersemicontinuity these asymptotics can go only down. The **MAIN THEOREM** ensures that these integers do not interact. So the maximum δ we can get by blow up is

$$dim(X) - 2.$$

- a contradiction.

Similarly we have [29].

Theorem 2.9 *Let X be a smooth Fano complete intersection of hypersurfaces of degrees $d_1, \dots, d_m$ in $\mathbb{P}^N$. Denote by d_t the sum $d_1 + \cdots + d_n$ and by d_m the minimal degree.*

In this case the Arnold number (the largest number of the noncommutative spectrum) is equal to:

$$\delta = dim(X) - 2((d_t - d_m)/d_m)$$

.

1. *Let X be 3 dimensional and $\delta > 1$. Then X is not rational.*
2. *Let X be 4 dimensional, $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}$ and $\delta > 2$. Then X is not rational.*

Let us assume uppersemicontinuity condition.

3. *Let X be of odd dimension and $\delta > dim(X) - 2$. Then X is not rational.*
4. *Let X be of even dimension $2k$, $H^{k,k}(X, \mathbb{Z}) = \mathbb{Z}$ and $\delta > dim(X) - 2$. Then X is not rational.*

The same result works for well formed complete intersection in weighted projective spaces. The formulae for δ is similar:

$$\delta = \dim X - 2\frac{\omega_{\text{sum}} - d_{\text{sum}}}{d_{\text{max}}}, \quad \omega_{\text{sum}} := \sum_j \omega_j \text{ for } \mathbb{P}^{\omega_0, \dots, \omega_n}$$

3 Application to Arithmetics

The GW invariants can be defined over algebraically nonclosed fields L. Therefore the techniques of noncommutative spectrum can be used to investigate nonrationality over algebraically nonclosed fields L. Of course changing the fields does not change the GW invariants but it changes algebraic cycles. Changing algebraic cycles affects deformations of LG models and as a result the spectrum of quantum multiplication by the canonical class. In this case we do not need an uppersemicontinuity—the restriction on deformation comes from algebraic cycles.

Recall the example from the introduction—the two dimensional cubic: $X : X_0^3 + \cdots + X_3^3 = 0$. Consider X over algebraically nonclosed field L s.t. Pic $X_L = 1$. After analyzing the Sarkisov links we conclude that X is not rational.

We will look at this example from the point of view of the spectrum. We begin with:

Theorem 3.1 *Let X be a Fano stack of dimension at most* 4 *over a field L such that image of* $CH(X)$ *in* $\sum_i H^i(X,\mathbb{Z})$ *is generated by powers of anticanonical class. Assume that Arnold constant (the highest number in the spectrum) is bigger than* $dim(X)-2$. *Then X is not rational.*

The same theorem works in the case when dimension of X is greater than four but with the assumption of uppersemicontinuity condition.

Proof We give a proof under assumption of an isomorphism between the quantum cohomologies and Jacobian ring proven in many cases. The quantum multiplication by the canonical class K corresponds to multiplication of the class of W.

$$\begin{array}{ccc} QH(H^r) & \cong & \mathrm{Jac}\,(W) \\ \text{multi } K & & \text{mult by } W \end{array}$$

$$\begin{array}{ccc} QH & \cong & \mathrm{Jac}\,(W) \\ \cup & & \cup \\ \text{subring} & & \text{subring} \\ \text{generated by } K & & \text{generated by } W \end{array}$$

$$\begin{array}{ccc} 2 \text{ def of } K & \to & P \text{ polynomial of } W \\ & & W + P(W) \\ & & \text{all deformations} \\ & & \text{have the same critical values} \\ & & \text{as } W \end{array}$$

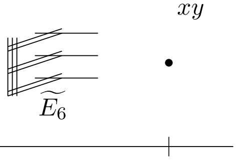
It follows that the spectrum of the most singular fiber of W does not go down since this most singular fiber does not split further under deformations. So we have $\delta > \dim X_L - 2 = 2$.

From another point the main assumption and the fact that we blow up points, curves and surfaces implies that $\delta = 2$—a contradiction. In the case of dimension higher than 4 the proof is the same.

We return to the case of cubic surface. We assume existence of a point in X_L over L. Its Landau–Ginzburg models is:

$$w = \frac{(x+y+1)^3}{xy} \text{ for cubic}$$

$$w = \frac{(x+y+1)^3}{xy} \text{ for cubic}$$



the deformation
does not change

If the $PicX_L = \mathbb{Z}$ then W have only two singular fibers.
We compute:

$$\delta = 2 - 2\frac{4-3}{3} = \frac{4}{3}$$

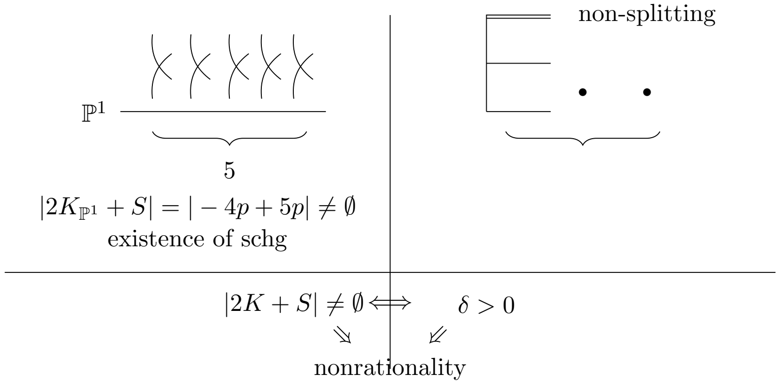
$$\Rightarrow X \text{ is not rational}$$

Since the Pic $X_L = \mathbb{Z}$ the deformation of W is restricted so we cannot morsify and δ does not go down to 0. So X_L is not rational. We move to considering a cubic with Pic $X_L = \mathbb{Z} + \mathbb{Z}$:

(1) In the case Pic $X_L = \mathbb{Z} + \mathbb{Z} \Rightarrow$ we get a conic bundle with 5 singular fibers. By Noether formulae:

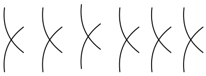
$$8 - S = k^2 = 3,$$

so we have 5 singular fibers. (The classical Iskovskikh criteria $|2K_{\mathbb{P}^1} + S| = |-4p + 5p| \neq \emptyset$ gives nonrationality.)
We will use spectrum in order to compute nonrationality. We compute the Bernstein polynomial for a cubic as an affine cone with a singularity at zero.
We have $8 - C = 3$. $C = 5$ pts.
$|2K + C| = |-4 + 5| = \mathcal{O}_{\mathbb{P}^1}(1) \neq \emptyset$
$f = a^5x^2 + b^5y^2 + c^5z^2$
$f = (s+1)^2(s+2)^2(s+\frac{3}{2})^2 \cdots (s+\frac{3}{10})$
So $\delta = \frac{3}{2} - \frac{3}{10} \neq 0$ and X_L is nonrational.



(2) We consider del Pezzo surface $X_L =$ of degree 4 in $\mathbb{P}^3(1,1,1,2)$ with Pic $X_L = \mathbb{Z} + \mathbb{Z}$ It is a conic bundle with 6 singular fibers. (The classical Iskovskikh criteria $|2K_{\mathbb{P}^1} + S| = |-4p + 6p| \neq \emptyset$ gives nonrationality.)
As before we use the Bernstein polynomial to show that $\delta > 0$ and X_L are not rational.

$$8 - S = K^2 = 2$$

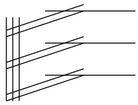


$$\underbrace{\qquad\qquad\qquad}_{6}$$

$$|2K + S| \neq \emptyset$$

2 points

$$\delta = 2 - 2\tfrac{5-4}{4}$$

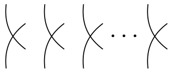


$\longrightarrow$ no splitting

(3) Consider del Pezzo surface X_L of degree 6 in $\mathbb{P}(1, 1, 2, 3)$.

$$8 - S = K^2 = 1$$

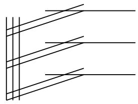
$$S = 7$$



$$\underbrace{\qquad\qquad\qquad}_{7}$$

$$|2K + 7| \neq \emptyset$$

$$\delta = 2 - 2\tfrac{7-6}{6}$$



$\longrightarrow$ no splitting

As before we use the Bernstein polynomial to show that $\delta > 0$ and X_L are not rational.

The above observations suggest the following conjecture.

Conjecture 3.2 Let X_L be a conic bundle over $\mathbb{P}^2$ (or another rational surface). Assume that the following holds:

$$\begin{tikzcd}
{|2K+S| \neq \emptyset} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & nonsplitting \arrow[dl, leftrightarrow] \\
& \delta > dim(X/L) - 2 &
\end{tikzcd}$$

Then X_L is not rational.

Let us consider a stack X/G. In this case the GW invariant of X are different from the ones of X/G. From another point the new contributions to cohomologies do form as twisted sectors which do not interact with the quantum span of the anticanonical divisor.

We denote the cohomologies associated to twisted sectors by $H_{\gamma_1}, ... + \cdots , H_{\gamma_k}$. We have the following splitting of quantum cohomologies.

$$QH(X)^G = H + H_{\gamma_1} + \cdots H_{\gamma_k}$$

It leads to the following conjecture.

Conjecture 3.3 Let X/G be a stack defined over a field L such that the image of $CH(X)$ in $\sum_i H^i(X, \mathbb{Z})$ is generated by powers of anticanonical class.
Assume that $\delta > dim(X/G) - 2$. Then X/G is not rational.

The proof is very similar to the proof of the previous theorem. As before we have:

$$\begin{array}{ccc} QH = H + H_{\gamma_1} + \cdots H_{\gamma_k} & \longrightarrow & \text{Jac}\,(W_m) + J_{\gamma_1} + \cdots J_{\gamma_k} \\ < 1, K(1)_1 > \text{ deformed} & \cong & < W_m > + P(W_m) \\ & & = \text{ no new eigenvalues} \end{array}$$

Here we denote by W_m the potential modified by the contributions of the age factors. As before we do not have further splitting of the cohomology and the inequality $\delta > dim(X/G) - 2$ implies nonrationality.

We will look at some examples of del Pezzo stacks.

Using this theorem we consider several examples of del Pezzo stacks—all hypersurfaces in weighted projective $\mathbb{P}^3$. Consider the case of weights: 3, 3, 5, 5 and a hypersurface of degree 15. In this case $\delta = 2 - 2(16 - 15)/15 = 28/15 > 0$ so we have nonrationality. We can compute the spectrum applying theorem 5.5. Using Singular we compute the Steenbrink spectrum of $Cone(X)$-(0, 1), ..., (28/15, 1). So $\delta = 48/15$. We obtain nonrationality.

Remark 3.4 Observe that choice of the field L and the condition $Im(CH \to H) = \langle 1, K(1), K^2(1), \cdots \rangle$ are essential. Without these assumptions the most singular fiber of W_m splits to singularities A_4, A_2, A_2 and further which makes $\delta = 0$.

Similarly consider the weights: 3, 5, 7, 11 and a hypersurface of degree 25. The Steenbrink spectrum of $Cone(X)$ is (0, 1), ..., (48/25, 1). So $\delta = 48/25$. We obtain nonrationality.

This methods work in all Johnson-Kollár examples as well as in higher dimension—for more see [35].

4 Low Dimensional Topology Invariants

We explain a parallel between quantum spectrum and classical 3-dim, 4-dim invariants. First we recall the classical theory. We start with theory of knots and Alexander polynomials. Consider the singular curve:

$$f(z, w) = z^p + w^q, (z, w) \in \mathbb{C}^2$$

$$S_\epsilon = \{|z|^2 + |w|^2 = \epsilon^2\} \subset \mathbb{C}^2, 0 < \epsilon << 1$$

$$K_{p,q} = f^{-1}(0) \cap S_\epsilon \text{ a knot}$$

Alexander polynomial of this torus knot is:

$$\Delta_{p,q} = t^{-\frac{(p-1)(q-1)}{2}} \cdot \frac{(t-1)(t^{pq}-1)}{(t^p-1)(t^q-1)}$$

We define $Sp(f) := \sum_{\alpha\in\mathbb{Q}} n_{f,\alpha} t^{\alpha}$ the *Steenbrink spectrum*

$$Steen = \{\alpha_1, \alpha_2, \ldots, \alpha_\mu\},\ \mu = (p-1)(q-1)$$

Fact $\Delta_{K_{p,q}} = t^{-\frac{\mu}{2}} \prod_{i=1}^{\mu} \Phi_{\alpha_i}(t)$, $\Phi_{\alpha_i}(t) = (t - e^{2\pi i \alpha_i})$

Example 7 *$((p,q) = (2,3))$*

$$\Delta_{K_{2,3}} = t^{-\frac{\mu}{2}} \frac{(t^6-1)(t-1)}{(t^2-1)(t^3-1)} = t^{-\frac{\mu}{2}} (t - e^{2\pi i \frac{5}{6}})(t - e^{2\pi i \frac{7}{6}})$$

$Steen = \{\frac{5}{6}, \frac{7}{6}\}$. Also using Thom-Sebastiani theorem we get:

$$Steen = \{Steen(z^2)\} + \{Steen(w^3)\} = \left\{\frac{1}{2}\right\} + \left\{\frac{1}{3}, \frac{2}{3}\right\} = \left\{\frac{5}{6}, \frac{7}{6}\right\}$$

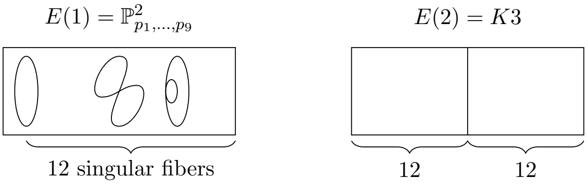
Example 8 *$((p,q) = (2,5))$*

$$\Delta_{K_{2,5}} = t^{-\frac{\mu}{2}} \frac{(t^{10}-1)(t-1)}{(t^2-1)(t^5-1)}$$

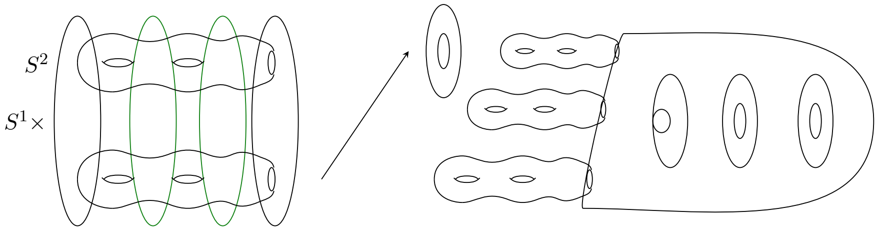
$Steen = \{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\}$. Using Thom-Sebastiani we get:

$$Steen(z^2 + w^5) = \{Steen(z^2)\} + \{Steen(w^5)\} = \left\{\frac{1}{2}\right\} + \left\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right\}$$

We move 1 dimension higher. Consider an elliptic surface $E(n)$: an elliptic fibration.



We describe fibered knot surgery and its connections with Seiberg Witten invariants SW.



$S^1 \times (\sum_g \times S^1)$ into $\qquad\qquad$ $E(n)$

Under surgery:

$$SW_{E_K(n)} = \sum_{K\in\mathbb{Z}} SW(K[F])t^K = SW_{E(n)}(t)\Delta_K(t), SW_{E(n)} = (t - t^{-1})^{n-2}$$

where F is the fiber of $E_K(n)$.

Theorem 4.1 (Gr = SW) *Coefficients of Δ_K count holomorphic curves $g = 1$ in the class $K[F]$ in $E_K(n)$.*

We explore the connection with spectra. Recall that:

$$\begin{array}{c} \sum_g \to S^3 - K \\ \downarrow \\ S_1 \end{array}$$

Φ the monodromy of the surgery (char polynomial of $\Delta_k(t)$) produces an endofunctor on $Fuk(\sum_g)$ and $Fuk(Sym^k \sum_g)$ (or $FS(\sum_g)$?).

Conjecture 4.2 Φ defines filtration on $HH(Fuk(\sum_g))$ which corresponds to Steen.

Conjecture 4.3 $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f)$ has a filtration

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f) \supset \mathcal{F}_{\alpha_1} \supset \mathcal{F}_{\alpha_2} \cdots$$

given by the spectra.

Let $\mathcal{F}$ be mirror of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(f)$. Consider the quantum differential Eq. 1

$$\{\text{asymptotics of 2.1}\} \leftrightarrow \{\text{Spectrum of } f\}$$

Conjecture 4.4 Entropy of Φ: $\eta(\Phi)$ is the first coefficient of $\Delta_K(t)$.

These simple observations suggest the following questions:

Question 4.5 Does the spectrum define canonical filtration on Floer homology?

Question 4.6 What is the symplectic meaning of this filtration? We expect it is connected with the structures of the Lagrangian skeleta.

We discuss further applications. We define modular spectrum of a link M - link of singularity $X_f \leftarrow Y_{1,q}$ as the Steenbrink $Steen(Y_{1,q})$. We give a brief example to fix notations.

Example 9 $M = \Sigma(2, 3, 5)$

$$Y_{1,q} - E_8$$

$$WRT(M) \leftrightarrow (1, 7, 11, 13, 17, 19, 23, 29)$$

Here $WRT(M)$ is the Witten-Reshetikhin-Turaev (WRT) invariant of the 3-manifold M.

We pose the following:

Question 4.7 Is there a categorical meaning of WRT?

We will discuss some of these questions in the next section.

4.1 Spectra and WRT

Let M be a smooth 3-manifold which is a link of an isolated normal surface singularity in $\mathbb{C}^3$. In the following sections, we study topological invariants of M and their relation to spectra. GPPV invariants[1] $\hat{Z}_b(q)$ [37, 38] are q-series that refine the WRT invariants.

Series $Z_b(q)$ can be expressed as a linear combination of false theta functions in the case of Seifert manifolds with 3 singular fibres. Corresponding theta functions can be conjecturally written as components of a vector-valued modular form, which is know for some examples, including links of ADE singularities [37]. Induced representation of $SL(2, \mathbb{Z})$ is a subrepresentation of $2m$-dimensional Weil representation for some integer m and θ functions are labelled by residue classes modulo $2m$. We are interested in these residue classes for all components of the modular form, not just those that correspond to $\hat{Z}_b$. We call this set *Modular spectrum* for convenience. A precise definition depends on the conjectural existence of a natural vector-valued modular form. It was posed as a question in [37] what is a deeper meaning of these residue classes.

[1] also called BPS q-series or homological blocks.

Example 10 The relation with the spectrum started with an observation about E_8 singularity, defined by the equation $x^2 + y^3 + z^5 = 0$. Its link is a Poincar-ÃƒÆ'Ã‚Â© homology sphere, Seifert manifold $M(-2, 1/2, 2/3, 4/5)$. WRT invariants of this manifold have been studied in [40]. Lawrence and Zagier defined two functions holomorphic inside the unit circle:

$$\theta_+(\tau) = q^{1/120}(1 + 11q + 19q^3 + 29q^7 - 31q^8 - 41q^{14} - \dots$$
$$\theta_-(\tau) = q^{49/120}(7 + 13q + 17q^2 + 23q^4 - 37q^{11} - 43q^{15} - \dots$$

The first function gives WRT as the radial limits at the roots of unity. Both functions together form a vector-valued modular form for $SL(2, \mathbb{Z})$.

Those functions can be written as a linear combination of theta functions assigned to residue classes modulo 60 (see Sect. 2):

$$\theta_+(\tau) = \theta^1_{30,1}(\tau) + \theta^1_{30,11}(\tau) + \theta^1_{30,19}(\tau) + \theta^1_{30,29}(\tau) + \dots$$
$$\theta_-(\tau) = \theta^1_{30,7}(\tau) + \theta^1_{30,13}(\tau) + \theta^1_{30,17}(\tau) + \theta^1_{30,23}(\tau) + \dots$$

The spectrum of E_8 singularity is

$$\{1/30, 7/30, 11/30, 13/30, 17/30, 19/30, 23/30, 29/30\}$$

and we can see that the numerators of the elements of spectrum correspond to residue classes of the theta functions while the denominator corresponds to the modulus.

This example can be generalized in two ways. One is the class of Brieskorn homology spheres $x^{p_1} + y^{p_2} + z^{p_3} = 0$ for a_0, a_1, a_2 pairwise coprime. An analogical relation of theta functions and spectrum is true for them as described in Sect. 3. It is remarkable since the spectrum contains negative numbers and this is reflected in topology.

Theorem 4.8 *Let M be a Brieskorn homology sphere, i.e. the link of the singularity X given by the equation* $x^{p_1} + y^{p_2} + z^{p_3} = 0$ *Then*

Modular spectrum of M = Steenbrink spectrum of X.

Another generalization is the class of ADE singularities. Here we need to take a spectrum of a different but related singularity—universal Abelian cover.

Theorem 4.9 *Let M be a link of ADE singularity X and Y be the corresponding maximal Abelian cover. Then*

Modular spectrum of M = Steenbrink spectrum of Y.

This phenomenon can be certainly generalized to Seifert manifolds, where $\hat{Z}_b$ have been explicitly computed recently. For more general plumbed 3-manifolds, the singularities to consider are splice-quotients and their universal covers, where the spectrum is difficult to compute, however much can be said about the topology itself using ideas from singularity theory and simpler invariants than spectrum. For these generalizations, see [32]. On the topology side, since the description of $\hat{Z}_b$ using false theta functions is limited to 3 singular fibres of Seifert fibration on M, we need to replace theta function labels by something more general. The poles of Borel plane [43] seem to be a good candidate.

4.1.1 Theta Functions

We will follow the notation in [37]. In particular we denote $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$.

Definition 4.10 Let m be a positive integer and r a residue class $\bmod 2m$. We define weight 1/2 theta function and weight 3/2 unary theta function as (respectively)

$$\theta_{m,r}(\tau, z) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} q^{\ell^2/4m} y^{\ell};\ \theta^1_{m,r}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} \ell\, q^{\ell^2/4m}, \tag{2}$$

Unary theta functions form a (rank $2m$) vector-valued modular form of weight 3/2. Its matrices S and T define *Weil representation* of $\widetilde{SL}(2, \mathbb{Z})$, the double cover of $SL(2, \mathbb{Z})$.

Definition 4.11 *False theta function (or Eichler integral)* of $\theta_{m,r}$ is

$$\Psi_{m,r}(\tau) = \sum_{\substack{\ell\in\mathbb{Z} \\ \ell\equiv r \pmod{2m}}} \mathrm{sgn}(\ell)\, q^{\ell^2/4m}. \tag{3}$$

False theta functions keep a weaker modular property—quantum modularity [41]. Note also the obvious relations:

$$\Psi_{m,r}(\tau) = \Psi_{m,-r}(\tau) \tag{4}$$

$$\Psi_{m,r+2m}(\tau) = \Psi_{m,r}(\tau) \tag{5}$$

The basic idea is the correspondence $\frac{r}{m}$ as an element of the spectrum of certain singularity related to the 3-manifold and $\Psi_{m,r}(\tau)$ as an Eichler integral of a certain theta function assigned to a 3-manifold.

4.1.2 GPPV Invariants

A plumbed 3-manifold M admits GPPV invariants [38], which are q-series $\hat{Z}_b(q)$ defined using plumbing graph of M and labeled by elements of $H_1(M)$ or $spin^c$ structures. These invariants can be computed by an explicit integral formula [37]. It is an intriguing question whether the series Z_b can be written as components of (quantum) modular forms.

The vector-valued modular forms described in [37] have usually more components than is the number of $Z_b(q)$ (as in the example E_8 in the introduction). It is not clear what is the meaning of these components for the 3-manifold and how to get an intrinsic definition of them.

4.1.3 Example of Brieskorn Homology Sphere $\Sigma(3, 4, 5)$

Here we give an example of theorem 4.8. Homology sphere $\Sigma(3, 4, 5)$ is the link of $x^3 + y^4 + z^5 = 0$. This case has been studied in [37], p. 67. They describe a representation of $\widetilde{SL}(2, \mathbb{Z})$ given by theta functions $\theta^1_{m,r}$ and corresponding false theta functions $\Psi_{m,r}$. The number m is $3 \cdot 4 \cdot 5 = 60$.

False theta functions:

$$\begin{gathered}
\Psi_{60,1} - \Psi_{60,31} - \Psi_{60,41} - \Psi_{60,49}\\
\Psi_{60,2} + \Psi_{60,22} + \Psi_{60,38} + \Psi_{60,58}\\
\Psi_{60,7} + \Psi_{60,17} + \Psi_{60,23} - \Psi_{60,47}\\
\Psi_{60,11} + \Psi_{60,19} + \Psi_{60,29} - \Psi_{60,59}\\
\Psi_{60,13} - \Psi_{60,37} - \Psi_{60,43} - \Psi_{60,53}\\
\Psi_{60,14} + \Psi_{60,26} + \Psi_{60,34} - \Psi_{60,46}
\end{gathered}$$

If we use the relation $\Psi_{m,2m+r} = \Psi_{m,r}$ and multiply first and fifth row by -1 (change of the basis of the representation) we obtain

$$\begin{gathered}
\Psi_{60,-1} + \Psi_{60,31} + \Psi_{60,41} + \Psi_{60,49}\\
\Psi_{60,2} + \Psi_{60,22} + \Psi_{60,38} + \Psi_{60,58}\\
\Psi_{60,7} + \Psi_{60,17} + \Psi_{60,23} + \Psi_{60,73}\\
\Psi_{60,11} + \Psi_{60,19} + \Psi_{60,29} + \Psi_{60,61}\\
\Psi_{60,-13} + \Psi_{60,37} + \Psi_{60,43} + \Psi_{60,53}\\
\Psi_{60,14} + \Psi_{60,26} + \Psi_{60,34} + \Psi_{60,46}
\end{gathered}$$

Now the labels r of $\Psi_{m,r}$ are exactly the numerators of the elements of Steenbrink spectrum of $x^3 + y^4 + z^5 = 0$. The terms in each sum correspond to the orbits of a natural action of $\mathbb{Z}_2^2$ on the spectrum. Note that since the theta functions only depend on $r \pmod{2m}$ the relevant spectrum is spectrum modulo 2 (we cannot hope to recover the full Hodge-theoretic information from topology).

The series $Z_0(q)$ is at the fifth row. It contains the term labelled by the smallest number in the spectrum: $-13/60$.

Remark 4.12 As conjectured in [37], components of the representation should correspond to non-abelian $SL(2, \mathbb{C})$ connections (it is true for Brieskorn spheres). If

Table 1 Labels of false theta functions for M, the link of singularity X, correspond to the spectrum of the universal Ab. cover Y of X

Manifold M	X	Y	False thetas of M	Spectrum of Y
Lens space	A_n	$\mathbb{C}^2$	No thetas	Empty
$M(-2; \frac{1}{2}, \frac{1}{2}, \frac{n-3}{n-2})$	D_n	A_{n-3}	$\Psi_{1,n-2}, \Psi_{2,n-2}, \ldots, \Psi_{n-3,n-2}$	$(1, 2, \ldots, n-3)/(n-2)$
$M(-2; \frac{1}{2}, \frac{2}{3}, \frac{2}{3})$	E_6	D_4	$\Psi_{6,1} + \Psi_{6,5}, 2\Psi_{6,3}$	$(1, 3, 3, 5)/6$
$M(-1; \frac{1}{2}, \frac{2}{3}, \frac{3}{4})$	E_7	E_6	$\Psi_{12,1} + \Psi_{12,7}, \Psi_{12,4}+$ $\Psi_{12,8}, \Psi_{12,5} + \Psi_{12,11}$	$(1, 4, 5, 7, 8, 11)/12$
$\Sigma(2, 3, 5)$	E_8	E_8	10, [40]	$(1, 7, 11, 13, 17, 19, 23, 29)/30$

we use this identification and restrict it to real connections, we recover the classical relation of the signature of Milnor fiber of the Brieskorn singularity and Casson invariant of M [44].

4.1.4 ADE Singularities

Before we get to the relation of GPPV and the spectrum, we need to recall the notion of universal Abelian cover of an isolated singularity (see, for example, [42]). Recall that a closed oriented 3-manifold M is a $\mathbb{Q}$-homology sphere if $H_*(M, \mathbb{Q}) = H_*(S^3, \mathbb{Q})$.

Definition 4.13 Let X be a germ of an isolated normal surface singularity whose link M is a $\mathbb{Q}$-homology sphere. The universal Abelian cover Y of X is a maximal Abelian cover of the germ ramified at the singular point.[2]

$\hat{Z}_b$ and modular forms of the links of ADE singularities were computed in [37], see also [39]. Using their results, we obtain Theorem 4.9. All ADE singularities, their Abelian covers and invariants are summarized in Table 1.

4.2 Topological Invariants of Plane Curve Singularity

We give some ideas of the categorical origin of these topological invariants. Let $C = \{f(x, y) = 0\}$ be a germ of a plane curve having an isolated singularity at the origin p and $L_{C,p}$ be an algebraic link of the plane curve singularity. There have been lots of works studying relations between algebraic geometry of C and topology of $L_{C,p}$. For example, the Alexander polynomial of $L_{C,p}$ can be computed via the ring of functions $\mathcal{O}_C$ thanks to the works of Campillo-Delgado-Gusein-Zade (cf. [5]) and the HOMFLY-PT polynomial of $L_{C,p}$ can be expressed in terms of Hilbert schemes of the plane curve singularity thanks to the works of Oblomkov-Shende (cf. [45]) and Maulik (cf. [21]). On the other hand, there have been lots of interests in mirror

[2] The covering group is then $H_1(M, \mathbb{Z})$.

symmetry of hypersurface singularities these days (see [15] and references therein for more details) and plane curve singularities again have provided natural testing grounds for mirror symmetry conjecture. Takahashi conjectured that for an invertible polynomial f, the category of graded matrix factorization $\mathrm{HMS}^{L_f}(f)$ will be equivalent to the Fukaya-Seidel category $\mathrm{Fuk}^{\to}(f^T)$ of the Berglund-Hübsch mirror polynomial f^T and recently there have been lots of works in this direction and both categories have been intensively studied. For example, it turns out that $\mathrm{HMS}^{L_f}(f)$ has a full exceptional collection and admits a Gepner type stability condition when f is of ADE type. Here, we will discuss the relation between Hilbert schemes of plane curve singularities, certain topological data of some algebraic links, and matrix factorizations.

To be more precise, we will consider the images of ideals which belong to certain Hilbert scheme $C_p^{[*]}$ in the category $\mathrm{HMF}^{L_f}(f)$ when $f = x^2 + y^3$. Then we can check that the images have interesting properties. For example, a natural stratification on (some parts of) the Hilbert scheme $C_p^{[*]}$ corresponds to an indecomposable object in $\mathrm{HMS}^{L_f}(f)$. We can also verify that the difference between the Alexander polynomial and the HOMFLY-PT polynomial of $L_{C,p}$ can be expressed in terms of $\mathrm{HMF}^{L_f}(f)$.

4.2.1 Hilbert Schemes

Let $C = \{f(x, y) = 0\}$ be the germ of a plane curve with an isolated singularity at the origin at $p = (0, 0)$.

Definition 4.14 Let $C_p^{[l]}$ be the Hilbert scheme of length l zero dimensional subschemes of C which are set-theoretically supported at p. And let $C_p^{[*]} := \bigcup_l C_p^{[l]}$.

The normalization induces an embedding $\mathcal{O}_C \to \mathbb{C}[[t]]$. And the natural valuation induces a valuation $\mathcal{O}_C \to \mathbb{N}$. Let $\Gamma = \nu(\mathcal{O})$ be the semigroup. Let $I \subset \mathcal{O}_C$ be a L_f-graded ideal. Then $\mathcal{O}_C/I$ gives an element in $D_{\mathrm{sg}}^{L_f}(R_f)$.

Proposition 4.15 *Let f be a weighted homogeneous polynomial. Then there is a $\mathbb{C}^*$-action on $C_p^{[*]}$. A $\mathbb{C}^*$-invariant ideal gives an $\mathbb{Z}$-graded ideal.*

Proof The obvious $\mathbb{C}^*$-action on f induces an action on $C_p^{[*]}$ and having a $\mathbb{C}^*$-action is equivalent to having a $\mathbb{Z}$-grading.

The following remark tells us that not all ideals of $\mathcal{O}_C$ give nontrivial elements in $\mathrm{HMF}^{L_f}(f)$.

Remark 4.16 Let g be a nonzero divisor in $\mathcal{O}_C$. Then $\mathcal{O}/(g)$ is a perfect complex.

Proof We have the following short exact sequence.

$$0 \to \mathcal{O}_C \to \mathcal{O}_C \to \mathcal{O}_C/(g) \to 0$$

Therefore $\mathcal{O}/(g)$ is a perfect complex.

4.2.2 Example $f = x^2 + y^3$

We can compute L_f as follows.

$$L_f = \mathbb{Z}\overrightarrow{x} \oplus \mathbb{Z}\overrightarrow{y} \oplus \mathbb{Z}\overrightarrow{f}/(\overrightarrow{f} - 2\overrightarrow{x} - 3\overrightarrow{y}) \cong \mathbb{Z}$$

$$R_f = \mathcal{O}_C = \mathbb{C}[[x, y]]/(x^2 + y^3) = \mathbb{C}[[t^2, t^3]]$$

There is a stratification on the Hilbert scheme as follows.

$$(1)$$

$$(t^i + ut^{i+1}), \quad i \geq 2, u \in \mathbb{C}$$

$$(t^i, t^{i+1}), \quad i \geq 2$$

The $\mathbb{C}^*$-invariant parts of the Hilbert scheme are as follows.

$$(1)$$

$$(t^i), \quad i \geq 2$$

$$(t^i, t^{i+1}), \quad i \geq 2$$

The semigroup Γ is $\{0, 2, 3, 4, 5, 6, 7, \cdots\}$.

The Koszul resolution of $\mathbb{C}[[x, y]]/(x, y)$ induces an L_f-graded matrix factorization $F = (F_0, F_1, f_0, f_1)$ of f where $P(\overrightarrow{f}) := S(-\overrightarrow{x}) \oplus S(-\overrightarrow{y})$ and

$$F_0 := S \oplus \wedge^2 P(\overrightarrow{f}), \quad F_1 := P(\overrightarrow{f}).$$

Proposition 4.17 *The matrix factorizations correspond to the ideal* (t^i, t^{i+1}) *is the image of the above matrix factorization under the autoequivalence* $(\overrightarrow{l})$ *for some* $\overrightarrow{l} \in L_f$.

Proof Let $M = \mathbb{C}[[x, y]]/(x, y)$. Let M^{stab} be the above matrix factorization. Note that (t^i, t^{i+1}) is isomorphic to (t^2, t^3) as an R_f-modules. The only difference between them is grading and hence we obtain the desired conclusion.

Proposition 4.18 *The ideal* (t^i, t^{i+1}) *is an exceptional object in* $\mathrm{HMF}^{L_f}(f)$.

Proof Because $\mathbb{C}[[x, y]]/(x, y)$ is an exceptional object (cf. [16]), we see that (t^i, t^{i+1}) is also exceptional.

Then we have the following.

Corollary 4.19 *The ideal* (t^i, t^{i+1}) *is an indecomposable object in* $\mathrm{HMF}^{L_f}(f)$.

It is well-known that there are only finitely many indecomposable objects in $\mathrm{HMF}^{L_f}(f)$ up to autoequivalences.

Theorem 4.20 *The difference between the Alexander polynomial and the HOMFLY-PT polynomial is a categorical invariant.*

Proof The difference between the Alexander polynomial and the HOMFLY-PT polynomial of $L_{C,p}$ is the integration over ideals of type (t^i, t^{i+1}). And every element of the form (t^i, t^{i+1}) can be obtained from (t^2, t^3) by applying translations. From the above discussion, we see that these ideals give nontrivial elements in $\mathrm{HMF}^{L_f}(f)$. Therefore, one can see that the difference can be written in terms of $\mathrm{HMF}^{L_f}(f)$.

5 Generalization of Spectra

We extend the connection of spectra with Alexander polynomial initiated in the previous section. We extend the correspondence:

$$\boxed{\text{Multivariable Alexander Polynomials}} \longleftrightarrow \boxed{\text{multispectra}}$$

Theorem of Libgober [26] says that we can associate to spectrum of $f_1, f_2, \ldots \leftrightarrow$ faces of quasiadjunction. We will give a categorical version of this process:

5.1 Splitting of a Potential

Consider a Landau–Ginzburg model with a potential $W = W_1 + W_2$ We consider the associated Fukaya-Seidel categories $FS(W_1)$, $FS(W_2)$, $FS(W)$.

We start with the tower:

$$\begin{tikzcd} FS(W_1+W_2) \arrow[r] \arrow[d] & FS(W_1) \arrow[d] \\ FS(W_1) \arrow[r] & FS(W_1 \cap W_2) \end{tikzcd}$$

Example 11 *($X_3^5 \subset \mathbb{P}^6$ 5-dim cubic)*

$$\begin{aligned} \mathrm{D^b}(X_3^5) &\cong FS(W_1+W_2) \\ \mathrm{D^b}(X_6^4) &\cong FS(W_1) \\ \mathrm{D^b}(X_6^4) &\cong FS(W_2) \end{aligned}$$

Conjecture 5.1 The NC spectra of X_3^5 is a superposition of X_6^4 and X_6^4.

We have the P.D.E.

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G$$

Conjecture 5.2 The P.D.E. of X_6^4 and P.D.E. of X_6^4 produce the P.D.E. of X_3^5 via convolution.

$$PDE(X_6^4) *_A PDE(X_6^4) \cong PDE(X_3^5)$$

We see that asymptotics are superposition of asymptotics.

Corollary 5.3 *Let $\widetilde{\mathbb{P}_X^N}$ is a blow-up of $\mathbb{P}^N$ along X. Then the faces of quasiadjuction contain*

$$(-(\dim X)/2, \ldots, -(\dim X)/2)$$

In general, we have

$$Spec(\{A_i\}) \twoheadrightarrow Spec(\{K\})$$

Here the algebra $\{K\}$ is the algebra generated by canonical bundle. $\{A_i\}$ is the algebra generated by algebraic cycles. The above epimorphism defines a deeper filtration.

Question 5.4 Is this new filtration a birational invariant?

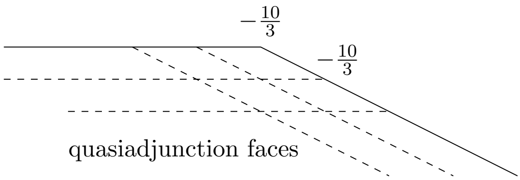
Question 5.5 Does the algebra defined by splitting produce birational invariants?

We consider the example of 5-dim cubics.

$$\begin{tikzcd}
 & \mathrm{D^b}(X_3^5) & \\
\mathrm{D^b}(X_{3,2}^4) \arrow[ur, hook] & & \mathrm{D^b}(X_{3,2}^4) \arrow[ul, hook']
\end{tikzcd}$$

$$\delta_1(X_3^5) = \frac{7}{3}$$
$$\delta_1(X_{3,2}^5) = 4 - 2\frac{6-3-2}{3} = \frac{10}{3}$$
$$\delta_1(X_3^5) = \frac{7}{3}$$
$$\delta_1(X_{3,2}^5) = 4 - 2\frac{6-3-2}{3} = \frac{10}{3}$$

We compute the quasiadjunction of the above splitting.



Observation
We notice that in the above spliting $-(\dim X)/2, \ldots, (\dim X)/2$ do not belong to quasiadjunction faces of the polygon. This suggest a different proof of the nonrationality 3-dimensional cubic.

5.2 *Category Filtrations*

For a category $\mathcal{C}$ and A, B and a noncommutative Hodge structure $\mathcal{H}$, ∇, $Herm > 0$, we define a sequence of stability conditions $\mathcal{J}_1, \ldots, \mathcal{J}_k$ corresponding to asymptotics of stability spectrum.

We consider the asymptotics of integral $\int_{\Gamma'(0)} \alpha_{(0)} \sim$ Asymptotics at $z = 0$. These asymptotics define

stability spectrum.

Example 12 Consider the category A_n—1 dimensional Fukaya-Seidel categories. So we have $x^j e^{\frac{p}{u}} dx$ is a stability condition. Here p is a polynomial of degree $<$ $(n-1)$.

<u>Step 1</u> We have $\alpha = dx$.
<u>Step 2</u> We move to define Kähler metric on moduli space of stability conditions. We begin with $K_{ij}(u, \bar{u}) = \iint_{\mathbb{C}} x^i x^j e^{\frac{p}{u} - \frac{\bar{p}}{\bar{u}}} dx d\bar{x}$

$$\Phi : |u| \leq 1 \to GL(n+1, \mathbb{C})$$

$$\forall |u| = 1,\ \Phi(u)\Phi^t(u) = K_{ij}$$

We define Hermitian form

$$H(u) = \Phi(u)\Phi^t(u)$$

$$\text{Asymptotics } \int x^i e^{\frac{p}{u}} dx$$

define asymptotics and the noncommutative spectrum.

As we saw the asymptotics of the integral $\lim\limits_{n \to 0} Z_n = \sum u^{\alpha_i}$ define stability and nc spectra. We move in to investigate the connection with analysis.

We have the following:

Theorem 5.6 *The stability conditions $\mathcal{J}_1, \ldots, \mathcal{J}_k$ define a filtration on $\mathcal{C}$:*

$$\mathcal{F}_{\leq i}(\mathcal{C}) = semistable\ Obj(\mathcal{E})$$

such that

$$Z_{\mathcal{J}_i}(\mathcal{E}) \leq \mathcal{O}(|\mathcal{J}|^j)$$

This theorem will be discussed in detail in [29]. We will make some use of this filtration in what follows. We consider a Fano X and a splitting of a canonical divisor $K_X = D_1 + D_2$.

$$X - \text{Fano}$$
$$K_X = D_1 + D_2$$

On the mirror side we have spitting of the potential $W = W_1 + W_2$.

$$\begin{tikzcd} FS(W_1) \arrow[r] & FS(W) \\ Fuk(CY) \arrow[u] \arrow[r] & FS(W_2) \arrow[u] \end{tikzcd}$$

Monodromy of W_1 gives a filtration:

$$FS(W_1) \supset \mathcal{F}_{\lambda_1} \supset \cdots \supset \mathcal{F}_{\lambda_n}$$

Monodromy of W_2 gives a filtration:

$$FS(W_2) \supset \mathcal{F}_{\mu_1} \supset \cdots \supset \mathcal{F}_{\mu_n}$$

giving a double filtration

$$FS(W) \supset \mathcal{F}_{\mu_1,\lambda_1} \supset \cdots$$

$$FS(W) \supset \mathcal{F}_{\nu_1} \supset \cdots$$

The behavior of λ_i, μ_j is of Thom Sebastiani type generalized

$$\nu_i \overset{ThomSebastiani}{=} (\lambda_i, \mu_i)$$

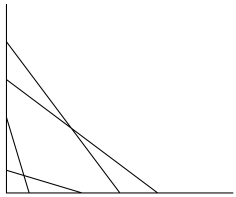
In fact, we have a correspondence:

$$\left\{\begin{matrix} Choices \\ of \\ W_1, W_2, \ldots \end{matrix}\right\} \to \left\{\begin{matrix} generalized \\ Thom Sebastiani \\ \lambda_i \ \mu_i \ \nu_i \\ \vdots \ \vdots \ \vdots \end{matrix}\right\}$$

Question 5.7 Can one produce out of λ_i, μ_i, ν_i new birational invariants?

We discuss briefly a couple of examples.

Example 13 (*Polytope of quasiadjunction* $(x^2 + y^3)(x^3 + y^2)$)



$(x^2+y^3)(x^3+y^2)$

$$2u + 3v = \tfrac{1}{2}\tfrac{3}{2}\tfrac{5}{2}$$

$$3u + 2v = \tfrac{1}{2}\tfrac{3}{2}\tfrac{5}{2}$$

The Alexander polynomial is:

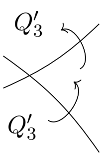
$$(t_1^2 t_2^3 + 1)(t_1^3 t_2^2 + 1)$$

Example 14 (*3-dim cubic*)

$$-K_X = 2H$$

$$f = Q_3' Q_3' \qquad \text{two cubics}$$

$$\begin{matrix} \lambda_1 = \dfrac{5}{3} \\ \lambda_2 = \dfrac{5}{3} \end{matrix} \quad \to \quad \delta = \frac{5}{3}$$



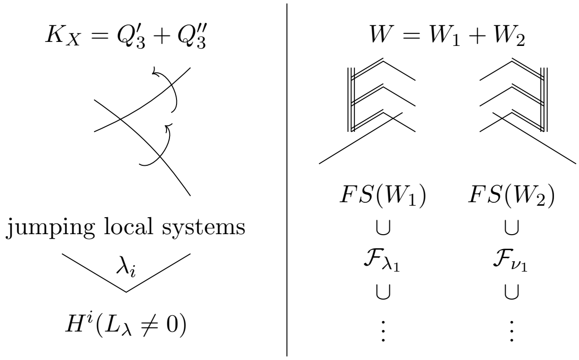
local Alexander polynomials

$$\Downarrow$$

$$\frac{5}{3}$$

Mirror

$$\begin{array}{ccc} W = W_1 & + W_2 & \\ | & | & \\ Q_3'' & Q_3'' & \text{no deformations} \\ \frac{5}{3} & \frac{5}{3} & \end{array}$$



6 Spectrum, Orbifoldization and Conformal Field Theory

In this section we propose a new point of view of noncommutative spectra. Details will appear elsewhere see e.g. [27, 32].

Our approach is based on the parallel between:

- Birkar's proof [1] of boundness of Fano's.
- Zamolodchikov's [7] c-theorem.

We combine these two directions with categorical resolution of singularities. The final outcome is creating theory of noncommutative spectra similar to Arnold-Varchenko-Steenbrink spectrum.

We will describe a procedure of computing noncommutative spectrum as equivariant part of Steenbrink spectrum of the corresponding affine cone.

Steenbrink Spectrum $\xrightarrow[Equivariant]{Elliptic}$ Noncommutative Spectrum.

We consider the following examples.

1. Let X be a hypersurface (Fermat) of degree d in $\mathbb{P}^N$

$$x_0^d + \cdots x_N^d$$

by Steenbrink $(y^{\frac{1}{d}} + \cdots + y^{\frac{d-1}{d}})^{N+1}$.

This is the fixed part of the Elliptic genus when applied to 5-dim. cubic.

Recall that

$$x_0^3 + \cdots x_6^3 = 0$$

has Steenbrink Spectrum

$$(y^{\frac{1}{3}} + y^{\frac{2}{3}})^7$$

We orbitalize using action of $\mathbb{Z}_3$

$$\frac{1}{3} y^{-\frac{7}{2}} \left(\sum_{0 \le a \le 3} \left(\frac{y^{\frac{1}{3}} - y\omega^{-a}}{y^{\frac{1}{3}} - \omega^{-a}} \right)^7 + \sum \left(y^{\frac{6}{3}} \right)^7 \right)$$

So after that, we get

$$-21(y^{-\frac{7}{2}} + y^{\frac{1}{2}}) + y^{-\frac{7}{6}} + y^{\frac{7}{6}}$$

$$\Rightarrow \left(-\frac{7}{6}, \frac{7}{6} \right) \text{ - noncommutative spectrum}$$

2. Similarly for 2-dim. cubic $y^{-\frac{2}{3}} + 2 + y^{\frac{2}{3}}$.
For K3 ($x_0^4 + \cdots + x_3^4 = 0$), we have $2y^{-1} + 20 + 2y$.

Proposition 6.1 *For CY, the procedure gives* $-\frac{\dim X}{2}, \ldots, \frac{\dim X}{2}$.

Proposition 6.2 *For general type, the procedure gives* $-\frac{\dim X}{2}, \ldots, \frac{\dim X}{2}$.

Proposition 6.3 *The uppersemicontinuity for Steenbrink spectrum brings uppersemicontinuity for noncommutative spectrum.*

We consider the Berglund-Hübsch Mirror Symmetry.

$$X^\vee = \mathbb{C}^{n+1} / \Gamma \xrightarrow{f} \mathbb{C}$$

where $X^\vee$ is the mirror of $X \subset \mathbb{P}^N$. So we have:

Conjecture 6.4 $\mathrm{D}^{\mathrm{b}}_{\mathrm{sing}}(X^\vee, f)^{\mathrm{eq}} = Fuk^0(X)$.

Now we present a program which not only explains Conjecture 6.1 but suggests a far going program of categorical resolutions. We begin by:

Conjecture 6.5 Let $r : X \to X_{\mathrm{sing}}$ be a resolution of singularity. There exists a category $\mathcal{C}_0$ which does not depend on r.

In the case of orbifold we can be more precise:

Conjecture 6.6 There exists a piece $\mathcal{H}_0 \subset H^i(X)$ which does not depend on r. Then $\mathcal{H}_0 \cong IH(X_{\mathrm{sing}})$.

We have:

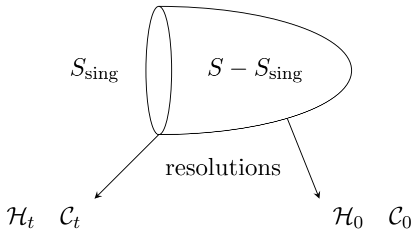
$$H^{\text{String}}(X_{\text{sing}}) = IH(X_{\text{sing}}) + T_{S_1} + \cdots T_{S_w}$$

Here IH are the intersection cohomologies of X. The noncommutative spectrum is defined over $IH(X_{\text{sing}})$. We can combine above conjecture with our orbifoldization procedure. We observe that the twisted sectors we need to take are precisely the ones on which the group acts with determinant equal to one. The above considerations can be lifted to categorical level.

Conjecture 6.7 Consider a resolution $S' \xleftarrow{\text{res}} S$ of terminal singularities. Assume $S - S_{\text{sing}}$ has a volume form. Then

(1) $\mathcal{H}_0$ is independent of r;
(2) $\mathcal{C}_0$ is a CY-category, subcategory of $Perf(X)$ is independent of r.

We would like to make a parallel between Birkar's theory and category theory.



In the above setting $S - S_{\text{sing}}$ determines $\mathcal{H}_0$ and S_{sing} the rest of semi-orthogonal decomposition.

We have a correspondence between classical and categorical notions:

$$K_X, B \longleftrightarrow S_{\text{sing}}$$

$$B'_{\text{complement}} \longleftrightarrow S/S_{\text{sing}}$$

$$\text{volumes} \longleftrightarrow \text{Categorical Entropy } h$$

Let $\mathcal{C}^d_{\mathcal{E}}$ be a log Calabi-Yau category. (We fix the biggest number in the spectra and d is the categorical dimension.)

Question 6.8 Φ is a functor of $\mathcal{C}^d_{\mathcal{E}}$. Are $h(\Phi)$ bounded?

Question 6.9 Is Aut($\mathcal{C}^d_{\mathcal{E}}$) of Jordan type? (Here Aut($\mathcal{C}^d_{\mathcal{E}}$) is the group of autoequivalences).

Question 6.10 Is F($\mathcal{C}^d_{\mathcal{E}}$) a bounding family? (Here F($\mathcal{C}^d_{\mathcal{E}}$) is the family parametrizing the categories with dimension d and bounded the biggest number of the spectra from below. Proper definition will take effort.)

Question 6.11 Consider the splitting

$$\mathcal{C} = \bigcup_{i \geq 0}^{\lambda(\mathcal{E},d)} \mathcal{C}_i$$

$$\mathcal{H} = \bigcup_{i \geq 0}^{\lambda(\mathcal{E},d)} \mathcal{H}_i$$

Show that $\lambda(\mathcal{E}, d)$ is finite.

Question 6.12 Are categorical dimensions of $\mathcal{C}^{\lambda_i}_{\mathcal{E},d}$ bounded?

The above considerations suggest the following parallels.

Fano	Category	CFT
Birkar's Theory $\mathcal{E}, d$ Boundness	σ, d Boundness of log CY theory	Behavior of σ, d theory
Jordan Property of Birational Aut	Jordan Property of Aut D^{b}	
	uppersemicontinuity of Spectra	Zamolodchikov Theorem

The Zamolodchikov's c theorem suggests semicontinuity of the noncommutative spectra—see [6, 8]. This correspondence will be discussed elsewhere.

Our findings in the previous sections suggest that in the case of X, an algebraic surface, we have the following correspondence.

$$\left\{ \begin{array}{c} \text{Additional basic} \\ \text{for } H^2(X) \text{ classes} \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{Phantoms} \\ \text{of } \mathrm{D}^{\mathrm{b}}(X) \text{ classes} \end{array} \right\} \qquad\qquad \left\{ \ \delta > 0 \ \right\}$$

The above findings suggest that new (A, B) structures can be used to define new invariants, A side invariants for the B side.

We have the following parallel:

Resolution of singularity	Surgery
Creation of Spectra	Creation of Spectra

Conjecture 6.13 Log transform (rational blow down) creates nontrivial $\delta > 0$.

This suggests the following questions.

Question 6.14 Can we have symplectic 4-fold with the same basic classes but different spectra?

We have a connection with k-spectra of CFT. This observations lead to: symplectic Poincare conjectures.

– Find a 4-dim symplectic manifold s.t. $X \overset{\text{homeo}}{\cong} \mathbb{P}^2$ and $\delta(X) > 0$.
– Find a 4-dim symplectic manifold s.t. $X \overset{\text{homeo}}{\cong} \mathbb{P}^1 \times \mathbb{P}^1$ and $\delta(X) > 0$.
– Find a $2n$-dim symplectic manifold s.t. $X \cong \mathbb{P}^n$ and $\delta(X) > 0$.

The parallel between RG flow and Kaehler Ricci flow suggests that the other R-charges can also lead to birational invariants.

Renormalisation group flow and defects lines in the LG model could lead to higher invariants. We investigate these phenomena further in [33].

Acknowledgements We are grateful to D. Auroux, I. Cheltsov and V. Prizhalkovskii, E. Lupercio, A. Libgober, S. Gukov, M. Kontsevich, I. Todorov for the help with the preparation of the paper.
L. K. was partially supported by Simons Investigators Award HMS, Simons Collaborations Grant, NSF Grant, HSE University Basic Research Program, National Science Fund of Bulgaria, National Scientific Program—Excellent Research and People for the Development of European Science—(VIHREN), Project No. KP-06-DV-7.
K.-S. Lee was supported by Simons Collaborations Grant.
J. S. was supported by Simons Investigator Grant.
A. P. was supported National Science Fund of Bulgaria, National Scientific Program—Excellent Research and People for the Development of European Science—(VIHREN), Project No. KP-06-DV-7.

References

1. Birkar, C.: Anti-pluricanonical Systems on Fano Varieties, vol. 190, Issue 2, pp. 345–463 (2019)
2. Hassett, B., Pirutka, A., Tschinkel, Yu.: A very general quartic double fourfold is not stably rational. Algebr. Geom. **6**(1), 64–75 (2019)
3. Iskovskikh, V., Pukhlikov, A.: Birational automorphisms of multidimensional algebraic manifolds. Algebraic Geometry, 1. J. Math. Sci. **82**(4), 3528–3613 (1996)
4. Colliot-Thelene, J.-L., Pirutka, A.: Hypersurfaces quartiques de dimension 3: non-rationalite stable. Ann. Sci. Ee. Norm. Super. (4) **49**(2), 371–397 (2016)
5. Campillo, A., Delgado, F., Gusein-Zade, S.M.: The Alexander polynomial of a plane curve singularity via the ring of functions on it. Duke Math. J. **117**(1), 125–156 (2003)
6. Cecotti, S., Vafa, C.: Nucl. Phys. B **367**, 359–461 (1991)
7. Graham Shor, M.: The c and a theorems and the local renormalization group. hep -th. arXiv:1601.06662
8. Gukov, S.: Counting RG flows, hepth arXiv:1503.01474
9. Iskovskih, V., Manin, Yu.: Three-dimensional quartics and counterexamples to the Luroth problem. Mat. Sb. (N.S.) **86**(128), 140–166 (1971)
10. Beauville, A.: Varietes de Prym et jacobiennes intermediaires. Ann. Sci. Ecole Norm. Sup. (4), **10**(3), 309–391 (1977)
11. Grinenko, M.: Mori structures on a Fano threefold of index 2 and degree l. Proc. Steklov Inst. Math. **246**(3), 103–128 (2004)
12. Voisin, C.: Unirational threefolds with no universal codimension 2 cycle. Invent. Math. **201**(1), 207–237 (2015)
13. Voisin, C.: Sur la jacobienne intermediaire du double solide d'indice deux. Duke Math. J. **57**(2), 629–646 (1988)
14. Clemens, H., Griffiths, Ph.: The intermediate Jacobian of the cubic threefold. Ann. Math. **2**(95), 281–356 (1972)

15. Ebeling, W.: Homological mirror symmetry for singularities . arXiv:1601.06027
16. Futaki, M., Ueda, K.: Homological mirror symmetry for Brieskorn-Pham singularities. Sel. Math. **17**(2), 435–452 (2011)
17. Okada, T.: Stable rationality of cyclic covers of projective spaces. Proc. Edinb. Math. Soc. (2) **62**(3), 667–682 (2019)
18. Totaro, B.: Hypersurfaces that are not stably rational. J. Am. Math. Soc. **29**(3), 883–891 (2016)
19. Schreieder, S.: Stably irrational hypersurfaces of small slopes (2018). arXiv:1801.05397
20. Kollar, J.: The rigidity theorem of Fano-Segre-Iskovskikh-Manin-PukhlikovCorti- Cheltsov-de Fernex- Ein- Mustata- Zhuang (2018). arXiv:1807.00863
21. Maulik, D.: Stable pairs and the HOMFLY polynomial. Invent. math. **204**, 787–831 (2016)
22. Nicaise, J., Ottem, J.: Tropical degenerations and stable rationality (2019). arXiv:1911.06138
23. Hassett, B., Pirutka, A., Tschinkel, Yu.: Intersections of three quadrics in $\mathbb{P}^7$ (2017). arXiv:1706.01371
24. Hassett, B., Pirutka, A., Tschinkel, Yu.: A very general quartic double fourfold is not stably rational. Algebr. Geom. **6**(1), 64–75 (2019)
25. Russo, F., Stagliano, G.: Explicit rationality of some cubic fourfolds (2018). arXiv:1811.03502
26. Libgober, A.: Elliptic genus of singular algebraic varieties and quotients. arXiv:1702.03580
27. Katzarkov, L., Lee, K.-S., Libgober, A.: In preparation
28. Katzarkov, L., Kontsevich, M., Pantev, T., Yu, T.: Singularity theory, for LG theory. In preparation
29. Katzarkov, L., Kontsevich, M., Pantev, T., Yu, T.: Blow up formulae. In preparation
30. Katzarkov, L., Kontsevich, M., Pantev, T.: Hodge theoretic aspects of mirror symmetry. From Hodge Theory to Integrability and TQFT: tt*-geometry. In: Proceedings of Symposia in Pure Mathematics, vol. 78 (2008)
31. Katzarkov, L., Liu, Y.: Categorical base loci and spectral gaps, via okounkov bodies and nevanlinna theory L Katzarkov, Y Liu - String-Math (2013)
32. Gukov, S., Katzarkov, L., Svoboda, J.: GPPV invariants and splicing. In preparation
33. Gukov, S., Katzarkov, L., Svoboda, J.: RG and birational invariants. In preparation
34. Katzarkov, L., Prizhalkovsky, V.: tLandau-Ginzburg models old new. In: Giokova Proceedings (2012)
35. Johnson, J., Kollar, J.: Kahler-Einstein metrics on log del Pezzo surfaces in weighted projective 3-spaces
36. Pukhlikov, A.: Birationally rigid Fano cyclic covers (2004). arXiv:0411198
37. Cheng, M.C., Chun, S., Ferrari, F., Gukov, S., Harrison, S.: 3d modularity. JHEP (10)(2018)
38. Gukov, S., Pei, D., Putrov, P., Vafa, C.: BPS spectra and 3-manifold invariants. J. Knot Theor. Ramificat. **2** (2020)
39. Hikami, K.: On the quantum invariants for the spherical seifert manifolds. Commun. Math. Phys. **2**, 285–319 (2006)
40. Lawrence, R., Zagier, D.: Modular forms and quantum invariants of 3-manifolds. Asian J. Math. **1**, 93–108 (1999)
41. Zagier, D.: Quantum Modular Forms. Clay Mathematics Proceedings Volume 12 (2010)
42. Neumann, W.D., Wahl, J.: Universal Abelian Covers of Surface Singularities. Birkhäuser Basel, pp. 181–190 (2002)
43. Gukov, S., Marino, M., Putrov, P.: Resurgence in complex Chern-Simons theory
44. Fintushel, R., Stern, R.J.: Instanton homology of Seifert fibred homology three spheres. Proc. Lond. Math. Soc. **3**, 61 (1990)
45. Oblomkov, A., Shende, V.: The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link. Duke Math. J. **161**(7), 1277–1303 (2012)
46. Singular - handbook
47. Tyurin, A.: On intersections of quadrics. Russ. Math. Surv. **30**(6), 51–105 (1975)
48. Bondal, A.I., Kapranov, M.M.: Representable functors, Serre functors, and mutations. Izv. Akad. Nauk SSSR Ser. Mat. **53**(6), 1183–1205 (1989)
49. Elagin, A., Lunts, V.A.: Three notions of dimension for triangulated categories. J. Algebra **569**, 334–376 (2021). Accessed from 1 March 2021