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Orlov spectra: bounds and gaps

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Abstract The Orlov spectrum is a new invariant of a triangulated category. It was introduced by D. Orlov, building on work of A. Bondal-M. Van den Bergh and R. Rouquier. The supremum of the Orlov spectrum of a triangulated category is called the ultimate dimension. In this work, we study Orlov spectra of triangulated categories arising in mirror symmetry. We introduce the notion of gaps and outline their geometric significance. We provide the first large class of examples where the ultimate dimension is finite: categories of singularities associated to isolated hypersurface singularities. Similarly, given any nonzero object in the bounded derived category of coherent sheaves on a smooth Calabi-Yau hypersurface, we produce a generator, by closing the object under a certain monodromy action, and uniformly bound this generator's generation time. In addition, we provide new upper bounds on the generation times of exceptional collections and connect generation time to braid group actions to provide a lower bound on the ultimate dimension of the derived Fukaya category of a symplectic surface of genus greater than one.

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1 Introduction

The spectrum of a triangulated category was introduced by D. Orlov in [42], building on work of A. Bondal, R. Rouquier, and M. Van den Bergh, [12, 47]. This categorical invariant, which we shall call the Orlov spectrum, is simply a list of non-negative integers. Each integer is the generation time of an object in the triangulated category. Roughly, the generation time of an object is the necessary number of exact triangles it takes to build the category using this object. If the triangulated category is of geometric origin, like the bounded derived category of coherent sheaves on a scheme, the Orlov spectrum encodes nontrivial geometric information. In this paper, we study how geometry influences the structure of Orlov spectra and we find geometric meaning in the gaps arising in Orlov spectra.

Although the (pre-)history of Orlov spectra extends back further, notably to work of A. Neeman, and Bondal-M. Kapranov, the fundamental background for this paper arose in [12]. In [12], Bondal and Van den Bergh lay out the foundations, introducing all of the notions necessary to define generation time. They apply their new notions to categories arising in algebraic geometry, proving a number of interesting and deep results that tie generators and geometry together. Let us emphasize the following one:

Theorem 1.1 *For a smooth scheme over a field*, *the bounded derived category of coherent sheaves admits a strong generator* (*i.e. a generator of finite generation time*).

In [47], Rouquier expanded on the foundations of [12]. He studied the minimal generation time amongst all strong generators, i.e. the minimum of the Orlov spectrum. This notion we shall call the Rouquier dimension of a triangulated category. Rouquier proved many interesting results in [47] concerning the Rouquier dimension. His results had deep applications in both geometry and pure algebra. Let us emphasize the following theorems which appear in loc. cit.:

Theorem 1.2 *For a reduced separated scheme of finite type over a field*, *the Rouquier dimension of derived category of coherent sheaves is bounded below by the Krull dimension*.

Theorem 1.3 *For smooth quasi-projective schemes over a field*, *the Rouquier dimension of the derived category of coherent sheaves is bounded above by twice the Krull dimension*.

The following generalizes the above-mentioned result of Bondal and Van den Bergh:

Theorem 1.4 *For any separated scheme of finite type over a field* (*not necessarily smooth*), *the derived category of coherent sheaves admits a strong generator.*

Rouquier also contends that the supremum amongst all generation times, which we shall call the ultimate dimension, should be studied in its own right.

In [42], Orlov utilizes results on the semi-stability of vector bundles on curves to prove the following interesting result:

Theorem 1.5 *The Rouquier dimension of the derived category of coherent sheaves on any smooth algebraic curve is one.*

Having proven the one dimensional case, he proposes the following general conjecture:

Conjecture 1 *For a smooth algebraic variety*, X, *the Krull dimension of* X *and the Rouquier dimension of* $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)$ *are equal.*

This conjecture asserts that Rouquier's notion of dimension of a triangulated category is deeply geometric. Furthermore, Orlov contends that in order to extract additional, more novel, geometric invariants from the category, one should study all possible generation times—the Orlov spectrum. With this in mind, he begins the analysis of the Orlov spectrum of a smooth algebraic curve proving:

Theorem 1.6 *The set*, $\{1, 2\}$, *is a subset of Orlov spectrum of the bounded derived category of coherent sheaves on a smooth proper algebraic curve*, *with equality if and only if the curve is rational.*

Orlov then poses the following questions:

- Is the Orlov spectrum of the bounded derived category of coherent sheaves on a smooth quasi-projective scheme bounded above? Is it bounded above for a non-smooth scheme?
- Does the Orlov spectrum of the bounded derived category of coherent sheaves on a (smooth) quasi-projective scheme form an integer interval?

Orlov's ideas were developed further by the first two authors in [5]. In loc. cit., the authors prove that calculating the generation time of a tilting object reduces to a very simple geometric computation. They use it to prove Orlov's conjecture in many new cases. Through examples, they illustrate some subtleties encoded in generation time, including how it can vary in certain moduli and its relationship with positivity of the anti-canonical bundle.

In addition to the papers discussed above, there are other works we should mention. Indeed, study of Orlov spectra, possibly proceeding under other names, seems a common endeavor across different algebraic fields. Rouquier's paper inspired further work in algebra by L.L. Avramov, P.A. Bergh, R.-O. Buchweitz, S. Iyengar, H. Krause, D. Kussin, C. Miller, and S. Oppermann, see [3, 9, 30, 37, 38]. Notably, [9] seems closely related to Sect. 4 of this work. D. Benson, J. Carlson, S. Chebolu, J.D. Christensen, M. Hovey, K. Lockridge, Y. Mináč, and G. Puninski, see [8, 14, 15, 21, 22, 33], are inspired by analogs of Freyd's Generating Hypothesis, which, in our language, seeks to determine whether an object has generation time zero.

Even with the wealth of knowledge detailed above, precise descriptions of Orlov spectra for, even simple, categories are still elusive. This paper builds on the growing understanding of the structure of Orlov spectra of categories of geometric origin, particularly categories of interest in mirror symmetry. The novelty of our current work lies in its approach to geometric themes encoded in the Orlov spectrum. Upper bounds on the ultimate dimension are closely tied to the Hochschild homology of the category. Lower bounds on the ultimate dimension are controlled by the complexity of braid groups actions. We expect that these phenomena, properly understood and synthesized, are universal.

We outline a new approach to decode the geometry found in the gaps of Orlov spectra. Gaps are simply the missing numbers in an Orlov spectrum. Their existence is precisely the content of Orlov's second question from above. Despite their simplicity, the authors expect that gaps are a deep geometric invariant related to monodromy and capturing motivic information in the case of the derived category of coherent sheaves on a smooth proper variety.

Let us highlight our predictions by discussing some of the major themes of this work:

(1) We provide the first large class of examples where the Orlov spectrum is bounded above: the category of singularities of an isolated hypersurface singularity. Our bound is expressed in terms of the embedding dimension and the nilpotence of the Tjurina algebra.

Theorem 1.7 *Let $(S, \mathfrak{m}_S)$ be an isolated hypersurface singularity. The Orlov spectrum of $\mathrm{D}_{\mathrm{sg}}(S)$ is bounded by $2(\dim S + 2)\,\mathrm{LL}(S/(\partial w)) - 1$, where* LL *denotes the Loewy length of an algebra.*

We also calculate the full Orlov spectrum when $(S, \mathfrak{m}_S)$ is an A_n singularity.

Theorem 1.8 *The Orlov spectrum of $\mathrm{D}_{\mathrm{sg}}(A_{n-1})$ is*

$$\left\{ \left\lceil \frac{\lfloor n/2 \rfloor}{s} \right\rceil - 1 \,:\, s \in \mathbb{N} \right\}$$

where $\lfloor\alpha\rfloor$ is the greatest integer less than α and $\lceil\alpha\rceil$ is the least integer greater than α.

Let us note that the results of [53] can be applied to deduce that the level, see Definition 2.2, of the residue field in $\mathrm{D}_{\mathrm{sg}}(S)$ with respect to any nonzero object of $\mathrm{D}_{\mathrm{sg}}(S)$ is bounded. This is an important step in the proof of Theorem 1.7.

(2) The most unexpected geometric consequence is the connection of the theory of gaps of Orlov spectra to questions of rationality. Based on work of Bondal, A. Kuznetsov and Orlov, rationality enters category theory by way of semi-orthogonal decompositions. We demonstrate that, in many cases, the maximal gap of the Orlov spectrum of a triangulated category is bounded above by the maximal Rouquier dimension of its semi-orthogonal components.

Theorem 1.9 *Suppose $\langle \mathscr{A}_1, \ldots, \mathscr{A}_n\rangle$ is a semi-orthogonal decomposition of $\mathscr{T}$ and $\mathscr{G} := G_1 \oplus \cdots \oplus G_n$ is a generator of $\mathscr{T}$ with $G_i \in \mathscr{A}_i$. By performing a series of mutations to dual decompositions, we get a set of generators. These generators give a subset of the Orlov spectrum, in which there is no gap greater than the maximum of the generation times of the G_i in $\mathscr{A}_i$.*

Remark 1.10 The generators obtained from the construction in the previous theorem only provide an upper bound on the length of the gaps. In general, it seems likely that the gaps are indeed smaller.

Now, in light of the above theorem, we propose the following conjectures:

Conjecture 2 *Let X be a smooth algebraic variety and $\langle \mathscr{A}_1, \ldots, \mathscr{A}_n\rangle$ be a semi-orthogonal decomposition of* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. *The length of any gap in the spectrum of* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *is at most the minimum of the maximal Rouquier dimension amongst the $\mathscr{A}_i$ and the maximal gap amongst the $\mathscr{A}_i$.*

Conjecture 3 *If $\mathscr{A}$ has a gap of length at least s, then so does* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.

These have the following nice corollaries:

Corollary 1.11 *Suppose Conjectures 1 and 2 hold. If X is a smooth variety then any gap of* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *has length at most the Krull dimension of X.*

Corollary 1.12 *Suppose Conjectures 1, 2, and 3 hold. Let X and Y be birational smooth proper varieties of dimension n. The category,* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, *has a gap of length n or $n-1$ if and only if* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ *has a gap of the same length i.e. the gaps of length greater than $n-2$ are a birational invariant.*

Corollary 1.13 *Suppose Conjectures 1, 2, and 3 hold. If X is a rational variety of dimension n, then any gap in* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *has length at most* $n-2$.

The above corollaries outline a new approach to dealing with questions of rationality with enormous potential towards applications. In particular, based on work of Kuznetsov, we believe this could lead to a proof of non-rationality for a generic cubic fourfold. The mirror interpretation of this framework is discussed in [24] and [25].

(3) The first two themes are related by work of Orlov (see [41]). For a smooth Fano or Calabi-Yau hypersurface, the graded category of singularities of its affine cone is a semi-orthogonal component of the derived category of coherent sheaves. Therefore, the Orlov spectrum of this component is also related to the Loewy length of the Tjurina algebra (which for a homogeneous polynomial is equal to the Milnor algebra) of the defining function. In this case, this Loewy length is just $(d(n+1)-2n-1)$ by Macaulay's theorem.

Theorem 1.14 *Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d and $A := k[x_0, \dots, x_n]/(f)$. Assume that A has an isolated singularity. For any non-zero object, M, in* $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, *the object,* $M \oplus M(1) \oplus \cdots \oplus M(d-1)$, *is a generator of* $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ *with generation time at most* $2(n+1)(d(n+1)-2n-1)-1$.

Orlov's work then provides us with the following geometric version:

Corollary 1.15 *Let X be a smooth hypersurface of degree $n+1$ in $\mathbb{P}^n$. Set $\{1\} := L_{\mathcal{O}} \circ (- \otimes_{\mathcal{O}} \mathcal{O}(1))$ where $L_{\mathcal{O}}$ is the Seidel-Thomas twist by $\mathcal{O}$. For any nonzero* $E \in \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, $E \oplus E\{1\} \oplus \cdots \oplus E\{n\}$ *is a generator of* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *with generation time bounded by* $2n^2(n+1)-1$.

In light of the above discussion, we expect that gaps in the derived category of coherent sheaves on a Fano or Calabi-Yau hypersurface in projective space are related to the structure of the corresponding Milnor algebra.

(4) We give a new upper bound on the generation time of any (full, not necessarily strong) exceptional collection. The upper bound comes from studying A_∞-enhancements of triangulated categories. It ties in quite nicely with Koszul duality.

Theorem 1.16 *Let $\mathscr{A}$ be a cohomologically-finite triangulated A_∞-category possessing a (full) exceptional collection $A_1, \dots, A_n$. The generation time of the dual collection in $H(\mathscr{A})$ is bounded above by* $\mathrm{LL}_\infty(A') - 1$ *where A' is a minimal A_∞-algebra quasi-isomorphic to the A_∞-endomorphism algebra of $\bigoplus_{i=1}^n A_i$. If the A_∞-endomorphism algebra of $\bigoplus_{i=1}^n A_i$ is formal (quasi-isomorphic to its cohomology), then the generation time of the dual collection* *is equal to one less than the Loewy length of the cohomology of the A_∞-endomorphism algebra of $\bigoplus_{i=1}^n A_i$.*

Here, LL_∞ is an extension of the notion of Loewy length to minimal A_∞-algebras. In addition, we also provide examples to demonstrate how generation time can depend on "higher homotopy" information of the endomorphism algebra of an object, and we compute the Orlov spectra of the bounded derived categories of finite-dimensional representations of A_n quivers.

Theorem 1.17 *Let Q be a quiver such that the underlying graph is a Dynkin diagram of type A_n. The Orlov spectrum of $\mathrm{D^b}(\mathrm{mod}\, kQ)$ is equal to the integer interval $\{0, \dots, n-1\}$.*

(5) From the symplectic perspective, one can consider the Orlov spectrum as a new invariant of a Fukaya category. Here we see that the correlation with monodromy theory is once again manifest by connecting generation time to braid group actions. An upper bound, comes from a more well known construction and occurs as follows:

Proposition 1.18 *Let $S_1, \dots, S_n$ be spherical objects in the homotopy category, $\mathcal{T}$, of a triangulated cohomologically-finite A_∞-category and assume we have $\mathrm{HH}^0(\mathcal{T}) = k$. Suppose there exists a relation among the corresponding spherical twists*:

$$L_{S_{a_1}} \cdots L_{S_{a_r}} \cong \mathrm{Id}_{\mathcal{T}}$$

with $1 \leq a_i \leq n$. Then $S_1 \oplus \cdots \oplus S_n$ strongly generates $\mathcal{T}$ with generation time at most $r - 1$.

Using a combination of braid relations and geometry, it is also possible to obtain lower bounds on generation time as in the following theorem:

Theorem 1.19 *The ultimate dimension of the derived Fukaya category of a symplectic surface of genus g is at least $4g$.*

For an elliptic curve we calculate the Orlov spectrum in its entirety. This result was also attained independently by Orlov (unpublished).

Theorem 1.20 *The Orlov spectrum of the bounded derived category of coherent sheaves on an elliptic curve is $\{1, 2, 3, 4\}$.*

We expect derived Fukaya categories of symplectic surfaces to have no gaps, as is the case for a Riemann surface of genus one via homological mirror symmetry. However, we suspect that there exist symplectic manifolds of real dimension four whose derived Fukaya categories have Orlov spectra with large gaps. This indicates that gaps of the Orlov spectrum of the derived Fukaya category is a nontrivial invariant of the symplectic motive.

The paper is organized as follows. In Sect. 2, we establish our notational conventions and define all necessary mathematical notions revolving around the Orlov spectrum. We proceed with a discussion of ghost maps, a theory originating in [28], and used, implicitly and explicitly, by many subsequent authors in connection to this subject. We illustrate a number of examples occurring in geometry, notably, spherical twists and monodromy of the quintic threefold. In Sect. 3, we remind the reader of the basics of semi-orthogonal decompositions and demonstrate how semi-orthogonal decompositions whose components have small Rouquier dimension limit the size of gaps. We then outline how gaps in the Orlov spectrum of the bounded derived category of a variety can be used to answer questions about rationality. Finally, we develop a method, distinct from [5] and fully general, to bound the generation time of exceptional collections using the Loewy length of the dual collection. We provide a handful of examples to illustrate the utility of the method. In Sect. 4, we discuss strong generators for categories of singularities of isolated singularities. We provide new proofs, from our perspective, and extensions of some of the known results in this area. We use these ideas to bound the Orlov spectrum of an isolated hypersurface singularity. In Sect. 5, we give a detailed recap of Orlov's theorem relating graded categories of singularities to bounded derived categories of coherent sheaves. We use our examination of Orlov's theorem and extensions of results from Sect. 4 to study the Orlov spectrum for hypersurfaces in projective space. Section 6, though connected to the other sections, can certainly be read independently. Here, we illustrate the relationship between generation time and braid group actions, by means of the derived Fukaya category of a symplectic surface. We compute the full Orlov spectrum of the elliptic curve and provide a lower bound on the ultimate dimension of the derived Fukaya category of a symplectic surface of higher genus.

2 Preliminaries

Throughout this article, k denotes an algebraically-closed field of characteristic zero. All categories will be k-linear. For a ring, R, $\operatorname{Mod} R$ denotes the category of right R-modules and $\mathrm{D}(\operatorname{Mod} R)$ denotes the unbounded derived category of right R-modules. The bounded derived category of right R-modules we denote by $\mathrm{D}^{\mathrm{b}}(\operatorname{Mod} R)$. For a Noetherian ring, R, $\operatorname{mod} R$ denotes the category of finitely-generated right R-modules and $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} R)$ denotes its bounded derived category. If X is a variety, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ denotes the bounded derived category of coherent sheaves on X.

Let $\mathscr{T}$ be a triangulated category. For a full subcategory, $\mathscr{I}$, of $\mathscr{T}$ we denote by $\langle \mathscr{I} \rangle$ the full subcategory of $\mathscr{T}$ whose objects are isomorphic to summands of finite coproducts of shifts of objects in $\mathscr{I}$. In other words, $\langle \mathscr{I} \rangle$ is the smallest full subcategory containing $\mathscr{I}$ which is closed under isomorphisms, shifting, and taking finite coproducts and summands. For two full subcategories, $\mathscr{I}_1$ and $\mathscr{I}_2$, we denote by $\mathscr{I}_1 * \mathscr{I}_2$ the full subcategory of objects, B, such that there is a distinguished triangle, $B_1 \to B \to B_2 \to B_1[1]$, with $B_i \in \mathscr{I}_i$. Set $\mathscr{I}_1 \diamond \mathscr{I}_2 := \langle \mathscr{I}_1 * \mathscr{I}_2 \rangle$, $\langle \mathscr{I} \rangle_0 := \langle \mathscr{I} \rangle$, and inductively define

$$\langle \mathscr{I} \rangle_n := \langle \mathscr{I} \rangle_{n-1} \diamond \langle \mathscr{I} \rangle.$$

Similarly we define

$$\langle \mathscr{I} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathscr{I} \rangle_n.$$

For an object, $E \in \mathscr{T}$, we notationally identify E with the full subcategory consisting of E in writing, $\langle E \rangle_n$. The reader is warned that, in some of the previous literature, $\langle \mathscr{I} \rangle_0 := 0$ and $\langle \mathscr{I} \rangle_1 := \langle \mathscr{I} \rangle$. We follow the notation in [5]. With our convention, the index equals the number of cones allowed. The operations, $*$ and $\diamond$, were introduced in [12] where their associativity is proven. From associativity, it follows that $\langle \mathscr{I} \rangle_n \diamond \langle \mathscr{I} \rangle_m = \langle \mathscr{I} \rangle_{n+m+1}$. We will use this fact implicitly.

We will need small modifications for the statement and proof of Proposition 4.4. Let $\bar{\mathscr{I}}$ denote the smallest subcategory of $\mathscr{T}$ containing $\mathscr{I}$ and closed under $\mathscr{T}$-coproducts of objects of $\mathscr{I}$. Let $\tilde{\mathscr{I}}$ denote the smallest subcategory of $\mathscr{T}$ containing coproducts of the form, $\bigoplus_{a \in A} I$, for a single $I \in \mathscr{I}$ whenever $\bigoplus_{a \in A} I$ exists in $\mathscr{T}$. We then set $\langle \bar{\mathscr{I}} \rangle_0 = \langle \bar{\mathscr{I}} \rangle$ and

$$\langle \bar{\mathscr{I}} \rangle_n := \overline{\langle \bar{\mathscr{I}} \rangle_{n-1} \diamond \langle \bar{\mathscr{I}} \rangle}.$$

We also set $\langle \tilde{\mathscr{I}} \rangle_0 = \langle \tilde{\mathscr{I}} \rangle$ and

$$\langle \tilde{\mathscr{I}} \rangle_n := \langle \tilde{\mathscr{I}} \widetilde{\rangle_{n-1} \diamond} \langle \tilde{\mathscr{I}} \rangle.$$

Definition 2.1 Let E be an object of a triangulated category, $\mathscr{T}$. If there is an n with $\langle E \rangle_n = \mathscr{T}$, we set

$$\circleddash_{\mathscr{T}}(E) := \min \{ n \geq 0 \mid \langle E \rangle_n = \mathscr{T} \}.$$

Otherwise, we set $\circleddash_{\mathscr{T}}(E) := \infty$. We call $\circleddash_{\mathscr{T}}(E)$ the *generation time* of E. When, $\mathscr{T}$ is clear from context, we omit it and simply write $\circleddash(E)$. If $\langle E \rangle_\infty$ equals $\mathscr{T}$, we say that E is a *generator*. If $\circleddash(E)$ is finite, we say that E is a *strong generator*. The *Orlov spectrum* of $\mathscr{T}$, denoted OSpec $\mathscr{T}$, is the set

$$\{ \circleddash(G) \mid G \in \mathscr{T}, \ \circleddash(G) < \infty \} \subset \mathbb{Z}_{\geq 0}.$$

The *Rouquier dimension* of $\mathscr{T}$, denoted $\operatorname{rdim} \mathscr{T}$, is the infimum of $\operatorname{OSpec} \mathscr{T}$; it is defined as ∞ when $\operatorname{OSpec} \mathscr{T}$ is empty. The *ultimate dimension* of $\mathscr{T}$, denoted $\operatorname{udim} \mathscr{T}$, is the supremum of $\operatorname{OSpec} \mathscr{T}$; it is defined as ∞ when $\operatorname{OSpec} \mathscr{T}$ is empty.

We shall denote the Orlov spectrum, Rouquier dimension, and ultimate dimension of $\mathrm{D^b}(\operatorname{coh} X)$ by $\operatorname{OSpec} X$, $\operatorname{rdim} X$, and $\operatorname{udim} X$, respectively. It is also convenient to recall the following definition which first appeared in [3].

Definition 2.2 Let E be an object of a triangulated category, $\mathscr{T}$. If there is an n with $A \in \langle E \rangle_n$, we set

$$\operatorname{Lvl}^E_{\mathscr{T}}(A) := \min \{ n \geq 0 \mid A \in \langle E \rangle_n \}.$$

Otherwise, we set $\operatorname{Lvl}^E_{\mathscr{T}}(A) = \infty$. This number is called the *level of A with respect to E*, or simply the level of A when E is implicit.

The case where $\mathscr{T}$ is $\mathrm{D^b}(\operatorname{mod} A)$, the bounded derived category of coherent modules, and $G = A$ is the free module, provides some insight into the formalism above. The following theorem is taken from [30]. We refer the reader to loc. cit. for the definition of coherent rings and modules. However, let us note that, if A is right-Noetherian, then it is right-coherent, and a right A-module is finitely-generated if and only if it is coherent. If A is finite-dimensional over k or commutative and essentially of finite type, the result is originally due to Rouquier, [47].

Theorem 2.3 *Let A be a right-coherent k-algebra. The generation time of A, as an object of* $\mathrm{D^b}(\operatorname{mod} A)$, *the bounded derived category of coherent A-modules, is equal to the global dimension of A.*

Remark 2.4 Using ideas from [3], one can extend the notion of global dimension to dg-algebras in a natural manner and check that the analog of Theorem 2.3 holds for dg-algebras. As noted in [47], for an enhanceable triangulated category, $\mathscr{T}$, each generator, G, allows one to construct an equivalence of $\mathscr{T}$ with the derived category of perfect dg-modules over the dg-endomorphisms of G. In this way, the Orlov spectrum can be viewed as a list of global dimensions of dg-algebras within a derived Morita equivalence class.

We have the following simple lemma, for a proof see Lemma 2.4 of [5]:

Lemma 2.5 *Let $F : \mathscr{T} \to \mathscr{R}$ be an exact functor between triangulated categories. Let G be an object of $\mathscr{T}$. If $B \in \langle G \rangle_n$, then $F(B) \in \langle F(G) \rangle_n$. Moreover, if F commutes with coproducts and $B \in \langle \overline{G} \rangle_n$, then $F(B) \in \langle \overline{F(G)} \rangle_n$.*

Let $F : \mathscr{T} \to \mathscr{R}$ be an exact functor between triangulated categories. If every object in $\mathscr{R}$ is isomorphic to a direct summand of an object in the essential image of F, we say that F is *dense*, or has dense image.

Lemma 2.6 *If $F : \mathscr{T} \to \mathscr{R}$ has dense image and G is a strong generator, then $\circleddash(G) \geq \circleddash(F(G))$. In particular,* $\dim \mathscr{T} \geq \dim \mathscr{R}$.

For a proof, see Lemma 2.5 of [5].

Example 2.7 Let V be a vector bundle in $\mathrm{D^b}(\mathrm{coh}\, X)$. Then the functor $(- \otimes_{\mathscr{O}} V) : \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$ is dense, as any object, F, is a summand of $(F \otimes_{\mathscr{O}} V^{\vee}) \otimes_{\mathscr{O}} V$.

Example 2.8 Consider a finite group Γ acting on an algebraic variety, X, and consider the derived category of coherent sheaves on X, $\mathrm{D^b}(\mathrm{coh}\, X)$, and the derived category of Γ-equivariant coherent sheaves on X, $\mathrm{D}^{\mathrm{b}}_{\Gamma}(\mathrm{coh}\, X)$. We have two exact functors: the forgetful functor, $\mathrm{For} : \mathrm{D}^{\mathrm{b}}_{\Gamma}(\mathrm{coh}\, X) \to \mathrm{D^b}(\mathrm{coh}\, X)$, and the inflation functor, $\mathrm{Inf} : \mathrm{D^b}(\mathrm{coh}\, X) \to \mathrm{D}^{\mathrm{b}}_{\Gamma}(\mathrm{coh}\, X)$, where, by definition, $\mathrm{Inf}(A) = \bigoplus_{g \in \Gamma} g^* A$, with the natural Γ action.

Notice that any $A \in \mathrm{D^b}(\mathrm{coh}\, X)$ is a summand of $\mathrm{For}(\mathrm{Inf}(A))$, hence the forgetful functor is dense. On the other hand, for any $B \in \mathrm{D}^{\mathrm{b}}_{\Gamma}(\mathrm{coh}\, X)$ and each $g \in \Gamma$, we have an isomorphism, $\phi_g : g^*\mathrm{For}(B) \to \mathrm{For}(B)$, in $\mathrm{D^b}(\mathrm{coh}\, X)$ coming from the equivariant structure on B. For is the left adjoint to Inf with adjunction morphism in $\mathrm{D}^{\mathrm{b}}_{\Gamma}(\mathrm{coh}\, X)$ defined by:

$$\sum_{g \in \Gamma} \phi_g : \mathrm{Inf}(\mathrm{For}(B)) \to B.$$

The map,

$$\frac{1}{|\Gamma|} \bigoplus_{g \in \Gamma} \phi_g^{-1} : B \to \mathrm{Inf}(\mathrm{For}(B)),$$

provides a splitting of the map above. Therefore, B is a summand of $\mathrm{Inf}(\mathrm{For}(B))$, and the functor Inf is also dense.

Hence, for any generator, G, of $\mathrm{D^b}(\mathrm{coh}\, X)$, we have:

$$\circleddash(\mathrm{For}(\mathrm{Inf}(G)) \leq \circleddash(\mathrm{Inf}(G)) \leq \circleddash(G).$$

It follows that $\mathrm{D^b}(\mathrm{coh}\, X)$ and $\mathrm{D}^{\mathrm{b}}_{\Gamma}(\mathrm{coh}\, X)$ have the same Rouquier dimension. Furthermore, for any generator, G, of $\mathrm{D^b}(\mathrm{coh}\, X)$ which is equivariant under the action of Γ, we have $\langle G \rangle_0 = \langle \mathrm{For}(\mathrm{Inf}(G)) \rangle_0$ hence $\circleddash(\mathrm{For}(\mathrm{Inf}(G))) = \circleddash(G)$ and thus $\circleddash(G) = \circleddash(\mathrm{Inf}(G))$.

Lemma 2.9 *If $\mathcal{T}$ is a triangulated category with finite Rouquier dimension, then any generator is a strong generator.*

For a proof, see Lemma 2.6 of [5].

The generation time of an object can be reinterpreted in terms of so called "ghost maps"; this reinterpretation turns out to be quite useful both for intuition about generation time and as a means of calculation.

Definition 2.10 Let $\mathcal{T}$ be a triangulated category, $f : X \to Y$ be a morphism, and $\mathcal{I}$ be a full subcategory. We say that $f : X \to Y$ is $\mathcal{I}$ *ghost* if, for all $I \in \mathcal{I}$, the induced map, $\mathrm{Hom}_{\mathcal{T}}(I, X) \to \mathrm{Hom}_{\mathcal{T}}(I, Y)$, is zero. We say that f is $\mathcal{I}$ *co-ghost* if, for all $I \in \mathcal{I}$, the induced map, $\mathrm{Hom}_{\mathcal{T}}(Y, I) \to \mathrm{Hom}_{\mathcal{T}}(X, I)$, is zero. If G is an object of $\mathcal{T}$, we will say that f is G ghost if f is $\langle G\rangle_0$ ghost and f is G co-ghost if f is $\langle G\rangle_0$ co-ghost.

Remark 2.11 Recall that an ideal, $\mathcal{J}$, in an additive category, $\mathcal{C}$, is a subset,

$$\mathcal{J}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

for each $X, Y \in \mathcal{C}$ such that: for any $g : Y \to Z$, $h : W \to X$, and any $f, f' : X \to Y$ in $\mathcal{J}(X, Y)$, $h \circ f \in \mathcal{J}(X, Z)$, $f \circ g \in \mathcal{J}(W, Y)$, and $f + f' \in \mathcal{J}(X, Y)$. Note that $\mathcal{I}$ ghosts and $\mathcal{I}$ co-ghosts both form ideals in $\mathcal{T}$.

The following lemmas relate generation time to ghost maps and are a crucial ingredient in our study of Orlov spectra. Lemma 2.12 first appeared in [28] and later appeared in many places, for example see [16, 47].

Lemma 2.12 *Let $\mathcal{T}$ be a triangulated category and let G be an object of $\mathcal{T}$. If there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \le i \le t$, in $\mathcal{T}$ where each f_i is G ghost and $f_t \circ \cdots \circ f_1 \neq 0$, then $X_0 \notin \langle G\rangle_{t-1}$.*

Proof Let us show that $f_t \circ \cdots \circ f_1$ is ghost for $\langle G\rangle_{t-1}$. For simplicity, set $f^t := f_t \circ \cdots \circ f_1$. We proceed by induction with the case, $t = 1$, clear. Assume we know f^t is $\langle G\rangle_{t-1}$ ghost for $t \le n-1$, and let us consider the case $t = n$. From the induction hypothesis, f^{n-1} is $\langle G\rangle_{n-2}$ ghost. Let Y be an object of $\mathcal{T}$ lying in a triangle

$$Z \xrightarrow{\alpha} Y \xrightarrow{\beta} Y_G \to Z[1]$$

with $Z \in \langle G\rangle_{n-2}$ and $Y_G \in \langle G\rangle_0$. Take any map $g : Y \to X_0$. As f^{n-1} is $\langle G\rangle_{n-2}$ ghost, the composition $f^{n-1} \circ g \circ \alpha$ vanishes. Thus, we have a map $h : Y_G \to X_{n-1}$ with $f^{n-1} \circ g = h \circ \beta$. As f_n is $\langle G\rangle_0$ ghost, $f_n \circ h \circ \beta = f^n \circ g$ vanishes. Thus, f^n is $\langle G\rangle_{n-2} * \langle G\rangle_0$ ghost. It is clear this implies that f^n is $\langle G\rangle_{n-1}$ ghost.

To finish the proof the lemma, note that, if X_0 lies in $\langle G\rangle_{t-1}$, then $f^t \circ \mathrm{id}_{X_0} = f^t$ vanishes. □

We also have the dual statement whose proof is the same.

Lemma 2.13 *Let $\mathcal{T}$ be a triangulated category and let G be an object of $\mathcal{T}$. If there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \le i \le t$, in $\mathcal{T}$ where each f_i is G co-ghost and $f_t \circ \cdots \circ f_1 \neq 0$, then $X_t \notin \langle G\rangle_{t-1}$.*

The following partial converse seems well-known, see [7].

Lemma 2.14 *Let $\mathcal{T}$ be a triangulated category and let G be an object of $\mathcal{T}$. Assume that for any object, X, of $\mathcal{T}$ there exists a morphism, $\nu_X : X_G \to X$, with $X_G \in \langle G\rangle_0$ and satisfying the following condition: for any morphism, $g : Y \to X$, with $Y \in \langle G\rangle_0$, there exists a morphism, $h : Y \to X_G$, with $g = \nu_X \circ h$. If $X \notin \langle G\rangle_{t-1}$, then there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \le i \le t$, in $\mathcal{T}$ where each f_i is G ghost, $X_0 = X$ and $f_t \circ \cdots \circ f_1 \neq 0$.*

Proof Complete $\nu_X : X_G \to X$ to a distinguished triangle

$$X_G \overset{\nu_X}{\to} X \overset{f_1}{\to} X_1 \to X_G[1].$$

f_1 is G ghost. Now iterate to get triangles

$$(X_i)_G \overset{\nu_{X_i}}{\to} X_i \overset{f_{i+1}}{\to} X_{i+1} \to (X_i)_G[1].$$

Let $f^t := f_t \circ \cdots \circ f_1$. If the composition, f^n, vanishes, then $X_0[1] \oplus X_n \cong C(f^n)$, where $C(f^t)$ denotes the cone over f^t. Once we know that $C(f^n)$ lies in $\langle G\rangle_{n-1}$, then we can conclude the proof.

Let us show that $C(f^t)$ lies in $\langle G\rangle_{t-1}$ by induction on t. The case $t = 1$ is clear. Assume the statement holds for $t \le m-1$ and consider the case $t = m$. We use the octrahedral axiom for the composition, $f^m = f_m \circ f^{m-1}$, to get a triangle

$$C(f^{m-1}) \to C(f^m) \to C(f_m) \to C(f^{m-1})[1].$$

$C(f_m) \in \langle G\rangle_0$, by construction, and $C(f^{m-1}) \in \langle G\rangle_{m-2}$, from the induction hypothesis, so $C(f^m) \in \langle G\rangle_{m-1}$. □

We also have the dual statement whose proof is the same.

Lemma 2.15 *Let $\mathcal{T}$ be a triangulated category and let G be an object of $\mathcal{T}$. Assume that for any object of X of $\mathcal{T}$ there exists a morphism, $\nu_X : X \to X_G$, with $X_G \in \langle G\rangle_0$ and satisfying the following condition: for any morphism,*

$g : X \to Y$, with $Y \in \langle G\rangle_0$, there exists a morphism, $h : X_G \to Y$, with $g = h \circ \nu_X$. If $X \notin \langle G\rangle_{t-1}$, then there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \le i \le t$, in $\mathscr{T}$ where each f_i is G co-ghost and $X_t = X$.

Remark 2.16 We can replace G by a general subcategory, $\mathscr{I}$, in each of these statements. However, we should note that it is necessary to assume the existence of an "$\mathscr{I}$-approximation" similar to the hypotheses of Lemmas 2.14 and 2.15. If X is a projective variety, then there are no $\operatorname{Perf} X$ ghosts in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, see [4], and $\operatorname{Perf} X$ is not dense in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, for a general X.

Recall that a triangulated category, $\mathscr{T}$, is Ext-finite, if for any pair of objects, A and B, of $\mathscr{T}$, we have

$$\dim_k\left(\bigoplus_{l\in\mathbb{Z}} \operatorname{Hom}_{\mathscr{T}}(A, B[l])\right) < \infty.$$

Combining the previous observations, we get the following corollary, which cannot be called anything other than a lemma:

Lemma 2.17 (Ghost/Co-ghost Lemma and Converse) *Let $\mathscr{T}$ be a k-linear Ext-finite triangulated category and let G and X_0 be objects in $\mathscr{T}$. The following are equivalent*:

(i) *$X_0 \in \langle G\rangle_n$ and $X_0 \notin \langle G\rangle_{n-1}$;*
(ii) *there exists a sequence,*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n,$$

of maps in $\mathscr{T}$ such that all the f_i are ghost for G and $f_n \circ \cdots \circ f_1 \neq 0$. Furthermore there is no such sequence for $n+1$.
(iii) *there exists a sequence,*

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_{n-1} \xrightarrow{f_1} X_0,$$

of maps in $\mathscr{T}$ such that all the f_i are co-ghost for G and $f_1 \circ \cdots \circ f_n \neq 0$. Furthermore there is no such sequence for $n+1$.
(iv) *there exists a sequence,*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_{n-1} \xrightarrow{f_n} X_n,$$

of maps in $\mathscr{T}$ with indecomposable objects, $X_i \in \mathscr{T}$, such that all the f_i are ghost for G and $f_n \circ \cdots \circ f_1 \neq 0$. Furthermore, there is no such sequence for $n+1$.

(v) *there exists a sequence*,

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} X_{n-1} \xrightarrow{f_1} X_0,$$

of maps in $\mathscr{T}$ *with indecomposable objects*, $X_i \in \mathscr{T}$, *such that all the* f_i *are co-ghost for* G *and* $f_1 \circ \cdots \circ f_n \neq 0$. *Furthermore*, *there is no such sequence for* $n+1$.

Proof $\mathscr{T}$ satisfies the hypothesis of Lemma 2.14. Let X be an object of $\mathscr{T}$. We set $X_G = \bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{T}}(G, X[l]) \otimes_k G[-l]$ and let $\nu_X : X_G \to X$ be the evaluation map. Similarly, $\mathscr{T}$ satisfies the hypothesis of Lemma 2.15. The equivalence of (i), (ii), (iii) is a combination of Lemmas 2.12, 2.13, 2.14, and 2.15. The only difference between (ii) and (iv) is that the objects are assumed to be indecomposable; their equivalence is clear. The same goes for (iii) and (v). □

We have an important special case. Recall that a hereditary abelian category is one where $\mathrm{Ext}^2(A, B) = 0$ for any two objects, A and B.

Lemma 2.18 *Let* $\mathscr{C}$ *be a hereditary abelian category with finite dimensional morphism spaces and let* G *be an object of* $\mathrm{D^b}(\mathscr{C})$ *and* X_0 *be an object of* $\mathscr{C}$. *The following are equivalent*:

(i) $X_0 \in \langle G\rangle_n$ *and* $X_0 \notin \langle G\rangle_{n-1}$;

(ii) n *is the largest integer such that there exists a sequence*,

$$X_0 \xrightarrow{g_1} \cdots \xrightarrow{g_s} X_s \xrightarrow{h_1} Y_1[1] \xrightarrow{h_2} \cdots \xrightarrow{h_t} Y_t[1],$$

of maps in $\mathrm{D^b}(\mathscr{C})$ *with* X_i *and* Y_i *indecomposable objects of* $\mathscr{C}$, $s+t=n$, *and such that all the* f_i *and* g_i *are ghost for* G *and* $h_t \circ \cdots \circ g_1 \neq 0$.

(iii) n *is the largest integer such that there exists a sequence*,

$$Y_t[-1] \xrightarrow{h_t} \cdots \xrightarrow{h_2} Y_0 \xrightarrow{g_s} X_s \xrightarrow{g_{s-1}} \cdots \xrightarrow{g_1} X_0,$$

of maps in $\mathrm{D^b}(\mathscr{C})$ *with* X_i *and* Y_i *indecomposable objects of* $\mathscr{C}$, $s+t=n$, *and such that all the* f_i *and* g_i *are co-ghost for* G *and* $g_1 \circ \cdots \circ h_t \neq 0$. *Furthermore*, *there is no such sequence for* $n+1$.

Recall that for a finite dimensional algebra, A, with nilradical, N, the *Loewy length*, denoted $\mathrm{LL}(A)$, is smallest n such that $N^n = 0$.

Corollary 2.19 *Suppose* $\mathscr{C}$ *is a* k*-linear hereditary abelian category with finite dimensional morphism spaces and finitely many isomorphism classes of indecomposable objects*. *Let* M_i *be chosen representatives of the isomorphism classes*. *Then*, $\mathrm{udim}\,\mathscr{T} \leq \mathrm{LL}(\mathbf{R}\mathrm{End}(\oplus M_i)) - 1$.

There is an important relationship between ghost maps and Serre functors. Let us recall the definition of a Serre functor, due to Bondal and M. Kapranov, [10]:

Definition 2.20 A k-linear exact autoequivalence, S, of $\mathscr{T}$, is called a *Serre functor* if for any pair of objects, X and Y, of $\mathscr{T}$, there exists an isomorphism of vector spaces,

$$\operatorname{Hom}_{\mathscr{T}}(Y, X)^{\vee} \cong \operatorname{Hom}_{\mathscr{T}}(X, S(Y)),$$

which is natural in X and Y.

A Serre functor, if it exists, is determined uniquely up to natural isomorphism. If $F : \mathscr{T} \to \mathscr{S}$ is an exact equivalence of triangulated categories possessing Serre functors, then F commutes with those Serre functors, loc. cit. Now, recall that a category is called *Karoubi closed* if all idempotents split. Suppose $\mathscr{T}$ is a k-linear Karoubi closed triangulated category with finite dimensional morphism spaces which admits a Serre functor, S. Let X be an indecomposable object of $\mathscr{T}$. In this situation, there is a natural map, $\epsilon_X : X \to S(X)$, corresponding to,

$$\operatorname{Hom}_{\mathscr{T}}(X, X) \to \operatorname{Hom}_{\mathscr{T}}(X, X)/\operatorname{Rad}_{\mathscr{T}}(X, X) \cong k,$$

where the isomorphism with the base field identifies the image of the identity with 1. By definition of a Serre functor, there is also a nondegenerate pairing,

$$\operatorname{Hom}_{\mathscr{T}}(A, B) \otimes_k \operatorname{Hom}_{\mathscr{T}}(B, S(A)) \to k.$$

Hence any nonzero morphism, $X \to A$, can be extended to a nonzero morphism, $X \to A \to S(X)$; the total morphism in this situation can be taken to be the natural map described above, see [46].

Proposition 2.21 *Let $\mathscr{T}$ be a k-linear triangulated Karoubi closed category with finite-dimensional morphism spaces. Assume $\mathscr{T}$ possesses a Serre functor, S. Let X be an indecomposable object in $\mathscr{T}$ and $f : X \to Y$ a morphism. There exists a morphism, $g : Y \to S(X)$, so that $g \circ f = \epsilon_X$.*

Given any nonzero ghost sequence, $X \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n$ with $f_n \circ \cdots \circ f_1 \neq 0$, we can extend it to a new sequence, $X \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n \xrightarrow{g} S(X)$, with $g \circ f_n \circ \cdots \circ f_1 = \epsilon_X$ and where only g is possibly non-ghost. Concatenating f_n with g, we get a ghost sequence of equal length beginning at X and terminating at $S(X)$.

Now, for any map, $G \to X$, consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(X, S(X)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(G, S(X)) \arrow[d, "\cong"] \\ \mathrm{Hom}(X, X)^{\vee} \arrow[r] & \mathrm{Hom}(X, G)^{\vee}. \end{tikzcd}$$

By duality, requiring that the image of ϵ_X is nonzero in $\mathrm{Hom}(X, G)^{\vee}$ is equivalent to requiring that $\mathrm{Hom}(X, G) \to \mathrm{Hom}(X, X)$ does not lie in $\mathrm{Rad}(X, X)$. Hence, $G \to X$ has a section. Meaning that if $G \to X \xrightarrow{\epsilon_X} S(X)$ is nonzero than X is a summand of G. One can similarly show that, for any map, $S(X) \to G$, if the composition, $X \to S(X) \to G$, is nonzero, then G is a summand of $S(X)$. Therefore, ϵ_X composed with any map besides a sequence of split epimorphisms and/or monomorphisms is zero. In other words, the natural map, ϵ_X, is G ghost and G co-ghost for any object, G, of which X is not a summand. Hence, given a ghost sequence whose total map is ϵ_X, it can not be extended any further (although it could be perhaps factored into more maps).

Ghost maps often have geometric origins. We collect some examples here.

Example 2.22 (Central actions as ghosts) Let $\mathcal{T}$ be a triangulated category. The center of $\mathcal{T}$, denoted $Z(\mathcal{T})$, is the space of natural transformations from $\mathrm{Id}_{\mathcal{T}}$ to $\mathrm{Id}_{\mathcal{T}}$. Let x be an element of $Z(\mathcal{T})$. If G is an object of $\mathcal{T}$ with $x(G) = 0$, then we say that x annihilates G. For any object, $A \in \mathcal{T}$, and any morphism, $\alpha : G \to A[i]$, we have the following commutative diagram:

$$\begin{tikzcd} G \arrow[r, "x(G)"] \arrow[d, "\alpha"'] & G \arrow[d, "\alpha"'] \\ A[i] \arrow[r, "x(A)"] & A[i]. \end{tikzcd}$$

If $x(G) = 0$, then $x(A) \circ \alpha = 0$ for all $\alpha \in \mathrm{Hom}_{\mathcal{T}}(G, A[i])$. In other words, $x(A)$ is G ghost for any object $A \in \mathcal{T}$. Similarly, $x(A)$ is G co-ghost. If $\mathcal{T} = \mathrm{D^b}(\mathrm{coh}\, X)$ for a quasi-projective variety, X, then $Z(\mathrm{D^b}(\mathrm{coh}\, X)) \cong \Gamma(X, \mathcal{O}_X)$, [48].

Example 2.23 (Divisors and ghosts) The choice of a divisor, $i : D \to X$, and a section, s, of $\mathcal{O}(D)$, gives a natural transformation, $\alpha : \mathrm{Id}_{\mathrm{D^b}(\mathrm{coh}\, X)} \to (- \otimes_{\mathcal{O}} \mathcal{O}(D))$. Let H be in the essential image of the functor $i_* : \mathrm{D^b}(\mathrm{coh}\, D) \to \mathrm{D^b}(\mathrm{coh}\, X)$. Then, for any object, $A \in \mathrm{D^b}(\mathrm{coh}\, X)$, $\alpha(A)$ is H ghost by adjunction.

Example 2.24 (Tangent vectors as ghosts) Let X be a variety of dimension n. Let G be any object of $\mathrm{D^b}(\mathrm{coh}\, X)$ and consider a smooth point, p, at which the cohomology sheaves of G are locally free. It is easily verified that any tangent vector, $\zeta \in \mathrm{Hom}(\mathcal{O}_p, \mathcal{O}_p[1]) \cong T_pX$, is G ghost. Now take a basis for the tangent space, $\zeta_1, \ldots, \zeta_n$. The composition is a nonzero ghost sequence:

$$\mathcal{O}_p \xrightarrow{\zeta_1} \mathcal{O}_p[1] \to \cdots \to \mathcal{O}_p[n-1] \xrightarrow{\zeta_n[n-1]} \mathcal{O}_p[n].$$

It follows from the Ghost Lemma, Lemma 2.17, that $n \leq \circleddash(G)$. Hence, $n \leq \operatorname{rdim} X$. This proof is due to Rouquier and can be found in [47].

Example 2.25 (Cycles and levels) We can extend the previous example a bit more. Let $i : V \to X$ be a smooth subvariety of X. By adjunction, the pushforward of any $\mathbf{L}i^*G$ ghost is G ghost. For any point, $p \in V$, take any $\mathbf{L}i^*G$ ghost sequence, $\mathcal{O}_p \to A_1 \to \cdots \to A_n$. By nondegeneracy of the Serre pairing (as mentioned above) we may assume $A_n = \mathcal{O}_p[\dim V]$. Consider the total composition, $f : \mathcal{O}_p \to \mathcal{O}_p[\dim V]$. The pushforward $i_* f$ is a nonzero element of the top exterior power of T_pV under the isomorphism,

$$\mathrm{Hom}_X(\mathcal{O}_p, \mathcal{O}_p[\dim V]) \cong \Lambda^{\dim V} T_pX.$$

Now take a collection of smooth subvarieties, $V_1, \ldots, V_s$, intersecting transversally at a point, $p \in X$. Denote the inclusion maps by $i_j : V_j \to X$. Let G be a generator of $\mathrm{D^b}(\operatorname{coh} X)$. By the Ghost Lemma, Lemma 2.17, for each V_j we can construct a ghost sequence for $\mathcal{O}_p$ whose length is the level of $\mathcal{O}_p$ with respect to $\mathbf{L}i_j^*G$. As noted above, we may assume this ghost sequence terminates at $\mathcal{O}_p[\dim V_j]$. Denote the total composition by $f_j : \mathcal{O}_p \to \mathcal{O}_p[\dim V_j]$. The pushforward, $i_{j*}f_j$, is a nonzero element of $\Lambda^{\dim V_j} T_pV_j \subset \Lambda^{\dim V_j} T_pX$. We may then construct a ghost sequence:

$$\mathcal{O}_p \xrightarrow{i_{1*}f_1} \mathcal{O}_p[\dim V_1] \to \cdots \to \mathcal{O}_p[n - \dim V_s] \xrightarrow{i_{s*}f_s[n-\dim V_s]} \mathcal{O}_p[n].$$

Each of the G ghosts, $i_{j*}f_j$, factors into $\mathrm{Lvl}_{V_j}^{\mathbf{L}i_j{}^*G}(\mathcal{O}_p)$ additional G ghosts. Hence we have:

$$\sum_{j=1}^{s} \mathrm{Lvl}_{V_j}^{\mathbf{L}i_j{}^*G}(\mathcal{O}_p) \leq \mathrm{Lvl}_X^G(\mathcal{O}_p).$$

Let us use this example to give a simple proof that the ultimate dimension of $\mathbb{P}^n$ is at least $2n$.

Proposition 2.26 $\operatorname{udim} \mathbb{P}^n \geq 2n$.

Proof We work by induction. Let $G_n = \mathcal{O} \oplus \mathcal{O}_{H_1} \oplus \cdots \oplus \mathcal{O}_{H_{n-1}} \oplus \mathcal{O}_p$ where H_i is a linear subspace of $\mathbb{P}^n$ of codimension i. The induction hypothesis is that, for any point, $q \in \mathbb{P}^n$, not lying in any H_i, the level of $\mathcal{O}_q$ is at least $2n$. Let us tackle the case of $\mathbb{P}^1$ first. Let q be a point distinct from p. The sequence

$$\mathcal{O}_q \to \mathcal{O}(-1)[1] \to \mathcal{O}_q[1]$$

is a ghost sequence for $\mathcal{O} \oplus \mathcal{O}_p$. Hence, $\mathcal{O}_q \notin \langle \mathcal{O} \oplus \mathcal{O}_p \rangle_1$ implying that,

$$\mathrm{Lvl}_{\mathcal{O}\oplus\mathcal{O}_p}(\mathcal{O}_q) \geq 2.$$

Now assume we know the result for $\mathbb{P}^j$ when $j \leq n-1$, and let us work on the case $j = n$. Take any point, q, not lying on each H_i so that G_n is free near q. Take a hyperplane, H, passing through q and intersecting each H_i transversally and a line, L, passing through q and intersecting each H_i and H transversally. Restricting G_n to H gives an element of $\langle G_{n-1} \rangle_0$ and restricting to L gives an element of $\langle G_1 \rangle_0$. By Example 2.25, the level of $\mathcal{O}_q$ is at least $2n$. □

Remark 2.27 A more careful analysis reveals that

$$\{n, n+1, \ldots, 2n-1, 2n\} \subset \mathrm{OSpec}\, \mathbb{P}^n.$$

We suspect this is in fact an equality. However, this is only known in the case $n = 1$.

When $\mathcal{T}$ is Ext-finite, we have a (weakly) universal G ghost from any object, $A \in \mathcal{T}$: we take the cone over the natural evaluation map

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G[-i], A) \otimes_k G[-i] \xrightarrow{\mathrm{ev}_A} A.$$

Denote, for the moment, the cone by $L_G(A)$. For a general $\mathcal{T}$ and G, the assignment, $A \to L_G(A)$, cannot necessarily be promoted to an endofunctor. To guarantee good behavior of L_G, we can assume that $\mathcal{T}$ is the homotopy category of a triangulated A_∞-category, $\mathcal{A}$, see Chap. 1, Sect. 3 of [50]. In this case, we have a cone construction on $\mathcal{A}$ which enhances the assignment, L_G, and guarantees functoriality. We record the definition of L_G and R_G for subsequent use.

Definition 2.28 Let $\mathcal{T}$ be an Ext-finite triangulated category that is the homotopy category of a triangulated A_∞-category, $\mathcal{A}$. For any pairs of objects, G and A, of $\mathcal{A}$, we have a natural evaluation map

$$\mathrm{Hom}_{\mathcal{A}}(G, A) \otimes_k G \xrightarrow{\mathrm{ev}_A} A.$$

Define $L_G : \mathscr{A} \to \mathscr{A}$ as the A_∞-endofunctor which takes A to the cone over ev_A. We also use the notation, $L_G : \mathscr{T} \to \mathscr{T}$, for the induced exact functor on $\mathscr{T}$, called the *left twist* by G. There is an natural transformation, $\lambda : \mathrm{Id}_{\mathscr{A}} \to L_G$, which descends to a natural transformation, $\lambda : \mathrm{Id}_{\mathscr{T}} \to L_G$. We have an exact triangle in $\mathscr{T}$, where the slashed arrow denotes a degree one morphism:

$$\begin{tikzcd}
A \arrow[rr, "\lambda(A)"] & & L_G(A) \arrow[dl, "|" marking] \\
& \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{T}}(G, A[i]) \otimes_k G[-i] \arrow[ul, "\mathrm{ev}_A"] &
\end{tikzcd}$$

Definition 2.29 Let $\mathscr{T}$ be an Ext-finite triangulated category that is the homotopy category of a triangulated A_∞-category, $\mathscr{A}$. For any pairs of objects, G and A, of $\mathscr{A}$, we have a natural co-evaluation map

$$A \stackrel{\mathrm{coev}_A}{\to} \mathrm{Hom}_{\mathscr{A}}(A, G)^\vee \otimes_k G.$$

Define $R_G : \mathscr{A} \to \mathscr{A}$ as the A_∞-endofunctor which takes A to the cone over $\mathrm{coev}_A[-1]$. We also use the notation, $R_G : \mathscr{T} \to \mathscr{T}$, for the induced exact functor on $\mathscr{T}$, called the *right twist* by G. There is an natural transformation, $\rho : R_G \to \mathrm{Id}_{\mathscr{A}}$, which descends to a natural transformation, $\rho : R_G \to \mathrm{Id}_{\mathscr{T}}$. We have an exact triangle in $\mathscr{T}$:

$$\begin{tikzcd}
R_G(A) \arrow[rr, "\rho(A)"] & & A \arrow[dl, "\mathrm{coev}_A"] \\
& \mathrm{Hom}_{\mathscr{T}}(A, G[i])^\vee \otimes_k G[-i] \arrow[ul, "|" marking] &
\end{tikzcd}$$

Example 2.30 In [52], Seidel and Thomas show that for a spherical object (see Definition 6.1) the associated left twist functor is an autoequivalence. For the derived Fukaya category of a symplectic manifold, the left twist functor along a Lagrangian sphere is precisely the autoequivalence given by taking a Dehn twist along this sphere. Seidel and Thomas also show that certain configurations of spherical objects induce the action of a braid group on the category. While one twist along a Lagrangian sphere provides a single ghost map, we will see in Sect. 6 that words in the braid group induce ghost sequences.

Example 2.31 (Global monodromy of the quintic as a ghost map) Let X be a quintic hypersurface in $\mathbb{P}^4$, Y be the family of Calabi Yau manifolds that is mirror to X according to Batyrev's construction. A loop around infinity, in the base $\mathbb{P}^1\backslash\{0, 1, \infty\}$, induces a categorical monodromy, $\{1\}$. This monodromy is a composition of autoequivalences, $\{1\} = L_{\mathscr{O}} \circ (- \otimes_{\mathscr{O}} \mathscr{O}(1))$.

If we choose a hyperplane section, H, we get a natural transformation, $\zeta_H : \mathrm{Id}_{\mathrm{D^b}(\operatorname{coh} X)} \to \{1\}$. For any object, A, the map, $\zeta_H(A)$, is ghost for $\mathscr{O}$ and for the essential image of $\mathrm{D^b}(\operatorname{coh} H)$ under inclusion. If we take any generator, N, of $\mathrm{D^b}(\operatorname{coh} H)$, then $\mathscr{O} \oplus N$ generates $\mathrm{D^b}(\operatorname{coh} X)$ and $\zeta_H(A)$ is $\mathscr{O} \oplus N$ ghost (see Examples 2.30 and 2.23).

In Sect. 5, we will see that $\{1\}$ is precisely the autoequivalence corresponding to twisting the grading in the associated category of graded singularities.

3 Semi-orthogonal decompositions, exceptional collections and birational geometry

3.1 Semi-orthogonal decompositions

Let $\mathscr{T}$ be a triangulated category and $\mathscr{I}$ a full subcategory. Recall that the left orthogonal, ${}^{\perp}\mathscr{I}$, is the full subcategory $\mathscr{T}$ consisting of all objects, $T \in \mathscr{T}$, with $\mathrm{Hom}_{\mathscr{T}}(T, I) = 0$ for any $I \in \mathscr{I}$. The right orthogonal, $\mathscr{I}^{\perp}$, is defined similarly.

Definition 3.1 A *semi-orthogonal decomposition* of a triangulated category, $\mathscr{T}$, is a sequence of full triangulated subcategories, $\mathscr{A}_1, \ldots, \mathscr{A}_m$, in $\mathscr{T}$ such that $\mathscr{A}_i \subset \mathscr{A}_j^{\perp}$ for $i < j$ and, for every object $T \in \mathscr{T}$, there exists a diagram:

$$\begin{tikzcd}
0 \arrow[rr] & & T_{m-1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & T_2 \arrow[rr] & & T_1 \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
& A_m \arrow[ul, "\times" marking] & & & & A_2 \arrow[ul, "\times" marking] & & A_1 \arrow[ul, "\times" marking] &
\end{tikzcd}$$

where all triangles are distinguished and $A_k \in \mathscr{A}_k$. We shall denote a semi-orthogonal decomposition by $\langle \mathscr{A}_1, \ldots, \mathscr{A}_m \rangle$.

A case of particular importance is if each $\mathscr{A}_i$ is equivalent to $\mathrm{D^b}(\operatorname{mod} k)$ as a triangulated category. Let A_i denote the object in $\mathscr{T}$ corresponding to k in $\mathscr{A}_i$. In this case, we call $A_1, \ldots, A_m$ an *exceptional collection*. If, in addition, $\mathrm{Hom}_{\mathscr{T}}(A_i, A_j[l]) = 0$ for $l \neq 0$, we say that the exceptional collection, $A_1, \ldots, A_n$, is *strong*.

As a warning to the reader. The notion of exceptional collection which appears here is often called a full exceptional collection in the literature. The distinction is that our exceptional collections always generate the triangulated category in question.

Remark 3.2 While not required in the definition, it is easy to see that T uniquely determines the diagram appearing in Definition 3.1.

The following lemma is clear from the definition of a semi-orthogonal decomposition:

Lemma 3.3 *Suppose $\langle \mathscr{A}_1, \ldots, \mathscr{A}_m \rangle$ is a semi-orthogonal decomposition of $\mathscr{T}$ and, for each i, G_i is a strong generator of $\mathscr{A}_i$. Then, $\bigoplus_{i=1}^m G_i$ is a strong generator of $\mathscr{T}$.*

In this section we will analyze how the generation time behaves when we have generators coming from semi-orthogonal decompositions.

Due to work of Bondal, Kuznetsov, and Orlov, it is widely believed that semi-orthogonal decompositions could play an important role in birational geometry. We have the following result due to Orlov, see [39]:

Theorem 3.4 *Let $\pi : \hat{X} \to X$ be the blow up of a smooth variety, X, along a smooth subvariety, Y, of codimension c. Let E denote the exceptional divisor on $\hat{X}$ and $\mathscr{O}_E(1)$ denote the relative twisting sheaf of $\pi|_E : E \to Y$. Denote the inclusion as $j : E \to \hat{X}$. There is a semi-orthogonal decomposition of* $\mathrm{D^b}(\operatorname{coh} \hat{X})$ *given by*

$$\big\langle \mathrm{D^b}(\operatorname{coh} Y), \ldots, \mathrm{D^b}(\operatorname{coh} Y), \mathrm{D^b}(\operatorname{coh} X) \big\rangle.$$

In this decomposition, the category $\mathrm{D^b}(\operatorname{coh} Y)$ *occurs $c-1$ times under the following equivalences for $-c+1 \leq l \leq -1$:*

$$\mathrm{D^b}(\operatorname{coh} Y) \cong j_*((\pi|_E)^* \mathrm{D^b}(\operatorname{coh} Y) \otimes_{\mathscr{O}} \mathscr{O}_E(l)),$$

and the category $\mathrm{D^b}(\operatorname{coh} X)$ *is equivalent to* $\mathbf{L}\pi^* \mathrm{D^b}(\operatorname{coh} X)$.

Based on the above theorem, and further work of his own, Kuznetsov has proposed the existence of a categorical analogue to the Clemens-Griffiths component of the intermediate Jacobian, [32]. Roughly, this is the component of a semi-orthogonal decomposition which is not equivalent to a component of the derived category of a variety of smaller dimension. In what follows, we hope to suggest that the Orlov spectrum can detect, in some cases, when Kuznetsov's Clemens-Griffiths component is nontrivial.

Definition 3.5 Let $\alpha : \mathscr{A} \to \mathscr{T}$ be the inclusion of a full triangulated subcategory of $\mathscr{T}$. The subcategory, $\mathscr{A}$, is called *right admissible* if the inclusion functor, α, has a right adjoint, $\alpha^!$, and *left admissible* if it has a left adjoint, α^*. A full triangulated subcategory is called *admissible* if it is both right and left admissible.

The proofs of the following lemmas can be found in [10]:

Lemma 3.6 *Let $\mathscr{A}$ be a full triangulated subcategory of a triangulated category, $\mathscr{T}$, with Serre functor. Then, the following are equivalent:*

(i) *$\mathscr{A}$ is left admissible*
(ii) *$\mathscr{A}$ is right admissible*
(iii) *$\mathscr{A}$ is admissible*

Lemma 3.7 *If $\langle \mathscr{A}_1, \ldots, \mathscr{A}_m \rangle$ is a semi-orthogonal decomposition of a triangulated category, $\mathscr{T}$, with Serre functor, then $\mathscr{A}_i$ is admissible for all i. Furthermore, if $\mathscr{T} = \langle \mathscr{A}, \mathscr{B} \rangle$ is a semi-orthogonal decomposition, then $\mathscr{B} = {}^{\perp}\mathscr{A}$.*

Let $\langle \mathscr{A}_1, \ldots, \mathscr{A}_m \rangle = \mathscr{T}$ be a semi-orthogonal decomposition of a triangulated category, $\mathscr{T}$, with Serre functor. Denote each inclusion functor by $\alpha_i : \mathscr{A}_i \to \mathscr{T}$. Let $\lambda_i : {}^{\perp}\mathscr{A}_i \to \mathscr{T}$ denote the inclusion of the left orthogonal and $\rho_i : \mathscr{A}_i^{\perp} \to \mathscr{T}$ denote the inclusion of the right orthogonal. For any $X \in \mathscr{T}$ we have the following exact triangles,

$$\alpha_i \alpha_i^! X \to X \to \rho_i \rho_i^* X, \tag{3.1}$$

and

$$\lambda_i \lambda_i^! X \to X \to \alpha_i \alpha_i^* X. \tag{3.2}$$

There is an action of the braid group on m strands on the set of all m-term semi-orthogonal decompositions of $\mathscr{T}$, [10]. The standard generators are given by either taking right mutations, $\mathbb{R}_i$, or left mutations, $\mathbb{L}_i$. Let us recall now the definition,

$$\mathbb{R}_i(\mathscr{A}_\bullet)_j = \begin{cases} \mathscr{A}_j & \text{if } j \neq i-1, i \\ \mathscr{A}_i & \text{if } j = i-1 \\ {}^{\perp}\langle \mathscr{A}_1, \ldots, \mathscr{A}_{i-2}, \mathscr{A}_i \rangle \cap \langle \mathscr{A}_{i+1}, \ldots, \mathscr{A}_m \rangle^{\perp} & \text{if } j = i \end{cases}$$

$$\mathbb{L}_i(\mathscr{A}_\bullet)_j = \begin{cases} \mathscr{A}_j & \text{if } j \neq i, i+1 \\ {}^{\perp}\langle \mathscr{A}_1, \ldots, \mathscr{A}_{i-1} \rangle \cap \langle \mathscr{A}_i, \mathscr{A}_{i+2} \ldots, \mathscr{A}_m \rangle^{\perp} & \text{if } j = i \\ \mathscr{A}_i & \text{if } j = i+1. \end{cases}$$

Given a generator, $\mathscr{G} := G_1 \oplus \cdots \oplus G_m$, with each $G_i \in \mathscr{A}_i$, we can define new generators,

$$\mathbb{L}_i \mathscr{G} := G_1 \oplus \cdots \oplus G_{i-1} \oplus \rho_i \rho_i^*(G_{i+1}) \oplus G_i \oplus \cdots \oplus G_m,$$

and

$$\mathbb{R}_i \mathscr{G} := G_1 \oplus \cdots \oplus G_i \oplus \lambda_i \lambda_i^!(G_{i-1}) \oplus G_{i+1} \oplus \cdots \oplus G_m.$$

Further, let us define: $\mathfrak{L}_i := \mathbb{L}_m \cdots \mathbb{L}_i$ and $\mathfrak{R}_i := \mathbb{R}_m \cdots \mathbb{R}_i$ so that,

$$\mathfrak{L}_i \mathscr{G} = G_1 \oplus \cdots \oplus G_{i-1} \oplus \rho_i \rho_i^*(G_{i+1}) \oplus \cdots \oplus \rho_i \rho_i^*(G_m) \oplus G_i$$

and

$$\mathfrak{R}_i \mathscr{G} = G_i \oplus \lambda_i \lambda_i^!(G_1) \oplus \cdots \oplus \lambda_i \lambda_i^!(G_{i-1}) \oplus G_{i+1} \oplus \cdots \oplus G_m.$$

Finally, set $\mathfrak{L}_D := \mathfrak{L}_1 \cdots \mathfrak{L}_{n-1}$ and $\mathfrak{R}_D := \mathfrak{R}_n \cdots \mathfrak{R}_2$.

Definition 3.8 Given a semi-orthogonal decomposition, $\langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle$, of a triangulated category, $\mathscr{T}$, with Serre functor, we define the *left dual semi-orthogonal decomposition* by,

$$\langle \mathscr{A}_1^\vee, \dots, \mathscr{A}_n^\vee \rangle := \mathfrak{L}_D \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle,$$

and the *right dual semi-orthogonal decomposition* by,

$$\langle {}^\vee\mathscr{A}_1, \dots, {}^\vee\mathscr{A}_n \rangle := \mathfrak{R}_D \langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle.$$

The following proposition is clear from the definition of mutation:

Proposition 3.9 *We have the following equalities*:

$$\mathscr{A}_i^\vee = \langle \mathscr{A}_1, \dots, \mathscr{A}_i, \mathscr{A}_{i+1}, \dots, \mathscr{A}_n \rangle^\perp$$
$${}^\vee\mathscr{A}_i = {}^\perp\langle \mathscr{A}_1, \dots, \mathscr{A}_i, \mathscr{A}_{i+1}, \dots, \mathscr{A}_n \rangle.$$

Lemma 3.10 *Let $\mathscr{T}$ be a triangulated category possessing a Serre functor, S, and suppose that $\mathscr{T}$ has a semi-orthogonal decomposition, $\langle \mathscr{A}_1, \dots, \mathscr{A}_n \rangle$. We have isomorphisms for any $X \in \mathscr{A}_i$*:

$$S(X) \cong \mathfrak{L}_D^2(S_{\mathscr{A}_i} X) \cong S_{\mathscr{A}_i^{\vee\vee}} \mathfrak{L}_D^2(X)$$

and

$$S^{-1}(X) \cong \mathfrak{R}_D^2(S_{\mathscr{A}_i}^{-1} X) \cong S_{{}^{\vee\vee}\mathscr{A}_i}^{-1} \mathfrak{R}_D^2(X),$$

where $S_{\mathscr{S}}$, respectively $S_{\mathscr{S}}^{-1}$, denotes the Serre functor, respectively the inverse to the Serre functor, for a subcategory, $\mathscr{S}$, of $\mathscr{T}$.

Proof Note that the effect of the application of $\mathfrak{L}_D^2$ is to project $\mathscr{A}_i$ to $\mathscr{A}_i^{\perp\perp}$. Similarly, $\mathfrak{R}_D^2$ is the projection from $\mathscr{A}_i$ to ${}^{\perp\perp}\mathscr{A}_i$. Proposition 3.7 of [10] states that $\mathfrak{L}_D^2$ commutes with Serre functors. Similarly, $\mathfrak{R}_D^2$ commutes with inverses to the Serre functors. $\square$

Definition 3.11 Let $[a, b]$ denote the integer interval with endpoints a and b in $\mathbb{Z}$. Despite the usual notation, we do not distinguish between $a \leq b$ and $a \geq b$ i.e. $[a, b] = [b, a]$. Furthermore, our intervals only contain integers. Let I be a subset of $\mathbb{Z}$. We say that I has a *gap* of length s if, for some a, $[a, a+s+1] \cap I = \{a, a+s+1\}$. We say that a triangulated category, $\mathscr{T}$, has a *gap* of length s if OSpec $\mathscr{T}$ has a gap of length s.

Theorem 3.12 *Suppose $\langle \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle$ is a semi-orthogonal decomposition of $\mathscr{T}$ and $\mathscr{G} := G_1 \oplus \cdots \oplus G_n$ is a generator of $\mathscr{T}$ with $G_i \in \mathscr{A}_i$. Let $M := \max_i\{\circleddash_{\mathscr{A}_i}(G_i)\}$. Any gap inside $[\circleddash_{\mathscr{T}}(\mathscr{G}), \circleddash_{\mathscr{T}}(\mathfrak{L}_D(\mathscr{G}))] \cap \operatorname{OSpec} \mathscr{T}$ has length at most M. In particular, if $\circleddash_{\mathscr{A}_i}(G_i)$ equals the Rouquier dimension of $\mathscr{A}_i$ for each i, then any gap inside $[\circleddash_{\mathscr{T}}(\mathscr{G}), \circleddash_{\mathscr{T}}(\mathfrak{L}_D(\mathscr{G}))] \cap \operatorname{OSpec} \mathscr{T}$ has length at most $\max_i \operatorname{rdim} \mathscr{A}_i$. The same statement is true passing to the right dual.*

Proof Let us state the following claim: let $\langle \mathscr{C}_1, \ldots, \mathscr{C}_m \rangle$ be any semi-orthogonal decomposition of $\mathscr{T}$ and let $\mathscr{H} = H_1 \oplus \cdots \oplus H_m$ be a generator with $H_i \in \mathscr{C}_i$. The generation time of $\mathfrak{L}_i \mathscr{H}$ is at most $\max_i\{\circleddash_{\mathscr{C}_i}(H_i)\} + \circleddash_{\mathscr{T}}(\mathscr{H}) + 1$. Additionally, the generation time of $\mathfrak{R}_i \mathscr{H}$ is at most $\max_i\{\circleddash_{\mathscr{C}_i}(H_i)\} + \circleddash_{\mathscr{T}}(\mathscr{H}) + 1$.

For the moment, we assume this claim. As a consequence, the generation time of $\mathfrak{L}_i \cdots \mathfrak{L}_{n-1}\mathscr{G}$ is at most $M + \circleddash_{\mathscr{T}}(\mathfrak{L}_{i+1} \cdots \mathfrak{L}_{n-1}\mathscr{G}) + 1$. The generation times of the mutations $\mathscr{G}, \mathfrak{L}_{n-1}\mathscr{G}, \ldots, \mathfrak{L}_2 \cdots \mathfrak{L}_{n-1}\mathscr{G}, \mathfrak{L}_D\mathscr{G}$ form a list of numbers which can increase in increments of at most $M+1$ (although there is no control over increments of decrease.) By Lemma 3.10, $\mathfrak{R}_D^2 \mathfrak{L}_D^2 \mathscr{G}$ is isomorphic to $\mathscr{G}$. The generation times of the set of full mutations, i.e. those coming from applications of $\mathfrak{L}_i$ or $\mathfrak{R}_i$, from $\mathscr{G}$ to $\mathfrak{R}_D^2 \mathfrak{L}_D^2 \mathscr{G}$ provide a (possibly) larger subset, V, of OSpec $\mathscr{T}$ which start and end at $\circleddash_{\mathscr{T}}(\mathscr{G})$ and can increase by at most $M+1$. Consequently, for any $a, b \in V$, $[a, b] \cap V \subset \operatorname{OSpec} \mathscr{T}$ has gaps of size at most M. Taking $a = \circleddash_{\mathscr{T}}(\mathscr{G})$ and $b = \circleddash_{\mathscr{T}}(\mathfrak{L}_D\mathscr{G})$ provides the statement of the theorem.

To finish the proof, let us verify the claim. Let $\mathscr{C} = \langle \mathscr{C}_1, \ldots, \mathscr{C}_m \rangle$ be any semi-orthogonal decomposition of $\mathscr{T}$ and let $\mathscr{H} = H_1 \oplus \cdots \oplus H_m$ be a generator with $H_i \in \mathscr{C}_i$. Since $\rho_i^* : \mathscr{T} \to \mathscr{C}_i^{\perp}$ is essentially surjective we have,

$$\circleddash_{\mathscr{C}_i^{\perp}}(\rho_i^* \mathscr{H}) \leq \circleddash_{\mathscr{T}}(\mathscr{H}).$$

Now by definition,

$$\mathfrak{L}_i \mathscr{H} = H_1 \oplus \cdots \oplus H_{i-1} \oplus \rho_i \rho_i^*(H_{i+1}) \oplus \cdots \oplus \rho_i \rho_i^*(H_n) \oplus H_i$$

and

$$\rho_i^*(\mathscr{H}) = H_1 \oplus \cdots \oplus H_{i-1} \oplus \rho_i^*(H_{i+1}) \oplus \cdots \oplus \rho_i^*(H_n).$$

Hence $\mathfrak{L}_i \mathscr{H} = \rho_i \rho_i^*(\mathscr{H}) \oplus H_i$. Therefore, $\mathfrak{L}_i \mathscr{H}$ generates the left orthogonal of $\mathscr{C}_i$ in at most $\circleddash_{\mathscr{T}}(\mathscr{H})$-steps. Furthermore, as H_i is a summand of $\mathfrak{L}_i \mathscr{H}$, $\mathfrak{L}_i \mathscr{H}$ generates $\mathscr{C}_i$ in at most $\circleddash_{\mathscr{C}_i}(H_i)$-steps. Now triangle (3.1) tells us that $\circleddash_{\mathscr{T}}(\mathfrak{L}_i \mathscr{H}) \leq \circleddash_{\mathscr{T}}(\mathscr{H}) + \circleddash_{\mathscr{C}_i}(H_i) + 1 \leq \circleddash(\mathscr{H}) + \max_i\{\circleddash_{\mathscr{C}_i}(H_i)\} + 1$. We have learned that the generation time increases in increments of at most $\max_i\{\circleddash_{\mathscr{C}_i}(H_i)\} + 1$ after application of a single $\mathfrak{L}_i$. A similar argument shows that, after applying the mutation $\mathfrak{R}_i$, the generation time does not increase by more than $\max_i\{\circleddash_{\mathscr{C}_i}(H_i)\} + 1$. □

3.2 A conjectural aside

The "results" in this subsection are all purely conjectural. However, nothing from this section will be used for further argument.

Recall from the introduction that the following conjecture appears in [42], where it is proven for curves.

Conjecture 1 *For a smooth algebraic variety, X, the Krull dimension of X and the Rouquier dimension of* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *are equal.*

Now, in light of Theorem 3.12, let us propose our own conjecture.

Conjecture 2 *Let X be a smooth algebraic variety and* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle$ *be a semi-orthogonal decomposition. The length of any gap in* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *is bounded above by the maximal Rouquier dimension amongst the $\mathscr{A}_i$ and the maximal gap size amongst the $\mathscr{A}_i$.*

Corollary 3.13 *Suppose Conjectures 1 and 2 hold. If X is a smooth variety, then any gap of* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *has length at most the Krull dimension of X.*

Let us propose another conjecture:

Conjecture 3 *If $\mathscr{A}$ has a gap of length at least s, then so does* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$.

Corollary 3.14 *Suppose Conjectures 2 and 3 hold. Let X be a smooth algebraic variety such that there exists a semiorthogonal decomposition,*

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) = \langle \mathscr{A}_1, \ldots, \mathscr{A}_n \rangle.$$

The maximal length of any gap of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *is equal to the largest gap length amongst the $\mathscr{A}_i$.*

Corollary 3.15 *Suppose Conjectures 1, 2, and 3 hold. Let X and Y be birational smooth proper varieties of dimension n. The category,* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$, *has a gap of length n or $n-1$ if and only if* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ *has a gap of the same length i.e. the gaps of length greater than $n-2$ are a birational invariant.*

Proof We may suppose that Y is the blow-up of X along Z. By Theorem 3.4, we have a semi-orthogonal decomposition,

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y) = \langle \mathrm{D}^{\mathrm{b}}(\operatorname{coh} X), \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z), \dots, \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z)\rangle.$$

By Corollary 3.13, the length of any gap in $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z)$ is at most $n-2$. Hence by Corollary 3.14, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$ has a gap of length greater than $n-2$ if and only if $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ has a gap of length greater than $n-2$. □

Corollary 3.16 *Suppose Conjectures 1, 2, and 3 hold. If X is a rational variety of dimension n, then any gap in* $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ *has length at most $n-2$.*

Proof It is well known that $\mathbb{P}^n$ has an exceptional collection. In particular, it has a semi-orthogonal decomposition into categories of Rouquier dimension zero. By Conjecture 2, $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} \mathbb{P}^n)$ has no gaps. The statement follows from Corollary 3.15. □

In Sect. 4 we will see that the category of singularities of an A_n-singularity in even dimension has gaps. In Sect. 5, we will explore semi-orthogonal decompositions for hypersurfaces in $\mathbb{P}^n$ and their Orlov spectra.

3.3 Bounds on generation time for exceptional collections

The following proposition is an immediate consequence of the main theorem of [5].

Proposition 3.17 *Let $A_1, \dots, A_n$ be a strong exceptional collection in an Ext-finite triangulated category, $\mathcal{T}$, that possesses an enhancement. The generation time of $G_A = A_1 \oplus \cdots \oplus A_n$ is bounded above by*

$$\max\,\{i \mid \operatorname{Hom}_{\mathcal{T}}(G_A, S^{-1}(G_A)[i]) \neq 0\}.$$

In this subsection, we establish a new bound for a general exceptional collection. We require the machinery of triangulated A_∞-categories. We will recall the bare necessities and refer the reader to [50] for a deeper discussion. We also follow the (slightly nonstandard) sign, ordering, and notational conventions found in loc. cit.

Recall that for an A_∞-category, $\mathcal{A}$, the morphism spaces are graded vector spaces and we have multi-compositions. For any sequence of objects, $X_0, \dots, X_n$, $n > 0$, there is a k-linear map

$$m_n : \operatorname{Hom}_{\mathcal{A}}(X_0, X_1) \otimes_k \cdots \otimes_k \operatorname{Hom}_{\mathcal{A}}(X_{n-1}, X_n) \to \operatorname{Hom}_{\mathcal{A}}(X_0, X_n)$$

of degree $2-n$. The ordering of the morphism spaces is as in loc. cit. These maps satisfy a hierarchy of quadratic relations. The first two of which state that m_1 is a differential on each $\mathrm{Hom}_{\mathscr{A}}(X_0, X_1)$ and m_2 is a map of complexes. The A_∞-category, $\mathscr{A}$, is called *minimal* if $m_1 = 0$.

The homotopy category, $H(\mathscr{A})$, of $\mathscr{A}$ is defined by taking the same objects as $\mathscr{A}$ but taking morphisms between X_0 and X_1 to be $H^0(\mathrm{Hom}_{\mathscr{A}}(X_0, X_1), m_1)$. We also have the graded category where we take the same objects but we take morphisms to be $H^*(\mathrm{Hom}_{\mathscr{A}}(X_0, X_1), m_1)$. This is denoted by $H^*(\mathscr{A})$. If $H^*(\mathscr{A})$ has finite dimensional morphisms spaces, i.e. if one has $\dim_k \mathrm{Hom}_{H^*(\mathscr{A})}(X, Y) < \infty$ for any pair of objects, $X, Y \in \mathscr{A}$, then $\mathscr{A}$ is called *cohomologically-finite*.

We shall always assume that $\mathscr{A}$ is strictly unital meaning, for each $A \in \mathscr{A}$, there is an element, $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{A}}(A, A)$, that passes to the identity on $H(\mathscr{A})$ and satisfies the following: for any $\phi : B \to A$ and $\psi : A \to B$, we have $m_2(\phi, \mathrm{id}_A) = \phi$, $m_2(\mathrm{id}_A, \psi) = \psi$ and any multi-composition $m_n(\phi_1, \otimes \cdots \otimes \mathrm{id}_A \otimes \cdots \otimes \phi_{n-1}) = 0$ for $n \geq 3$.

A right module over $\mathscr{A}$ is an A_∞-functor from $\mathscr{A}^{\mathrm{op}}$ to the dg-category of chain complexes of k-modules. Right modules over $\mathscr{A}$ form an A_∞-category. An A_∞ category, $\mathscr{A}$, is called *triangulated*, or often *pretriangulated* [11], if its essential image, under the Yoneda embedding, in $H(\text{Mod-}\mathscr{A})$ is a triangulated category.

Given a generator, G, of $H(\mathscr{A})$, twisted complexes concretely express how any object in $\mathscr{A}$ is built from G using cones, shifts, and summands. They are a useful tool in analyzing generation time. We recall the definition now, so that it may be used in what follows.

First, we additively enlarge to create a new A_∞-category. Let $\mathscr{B}$ be an A_∞-category. Its additive enlargement is the A_∞-category, $\Sigma\mathscr{B}$, whose are objects are denoted by

$$\bigoplus_{i \in I} V_i \otimes_k Y_i,$$

with I a finite set, V_i finite-dimensional graded vector spaces, and Y_i objects of $\mathscr{B}$. The morphism space in $\Sigma\mathscr{B}$ between $C := \bigoplus_i V_i \otimes_k Y_i$ and $D := \bigoplus_i W_i \otimes_k Y_i$ is

$$\mathrm{Hom}_{\mathrm{Tw}\text{-}\mathscr{B}}(C, D) := \bigoplus_{i,j} \mathrm{Hom}_k(V_i, W_j) \otimes_k \mathrm{Hom}_{\mathscr{B}}(Y_i, Y_j)$$

with the natural associated grading. The multi-compositions in $\Sigma\mathscr{B}$ are natural linear extensions of those in $\mathscr{B}$.

A twisted complex over $\mathscr{B}$ is a pair, (C, δ_C), where C is an object of $\Sigma\mathscr{B}$ and where δ_C is an endomorphism of C in $\Sigma\mathscr{B}$ of degree one. We require that δ_C satisfies the following conditions: one, there is a finite decreasing filtration of the V_i's that is preserved under the action of δ_C and so that the map induced by δ_C on the associated graded pieces is zero, and, two, the sum

$$\sum_{i=1}^{\infty} m_r(\delta_C^{\otimes r}) = 0 \tag{3.3}$$

where m_r is the r-th composition in ΣB. Note that finiteness of the sum in (3.3) is a consequence of the first condition on δ_C. We will often suppress the δ_C from the notation of a twisted complex. Such a twisted complex was called a one-sided twisted complex in [11].

Twisted complexes over $\mathscr{B}$ form an A_∞-category, denoted by Tw-$\mathscr{B}$. The graded vector space of morphisms between two twisted complexes (C, δ_C) and (D, δ_D) with $C = \bigoplus_i V_i \otimes_k Y_i$ and $D = \bigoplus_i W_i \otimes_k Y_i$ is

$$\mathrm{Hom}_{\mathrm{Tw}\text{-}\mathscr{B}}(C, D) := \bigoplus_{i,j} \mathrm{Hom}_k(V_i, W_j) \otimes_k \mathrm{Hom}_{\mathscr{B}}(Y_i, Y_j)$$

with the natural associated grading.

If we have n twisted complexes, (C_i, δ_{C_i}), $0 \leq i \leq n$, then the n-order multi-composition on Tw-$\mathscr{B}$ is given by

$$\begin{aligned} \phi_1 \otimes \cdots \otimes \phi_n \mapsto \sum_{i_0,\ldots,i_n \geq 0} & m_{n+i_0+\cdots+i_n}(\delta_{C_0}^{\otimes i_0} \otimes \phi_1 \otimes \delta_{C_1}^{\otimes i_1} \otimes \cdots \\ & \otimes \delta_{C_{n-1}}^{\otimes i_{n-1}} \otimes \phi_n \otimes \delta_{C_n}^{\otimes i_n}). \end{aligned} \tag{3.4}$$

The multi-compositions in Tw-$\mathscr{B}$ satisfy the A_∞-relations as a result of (3.3).

We say that $A_1, \ldots, A_n$ is an *exceptional collection* in $\mathscr{A}$ if $A_1, \ldots, A_n$ is an exceptional collection in $H(\mathscr{A})$. Similarly, $A_1, \ldots, A_n$ is strong in $\mathscr{A}$ if $A_1, \ldots, A_n$ is strong in $H(\mathscr{A})$. We will say that $A_1, \ldots, A_n$ is minimal when the A_∞-endomorphism algebra, $\mathrm{Hom}_{\mathscr{A}}(\bigoplus_{i=1}^n A_i, \bigoplus_{i=1}^n A_i)$, of the A_i's is minimal. When $\mathscr{A}$ has an exceptional collection, we can provide a normalized form for objects of $\mathscr{A}$.

Definition 3.18 Let A denote the full subcategory of $\mathscr{A}$ consisting of $A_1, \ldots, A_n$ and Tw-A denote the category of twisted complexes over A. Let (C, δ_C) be a twisted complex over A and let

$$C = \bigoplus_{i=0}^{n} V_i \otimes_k A_i.$$

Consider the filtration $F^l C = \bigoplus_{i=l}^n V_i \otimes_k A_i$. We say that (C, δ_C) is *normalized* if δ_C respects the filtration and vanishes on the associated graded pieces, $F^l C / F^{l+1} C$.

Lemma 3.19 *Let $\mathscr{A}$ be a cohomologically-finite triangulated A_∞-category with $A_1, \ldots, A_n$ an exceptional collection. Every object of $\mathscr{A}$ is isomorphic to a normalized twisted complex over A in $H(\mathscr{A})$.*

Proof This is essentially Lemma 5.13 of [50]. For any object, Y of $\mathscr{A}$, we set $Y_n = Y$ and $Y_{i-1} = L_{A_i} Y_i$. As Y_0 lies in the left orthogonal to each of the A_i in $H(\mathscr{A})$, it follows that it lies in the left orthogonal to the category generated by $A_1 \oplus \cdots \oplus A_n$, which by assumption is all of $H(\mathscr{A})$. Hence, Y_0 is acyclic. Choose a basis for $\mathrm{Hom}_{H^*(\mathscr{A})}(A_i, Y_i)$ and lifts of this basis to cycles in $\mathrm{Hom}_{\mathscr{A}}(A_i, Y_i)$. Denote by V_i the span of these cycles. This provides a splitting $V_i \hookrightarrow \mathrm{Hom}_{\mathscr{A}}(A_i, Y_i) \twoheadrightarrow V_i$. Now, we work backwards to get a normalized twisted complex quasi-isomorphic to Y. Since Y_0 is trivial in $H(\mathscr{A})$, Y_1 is quasi-isomorphic to $V_1 \otimes_k A_1$. Now, Y_2 is quasi-isomorphic to the cone over the composition of morphisms,

$$V_1 \otimes_k A_1 \to \mathrm{Hom}_{\mathscr{A}}(A_1, Y_1) \otimes_k A_1 \to Y_1 \to \mathrm{Hom}_{\mathscr{A}}(A_2, Y_2) \otimes_k A_2[1] \\ \to V_2 \otimes_k A_1[1],$$

which we denote by X_2. As a cone, X_2 is a normalized twisted complex with

$$X_2 = V_1 \otimes_k A_1[1] \oplus V_2 \otimes_k A_2[1].$$

Applying induction, we see that X_i is the cone over a map from a normalized twisted complex, X_{i-1} of the form

$$X_{i-1} = \bigoplus_{l=1}^{i-1} V_l \otimes_k A_l[i-1],$$

to $V_i \otimes_k A_i[i]$. Thus, X_i is a normalized twisted complex quasi-isomorphic to Y_i. Setting $C = X_n$ gives the desired twisted complex. □

Remark 3.20 As noted in [50], the Y_i constructed in Lemma 3.19 fit into a Postnikov tower:

$$\begin{tikzcd}
Y_n \arrow[rr] & & Y_{n-1} \arrow[dl, "/" marking] \arrow[r] & \cdots \arrow[r] & Y_2 \arrow[rr] & & Y_1 \arrow[dl, "/" marking] \arrow[rr] & & Y_0 \arrow[dl, "/" marking] \\
& V_n \otimes_k A_n \arrow[ul] & & & & V_2 \otimes_k A_2 \arrow[ul] & & V_1 \otimes_k A_1 \arrow[ul] &
\end{tikzcd}$$

One can also prove Lemma 3.19 by realizing the diagonal bi-module as a normalized twisted complex over the category of bi-modules consisting of $A_i \boxtimes B_j$ and then convolving. See Proposition 3.8 of [31] for a particular example. We thank Kuznetsov for pointing this out.

As in the case of a triangulated category, there is a left dual collection to $A_1, \dots, A_n$ in $\mathscr{A}$. We set

$$B_{n+1-k} := L_{A_1} L_{A_2} \cdots L_{A_{k-1}}(A_k).$$

It is straightforward to check that $B_1, \dots, B_n$ descends to the left dual collection of $A_1, \dots, A_n$ in $H(\mathscr{A})$ as defined in Definition 3.8.

Lemma 3.21 *Let $\phi : X \to Y$ be a morphism in $H(\mathscr{A})$. Denote the following induced morphisms by*:

$$\begin{aligned}\phi_i^t : \operatorname{Hom}_{H(\mathscr{A})}&(A_i, L_{A_n} \cdots L_{A_{i+1}}(X)[t]) \\ &\to \operatorname{Hom}_{H(\mathscr{A})}(A_i, L_{A_n} \cdots L_{A_{i+1}}(Y)[t]).\end{aligned}$$

The morphism, ϕ, is co-ghost for $G_B = \bigoplus_{i=1}^n B_i$ if and only if ϕ_i^t vanishes for $1 \le i \le n$ and any $t \in \mathbb{Z}$.

Proof Take any B_{n+1-i}. From Proposition 3.9, $\operatorname{Hom}_{H(\mathscr{A})}(A_l, B_{n+1-i}[t])$ is zero for $l \neq i$ for any t. Note that, because of this orthogonality,

$$\operatorname{Hom}_{H(\mathscr{A})}(X, B_{n+1-i}[t]) \cong \operatorname{Hom}_{H(\mathscr{A})}(X_i, B_{n+1-i}[t]),$$

where $X_i = L_{A_{i+1}} \cdots L_{A_n}(X)$. Similarly, the evaluation map

$$\bigoplus_j \operatorname{Hom}_{H(\mathscr{A})}(A_i[j], X_i) \otimes_k A_i[j] \to X_i$$

induces an isomorphism,

$$\begin{aligned}&\operatorname{Hom}_{H(\mathscr{A})}(X_i, B_{n+1-i}[t]) \\ &\quad\cong \operatorname{Hom}_{H(\mathscr{A})}\left(\bigoplus_j \operatorname{Hom}_{H(\mathscr{A})}(A_i[j], X_i) \otimes_k A_i[j], B_{n+1-i}[t]\right) \\ &\quad\cong (\operatorname{Hom}_{H(\mathscr{A})}(A_i[t], X_i))^\vee.\end{aligned}$$

The same statement is true for Y. We see that the map,

$$\begin{aligned}\operatorname{Hom}_{H(\mathscr{A})}(\phi, B_{n+1-i}[t]) : &\operatorname{Hom}_{H(\mathscr{A})}(Y, B_{n+1-i}[t]) \\ &\to \operatorname{Hom}_{H(\mathscr{A})}(X, B_{n+1-i}[t]),\end{aligned}$$

coincides with the map,

$$\begin{aligned}(\phi_i^{-t})^\vee : &\operatorname{Hom}_{H(\mathscr{A})}(A_i[t], L_{A_n} \cdots L_{A_{i+1}}(Y))^\vee \\ &\to \operatorname{Hom}_{H(\mathscr{A})}(A_i[t], L_{A_n} \cdots L_{A_{i+1}}(X))^\vee,\end{aligned}$$

under the isomorphisms above. This implies the claim. □

We have the following corollary:

Corollary 3.22 *Assume we have a minimal exceptional collection,* $A_1, \ldots, A_n$, *in* $\mathscr{A}$. *Let* $X = \bigoplus V_i \otimes_k A_i$ *and* $Y = \bigoplus W_i \otimes_k A_i$ *be twisted complexes. A cocycle,* $\phi \in \mathrm{Hom}_{\mathrm{Tw}\text{-}A}(X, Y)$, *is co-ghost for* B_{n+1-i} *if and only if the component* $\phi^{ii} : V_i \otimes_k A_i \to W_i \otimes_k A_i$ *is zero in* $H(\mathscr{A})$.

Proof Note that, by minimality, ϕ^{ii} must be some matrix in $\mathrm{Hom}_k(V_i, W_i)$ tensored with the identity on A_i. In particular, it is a cocycle.

Let $\phi : X \to Y$ be a map of normalized twisted complexes over A. We say that X has length l if $X = \bigoplus_{i=1}^{l} V_i \otimes_k A_i$. We proceed by induction on the length of the twisted complexes X and Y. The case $n = 1$ is clear.

Let us assume we know the claim is true when the lengths of X and Y are less than n and assume we have an exceptional collection of length n. For notation, let $X = \bigoplus_{i=1}^{n} V_i \otimes_k A_i$ and $Y = \bigoplus_{i=1}^{n} W_i \otimes_k A_i$. Note that the inclusion, $V_n \otimes_k A_n \hookrightarrow X$, is a cocycle in $\mathrm{Hom}_{\mathscr{A}}(V_n \otimes_k A_n, X)$. Let $X_{n-1} = \bigoplus_{i=1}^{n-1} V_i \otimes_k A_i$ with $\delta^{ij}_{X_{n-1}} = \delta^{ij}_X$ for $0 \le i, j \le n-1$. The cone over $V_n \otimes_k A_n \hookrightarrow X$ is the twisted complex, $X \oplus V_n \otimes_k A_n[1]$, with twisting cochain, $\begin{pmatrix} \delta_X & \mathrm{id}_{V_n \otimes_k A_n} \\ 0 & 0 \end{pmatrix}$. The projection, $X \oplus V_n \otimes_k A_n[1] \to X_{n-1}$, is a cocycle and induces a quasi-isomorphism of X_{n-1} with $L_{A_n}(X)$. The map, $\phi : X \to Y$, induces a commutative diagram,

$$\begin{tikzcd} X \oplus V_n \otimes_k A_n[1] \arrow[r, "\begin{pmatrix} \phi & 0 \\ 0 & \phi^{nn}[1] \end{pmatrix}"] \arrow[d] & Y \oplus W_n \otimes_k A_n[1] \arrow[d] \\ X_{n-1} \arrow[r, "\phi_{n-1}"] & Y_{n-1} \end{tikzcd}$$

where $\phi^{ij}_{n-1} = \phi^{ij}$ for $1 \le i, j \le n-1$. For $1 \le i \le n-1$, ϕ is B_{n+1-i} co-ghost if and only if ϕ_{n-1} is B_{n+1-i} co-ghost. Also, ϕ^{ii} vanishes if and only ϕ^{ii}_{n-1} vanishes. So, to verify the claim in the case that $1 \le i \le n-1$, we can pass to $\phi_{n-1} : X_{n-1} \to Y_{n-1}$ and apply the induction hypothesis. When $i = n$, we have the commutative diagram

$$\begin{tikzcd} V_n \otimes_k A_n \arrow[r] \arrow[d, "\phi^{nn}"] & X \arrow[d, "\phi"] \\ W_n \otimes_k A_n \arrow[r] & Y \end{tikzcd}$$

The inclusions induce isomorphisms,

$$\mathrm{Hom}_{H(\mathscr{A})}(A_n, V_n \otimes_k A_n[t]) \cong \mathrm{Hom}_{H(\mathscr{A})}(A_n, X[t])$$

and

$$\mathrm{Hom}_{H(\mathscr{A})}(A_n, W_n \otimes_k A_n[t]) \cong \mathrm{Hom}_{H(\mathscr{A})}(A_n, Y[t]).$$

Hence, $\mathrm{Hom}_{H(\mathscr{A})}(A_n, \phi[t]) = 0$ if and only if $\mathrm{Hom}_{H(\mathscr{A})}(A_n, \phi^{nn}[t])$ is trivial. Precomposing with the identity on $V_n \otimes_k A_n$ shows that $\mathrm{Hom}_{H(\mathscr{A})}(A_n, \phi^{nn}[t])$ vanishes for all t if and only if ϕ^{nn} vanishes. □

Let $A_1, \dots, A_n$ be a minimal exceptional collection and let $B_1, \dots, B_n$ be the left dual collection. To compress notation, set $\mathrm{End}(A) = \mathrm{Hom}_{\mathscr{A}}(\bigoplus_{i=1}^n A_i, \bigoplus_{i=1}^n A_i)$. Let I be the subspace of $\mathrm{End}(A)$ consisting of $\phi \in \mathrm{End}(A)$ for which $\mathrm{Hom}_{H(\mathscr{A})}(\phi, B_i)$ is zero for each i. Let us set $I^1 = I$ and define I^n as the following vector space:

$$\big\langle m_t(i_1, \dots, i_t) : i_j \in I^{s_j} \text{ with } 1 \leq s_j \leq n-1, s_1 + \cdots + s_t - t \geq n-1, \\ \text{and } t \geq 2\big\rangle.$$

Definition 3.23 We set $\mathrm{LL}_\infty(A) := \min\{n \mid I^n = 0\}$. We call $\mathrm{LL}_\infty(A)$ the *Loewy length* of A.

In the case that $m_i = 0$ for $i \neq 2$, $\mathrm{End}(A)$ is an algebra and I^n is the standard nth power of I as an ideal of A. So, $\mathrm{LL}_\infty(A)$ equals the minimal n for which any product of elements of I of length n is zero.

Proposition 3.24 *Let $\mathscr{A}$ be a cohomologically-finite triangulated A_∞-category possessing an exceptional collection, $A_1, \dots, A_n$, and let $B_1, \dots, B_n$ be the (left) dual collection. Set $G_B = \bigoplus_{i=1}^n B_i$. The generation time of G_B in $H(\mathscr{A})$ is bounded above by* $\mathrm{LL}_\infty(A') - 1$ *where A' is a minimal A_∞-algebra quasi-isomorphic to* $\mathrm{End}(A)$.

Proof Let $\phi_i : X_{i-1} \to X_i$, for $1 \leq i \leq s$, be a chain of G_B co-ghosts. By Lemma 3.19, we can assume each δ_{X_i} has components lying in I. By Corollary 3.22, the components of ϕ_i must lie in I. From the formula in (3.4), we see that all components of $\phi_s \circ \cdots \circ \phi_1$ lie in I^s. If $s \geq \mathrm{LL}_\infty(A')$, then $\phi_s \circ \cdots \circ \phi_1$ is zero. □

Proposition 3.25 *Let $\mathscr{T}$ be a triangulated category possessing an exceptional collection, $A_1, \dots, A_n$, with $B_1, \dots, B_n$ being the (left) dual collection. Set $G_B = \bigoplus_{i=1}^n B_i$. The generation time of G_B is bounded below by*

$$\mathrm{LL}_\infty\Bigg(\bigoplus_{l \in \mathbb{Z}, 1 \leq i \leq n} \mathrm{Hom}_{\mathscr{T}}(A_i, A_j[l])\Bigg) - 1.$$

Proof For $\bigoplus_{l\in\mathbb{Z},i} \operatorname{Hom}_{\mathcal{T}}(A_i, A_j[l])$, the ideal I consists of all maps between distinct objects in the exceptional collection. By the orthogonality properties of the right and left dual, any element of I is ghost for the right dual and co-ghost for the left dual. The Ghost Lemma, Lemma 2.17, gives the lower bound. □

Corollary 3.26 *Let $A_1, \dots, A_n$ be an exceptional collection in $\mathcal{A}$. Assume that A is formal, i.e. A is quasi-isomorphic to $H(A)$, with $m_i = 0$ for $i \neq 2$. The generation time of G_B is equal to the Loewy length of $H(A)$.*

Proof Note that we can apply Proposition 3.24 using $H(A)$ as A', and the upper bound from Proposition 3.24 and the lower bound from Proposition 3.25 coincide. □

Example 3.27 Let us consider the quiver

$$\begin{tikzcd} \bullet \arrow[r, "a_1"] & \bullet \arrow[r, "a_2"] & \cdots\cdots \arrow[r, "a_{n-1}"] & \bullet \arrow[r, "a_n"] & \bullet \end{tikzcd}$$

with the relation $a_n \cdots a_1 = 0$. Let A denote the path algebra modulo this relation. The right dual collection to the exceptional collection formed by the projective summands of A is the collection of the simple modules, $S_0, \dots, S_n$ (up to shifting the objects). Let $S := S_0 \oplus \cdots \oplus S_n$. From [26] $A^! = \mathbf{R}\operatorname{End}_A(S)$ can be represented by the graded quiver

$$\begin{tikzcd} \bullet \arrow[r, "b_1"] \arrow[rrrr, bend left, "z"] & \bullet \arrow[r, "b_2"] & \cdots\cdots \arrow[r, "b_{n-1}"] & \bullet \arrow[r, "b_n"] & \bullet \end{tikzcd}$$

with each b_i of degree one and z of degree two subject to the relations $b_{i+1}b_i = 0$ with the single multi-composition $m_n(b_n, \dots, b_1) = z$. We have $\mathrm{LL}_\infty(A^!) = 3$. As the right dual differs from the left dual by an application of the Serre functor, we have $\circleddash(A) \leq 2$ by Proposition 3.24. Consider the twisted complex over $A^!$

$$(C, \delta_C) = \left(S_1 \oplus \cdots \oplus S_{n-1}, \begin{pmatrix} 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ & \vdots & & \vdots & \\ 0 & \cdots & 0 & 0 & b_{n-1} \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \right)$$

Then the maps, $b_1 : S_0 \to C[1]$ and $b_n : C \to S_n[1]$, are A ghost and their composition is nonzero. So $\circleddash(A) \geq 2$ and hence $\circleddash(A) = 2$. This demonstrates that one can have a strict inequality in Proposition 3.25. One can also construct examples where the upper bound of Proposition 3.24 is strict. Furthermore, one can apply Corollary 3.26 to see that $\circleddash(A^!) = n-1$.

Example 3.28 In this example, we demonstrate that the supremum of the ultimate dimension over a given birational class is infinite. We first demonstrate the method on $\mathbb{P}^2$ as we get a slightly sharper statement than in the general case. Let X_1 be the blow-up of $\mathbb{P}^2$ at a point, p, E_1 denote the exceptional curve, and $\mathcal{O}(H)$ the pullback of $\mathcal{O}(1)$ on $\mathbb{P}^2$. Let X_2 denote the blow-up of X_1 at a point on E_1. Let E_2 be the exceptional curve of this blow-up and, abusing notation, let E_1 be the total transform of E_1, i.e. the union of the strict transform of E_1 and E_2. Also, set $\mathcal{O}(H)$ equal to the pullback of $\mathcal{O}(H)$ on X_1. We define X_n inductively as the blow-up of X_{n-1} at a point on the exceptional curve of the blow-up, $X_{n-1} \to X_{n-2}$. We denote by E_n the exceptional curve of the blow-up, $X_n \to X_{n-1}$ and by E_i, for $1 \le i \le n-1$, the total transforms of the E_i on X_{n-1}. We continue to write $\mathcal{O}(H)$ for the pullback of $\mathcal{O}(H)$ to X_n. Consider the object, $G_n = \mathcal{O}(-2H) \oplus \mathcal{O}(-H) \oplus \mathcal{O} \oplus \mathcal{O}_{E_1} \oplus \cdots \oplus \mathcal{O}_{E_n}$. From Theorem 3.4, G_n is a generator, and it is simple to check that $\mathcal{O}(-2H), \ldots, \mathcal{O}_{E_n}$ is an exceptional collection. Note that there is a nonzero composition of length $n+2$ in $\mathrm{End}_{X_n}(G_n)$ which corresponds to taking two sections, s_1, s_2, of $\mathcal{O}(1)$ on $\mathbb{P}^2$ not vanishing at p, pulling them back to X_n, and restricting down the chain

$$\mathcal{O}(-2H) \overset{\pi^* s_1}{\to} \mathcal{O}(-H) \overset{\pi^* s_2}{\to} \mathcal{O} \to \mathcal{O}_{E_1} \to \mathcal{O}_{E_2} \to \cdots \to \mathcal{O}_{E_n}.$$

By Proposition 3.25, the generation time of the dual collection is bounded below by $n+2$. In fact, this is an equality as the exceptional collection consists of $n+3$ objects. Thus, $n+2 \in \mathrm{OSpec}\, X_n$ and $\mathrm{udim}(X_n) \ge n+2$.

On any variety of dimension at least two, by blowing-up points iteratively, one can construct an exceptional collection with arbitrarily high Loewy length. In doing so, one obtains a generator of an admissible subcategory of some blowup with arbitrarily large generation time. Extending this generator by the pullback of a generator from the base, gives a generator of some blowup with arbitrarily large generation time.

Proposition 3.29 *Suppose $A_1, \ldots, A_n$ is a strong exceptional collection in a triangulated category, $\mathcal{T}$, which is the homotopy category of a triangulated A_∞-category. Let r be the projective dimension of $\mathrm{End}_{\mathcal{T}}(G_A)$ and s be the Loewy length. Then $[r, s]$ is contained in the Orlov spectrum of $\mathcal{T}$.*

Proof The generation time of G_A is r by Theorem 2.3. Hence, r is in the Orlov spectrum. The generator G_B corresponding to the dual collection, $B_1, \ldots, B_n$, has generation time equal to the Loewy length of $\mathrm{End}_{\mathcal{T}}(G_A)$

by Corollary 3.26. Hence, s is also in the Orlov spectrum. As $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ is a semi-orthogonal orthogonal decomposition consisting of subcategories of Rouquier dimension zero, the result follows from Theorem 3.12. □

Lemma 3.30 *Let* Q *be a quiver such that the underlying graph is a Dynkin diagram of type* A_n. *For each isomorphism class of indecomposable objects in* $\mathrm{D^b}(\mathrm{mod}\, kQ)$ *choose a representative,* M_i. *The Loewy length of the graded algebra* $\mathbf{R}\mathrm{End}_{kQ}(\oplus M_i)$ *is* n.

Proof All such quivers are derived Morita equivalent so we may assume all the arrows point to the right. Let us denote the right module generated by the i^{th} vertex by P_i. Then one can label the indecomposable objects by $M_{ij} := P_i/P_{j+1}$, $1 \le i, j \le n$ where $M_{in} = P_i$. If one prefers, this object can be identified with a string of 1-dimensional vector spaces beginning at the i^{th} vertex and ending at the j^{th} vertex with chosen isomorphisms in between. For $j < n$, the Serre functor S acts on objects which are not projective ($j < n$) by $S(M_{ij}) \cong M_{(i+1)(j+1)}[1]$ (this is merely a computation of Auslander-Reiten translation, see [2, 46]).

The morphism,

$$P_n \to \cdots \to P_1,$$

is a nontrivial composition of $n - 1$ nilpotent elements in $\mathbf{R}\mathrm{End}_{kQ}(\oplus M_i)$. This gives the lower bound.

Now for any nonzero morphism from M_{ij} to M_{st} one has $s \le i \le t \le j$, in order for it to not be an isomorphism, either $s < i$ or $t < j$. Now, consider a nonzero sequence of morphisms in the nilradical of $\mathbf{R}\mathrm{End}_{kQ}(\oplus M_i)$:

$$M_{i_1 j_1} \to \cdots \to M_{i_a j_a}.$$

We have $i_1 \le i_m \le j_1 \le j_m$ for all m and either i_m or j_m decreases. Thus, the total length of such a sequence is at most $j_1 - j_a + i_1 - i_a$. Now, let's add a morphism of degree one. By Proposition 2.21, we can assume a sequence of maximal length looks like:

$$M_{i_1 j_1} \to \cdots \to M_{i_a j_a} \to M_{st}[1] \to \cdots \to M_{(i_1+1)(j_1+1)}[1].$$

Hence the total length is at most,

$$\begin{aligned} & j_1 - j_a + i_1 - i_a + s - (i_1 + 1) + t - (j_1 + 1) + 1 \\ & \le i_1 + 1 - j_a + t - i_a - 1 \le -j_a + t < n. \end{aligned}$$

This is the desired upper bound. □

Theorem 3.31 *Let Q be a quiver such that the underlying graph is a Dynkin diagram of type A_n. The Orlov spectrum of* $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} kQ)$ *is equal to the integer interval* $\{0, \ldots, n-1\}$.

Proof The upper bound is from Corollary 2.19 and Lemma 3.30. The set $\{1, \ldots, n-1\}$ is contained in the Orlov spectrum from Proposition 3.29. Zero is in the Orlov spectrum since the category has finitely isomorphism classes of many indecomposable objects. □

4 Isolated singularities: the ungraded case

One can extract a fair bit of information about the structure of the Orlov spectrum for isolated singularities in both the graded and ungraded cases. In this section, we tackle the ungraded case leaving the graded case to the next section. Let us recall the necessary ideas.

Let S be a commutative Noetherian k-algebra.

Definition 4.1 The *category of singularities*, or *stable derived category*, of S is the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} S)$ by the subcategory consisting of all bounded complexes of finitely-generated projective modules. This is denoted by $\mathrm{D}_{\mathrm{sg}}(S)$.

Now let us assume that $(S, \mathfrak{m}_S)$ is a local Noetherian k-algebra. We say that $(S, \mathfrak{m}_S)$ is an *isolated singularity* if $S_{\mathfrak{p}}$ is a regular ring for any prime ideal, $\mathfrak{p} \neq \mathfrak{m}_S$, of S. The following proposition characterizes an isolated singularity purely in terms of its categories of singularities:

Proposition 4.2 *Let $(S, \mathfrak{m}_S)$ be a local commutative Noetherian k-algebra. The following are equivalent*:

(i) *$(S, \mathfrak{m}_S)$ is an isolated singularity*
(ii) *The residue field, k, is a generator of* $\mathrm{D}_{\mathrm{sg}}(S)$.

This is the content of Proposition A.2 of [27]. The implication (i) $\Rightarrow$ (ii) also follows immediately from the work in [49] or the work in [43]. A special case of this implication is contained in [17].

Let us now provide a criterion for when k strongly generates.

Proposition 4.3 *Let $(S, \mathfrak{m}_S)$ be a local commutative Noetherian k-algebra. The following are equivalent*:

(i) *k is a strong generator of* $\mathrm{D}_{\mathrm{sg}}(S)$.
(ii) *The natural homomorphism $S \to Z(\mathrm{D}_{\mathrm{sg}}(S))$ factors through $S/\mathfrak{m}_S^l$ for some l.*

Proof Let us assume that k is a strong generator of $\mathrm{D}_{\mathrm{sg}}(S)$. From Example 2.22, we see that $s(M)$ is k ghost and k co-ghost for any $M \in \mathrm{D}_{\mathrm{sg}}(S)$ and $s \in \mathfrak{m}_S$. Therefore any element of the form $s_1 \cdots s_l \in \mathfrak{m}_S^l$ gives a ghost sequence for k of length l. Since k strongly generates, $\mathrm{D}_{\mathrm{sg}}(S) = \langle k \rangle_{l-1}$ for some $l-1$, it follows from the Ghost Lemma, Lemma 2.17, that $s_1 \cdots s_l(M) = s_1(M) \circ \cdots \circ s_l(M) = 0$. Therefore, $\mathfrak{m}_S^l$ lies in the kernel of the map $S \to Z(\mathrm{D}_{\mathrm{sg}}(S))$.

Now, assume that $\mathfrak{m}_S^l$ lies in the kernel of the map $S \to Z(\mathrm{D}_{\mathrm{sg}}(S))$. For an element $s \in S$, let $K(s)$ denote the complex $S \xrightarrow{s} S$. Given a collection of elements $s_1, \ldots, s_m \in S$, consider the Koszul complex associated to this collection,

$$K(s_1, \ldots, s_m) = \bigotimes_{i=1}^{m} K(s_i).$$

Choose generators, $x_1, \ldots, x_m$, of the maximal ideal, $\mathfrak{m}_S$. For some l, the cohomology of $K(x_1^l, \ldots, x_n^l)$ is annihilated by $\mathfrak{m}_S^{nl}$ as every element of $\mathfrak{m}_S^{nl}$ is divisible by x_i^l for some i. Therefore, the cohomology modules of $K(x_1^1, \ldots, x_n^l) \otimes_S M$ are annihilated by $\mathfrak{m}_S^{nl}$ for any M from $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} S)$. This implies that $K(x_1^1, \ldots, x_n^l) \otimes_S M$ lies in $\langle k \rangle_{(n+1)(ln+1)-1}$, here taken in $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} S)$. In $\mathrm{D}_{\mathrm{sg}}(S)$, M is a summand of $K(x_1^1, \ldots, x_n^l) \otimes_S M$ and, hence, lies in $\langle k \rangle_{(n+1)(ln+1)-1}$. □

For a general ring (not necessarily of finite-type over k), it is unclear whether or not k is always a strong generator of $\mathrm{D}_{\mathrm{sg}}(S)$. However, the following proposition covers many examples originating from algebraic geometry. Recall that S is said to be *essentially of finite type* if it is the localization of a finitely-generated k-algebra.

Proposition 4.4 *Let S be a commutative k-algebra that is essentially of finite type. There exists a finitely-generated S-module, E, and an $l \in \mathbb{Z}_{\geq 0}$ so that*

$$\mathrm{D}(\operatorname{Mod} S) = \langle \bar{E} \rangle_l, \qquad \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} S) = \langle \tilde{E} \rangle_l, \quad \text{and} \quad \mathrm{D}^{\mathrm{b}}(\operatorname{mod} S) = \langle E \rangle_l.$$

Proof Let us recall the generation notions appearing in the statement of theorem, as they have laid dormant since Sect. 2. Given a subcategory, $\mathscr{S}$, of triangulated category, $\mathscr{T}$, the subcategory, $\langle \overline{\mathscr{S}} \rangle$, is the smallest full subcategory of $\mathscr{T}$, containing $\mathscr{S}$ and closed under isomorphisms, sums, shifts, and set-indexed $\mathscr{T}$-coproducts (that exist). Given an object, X, of $\mathscr{T}$ and a set, A, the A-multiple of X is $\bigoplus_{a \in A} X$, if it exists in $\mathscr{T}$. $\langle \widetilde{\mathscr{S}} \rangle$ is the smallest full subcategory of $\mathscr{T}$ containing $\mathscr{S}$ and closed under isomorphisms, sums, shifts, and multiples. One then defines $\langle\overline{\mathscr{S}}\rangle_n$ and $\langle\widetilde{\mathscr{S}}\rangle_n$ inductively with

$$\langle\overline{\mathscr{S}}\rangle_0 := \langle\overline{\mathscr{S}}\rangle, \qquad \langle\overline{\mathscr{S}}\rangle_n := \overline{\langle\overline{\mathscr{S}}\rangle_{n-1} \diamond \langle\overline{\mathscr{S}}\rangle}$$

and

$$\langle\widetilde{\mathscr{S}}\rangle_0 := \langle\widetilde{\mathscr{S}}\rangle, \qquad \langle\widetilde{\mathscr{S}}\rangle_n := \langle\widetilde{\mathscr{S}}\widetilde{\rangle_{n-1} \diamond \langle}\widetilde{\mathscr{S}}\rangle.$$

Recall that Theorem 7.39 of [47] states that such an E exists for the derived categories associated to any finitely-generated k-algebra. We will follow and use the proof of Theorem 7.39 in loc. cit. The proofs are very similar for $\mathrm{D}(\mathrm{Mod}\, S)$ and $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, S)$, so we will only provide the proof of the latter and leave the proof for $\mathrm{D}(\mathrm{Mod}\, S)$ as an exercise to the reader. The statement for $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, S)$ is an immediate consequence of Corollary 6.16 and Corollary 3.13 of loc. cit.

Let R be a finitely-generated k-algebra and I a multiplicative subset of R so that $S = R_I$. Let U be a smooth open subset of $\operatorname{Spec} R$ with complement determined by the ideal J. Let us proceed by induction on the Krull dimension of R. When R has Krull dimension zero, the statement is a consequence of Theorem 7.39 of loc. cit. as R_I is finitely-generated over k.

From the proof of Theorem 7.39 of loc. cit., one has the following exact triangle in $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, R^e)$,

$$C \to R \oplus R[1] \to D,$$

where C is a perfect R^e-module and D is a $R/J^n \otimes_k R$-module. If we localize C and D on the left and right by I, we get a triangle,

$$C_I \to R_I \oplus R_I[1] \to D_I, \tag{4.1}$$

where C_I is a perfect R_I^e-module and D_I is a $R_I/J^n R_I \otimes_k R_I$-module.

Let M be any object of $\mathrm{D}^{\mathrm{b}}(\text{Mod-}\, R_I)$ and apply $- \overset{\mathbf{L}}{\otimes}_{R_I} M$ to (4.1):

$$C_I \overset{\mathbf{L}}{\otimes}_{R_I} M \to M \oplus M[1] \to D_I \overset{\mathbf{L}}{\otimes}_{R_I} M.$$

As C_I is perfect, $C_I \overset{\mathbf{L}}{\otimes}_{R_I} M$ has bounded cohomology. From the long exact sequence of cohomology modules, we see that $D_I \overset{\mathbf{L}}{\otimes}_{R_I} M$ has bounded cohomology.

From the induction hypothesis, there exists a finitely-generated $R_I/JR_I = (R/J)_I$-module, E', for which $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, R_I/JR_I) = \langle\tilde{E}'\rangle_l$ for some $l \in \mathbb{Z}_{\geq 0}$. Furthermore, $D_I \overset{\mathbf{L}}{\otimes}_{R_I} M$ lies in,

$$\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, R_I/J^n R_I) = \langle\tilde{E}'\rangle_{(n+1)(l+1)-1}.$$

As C_I lies in $\langle \widetilde{R_I \otimes_k R_I} \rangle_t$ for some t, $C_I \overset{\mathbf{L}}{\otimes}_{R_I} M$ lies in $\langle \tilde{R}_I \rangle_t$. This implies that $M \in \langle R_I \oplus E' \rangle_{(t+1)(n+1)(l+1)-1}$. We can take $E = R_I \oplus E'$. $\square$

Proposition 4.5 *Let $(S, \mathfrak{m}_S)$ be a local commutative k-algebra that is essentially of finite type over k. There exists a finitely-generated $\hat{S}$-module, E, and an $l \in \mathbb{Z}_{\geq 0}$ so that*

$$\mathrm{D}(\mathrm{Mod}\,\hat{S}) = \langle \bar{E} \rangle_l, \qquad \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,\hat{S}) = \langle \tilde{E} \rangle_l, \quad \textit{and} \quad \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\hat{S}) = \langle E \rangle_l,$$

where $\hat{S}$ is the completion of S at $\mathfrak{m}_S$.

Proof The argument is the same as in the proof of Proposition 4.4 above. $\square$

Corollary 4.6 *If $(S, \mathfrak{m}_S)$ is a local commutative k-algebra essentially of finite type over k, then $\mathrm{D}_{\mathrm{sg}}(S)$ has finite Rouquier dimension. The same is true for $\mathrm{D}_{\mathrm{sg}}(\hat{S})$.*

Combining the results above, we get the following characterization of an isolated singularity when the ring is essentially of finite type:

Theorem 4.7 *Let $(S, \mathfrak{m}_S)$ be a local commutative k-algebra essentially of finite type over k. The following are equivalent:*

(i) *$(S, \mathfrak{m}_S)$ is an isolated singularity.*
(ii) *k is a strong generator for $\mathrm{D}_{\mathrm{sg}}(S)$.*
(iii) *The natural map $S \to Z(\mathrm{D}_{\mathrm{sg}}(S))$ factors through $S/\mathfrak{m}_S^d$ for some $d \in \mathbb{N}$.*

Proof We know that (ii) and (iii) are equivalent by Proposition 4.3. Since S is essentially of finite type, Proposition 4.4 says we have a strong generator. Thus, if k is a generator, k must be a strong generator. $\square$

While Theorem 4.7 is an interesting characterization of an isolated singularity, it provides no control over the generation time of k or over the Orlov spectrum of $\mathrm{D}_{\mathrm{sg}}(S)$. To get such information, we restrict to the case of an isolated hypersurface singularity.

A local Noetherian k-algebra, $(S, \mathfrak{m}_S)$, is called a *hypersurface singularity* if S is isomorphic to $R/(w)$ with $(R, \mathfrak{m}_R)$ a Noetherian, regular local k-algebra and w lies in $\mathfrak{m}_R$. The multiplicity of w will be the minimal l so that $w \in \mathfrak{m}_R^l$. If $(S, \mathfrak{m}_S)$ is a hypersurface singularity, it is Gorenstein, and in particular, Cohen-Macaulay.

There are two additional constructions of $\mathrm{D}_{\mathrm{sg}}(S)$ which are useful to consider. Recall that a module, M, over S is called a *maximal Cohen-Macaulay* module, or a MCM module for short, if the depth of M is equal to the Krull dimension of S.

For the first construction, let $\mathrm{MCM}(S)$ be the full subcategory of $\operatorname{mod} S$ consisting of MCM modules. $\underline{\mathrm{MCM}}(S)$ is a category with the same objects as $\mathrm{MCM}(S)$ but with

$$\mathrm{Hom}_{\underline{\mathrm{MCM}}(S)}(M, N) = \mathrm{Hom}_S(M, N)/\sim$$

where $f \sim g$ if there exists maps $p : M \to P$ and $q : P \to N$ with $f - g = qp$ and P projective.

In the second construction, the objects are sequences of R-modules,

$$P_0 \xrightarrow{A} P_1 \xrightarrow{B} P_0,$$

with P_i finitely-generated projective R-modules, $AB = w\,\mathrm{id}_{P_1}$, and $BA = w\,\mathrm{id}_{P_0}$. Such sequences were introduced by D. Eisenbud, [19], who named them *matrix factorizations*. For simplicity, we denote a matrix factorization (P_0, P_1, A, B) by P and let A_P and B_P denote the maps in the matrix factorization. A morphism between two matrix factorizations, P and Q, consists of R-module maps, $f_0 : P_0 \to Q_0$ and $f_1 : P_1 \to Q_1$, making the following diagram commutative:

$$\begin{tikzcd}
P_0 \arrow[r, "A_P"] \arrow[d, "f_0"] & P_1 \arrow[r, "B_P"] \arrow[d, "f_1"] & P_0 \arrow[d, "f_0"] \\
Q_0 \arrow[r, "A_Q"] & Q_1 \arrow[r, "B_Q"] & Q_0
\end{tikzcd}$$

A homotopy between two morphisms, $f, g : P \to Q$, is a pair of maps $h_0 : P_0 \to Q_1$ and $h_1 : P_1 \to Q_0$ so that $f_0 - g_0 = B_Q h_0 + h_1 A_P$ and $f_1 - g_1 = A_Q h_1 + h_0 B_P$. The category of matrix factorization of w, $\mathrm{MF}(w)$, has matrix factorizations as objects and has homotopy classes of morphisms between P and Q as morphism sets.

In both of these descriptions, the resulting category is naturally triangulated. We have the following result, see [13] or [40]:

Theorem 4.8 *For an isolated hypersurface singularity, S, the categories $\mathrm{D}_{\mathrm{sg}}(S)$, $\underline{\mathrm{MCM}}(S)$, and $\mathrm{MF}(w)$ are all equivalent as triangulated categories.*

We draw from this two useful corollaries.

Corollary 4.9 *Every object in $\mathrm{D}_{\mathrm{sg}}(S)$ is isomorphic to a MCM module.*

Proof The equivalence of $\mathrm{D}_{\mathrm{sg}}(S)$ and $\underline{\mathrm{MCM}}(S)$ is induced by the inclusion,

$$\mathrm{MCM}(S) \hookrightarrow \mathrm{Ch}(\operatorname{mod} S),$$

which sends an MCM module, M, to the complex

$$\cdots \to 0 \to M \to 0 \to \cdots$$

with M in degree zero. □

For a choice of basis of $\Omega_{R/k}$, $e_1, \ldots, e_n$, we can write $dw = \partial_1 w e_1 + \cdots + \partial_n w e_n$ for $\partial_i w \in R$. Let $(\partial w) = (\partial_1 w, \ldots, \partial_n w)$. Note that the ideal is independent of the choice of basis for $\Omega_{R/k}$.

Corollary 4.10 *The natural map $S \to Z(\mathrm{D}_{\mathrm{sg}}(S))$ factors through the projection $S \to S/(\partial w)$.*

Proof We consider the category $\mathrm{MF}(w)$. If P is a matrix factorization, then, taking the i-th derivatives of $AB = w\,\mathrm{id}_{P_1}$ and $BA = w\,\mathrm{id}_{P_0}$, we get $\partial_i AB + A\partial_i B = \partial_i w\,\mathrm{id}_{P_1}$ and $\partial_i BA + B\partial_i A = \partial_i w\,\mathrm{id}_{P_0}$. This means that $(\partial_i A, \partial_i B)$ is a homotopy between $\partial_i w$ and 0. □

Recall the *Loewy length* of a local Artinian ring, R, is the minimal l for which $\mathfrak{m}_R^l = 0$. Denote this as $\mathrm{LL}(R)$. For an isolated hypersurface singularity, the Tjurina algebra, $S/(\partial w)$, is Artinian. We can apply the ideas of Proposition 4.3 to prove the following:

Proposition 4.11 *Let $(S, \mathfrak{m})$ be an isolated hypersurface singularity. The generation time of k in $\mathrm{D}_{\mathrm{sg}}(S)$ is bounded above by $2\mathrm{LL}(S/(\partial w)) - 1$. In particular, $\mathrm{D}_{\mathrm{sg}}(S)$ has finite Rouquier dimension.*

Proof Let M be any MCM module over S and consider the Koszul complex,

$$K(\partial w) := K(\partial_1 w, \ldots, \partial_n w).$$

As the Krull dimension of $S/(\partial w)$ is zero and M is MCM-module, there is an M-sequence of length $n-1$ in (∂w). By [35] Theorem 16.8, the cohomology of,

$$K(\partial w) \otimes_S M =: K(M, \partial w),$$

vanishes except for degrees zero and one. Furthermore, $H_i(K(M, \partial w))$ is a module over $S/(\partial w)$. For any $S/(\partial w)$-module, L, we have a filtration:

$$0 = \mathfrak{m}_{S/(\partial w)}^{\mathrm{LL}(S/(\partial w))} L \subseteq \cdots \subseteq \mathfrak{m}_{S/(\partial w)} L \subseteq L.$$

The quotients of this filtration are direct sums of the residue field. Therefore, we have

$$H_i(K(M, \partial w)) \in \langle k \rangle_{\mathrm{LL}(S/(\partial w))-1} \quad \text{and} \quad K(M, \partial w) \in \langle k \rangle_{2\mathrm{LL}(S/(\partial w))-1}$$

in $\mathrm{D^b}(\mathrm{mod}\, S)$. In $\mathrm{D_{sg}}(S)$, by Corollary 4.10, the partial derivatives of w vanish. Hence, M is a summand of $K(M, \partial w)$. Thus, $\mathrm{D_{sg}}(S) = \langle k \rangle_{2\,\mathrm{LL}(S/(\partial w))-1}$. ☐

Remark 4.12 Strong generation of k also follows from work in [17].

Our next goal is to study the Orlov spectrum of $\mathrm{D_{sg}}(S)$. Before we wade into the case of a general hypersurface, let us fully analyze the stable derived category of the ring $A_{n-1} = k[u]/(u^n)$. See also [40]. From the classification of modules over a PID, we know the only indecomposable modules are

$$k[u]/(u^n), k[u]/(u^{n-1}), \dots, k[u]/(u), 0.$$

Any morphism in $\mathrm{mod}\, A_{n-1}$ from $k[u]/(u^i)$ to $k[u]/(u^j)$ is a linear combination of the maps

$$\begin{aligned} \alpha^l_{i,j} : k[u]/(u^i) &\to k[u]/(u^j) \\ 1 &\mapsto u^l \end{aligned}$$

for $\max(0, j-i) \le l < j$. The map, $\alpha^l_{i,j}$, factors through $k[u]/(u^n)$ if and only if $l \ge n-i$. In $\mathrm{D_{sg}}(A_{n-1})$, we let V_i stand for the image of $k[u]/(u^i)$. The morphism space between V_i and V_j is spanned by the images of $\alpha^l_{i,j}$ with $\max(0, j-i) \le l < \min(j, n-i)$. Let us compute the cones. We have an exact sequence:

$$0 \to k[u]/(u^{\max(0,i-j+l)}) \to k[u]/(u^i) \xrightarrow{\alpha^l_{i,j}} k[u]/(u^j) \to k[u]/(u^l) \to 0. \tag{4.2}$$

Lemma 4.13 *The extension in* (4.2) *is trivial.*

Proof We can assume that $i-j+l$ is non-negative. Let us take a free resolution of $k[u]/(u^l)$ and choose a homotopy class of chain maps between the free resolution and the exact sequence (4.2).

$$\begin{tikzcd}
k[u]/(u^n) \arrow[r, "\alpha^{n-l}_{n,n}"] \arrow[d, "\lambda"] & k[u]/(u^n) \arrow[r, "\alpha^l_{n,n}"] \arrow[d, "\alpha^0_{n,i}"] & k[u]/(u^n) \arrow[r, "\alpha^0_{n,l}"] \arrow[d, "\alpha^0_{n,j}"] & k[u]/(u^l) \arrow[r] \arrow[d, "\alpha^0_{l,l}"] & 0 \\
k[u]/(u^{i-j+l}) \arrow[r, "\alpha^{j-l}_{i-j+l,i}"] & k[u]/(u^i) \arrow[r, "\alpha^l_{i,j}"] & k[u]/(u^j) \arrow[r, "\alpha^0_{j,l}"] & k[u]/(u^l) \arrow[r] & 0
\end{tikzcd}$$

Since $l < n-i$, $\alpha^0_{n,i} \circ \alpha^{n-l}_{n,n} = \alpha^{n-l}_{n,i}$ is zero. We can take λ to be zero which proves the claim. ☐

In $\mathrm{D}_{\mathrm{sg}}(A_{n-1})$, we get triangles

$$\begin{tikzcd}
V_i \arrow[rr, "\alpha^l_{i,j}"] & & V_j \arrow[dl] \\
& V_{\max(0,i-j+l)}[1] \oplus V_l \arrow[ul, "\shortmid" marking] &
\end{tikzcd}$$

We also have isomorphisms, $k[u]/(u^i) \cong k[u]/(u^{n-i})[1]$, coming from the short exact sequences,

$$0 \to k[u]/(u^{n-i}) \overset{\alpha^i_{n-i,n}}{\to} k[u]/(u^n) \to k[u]/(u^i) \to 0.$$

Theorem 4.14 *The Orlov spectrum of* $\mathrm{D}_{\mathrm{sg}}(A_{n-1})$ *is*

$$\left\{0, 1, \ldots, \left\lceil \frac{\lfloor n/2 \rfloor}{s} \right\rceil - 1, \ldots, \left\lceil \frac{\lfloor n/2 \rfloor}{2} \right\rceil - 1, \lfloor n/2 \rfloor - 1\right\},$$

where $\lfloor \alpha \rfloor$ is the greatest integer less than α and $\lceil \alpha \rceil$ is the least integer greater than α.

Proof Let G be a generator for $\mathrm{D}_{\mathrm{sg}}(A_{n-1})$. Without loss of generality, we can assume that

$$G = \bigoplus_{i \in I \subset \{1, \ldots, \lfloor n/2 \rfloor\}} V_i.$$

Let

$$\delta(t) = \max\{j | V_j \in \langle G \rangle_t, 0 \le j \le \lfloor n/2 \rfloor\}.$$

We first show that

$$\circleddash(G) \le \begin{cases} \max\{\lceil \frac{\lfloor n/2 \rfloor}{\delta(0)} \rceil - 1, 1\} & \langle G \rangle_0 \ne D_{\mathrm{sg}}(A_{n-1}) \\ 0 & \langle G \rangle_0 = D_{\mathrm{sg}}(A_{n-1}). \end{cases}$$

Assume that V_j, $j \le \lfloor n/2 \rfloor$, lies in $\langle G \rangle_t$. Without loss of generality we can assume that $j \ge \delta(0)$. To make new indecomposables, the possible cones we could take involve the pairs (i, j), $(i, n-j)$, $(n-i, j)$, $(n-i, n-j)$ with $i \in I$. If we use the pair (i, j), we get indecomposable objects V_t with $\max(0, j-i) \le t < j$ and $\max(0, i-j) \le t < i$ in the next step. If we use the pair $(i, n-j)$, we get the indecomposable objects V_t with $n-j-i \le t < \min(n-j, n-i)$ and $0 \le t < \min(i, j)$.

We see that $V_0, \ldots, V_{i+j}$ lies in $\langle G \rangle_{t+1}$. Therefore,

$$\delta(t+1) \geq \min(\delta(t) + \delta(0), \lfloor n/2 \rfloor),$$

and after the zeroth step, if $\langle G \rangle_t$ contains V_j for $j \leq \lfloor n/2 \rfloor$, then it contains V_s for $1 \leq s \leq j$. This gives the claimed upper bound.

To demonstrate that the lower bound holds, we note that x^l annihilates G when $l \geq \delta(0)$. By Example 2.22, $x^l(V_{\lfloor n/2 \rfloor})$ is G ghost. Furthermore, $(x^l)^{\lceil \frac{\lfloor n/2 \rfloor}{l} \rceil - 1}(V_{\lfloor n/2 \rfloor})$ is nonzero. Therefore, by the Ghost Lemma, Lemma 2.17, $\lceil \frac{\lfloor n/2 \rfloor}{l} \rceil - 1$ is a lower bound for the generation time of G.

Consequently,

$$\circleddash(G) = \begin{cases} \max\{\lceil \frac{\lfloor n/2 \rfloor}{\delta(0)} \rceil - 1, 1\} & \langle G \rangle_0 \neq D_{\mathrm{sg}}(A_{n-1}) \\ 0 & \langle G \rangle_0 = D_{\mathrm{sg}}(A_{n-1}). \end{cases} \qquad \square$$

Let us return to the case of a general isolated hypersurface singularity, see also [53] Sect. 5.

Lemma 4.15 *Let $(S, \mathfrak{m}_S)$ be a hypersurface singularity, $S = R/(w)$ with R regular, and let M be a MCM module over S. For a generic choice of a regular system of parameters on R, $y_1, \ldots, y_n$, the first $n-1$ parameters, $y_1, \ldots, y_{n-1}$, form both a S-regular and a M-regular sequence and the quotient $S/(y_1, \ldots, y_{n-1})S$ is isomorphic to a zero dimensional hypersurface singularity. Moreover, the multiplicity of w in R is the same as the multiplicity of $\bar{w}$ in $R/(y_1, \ldots, y_{n-1})$.*

Proof Recall that a sequence of elements, $s_1, \ldots, s_i$, is M-regular if s_j has zero annihilator in $M/(s_1, \ldots, s_{j-1})M$. Now, $x_1, \ldots, x_n$ is a regular system of parameters for R if $(x_1, \ldots, x_n) = \mathfrak{m}_R$ with n equal to the Krull dimension of R. Recall that $x_1, \ldots, x_n$ is a regular system of parameters for R if and only if the images of $x_1, \ldots, x_n$ form a basis for $\mathfrak{m}_R/\mathfrak{m}_R^2$, see [35] Theorem 14.2.

We prove the results involving S and then note that the same choices work to establish the result about M. We proceed by induction on n. The case $n = 1$ is clear.

Assume we know the result below $n-1$ and consider the case of n. w has a unique factorization (in R) into irreducible elements. Let us denote them by $w_1, \ldots, w_t$. Let x be an element of R that projects to a nonzero vector in $\mathfrak{m}_R/\mathfrak{m}_R^2$. It is clear that x is irreducible and is a zero divisor in S if and only if it equals some w_i. The associated graded ring, $\mathrm{gr}_{\mathfrak{m}_R}(R)$, is isomorphic to a polynomial ring over k in n variables, [35] Theorem 17.10. Let d be the multiplicity of w and denote the image of w in $\mathrm{gr}_{\mathfrak{m}_R}(R)$ by w_d.

If n is greater than one, we can choose an element u of R with nonzero image in $\mathfrak{m}_R/\mathfrak{m}_R^2$ so u is a not a zero-divisor in S and the image of u in $\mathrm{gr}_{\mathfrak{m}_R}(R)$

does not divide w_d. Now, complete u to a regular system of parameters for R, $u, u_2, \dots, u_n$. We find that S/uS is another hypersurface singularity to which we can apply the induction hypothesis.

Let $\mathfrak{p}$ be an associated prime for M. The depth of M is bounded above by the dimension of $A/\mathfrak{p}$, Theorem 17.2 [35]. If M is a MCM module, then the height of $\mathfrak{p}$ cannot be more than zero. By Krull's theorem, $\mathfrak{p}$ cannot contain a non-zerodivisor. Thus, $\mathfrak{p}$ is in the ideal generated by the $w_1, \dots, w_t$. Since our choices of a regular system of parameters avoids each w_i, they also provide an M-sequence. □

Lemma 4.16 *Let $(S, \mathfrak{m}_S)$ be an isolated hypersurface singularity and M be a module of infinite projective dimension over S. If $x \in S$ is a nonunit and S and M-regular, then M/xM is a module of infinite projective dimension over $S/(x)$.*

Proof Note that $S/(x)$ vanishes in $\mathrm{D}_{\mathrm{sg}}(S)$ as it is quasi-isomorphic to the cone of $x(S) : S \to S$ and hence perfect. Also note that the morphism, $x(M) : M \to M$, in $\mathrm{D}_{\mathrm{sg}}(S)$ is nilpotent by Proposition 4.3.

Assume that M/xM has finite projective dimension as an $S/(x)$ module. Then, M/xM vanishes in $\mathrm{D}_{\mathrm{sg}}(S)$. As M/xM is quasi-isomorphic to the cone of $x(M)$, we see that $x(M)$ is an isomorphism in $\mathrm{D}_{\mathrm{sg}}(S)$ and cannot be nilpotent. □

Lemma 4.17 *Any zero dimensional hypersurface singularity, $S = R/(w)$, is isomorphic to A_{d-1}, where d is the multiplicity of w.*

Proof As S is zero dimensional, completion does not change the ring. Thus, S is isomorphic to $\hat{R}/(w)$. Any complete, regular, local, Noetherian ring of dimension one is isomorphic to the formal power series ring in one variable $k[[u]]$ with the uniformizing parameter of $\hat{R}$ getting sent to u, [35] Theorem 29.7. A simple change of variables takes w to u^d. □

We now use these lemmas to facilitate a reduction from a general isolated hypersurface singularity to an A_n-singularity.

Lemma 4.18 *Let $(S, \mathfrak{m}_S)$ be an isolated hypersurface singularity and let M be any non-zero object of $\mathrm{D}_{\mathrm{sg}}(S)$. The level of the residue field of $(S, \mathfrak{m}_S)$ with respect to M is at most $\dim S + 1$, i.e. $k \in \langle M \rangle_n$ with $n \leq \dim S + 1$. In particular, M is a strong generator of $\mathrm{D}_{\mathrm{sg}}(S)$.*

Proof Let S be isomorphic to $R/(w)$. From Lemmas 4.15 and 4.17, we know we can choose a regular system of parameters, $x_1, \dots, x_n$, with $x_1, \dots, x_{n-1}$

a S-regular and a M-regular sequence and so that $S/(x_1, \dots, x_{n-1})$ is isomorphic to $A_{d-1} = k[u]/(u^d)$ where d is multiplicity of w. Note that $M/(x_1, \dots, x_{n-1})M$ cannot be free by Lemma 4.16.

Let $K(x) = K(x_1, \dots, x_n)$ and $K(M, x) = K(x) \otimes_S M$. Notice that $K(M, x)$ is quasi-isomorphic to the complex $M/(x_1, \dots, x_{n-1})M \xrightarrow{x_n} M/(x_1, \dots, x_{n-1})M$. Writing x_n as $\alpha_1 u + \cdots + \alpha_m u^{d-1}$, one sees that $M/(x_1, \dots, x_{n-1})M \xrightarrow{x_n} M/(x_1, \dots, x_{n-1})M$ is the composition of $M/(x_1, \dots, x_{n-1})M \xrightarrow{u} M/(x_1, \dots, x_{n-1})M$ and an automorphism of $M/(x_1, \dots, x_{n-1})M$. The octahedral axiom tells us that the cone of,

$$M/(x_1, \dots, x_{n-1})M \xrightarrow{u} M/(x_1, \dots, x_{n-1})M,$$

is isomorphic to the cone of,

$$M/(x_1, \dots, x_{n-1})M \xrightarrow{x_n} M/(x_1, \dots, x_{n-1})M.$$

As $M/(x_1, \dots, x_{n-1})M$ is nonfree, Lemma 4.13 implies that the cone of,

$$M/(x_1, \dots, x_{n-1})M \xrightarrow{u} M/(x_1, \dots, x_{n-1})M$$

is quasi-isomorphic to a sum of shifts of k. Hence k is a summand of $K(M, x)$ which manifestly lies in $\langle M \rangle_n$.

The above tells us that M generates k, and, by Theorem 4.7, k generates $\mathrm{D}_{\mathrm{sg}}(S)$. It follows that M is a strong generator. □

Remark 4.19 Lemma 4.18 is not true for complete intersections. For example, consider the ring $S = k[x, y]/(x^2, y^2)$. The module $k[x]/(x^2)$ is nonzero in $\mathrm{D}_{\mathrm{sg}}(S)$ but $k[y]/(y^2)$ is orthogonal to it.

Remark 4.20 Let M be a MCM module. The arguments in the proof of Lemma 4.18 give the following statement: M is a generator of $\mathrm{D}_{\mathrm{sg}}(S)$ if and only $M \overset{\mathbf{L}}{\otimes}_R k \in \langle k \rangle_0$ in $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} S)$. Does this statement hold for complete intersections? The authors know of no counterexample.

Combining Proposition 4.11 and Lemma 4.18 gives us the following theorem:

Theorem 4.21 *Let $(S, \mathfrak{m}_S)$ be an isolated hypersurface singularity. The ultimate dimension of $\mathrm{D}_{\mathrm{sg}}(S)$ is bounded by $2(\dim S + 2)\,\mathrm{LL}(S/(\partial w)) - 1$.*

Remark 4.22 The upper bound in the theorem above is a rough estimate. For example, it is not achieved for example for the A_n singularity, see Theorem 4.14. In addition, the map from $S/(\partial w)$ to natural transformations of the identity is rarely injective, and the bound can easily be improved to depend only on the nilpotence of this image. Consequently, we would be very surprised if this bound is saturated in any example.

For an example, let us consider the ring,

$$S_g = k[x, y, z]/(x^{2g+1} + y^{2g+1} + z^{2g+1} - xyz)$$

for $g > 1$. Let $w_g = x^{2g+1} + y^{2g+1} + z^{2g+1} - xyz$. We can take $x - z, y - z$ as a regular sequence and $S_g/(x - z, y - z)$ is isomorphic to A_2. The level of residue field is at most two for any generator of $\mathrm{D}_{\mathrm{sg}}(S_g)$. The Jacobian ideal of S_g is $((2g + 1)x^{2g} - yz, (2g + 1)y^{2g} - xz, (2g + 1)z^{2g} - xy)$. The Loewy length of $S_g/(\partial w_g)$ is $2g + 1$.

There is a $\mathbb{Z}/(2g + 1)\mathbb{Z}$ action on S_g with which it is proven in [51], for $g = 2$, and in [18], for $g \geq 2$, that the idempotent-completion of the $\mathbb{Z}/(2g + 1)\mathbb{Z}$-equivariant singularity category, $\mathrm{D}_{\mathrm{sg}}^{\mathbb{Z}/(2g+1)\mathbb{Z}}(S_g)$, is equivalent to the idempotent-completion of the derived Fukaya category of a genus g Riemann surface, $\mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)$. In light of Example 2.8, we can use our results to control the generation time of certain generators of $\mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)$ (the notation in the following proof can be found in this example). More precisely, recall that symplectically, the surface, Σ_g, admits a $\mathbb{Z}/(2g + 1)\mathbb{Z}$-branched cover over an orbifold $\mathbb{P}^1$. Let $\psi : \Sigma_g \to \Sigma_g$ be a generator of the covering group. We now have the following result:

Proposition 4.23 *Let M be any nonzero object of $\mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)$. Then, $\bigoplus_{i=0}^{2g} \psi^i(M)$ is a generator of $\mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)$ and its generation time is bounded by* $12g + 5$.

Proof By Lemma 4.18, $\operatorname{For}(M)$ generates $\mathrm{D}_{\mathrm{sg}}(S_g)$. By Example 2.8, the functor, Inf, is dense and hence $\bigoplus_{i=0}^{2g} \psi^i(M) \cong \operatorname{Inf}(\operatorname{For}(M))$ generates with,

$$\circleddash\left(\bigoplus_{i=0}^{2g} \psi^i(M)\right) = \circleddash\left(\operatorname{For}\left(\bigoplus_{i=0}^{2g} \psi^i(M)\right)\right).$$

The level of k with respect to any object of $D_{\mathrm{sg}}(S_g)$ is at most two and the generation time of k is at most $4g + 1$. Thus, $\circleddash(\operatorname{For}(\bigoplus_{i=0}^{2g} \psi^i(M))) \leq 12g + 5$. □

5 Isolated singularities: the graded case

Most of the results in Sect. 4 can be adapted to the graded case in a straightforward manner. When we combine these results with Orlov's results relating derived categories of coherent sheaves to graded categories of singularities, many interesting and nontrivial statements emerge. So, let us begin by recalling Orlov's results from [41]. We let $A = \bigoplus_{n \geq 0} A_n$ be a graded Noetherian k-algebra with $A_0 = k$. Recall that such a graded algebra is called *connected*. We write, $\operatorname{gr} A$, for the abelian category of finitely-generated graded A-modules. The morphisms in this category are taken to be degree zero A-module homomorphisms. The category has an internal Hom denoted $\underline{\mathrm{Hom}}$. For any graded module, M, we can form a new graded module, $M(1)$, with $M(1)_l = M_{l+1}$. Recall that A is *AS-Gorenstein*, or often just Gorenstein, if A has finite injective dimension, n, and $\underline{\mathrm{Ext}}^i_{\operatorname{gr} A}(k, A) = 0$ for $i \neq 0$ and $\underline{\mathrm{Ext}}^n_{\operatorname{gr} A}(k, A) = k(a)$. We call, a, the *Gorenstein parameter* of A. We have the maximal ideal, $\mathfrak{m}_A = \bigoplus_{l>0} A_l$.

Sitting inside of $\operatorname{gr} A$, we have the full subcategory of finite-dimensional modules (over k), $\operatorname{tors} A$. Inside of $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$, we have two thick triangulated subcategories: $\operatorname{perf} A$, the full subcategory consisting of all bounded complexes of finite rank free A-modules, and, $\mathrm{D}^{\mathrm{b}}(\operatorname{tors} A)$, the full subcategory consisting of all complexes quasi-isomorphic to a bounded complex of torsion modules.

Definition 5.1 Let $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$ denote the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$ by $\mathrm{D}^{\mathrm{b}}(\operatorname{tors} A)$. Let $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ denote the Verdier quotient of $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$ by $\operatorname{perf} A$. We call, $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, the *graded category of singularities* of A.

In [41], Orlov proves the following useful theorem relating $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$ and $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$:

Theorem 5.2 *Let A be a connected graded, Noetherian k-algebra and assume that A is AS-Gorenstein with Gorenstein parameter a. For any $i \in \mathbb{Z}$, we have the following statements*:

(i) *If* $a > 0$, *there is a fully-faithful functor*, $\Psi_i : \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A) \to \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$, *and a semi-orthogonal decomposition*,

$$\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A) \cong \big\langle A(-i-a+1), \dots, A(-i), \Psi_i(\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A))\big\rangle.$$

(ii) *If* $a = 0$, *there is an equivalence of triangulated categories*,

$$\Phi_i : \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A) \to \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A).$$

(iii) *If* $a < 0$, *there is a fully-faithful functor*, $\Phi_i : \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A) \to \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, *and a semi-orthogonal decomposition*,

$$\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A) \cong \big\langle k(-i), \dots, k(-i+a+1), \Phi_i(\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A))\big\rangle.$$

Recall that, in the case $A = k[x_0, \ldots, x_n]/I$, a well-known theorem of Serre states that $\mathrm{D^b}(\mathrm{qgr}\,A) \cong \mathrm{D^b}(\mathrm{coh}\,X)$ where $X = \mathrm{Proj}(A)$. If I is generated by an $k[x_0, \ldots, x_n]$-regular sequence, $f_1, \ldots, f_c$, then the algebra, A, is AS-Gorenstein with Gorenstein parameter equal to $\sum_{i=1}^{c} \deg f_i - (n+1)$. So, if we can control the Orlov spectra of $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, we can also control the Orlov spectra of $\mathrm{D^b}(\mathrm{coh}\,X)$ where X is a complete intersection. To translate statements about the category of singularities into statements about the derived category of coherent sheaves, we first need to understand what the grading shifts corresponds to on either side. We have the following lemma:

Lemma 5.3 *Let $\mathscr{T}$ be a triangulated category with $\mathscr{I}$ a thick subcategory. If we have an endofunctor, $F : \mathscr{T} \to \mathscr{T}$, so that, for any $I \in \mathscr{I}$, $F(I)$ is isomorphic to an object in $\mathscr{I}$, then F descends to an endofunctor, $\bar{F}$, of $\mathscr{T}/\mathscr{I}$. Up to natural isomorphism, $\bar{F}$ is the unique functor making the following diagram commute*:

$$\begin{tikzcd} \mathscr{T} \arrow[r, "F"] \arrow[d, "p"] \arrow[dr, phantom, "\circlearrowleft"] & \mathscr{T} \arrow[d, "p"] \\ \mathscr{T}/\mathscr{I} \arrow[r, "\bar{F}"] & \mathscr{T}/\mathscr{I} \end{tikzcd}$$

Moreover, if F is an autoequivalence, then $\bar{F}$ is also.

Proof This is a direct application of the universal property of the Verdier quotient, [36] Theorem 2.1.8. □

The autoequivalence, $(1) : \mathrm{D^b}(\mathrm{gr}\,A) \to \mathrm{D^b}(\mathrm{gr}\,A)$, preserves both $\mathrm{D^b}(\mathrm{tors}\,A)$ and $\mathrm{perf}\,A$ and, therefore, descends uniquely to an autoequivalence of both $\mathrm{D^b}(\mathrm{qgr}\,A)$ and $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, both of which shall be denoted by (1). However, under the semi-orthogonal decompositions of Theorem 5.2, the two distinct versions of (1) do not agree. Our first goal is to identify what operation on $\mathrm{D^b}(\mathrm{qgr}\,A)$ corresponds to (1) on $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$. To do this, we need to delve a bit deeper into the proof of Theorem 5.2. Let us now recollect the details of Orlov's work.

Let $\pi : \mathrm{D^b}(\mathrm{gr}\,A) \to \mathrm{D^b}(\mathrm{qgr}\,A)$ and $q : \mathrm{D^b}(\mathrm{gr}\,A) \to \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ denote the projections coming from Verdier localization. While π admits a right adjoint, usually denoted by $\mathbf{R}\omega$, q admits neither a right nor a left adjoint. To fix this, Orlov passes to a subcategory of $\mathrm{gr}\,A$. Namely he considers, $\mathrm{gr}\,A_{\geq i}$, the full subcategory of objects, M, of gr-A with $M_j = 0$ for $j < i$. Note that the (stupid) truncation functor, $\sigma_{\geq i} : \text{gr-}A \to \mathrm{gr}\,A_{\geq i}$, is right adjoint to the natural inclusion, $\mathrm{gr}\,A_{\geq i} \hookrightarrow \mathrm{gr}\,A$.

Denote the composition of the natural inclusion, $\mathrm{D^b}(\mathrm{gr}\,A_{\geq i}) \hookrightarrow \mathrm{D^b}(\mathrm{gr}\,A)$, and the projection, $\pi : \mathrm{D^b}(\mathrm{gr}\,A) \to \mathrm{D^b}(\mathrm{qgr}\,A)$, by $\pi_i : \mathrm{D^b}(\mathrm{gr}\,A_{\geq i}) \to$

$\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$. The functor, π_i, admits a right adjoint, $\sigma_{\geq i} \circ \mathbf{R}\omega =: \omega_i$. For any graded module, M, we have an exact sequence,

$$0 \to M_{\geq i} \to M \to M/M_{\geq i} \to 0.$$

For any object, X, of $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$, this induces an exact triangle,

$$\sigma_{\geq i} X \to X \to C_X,$$

with C_X lying in $\mathrm{D}^{\mathrm{b}}(\operatorname{tors} A)$. Thus, the images of X and $\sigma_{\geq i} X$ are isomorphic in $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$.

Denote the composition of the natural inclusion, $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A_{\geq i}) \hookrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$, and the projection, $q : \mathrm{D}^{\mathrm{b}}(\operatorname{gr} A) \to \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, by $q_i : \mathrm{D}^{\mathrm{b}}(\operatorname{gr} A_{\geq i}) \to \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$. For any object, X, of $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$, we can take a minimal graded free resolution $P \to X$. Recall that minimal in this context means that $d_P(P) \subset m_A P$. As P is minimal and $A_0 = k$, P_l must be generated by a free basis e_l^i with $\min_i \deg(e_l^i) \geq 1 + \min_i \deg(e_{l-1}^i)$. Thus, P_l must be concentrated in degrees above i for large enough l. We have an exact sequence of complexes,

$$0 \to P_{<i} \to P \to P_{\geq i} \to 0,$$

corresponding to splitting the free bases for each P_l into those of degree less than i and those of degree at least i. Since $P_{<i} \in \operatorname{perf} A$ it follows that, in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, we have isomorphisms, $P_{\geq i} \cong P \cong X$.

From this, Orlov deduces that the left orthogonal to $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A_{\geq i})$, in $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$, is the full subcategory of torsion complexes concentrated in degrees less than i, denoted by $\mathscr{S}_{<i}$, while the right orthogonal to $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A_{\geq i})$ consists of bounded complexes of free modules concentrated in degrees less than i, denoted by $\mathscr{P}_{<i}$. As ω_i is right adjoint to π_i, we see that the right orthogonal to the image of ω_i is the full subcategory of torsion complexes concentrated in degrees at least i, $\mathscr{S}_{\geq i}$. The image of ω_i, denoted by $\mathscr{D}_i$, is equivalent to $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$. The functor, ω_i, is a quasi-inverse to the functor, $\pi_i|_{\mathscr{D}_i}$.

Now, the kernel of q_i consists of bounded complexes of graded free modules concentrated in degree at least i, denoted by $\mathscr{P}_{\geq i}$. We now also have a nontrivial right orthogonal to the kernel of q_i, denote it by $\mathscr{T}_i$. The restriction of q_i to $\mathscr{T}_i$ is an equivalence with $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$. The quasi-inverse is the left adjoint to q_i.

From here, Orlov analyzes how the left and right orthogonals to $\mathscr{D}_i$ and $\mathscr{T}_i$ compare for different values of a to prove Theorem 5.2. He finds that for $a \geq 0$, $T_i \subset D_i$ and, for $a \leq 0$, $D_i \subset T_i$. In the case, $a \geq 0$, the left orthogonal to T_i in D_i is generated by objects isomorphic to $A(-i-a+1), \dots, A(-i)$ in $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$. In the case, $a \leq 0$, the left orthogonal to D_i in T_i is generated by objects isomorphic to $k(-i), \dots, k(-i+a+1)$ in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$.

We now begin listing a few observations about the constructions above. The autoequivalence, $(1) : \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A) \to \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, admits a nice description as an autoequivalence of D_i.

Lemma 5.4 *$L_{A(-i+1)}$ descends to the identity functor on $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$.*

Proof It is clear that $L_{A(-i+1)}$ preserves $\operatorname{perf} A$ and descends to a functor on $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$. The cone of the natural transformation, $\eta : \mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)} \to L_{A(-i+1)}$, lies in $\operatorname{perf} A$. Thus, $\bar{\eta} : \mathrm{Id}_{\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)} \to \bar{L}_{A(-i+1)}$ is an isomorphism. $\square$

Lemma 5.5 *$\pi_i \circ L_{A(-i+1)} \circ (1) \circ \omega_i$ is isomorphic to $L_{\pi A(-i+1)} \circ (1)$ on $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$.*

Proof As ω_i is right adjoint to π_i, if we apply π_i to the morphism,

$$\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)}(A(-i+1), \omega_i F(1)[j]) \otimes_k A(-i+1)[j] \stackrel{\mathrm{ev}_{\omega_i F(1)}}{\to} \omega_i F(1),$$

we get the morphism,

$$\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)}(\pi A(-i+1), F(1)[j]) \otimes_k \pi A(-i+1)[j] \stackrel{\mathrm{ev}_{F(1)}}{\to} \mathscr{F}(1).$$

Thus, $L_{\pi A(-i+1)}(F(1))$ is isomorphic to $\pi_i \circ L_{A(-i+1)} \circ (1) \circ \omega_i(F)$ for each F. We can take the dg-enhancements of $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$ and $\mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A)$ given by bounded complexes of injectives. On the level of the dg-enhancements, the adjunctions, $\pi \vdash \omega$ and $\pi_i \vdash \sigma_{\geq i}\omega$, on the abelian categories give a natural quasi-isomorphism of $L_{\pi A(-i+1)}(F(1))$ and $\pi_i \circ L_{A(-i+1)} \circ (1) \circ \omega_i(F)$. $\square$

Remark 5.6 This lemma was first noted in [27] as Lemma 5.2.1.

Consider the functors,

$$\{1\}_i := (-i+1) \circ L_{\pi A} \circ (1) \circ (i-1) : \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A) \to \mathrm{D}^{\mathrm{b}}(\operatorname{qgr} A).$$

Let $\{1\} := \{1\}_1$.

Lemma 5.7 *For any $X \in D_i$, one has $L_{A(-i+1)}(X(1)) \in D_i$.*

Proof There is a triangle in $\mathrm{D}^{\mathrm{b}}(\operatorname{gr} A)$,

$$\mathrm{RHom}_A(A(-i+1), X(1)) \otimes_k A(-i+1) \stackrel{\mathrm{ev}}{\to} X(1) \to L_{A(-i+1)}(X(1)).$$

We know that X lies in the intersection of ${}^{\perp}\mathscr{P}_{\geq i}$ and $\mathscr{P}_{<i}^{\perp}$ so $X(1)$ lies in the intersection of ${}^{\perp}\mathscr{P}_{\geq i-1}$ and $\mathscr{P}_{<i-1}^{\perp}$. As ${}^{\perp}\mathscr{P}_{\geq i} \subset {}^{\perp}\mathscr{P}_{\geq i-1}$ and $A(-i+1) \in {}^{\perp}\mathscr{P}_{\geq i}$, we see that $L_{A(-i+1)}(X(1))$ lies in ${}^{\perp}\mathscr{P}_{\geq i}$. As $A(-i+1)$ is an exceptional object in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\, A)$, $L_{A(-i+1)}(X(1))$ lies in $\langle A(-i+1)^{\perp}, \mathscr{P}_{<i-1}^{\perp}\rangle = \mathscr{P}_{<i}^{\perp}$. □

We saw above that $\omega_i : \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}\, A_{\geq i})$ is full and faithful onto D_i. Let us denote a quasi-inverse to $q_i : T_i \to \mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ by $\nu_i : \mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A) \to T_i$.

Proposition 5.8 *If $a \geq 0$, then $q_i \circ \omega_i \circ \{1\}_i \circ \pi_i \circ \nu_i$ is isomorphic to*

$$(1) : \mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A) \to \mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A).$$

If $a \leq 0$, then $\pi_i \circ \nu_i \circ (1) \circ q_i \circ \omega_i$ is isomorphic to

$$\{1\}_i : \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A).$$

Proof It is easy to see that $\{1\}_i$ is isomorphic to $L_{\pi A(-i+1)} \circ (1)$. Let us commence the manipulation proper. Assume that $a \geq 0$.

$$q_i \circ \omega_i \circ \{1\}_i \circ \pi_i \circ \nu_i \cong q_i \circ \omega_i \circ \pi_i \circ L_{A(-i+1)} \circ (1) \circ \nu_i$$

by Lemma 5.5. By Lemma 5.7, the image of $L_{A(-i+1)} \circ (1) \circ \nu_i$ lies in $T_i \subset D_i$. So

$$q_i \circ \omega_i \circ \pi_i \circ L_{A(-i+1)} \circ (1) \circ \nu_i \cong q_i \circ L_{A(-i+1)} \circ (1) \circ \nu_i,$$

as $\omega_i \circ \pi_i$ is isomorphic to the identity on D_i.

$$q_i \circ L_{A(-i+1)} \circ (1) \circ \nu_i \cong (1)$$

by Lemma 5.4.

Assume that $a \leq 0$.

$$\pi_i \circ \nu_i \circ (1) \circ q_i \circ \omega_i \cong \pi_i \circ \nu_i \circ q_i \circ L_{A(-i+1)} \circ (1) \circ \omega_i$$

by Lemma 5.4. As the image of $L_{A(-i+1)} \circ (1) \circ \omega_i$ lies in T_i, by Lemma 5.7 and $\nu_i \circ q_i$ is isomorphic to the identity on T_i , we have

$$\pi_i \circ \nu_i \circ q_i \circ L_{A(-i+1)} \circ (1) \circ \omega_i \cong \pi_i \circ L_{A(-i+1)} \circ (1) \circ \omega_i \cong \{1\}_i$$

where the last isomorphism comes from Lemma 5.5. □

Remark 5.9 Note that $\pi_i \circ \nu_i$ is Φ_i and $q_i \circ \omega_i$ is Ψ_i from Theorem 5.2. Thus, Proposition 5.8 roughly states that (1) on $\mathrm{D}_{\mathrm{sg}}^{\mathrm{gr}}(A)$ and $\{1\}_i$ on $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A)$ correspond under the semi-orthogonal decompositions of Theorem 5.2.

The previous lemma becomes even more useful in the hypersurface case. To see why, we must recall the notion of a graded matrix factorization, see [41]. The definition is a repetition of that of the category of matrix factorization while taking care of the grading. Let $A = k[x_0, \ldots, x_n]/(f)$ with f homogeneous of degree d. A graded matrix factorization is pair of graded free A-modules is a diagram,

$$P_0 \overset{p_0}{\to} P_1 \overset{p_1}{\to} P_0(d),$$

of morphisms in $\operatorname{gr} A$ so that $p_0 p_1 = f$ and $p_1 p_0 = f$. We such just denote the collection as P. Morphisms from P to Q are pairs of maps, $f_0 : P_0 \to Q_1$ and $f_1 : P_1 \to Q_1$, so that squares in the diagram

$$\begin{tikzcd}
P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0(d) \arrow[d, "f_0(d)"] \\
Q_0 \arrow[r, "q_0"] & Q_1 \arrow[r, "q_1"] & Q_0(d)
\end{tikzcd}$$

commute. A homotopy between $f : P \to Q$ and $g : P \to Q$ is a pair of maps $h_0 : P_0 \to P_1(-d)$ and $h_1 : P_1 \to Q_0$ so that $f_0 - g_0 = q_1 h_0 + h_1 p_0$ and $f_1 - g_1 = q_0 h_1 + h_0 p_1$. We also have a shift, [1], which takes P to matrix factorization,

$$P_1 \overset{p_1}{\to} P_0(d) \overset{p_0(d)}{\to} P_1(d).$$

Let $\operatorname{GrMF}(f)$ denote the homotopy category of the category of graded matrix factorizations. In [41], Orlov proves the following:

Theorem 5.10 *There is an equivalence of triangulated categories between* $\operatorname{GrMF}(f)$ *and* $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$.

We record the following elementary observations about $\operatorname{GrMF}(f)$:

Lemma 5.11 *Let A be the homogeneous coordinate ring of a hypersurface of degree d. Then,* $[2] \cong (d)$.

Remark 5.12 Combining the above lemma with Proposition 5.8 and Theorem 5.2, we see that for any smooth hypersurface of degree $n+1$ in $\mathbb{P}^n$, one has:

$$(L_{\mathscr{O}} \circ (- \otimes_{\mathscr{O}} \mathscr{O}(1)))^{n+1} \cong [2].$$

This isomorphism was first noticed by M. Kontsevich, [29], based on the relationship with the symplectic monodromy of the mirror Calabi-Yau family, see Example 2.31. The isomorphism can also be verified without reference to matrix factorizations, see [1].

Lemma 5.13 *Let A be the homogeneous coordinate ring of a hypersurface. The natural map from A to the ring of natural transformations, $\bigoplus_{i\in\mathbb{Z}} \mathrm{Nat}(\mathrm{Id}_{\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)}, (i))$, factors through $A/(\partial f)$.*

Proof This is entirely analogous to the proof of Corollary 4.10. □

Remark 5.14 A natural question in light of Lemma 5.13 is the following: is the Jacobian ring, $A/(\partial f)$, isomorphic as a graded ring to the ring of derived natural transformations from the identity to the twists, $\{i\}$. This is almost the case. Once one accounts for the twisted sectors associated to the $\mathbb{Z}/d\mathbb{Z}$-symmetry of f, there is an isomorphism, see [6] for a complete description.

We can translate these into more geometric statements. Let $\langle t\rangle := \pi_1 \circ (L_A \circ (1))^t \circ \omega_1$.

Proposition 5.15 *Let X be a hypersurface of degree d in $\mathbb{P}^n$ determined by a homogeneous polynomial, f, of degree d with $A = k[x_0, \ldots, x_n]/(f)$. If $d \geq n+1$, then the natural map, $A \to \bigoplus_{i\in\mathbb{Z}} \mathrm{Nat}(\mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)}, \{i\})$, factors through $A/(\partial f)$. If $d < n+1$, then the natural map, $A \to \bigoplus_{i\in\mathbb{Z}} \mathrm{Nat}(\mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)}, \{i\})$, factors through $A/(\partial f \cdot \mathfrak{m}_A^a)$ where $a = n+1-d$.*

Proof Assume that $d \geq n+1$. Choose an $\alpha_i \in A_1$ for $1 \leq i \leq t$. Denote the associated natural transformation in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ from $\mathrm{Id}_{\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)}$ to (1) by η_{α_i} and the natural transformation from $\mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A)}$ to $\{1\}$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{qgr}\, A)$ by $\bar{\eta}_{\alpha_i}$. Note that $q_1 \circ \omega_1$ has $\pi_1 \circ \nu_1$ as its left adjoint. To simplify notation, let us set $\Psi = q_1 \circ \omega_1$ and $\Psi^* = \pi_i \circ \nu_i$. Let $Q = \Psi \circ \Psi^*$ and denote the unit of adjunction by $e : \mathrm{Id}_{\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)} \to Q$. The composition

$$\bar{\eta}_{\alpha_t} \circ \cdots \circ \bar{\eta}_{\alpha_1} : \mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)} \to \Psi^* \circ (1) \circ Q \circ \cdots \circ Q \circ (1) \circ \Psi = \{t\}$$

factors

$$\begin{tikzcd}
 & \langle t\rangle \arrow[d] \\
\mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X)} \arrow[ur, "\Psi^* \circ \eta_{\alpha_t} \circ \cdots \circ \eta_{\alpha_1} \circ \Psi"] \arrow[r, "\bar{\eta}_{\alpha_t} \circ \cdots \circ \bar{\eta}_{\alpha_1}"'] & \{t\}
\end{tikzcd}$$

where the map, $\langle t\rangle \to \{t\}$, comes from insertions of e. If $\eta_{\alpha_t} \circ \cdots \circ \eta_{\alpha_1}$ vanishes, then so does $\bar{\eta}_{\alpha_t} \circ \cdots \circ \bar{\eta}_{\alpha_1}$. The claim now follows from Lemma 5.13.

When $d < n+1$, we have the semi-orthogonal decompositions,

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X) \cong \langle \mathcal{O}(-a), \ldots, \mathcal{O}(-1), \mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)\rangle.$$

Note that, for $1 \le j \le a$,

$$\mathcal{O}(-j)\{1\} = \begin{cases} 0 & j = 1 \\ \mathcal{O}(-j+1) & \text{otherwise.} \end{cases}$$

Let F be an object of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. We can decompose F via the exact triangle,

$$F_s \to F \to F_e$$

using the semiorthogonal decomposition above. Now let α be a polynomial of degree $i \ge a$ and β be a polynomial of degree j in ∂f. We have the following commutative diagram:

$$\begin{tikzcd}
F_s \arrow[r] \arrow[d, "\alpha"] & F \arrow[r] \arrow[d, "\alpha"] \arrow[dl, dotted] & F_e \arrow[d, "0"] \\
F_s\{i\} \arrow[r] \arrow[d, "0"] & F\{i\} \arrow[r] \arrow[d, "\beta"] & F_e\{i\} \arrow[d, "\beta"] \\
F_s\{i+j\} \arrow[r] & F\{i+j\} \arrow[r] & F_e\{i+j\}
\end{tikzcd}$$

Let us justify the diagram above. When $i \ge a$, the functor, $\{i\}$, kills all objects in $\langle \mathcal{O}(-a), \ldots, \mathcal{O}(-1)\rangle$. Hence, $F_e\{i\}$ is zero, which gives us the right hand zero. This tells us that $F \xrightarrow{\alpha} F\{i\}$ factors through $F_s\{i\}$, represented by the dotted arrow. Now, $F_s\{i\} \xrightarrow{\beta} F_s\{i+j\}$ vanishes by Lemma 5.13, which gives us the left hand zero.

Now from the diagram, we see that $F \xrightarrow{\alpha\beta} F\{i+j\}$ factors through zero and thus vanishes on the arbitrary object, F. Therefore, $\alpha\beta$ lies in the kernel of the natural map, $A \to \bigoplus_{i\in\mathbb{Z}} \operatorname{Nat}(\mathrm{Id}_{\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)}, \{i\})$. The product ideal, $(\partial f \cdot \mathfrak{m}_A^a)$, is generated by elements of this type. □

Theorem 5.16 *Let $f \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d and $A := k[x_0, \ldots, x_n]/(f)$. Assume that A has an isolated singularity. For any non-zero object, M, in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$, the object, $M \oplus M(1) \oplus \cdots \oplus M(d-1)$, is a generator of $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ and*

$$\circleddash(M \oplus M(1) \oplus \cdots \oplus M(d-1)) \le 2(n+1)(d(n+1) - 2n - 1) - 1.$$

Proof After a change of basis, we can assume that $A/(x_0,\dots,x_{n-1})$ is isomorphic to $k[u]/(u^d)$ as a graded ring. Replacing M with $\operatorname{grsyz}^n(M)$, if necessary, we can assume that M is a MCM module over A, in $\operatorname{mod} A$. Here, $\operatorname{grsyz}^n(M)$ is a choice of nth graded syzygy. Lemma 4.15 says that $M/(x_0,\dots,x_{n-1})$ is nonzero in $\mathrm{D}_{\mathrm{sg}}(A_{d-1})$. Thus, it must be nonzero in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A_{d-1})$. The proof of Theorem 5.16 is concluded by Lemma 5.17 and Lemma 5.18. □

Lemma 5.17 *Let N be any nonzero object of $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A_{d-1})$. The level of $k(0)\oplus\cdots\oplus k(d-1)$ with respect to $N\oplus N(1)\oplus\cdots\oplus N(d-1)$ is at most one.*

Proof Since $(d)\cong[2]$, we can assume that we have $N(i)$ for any $i\in\mathbb{Z}$. For some l and for all i, we have $k[u]/(u^l)(i)$ in $\langle\{N(j)\}_{j\in\mathbb{Z}}\rangle_0$. The short exact sequences,

$$0\to k[u]/(u^{l-1})(i)\to k[u]/(u^l)(i)\xrightarrow{u} k[u]/(u^l)(i+1)\to k(i+1)\to 0,$$

split by Lemma 4.13. More precisely, one can choose gradings for all the modules in the proof of Lemma 4.13 so that the maps are degree zero. □

Lemma 5.18 *The generation time of $k(0)\oplus\cdots\oplus k(d-1)$ in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ is bounded above by $2(d(n+1)-2n-1)-1$.*

Proof The proof is completely analogous to the proof of Proposition 4.11. Furthermore, by Macaulay's theorem, the nilpotence of $A/(\partial f)$ is $d(n+1)-2n-1$. □

Remark 5.19 Theorem 5.16 does not hold in the case of a general complete intersection, even if we allow all grading shifts, as we have already seen in Remark 4.19.

We can translate this into a more geometric statement.

Corollary 5.20 *Let X be a smooth hypersurface of degree d in $\mathbb{P}^n$.*

(i) *Assume $1<d<n+1$. Let $F\in {}^{\perp}\langle\mathcal{O}(d-n-1),\dots,\mathcal{O}(-1)\rangle$ be nonzero. The object,*

$$\mathcal{O}(d-n-1)\oplus\cdots\oplus\mathcal{O}(-1)\oplus F\oplus\cdots\oplus F\{n+1\},$$

is a generator of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$ with generation time bounded by $2(n+1)(d(n+1)-2n-1)+n-d$.

(ii) *Assume $d=n+1$. Let F be a nonzero object of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} X)$. The object,*

$$F\oplus\cdots\oplus F\{n\},$$

is a generator of $\mathrm{D^b}(\operatorname{coh} X)$ *with generation time bounded by* $2(n+1)((n+1)^2-2n-1)-1$.

(iii) *Assume* $d > n+1$. *Let* F *be a nonzero object of* $\mathrm{D^b}(\operatorname{coh} X)$. *The object*,

$$F \oplus F\langle 1\rangle \oplus \cdots \oplus F\langle d-1\rangle,$$

is a generator of $\mathrm{D^b}(\operatorname{coh} X)$ *with generation time bounded by* $2(n+1)(d(n+1)-2n-1)-1$.

Proof Both (1) and (2) are straightforward consequences of Theorem 5.2 and Theorem 5.16 so let us assume that $d \geq n+1$ and take $F \in \mathrm{D^b}(\operatorname{coh} X)$ nonzero. From Theorem 5.16, we know that $\omega_1(F) \oplus \omega_1(F)(1) \oplus \cdots \omega_1(F)(d-1)$ is a generator of $\mathrm{D^{gr}_{sg}}(A)$ of generation time at most $2(n+1)(d(n+1)-2n-1)$. Since $L_A \circ (1)$ is isomorphic to (1) on $\mathrm{D^{gr}_{sg}}(A)$, we have

$$\pi_1(\omega_1(F) \oplus \omega_1(F)(1) \oplus \cdots \oplus \omega_1(F)(d-1)) \cong F \oplus F\langle 1\rangle \oplus \cdots \oplus F\langle d-1\rangle.$$

$\square$

Remark 5.21 We will get a comparison point for the bound in part (2) of Corollary 5.20 in Sect. 6.1 where we find that the ultimate dimension of a smooth degree three hypersurface in $\mathbb{P}^2$ is 4. Our bound above is 23.

The only obstacle to bounding the Orlov spectrum of $\mathrm{D^b}(\operatorname{coh} X)$ is controlling the ultimate dimension under semi-orthogonal decompositions. We state the following hope:

Conjecture 4 *Let* $\mathscr{T}$ *be a triangulated category with a semi-orthogonal decomposition,* $\mathscr{T} = \langle \mathscr{A}, \mathscr{B}\rangle$. *If the ultimate dimensions of* $\mathscr{A}$ *and* $\mathscr{B}$ *are finite, then the ultimate dimension of* $\mathscr{T}$ *is finite.*

Proposition 5.22 *If Conjecture* 4 *is true, then the ultimate dimension of* $\mathrm{D^b}(\operatorname{coh} X)$ *is bounded for any smooth hypersurface*, X.

Proof From Conjecture 4 and Theorem 5.2, we only have to bound the ultimate dimension of $\mathrm{D^{gr}_{sg}}(A)$ where $A = k[x_0, \ldots, x_n]/(f)$ is the homogeneous coordinate ring of X. Let d be the degree of f. Since X is smooth, A is an isolated singularity. By the graded version of Proposition 4.3, we know that the natural map, $A \to \bigoplus_{i\in\mathbb{Z}} \operatorname{Nat}(\operatorname{Id}_{\mathrm{D^{gr}_{sg}}(A)}, (i))$, factors through $\mathfrak{m}_A^s$ for some s. Let l be divisible by d and larger than s. We can change coordinates so that $x_0, \ldots, x_{n-1}$ is an A-regular sequence. Then, $x_0^l, \ldots, x_{n-1}^l$ is also an A-regular sequence and $A/(x_0^l, \ldots, x_{n-1}^l)$ is graded

Artinian complete intersection singularity. Let M be any object of $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$. $M/(x_0^l, \dots, x_{n-1}^l)M$ in $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A)$ lies in $\langle M \rangle_0$. Hence, if M is a generator, then so is $M/(x_0^l, \dots, x_{n-1}^l)M$. Using Theorem 5.2 for $A/(x_0^l, \dots, x_{n-1}^l)$, we see that the category, $\mathrm{D}^{\mathrm{gr}}_{\mathrm{sg}}(A/(x_0^l, \dots, x_{n-1}^l))$, has a full exceptional collection and thus, by Conjecture 4, has bounded Orlov spectrum. So $\mathrm{Lvl}_M(k(i))$ is uniformly bounded for all $i \in \mathbb{Z}$. Since A is isolated, the category consisting of the $k(i)$ generates. This bounds the Orlov spectrum. □

Remark 5.23 Note that we only need Conjecture 4 to hold for case where $\mathscr{A}$ is equivalent to $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} k)$.

6 Spherical collections

In this section we explore the generation time of collections of spherical objects in triangulated categories, specifically the bounded derived category of an elliptic curve and the derived Fukaya category of a genus g surface. By homological mirror symmetry for higher genus curves (see [18, 51]), we can compare this to our results from Sect. 4. However, the method of approach is fairly different from that in Sect. 4. Here we use the observation of Example 2.30, that spherical twists induce ghost maps, to produce ghost sequences from certain words in a braid group. For the reader's convenience we now recall some definitions.

Definition 6.1 Let $\mathscr{T}$ be the homotopy category of a triangulated A_∞-category. Assume that $\mathscr{T}$ possesses a Serre functor, S. An object, $\mathscr{E} \in \mathscr{T}$, is called *spherical* if,

- $S(\mathscr{E}) \cong \mathscr{E}[n]$
- $\mathrm{Hom}_{\mathscr{T}}(\mathscr{E}, \mathscr{E}[i]) \cong \begin{cases} k & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$

Definition 6.2 Let $\mathscr{T}$ be the homotopy category of a triangulated A_∞-category. A collection of m spherical objects, $\mathscr{E}_1, \dots, \mathscr{E}_m$, is called an A_m-*configuration* if,

$$\dim\left(\bigoplus_{l \in \mathbb{Z}} \mathrm{Hom}_{\mathscr{T}}(\mathscr{E}_i, \mathscr{E}_j[l])\right) = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| \geq 2. \end{cases}$$

In Definitions 2.28, 2.29, we already discussed the notions of a left and right twist functors. When we take the left twist functor with respect to a spherical object, we shall call this a *spherical twist*. The following result can be found in [52]:

Theorem 6.3 *A spherical twist is an exact autoequivalence. Moreover, if* $\mathscr{E}_1, \dots, \mathscr{E}_m$ *is an* A_m*-configuration, then the spherical twists,* $L_{\mathscr{E}_i}$*, satisfy the braid relations*:

$$\begin{aligned} L_{\mathscr{E}_i} L_{\mathscr{E}_{i+1}} L_{\mathscr{E}_i} &\cong L_{\mathscr{E}_{i+1}} L_{\mathscr{E}_i} L_{\mathscr{E}_{i+1}} \quad i = 1, \dots, m-1, \\ L_{\mathscr{E}_i} L_{\mathscr{E}_j} &\cong L_{\mathscr{E}_j} L_{\mathscr{E}_i} \quad |i-j| \geq 2. \end{aligned}$$

The following proposition will allow us to control the generation times of spherical collections:

Proposition 6.4 *Let* $S_1, \dots, S_n$ *be spherical objects in the homotopy category,* $\mathscr{T}$*, of a triangulated cohomologically-finite* A_∞*-category and assume we have* $\mathrm{HH}^0(\mathscr{T}) = k$*. Suppose there exists a relation,*

$$L_{S_{a_1}} \cdots L_{S_{a_r}} \cong \mathrm{Id}_{\mathscr{T}}$$

with $1 \leq a_i \leq n$*. Then* $S_1 \oplus \cdots \oplus S_n$ *strongly generates* $\mathscr{T}$ *with generation time at most* $r-1$*. Furthermore, if we partition the relation into intervals containing mutually orthogonal spherical objects, then the generation time is at most the number of intervals minus one.*

Proof For any object, X, in $\mathscr{T}$, the left twist by X comes equipped with a natural transformation, $\mathrm{Id}_{\mathscr{T}} \to L_X$, which descends from a morphism of A_∞-bimodules. Composing these natural transformations yields a natural transformation, $\zeta : \mathrm{Id}_{\mathscr{T}} \to L_{S_{a_1}} \cdots L_{S_{a_r}} = \mathrm{Id}_{\mathscr{T}}$. As this descends from a morphism of A_∞-bimodules, we have $\zeta \in \mathrm{HH}^0(\mathscr{T})$. By assumption, ζ must be a scalar multiple of the identity natural transformation. Since ζ vanishes on S_{a_1} it must be zero. Hence, for any object, $X \in \mathscr{T}$ we get a sequence of r morphisms,

$$X \to L_{S_{a_1}}(X) \to \cdots \to L_{S_{a_2}} \cdots L_{S_{a_r}}(X) \to X.$$

The total map must be zero and the cones of each map lie in $\langle S_1 \oplus \cdots \oplus S_n \rangle_0$. Repeated application of the octahedral axiom, as in the proof of Lemma 2.14, reveals that X is constructed in at most $r-1$ steps.

Now if $S_1, \dots, S_l$ are mutually orthogonal, then $L_{S_1} \cdots L_{S_l} = L_{S_1 \oplus \cdots \oplus S_l}$. Thus the sequence of l cones can be replaced by a single cone. The result follows. □

Remark 6.5 The analogous statement, with the assumption that $\mathscr{A}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded A_∞ category, is also true. The proof remains the same.

In addition, one can assume that the relation is of the form,

$$L_{S_{a_1}} \cdots L_{S_{a_r}} \cong [s],$$

and that any nonzero element of $\mathrm{HH}^s(\mathscr{T})$ does not vanish on S_i for some i (in particular one can take $\mathrm{HH}^s(\mathscr{T}) = 0$).

6.1 The Orlov spectrum of an elliptic curve

In this section, we study the Orlov spectrum of a smooth proper curve of genus one over an algebraically closed field of characteristic zero. Although this is a slight abuse of terminology, we refer to such a curve simply as an elliptic curve. Our goal will be to prove the following theorem in a series of lemmas:

Theorem 6.6 *The Orlov spectrum of the bounded derived category of coherent sheaves on an elliptic curve is* $\{1, 2, 3, 4\}$.

Proof This follows from Lemma 6.9 and Lemma 6.11 proven below. □

Lemma 6.7 *Let E be an elliptic curve and G be a generator of* $\mathrm{D^b}(\mathrm{coh}\, E)$. *Then up to shifting summands, G is either a vector bundle which is not semi-stable or a vector bundle plus a torsion sheaf.*

Proof Since $\mathrm{Coh}\, E$ is hereditary, all complexes are isomorphic to their cohomology, (see for example [23]). Hence, after shifting the summands, any generator, G, is a sheaf. From Atiyah's classification of vector bundles on an elliptic curve, we may assume $G = V_1 \oplus \cdots \oplus V_n$, where the V_i are indecomposable sheaves of slope μ_i. Here we follow the convention that a torsion sheaf has infinite slope. If $\mu_1 = \cdots = \mu_n \neq \infty$, then G is a vector bundle and by a well-known result of Faltings, [20], there exists a vector bundle which is orthogonal to G in $\mathrm{D^b}(\mathrm{coh}\, E)$. If $\mu_1 = \cdots = \mu_n = \infty$, then clearly they cannot generate $\mathrm{D^b}(\mathrm{coh}\, E)$ as all the objects generated by this object must be torsion sheaves. Therefore, we may assume $\mu_1 \neq \mu_2$. As there exists an autoequivalence, F, of $\mathrm{D^b}(\mathrm{coh}\, E)$ such that the slope of $F(V_2)$ has infinite slope, we may assume that V_2 is a torsion sheaf. Let D be the support of V_2. From V_2 and a the vector bundle V_1 we can get $V_1(nD)$ for all n. Since the full subcategory consisting of the objects $\{\mathscr{O}(nD)\}|_{n\in\mathbb{Z}}$ generates $\mathrm{D^b}(\mathrm{coh}\, E)$ and $- \otimes_{\mathscr{O}} V_1$ is dense (see Example 2.7), it follows that $V_1 \oplus V_2$ generates. □

Lemma 6.8 *Let E be a smooth curve of genus one. Let V be a vector bundle on E and T be a torsion sheaf. Then the generation time of $V \oplus T$ is bounded above by the generation time of $\mathscr{O} \oplus T$.*

Proof The functor, $- \otimes_{\mathscr{O}} V$, is dense (see Example 2.7). By Lemma 2.6, for any generator, G, one has

$$\circleddash(G \otimes_{\mathscr{O}} V) \leq \circleddash(G).$$

Letting $G = \mathscr{O} \oplus T$, we obtain

$$\circleddash((\mathscr{O} \oplus T) \otimes_{\mathscr{O}} V) = \circleddash(V \oplus T^{\oplus \mathrm{rk}(V)}) = \circleddash(V \oplus T) \leq \circleddash(\mathscr{O} \oplus T)$$

as desired. □

Lemma 6.9 *Let E be an elliptic curve with identity element e. Let G be a generator of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, E)$. Then the generation time of G is bounded above by the generation time of $\mathscr{O} \oplus \mathscr{O}_e$.*

Proof Write $G = V_1 \oplus \cdots \oplus V_n$ where the V_i are indecomposable sheaves of slope μ_i. By Lemma 6.7 at least two of these objects have different slope and by reordering, we may assume $\mu_1 \neq \mu_2$. Notice then that $V_1 \oplus V_2$ also generates and we have

$$\circleddash(G) \leq \circleddash(V_1 \oplus V_2).$$

As there exists an autoequivalence, F, of $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, E)$ such that the slope of $F(V_2)$ is infinite, we may also assume that V_2 is a torsion sheaf.

Let $\mathscr{P}$ be the Poincaré line bundle on $E \times E$. Now we have the following inequalities,

$$\circleddash(G) \leq \circleddash(V_1 \oplus V_2) \leq \circleddash(\mathscr{O}_E \oplus V_2) = \circleddash(\Phi_{\mathscr{P}}(V_2) \oplus \mathscr{O}_e) \leq \circleddash(\mathscr{O}_E \oplus \mathscr{O}_e).$$

The first inequality is above. The second is from Lemma 6.8. The equality in the middle comes from applying the autoequivalence $\Phi_{\mathscr{P}}$ given by the Fourier-Mukai transform through the Poincaré line bundle. The last inequality is achieved by applying Lemma 6.8 once again. □

In order to calculate the generation time of objects on an elliptic curve, we appeal to Proposition 4.3 of [38]. Let $\mathrm{coh}_I\, E$ denote the subcategory of $\mathrm{coh}\, X$ consisting of sheaves of slope, $\mu \in I \subset \mathbb{R}$. Following Oppermann, for an indecomposable vector bundle, V, on an elliptic curve, we define,

$$\delta(V) = \frac{q(V)}{(\mathrm{rk}\ (V))^2},$$

where $q(V)$ is the number of terms in the Jordan-Holder filtration of V.

Proposition 6.10 *Let V_1 and V_2 be semi-stable vector bundles of slope, μ_1 and μ_2 respectively. Suppose $\mu_1 < \mu_2$ and $\Delta = \mu_2 - \delta(V_2) - (\mu_1 + \delta(V_1)) > 0$. Then, any coherent sheaf in*

(i) $\mathrm{coh}_{\leq \mu_1 - \delta(V_1) - \frac{\delta(V_1)}{\Delta}}\, E$, *or*
(ii) $\mathrm{coh}_{> \mu_2 + \delta(V_2) + \frac{\delta(V_2)}{\Delta}}\, E$, *or*

(iii) $\mathrm{coh}_{>\mu_1+\delta(V_1)} E \cap \mathrm{coh}_{\leq \mu_2-\delta(V_2)} E$,

is a summand of the cone over a map from an object of $\langle V_1\rangle_0$ *to an object of* $\langle V_2\rangle_0$.

It is proven in [42], that $\{1,2\} \subsetneq \mathrm{D^b}(\mathrm{coh}\, C)$ for a smooth proper curve of genus at least one. However, for completeness, let us give explicit generators of $\mathrm{D^b}(\mathrm{coh}\, E)$ achieving the set, $\{1,2,3,4\}$.

Lemma 6.11 *We have the following*:

(i) $\circleddash(\mathscr{O}(-3e)\oplus\mathscr{O}\oplus\mathscr{O}(3e)) = 1$,
(ii) $\circleddash(\mathscr{O}\oplus\mathscr{O}(3e)) = 2$,
(iii) $\circleddash(\mathscr{O}\oplus\mathscr{O}_{2e}) = 3$, *and*
(iv) $\circleddash(\mathscr{O}\oplus\mathscr{O}_{e}) = 4$.

Proof The fact that $\circleddash(\mathscr{O}(-3e)\oplus\mathscr{O}\oplus\mathscr{O}(3e)) = 1$ follows directly from Proposition 6.10 and the fact that all torsion sheaves are obtained from $\mathscr{O}$ and $\mathscr{O}(3e)$ (see also [38] Example 4.6 and [42] Lemma 7).

To show $\circleddash(\mathscr{O}\oplus\mathscr{O}(3e)) = 2$, first note that $\mathscr{O}(-3e)\in\langle\mathscr{O}\oplus\mathscr{O}(3e)\rangle_1$. As we have already shown that $\circleddash(\mathscr{O}(-3e)\oplus\mathscr{O}\oplus\mathscr{O}(3e)) = 1$, we obtain $\circleddash(\mathscr{O}\oplus\mathscr{O}(3e)) \leq 2$. For the lower bound, note that, if $p\neq q$, $\mathscr{O}(p-q)$ is both left and right orthogonal to $\mathscr{O}$. Hence, $\mathscr{O}(p-q)$ can not be obtained in one step as it can not be obtained from $\mathscr{O}(3e)$ alone (this is the argument from [42]).

To prove $\circleddash(\mathscr{O}\oplus\mathscr{O}_{2e}) = 3$, begin by noting that $\mathscr{O}(-2e), \mathscr{O}(2e)\in\langle\mathscr{O}\oplus\mathscr{O}_{2e}\rangle_1$ and $\mathscr{O}(-4e), \mathscr{O}(4e)\in\langle\mathscr{O}\oplus\mathscr{O}_{2e}\rangle_2$. Applying Proposition 6.10 part (iii) with $V_1 = \mathscr{O}(-2e)$ and $V_2 = \mathscr{O}(2e)$, we obtain all semi-stable bundles of slope $-1<\mu\leq 1$ in three steps. Using $V_1 = \mathscr{O}$ and $V_2 = \mathscr{O}(4e)$, from part (iii), we get all semi-stable bundles of slope $1<\mu\leq 3$ and from part (i) we get all semi-stable bundles with slope $\mu\leq 2$. Now as the generator is self dual, we see that we get all possible slopes are achieved in three steps. The torsion sheaves are obtained in one step using $\mathscr{O}$ and $\mathscr{O}(4e)$. Hence, $\circleddash(\mathscr{O}\oplus\mathscr{O}_{2e})\leq 3$. For the lower bound, let q be a point of order two and consider the following sequence:

$$\mathscr{O}_q \to \mathscr{O}(-q)[1] \to \mathscr{O}(2e-q)[1] \to \mathscr{O}_q[1].$$

One easily verifies that all these maps are ghost for $\mathscr{O}\oplus\mathscr{O}_{2e}$, hence by Lemma 2.17 we obtain the lower bound.

Finally, to show $\circleddash(\mathscr{O}\oplus\mathscr{O}_{e}) = 4$, we use the same methods. Note that $\mathscr{O}(-e)\in\langle\mathscr{O}\oplus\mathscr{O}_e\rangle_1$ and $\mathscr{O}(2e)\in\langle\mathscr{O}\oplus\mathscr{O}_e\rangle_2$. Therefore, by Proposition 6.10 part (iii) with $V_1 = \mathscr{O}(-e)$ and $V_2 = \mathscr{O}(2e)$, all semi-stable bundles of slope μ with $0<\mu\leq 1$ are obtained in four steps. As above, all torsion sheaves are also achieved using these two objects. Since the generator is self dual we see that all objects of slope, $-1\leq\mu<0$, are achieved in four steps as well.

Furthermore, as the generator is fixed under the autoequivalence given by the Poincaré line bundle which inverts the slopes, we see that all objects of slope $\mu = 0$ or $|\mu| \geq 1$ are obtained in four steps as well. This covers all possible slopes. For the lower bound, let q be a point of order two and consider the following sequence:

$$\mathcal{O}_q \to \mathcal{O}(-q)[1] \to \mathcal{O}(e-q)[1] \to \mathcal{O}(2e-q)[1] \to \mathcal{O}_q[1].$$

One easily verifies that all these maps are ghost for $\mathcal{O} \oplus \mathcal{O}_e$ (see also Proposition 6.16 below), hence by Lemma 2.17 we obtain the lower bound. □

6.2 The Orlov spectrum of the Fukaya category of a Riemann surface of higher genus

In the previous section, we showed that the Orlov spectrum of the bounded derived category of coherent sheaves on an elliptic curve is $\{1, 2, 3, 4\}$. Via homological mirror symmetry, we could equally well view this category as the derived Fukaya category of an elliptic curve, see [45]. In this case, the generator with maximal generation time can be described by two loops on a torus which generate the fundamental group.

Let us outline the construction of the Fukaya category of a higher genus surface appearing in [51]. Let Σ_g be a closed, connected smooth surface of genus g. Let ω be a symplectic form on Σ_g and consider the circle tangent bundle, $\pi : S(T\Sigma_g) \to \Sigma_g$. To avoid the use of a metric, one can define $S(T\Sigma_g)$ as the bundle of oriented real lines in $T\Sigma_g$. The pullback of ω vanishes in de Rham cohomology so we can choose a one-form, θ, on $S(T\Sigma_g)$ with $d\theta = \pi^*\omega$.

Given a connected Lagrangian submanifold, L, we get a section, $\sigma : L \to S(TM)|_L$, from a choice of orientation on L. We say that L is *balanced* if $\int_L \sigma^*\theta = 0$. As σ and $-\sigma$ are fiber-wise homotopic in $S(TM)|_L$, the choice of orientation does not affect balance. As noted in loc. cit., each isotopy class of curves, with no contractible member, has a unique balanced representative up to Hamiltonian isotopy.

The Fukaya category, $\mathrm{Fuk}(\Sigma_g)$, of Σ_g is a $\mathbb{Z}/2\mathbb{Z}$-graded A_∞-category linear over $\mathbb{C}$ with balanced Lagrangians equipped with orientations and Spin structures as objects. The morphism complexes and multi-compositions are Morse-Bott variants of the usual Lagrangian Floer complexes and multi-compositions. We refer the reader to Sect. 7 of loc. cit. for full details. The dependence on ω and θ vanishes up to quasi-equivalence, see Sect. 6 of loc. cit. Thus, if we define the derived Fukaya category, $\mathrm{D}^\pi \mathrm{Fuk}(\Sigma_g)$, as the idempotent competition of the homotopy category of the category of perfect $\mathrm{Fuk}(\Sigma_g)$-modules, $\mathrm{H}^0(\mathrm{Perf}(\mathrm{Fuk}(\Sigma_g)))$, then $\mathrm{D}^\pi \mathrm{Fuk}(\Sigma_g)$ does not depend on the choice of ω or θ. As a consequence, the mapping class group of Σ_g acts on $\mathrm{D}^\pi \mathrm{Fuk}(\Sigma_g)$.

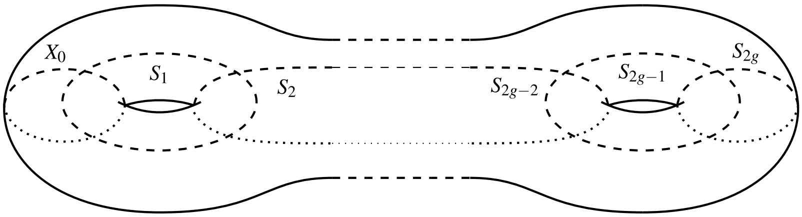

Fig. 1 A choice of the S_i and X_0 from the proof of Proposition 6.16

Remark 6.12 The previous discussion is slightly inaccurate as the A_∞-category also depends on a countable choice of balanced Lagrangian submanifolds, equipped with orientations and Spin structures, satisfying a small list of technical conditions. However, different countable sets lead to quasi-equivalent Fukaya categories so the distinctions disappear at the level of $\mathrm{D}^\pi\,\mathrm{Fuk}(\Sigma_g)$, again see loc. cit.

There is a particular geometric picture relevant to the study of $\mathrm{D}^\pi\,\mathrm{Fuk}(\Sigma_g)$. The Riemann surface, Σ_g, admits a double branched over $\mathbb{P}^1$. Let $\tau : \Sigma_g \to \Sigma_g$ denote the corresponding hyperelliptic involution. Let $S_1, \dots, S_{2g}$ be a choice of an A_{2g}-configuration of Lagrangian spheres, generating $H_1(\Sigma_g, \mathbb{Z})$ and anti-invariant under τ, up to a Hamiltonian isotopy, i.e. $\tau(S_i)$ is S_i with orientation reversed up to Hamiltonian isotopy. Figure 1 describes the situation.

To fix notation, we denote the morphism space from X to Y in $\mathrm{D}^\pi\,\mathrm{Fuk}(\Sigma_g)$ by $\mathrm{Hom}_{\Sigma_g}(X, Y)$. We can shift the gradings of the S_i so that $\mathrm{Hom}_{\Sigma_g}(S_i, S_j[1]) = 0$ for $i < j$. Let $L_i = L_{S_i}$ and let s_i denote the symplectic Dehn twist about S_i. As mentioned in Example 2.30, the endofunctor on $\mathrm{D}^\pi\,\mathrm{Fuk}(\Sigma_g)$ induced by s_i is L_i.

In order to proceed, we will need to use the following relation in the mapping class group due to M. Matsumoto (see [34] Theorem 1.5):

Theorem 6.13 *In the mapping class group of Σ_g, we have the equality,*

$$(s_1 \cdots s_{2g})^{2g+1} = \tau.$$

This gives the following corollary:

Corollary 6.14 *We have an isomorphism of endofunctors*, $(L_1 \cdots L_{2g+1})^{2g+1} \cong \tau$, *of* $\mathrm{D}^\pi\,\mathrm{Fuk}(\Sigma_g)$.

We shall also need to know the zeroth, $\mathbb{Z}/2\mathbb{Z}$-graded Hochschild cohomology group, $\mathrm{HH}^0(\mathrm{D}^\pi\,\mathrm{Fuk}(\Sigma_g))$.

Lemma 6.15 $\mathrm{HH}^i(\mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)) = \begin{cases} k & i=0 \\ k^{\oplus 2g} & i=1. \end{cases}$

Proof By Homological Mirror Symmetry, as proven in [18, 51], $\mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)$ is equivalent to the idempotent completion of $\mathrm{D}_{\mathrm{sg}}^{\mathbb{Z}/(2g+1)\mathbb{Z}}(S_g)$. As Hochschild cohomology is invariant under idempotent completion, it is sufficient to compute the Hochschild cohomology of $\mathrm{D}_{\mathrm{sg}}^{\mathbb{Z}/(2g+1)\mathbb{Z}}(S_g)$ itself.

The Serre functor on $\mathrm{D}_{\mathrm{sg}}^{\mathbb{Z}/(2g+1)\mathbb{Z}}(S_g)$ is [1]. This can be seen by dualizing the explicit diagonal factorization of [44] and noting it is quasi-isomorphic to itself shifted by the parity of the dimension of the ambient ring, as in Lemma 6.8 of [17].

Consequently, the Hochschild cohomology and homology of $\mathrm{D}_{\mathrm{sg}}^{\mathbb{Z}/(2g+1)\mathbb{Z}}(S_g)$ differ by a shift, and a computation of the Hochschild homology of $\mathrm{D}_{\mathrm{sg}}^{\mathbb{Z}/(2g+1)\mathbb{Z}}(S_g)$ suffices. This computation can be done using the formula in Theorem 2.5.4 of [44]. The computation is straightforward. We leave the details to the reader. □

Proposition 6.16 *Let $G = S_1 \oplus \cdots \oplus S_{2g}$. Then, $4g \leq \circleddash(G) \leq 8g+3$.*

Proof To prove the lower bound, we construct a ghost sequence for G of length $4g$. Namely, consider a simple loop, $X_0 \in \mathrm{D}^\pi \operatorname{Fuk}(\Sigma_g)$, which is orthogonal to $S_2, \dots, S_{2g}$ and is anti-invariant under the hyperelliptic involution, see Fig. 1. For $0 < i \leq 2g$, define X_i inductively by $X_i = L_i(X_{i-1})$ and for $2g < i \leq 4g$ by $X_i = L_{4g+1-i}(X_{i-1})$. We also have a map, $f_i : X_i \to X_{i+1}$, given by the exact triangle,

$$\operatorname{Hom}_{\Sigma_g}(S_j[1], X_i) \otimes_k S_j[1] \oplus \operatorname{Hom}_{\Sigma_g}(S_j, X_i) \otimes_k S_j \to X_i \to X_{i+1}$$

with $j = i+1$ for $0 < i \leq 2g$ and $j = 4g+1-i$ for $2g < i \leq 4g$.

Our ghost sequence for G will be the following:

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{4g-1}} X_{4g}.$$

In order to apply Lemma 2.17, we will need to show that the total map is non-zero and f_i is ghost for G for all i.

We begin our proof by showing that for all i, the map f_i is G ghost. Equivalently, we must show that f_i is ghost for S_j for all j and all i. For notational simplicity, we will consider the case where $0 < i \leq 2g$, though the proof is the same for $2g < i \leq 4g$.

The first step is to consider the triangle,

$$\operatorname{Hom}_{\Sigma_g}(S_{i+1}[1], X_i) \otimes_k S_{i+1}[1] \oplus \operatorname{Hom}_{\Sigma_g}(S_{i+1}, X_i) \otimes_k S_{i+1} \to X_i \to X_{i+1}.$$

Since any map from S_{i+1} to X_i factors through,

$$\mathrm{Hom}_{\Sigma_g}(S_{i+1}[1], X_i) \otimes_k S_{i+1}[1] \oplus \mathrm{Hom}_{\Sigma_g}(S_{i+1}, X_i) \otimes_k S_{i+1},$$

it follows that f_i is ghost for S_{i+1}.

To show that f_i is ghost for S_j with $j \neq i+1$ we will show that

$$S_j \in {}^{\perp}\langle X_i \rangle \text{ unless } j = i \text{ or } j = i+1. \tag{6.1}$$

From this equation, it follows that f_i is ghost for S_j for $j \neq i+1$ because, in this case, either $\mathrm{Hom}_{\Sigma_g}(S_j, X_i) = 0$ or $\mathrm{Hom}_{\Sigma_g}(S_j, X_{i+1}) = 0$.

Hence, in order to finish showing that all the f_i are ghost for G, we must prove the orthogonality conditions of (6.1). To achieve this, we proceed by induction on i. Assume (6.1) holds for $i-1$. Now consider the triangle,

$$\mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}) \otimes_k S_i \oplus \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}[1]) \otimes_k S_i[1] \to X_{i-1} \to X_i.$$

Let

$$\begin{aligned} H_1 := {} & \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}) \otimes_k \mathrm{Hom}_{\Sigma_g}(S_j, S_i) \oplus \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}[1]) \\ & \otimes_k \mathrm{Hom}_{\Sigma_g}(S_j, S_i[1]), \\ H_2 := {} & \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}) \otimes_k \mathrm{Hom}_{\Sigma_g}(S_j, S_i[1]) \oplus \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}[1]) \\ & \otimes_k \mathrm{Hom}_{\Sigma_g}(S_j, S_i). \end{aligned}$$

Applying the functor, $\mathrm{Hom}_{\Sigma_g}(S_j, -)$, one obtains a long exact sequence,

$$\begin{array}{l} \cdots \to H_1 \to \mathrm{Hom}_{\Sigma_g}(S_j, X_{i-1}) \to \mathrm{Hom}_{\Sigma_g}(S_j, X_i) \to \\ \to H_2 \to \mathrm{Hom}_{\Sigma_g}(S_j, X_{i-1}[1]) \to \mathrm{Hom}_{\Sigma_g}(S_j, X_i[1]) \to \\ \to H_1 \to \mathrm{Hom}_{\Sigma_g}(S_j, X_{i-1}) \to \mathrm{Hom}_{\Sigma_g}(S_j, X_i) \to \cdots \end{array}$$

For $j \neq i-1, i,$ or $i+1$, S_j is orthogonal to S_i hence $H_1 = H_2 = 0$. By the induction hypothesis, S_j is orthogonal to X_{i-1}, hence the terms in the middle in the long exact sequence above vanish as well. Therefore, S_j is orthogonal to X_i.

The only case which remains to show is when $j = i-1$. In this case,

$$\mathrm{Hom}_{\Sigma_g}(S_{i-1}, S_i[1]) = 0.$$

Hence,

$$H_1 = \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}) \otimes \mathrm{Hom}_{\Sigma_g}(S_{i-1}, S_i)$$

and

$$H_2 = \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}[1]) \otimes_k \mathrm{Hom}_{\Sigma_g}(S_{i-1}, S_i).$$

Therefore, applying the long exact sequence, one must show that the maps,

$$\alpha : \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}) \otimes_k \mathrm{Hom}_{\Sigma_g}(S_{i-1}, S_i) \to \mathrm{Hom}_{\Sigma_g}(S_{i-1}, X_{i-1})$$

and

$$\beta : \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}[1]) \otimes_k \mathrm{Hom}_{\Sigma_g}(S_{i-1}, S_i) \to \mathrm{Hom}_{\Sigma_g}(S_{i-1}, X_{i-1}[1]),$$

are isomorphisms. To this end, consider the following exact triangle:

$$S_{i-1} \to S_i \to L_{i-1}(S_i).$$

Notice that

$$\begin{aligned} \mathrm{Hom}_{\Sigma_g}(L_{i-1}(S_i), X_{i-1}[k]) &= \mathrm{Hom}_{\Sigma_g}(L_{i-1}(S_i), L_{i-1}(X_{i-2})[k]) \\ &= \mathrm{Hom}_{\Sigma_g}(S_i, X_{i-2}[k]). \end{aligned}$$

Hence, this morphism space vanishes by the induction hypothesis. Therefore, when we apply the functor $\mathrm{Hom}_{\Sigma_g}(-, X_{i-1})$, we get two isomorphisms,

$$\mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}) \to \mathrm{Hom}_{\Sigma_g}(S_{i-1}, X_{i-1})$$

and

$$\mathrm{Hom}_{\Sigma_g}(S_i, X_{i-1}[1]) \to \mathrm{Hom}_{\Sigma_g}(S_{i-1}, X_{i-1}[1]).$$

Since $\mathrm{Hom}_{\Sigma_g}(S_{i-1}, S_i)$ is one dimensional, these two isomorphisms can be identified with α and β.

In summary, we have proven the validity of (6.1) and from this we were able to deduce that all maps in this sequence are ghost for G.

Next, we would like to show that the total map, $X_0 \to X_{4g}$, is non-zero. To get this result for the map from X_0 to X_{4g-1}, we proceed once again by induction. To establish the base case, notice that as X_0 is not a summand of S_1, the map, $X_0 \to X_1$, is non-zero. Now, consider the triangle,

$$\mathrm{Hom}_{\Sigma_g}(S_j, X_i) \otimes_k S_j \oplus \mathrm{Hom}_{\Sigma_g}(S_j, X_i[1]) \otimes_k S_j[1] \to X_i \to X_{i+1}.$$

Applying the functor $\mathrm{Hom}_{\Sigma_g}(X_0, -)$ to the above triangle and using the fact that S_j is orthogonal to X_0 for $j \geq 2$, we obtain that this map in non-zero until $i = 4g - 1$ i.e. $X_0 \to X_{4g-1}$ is non-zero.

Now we have,

$$X_{4g} = L_1 \cdots L_{2g} L_{2g} \cdots L_1(X_0) \cong (L_1 \cdots L_{2g})^{2g+1}(X_0) \cong \tau(X_0) \cong X_0[1].$$

The second equality follows from (6.1), the third equality comes from the relation in Corollary 6.14 and the last equality comes from the fact that X_0 was chosen to be τ-anti-invariant. From the above equation, it follows that $X_{4g-1} = L_1^{-1}(X_0)[1]$. This allows us to easily calculate morphisms from S_1 to X_{4g-1}. Namely,

$$\mathrm{Hom}_{\Sigma_g}(S_1, L_1^{-1}(X_0)) = \mathrm{Hom}_{\Sigma_g}(S_1, X_0)$$

is one dimensional and $\mathrm{Hom}_{\Sigma_g}(S_1, L_1^{-1}(X_0)[1]) = \mathrm{Hom}_{\Sigma_g}(S_1, X_0[1]) = 0$. Hence the map from X_{4g-1} to X_{4g} fits into the following triangle:

$$S_1 \to X_{4g-1}[1] \to X_{4g}[1].$$

Applying the functor $\mathrm{Hom}_{\Sigma_g}(X_0, -)$, one obtains a long exact sequence,

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}_{\Sigma_g}(X_0, X_{4g}[1]) \arrow[r] & \mathrm{Hom}_{\Sigma_g}(X_0, S_1[1]) \arrow[dll, rounded corners] \\ \mathrm{Hom}_{\Sigma_g}(X_0, X_{4g-1}) \arrow[r] & \mathrm{Hom}_{\Sigma_g}(X_0, X_{4g}) \arrow[r] & \mathrm{Hom}_{\Sigma_g}(X_0, S_1) \end{tikzcd}$$

Since $X_{4g} = X_0[1]$, we deduce that $\mathrm{Hom}_{\Sigma_g}(X_0, X_{4g}[1]) = \mathrm{Hom}_{\Sigma_g}(X_0, X_0)$. Thus, the first map in the above sequence is nonzero because the identity cannot lie in the kernel. Furthermore, as $\mathrm{Hom}_{\Sigma_g}(X_0, S_1[1])$ is one dimensional, the first map must be a surjection. We conclude that the map, $\mathrm{Hom}_{\Sigma_g}(X_0, X_{4g-1}) \to \mathrm{Hom}_{\Sigma_g}(X_0, X_{4g})$, is an inclusion. As we have already deduced that our map, $X_0 \to X_{4g-1}$, is nonzero, it follows that the total map, $X_0 \to X_{4g}$, is nonzero.

Ultimately, we have produced a nonzero map which factors as $4g$ ghost maps for G. By Lemma 2.17 we get $4g \leq \ominus(G)$.

For the upper bound one notes that,

$$(L_1 L_3 \cdots L_{2g-1} L_2 L_4 \cdots L_{2g})^{4g+2} \cong (L_1 \cdots L_{2g})^{4g+2} \cong \mathrm{Id}_{\mathrm{D}^\pi \mathrm{Fuk}(\Sigma_g)}.$$

The first equality is just a formal relation in the braid group, and the second comes from squaring the relation in Corollary 6.14. By Lemma 6.15, we can apply Proposition 6.4, see also Remark 6.5, which yields the upper bound, $\ominus(G) \leq 8g + 3$. $\square$

Remark 6.17 The beginning of the proof above works in the abstract setting. That is, if $S_0, \dots, S_n$ is an A_{n+1}-configuration of spherical objects in $\mathscr{T}$ such that the A_n-configuration, $S_1 \oplus \cdots \oplus S_n$, generates, then $2n - 1 \leq \circleddash(S_1 \oplus \cdots \oplus S_n)$.

Remark 6.18 The lower bound of 4 for the generator $\mathscr{O} \oplus \mathscr{O}_e$ on an elliptic curve is a special case of the proposition above when $\mathrm{D^b}(\operatorname{coh} E)$ is viewed as a derived Fukaya category via mirror symmetry. In this case, using various algebraic techniques, we were able to achieve an upper bound of 4 as well (see Sect. 6.1). The authors believe that for curves of higher genus, the above lower bound is in fact an equality, i.e. this generator has generation time $4g$. Furthermore, we suspect that $4g$ is the ultimate dimension of the derived Fukaya category of a genus g symplectic surface, like in the genus one case.

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