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Hodge theoretic aspects of mirror symmetry

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Abstract. We discuss the Hodge theory of algebraic non-commutative spaces and analyze how this theory interacts with the Calabi-Yau condition and with mirror symmetry. We develop an abstract theory of non-commutative Hodge structures, investigate existence and variations, and propose explicit construction and classification techniques. We study the main examples of non-commutative Hodge structures coming from a symplectic or a complex geometry possibly twisted by a potential. We study the interactions of the new Hodge theoretic invariants with mirror symmetry and derive non-commutative analogues of some of the more interesting consequences of Hodge theory such as unobstructedness and the construction of canonical coordinates on moduli.

Contents

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1. Introduction

This paper is the first in a series aiming to develop a general procedure associating a 2-dimensional cohomological field theory in the sense of [**KM94**] (CohFT for short) to a certain structure in derived algebraic geometry. More precisely, for any Calabi-Yau A_∞-category satisfying appropriate finiteness conditions (smoothness and compactness), and such that a noncommutative analog of the Hodge $\Rightarrow$ de Rham spectral sequence collapses, we associate an infinite-dimensional family of CohFTs. The additional parameters needed to specify the CohFT are of a purely cohomological nature. Conjecturally, our procedure applied to the Fukaya category should give (higher genus) Gromov-Witten invariants of the underlying symplectic manifold.

This program was first outlined by the second author in several talks given in 2003-2004, and some aspects of it were later developed in depth by K.Costello [**Cos07b, Cos05, Cos07c, Cos07a**]. The whole picture turns out to be very intricate, and in the process of writing we realized that we have to address a large variety of problems. In this installment we do not discuss the general plan of our approach but rather focus on those features of A_∞ or dg categories that can be captured by Hodge theoretic constructions. We propose a formalism that starts with Homological Mirror Symmetry and extrapolates a geometric picture for the requisite categories that makes them amenable to study via old and new Hodge theory. Our hope is that this geometric treatment will provide new invariants and will expand the scope of possible applications in symplectic geometry and algebraic geometry.

Mirror symmetry was introduced in physics as a special duality between two $N = 2$ superconformal field theories. Traditionally a $N = 2$ superconformal field theory is constructed as a quantization of a non-linear σ-model with target a compact Calabi-Yau manifold equipped with a Ricci-flat Kähler metric and a closed 2-form - the so called B-field. Two Calabi-Yau manifolds X and Y form a *mirror pair* $X|Y$ if the associated $N = 2$ superconformal field theories are mirror dual to each other [**CK99**].

Homological Mirror Symmetry was introduced in 1994 by the second author for the case of Calabi-Yau manifolds, but today the realm of its applicability is much broader. In particular many of our considerations in the present work are governed by an analogue of Homological Mirror Symmetry for geometries with potentials. We study the effect of such mirror symmetry on the associated categories of D-branes and especially on the associated non-commutative Hodge structures on homological invariants, i.e. on the Hochschild and cyclic homology and cohomology of such categories. We study mirror pairs consisting of a compact manifold on one side, and of a Landau-Ginzburg model with a proper potential on a non-compact manifold having $c_1 = 0$ on the other. We formulate the mirror symmetry conjecture on the Hodge theoretic level in both directions. That is, we compare the non-commutative Hodge structures associated with a compact complex manifold and a mirror holomorphic Landau-Ginzburg model, and also the non-commutative Hodge structures associated with a compact complex manifold with a chosen smooth anticanonical divisor and with the mirror symplectic Landau-Ginzburg model. This picture is clearly non-symmetric and has to be generalized. In order to completely understand the Hodge theoretic aspect of mirror symmetry, one will have to allow for non-trivial potentials on both sides of the duality and include the novel toric dualities between formal Landau-Ginzburg models of Clarke [**Cla08**] and the new smooth variations of non-commutative Hodge structures of Calabi-Yau type that we attach to anticanonical $\mathbb{Q}$-divisors in Section 4.3. We plan to return to such a generalization in a future work.

Due to its foundational nature the paper comes out somewhat long-winded and technical, for which we apologize. It is organized in three major parts:

The first part introduces and develops the abstract theory of non-commutative (**nc**) Hodge structures. This theory is a variant of the formalism of semi-infinite Hodge structures that was introduced by Barannikov [**Bar01, Bar02a, Bar02b**]. We discuss the general theory of **nc**-Hodge structures in the abstract and analyze the various ways in which the Betti, de Rham and Hodge filtration data can be specified. In particular we compare **nc** and ordinary Hodge theory and explain how **nc**-Hodge theory fits within the setup of categorical non-commutative geometry. We also pay special attention to the **nc**-aspect of Hodge theory and its interaction with the classification of irregular connections on the line via topological data. One of the most useful technical results in this part is the gluing Theorem 2.35 which allows us to assemble **nc**-Hodge structures out of some simple geometric ingredients. This theorem is used later in the paper for constructing **nc**-Hodge structures attached to geometries with a potential.

The second part explains how symplectic and complex geometry give rise to **nc**-Hodge structures and how these structures can be viewed as interesting invariants of Gromov-Witten theory, projective geometry and the theory of algebraic cycles. In particular we analyze the Betti part of the **nc**-Hodge theory of a projective space (viewed as a symplectic manifold) and use this analysis to propose a general conjecture for the integral structure on the cohomology of the Fukaya category of a general compact symplectic manifold. The formula for the integral structure uses only genus zero Gromov-Witten invariants and characteristic classes of the tangent bundle. Our conjecture is in complete agreement with the recent work of Iritani [**Iri07**] who made a similar proposal based on mirror symmetry for toric Fano orbifolds. We also discuss in detail the origin of the Stokes data for holomorphic geometries with potentials and investigate the possible categorical incarnations of these data.

In the third part we study **nc**-Hodge structures and their variations under the Calabi-Yau condition. We extend and generalize the standard treatment of the deformation theory of Calabi-Yau spaces in order to get a theory which works equally well in the **nc**-context and to be able to properly define the canonical coordinates in Homological Mirror Symmetry. We approach the deformation-obstruction problem both algebraically and by Hodge theoretic means and we obtain unobstructedness results, generalized pre-Frobenius structures and some interesting geometric properties of period domains for **nc**-Hodge structures. We also study global and infinitesimal deformations and describe different constructions of Betti and de Rham **nc**-Hodge data for ordinary geometry, relative geometry, geometry with potentials and abstract **nc**-geometry.

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2. Non-commutative Hodge structures

In this section we will discuss the notion of a pure non-commutative (**nc**) Hodge structure. The **nc**-Hodge structures are analogues of the classical notion of a pure Hodge structure on a complex vector space. Both the **nc**-Hodge structures discussed presently and Simpson's non-abelian Hodge structures [**Sim97a**] generalize classical Hodge theory. In Simpson's theory, one allows for non-linearity in the substrate of the Hodge structure: the non-abelian Hodge structures of [**Sim97a**] are given by imposing Hodge and weight filtrations on non-linear topological invariants of a Kähler space, e.g. on cohomology with non-abelian coefficients, or on the homotopy type. In contrast the **nc**-Hodge structures discussed in this paper consist of a novel filtration-type data (the twistor structure of [**Sim97b, Her03, Sab05b**]) which are still specified on a vector space, e.g. on the periodic cyclic homology of an algebra.

Similarly to ordinary Hodge theory **nc**-Hodge structures arise naturally on the de Rham cohomology of non-commutative spaces of categorical origin.

2.1. Hodge structures. We will give several different descriptions of an **nc**-Hodge structure in terms of local data. We begin with the notion of rational and unpolarized **nc**-Hodge structures, ignoring for the time being the existence of polarizations and integral lattices.

2.1.1. ***Notation.*** The **nc**-Hodge structures will be described in terms of geometric data on the punctured complex line, so we fix once and for all a coordinate u on $\mathbb{C}$ and the compactification $\mathbb{C} \subset \mathbb{P}^1$. We will write $\mathbb{C}[[u]]$ for the algebra of formal power series in u, and $\mathbb{C}((u))$ for the field of formal Laurent series in u. Similarly, we will write $\mathbb{C}\{u\}$ for the algebra of power series in u having positive radius of convergence, and $\mathbb{C}\{u\}[u^{-1}]$ for the field of meromorphic Laurent series in u with a pole at most at $u = 0$.

2.1.2. ***Meromorphic connections on the complex line.*** We will need some standard notions and facts related to meromorphic connections on Riemann surfaces. We briefly recall those next. More details can be found in e.g. [**Sab02**, chapter II].

Let $\mathscr{M}$ be a finite dimensional vector space over $\mathbb{C}\{u\}[u^{-1}]$, and let ∇ be a ***meromorphic connection*** on $\mathscr{M}$. Explicitly, ∇ is given by a $\mathbb{C}$-linear map $\nabla_{\frac{d}{du}} : \mathscr{M} \to \mathscr{M}$ which satisfies the Leibniz rule for multiplication by elements in $\mathbb{C}\{u\}[u^{-1}]$. A ***holomorphic extension of*** $\mathscr{M}$ is a free finitely generated $\mathbb{C}\{u\}$-submodule $\mathscr{H} \subset \mathscr{M}$, such that

$$\mathscr{H} \otimes_{\mathbb{C}\{u\}} \mathbb{C}\{u\}[u^{-1}] = \mathscr{M}.$$

Traditionally (see e.g. [**Sab02**, section 0.8]) a holomorphic extension is called a *lattice*. We will avoid this classical terminology since it may lead to confusion with the integral lattice structures that we need.

As usual the data $(\mathscr{M}, \nabla)$ or $(\mathscr{H}, \nabla)$ should be viewed as local models for geometric data on a Riemann surface: $(\mathscr{M}, \nabla)$ is the local model of a germ of a meromorphic bundle with connection on a Riemann surface, and $(\mathscr{H}, \nabla)$ is the local model of a holomorphic bundle with meromorphic connection on a Riemann surface.

Suppose $(\mathscr{M}, \nabla)$ is a meromorphic bundle with connection over $\mathbb{C}\{u\}[u^{-1}]$ and let $\mathscr{H} \subset \mathscr{M}$ be a holomorphic extension. We say that $\mathscr{H}$ is **logarithmic** with respect to ∇ if $\nabla(\mathscr{H}) \subset \mathscr{H}\frac{du}{u}$. We say that $(\mathscr{M}, \nabla)$ has at most a **regular singularity at** 0 if we can find a holomorphic extension $\mathscr{H} \subset \mathscr{M}$ which is logarithmic with respect to ∇.

REMARK 2.1. The Riemann-Hilbert correspondence implies (see e.g. [**Sab02**, II.3.7]) that the functor of taking the germ at $0 \in \mathbb{P}^1$:

$$\left(\begin{array}{l}\text{finite rank algebraic vector bun-}\\ \text{dles with connections on } \mathbb{A}^1 - \{0\}\\ \text{with a regular singularity at } \infty\end{array}\right) \xrightarrow{\mathfrak{G}_0} \left(\begin{array}{l}\text{finite dimensional } \mathbb{C}\{u\}[u^{-1}]\text{-}\\ \text{vector spaces with meromor-}\\ \text{phic connections}\end{array}\right)$$

is an equivalence of categories. For future reference we choose once and for all an inverse $\mathfrak{B}_0$ of $\mathfrak{G}_0$. We will call $\mathfrak{B}_0$ the **algebraization functor** or the **Birkhoff extension functor**.

Suppose $\mathscr{H}$ is a *holomorphic bundle* over $\mathbb{C}\{u\}$ equipped with a meromorphic connection ∇. Let $\mathscr{M} = \mathscr{H} \otimes_{\mathbb{C}\{u\}} \mathbb{C}\{u\}[u^{-1}]$ and let $(M, \nabla) = \mathfrak{B}_0(\mathscr{M}, \nabla)$ be the corresponding Birkhoff extension. The algebraic bundle M on $\mathbb{A}^1 - \{0\}$ admits a natural extension to a holomorphic bundle H on $\mathbb{A}^1$: on a small punctured disc centered at $0 \in \mathbb{A}^1$, the bundle M is analytically isomorphic to $\mathscr{M}$, and so $\mathscr{H}$ gives us an extension to the full disc. In particular $\mathfrak{G}_0$ and $\mathfrak{B}_0$ can be promoted to a pair of inverse equivalences

$$\left(\begin{array}{l}\text{finite rank algebraic vector bun-}\\ \text{dles on } \mathbb{A}^1 \text{ equipped with a mero-}\\ \text{morphic connection with poles at}\\ \text{most at } 0 \text{ and } \infty\text{, and a regular}\\ \text{singularity at } \infty\end{array}\right) \underset{\mathfrak{B}_0}{\overset{\mathfrak{G}_0}{\rightleftarrows}} \left(\begin{array}{l}\text{finite rank free } \mathbb{C}\{u\}\text{-modules}\\ \text{equipped with a meromorphic}\\ \text{connection}\end{array}\right)$$

which we will denote again by $\mathfrak{G}_0$ and $\mathfrak{B}_0$.

2.1.3. ***Stokes data.*** Let $(\mathscr{H}, \nabla)$ be a holomorphic bundle with meromorphic connection over $\mathbb{C}\{u\}$. We will need the Deligne-Malgrange description of the associated meromorphic connection $(\mathscr{M}, \nabla)$ via a filtered sheaf on the circle. We briefly recall this description next. More details can be found in [**Mal83**] and [**BV85**]. Let $(M, \nabla) := \mathfrak{B}_0((\mathscr{M}, \nabla))$ be the Birkhoff extension of $(\mathscr{M}, \nabla)$ to $\mathbb{P}^1$. Consider the circle $\boldsymbol{S}^1 := \mathbb{C}^\times/\mathbb{R}_+^\times$. The sheaf of local ∇-horizontal sections of M^{an} on $\mathbb{C}^\times$ is a locally constant sheaf on $\mathbb{C}^\times$, which by contractibility of $\mathbb{R}_+^\times$ induces a locally constant sheaf S of $\mathbb{C}$-vector spaces on $\boldsymbol{S}^1$.

The sheaf S is equipped with a natural local filtration by subsheaves $\{\mathsf{S}_{\leq\omega}\}_{\omega\in\mathsf{Del}}$, where

(i) Del is the complex local system on $\boldsymbol{S}^1$ for which for every open $U \subset \boldsymbol{S}^1$ the space of sections $\mathsf{Del}(U)$ is defined to be the space of all holomorphic one-forms ω on the sector

$$\operatorname{Sec}(U) := \left\{ re^{i\varphi} \middle| r > 0, \varphi \in U \right\}$$

which are of the form

$$\omega = \left(\sum_{\substack{a\in\mathbb{Q} \\ a<-1}} c_a u^a \right) du,$$

where at most finitely many $c_a \neq 0$ and the branches u^a are chosen arbitrarily.

Note that the germs of sections of Del are naturally ordered: if $\omega', \omega'' \in \mathsf{Del}(U)$, $\varphi \in U$, and if

$$\omega' - \omega'' = c_a u^a + \begin{pmatrix} \text{higher} \\ \text{order terms} \end{pmatrix},$$

then one sets

$$\omega' <_\varphi \omega'' \quad \Leftrightarrow \quad \operatorname{Re}\left(\frac{c_a e^{i\varphi(a+1)}}{a+1} \right) < 0.$$

(ii) For every $\varphi \in \boldsymbol{S}^1$ and every $\omega \in \mathsf{Del}_\varphi$ the stalk

$$(\mathsf{S}_{\leq\omega})_\varphi \subset \mathsf{S}_\varphi$$

is defined to be the subspace

$$(\mathsf{S}_{\leq\omega})_\varphi := \left\{ s \in \mathsf{S}_\varphi = \Gamma\left(\mathbb{R}_+^\times e^{i\varphi}, M^{\mathrm{an}}\right)^\nabla \;\middle|\; \begin{array}{l} e^{-\int\omega}s \text{ has moderate growth in} \\ \text{the direction } \varphi\text{, i.e.} \\ \quad \left\| e^{-\int\omega}s \right\|_{|\mathbb{R}_+^\times e^{i\varphi}} = \boldsymbol{O}\left(r^{-N}\right), \\ \text{when } r \to 0,\ N \gg 0. \end{array} \right\}$$

Here $\|\bullet\|$ is the Hermitian norm of a section of M computed in some (any) meromorphic trivialization of M^{an} near $u = 0$.

DEFINITION 2.2. *The filtration we just defined is the* ***Deligne-Malgrange-Stokes filtration****, and the* Del*-filtered sheaf* S *is called the* ***Stokes structure*** *associated to* $(\mathscr{M}, \nabla)$.

REMARK 2.3. The Deligne-Malgrange-Stokes filtration satisfies the following property. First of all, there exists a covariantly local system of finite sets $\mathsf{Del}_{(\mathscr{M},\nabla)} \subset \mathsf{Del}$ canonically associated with $(\mathscr{M}, \nabla)$ such that the filtration by all of Del is determined by a filtration by all $\mathsf{Del}_{(\mathscr{M},\nabla)}(U)$ and all consecutive factors are non-trivial at all points of $\boldsymbol{S}^1$ except finitely many (called the directions of the Stokes rays). Outside the Stokes rays the set $\mathsf{Del}_{\mathscr{M},\nabla}(\phi)$ is totally ordered. It is easy to see that when we cross a Stokes ray then the order changes by flipping the order on several disjoint intervals (e.g. $\{1,2,3,4,5,6\} \to \{2,1,3,6,5,4\}$). Moreover, on the subquotients corresponding to these intervals, the two filtrations coming from the left and from the right of the anti-Stokes ray are opposed to each other. This implies that the graded pieces with respect to the Deligne-Malgrange-Stokes filtration are locally constant systems of vector spaces on the total space of stalks of the sheaf $\mathsf{Del}_{(\mathscr{M},\nabla)}$ (which is a disjoint union of finite coverings of $\boldsymbol{S}^1$).

REMARK 2.4. A fundamental theorem of Deligne and Malgrange [**Mal83**, Theorem 4.2], [**BV85**, Theorem 4.7.3] asserts that the functor $(\mathscr{M}, \nabla) \longmapsto (\mathsf{S}, \{\mathsf{S}_{\leq\omega}\}_{\omega\in\mathsf{Del}})$ is an equivalence between the category of meromorphic connections over $\mathbb{C}\{u\}[u^{-1}]$ and the category of Del-filtered local systems on $\boldsymbol{S}^1$ satisfying the conditions described in Remark 2.3. We will use this equivalence to define the Betti part of an **nc**-Hodge structure.

2.1.4. *The definition of an* nc*-Hodge structure.*

After these preliminaries we are now ready to define **nc**-Hodge structures.

DEFINITION 2.5. *A* ***rational pure* nc*-Hodge structure*** *consists of the data* $(H, \mathscr{E}_B, \mathsf{iso})$*, where*

- H *is a* $\mathbb{Z}/2$*-graded algebraic vector bundle on* $\mathbb{A}^1$.
- $\mathscr{E}_B$ *is a local system of finite dimensional* $\mathbb{Z}/2$*-graded* $\mathbb{Q}$*-vector spaces on* $\mathbb{A}^1 - \{0\}$.
- iso *is an* analytic *isomorphism of holomorphic vector bundles on* $\mathbb{A}^1 - \{0\}$:

$$\mathsf{iso} : \mathscr{E}_B \otimes \mathcal{O}_{\mathbb{A}^1-\{0\}} \xrightarrow{\cong} H_{|\mathbb{A}^1-\{0\}}.$$

Note: *The isomorphism* iso *induces a natural flat holomorphic connection* ∇ *on* $H_{|\mathbb{A}^1-\{0\}}$.

These data have to satisfy the following axioms:

(nc-filtration axiom) ∇ *is a meromorphic connection on* H *with a pole of order* ≤ 2 *at* $u = 0$ *and a regular singularity at* ∞. *More precisely, there exist:*

- *a holomorphic frame of* H *near* $u = 0$ *in which*

$$\nabla = d + \left(\sum_{k\geq -2} A_k u^k\right) du$$

with $A_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$, $r = \mathrm{rank}\ H$.

- *a meromorphic frame of H near $u = \infty$ in which*

$$\nabla = d + \left(\sum_{k \geq -1} B_k u^{-k} \right) d(u^{-1})$$

and $B_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$.

($\mathbb{Q}$-structure axiom) *The $\mathbb{Q}$-structure $\mathscr{E}_B$ on (H, ∇) is compatible with the Stokes data. More precisely, let $\left(\mathsf{S}, \{\mathsf{S}_{\leq \omega}\}_{\omega \in \mathsf{Del}}\right)$ be the Stokes structure corresponding to the germ $(\mathscr{H}, \nabla) := \mathfrak{G}_0(H, \nabla)$, and let $\mathsf{S}_B \subset \mathsf{S}$ be the $\mathbb{Q}$-structure on S induced from $\mathscr{E}_B$ via the isomorphism* **iso**. *We require that the Deligne-Malgrange-Stokes filtration on S is defined over $\mathbb{Q}$, i.e.*

$$(\mathsf{S}_{\leq\omega} \cap \mathsf{S}_B) \otimes_{\mathbb{Q}} \mathbb{C} = \mathsf{S}_{\leq \omega}$$

for all local sections $\omega \in \mathsf{Del}$.

(opposedness axiom) *The $\mathbb{Q}$-structure S_B induces a real structure on S and hence a complex conjugation $\tau : \mathsf{S} \to \mathsf{S}$. Let $\widehat{H}$ be the holomorphic bundle on $\mathbb{P}^1$ obtained as the gluing of $H^{\mathrm{alg}}_{|\{|u|\leq 1\}}$ and $\left(\gamma^* \overline{H^{\mathrm{alg}}}\right)_{|\{|u|\geq 1\}}$ via τ, where $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ is the real structure on $\mathbb{P}^1$ which fixes the unit circle, i.e. $\gamma(u) := 1/\bar{u}$. Then we require that $\widehat{H}$ be holomorphically trivial, i.e. $\widehat{H} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$.*

A morphism $\boldsymbol{f} : (H_1, \mathscr{E}_{B,1}, \mathsf{iso}_1) \to (H_2, \mathscr{E}_{B,2}, \mathsf{iso}_2)$ of **nc***-Hodge structures is a pair $\boldsymbol{f} = (f, f_B)$, where $f : H_1 \to H_2$ is an algebraic map of vector bundles which intertwines the connections, and $f_B : \mathscr{E}_{B,1} \to \mathscr{E}_{B,2}$ is a map of $\mathbb{Q}$-local systems, such that $f \circ \mathsf{iso}_1 = \mathsf{iso}_2 \circ (f_B \otimes \mathrm{id}_{\mathcal{O}})$. We will write* ($\mathbb{Q}$-**nc**HS) *for the category of pure* **nc***-Hodge structures.*

REMARK 2.6. The meromorphic connection (M, ∇) where $M = H \otimes_{\mathbb{C}[u]} \mathbb{C}[u, u^{-1}]$ can be thought of as the de Rham data of the **nc**-Hodge structure, the local system S_B of rational vector spaces over $\boldsymbol{S}^1$ endowed with the rational Stokes filtration (see $\mathbb{Q}$-structure axiom) can be thought of as the Betti data, and the holomorphic extension H of M can be thought of as the analogue of the Hodge filtration.

2.1.5. ***Variations of* nc*-Hodge structures.*** One can also define variations of **nc**-Hodge structures:

DEFINITION 2.7. *Let S be a complex manifold. A* ***variation of pure* nc*-Hodge structures*** *over S is a triple $(H, \mathscr{E}_B, \mathsf{iso})$, where*

- *H is a holomorphic $\mathbb{Z}/2$-graded vector bundle on $\mathbb{A}^1 \times S$ which is algebraic in the $\mathbb{A}^1$-direction.*
- *$\mathscr{E}_B$ is a local system of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on $(\mathbb{A}^1 - \{0\}) \times S$.*
- **iso** *is an analytic isomorphism of holomorphic vector bundles*

$$\mathsf{iso} : \mathscr{E}_B \otimes \mathcal{O}_{(\mathbb{A}^1 - \{0\})\times S} \xrightarrow{\cong} H_{|(\mathbb{A}^1-\{0\})\times S}.$$

Let ∇ be the induced meromorphic connection on H. The data $(H, \mathscr{E}_B, \mathsf{iso})$ should satisfy:

(nc-filtration axiom) *The connection* ∇ *has a regular singularity along* $\{\infty\} \times S$ *and Poincaré rank* ≤ 1 *along* $\{0\} \times S$, *i.e.*

$$u^2 \cdot \nabla_{\frac{\partial}{\partial u}} : H \to H$$

is a holomorphic differential operator on H *of order* ≤ 1.

(Griffiths transversality axiom) *For every locally defined vector field* $\xi \in T_S$ *we have that*

$$u \cdot \nabla_{\xi} : H \to H,$$

is a holomorphic differential operator on H *of order* ≤ 1.

($\mathbb{Q}$-structure axiom) *The Stokes structure on the local system* S *on* $\boldsymbol{S}^1 \times S$ *is well defined, i.e. the steps in the Deligne-Malgrange-Stokes filtration on* S *are sheaves on* $\boldsymbol{S}^1 \times S$. *Furthermore the* $\mathbb{Q}$*-structure* $\mathscr{E}_B$ *is compatible with the Stokes data as in Definition 2.5.*

(opposedness axiom) *The relative version of the gluing construction for* **nc***-Hodge structures gives a globally defined complex vector bundle* $\widehat{H}$ *on* $\mathbb{P}^1 \times S$, *which is holomorphically trivial in the* $\mathbb{P}^1$ *direction. Moreover, with respect to the extension* $\widehat{H}$ *the connection* ∇ *is meromorphic with Poincaré rank one along* $(\{0\} \times S) \cup (\{\infty\} \times S)$.

2.1.6. ***Relation to other definitions.*** Various special cases and partial versions of our notion of an **nc**-Hodge structure have been studied before in slightly different but related setups. We list a few of the relevant notions and references without going into detailed comparisons:

- A version of ($\mathbb{Z}$-graded) **nc**-Hodge structures appears in the fundamental work of K. Saito (see [**Sai83, Sai98b, Sai98a**] and references therein) on the Hodge theoretic invariants of quasi-homogeneous hypersurface singularities under the name ***weight system***.
- A version of the notion of a variation of (complex) **nc**-Hodge structure appears in the work of Cecotti-Vafa in Conformal Field Theory [**CV91, CV93a, CV93b, BCOV94**] under the name ***tt*-geometry***.
- Various versions of the notion of a (complex, polarized) **nc**-Hodge structure appear in algebraic geometry and non-abelian Hodge theory in the works of C. Simpson [**Sim97a, Sim97b**] and T. Mochizuki [**Moc06a, Moc06b, Moc07a, Moc07b**] under the names of ***(tame or wild) harmonic bundle*** or ***pure twistor structure***, and in the work of C. Sabbah [**Sab05b**] under the name ***integrable pure twistor D-module***.
- The analytic germ of a (complex) variation of **nc**-Hodge structures appears in mirror symmetry in the work of Barannikov [**Bar01, Bar02a, Bar02b**] and Barannikov and the second author [**BK98**] under the name ***semi-infinite Hodge structure***. The integral structures on semi-infinite Hodge structure were recently introduced and studied in the work of Iritani [**Iri07**].
- A version of the notion of a (real) **nc**-Hodge structure appears in singularity theory in the work of Hertling [**Her03, Her06**] and Hertling-Sevenheck [**HS07**] under the name ***TER structure***. Hertling and Sevenheck also

consider polarized and mixed **nc**-Hodge structures. Those appear under the names ***TERP structure*** and ***mixed TERP structure***, respectively. In particular in [**HS07**] Hertling and Sevenheck study the degenerations of TERP structures and prove a version of Schmid's nilpotent orbit theorem which gives rise to the notion of a limiting mixed TERP structure. Degenerations of variants of **nc**-Hodge structures, as well as limiting mixed **nc**-Hodge structures appear also in the works of C. Sabbah [**Sab05a**] and S. Szabo [**Sza07**].

2.1.7. ***Relation to usual Hodge theory.*** Recall (see e.g. [**Del71**]) that a ***pure rational Hodge structure*** of weight w is a triple $(V, F^{\bullet}V, V_{\mathbb{Q}})$ where:

- V is a complex vector space,
- $V_{\mathbb{Q}} \subset V$ is a $\mathbb{Q}$-subspace such that $V = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, and
- $F^{\bullet}V$ is a ***Hodge filtration*** of weight w on V, i.e. $F^{\bullet}V$ is a decreasing finite exhaustive filtration by complex subspaces which satisfies $F^pV \oplus \overline{F^{w+1-p}V} = V$, where the complex conjugation on V is the one given by the real structure $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes \mathbb{R} \subset V$.

A ***pure Hodge structure*** is a direct sum of pure Hodge structures of various weights, and a morphism of pure Hodge structures is a linear map of complex vector spaces which maps the rational structures into each other and is strictly compatible with the filtrations. We will write ($\mathbb{Q}$-HS) for the category of pure rational Hodge structures. It is well known [**Del71**] that ($\mathbb{Q}$-HS) is an abelian $\mathbb{Q}$-linear tensor category. For every $w \in \mathbb{Z}$ we have a $\otimes$-invertible object in ($\mathbb{Q}$-HS) of pure weight $2w$: the ***Tate Hodge structure*** $\mathbb{Q}(w)$ given by $\mathbb{Q}(w) := (\mathbb{C}, F^{\bullet}, \mathbb{Q})$, where $F^i = \mathbb{C}$ for $i \leq w$ and $F^i = \{0\}$ for $i > w$.

It turns out that pure Hodge structures can be viewed as **nc**-Hodge structures. This is achieved through a version of the Rees module construction (see e.g. [**Sim97a**]) which converts a filtered vector space into a bundle over the affine line $\mathbb{A}^1$. Specifically, given a pure Hodge structure $(V, F^{\bullet}V, V_{\mathbb{Q}})$ of weight w we consider the rank one meromorphic bundle with connection

$$\boldsymbol{\mathcal{T}}_{\frac{w}{2}} := \left(\mathbb{C}\{u\}[u^{-1}], d - \frac{w}{2} \cdot \frac{du}{u}\right)$$

and we set

- $\mathscr{H} := \mathscr{H}^{w \bmod 2} := \sum_i u^{-i} F^i V\{u\}$ viewed as a $\mathbb{C}\{u\}$-submodule in $\mathbb{C}\{u\}[u^{-1}] \otimes_{\mathbb{C}} V$. Clearly, this submodule is preserved by the operator $\nabla_{u\frac{d}{du}}$ for the connection $\nabla := \left(d - \frac{w}{2} \cdot \frac{du}{u}\right) \otimes \mathrm{id}_V$, i.e. $(\mathscr{H}, \nabla)$ is a logarithmic holomorphic extension of the meromorphic bundle with connection $\boldsymbol{\mathcal{T}}_{\frac{w}{2}} \otimes_{\mathbb{C}} V$.
  Note: Consider the algebraization $(H, \nabla) = \mathfrak{B}_0(\mathscr{H}, \nabla)$ of $(\mathscr{H}, \nabla)$. The fiber $H_1 := H/(u-1)H$ of H at $1 \in \mathbb{A}^1$ is canonically identified with V. By definition the connection ∇ on H has monodromy $(-1)^w \mathrm{id}_V$ and so preserves any rational subspace in V.
- $\mathscr{E}_B := \mathscr{E}_B^{w \bmod 2}$– the $\mathbb{Q}$-local system on $\mathbb{A}^1 - \{0\}$ defined as the subsheaf $\mathscr{E}_B \subset H$ consisting of sections whose value at 1 is in $V_{\mathbb{Q}} \subset V = H/(u-1)H$. In other words $\mathscr{E}_B$ is the locally constant sheaf on $\mathbb{A}^1 - \{0\}$ with fiber $V_{\mathbb{Q}}$ and monodromy $(-1)^w \mathrm{id}_{V_{\mathbb{Q}}}$.

- **iso** is the isomorphism of complex local systems, corresponding to the embedding $\mathscr{E}_B \subset H$.

REMARK 2.8. On every simply connected open (in the analytic topology) subset $U \subset \mathbb{A}^1 - \{0\}$ the bundle with connection $\boldsymbol{\mathcal{T}}_{\frac{w}{2}}$ has a horizontal section $u^{w/2}$. In particular on such opens we have $H_{|U} = \sum_i u^{w/2}u^{-i}F^i[u]$.

The data $(H, \mathscr{E}_B, \textsf{iso})$ satisfy tautologically the **($\mathbb{Q}$-structure axiom)** and the **(opposedness axiom)** from Definition 2.5. Indeed, the **($\mathbb{Q}$-structure axiom)** is satisfied since by definition ∇ has a regular singularity at 0 and so $\mathsf{S}_{\leq\omega} = \mathsf{S}$ or 0 for all ω. The **(opposedness axiom)** is satisfied as it is equivalent in the case of regular singularities to the oposedness property in the definition of the usual Hodge structures.

Thus, the assignment $(V, F^\bullet V, V_\mathbb{Q}) \to (H, \mathscr{E}_B, \textsf{iso})$ gives a functor

$$\mathfrak{n} : (\mathbb{Q}\text{-HS}) \to (\mathbb{Q}\text{-}\textsf{nc}\textsf{HS})$$

which by definition factors through the orbit category (see e.g. [**Kel05**] for the definition of an orbit category)

$$\pi : (\mathbb{Q}\text{-HS}) \to (\mathbb{Q}\text{-HS})/(\bullet \otimes \mathbb{Q}(1)),$$

i.e we have $\mathfrak{N} = \mathfrak{n} \circ \pi$ for a functor

$$\mathfrak{N} : (\mathbb{Q}\text{-HS})/(\bullet \otimes \mathbb{Q}(1)) \to (\mathbb{Q}\text{-}\textsf{nc}\textsf{HS}).$$

The proof of the following statement is an immediate consequence of the definition.

LEMMA 2.9. *The functor $\mathfrak{N}$ is fully faithful and its essential image consists of all* **nc***-Hodge structures that have regular singularities and monodromy* $=$ id *on* H^0 *and* $= -$ id *on* H^1.

REMARK 2.10. It is straightforward to check that the functor $\mathfrak{N}$ can also be defined in families and embeds the category of variations of Hodge structures (modulo the Tate twist) into the category of variations of **nc**-Hodge structures.

2.1.8. **nc-*Hodge structures of exponential type.*** As we saw in Section 2.1.7 the usual Hodge structures give rise to special **nc**-Hodge structures with regular singularities. The **nc**-Hodge structures with regular singularities are also important because they can serve as building blocks of general **nc**-Hodge structures. Let $(H, \mathscr{E}_B, \textsf{iso})$ be an **nc**-Hodge structure, let $(\mathscr{H}, \nabla) = \mathfrak{G}_0((H, \nabla))$ be the germ of (H, ∇) at zero, and assume that $A_{-2} \neq 0$, i.e. ∇ has a second order pole. According to the Turrittin-Levelt formal decomposition theorem (see e.g. [**Mal79**], [**BV85**], [**Sab02**, II.5.7 and II.5.9]) we can find a finite base change $p_N : \mathbb{C} \to \mathbb{C}$, $p_N(t) := t^N = u$, so that $p_N^*(\mathscr{H}, \nabla)[t^{-1}]$ is formally isomorphic to a direct sum of regular singular connections on meromorphic bundles multiplied by exponents of Laurent polynomials. More precisely we can find polynomial tails $g_i(t) \in \mathbb{C}[t^{-1}]$, $\mathbb{C}\{t\}[t^{-1}]$-vector spaces $\mathscr{R}_i$ and meromorphic connections

$$(\nabla_i)_{\frac{d}{dt}} : \mathscr{R}_i \to \mathscr{R}_i,$$

each with at most a regular singularity at 0, so that we have an isomorphism of formal meromorphic connections over $\mathbb{C}((t))$:

$$\Psi : p_N^*(\mathscr{H}, \nabla) \bigotimes_{\mathbb{C}\{t\}[t^{-1}]} \mathbb{C}((t)) \xrightarrow{\cong} \left(\bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{g_i} \bigotimes_{\mathbb{C}\{t\}[t^{-1}]} (\mathscr{R}_i, \nabla_i) \right) \bigotimes_{\mathbb{C}\{t\}[t^{-1}]} \mathbb{C}((t)).$$

Here $\boldsymbol{\mathcal{E}}^f$ denotes the rank one holomorphic bundle with meromorphic connection $(\mathbb{C}\{t\}, d - df)$, and $(\mathscr{R}_i, \nabla_i)$ denote meromorphic bundles with connections having regular singularities.

REMARK 2.11. The bundle $\boldsymbol{\mathcal{E}}^f$ has a non-vanishing horizontal section, namely e^f. In particular the multivalued flat sections of $\boldsymbol{\mathcal{E}}^{g_i} \otimes (\mathscr{R}_i, \nabla_i)$ are given by multiplying multivalued flat sections of $(\mathscr{R}_i, \nabla_i)$ by e^{g_i}.

In the examples coming from Mirror Symmetry that we are interested in, the base change p_N is not needed for the decomposition to work. In this case we can take $g_i(u) = \boldsymbol{c}_i/u$ where $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ denote the distinct eigenvalues of A_{-2}. Because of this we introduce the following definition (see also [**HS07**, Definition 8.1]):

DEFINITION 2.12. *We say that an* **nc***-Hodge structure* $(H, \mathscr{E}_B, \mathsf{iso})$ *is of* ***exponential type*** *if there exists a formal isomorphism*

$$\Psi : (\mathscr{H} \otimes_{\mathbb{C}\{u\}} \mathbb{C}[[u]], \nabla) \xrightarrow{\cong} \bigoplus_{i=1}^{m} \left(\boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i) \right) \otimes_{\mathbb{C}\{u\}} \mathbb{C}[[u]]$$

where $(\mathscr{R}_i, \nabla_i)$ *are meromorphic bundles with connections with regular singularities and* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ *denote the distinct eigenvalues of* A_{-2}.

REMARK 2.13. • There are various sufficient conditions that will guarantee that a given **nc**-Hodge structure is decomposable without base change. For instance, this will be the case if A_{-2} has distinct eigenvalues, or if $A_{-1} = 0$. More generally, it suffices to require that we can find holomorphic functions $\ell_i(u) \in \mathbb{C}\{u\}$ so that $\ell_i(0) = \boldsymbol{c}_i$ for $i = 1, \ldots, m$ and the characteristic polynomial of $u^2 A(u)$ is $\det\left(\boldsymbol{c} \cdot \mathrm{id} - u^2 A(u)\right) = \prod_{i=1}^m (\boldsymbol{c} - \ell_i)^{\nu_i}$.

• Not every irregular connection with a pole of order two is of exponential type. Indeed, the rank two connection

$$\nabla = d - \begin{pmatrix} 0 & u^{-2} \\ u^{-1} & \frac{u^{-1}}{2} \end{pmatrix}$$

has a horizontal section

$$\begin{pmatrix} e^{-2u^{-\frac{1}{2}}} \\ u^{\frac{1}{2}} e^{-2u^{-\frac{1}{2}}} \end{pmatrix},$$

and so one needs a quadratic base change for the formal decomposition to work for this connection.

• If an **nc**-Hodge structure $(H, \mathscr{E}_B, \mathsf{iso})$ is of exponential type, then one can check (see [**HS07**, Lemma 8.2]) that for each $i = 1, \ldots, m$ we can find a unique holomorphic extension $\mathscr{H}_{\boldsymbol{c}_i} \subset \mathcal{R}_i$ in which the connection has a second order pole and so that Ψ induces a formal isomorphism of holomorphic bundles with meromorphic connections

$$\Psi : (\mathscr{H}, \nabla) \otimes \mathbb{C}[[u]] \xrightarrow{\cong} \left(\bigoplus_{i=1}^{m} \mathcal{E}^{c_i/u} \bigotimes_{\mathbb{C}\{u\}} (\mathscr{H}_{c_i}, \nabla_i) \right) \otimes \mathbb{C}[[u]],$$

over $\mathbb{C}[[u]]$.

The **nc**-Hodge structures with regular singularities or the **nc**-Hodge structures of exponential type comprise full subcategories

$$(\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})^{\mathrm{reg}} \subset (\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})^{\mathrm{exp}} \subset (\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})$$

in $(\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})$. In fact, in the exponential type case one can state the **nc**-Hodge structure axioms in an easier way. The simplification comes from the fact that in this case the Deligne-Malgrange-Stokes filtration is given by subsheaves $\mathbf{S}_{\leq\lambda}$ of $\mathbf{S}$ that are labeled by $\lambda \in \mathbb{R}$ and consisting of solutions decaying faster than $\boldsymbol{O}\left(\exp\left(\frac{\lambda+o(1)}{r}\right)\right)$, $r = |u|$. Indeed, tracing through the definition one sees that in the exponential case for a ray defined by φ the jumps of the steps of the Deligne-Malgrange-Stokes filtration occur exactly at the numbers $\mathrm{Re}(\boldsymbol{c}_i e^{-i\varphi})$. Furthermore, the associated graded pieces for the filtration are local systems on the circle and in fact coincide with the regular pieces $(\mathscr{R}_i, \nabla_i)$ that appear in the formal decomposition of the connection. Hence one arrives at the following

DEFINITION 2.14. *A* ***rational pure*** **nc*-Hodge structure of exponential type*** *consists of the data* $(H, \mathscr{E}_B, \mathsf{iso})$, *where*

- H *is a* $\mathbb{Z}/2$*-graded algebraic vector bundle on* $\mathbb{A}^1$.
- $\mathscr{E}_B$ *is a local system of finite dimensional* $\mathbb{Z}/2$*-graded* $\mathbb{Q}$*-vector spaces on* $\mathbb{A}^1 - \{0\}$.
- **iso** *is an* analytic *isomorphism of holomorphic vector bundles on* $\mathbb{A}^1 - \{0\}$:

$$\mathsf{iso} : \mathscr{E}_B \otimes \mathcal{O}_{\mathbb{A}^1-\{0\}} \xrightarrow{\cong} H_{|\mathbb{A}^1-\{0\}}.$$

These data have to satisfy the following axioms:

(nc-filtration axiom)$^{\mathrm{exp}}$ *The connection* ∇ *induced from* **iso** *is a meromorphic connection of exponential type on* H *with a pole of order* ≤ 2 *at* $u = 0$ *and a regular singularity at* ∞. *More precisely, there exist:*

- *a holomorphic frame of* $\mathscr{H}$ *near* $u = 0$ *in which*

$$\nabla = d + \left(\sum_{k\geq -2} A_k u^k \right) du$$

with $A_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$, $r = \mathrm{rank}_{\mathbb{C}\{u\}} \mathscr{H}$.

- *a holomorphic frame of* $\mathscr{H}$ *near* $u = \infty$ *in which*

$$\nabla = d + \left(\sum_{k\geq -1} B_k u^{-k} \right) d(u^{-1})$$

and $B_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$.

- *a formal isomorphism over* $\mathbb{C}((u))$*:*

$$(\mathscr{H}[u^{-1}], \nabla) \stackrel{\cong}{\to} \bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i)$$

where $(\mathscr{R}_i, \nabla_i)$ *are meromorphic bundles with connections with regular singularities and* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ *denote the distinct eigenvalues of* A_{-2}*.*

($\mathbb{Q}$-structure axiom)$^{\mathbf{exp}}$ *The* $\mathbb{Q}$*-structure* $\mathscr{E}_B$ *on* (H, ∇) *is compatible with Stokes data in the following sense. The filtration* $\{\mathsf{S}_{\leq\lambda}\}_{\lambda\in\mathbb{R}}$ *of* S *by the subsheaves* $\mathsf{S}_{\leq\lambda}$*, whose stalk at* $\varphi \in \boldsymbol{S}^1$ *is given by*

$$(\mathsf{S}_{\leq\lambda})_{\varphi} := \left\{ s \in \mathsf{S}_{\varphi} = \Gamma\left(\mathbb{R}_{+}^{\times} e^{i\varphi}, H\right) \;\middle|\; \begin{array}{l} s \text{ is a } \nabla\text{-horizontal section of } H \text{ over the} \\ \text{ray } \mathbb{R}_{+}^{\times} e^{i\varphi}\text{, for which} \\ \qquad \left\| s\left(re^{i\varphi}\right)\right\| = \boldsymbol{O}\left(\exp\left(\dfrac{\lambda + \boldsymbol{o}(1)}{r}\right)\right) \\ \text{when } r \to 0. \end{array} \right\}$$

is defined over $\mathbb{Q}$*, i.e.*

$$(\mathsf{S}_{\leq\lambda} \cap \mathsf{S}_B) \otimes_{\mathbb{Q}} \mathbb{C} = \mathsf{S}_{\leq\lambda}$$

for all $\lambda \in \mathbb{R}$*.*

(opposedness axiom)$^{\mathbf{exp}}$ = (opposedness axiom)

REMARK 2.15. It is instructive to understand more explicitly the behavior of the Deligne-Malgrange-Stokes filtration for **nc**-Hodge structures (or more generally irregular connections) of exponential type. As before we denote by S the complex local system on the circle $\boldsymbol{S}^1$ corresponding to an **nc**-Hodge structure for which A_{-2} has distinct eigenvalues $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m$.

By definition, for every φ, the steps in the Deligne-Malgrange-Stokes filtration $(\mathsf{S}_{\leq\lambda})_{\varphi}$ jump exactly when λ crosses one of the numbers $\mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})$. More invariantly, the assignment $\varphi \in \boldsymbol{S}^1 \mapsto \{\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})\} \subset \mathbb{R}$ is a sheaf Λ of finite sets of real numbers (possibly with repetitions) on $\boldsymbol{S}^1$. For a general value of φ, the real numbers $\{\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})\}$ are all distinct but for finitely many special values of φ some of $\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})$ will coalesce. More precisely we have the Stokes rays $\mathbb{R}_{>0} \cdot i(\boldsymbol{c}_b - \boldsymbol{c}_a)$ and the associated set $\mathsf{SD} \subset [0, 2\pi)$ of Stokes directions: i.e. $\varphi \in \mathsf{SD}$ if and only if there is some pair $a \neq b$ such that $\boldsymbol{c}_a - \boldsymbol{c}_b = re^{i\left(\frac{\pi}{2}+\varphi\right)}$ for some $r > 0$. Clearly for every open arc $U \subset \boldsymbol{S}^1$ which does not intersect SD, the restriction $\Lambda_{|U}$ is a local system of finite sets of cardinality m. Moreover the values $\varphi \in \mathsf{SD}$ are precisely the ones for which some of $\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_k e^{-i\varphi})$ become equal to each other.

Now recall that for any given $\varphi \in \boldsymbol{S}^1$, the subspaces $(\mathsf{S}_{\leq\lambda})_{\varphi} \subset \mathsf{S}_{\varphi}$ do not change if we move $\lambda \in \mathbb{R}$ continuously without passing through some element of Λ_{φ}. In other words, we can label the steps of the Deligne-Malgrange-Stokes filtration by local sections of Λ, and so that at each $\varphi \in \boldsymbol{S}^1$ the steps are ordered according to the order on Λ_{φ} induced from the embedding $\Lambda_{\varphi} \subset \mathbb{R}$. The finite set $\mathsf{SD} \subset \boldsymbol{S}^1$ of Stokes directions breaks the circle into disjoint arcs. Over each such arc U we have that $\Lambda_{|U}$ is a local system of finite sets of real numbers with m linearly ordered flat sections, and the steps of the Deligne-Malgrange-Stokes filtration of $\mathsf{S}_{|U}$ are labeled naturally by these sections. If we move from U to an adjacent arc U' by passing across a Stokes direction $\phi \in \mathsf{SD}$, then some of the elements in the labelling set get identified at ϕ and get reordered when we cross over to U' (see Figure 1).

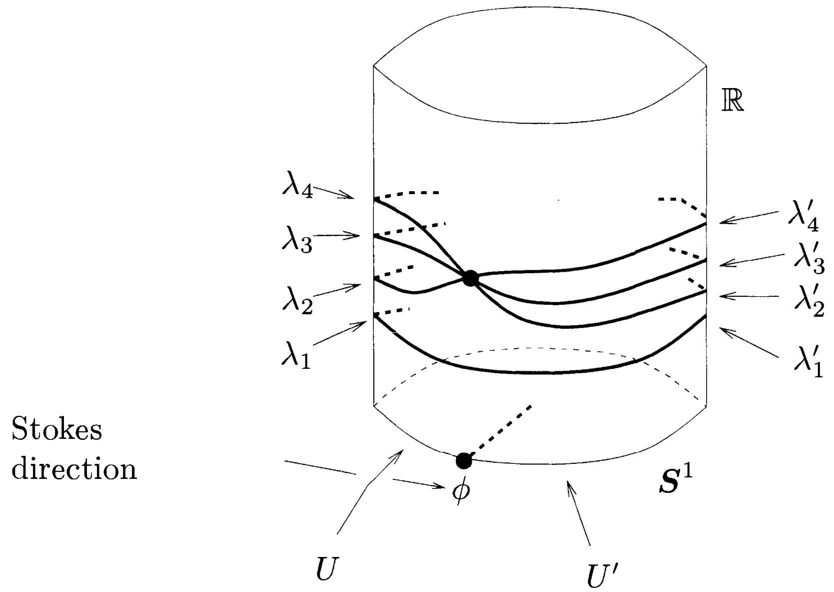


FIGURE 1. The system of labels for the Deligne-Malgrange-Stokes filtration.

In fact, if $\lambda_1 < \ldots < \lambda_m$ are the ordered flat sections of $\Lambda_{|U}$, and $\lambda'_1 < \ldots < \lambda'_m$ are the ordered flat sections of $\Lambda_{|U'}$, then the transition from the λ's to the λ''s is always such that certain groups of consecutive λ's are totally reordered into groups of consecutive λ''s. For instance in Figure 1 the passage from $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ to $\{\lambda'_1, \lambda'_2, \lambda'_3, \lambda'_4\}$ across the Stokes point $\phi \in \mathsf{SD}$ has the effect of relabelling: $\lambda_1 \mapsto \lambda'_1$, $\lambda_2 \mapsto \lambda'_4$, $\lambda_3 \mapsto \lambda'_3$, and $\lambda_4 \mapsto \lambda'_2$.

This behavior of the labelling set and the behavior of the associated filtration can be systematized in the following:

DEFINITION 2.16. *Let* S *be a finite dimensional local system of* $\mathbb{Z}/2$*-graded complex vector spaces over* $\boldsymbol{S}^1$. *Let* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m$ *be distinct complex numbers, let* Λ *be the sheaf of finite sets of real numbers on* $\boldsymbol{S}^1$ *given by* $\varphi \mapsto \{\mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi}), \ldots, \mathrm{Re}(\boldsymbol{c}_1 e^{-i\varphi})\}$, *and let* $\mathsf{SD} \subset \boldsymbol{S}^1$ *be the associated set of Stokes directions.*

An ***abstract Deligne-Malgrange-Stokes filtration of*** S ***of exponential type and exponents*** $(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m)$ *is a filtration by subsheaves* $\mathsf{S}_{\leq \lambda}$ *such that:*

- $\mathsf{S}_{\leq \lambda}$ *is labelled by local continuous sections* λ *of* Λ *and is locally constant on any arc which does not intersect* SD.
- *Suppose* $\varphi \in \mathsf{SD}$, *and let* $U, U' \subset \boldsymbol{S}^1 - \mathsf{SD}$ *be the two arcs adjacent at* φ. *Let* $\lambda_1 < \cdots < \lambda_m$ *and* $\lambda'_1 < \cdots < \lambda'_m$ *be the ordered flat sections of* $\Lambda_{|U}$ *and* $\Lambda_{|U'}$, *respectively. Trivialize* S *on* $U \cup U' \cup \{\varphi\}$ *by identifying the flat sections with the elements of the fiber* S_φ, *and let*

$$0 \subset F_{\leq \lambda_1} \subset \cdots \subset F_{\leq \lambda_m} \subset \mathsf{S}_\varphi, \quad \textit{and} \quad 0 \subset F'_{\leq \lambda'_1} \subset \cdots \subset F'_{\leq \lambda'_m} \subset \mathsf{S}_\varphi$$

be the filtrations corresponding to this trivialization and the filtrations $\mathsf{S}_{\leq\lambda}$ *on* U *and* U'*, respectively, i.e.*

$$F_{\leq\lambda_i} := \lim_{\substack{\psi\in U\\ \psi\to\varphi}} (\mathsf{S}_{\leq\lambda_i})_\psi \qquad \text{and} \qquad F'_{\leq\lambda'_i} := \lim_{\substack{\psi\in U'\\ \psi\to\varphi}} \left(\mathsf{S}_{\leq\lambda'_i}\right)_\psi$$

Let $1 \leq i_1 < j_1 \leq i_2 < j_2 \leq \cdots \leq i_k < j_k \leq m$ *be the sequence of integers such that* $\lambda_a = \lambda'_a$ *for* $a \notin [i_1, j_1] \cup [i_2, j_2] \cup \cdots \cup [i_k, j_k]$*, and for each interval* $[i_s, j_s]$ *we have that* $\lambda'_{j_s} = \lambda_{i_s}$*,* $\lambda_{j_s-1} = \lambda_{i_s+1}$*,* $\ldots \lambda'_{i_s} = \lambda_{j_s}$*. Then we require that:*

- *for each* $a \notin [i_1, j_1] \cup [i_2, j_2] \cup \cdots \cup [i_k, j_k]$ *we have* $F_{\leq\lambda_a} = F'_{\lambda'_a}$*;*
- *for each* $s = 1, \ldots, k$*,* $F_{\leq\lambda_{j_s}} = F_{\leq\lambda'_{j_s}}$ *and the filtrations*

$$\begin{array}{ccccccc}
F_{\leq\lambda_{i_s}}/F_{\leq\lambda_{i_s-1}} & \subset & F_{\leq\lambda_{i_s+1}}/F_{\leq\lambda_{i_s-1}} & \subset & \cdots & \subset & F_{\leq\lambda_{j_s}}/F_{\leq\lambda_{i_s-1}} \\
 & & & & & & \| \\
F'_{\leq\lambda'_{i_s}}/F'_{\leq\lambda'_{i_s-1}} & \subset & F'_{\leq\lambda'_{i_s+1}}/F'_{\leq\lambda'_{i_s-1}} & \subset & \cdots & \subset & F'_{\leq\lambda'_{j_s}}/F'_{\leq\lambda'_{i_s-1}}
\end{array}$$

are $(j_s - i_s)$*-opposed.*

REMARK 2.17. The above discussion generalizes immediately from connections of exponential type to arbitrary meromorphic connections (see remark 2.3). One gets a collection of curves drawn on the boundary of the cylinder which can be interpreted as a projection to 0-jets of a Legendrian link in the contact manifold of 1-jets of functions on $\boldsymbol{S}^1$.

The categories of **nc**-Hodge structures, of **nc**-Hodge structures of exponential type, or of **nc**-Hodge structures with regular singularities all behave similarly to ordinary Hodge structures. For instance one can introduce the notion of polarization on **nc**-Hodge structures, which specializes to the usual notion in the case of ordinary Hodge structures. (This will not be needed for our discussion so we will not spell it out here. The interested reader may wish to consult [**Her06, HS07, Kon08**] for the details of the definition.) In fact we have the following

LEMMA 2.18. *The categories* $(\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})^{\mathrm{reg}} \subset (\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})^{\exp} \subset (\mathbb{Q}\text{-}\mathbf{nc}\mathsf{HS})$ *are* $\mathbb{Q}$*-linear abelian categories. The respective categories of polarizable* **nc***-Hodge structures are semi-simple.*

Proof: The statement is a manifestation of Simpson's **Meta-Theorem** from [**Sim97b**]. The opposedness axiom implies that the respective categories are abelian and the existence of polarizations implies the semi-simplicity. The proofs follow verbatim the argument in usual Hodge theory or the argument in [**Sim97b**]. Alternatively one can use the comparison statement [**HS07**, Lemma 3.9] identifying the **nc**-Hodge structures with pure twistor structures and then invoke [**Sim97b**, Lemma 1.3 and Lemma 3.1]. □

The bundles with connections $(\mathscr{H}_{\mathbf{c}_i}, \nabla_i)$ can be thought of as the regular singular constituents of the **nc**-Hodge structure $(H, \mathscr{E}_B, \mathsf{iso})$. The $(\mathscr{H}_{\mathbf{c}_i}, \nabla_i)$'s are invariants of the **nc**-Hodge structure but of course they do not give a complete set of invariants (see the third point in 2.13). As usual we need additional Stokes data (see e.g. [**Sab02**]) in order to reconstruct the pair $(\mathscr{H}, \nabla)$ from its regular constituents. To understand how the rest of **nc**-Hodge structure arises from the constituents we need to understand how the rational structure $\mathscr{E}_B$ interacts with the Stokes data. This process is very similar to the interaction between Betti, de Rham and Dolbeault cohomology in ordinary Hodge theory and we will describe it in detail in section 2.3.

The **nc**-Hodge structures one finds in geometric examples are very often regular (e.g. in the case of ordinary Hodge structures) or at worst have exponential type. It is also expected that the **nc**-Hodge structures arising in mirror symmetry will always be of exponential type but at the moment this is only supported by experimental evidence.

We will discuss in detail some of this evidence in the subsequent sections. Before we get to the examples however, it will be instructive to comment on the reason for introducing the **nc**-Hodge structures at the first place. The geometric significance of these structures stems from the fact that they appear naturally on the cohomology of non-commutative spaces of categorical nature.

2.2. Hodge structures in nc geometry. The version of non-commutative geometry that is most relevant to **nc**-Hodge structures is the one in which a proxy for the notion of a non-commutative space (**nc**-space) is a category, thought of as the (unbounded) derived category of quasi-coherent sheaves on that space.

2.2.1. ***Categorical* nc*-geometry.*** The basic notion here is:

DEFINITION 2.19. *A **graded complex nc-space** (respectively a **complex nc-space**) is a $\mathbb{C}$-linear differential graded (respectively $\mathbb{Z}/2$-graded) category C which is homotopy complete and cocomplete.*

Notation: We will often write C_X for the category to signify that it describes the sheaf theory of some **nc**-space X, even when we do not have a geometric construction of X.

The categorical point of view on non-commutative geometry goes back to the works of Bondal [**Bon93**], Bondal-Orlov [**BO01, BO02**] with many non-trivial examples computed in the later works of Orlov [**Orl04, Orl05b, Orl05a**], Caldero-Keller [**CK05, CK06**], Auroux, Orlov, and the first author, [**AKO04, AKO06**], Kuznetsov [**Kuz05b, Kuz05a, Kuz06**], etc. More recently this approach to **nc**-geometry became the central part of a long term research program initiated by the second author and was studied systematically in the works of the second author and Soibelman [**KS06b, Kon08**], Toën [**Toë07a**], and Toën-Vaquie [**TV05**].

REMARK 2.20. **(i)** We do not spell out here the notions of homotopy completeness and cocompleteness in dg categories since on the one hand they are quite technical and on the other hand they will not be used later in the paper. It is worth mentioning though that some effort is required to define these notions. In the original approach of the second author described in his 2005 IAS lectures and in his 2007 course at the University of Miami the homotopy completeness and cocompletness in C was defined by a universal property for homotopy coherent diagrams of objects in the dg category labeled by simplicial sets. Alternatively [**Toë07c**], one may use the model category $(C^{\mathrm{op}} - \mathsf{mod})$ of C^{op}-dg modules, whose equivalences are the quasi-isomorphisms, and whose fibrations are the epimorphisms. In these

terms one says that C is homotopy complete if the full subcategory of $(C^{\mathrm{op}} - \mathsf{mod})$ consisting of quasi-representable objects is preserved by all small homotopy limits (defined via the given model structure). Similarly we say that C is homotopy cocomplete if C^{op} is homotopy complete.
(ii) Note that in the above definition the category C is automatically triangulated as follows already from the existence of *finite* homotopy limits, and Karoubi closed by the standard mapping telescope construction [**BN93**].

EXAMPLE 2.21. The two main types of **nc**-spaces are the following:

usual schemes: Usual complex schemes can be viewed as (graded) **nc**-spaces. Given a scheme X over $\mathbb{C}$, the corresponding category C_X is the derived category $D(\mathsf{Qcoh}(X)w)$ of quasi-coherent sheaves on X taken with an appropriate dg enhancement (see [**BK91**]). In particular, the closed point $\mathrm{pt} = \mathrm{Spec}(\mathbb{C})$ corresponds to the category C_{pt} of complexes of $\mathbb{C}$-vector spaces.

modules over an algebra: If A is a differential graded (or $\mathbb{Z}/2$-graded) unital associative algebra over $\mathbb{C}$, then we get an **nc**-space $\mathbf{ncSpec}(A)$ such that $C_{\mathbf{ncSpec}(A)} = (A - \mathsf{mod})$ is the category of dg modules over A which admit an exhaustive increasing filtration whose associated graded are sums of shifts of A.

To illustrate how the above notion of an **nc**-space fits with the **nc**Hodge structures we will concentrate on the case of **nc**-affine spaces, i.e. **nc**-spaces equivalent to $\mathbf{ncSpec}(A)$ for some differential $\mathbb{Z}/2$-graded algebra A over $\mathbb{C}$. Note that because of derived Morita equivalences an affine **nc**-space X does not determine an algebra A uniquely, i.e. different algebras can give rise to the same **nc**-space.

REMARK 2.22. The condition is not as restrictive as it appears at first glance. In fact almost all **nc**-spaces that one encounters in practice are affine. For instance usual quasi-compact quasi-separated schemes of finite type over $\mathbb{C}$ are affine when viewed as **nc**-spaces. This follows from a deep theorem of Bondal and van den Bergh [**BvdB03**] which asserts that for such a scheme X the category $C_X = D(\mathsf{Qcoh}(X))$ has a compact generator $\mathcal{E}$. That is, we can find an object $\mathcal{E} \in C_X$ so that

$$\mathrm{Hom}(\mathcal{E}, \bullet) : C_X \to C_{\mathrm{pt}}$$

commutes with homotopy colimits and has a zero kernel. In particular the dg algebra computing the category C_X is given in terms of the generator $\mathcal{E}$, i.e.

$$C_X \cong (\mathrm{Hom}(\mathcal{E}, \mathcal{E})^{\mathrm{op}} - \mathsf{mod}).$$

Suppose now that $X = \mathbf{ncSpec}(A)$. Recall that an object $\mathcal{E} \in C_X = (A - \mathsf{mod})$ is ***perfect*** if $\mathrm{Hom}(\mathcal{E}, \bullet)$ preserves small homotopy colimits. We will write Perf_X for the full subcategory of perfect objects in C_X. We now have the following definition (see e.g. [**KS06b, Kon08**] or [**TV05**]):

DEFINITION 2.23. *A complex differential $\mathbb{Z}/2$-graded algebra is called*

smooth: *if* $A \in \mathsf{Perf}_{\mathbf{ncSpec}(A \otimes A^{\mathrm{op}})}$;
compact: *if* $\dim_{\mathbb{C}} H^{\bullet}(A, d_A) < +\infty$ *or equivalently if* $A \in \mathsf{Perf}_{\mathrm{pt}}$.

Note: One can check (see e.g. [**KS06b**] or [**TV05**]) that the properties of X being smooth and compact do not depend on the choice of the algebra A which computes

C_X. Also, for a usual scheme X of finite type over $\mathbb{C}$, smoothness and compactness in the scheme-theoretic sense are equivalent to smoothness and compactness in the **nc**-sense.

2.2.2. ***The main conjecture.*** The analogy with commutative geometry suggests that one should look for pure **nc**-Hodge structures on the cohomology of smooth and proper **nc**-spaces. More precisely we have the following basic conjecture

CONJECTURE 2.24. *Let X be a smooth and compact* **nc***-space over $\mathbb{C}$. Then the periodic cyclic homology $HP_\bullet(C_X)$ of C_X carries a natural functorial pure $\mathbb{Q}$-***nc***-Hodge structure with regular singularities.*

Furthermore, if the $\mathbb{Z}/2$-grading on X can be refined to a $\mathbb{Z}$-grading, then the **nc***-Hodge structure on $HP_\bullet(C_X)$ is an ordinary pure Hodge structure, i.e. belongs to the essential image of the functor $\mathfrak{N}$.*

2.2.3. ***Cyclic homology.*** There are some natural candidates for the various ingredients of the conjectural **nc**-Hodge structure on $HP_\bullet(C_X)$. Assuming that $X \cong \mathbf{ncSpec}(A)$ is **nc**-affine, we can compute $HP_\bullet(C_X)$ in terms of A. Namely

$$HP_\bullet(C_X) = HP_\bullet(A) = HP_\bullet\left(C_\bullet^{\mathrm{red}}(A, A)((u)), \partial + u \cdot B\right),$$

where

- u is an even formal variable (of degree 2 in the $\mathbb{Z}$-graded case);
- $C_{-k+1}^{\mathrm{red}}(A, A)((u)) := A \otimes (A/\mathbb{C} \cdot 1_A)^{\otimes k} \otimes \mathbb{C}((u))$, for all $k \geq 0$;
- $\partial = b + \delta$, where

$$b(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^{\deg(a_0 \otimes \cdots \otimes a_i)} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$
$$+ (-1)^{\deg(a_0 \otimes \cdots \otimes a_n)(\deg(a_n)+1)+1} a_n a_0 \otimes \cdots \otimes a_{n-1},$$

  is the Hochschild differential, and

$$\delta(a_0 \otimes \cdots \otimes a_n) := \sum_{i=0}^{n} (-1)^{\deg(a_0 \otimes \cdots \otimes a_{i-1})} a_0 \otimes \cdots \otimes d_A a_i \otimes \cdots \otimes a_n$$

  is the differential induced by d_A via the Leibniz rule;
-

$$B(a_0 \otimes \cdots \otimes a_n) \quad := \quad \sum_{i=0}^{n} (-1)^{(\deg(a_0 \otimes \cdots \otimes a_i) - 1)(\deg(a_{i+1} \otimes \cdots \otimes a_n) - 1)}$$
$$\cdot 1_A \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i$$

  is Connes' cyclic differential.

2.2.4. ***The degeneration conjecture and the vector bundle part of the*** **nc*-Hodge structure.*** Note that by construction $HP_\bullet(C_X)$ is a module over $\mathbb{C}((u))$. We can also look at the negative cyclic homology $HC_\bullet^-(C_X)$ of C_X. By definition $HC_\bullet^-(C_X)$ is the cohomology of the complex

$$\left(C_\bullet^{\mathrm{red}}(A, A)[[u]], \partial + u \cdot B\right),$$

and so is a module over $\mathbb{C}[[u]]$. The specialization

$$HC_\bullet^-(C_X)/uHC_\bullet^-(C_X)$$

of this module at $u = 0$ maps to the cohomology of the complex

$$(C_\bullet^{\mathrm{red}}(A, A), \partial)$$

of reduced Hochschild chains for A which by definition is the Hochschild homology $HH_\bullet(A)$ of A. The Hochschild-to-cyclic spectral sequence implies that

$$\dim_{\mathbb{C}((u))} HP_\bullet(A) \leq \dim_{\mathbb{C}} HH_\bullet(A). \tag{2.2.1}$$

If X is a smooth and compact **nc**-space, the Hochschild chain complex of C_X is the derived tensor product over $A \otimes A^{\mathrm{op}}$ of a perfect complex with finite dimensional cohomology with itself. In particular $HH_\bullet(C_X) := HH_\bullet(A)$ is a finite dimensional $\mathbb{C}$-vector space, and so by (2.2.1) we have that $HP_\bullet(C_X)$ is finite dimensional over $\mathbb{C}((u))$. Thus the $\mathbb{C}[[u]]$-module $HC_\bullet^-(C_X)$ is finitely generated and so corresponds to the formal germ at $u = 0$ of an algebraic $\mathbb{Z}/2$-graded coherent sheaf on $\mathbb{A}^1_{\mathbb{C}}$. The fiber of this sheaf at $u = 0$ is $HH_\bullet(C_X)$ and the generic fiber is $HP_\bullet(C_X)$. In [**KS06b, Kon08**] the second author proposed the so called ***degeneration conjecture*** asserting that for a smooth and compact **nc**-space $X = \mathbf{ncSpec}(A)$ we have an equality of dimensions in (2.2.1). In other words the degeneration conjecture asserts that for a smooth and compact **nc**-space the $\mathbb{C}[[u]]$-module $HC_\bullet^-(C_X)$ is free of finite rank and thus corresponds to an algebraic vector bundle on the one-dimensional formal disc $\mathbb{D} := \mathrm{Spf}(\mathbb{C}[[u]])$.

REMARK 2.25. There is a lot of evidence supporting the validity of this conjecture. The work of Weibel [**Wei96**] shows that if X is a usual quasi-compact and quasi-separated complex scheme the Hochschild and periodic cyclic homology of X viewed as a **nc**-spaces can be identified with the algebraic de Rham and Dolbeault cohomology of X, respectively. Combined with the degeneration of the Hodge-to-de Rham spectral sequence in the smooth proper case this shows that the degeneration conjecture holds true for usual schemes. Also recently in a very exciting sequence of papers [**Kal07a, Kal06**] Kaledin proved the degeneration conjecture for graded **nc**-spaces $X = \mathbf{ncSpec}(A)$ for which A is concentrated in non-negative degrees. The case of graded **nc**-spaces $X = \mathbf{ncSpec}(A)$ for which A is concentrated in non-positive degrees was also settled by Shklyarov [**Shk08**]. The general graded case and the $\mathbb{Z}/2$-graded case are still wide open.

2.2.5. ***The meromorphic connection in the u-direction.*** The next observation is that the $\mathbb{C}\{u\}[u^{-1}]$-module $HP_\bullet(C_X)$ comes equipped with a natural meromorphic connection. Indeed, recall that by the work of Getzler [**Get93**] there is a version of the Gauss-Manin connection which exists on the periodic cyclic homology of any flat family of differential graded algebras (see also [**Tsy07, Kal07b**]). An analogous statement holds in the $\mathbb{Z}/2$-graded case as explained e.g. in [**KS06b**, Section 11.5]. The Gauss-Manin connection for any family of dg algebras $\mathcal{A}_x$ over the formal disc $\mathrm{Spf}\,\mathbb{C}[[x]]$ with a formal parameter x is an operator

$$\begin{aligned}\nabla^{\mathrm{GM}}_{u\frac{\partial}{\partial x}} : H^\bullet(C^{\mathrm{red}}(\mathcal{A}_x, \mathcal{A}_x)[[u,x]], \partial_{\mathcal{A}_x} + u \cdot B_{\mathcal{A}_x}) \\ \to H^\bullet(C^{\mathrm{red}}(\mathcal{A}_x, \mathcal{A}_x)[[u,x]], \partial_{\mathcal{A}_x} + u \cdot B_{\mathcal{A}_x})\end{aligned}$$

satisfying the Leibniz rule with respect to the multiplications by u and x (compare this with the **(Griffiths transversality axiom)** in Definition 2.7 from Section 2.1.5).

Suppose now that A is a differential $\mathbb{Z}/2$-graded (d $(\mathbb{Z}/2)$g) algebra with product m_A, differential d_A, and a strict unit 1_A. Then we can form a flat family $\mathcal{A} \to \mathbb{A}^1 - \{0\}$ of differential $\mathbb{Z}/2$-graded algebras parameterized by the punctured affine line $\mathbb{A}^1 - \{0\}$. The fiber $\mathcal{A}_t$ of $\mathcal{A}$ over a point $t \in \mathbb{A}^1_{\mathbb{C}} - \{0\}$ is the d$(\mathbb{Z}/2)$g algebra for which the underlying $\mathbb{Z}/2$-graded vector space is A and

$$\begin{aligned} m_{\mathcal{A}_t} &:= t \cdot m_A, \\ d_{\mathcal{A}_t} &:= t \cdot d_A, \\ 1_{\mathcal{A}_t} &:= t^{-1} \cdot 1_A. \end{aligned}$$

Looking at the scaling properties of ∂ and B we see that the identity morphism on the level of cochains induces a natural isomorphism
(2.2.2)

$$H^{\bullet}\left(C^{\mathrm{red}}_{\bullet}(\mathcal{A}_t, \mathcal{A}_t)[[u]], \partial_{\mathcal{A}_t} + u \cdot B_{\mathcal{A}_t}\right) \xrightarrow{\cong} H^{\bullet}(C^{\mathrm{red}}_{\bullet}(A, A)[[u]], \partial + ut^{-2} \cdot B).$$

This isomorphism does not come from a quasi-isomorphism of complexes, as the identity map is not a morphism of complexes: the differentials do not coincide but differ by the factor t. If A is smooth and compact, then the negative cyclic homology of the family of algebras $\mathcal{A}_t$ gives rise to an algebraic vector bundle $\widetilde{HC}^{-}$ on the product $(\mathbb{A}^1 - \{0\}) \times \mathbb{D}$. Here $\mathbb{D} := \operatorname{Spf} \mathbb{C}[[u]]$ denotes the one-dimensional formal disc. We will write (t, u) for the coordinates on $(\mathbb{A}^1 - \{0\}) \times \mathbb{D}$. We will be interested in fact only in the formal neighborhood of the point $t = 1$ where we can choose as a local coordinate $x := \log(t)$. The Getzler-Gauss-Manin connection can then be viewed as a relative holomorphic connection ∇^{GM} on $\widetilde{HC}^{-}$ which differentiates only along $\mathbb{A}^1_{\mathbb{C}} - \{0\}$. On the other hand the formal completion of the group $\mathbb{C}^{\times}$ at 1 acts on $(\mathbb{A}^1_{\mathbb{C}} - \{0\}) \times \mathbb{D}$ by $(t, u) \mapsto (\mu t, \mu^2 u)$ for $\mu \in \mathbb{C}^{\times}$. The isomorphism (2.2.2) gives rise to a $\mathbb{C}^{\times}$-equivariant structure on the vector bundle $\widetilde{HC}^{-}$ and the infinitesimal action of $d/d\mu$ associated with this equivariant structure gives a holomorphic differential operator $\mathbf{\Lambda} \in \mathrm{Diff}^{\leq 1}(\widetilde{HC}^{-}, \widetilde{HC}^{-})$ with symbol equal to

$$\left(t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}\right) \cdot \mathrm{id}_{\widetilde{HC}^{-}}.$$

Hence

$$\nabla_{u^2 \frac{\partial}{\partial u}} := \frac{u}{2} \cdot \mathbf{\Lambda} - \nabla^{\mathrm{GM}}_{\frac{ut}{2}\frac{\partial}{\partial t}}$$

is a first order differential operator on $\widetilde{HC}^{-}$ with symbol

$$u^2 \frac{\partial}{\partial u} \cdot \mathrm{id}_{\widetilde{HC}^{-}}$$

and so after restricting $\widetilde{HC}^{-}$ to $\{1\} \times \mathbb{D}$ this operator gives a meromorphic connection ∇ on the $\mathbb{C}[[u]]$-module $HC^{-}_{\bullet}(C_X)$ with at most a second order pole at $u = 0$. Note also that if the algebra A is $\mathbb{Z}$-graded, then the family $\mathcal{A}_t$ is easily seen to be trivial and the connection ∇ has a first order pole at $u = 0$ with monodromy equal to $(-1)^{\mathrm{parity}}$.

2.2.6. ***The $\mathbb{Q}$-structure.*** The categorical origin of the rational (or integral) structure of the conjectural **nc**-Hodge structure is more mysterious. Conceptually, the correct rational structure should come from the Betti cohomology or, say, the topological K-theory of the **nc**-space. There are two natural approaches to constructing the rational structure $\mathscr{E}_B \subset HP_\bullet(C_X)$:

(a) The soft algebra approach ([Kon08]). Let again $X = \mathbf{ncSpec}(A)$ be an affine **nc**-space, and assume X is compact. By analogy with the classical geometric case one expects that there should exist a nuclear Fréchet d($\mathbb{Z}/2$)g algebra A_{C^∞} so that

- The K-theory of A_{C^∞} satisfies Bott periodicity, i.e. $K_i(A_{C^\infty}) = K_{i+2}(A_{C^\infty})$ for all $i \geq 0$.
- There is a homomorphism $\varphi : A \to A_{C^\infty}$ of d($\mathbb{Z}/2$)g algebras for which $\varphi_* : HP_\bullet(A) \to HP_\bullet(A_{C^\infty})$ is an isomorphism, and the image of the Chern character map
$$\mathrm{ch} : K_\bullet(A_{C_\infty}) \to HP_\bullet(A_{C^\infty})$$
is an integral lattice, and hence gives a rational structure $\mathscr{E}_B \subset HP_\bullet(A)$.

Note: If X is a smooth and compact complex variety and if $\mathcal{E} \in \mathsf{Perf}(X)$ is a vector bundle generating C_X, then one may take
$$\begin{aligned} A &:= A^{0,\bullet}(X, \mathcal{E}^\vee \otimes \mathcal{E}), \bar{\partial}) \\ A_{C^\infty} &:= A^{0,0}(X, \mathcal{E}^\vee \otimes \mathcal{E}). \end{aligned}$$
Note that the algebra A_{C^∞} is Morita equivalent to $C^\infty(X)$.

(b) The semi-topological K-theory approach (Bondal, Toën, [Toë07b]). Assume again that $X = \mathbf{ncSpec}(A)$ is a smooth and compact graded **nc**-affine **nc**-space. Consider the moduli stack $\mathscr{M}_X$ of all objects in Perf_X. This is an ∞-stack which by a theorem of Toën and Vaquie [**TV05**] is locally geometric and locally of finite presentation. Moreover, for any $a, b \in \mathbb{N}$ the substack $\mathscr{M}_X^{[a,b]} \subset \mathscr{M}_X$ consisting of objects of amplitude in the interval $[a, b]$ is a geometric $(b - a + 1)$-stack. The functor sending a complex scheme to the underlying topological space in the analytic topology gives rise by a left Kan extension to a topological realization functor
$$|\bullet| : \mathrm{Ho}(\mathsf{Stacks}/\mathbb{C}) \to \mathrm{Ho}(\mathsf{Top})$$
from the homotopy category of stacks to the homotopy category of complex spaces. Following Friedlander-Walker [**FW05**] we define the semi-topological K-group of the **nc**-space X to be
$$K_0^{st}(X) := \pi_0(|\mathscr{M}_X|).$$
The group structure here is induced by the direct sum $\oplus$ of A-modules: the monoid $(\pi_0(|\mathscr{M}_X|), \oplus)$ is actually a group. To see this note that for any A-module E we have that $[E \oplus E[1]] = 0$ in $\pi_0(|\mathscr{M}_X|)$. Indeed we have distinguished triangles
$$E \longrightarrow 0 \longrightarrow E[1] \longrightarrow E[1]$$
$$E \longrightarrow E \oplus E[1] \longrightarrow E[1] \longrightarrow E[1]$$

the first of which corresponds to $\mathrm{id} \in \mathrm{Ext}^1(E[1], E) = \mathrm{Hom}(E, E)$, and the second of which corresponds to $0 \in \mathrm{Ext}^1(E[1], E) = \mathrm{Hom}(E, E)$. Since $\mathrm{Ext}^1(E[1], E) = \mathrm{Hom}(E, E)$ is a vector space, it follows that id deforms to 0 continuously and so the second terms in the above triangles represent the same point in $\pi_0(|\mathscr{M}_X|)$.

More generally $\oplus$ makes $|\mathscr{M}_X|$ into an H-space $\mathbb{K}^{st}(X)$ which is the degree zero part of a natural spectrum. Using this one can define $K_i^{st}(X)$ for all $i \geq 0$.

Next note that since C_X is triangulated it is a module over the category $\mathsf{Perf}_{\mathrm{pt}}$ of complexes of $\mathbb{C}$-vector spaces with finite dimensional total cohomology. In particular $K_\bullet^{st}(X)$ is a graded module over $K_\bullet^{st}(\mathrm{pt})$. It can be checked that

$$\mathbb{K}^{st}(\mathrm{pt}) = BU = \mathbb{K}^{\mathrm{top}}(\mathrm{pt}),$$

and so $K_\bullet^{st}(X)$ is a graded $\mathbb{Z}[u]$-module ($\deg u = 2$).

Now we can define

$$K_\bullet^{\mathrm{top}}(X) := K_\bullet^{st}(X)[u^{-1}] = K_\bullet^{st}(X) \otimes_{\mathbb{Z}[u]} \mathbb{Z}[u, u^{-1}].$$

Again one expects that there is a Chern character map

$$\mathrm{ch} : K_\bullet^{\mathrm{top}}(X) \to HP_\bullet(C_X)$$

whose image gives a rational structure $\mathscr{E}_B$ on $HP_\bullet(C_X)$.

Note: If X is a smooth and compact complex variety, then the Friedlander-Walker comparison theorem [**FW01**] implies that $K^{\mathrm{top}}(D(\mathsf{QCoh}(X))) \cong K^{\mathrm{top}}(X^{\mathrm{top}})$, where X^{top} is the topological space underlying X.

2.2.7. ***Questions.*** Even though we have some good candidates for the ingredients H, ∇, $\mathscr{E}_B$ of the conjectural **nc**-Hodge structure associated with an **nc**-space, there are several important problems that need to be addressed before one can prove Conjecture 2.24:

- show that the connection ∇ has regular singularities (this is automatically true in the $\mathbb{Z}$-graded case);
- show that ∇ preserves the rational structure;
- show that the opposedness axiom holds.

One can show that for a smooth compact nc-space the coefficient A_{-2} in the u-connection is a nilpotent operator, which gives some evidence in favor of the regular singularity question.

In fact Conjecture 2.24 and the above questions are special cases of a general conjecture which predicts the existence of a general **nc**-Hodge structure on the periodic cyclic homology of a curved d($\mathbb{Z}/2$)g category which is formally smooth and compact. We will not discuss the general conjecture or the relevant constructions here but we will revisit these questions in some interesting geometric examples in Section 3.

2.3. Gluing data. In this section we discuss how general **nc**-Hodge structures of exponential type can be glued together out of **nc**-Hodge structures with regular singularities and additional gluing data.

2.3.1. **nc-*de Rham data.*** The de Rham part of an **nc**-Hodge structure of exponential type can be prescribed in three equivalent ways:

ncdR(i) A pair $(\mathscr{M}, \nabla)$, where $\mathscr{M}$ is a finite dimensional vector space over $\mathbb{C}\{u\}[u^{-1}]$ and ∇ is a meromorphic connection. These data should satisfy the following

Property ncdR(i): There exist:

- a frame $\underline{e} = (e_1, \ldots, e_r)$ of $\mathscr{M}$ over $\mathbb{C}\{u\}[u^{-1}]$ in which
$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du$$
with $A_k \in \mathrm{Mat}_{r\times r}(\mathbb{C})$, $r = \mathrm{rank}_{\mathbb{C}\{u\}[u^{-1}]} \mathscr{M}$. In other words, there is a holomorphic extension $\mathscr{H} = \mathbb{C}\{u\}e_1 \oplus \cdots \oplus \mathbb{C}\{u\}e_r$ in which ∇ has at most a second order pole.

- a formal isomorphism over $\mathbb{C}((u))$:
$$(\mathscr{M}, \nabla) \otimes_{\mathbb{C}\{u\}[u^{-1}]} \mathbb{C}((u)) \xrightarrow{\cong} \bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i)$$
where $(\mathscr{R}_i, \nabla_i)$ are meromorphic bundles with connections with regular singularities and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ denote the distinct eigenvalues of A_{-2}.

ncdR(ii) A pair (M, ∇), where M is an algebraic vector bundle on $\mathbb{A}^1 - \{0\}$ and ∇ is a connection on M. These data should satisfy the following

Property ncdR(ii): M can be extended to an algebraic vector bundle $\widetilde{M}$ on $\mathbb{P}^1$, and

- with respect to this extension and appropriate local trivializations at zero and infinity we must have
$$\nabla = d + \left(\sum_{k \geq -2} A_k u^k \right) du \qquad \text{near } 0,$$
$$\nabla = d + \left(\sum_{k \geq -1} B_k u^{-k} \right) d(u^{-1}) \qquad \text{near } \infty.$$
In other words $\nabla : \widetilde{M} \to \widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \Omega^1_{\mathbb{P}^1}(2 \cdot \{0\} + \{\infty\})$.

- There is a formal isomorphism over $\mathbb{C}((u))$:
$$(M, \nabla) \otimes_{\mathbb{C}[u,u^{-1}]} \mathbb{C}((u)) \xrightarrow{\cong} \bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i, \nabla_i)$$
where $(\mathscr{R}_i, \nabla_i)$ are meromorphic bundles with connections with regular singularities and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_m \in \mathbb{C}$ denote the distinct eigenvalues of A_{-2}.

ncdR(iii) An algebraic holonomic $\mathcal{D}$-module $\boldsymbol{M}$ on $\mathbb{A}^1$. The $\mathcal{D}$-module $\boldsymbol{M}$ should also satisfy the following

Property ncdR(iii): $\boldsymbol{M}$ has regular singularities and $H^{\bullet}_{\mathrm{DR}}(\mathbb{A}^1, \boldsymbol{M}) = 0$.

The **nc**-de Rham data of types **ncdR(i)**, **ncdR(ii)**, and **ncdR(iii)** form obvious full subcategories in the categories of meromorphic connections over $\mathbb{C}\{u\}[u^{-1}]$, algebraic vector bundles with connections on $\mathbb{A}^1 - \{0\}$, and coherent algebraic $\mathcal{D}$-modules on $\mathbb{A}^1$, respectively. We have the following

LEMMA 2.26. *The categories of* **nc***-de Rham data of types* **ncdR(i)**, **ncdR(ii)**, *and* **ncdR(iii)** *are all equivalent.*

Proof. In essence we have already discussed the equivalence **ncdR(i)** $\Longleftrightarrow$ **ncdR(ii)** in Remark 2.1. Explicitly we have $(\mathscr{M},\nabla) = \mathfrak{G}_0((M,\nabla)) = (M \otimes_{\mathbb{C}[u,u^{-1}]} \mathbb{C}\{u\}[u^{-1}],\nabla)$.

We define a functor $\mathfrak{F}$: (data **(iii)**) $\to$ (data **(ii)**) as follows. Let $\boldsymbol{M}$ be a regular holonomic algebraic $\mathcal{D}$-module on $\mathbb{A}^1$ with trivial de Rham cohomology. Denote the coordinate on $\mathbb{A}^1$ by v. The vanishing of de Rham cohomology means that the action $\frac{d}{dv} : \boldsymbol{M} \to \boldsymbol{M}$ is an invertible operator. Consider the algebraic Fourier transform $\boldsymbol{\Phi M}$ which is the same vector space as $\boldsymbol{M}$ endowed with an action of the Weyl algebra defined by

$$\tilde{v} := \frac{d}{dv}$$

$$\frac{d}{d\tilde{v}} := -v$$

where $\tilde{v}$ is the coordinate on the dual line. By our assumptions $\boldsymbol{\Phi M}$ is a holonomic $\mathcal{D}$-module on which $\tilde{v}$ acts invertibly. Hence $\boldsymbol{\Phi M}$ is the direct image of a holonomic $\mathcal{D}$-module $\boldsymbol{\Phi M}'$ on $\mathbb{A}^1 - \{0\}$ under the embedding

$$\left(\mathbb{A}^1 - \{0\}\right) \hookrightarrow \mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[\tilde{v}])$$

Finally, making the change of coordinates $u = 1/\tilde{v}$ we obtain a $\mathcal{D}$-module M on $\mathbb{A}^1 - \{0\}$ with coordinate u.

We claim that $\mathfrak{F}(\boldsymbol{M}) := M$ obtained in this way satisfies the property **ncdR(ii)**, and that by this construction one obtains all such modules. It follows from the well-known properties of the Fourier transform that $\boldsymbol{\Phi M}$ has no singularities in $\mathbb{A}^1 - \{0\}$ and that its singularity at $\tilde{v} = 0$ is regular. Hence M is a vector bundle on $\mathbb{A}^1 - \{0\}$ endowed with a connection with regular singularity at ∞. It only remains to use the well-known fact (see e.g. [**Mal91**, Chapters IX-XI] or [**Kat90**, Theorem 2.10.16]) that the exponential type property for M is equivalent to the property of $\boldsymbol{M}$ to have only regular singularities. $\square$

REMARK 2.27. The characterization of the exponential type property in terms of the Fourier transform can be stated more precisely (see [**Mal91**, Chapters IX-XI] or [**Kat90**, Theorem 2.10.16]): For an algebraic holonomic $\mathcal{D}$-module M on the complex affine line, the following two conditions are equivalent:

1) M has regular singularities;
2) the Fourier transform $\boldsymbol{\Phi}M$ of M has no singularities outside 0, its singularity at 0 is regular, and its singularity at infinity is of exponential type.

Explicitly $\boldsymbol{\Phi}M$ being of exponential type at infinity means that if x is a coordinate on $\mathbb{A}^1$ centered at 0, then after passing to the formal completion

$(\mathbf{\Phi} M) \otimes_{\mathbb{C}[x]} \mathbb{C}((x^{-1}))$ the resulting module will be isomorphic to a finite sum

$$\bigoplus_{i=1}^{m} \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i x} \otimes (\mathscr{R}_i, \nabla_i)$$

where $(\mathscr{R}_i, \nabla_i)$ are $\mathcal{D}$-modules with a regular singularity at infinity.

REMARK 2.28. Note that the de Rham data **ncdR(i)** are analytic in nature, whereas **ncdR(ii)** and **ncdR(iii)** are algebraic. In fact from the proof it is clear that **ncdR(ii)** and **ncdR(iii)** and their equivalence still make sense if we replace $\mathbb{C}$ with any field of characteristic zero.

2.3.2. ***nc-Betti data.*** The (rational) Betti part of an **nc**-Hodge structure of exponential type can be prescribed in four ways:

ncB(i) A (middle perversity) perverse sheaf $\mathscr{G}^\bullet$ of $\mathbb{Q}$-vector spaces on the Riemann surface $\mathbb{C}$ (taken with the analytic topology) satisfying the following

Property ncB(i): $R\Gamma(\mathbb{C}, \mathscr{G}^\bullet) = 0$.

ncB(ii) A constructible sheaf $\mathscr{F}$ of $\mathbb{Q}$-vector spaces on the Riemann surface $\mathbb{C}$ (taken with the analytic topology) satisfying the following

Property ncB(ii): $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$.

ncB(iii) A finite collection of distinct points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset \mathbb{C}$, and

- a collection $U_1, U_2, \ldots, U_n$ of finite dimensional non-zero $\mathbb{Q}$-vector spaces,
- a collection of linear maps $T_{ij} : U_j \to U_i$, for all $i, j = 1, \ldots, n$,

satisfying the following

Property ncB(iii): $T_{ii} \in GL(U_i)$.

ncB(iv) A local system S of $\mathbb{Q}$-vector spaces on $\boldsymbol{S}^1$ equipped with a filtration $\{\mathsf{S}_{\leq\lambda}\}_{\lambda\in\mathbb{R}}$ by subsheaves of $\mathbb{Q}$-vector spaces, satisfying the following

Property ncB(iv): The filtration $\{\mathsf{S}_{\leq\lambda} \otimes \mathbb{C}\}_{\lambda\in\mathbb{R}}$ of $\mathsf{S} \otimes \mathbb{C}$ is a Deligne-Malgrange-Stokes filtration of exponential type. In other words, there exist complex numbers $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n \in \mathbb{C}$ so that:

- For every $\varphi \in \boldsymbol{S}^1$, the filtration $\{(\mathsf{S}_{\leq\lambda}\otimes\mathbb{C})_\varphi\}_{\lambda\in\mathbb{R}}$ of the stalk $(\mathsf{S}\otimes\mathbb{C})_\varphi$ jumps exactly at the real numbers $\{\mathrm{Re}\left(\boldsymbol{c}_k e^{-i\varphi}\right)\}_{k=1}^n$.
- The associated graded sheaves of $\mathsf{S}\otimes\mathbb{C}$ with respect to $\{\mathsf{S}_{\leq\lambda}\otimes\mathbb{C}\}_{\lambda\in\mathbb{R}}$ are local systems on $\boldsymbol{S}^1$.

Again there are natural equivalences of the different types of Betti data (for **ncB(iii)** the equivalence depends on certain choices of paths as one can see from the proof of Theorem 2.29 and the statement of Lemma 2.30). Consider the full subcategories $(\mathbf{ncB(i)})$ and $(\mathbf{ncB(ii)})$ of **nc**-Betti data of types **ncB(i)** and **ncB(ii)** in the category of perverse sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{C}$ and in the category of constructible sheaves of $\mathbb{Q}$-vector spaces on $\mathbb{C}$, respectively. We have the following

THEOREM 2.29. *The categories of* **nc**-*Betti data of types* **ncB(i)** *and* **ncB(ii)** *are naturally equivalent. More precisely, the natural functors*

$$\mathcal{H}^{-1} : D^b_{\mathrm{constr}}(\mathbb{C}, \mathbb{Q}) \to \mathsf{Constr}(\mathbb{C}, \mathbb{Q}) \quad \textit{and} \quad [1] : \mathsf{Constr}(\mathbb{C}, \mathbb{Q}) \to D^b_{\mathrm{constr}}(\mathbb{C}, \mathbb{Q})$$

induce mutually inverse equivalences of the full subcategories $(\mathbf{ncB(i)}) \subset D^b_{\text{constr}}(\mathbb{C}, \mathbb{Q})$ *and* $(\mathbf{ncB(ii)}) \subset \mathsf{Constr}(\mathbb{C}, \mathbb{Q})$.

Proof. First we look at the data $\mathbf{ncB(i)}$ more closely. Suppose X is a complex analytic space underlying a complex quasi-projective variety. Recall (see e.g. [**BBD82, KS94, Dim04**]) that a bounded complex $\mathscr{G}^\bullet$ of sheaves of $\mathbb{C}$-vector spaces on X is called a (middle perversity) perverse sheaf if it has constructible cohomology sheaves $\mathcal{H}^k(\mathscr{G}^\bullet)$ and if

- for all $k \in \mathbb{Z}$, we have $\dim_{\mathbb{R}}\{x \in X | \, \mathcal{H}^{-k}(\boldsymbol{i}_x^* \mathscr{G}^\bullet) \neq 0\} \leq 2k$,
- for all $k \in \mathbb{Z}$, we have $\dim_{\mathbb{R}}\{x \in X | \, \mathcal{H}^{k}(\boldsymbol{i}_x^! \mathscr{G}^\bullet) \neq 0\} \leq 2k$.

Here $\boldsymbol{i}_x : x \hookrightarrow X$ denotes the inclusion of the point x in X.

For future reference we will write $D^b_{\text{constr}}(X, \mathbb{Q})$ for the derived category of complexes of $\mathbb{Q}$-vector spaces on X with constructible cohomology, $\mathsf{Perv}(X, \mathbb{Q}) \subset D^b_{\text{constr}}(X, \mathbb{Q})$ for the full subcategory of middle perversity perverse sheaves, and $\mathsf{Constr}(X, \mathbb{Q}) \subset D^b_{\text{constr}}(X, \mathbb{Q})$ for the full subcategory of constructible sheaves.

From the definition it is clear that if $\mathscr{G}^\bullet$ is a perverse sheaf on $\mathbb{C}$, then $\mathscr{G}^\bullet$ has at most two non-trivial cohomology sheaves $\mathcal{H}^{-1}(\mathscr{G}^\bullet)$ and $\mathcal{H}^0(\mathscr{G}^\bullet)$. Moreover the support of $\mathcal{H}^0(\mathscr{G}^\bullet)$ has dimension ≤ 0. Now the cohomology $R\Gamma^\bullet(\mathscr{G}^\bullet) = \mathbb{H}^\bullet(\mathbb{C}, \mathscr{G}^\bullet)$ can be computed via the hypercohomology spectral sequence

$$E_2^{pq} = H^p(\mathbb{C}, \mathcal{H}^q(\mathscr{G}^\bullet)) \Rightarrow \mathbb{H}^{p+q}(\mathbb{C}, \mathscr{G}^\bullet).$$

Since $\mathscr{G}^\bullet$ has only two cohomology sheaves, the E_2 level of this spectral sequence is

$$\begin{tikzcd}
0 & H^0(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) \arrow[drr] & H^1(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) \arrow[drr] & H^2(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) & \cdots \\
-1 & H^0(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)) & H^1(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)) & H^2(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)) & \cdots \\
 & 0 & 1 & 2 &
\end{tikzcd}$$

By Artin's vanishing theorem for constructible sheaves [**Art73**, Corollary 3.2] we have $H^p(\mathbb{C}, \mathcal{H}^q(\mathscr{G}^\bullet)) = 0$ for all q and all $p > 1$. Furthermore, since $\mathcal{H}^0(\mathscr{G}^\bullet)$ has at most zero-dimensional support we have $H^1(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) = 0$. In particular the spectral sequence degenerates at E_2 and the only potentially non-trivial cohomology groups of $\mathscr{G}^\bullet$ are

$$\begin{aligned}
\mathbb{H}^{-1}(\mathbb{C}, \mathscr{G}^\bullet) &= H^0(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)), \text{ and} \\
\mathbb{H}^{0}(\mathbb{C}, \mathscr{G}^\bullet) &= H^1(\mathbb{C}, \mathcal{H}^{-1}(\mathscr{G}^\bullet)) \oplus H^0(\mathbb{C}, \mathcal{H}^{0}(\mathscr{G}^\bullet)).
\end{aligned}$$

Thus under the assumption that $R\Gamma(\mathscr{G}^\bullet) = 0$ we get that $H^0(\mathbb{C}, \mathcal{H}^0(\mathscr{G}^\bullet)) = 0$, i.e. that $\mathcal{H}^0(\mathscr{G}^\bullet) = 0$. In other words $\mathscr{G}^\bullet = \mathscr{F}[1]$ for some constructible sheaf $\mathscr{F}$ with $R\Gamma(\mathscr{F}) = 0$.

To finish the proof of the theorem we need to show that for every constructible sheaf $\mathscr{F}$ with $R\Gamma(\mathscr{F}) = 0$, the object $\mathscr{F}[1]$ will be perverse (for the middle perversity). For this we will have to look more closely at constructible sheaves on the complex line.

Suppose $\mathscr{F}$ is a constructible sheaf of $\mathbb{Q}$-vector spaces on $\mathbb{C}$. Then there is a finite set $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ of points in $\mathbb{C}$ so that $\mathbb{C} - S$ is the maximal open set on which $\mathscr{F}$ restricts to a local system. Let $\mathbb{F} := \mathscr{F}_{|\mathbb{C}-S}$ denote this local system. Let $\mathbb{C} - S \overset{\boldsymbol{j}}{\hookrightarrow} \mathbb{C} \overset{\boldsymbol{i}}{\hookleftarrow} S$ be the natural inclusions and let $\varphi : \mathscr{F} \to j_* j^* \mathscr{F} = j_* \mathbb{F}$ be the adjunction homomorphism.

Before we can describe $\mathbb{F}$ and $\mathscr{F}$ via the quiver-like data of type **ncB(iii)** we will need to make some rigidifying choices. First we fix a base point $\boldsymbol{c}_0 \in \mathbb{C} - S$. For $i = 1, \ldots, n$ we choose a collection of small disjoint discs $\boldsymbol{D}_i \subset \mathbb{C}$, each $\boldsymbol{D}_i$ centered at $\boldsymbol{c}_i$. For each disc we fix a point $\boldsymbol{o}_i \in \partial \boldsymbol{D}_i$ and denote by $\mathfrak{l}_i$ the loop starting and ending at $\boldsymbol{o}_i$ and tracing $\partial \boldsymbol{D}_i$ once in the counterclockwise direction. We fix an ordered system of non-intersecting paths $\{a_i\}_{i=1}^n \subset \mathbb{C} - (\cup_{i=1}^n \boldsymbol{D}_i)$ which connect the base point $\boldsymbol{c}_0$ with each of the $\boldsymbol{o}_i$ as in Figure 2.

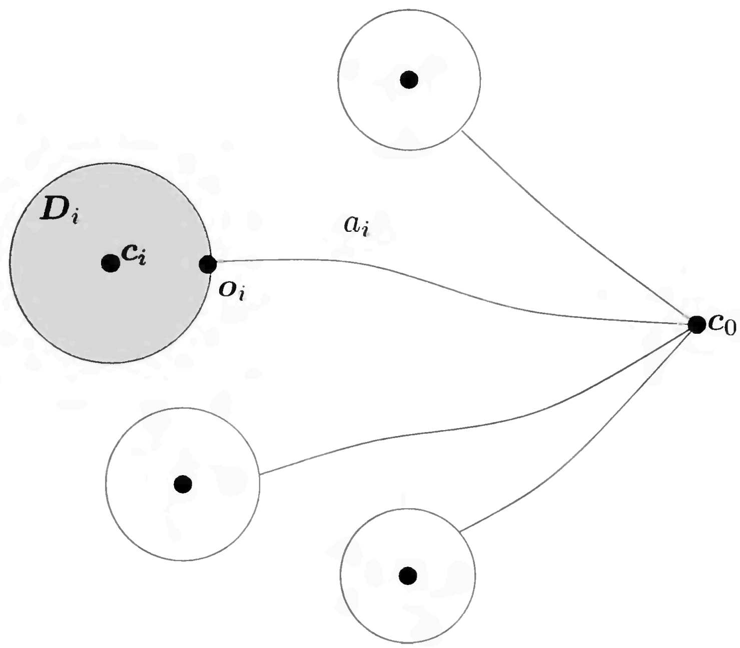


FIGURE 2. A system of paths for $S \subset \mathbb{C}$.

Let $\mathrm{mon}_{\mathfrak{l}_i} : \mathbb{F}_{\boldsymbol{o}_i} \to \mathbb{F}_{\boldsymbol{o}_i}$ be the monodromy operator associated with the local system $\mathbb{F}$ and the loop $\mathfrak{l}_i$. The stalk $(\boldsymbol{j}_*\mathbb{F})_{\boldsymbol{c}_i}$ of the constructible sheaf $\boldsymbol{j}_*\mathbb{F}$ at $\boldsymbol{c}_i$ can be naturally identified with the subspace $\mathbb{F}_{\boldsymbol{o}_i}^{\mathrm{mon}_{\mathfrak{l}_i}}$ of invariants for the local monodromy. Taking stalks at each $\boldsymbol{c}_i \in S$ we get $\mathbb{Q}$-vector spaces $\mathscr{F}_{\boldsymbol{c}_i}$ and the adjunction map $\varphi : \mathscr{F} \to \boldsymbol{j}_*\mathbb{F}$ induces linear maps

$$\varphi_{\boldsymbol{c}_i} : \mathscr{F}_{\boldsymbol{c}_i} \to \mathbb{F}_{\boldsymbol{o}_i}^{\mathrm{mon}_{\boldsymbol{c}_i}} \subset \mathbb{F}_{\boldsymbol{o}_i}.$$

Note that, by descent, specifying the constructible sheaf $\mathscr{F}$ is equivalent to specifying the collection of points $S \subset \mathbb{C}$, the local system $\mathbb{F}$ on $\mathbb{C} - S$, the collection of vector spaces $\{\mathscr{F}_{\boldsymbol{c}_i}\}_{i=1}^n$ and the collection of linear maps $\{\varphi_{\boldsymbol{c}_i}\}_{i=1}^n$. In particular, the compactly supported pullback of $\mathscr{F}[1]$ via the inclusion $\boldsymbol{i}_{\boldsymbol{c}_i} : \{\boldsymbol{c}_i\} \hookrightarrow \mathbb{C}$ can be computed in terms of these linear algebraic data and is given explicitly by the complex

$$\boldsymbol{i}_{\boldsymbol{c}_i}^!(\mathscr{F}[1]) = [\ \underset{-1}{\mathscr{F}_{\boldsymbol{c}_i}} \xrightarrow{\varphi_{\boldsymbol{c}_i}} \underset{0}{\mathbb{F}_{\boldsymbol{o}_i}} \xrightarrow{1-\mathrm{mon}_{\mathfrak{l}_i}} \underset{1}{\mathbb{F}_{\boldsymbol{o}_i}}\ ].$$

By definition $\mathscr{F}[1]$ is a perverse sheaf iff for all $\boldsymbol{c}_i \in S$ the complex of vector spaces $\boldsymbol{i}^!_{\boldsymbol{c}_i}(\mathscr{F}[1])$ has no cohomology in strictly negative degrees, i.e. iff $\varphi_{\boldsymbol{c}_i}$ is injective for all $i = 1, \ldots, n$.

Next we rewrite the condition $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$ in terms of the descent data $(\mathbb{F}, \{\mathscr{F}_{\boldsymbol{c}_i}\}, \{\varphi_{\boldsymbol{c}_i}\})$. To simplify notation let $U := \mathscr{F}_{\boldsymbol{c}_0}$, $V_i = \mathscr{F}_{\boldsymbol{c}_i}$ for $i = 1, \ldots, n$. Let $T_i : U \to U$ be the monodromy operator for the local system $\mathbb{F}$ and the $\boldsymbol{c}_0$-based loop γ_i obtained by first tracing the path a_i from $\boldsymbol{c}_0$ to $\boldsymbol{o}_i$, then tracing the loop $\mathfrak{l}_i$, and then tracing back a_i in the opposite direction. Similarly we have linear maps $\psi_i : V_i \to U^{T_i} \subset U$ obtained by conjugating $\varphi_{\boldsymbol{c}_i} : V_i \to \mathbb{F}_{\boldsymbol{o}_i}$ with the operator of parallel transport in $\mathbb{F}$ along the path a_i.

The descent data for $\mathscr{F}$ with respect to the open cover $\mathbb{C} = (\mathbb{C} - S) \cup (\cup_{i=1}^n \boldsymbol{D}_i)$ are now completely encoded in the linear algebraic data $(U, \{V_i\}_{i=1}^n, \{T_i\}_{i=1}^n, \{\psi_i\}_{i=1}^n)$. Cover $\mathbb{C}$ by the two opens $\mathbb{C} - S$ and $\cup_{i=1}^n \boldsymbol{D}_i$. The intersection of these two opens is the disjoint union of punctured discs $\coprod_{i=1}^n (\boldsymbol{D}_i - \boldsymbol{c}_i)$. The Mayer-Vietoris sequence for $\mathscr{F}$ and this cover identifies $R\Gamma(\mathbb{C}, \mathscr{F})$ with the complex:

$$\begin{tikzcd}[column sep=huge]
\bigoplus_{i=1}^n V_i \arrow[r, "\oplus_{i=1}^n \psi_i"] & U^{\oplus n} \arrow[r, "\oplus_{i=1}^n (1-T_i)"] & U^{\oplus n} \\
\oplus & \oplus & \\
U \arrow[uur, "\mathrm{id}_U^{\oplus n}" near start] \arrow[r, "\oplus_{i=1}^n (1-T_i)"'] & U^{\oplus n} \arrow[uur, "-\mathrm{id}_U^{\oplus n}"'] & \\
0 & 1 & 2
\end{tikzcd}$$

In other words we have a quasi-isomorphism of complexes of $\mathbb{Q}$-vector spaces:

$$R\Gamma(\mathbb{C}, \mathscr{F}) \cong \left[\begin{tikzcd}[column sep=huge]
\bigoplus_{i=1}^n V_i \arrow[dr, "\oplus_{i=1}^n \psi_i"] & \\
\oplus & U^{\oplus n} \\
U \arrow[ur, "\mathrm{id}_U^{\oplus n}"'] & \\
0 & 1
\end{tikzcd}\right]$$

The acyclicity of this complex is equivalent to the conditions

(a) the maps $\psi_i : V_i \to U$ are injective for all $i = 1, \ldots, n$, and
(b) the map $U \to \oplus_{i=1}^n U/V_i$ is an isomorphism.

Thus the acyclicity of $R\Gamma(\mathbb{C}, \mathscr{F})$ implies the perversity of $\mathscr{F}[1]$. The theorem is proven. $\square$

The conditions (a) and (b) from the proof of Theorem 2.29 suggest a better way of recording the linear algebraic content of $\mathscr{F}$. Namely, if we set $U_i := U/V_i$, then we can use (b) to identify U with $\oplus_{i=1}^n U_i$, V_i with $\oplus_{j \neq i} U_j$ and the map $\psi_i : V_i \hookrightarrow U$ with the natural inclusion $\oplus_{j \neq i} U_j \subset \oplus_{i=1}^n U_i$. The only thing left is the data of the monodromy operators $T_i \in GL(U)$, $i = 1, \ldots, n$. However for each i we have the embedding

$$V_i \xhookrightarrow{\psi_i} \mathrm{Ker}\left[U \xrightarrow{(1-T_i)} U\right]$$

and so under the decomposition $U = \oplus_{i=1}^{n} U_i$ the automorphism T_i has a block form

$$T_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & T_{1i} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & T_{2i} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & T_{ii} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & T_{i+1,i} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & T_{ni} & 0 & \cdots & 1 \end{pmatrix}$$

where $T_{i|U_i} = \sum_{j=1}^{n} T_{ji}$, and $T_{ji} : U_i \to U_j$. The linear maps T_{ji} are unconstrained except for the obvious condition that for all i the map T_i should be invertible, which is equivalent to $T_{ii} : U_i \to U_i$ being invertible for all $i = 1, \ldots, n$. Also since S was chosen to be such that $\mathbb{C} - S$ is the maximal open on which $\mathscr{F}$ is a local system, it follows that $U_i \neq \{0\}$ for all $i = 1, \ldots, n$.

In other words we have proven the following

LEMMA 2.30. *Fix the set of points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ and choose the discs $\{\boldsymbol{D}_i\}_{i=1}^{n}$ and the system of paths $\{a_i\}_{i=1}^{n}$. The functor assigning to a constructible sheaf $\mathscr{F}$ with singularities at S the data $(\{U_i\}_{i=1}^{n}, \{T_{ij}\})$ establishes an equivalence between the groupoid of all data of type* **ncB(ii)** *with singularities exactly at S and all data of type* **ncB(iii)** *with the given S.*

The bridge between the **nc**- de Rham and Betti data is provided as usual by the Riemann-Hilbert correspondence. This is tautological but we record it for future reference:

LEMMA 2.31. *The de Rham functor:*

$$\boldsymbol{M} \to \operatorname{cone}\left(\boldsymbol{M} \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{A}^1}^{\mathrm{an}} \xrightarrow{\frac{\partial}{\partial u}} \boldsymbol{M} \otimes_{\mathbb{C}[u]} \mathcal{O}_{\mathbb{A}^1}^{\mathrm{an}} \right)$$

establishes an equivalence between the categories (**ncdR(iii)**) *and* (**ncB(i)**)$\otimes\mathbb{C}$.

Finally, note that Theorem 2.29, together with Lemma 2.31, and Deligne's classification [**BV85**, Theorem 4.7.3] of germs of irregular connections give immediately:

LEMMA 2.32. *The data* (**ncB(ii)**) *and* (**ncB(iv)**) *are equivalent.*

Proof. Let $\mathscr{F}$ be a constructible sheaf of $\mathbb{Q}$-vector spaces on $\mathbb{C}$. Define a local system S of $\mathbb{Q}$-vector spaces on $\boldsymbol{S}^1$ as the restriction of $\mathscr{F}$ to the circle "at infinity", i.e. define the stalk of S at $\varphi \in \boldsymbol{S}^1$ to be

$$\mathsf{S}_\varphi := \lim_{r \to +\infty} \mathscr{F}_{re^{i\varphi}}.$$

Next, for any $\lambda \in \mathbb{R}$ and any $\varphi \in \boldsymbol{S}^1$ consider the half-plane

$$\mathfrak{H}_{\varphi,\lambda} := (\lambda + \{u \in \mathbb{C} |\ \operatorname{Re}(u) \geq 0\}) \cdot e^{i\varphi},$$

as shown in Figure 3.

Now suppose that $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$. By the long exact sequence for the cohomology of the pair $\mathfrak{H}_{\varphi,\lambda} \subset \mathbb{C}$ we get that $H^i(\mathbb{C}, \mathfrak{H}_{\varphi,\lambda}; \mathscr{F}) = 0$ unless $i = 1$. The Deligne-Malgrange-Stokes filtration on S is then given explicitly by

$$\mathsf{S}_{\varphi,\leq\lambda} := H^1(\mathbb{C}, \mathfrak{H}_{\varphi,\lambda}; \mathscr{F}) \subset \mathsf{S}_\varphi.$$

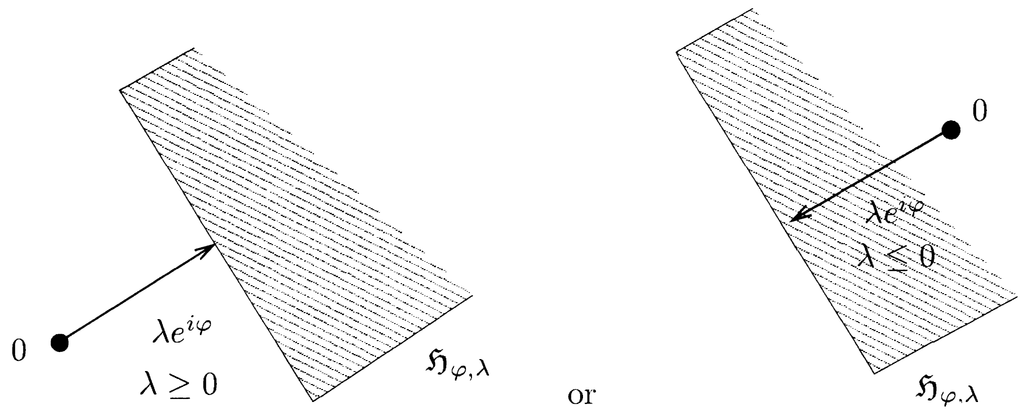


FIGURE 3. The half-plane $\mathfrak{H}_{\varphi,\lambda}$.

□

For the purposes of **nc**-Hodge theory all these statements can be summarized in the following

THEOREM 2.33. *There is a natural equivalence of categories*

(triples $(H, \mathscr{E}_B, \textsf{iso})$ satisfying the **(nc-filtration axiom)$^{\textsf{exp}}$** and the **($\mathbb{Q}$-structure axiom)$^{\textsf{exp}}$**) $\leftrightarrow$ (quadruples $((H,\nabla), \mathscr{F}_B, \boldsymbol{f})$, where
- H is an algebraic $\mathbb{Z}/2$-graded vector bundle on $\mathbb{C}$ and ∇ is a meromorphic connection on H satisfying the **(nc-filtration axiom)$^{\textsf{exp}}$**;
- $\mathscr{F}_B \in \mathsf{Constr}(\mathbb{C}, \mathbb{Q})$, satisfying $R\Gamma(\mathbb{C}, \mathscr{F}_B) = 0$;
- $\boldsymbol{f}$ is an isomorphism

$\boldsymbol{f} : \mathscr{F}_B \otimes \mathbb{C} \to DR\left(\boldsymbol{\Phi}\left[\iota_*\left((H,\nabla)_{|\mathbb{A}^1-\{0\}}\right)\right]\right)$ in $D^b_{\mathrm{constr}}(\mathbb{C}, \mathbb{C})$)

Here as before

> *DR is the de Rham complex functor from the derived category of regular holonomic $\mathbb{D}$-modules to the derived category of constructible sheaves,*
>
> *ι is the inclusion map $\iota : \mathbb{A}^1 - \{0\} \hookrightarrow \mathbb{A}^1$ given by $\iota(v) = v^{-1}$, and*
>
> *$\boldsymbol{\Phi}(\bullet)$ is the Fourier-Laplace transform for $\mathcal{D}$-modules on $\mathbb{A}^1$.*

Proof. Follows immediately from previous equivalences. □

2.4. Structure results. In this section we collect a few results clarifying the structure properties of the **nc**-Hodge structures of exponential type.

2.4.1. ***A quiver description of* nc*-Betti data.*** Since the gluing data **ncB(iii)** are of essentially combinatorial nature, it is natural to look for a quiver interpretation of these data. To that end consider the algebra

$$(2.4.1) \qquad \mathscr{A}_n := \left\langle \begin{matrix} \boldsymbol{p}_1, \ldots, \boldsymbol{p}_n \\ T, T_{11}^{-1}, \ldots, T_{nn}^{-1} \end{matrix} \;\middle|\; \begin{matrix} \boldsymbol{p}_1 + \boldsymbol{p}_2 + \cdots + \boldsymbol{p}_n = 1 \\ \boldsymbol{p}_i \boldsymbol{p}_j = \boldsymbol{p}_j \boldsymbol{p}_i \text{ for } i \neq j, \, \boldsymbol{p}_i^2 = \boldsymbol{p}_i \\ T_{ii}^{-1} \boldsymbol{p}_i T \boldsymbol{p}_i = \boldsymbol{p}_i T \boldsymbol{p}_i T_{ii}^{-1} = \boldsymbol{p}_i \end{matrix} \right\rangle$$

This is the path algebra of the complete quiver having n ordered vertices, $n^2 - n$ arrows connecting all pairs of distinct vertices, and $2n$-loops - two at each vertex, with the only relations being that the two loops at every given vertex are inverses of each other.

Note that our description of the gluing data **ncB(iii)** now immediately gives the following

LEMMA 2.34. *For a given set of points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset \mathbb{C}$, the category of gluing data* **nc*B(iii)*** *with singularities at S is equivalent to the category of finite dimensional representations of $\mathscr{A}_n$.*

In particular since the braid group $\boldsymbol{B}_n$ on n strands acts naturally on the data **ncB(iii)** we get a homomorphism $\boldsymbol{B}_n \to \mathrm{Aut}(\mathscr{A}_n)$ from the braid group to the group of algebra automorphisms of $\mathscr{A}_n$.

2.4.2. ***Gluing of* nc*-Hodge structures.*** It is natural to expect that the usual classification of connections with second order poles in terms of formal regular type and Stokes multipliers can be promoted to a similar classification of **nc**-Hodge structures. The search for such a classification leads naturally to the following theorem:

THEOREM 2.35. *Let $\{(H, \mathscr{E}_B, \mathsf{iso})\}$ be an* **nc***-Hodge structure of exponential type. Then specifying $\{(H, \mathscr{E}_B, \mathsf{iso})\}$ is equivalent to specifying the following data:*

- **(regular type):** *A finite set $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset \mathbb{C}$ and a collection $\{((\mathscr{R}_i, \nabla_i), \mathscr{E}_{B,i}, \mathsf{iso}_i)\}_{i=1}^n$ of* **nc***-Hodge structures with regular singularities.*
- **(gluing data):** *A base point $\boldsymbol{c}_0 \in \mathbb{C} - S$, a collection of discs $\{\boldsymbol{D}_i\}_{i=1}^n$ and paths $\{a_i\}_{i=1}^n$, chosen as in the proof of Theorem 2.29, and for every $i \neq j$, $i, j \in \{1, \ldots, n\}$ a map of rational vector spaces*

$$T_{ij} : (\mathscr{E}_{B,j})_{\boldsymbol{c}_0} \longrightarrow (\mathscr{E}_{B,i})_{\boldsymbol{c}_0}$$

Proof. It will be convenient to introduce formal counterparts to the de Rham parts of the **nc**-Hodge structures appearing in the statement of the theorem. We consider the following:

formal(a) A pair $(\mathscr{M}^{\mathrm{for}}, \nabla^{\mathrm{for}})$, where $\mathscr{M}^{\mathrm{for}}$ is a finite dimensional vector space over $\mathbb{C}((u))$ and ∇^{for} is a meromorphic connection on $\mathscr{M}^{\mathrm{for}}$ of exponential type.

formal(b) A finite set of points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset \mathbb{C}$ and a collection $\{(\mathscr{R}_i^{\mathrm{for}}, \nabla_i^{\mathrm{for}})\}_{i=1}^n$ where each $\mathscr{R}_i^{\mathrm{for}}$ is a non-zero finite dimensional vector space over $\mathbb{C}((u))$ and each ∇_i^{for} is a meromorphic connection on $\mathscr{R}_i^{\mathrm{for}}$ with a regular singularity.

formal(c) A finite collection of points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset \mathbb{C}$, and

- a collection $U_1, U_2, \ldots, U_n$ of finite dimensional non-zero $\mathbb{Q}$-vector spaces,
- a collection of linear maps $T_{ii} \in GL(U_i)$, for all $i = 1, \ldots, n$,

By Remark 2.13 the natural functor from the category of data **formal(b)** to the category of data **formal(a)**, which is given by

$$\begin{array}{ccc} (\mathbf{formal(b)}) & \longrightarrow & (\mathbf{formal(a)}) \\ \left(S; \{(\mathscr{R}_i^{\text{for}}, \nabla_i^{\text{for}})\}_{i=1}^n\right) & \longrightarrow & \bigoplus_{i=1}^n \boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u} \otimes (\mathscr{R}_i^{\text{for}}, \nabla_i^{\text{for}}) =: (\mathscr{M}^{\text{for}}, \nabla^{\text{for}}) \end{array}$$

is an equivalence of categories.

Also we have the following

LEMMA 2.36. *The categories of data* **formal(b)** *and* **formal(c)** *are naturally equivalent.*

Proof. Indeed, consider the category $\mathcal{C}$ of all data consisting of a finite set of points $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\} \subset \mathbb{C}$ and a collection $\{(\mathscr{R}_i, \nabla_i)\}_{i=1}^n$ where each $\mathscr{R}_i$ is a non-zero finite dimensional vector space over $\mathbb{C}\{u\}[u^{-1}]$ and each ∇_i is a meromorphic connection on $\mathscr{R}_i$ with a regular singularity and non-trivial monodromy. Then we have natural functors

$$\begin{tikzcd} & (\mathbf{formal(b)}) \\ \mathcal{C} \arrow[ur, "(\bullet)\otimes\mathbb{C}((u))"] \arrow[dr, "\mathsf{mon}"'] & \\ & (\mathbf{formal(c)}) \end{tikzcd}$$

where $(\bullet) \otimes \mathbb{C}((u))$ is the passage to a formal completion and mon is given by assigning to each $(\mathscr{R}_i, \nabla_i)$ the pair (U_i, T_i), where U_i is the fiber of the Birkhoff extension $\mathfrak{B}_0(\mathscr{R}_i, \nabla_i)$ of $(\mathscr{R}_i, \nabla_i)$ at $1 \in \mathbb{A}^1$, and T_i is the monodromy of $\mathfrak{B}_0(\mathscr{R}_i, \nabla_i)$ around the unit circle traced in the positive direction.

This proves the lemma since mon is an equivalence by the Riemann-Hilbert correspondence and $(\bullet) \otimes \mathbb{C}((u))$ is an equivalence by the formal decomposition theorem [**Sab02**, II.5.7]. □

Note that these equivalences are compatible with the corresponding equivalence of analytic de Rham data and Betti data. More precisely we have a commutative diagram of functors

(2.4.2)
$$\begin{tikzcd} (\mathbf{ncdR(i)}) \arrow[r] \arrow[d, leftrightarrow, dotted] & (\mathbf{formal(a)}) \arrow[d, leftrightarrow, dotted] \\ (\mathbf{ncB(iii)}) \arrow[r] & (\mathbf{formal(c)}) \end{tikzcd}$$

Here the right vertical equivalence is the composition of the equivalences $(\mathbf{formal(a)}) \cong (\mathbf{formal(b)}) \cong (\mathbf{formal(c)})$ that we just discussed. The left vertical equivalence is the composition of the equivalence $(\textsf{\textbf{nc}}\mathrm{dR(i)}) \cong (\textsf{\textbf{nc}}\mathrm{dR(iii)})$ given in Lemma 2.26, the equivalence $(\textsf{\textbf{nc}}\mathrm{dR(iii)}) \cong (\textsf{\textbf{nc}}\mathrm{B(i)})$ from Lemma 2.31, the equivalence $(\textsf{\textbf{nc}}\mathrm{B(i)}) \cong (\textsf{\textbf{nc}}\mathrm{B(ii)})$ given in Theorem 2.29, and the equivalence $(\textsf{\textbf{nc}}\mathrm{B(ii)}) \cong (\textsf{\textbf{nc}}\mathrm{B(iii)})$ from Lemma 2.30.

Horizontally we have the forgetful functors

$$\begin{aligned}(\mathbf{ncdR(i)}) &\longrightarrow (\mathbf{formal(a)})\\ (\mathscr{M},\nabla) &\longmapsto (\mathscr{M},\nabla)\otimes_{\mathbb{C}\{u\}[u^{-1}]}\mathbb{C}((u)),\end{aligned}$$

and

$$\begin{aligned}(\mathbf{ncB(iii)}) &\longrightarrow (\mathbf{formal(c)})\\ \left(S;\{U_i\}_{i=1}^n,\{T_{ij}\}_{i,j=1}^n\right) &\longmapsto \left(S;\{U_i\}_{i=1}^n,\{T_{ii}\}_{i=1}^n\right).\end{aligned}$$

Next we need the following

LEMMA 2.37. *Suppose that* $(\mathscr{M},\nabla)$ *is some de Rham data of type* $\mathbf{ncdR(i)}$ *and let*

$$(\mathscr{M}^{\mathrm{for}},\nabla^{\mathrm{for}}) = (\mathscr{M},\nabla)\otimes_{\mathbb{C}\{u\}[u^{-1}]}\mathbb{C}((u))$$

be the corresponding formal data. Then:

(a) *the map*

$$\begin{pmatrix}\text{$\mathbb{C}\{u\}$-submodules $\mathscr{H}\subset$}\\ \text{$\mathscr{M}$, on which ∇ has a pole}\\ \text{of order ≤ 2}\end{pmatrix} \xrightarrow{(\bullet)\otimes\mathbb{C}[[u]]} \begin{pmatrix}\text{$\mathbb{C}[[u]]$-submodules $\mathscr{H}^{\mathrm{for}}\subset$}\\ \text{$\mathscr{M}^{\mathrm{for}}$, on which ∇^{for} has a}\\ \text{pole of order ≤ 2}\end{pmatrix},$$

is bijective.

(b) *If* $\Psi : (\mathscr{M}^{\mathrm{for}},\nabla^{\mathrm{for}}) \to \bigoplus_{i=1}^n \boldsymbol{\mathcal{E}}^{c_i/u}\otimes(\mathscr{R}_i^{\mathrm{for}},\nabla_i^{\mathrm{for}})$ *is a formal isomorphism, then the map*

$$\begin{pmatrix}\text{$\mathbb{C}[[u]]$-submodules $\mathscr{H}^{\mathrm{for}}\subset$}\\ \text{$\mathscr{M}^{\mathrm{for}}$, on which ∇^{for} has a}\\ \text{pole of order ≤ 2}\end{pmatrix} \xleftarrow[\Psi]{} \begin{pmatrix}\text{$\mathbb{C}[[u]]$-submodules $\mathscr{H}_i^{\mathrm{for}}\subset$}\\ \text{$\mathscr{R}_i^{\mathrm{for}}$, for all $i=1,\dots,n$,}\\ \text{on which ∇_i^{for} has a pole of}\\ \text{order ≤ 2}\end{pmatrix},$$

is bijective.

Proof. (a) Pick some frame $\underline{e}$ of $\mathscr{M}$ over $\mathbb{C}\{u\}[u^{-1}]$ and let $\mathscr{H}^0 := \mathbb{C}\{u\}\cdot\underline{e}\subset\mathscr{M}$ be the submodule of all sections in $\mathscr{M}$ that are holomorphic in this frame. Now any $\mathbb{C}\{u\}$-submodule $\mathscr{H}\subset\mathscr{M}$ on which ∇ has a pole of order ≤ 2 will be a $\mathbb{C}\{u\}$-submodule of $\mathscr{M}$ which is commensurable with $\mathscr{H}^0$, i.e. we will have $u^N\mathscr{H}^0\subset\mathscr{H}\subset u^{-N}\mathscr{H}^0$ for $N\gg 1$. However the formal completion functor $(\bullet)\otimes_{\mathbb{C}\{u\}}\mathbb{C}[[u]]$ establishes an isomorphism between the Grassmannian $GL_r(\mathbb{C}\{u\}[u^{-1}])/GL_r(\mathbb{C}\{u\})$ and the affine Grassmannian $GL_r(\mathbb{C}((u))/GL_r(\mathbb{C}[[u]])$. But this map preserves the condition that a submodule $\mathscr{H}$ is invariant under $\nabla_{u^2d/du}$ which proves **(a)**.

(b) As already mentioned in Remark 2.13 this is proven in [**HS07**, Lemma 8.2]. Alternatively we can reason as in the proof of part **(a)**. Let $\mathscr{H}$ be a $\mathbb{C}[[u]]$-submodule in $\mathscr{M}^{\mathrm{for}}$ which is commensurable with $\mathscr{H}^{0,\mathrm{for}}$ and preserved by $\nabla_{u^2\frac{d}{du}}$. The operator $\nabla_{u^2\frac{d}{du}}$ acts on the infinite-dimensional topological complex vector space $\mathscr{M}^{\mathrm{for}}$ with finitely many infinite Jordan blocks with eigenvalues $\{\boldsymbol{c}_1,\dots,\boldsymbol{c}_n\}$. The corresponding generalized eigenspaces are exactly the modules $\boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u}\mathscr{R}_i^{\mathrm{for}}$. Hence

$$\mathscr{H}^{\mathrm{for}} = \oplus_i\left(\mathscr{H}^{\mathrm{for}}\cap\boldsymbol{\mathcal{E}}^{\boldsymbol{c}_i/u}\mathscr{R}_i^{\mathrm{for}}\right)$$

Therefore we obtain extensions $\mathscr{R}_i^{\mathrm{for}}$ with second order poles and regular singularity.

Combining the previous lemma with the equivalences in diagram (2.4.2) and the description of **nc**-Hodge structures from Section 2.1.8 gives the theorem. □

2.5. Deformations of nc-spaces and gluing. In this section we will briefly examine how the gluing construction for **nc**-Hodge structures varies with parameters. In particular, we will look at deformations of **nc**-spaces and the way the gluing data for the **nc**-Hodge structures on the cohomology of these spaces interacts with the appearance of a curvature in the d$(\mathbb{Z}/2)$g algebra computing the sheaf theory of the space.

2.5.1. ***The cohomological Hochschild complex.*** Suppose $X = \mathbf{ncSpec}\, A$ is an **nc**-affine **nc**-space. Recall that the cohomological Hochschild complex is defined as

$$C^\bullet(A, A) := \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{C}-\mathsf{Vect}}\left((\mathbf{\Pi}A)^{\otimes n}, A\right),$$

Its shift $\mathbf{\Pi}C^\bullet(A, A)$ is a Lie superalgebra with respect to the Gerstenhaber bracket [**Ger64**], and can be interpreted as the Lie algebra of continuous derivations of the free topological algebra $\prod_{n\geq 0}\left((\mathbf{\Pi}A)^{\otimes n}\right)^\vee$. The multiplication m_A and differential d_A of A combine into a cochain $\gamma_A := m_A + d_A \in C^\bullet(A, A)$ satisfying $[\gamma_A, \gamma_A] = 0$.

The formal deformation theory of X is controlled by a d$(\mathbb{Z}/2)$g Lie algebra structure $\mathbf{\Pi}C^\bullet(A, A)$ endowed with the differential $[\gamma_A, \bullet]$. It is convenient to consider also the reduced Hochschild complex

$$C^\bullet_{\mathrm{red}}(A, A) := \prod_{n\geq 0} \mathrm{Hom}_{\mathbb{C}-\mathsf{Vect}}\left(\left(\mathbf{\Pi}\left(A/\mathbb{C}\cdot 1_A\right)\right)^{\otimes n}, A\right),$$

which is naturally a subspace of $C^\bullet(A, A)$. The reduced complex is (after the parity change) a dg Lie subalgebra in $\mathbf{\Pi}C^\bullet(A, A)$. Moreover it is quasi-isomorphic to $\mathbf{\Pi}C^\bullet(A, A)$. Hence, for deformation theory purposes one can replace $\mathbf{\Pi}C^\bullet(A, A)$ by $\mathbf{\Pi}C^\bullet_{\mathrm{red}}(A, A)$.

Let $\gamma = \sum_{i\geq 1}\gamma_i t^i \in tC^{\mathrm{even}}_{\mathrm{red}}(A, A)[[t]]$ be a formal path consisting of solutions of the Maurer-Cartan equation, i.e.

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0 \quad (\Leftrightarrow\ [\gamma + \gamma_A, \gamma + \gamma_A] = 0).$$

Such a solution defines so-called formal deformation of the d$(\mathbb{Z}/2)$g algebra A as a weak (or curved) A_∞-algebra (see e.g. [**LH03**] for the definition and [**Sch03**] for a more detailed analysis). We can use the cochain $\gamma + \gamma_A \in C^{\mathrm{even}}(A, A)[[t]]$ to twist the notion of an A-module. We will write A_γ for the (weak) A_∞-algebra over $\mathbb{C}[[t]]$ corresponding to A and $\gamma + \gamma_A$ and $(A_\gamma - \mathsf{mod})$ for the $\mathbb{C}[[t]]$-linear dg category of all modules over A_γ. By definition $(A_\gamma - \mathsf{mod})$ is the category of dg modules over a bar-type resolution of A_γ [**KS06b**]. As an algebra the relevant bar dg algebra is the completed tensor product

$$\prod_{n\geq 0}\left((\mathbf{\Pi}A)^{\otimes n}\right)^\vee \widehat{\otimes}\mathbb{C}[[t]] \tag{2.5.1}$$

where the algebra structure comes from the usual algebra structure on $\mathbb{C}[[t]]$ and the tensor algebra structure on $\prod_{n\geq 0}\left((\mathbf{\Pi}A)^{\otimes n}\right)^\vee$. Thus for every $\gamma \in tC^{\mathrm{even}}_{\mathrm{red}}(A, A)[[t]]$

which solves the Mauer-Cartan equation we get a differential $\boldsymbol{\gamma} + \gamma_A$ on the graded algebra (2.5.1). The bar dg algebra of $A_{\boldsymbol{\gamma}}$ is now defined as the dg algebra

$$B_{\boldsymbol{\gamma}} := \left(\prod_{n \geq 0} \left((\mathbf{\Pi} A)^{\otimes n} \right)^{\vee} \widehat{\otimes} \mathbb{C}[[t]], \boldsymbol{\gamma} + \gamma_A \right).$$

The dg category $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ is by definition the category of dg modules over $B_{\boldsymbol{\gamma}}$ which are topologically free as modules of the underlying algebra, i.e. after forgetting the differential, and also satisfying the condition of unitality at $t = 0$.

As before this category can be viewed as the category $C_{\mathbb{X}_{\boldsymbol{\gamma}}} := (A_{\boldsymbol{\gamma}} - \mathsf{mod})$ of quasi-coherent sheaves on an **nc**-affine **nc**-space $\mathbb{X}_{\boldsymbol{\gamma}} \to \mathbb{D}$ defined over the formal disc $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[t]])$. More generally we will get an **nc**-space $\mathbb{X}$ over the formal scheme of solutions to the Maurer-Cartan equation and $\mathbb{X}_{\boldsymbol{\gamma}} \to \mathbb{D}$ is the restriction of $\mathbb{X}$ to the formal path $\boldsymbol{\gamma} + \gamma_A$ sitting inside that formal scheme.

Similarly we can use $\boldsymbol{\gamma}$ to twist the notion of a Hochschild cohomology class for A. Namely we can consider the Hochschild cohomology of the A_∞-algebra $A_{\boldsymbol{\gamma}}$. It is given explicitly as the cohomology

$$HH^\bullet_{\boldsymbol{\gamma}}(A) := H^\bullet\left(C^\bullet(A, A)[[t]], [\boldsymbol{\gamma} + \gamma_A, \bullet]\right),$$

and is a commutative algebra with respect to the cup product. Note also that the algebra $HH^\bullet_{\boldsymbol{\gamma}}(A)$ comes equipped with a unit $[1_A]$ and a distinguished even element $[\boldsymbol{\gamma} + \gamma_A]$, i.e. a structure similar to the one discussed in Section 2.2.5.

REMARK 2.38. • If $\boldsymbol{\gamma}$ has no component of degree zero, i.e. if

$$\boldsymbol{\gamma} \in tC^{\mathrm{even}}_{\mathrm{red.+}}(A, A)[[t]], \text{ where } C^\bullet_{\mathrm{red},+}(A, A) = \prod_{n \geq 1} \mathrm{Hom}_{\mathbb{C}-\mathsf{Vect}}\left((\mathbf{\Pi}\,(A/\mathbb{C} \cdot 1_A))^{\otimes n}, A \right),$$

then $A_{\boldsymbol{\gamma}}$ is an honest (strong) A_∞-algebra, and the category $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ will typically have many interesting objects. Furthermore, in this case smoothness and compactness are stable under deformations. That is, if A is smooth (respectively compact) over $\mathbb{C}$, then $A_{\boldsymbol{\gamma}}$ is smooth (respectively compact) over $\mathbb{C}[[t]]$.

• If the $n = 0$ component of $\boldsymbol{\gamma}$ is non-trivial, i.e. if the corresponding A_∞ structure has a non-trivial m_0, then the category $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ may have no non-zero objects. The basic example of this is when $A = \mathbb{C}$ and $\boldsymbol{\gamma} = t \cdot 1_A$.

If the original algebra A has the degeneration property, then it is easy to see that the Hodge-to-de Rham spectral sequence will degenerate for the periodic cyclic homology of $A_{\boldsymbol{\gamma}}$. In other words the formal **nc**-space $\mathbb{X}$ will give rise to a variation of **nc**-Hodge structures over the formal scheme of solutions of the Maurer-Cartan equation for A. When we have a non-trivial $n = 0$ component in $\boldsymbol{\gamma}$ this may lead to a paradoxical situation in which we have a family of **nc**-spaces over $\mathbb{D}$ which has no sheaves over the generic point but has non-trivial de Rham cohomology (i.e. periodic cyclic homology) generically. This suggests the following important

QUESTION 2.39. What is the geometrical meaning of $HH^\bullet_{\boldsymbol{\gamma}}(A)$, $HH_\bullet(A_{\boldsymbol{\gamma}})$, $HH^-_\bullet(A_{\boldsymbol{\gamma}})$, and $HP_\bullet(A_{\boldsymbol{\gamma}})$, when $\boldsymbol{\gamma}$ has non-trivial $n = 0$ component and the objects of $(A_{\boldsymbol{\gamma}} - \mathsf{mod})$ disappear over $\mathbb{D}^\times$?

REMARK 2.40. Note that if γ solves the Maurer-Cartan equation, then for any $c \in t\mathbb{C}[[t]]$, the cochain $\gamma + c \cdot 1_A$ will also solve the Maurer-Cartan equation[1]. So we have a natural mechanism for modifying formal paths of solutions of the Maurer-Cartan equation. We will exploit this mechanism in the next section.

2.5.2. ***Corrections by constants.*** The unpleasant phenomenon of having **nc**-spaces with no sheaves and non-trivial cohomology at the generic point is related to the gluing description for **nc**-Hodge structures. The idea is that the A-modules that disappear at the generic point of $\mathbb{D}$ may reappear again if we modify the weak A_∞-algebra A_γ appropriately. The periodic cyclic homologies of the different admissible modifications of A_γ then correspond to the regular pieces in the gluing description of the **nc**-de Rham data given by $HP_\bullet(A_\gamma)$. More precisely we have the following

CONJECTURE 2.41. *Suppose that A is a smooth and compact* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *algebra. Let $\gamma \in tC^{\text{even}}_{\text{red}}(A,A)[[t]]$ be a formal even path of solutions of the Maurer-Cartan equation for A. Then the periodic cyclic homology $HP_\bullet(A_\gamma)$ carries a canonical functorial structure of a variation of $\mathbb{Q}$-**nc**-Hodge structures of exponential type over* $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[t]])$. *Furthermore there exists a positive integer N and a finite collection of pairwise distinct Puiseux series*

$$\boldsymbol{c}_i = \sum_{j\geq 1} c_{i,j} t^{\frac{j}{N}}, \quad c_{i,j} \in \mathbb{C}$$

such that:

- *The series $\boldsymbol{c}_i$ are the distinct eigenvalues of the operator of multiplication by the class $[\gamma + \gamma_A]$ in the supercommutative algebra $HH^\bullet_\gamma(A)\widehat{\otimes}_{\mathbb{C}[[t]]}\mathbb{C}((t))$.*
- *For each i the category $(A_{\gamma+\boldsymbol{c}_i\cdot 1_A} - \mathsf{mod})$ is a non-trivial $\mathbb{C}\left[\left[t^{1/N}\right]\right]$-linear* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *category which are smooth and compact over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$ and is computed by a $d(\mathbb{Z}/2)g$ algebra B_i defined over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$ and quasi-isomorphic to the (weak) A_∞-algebra $A_{\gamma+\boldsymbol{c}_i\cdot 1_A}$.*
- *The Hochschild homologies $HH_\bullet(B_i)$ are flat $\mathbb{C}\left[\left[t^{1/N}\right]\right]$-modules and we have*

$$\sum_i \mathrm{rk}_{\mathbb{C}[[t^{1/N}]]}(HH_\bullet(B_i)) = \mathrm{rk}_{\mathbb{C}[[t^{1/N}]]} HH_\bullet(A_\gamma) = \dim_{\mathbb{C}} HH_\bullet(A).$$

- *The variation of **nc**-Hodge structures $HP_\bullet(A_\gamma)$ viewed as a variation over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$ has as regular constituents the variations of **nc**-Hodge structures on $HP_\bullet(B_i)$ whose existence is predicted by Conjecture 2.24.*

In particular Conjecture 2.41 says that the categorical and Hodge theoretic content of the algebra A_γ consists of the following data:

(categories): A finite collection of smooth and compact $\mathbb{C}\left[\left[t^{1/N}\right]\right]$-linear $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ categories $(B_i - \mathsf{mod})$.

[1]In fact this is the main reason for all the hassle with the unit and the reduced complex in this section.

(gluing): A finite collection of distinct Puiseux series $\boldsymbol{c}_i \in \mathbb{C}\left[\left[t^{1/N}\right]\right]$, and formal **nc**-gluing data which glues the variations of regular **nc**-Hodge structures on $HP_\bullet(B_i)$ into a variation of **nc**-Hodge structure of exponential type over $\mathbb{C}\left[\left[t^{1/N}\right]\right]$.

In the above discussion we have tacitly replaced the analytic setting from Section 2.3 by a formal setting. One can check that both the de Rham and Betti data make sense here, e.g. one can speak about homotopy classes of non-intersecting paths to points $\boldsymbol{c}_i$ thinking about t as a small real positive parameter.

REMARK 2.42. This situation is analogous to a well known setup in singularity theory. Namely, if we have a germ of an isolated hypersurface singularity given by an equation $f = 0$, and if we have a deformation of f which has several critical values, then the Milnor number of the original singularity is equal to the sum of the Milnor numbers of the simpler critical points of the deformed function. In fact, as we will see in Section 3.2 the singularity setup is a rigorous manifestation of the above conjectural picture.

2.5.3. ***Singular deformations.*** Suppose next that A is compact but not smooth (or smooth but non-compact) d($\mathbb{Z}/2$)g algebra and let again γ with $\gamma \in tC^{\text{even}}_{\text{red}}(A, A)[[t]]$ be a formal path of solutions of the Maurer-Cartan equation. We expect that the usual definition of smoothness and compactness can be modified to give a notion of smoothness together with compactness of A_γ at the generic point, i.e. over $\mathbb{C}((t))$, even when the objects in $(A_\gamma - \mathsf{mod})$ disappear over $\mathbb{C}((t))$.

In the case when A_γ is smooth and compact over $\mathbb{C}((t))$, i.e. when the deformation given by γ is a smoothing deformation, we also expect Conjecture 2.41 to hold at the generic point. More precisely, we expect to have Puiseux series $\boldsymbol{c}_i$ as above for which the associated categories $(A_{\gamma+\boldsymbol{c}_i 1_A} - \mathsf{mod})$ are non-trivial and smooth and compact over $\mathbb{C}\left(\left(t^{1/N}\right)\right)$. We also expect that the periodic cyclic homology $HP_\bullet\left(A_\gamma\right)$ is equipped with a variation of **nc**-Hodge structures of exponential type over $\mathbb{C}((t))$ so that the periodic cyclic homologies of the categories $(A_{\gamma+\boldsymbol{c}_i 1_A} - \mathsf{mod})$ are the regular pieces of this variation after we base change to $\mathbb{C}\left(\left(t^{1/N}\right)\right)$. Finally, the Puiseux series $\{\boldsymbol{c}_i\}$ should be the eigenvalues of the operator of multiplication by $[\gamma + \gamma_A] \in HH^\bullet\left(A_\gamma\right) \widehat{\otimes}_{\mathbb{C}[[t]]} \mathbb{C}((t))$.

3. Examples and relation to mirror symmetry

In this section we discuss examples of **nc**-Hodge structures arising from smooth and compact Calabi-Yau geometries and we study how these structures are affected by mirror symmetry. Specifically we look at a generalization of Homological Mirror Symmetry which relates categories of boundary topological field theories (or D-branes) associated with the following two types of geometric backgrounds:

A-model backgrounds: Pairs (X, ω), where X is a compact C^∞-manifold, and ω is a symplectic form on X satisfying a convergence property (see below).

B-model backgrounds: Pairs $\mathsf{w} : Y \to \text{disc} \subset \mathbb{C}$, where Y is a complex manifold with trivial canonical class, and w is a proper holomorphic map.

We will explain how each such background (both in the A and the B model) gives rise to the geometric and Hodge theoretic data described in Section 2.5.2. Namely we get:

- A finite collection $\{Z_i^{A/B}\}$ of smooth compact **nc**-spaces. In fact $\{Z_i^{A/B}\}$ will be (see Section 4.4.1 for the definition) odd/even Calabi-Yau **nc**-spaces of dimension $\left(\frac{\dim_{\mathbb{R}} X}{2} \mod 2\right) / \left(\dim_{\mathbb{C}} Y \mod 2\right)$.
- Complex numbers $\boldsymbol{c}_i^{A/B}$ and Betti gluing data $\left\{T_{ij}^{A/B}\right\}$ for the regular **nc**-Hodge structures on the periodic cyclic homology of $Z_i^{A/B}$.

In particular the data $\left(HC_\bullet^-\left(Z_i^A\right), \{\boldsymbol{c}_i^A\}, \{T_{ij}^A\}\right)$ and $\left(HC_\bullet^-\left(Z_i^B\right), \{\boldsymbol{c}_i^B\}, \{T_{ij}^B\}\right)$ each glue into a **nc**-Hodge structure of exponential type. The **generalized Homological Mirror Symmetry Conjecture** now asserts that if two A/B-model backgrounds $(X,\omega)/(Y,\mathsf{w})$ are mirror to each other, then the associated **nc**-geometry and **nc**-Hodge structure packages are isomorphic:

$$\left(Z_i^A, \{\boldsymbol{c}_i^A\}, \{T_{ij}^A\}\right) \cong \left(Z_i^B, \{\boldsymbol{c}_i^B\}, \{T_{ij}^B\}\right).$$

3.1. A-model Hodge structures: symplectic manifolds. Suppose (X,ω) is a compact symplectic manifold of dimension $\dim_{\mathbb{R}} X = 2d$. In the case when X is a Calabi-Yau variety (in particular $c_1(X) = 0$) one has a family of superconformal field theories attached to X in the large volume limit (i.e. after the rescaling $\omega \to \omega/\hbar$ where $0 < \hbar \ll 1$), and the A-twist gives a topological quantum field theory (see [**HKK$^+$03**]). In mathematical terms this means that we have Gromov-Witten invariants and a $\mathbb{Z}$-graded Fukaya category associated to $(X,\omega/\hbar)$. On the other side, Gromov-Witten invariants can be defined for an arbitrary compact symplectic manifold, not necessarily one with $c_1(X) = 0$. Our goal in this section is to describe what is an analog of the Fukaya category for general (X,ω).

Namely, it is expected that for (X,ω) of large volume the Fukaya category of (X,ω) is a weak $\mathbb{Z}/2$-graded A_∞-category which will satisfy the generalized smoothness and compactness properties conjectured in Section 2.5.3. Briefly this should work as follows. Following Fukaya-Oh-Ohta-Ono [**FOOO07**] consider a finite collection $\mathfrak{L} = \{L_i\}$ of transversal oriented spin Lagrangian submanifolds in X and form a "degenerate" version $\mathsf{Fuk}_{\mathfrak{L}}$ of Fukaya's category which only involves the L_i. More precisely we take $\mathsf{Ob}\left(\mathsf{Fuk}_{\mathfrak{L}}\right) = \{L_i\}$, and define

$$\operatorname{Hom}_{\mathsf{Fuk}_{\mathfrak{L}}}\left(L_i, L_j\right) = \begin{cases} \mathbb{C}^{L_i \cap L_j}, & i \neq j, \\ A^\bullet(L_i, \mathbb{C}), & i = j. \end{cases}$$

Here $\mathbb{C}^{L_i \cap L_j}$ is taken with the ordinary algebra structure but is put in degree equal to the Maslov grading mod 2, and $A^\bullet(L_i,\mathbb{C})$ is the dg algebra of C^∞ differential forms on L_i.

We consider a 1-parameter family of symplectic manifolds

$$\left(X, \frac{\omega}{\hbar}\right), \quad \hbar \in \mathbb{R}_{>0}, \quad \hbar \to 0. \tag{3.1.1}$$

It will be convenient to introduce a new parameter $q := \exp(-1/\hbar)$ (note that $q \to 0$ when $\hbar \to 0$). Denote by $\mathbb{C}_q$ the usual Novikov ring:

$$\mathbb{C}_q := \left\{ \sum_{i=0}^{\infty} a_i q^{E_i} \,\middle|\, \begin{array}{l} \text{formal series where } a_i \in \mathbb{C} \text{ and } E_i \in \mathbb{R} \\ \text{with } \lim_{i\to\infty} E_i = +\infty \end{array} \right\}$$

In the case $[\omega] \in H^2(X, \mathbb{Z})$ one can replace the Novikov ring $\mathbb{C}_q$ by the more familiar algebra $\mathbb{C}((q))$ of Laurent series. The three-point genus zero Gromov-Witten invariants of the symplectic family (3.1.1) give rise (see e.g. [**KM94, LT98, Sie99, CK99, FO01**]) to a $\mathbb{C}_q$-valued (small) quantum deformation of the cup product on $H^\bullet(X, \mathbb{C})$:

$$*_q : H^\bullet(X, \mathbb{C})^{\otimes 2} \to H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}_q$$

Conjecturally the series for the quantum product is absolutely convergent for sufficiently small q.

What is constructed in [**FOOO07**] is a solution $\boldsymbol{\gamma}$ of the Maurer-Cartan equation in the cohomological Hochschild complex of $\mathsf{Fuk}_{\mathfrak{L}}$ with coefficients in the series in $\mathbb{C}_q$ with strictly positive exponents (equal to the areas of non-trivial pseudo-holomorphic discs). The meaning of the quantum product is the cup-product in the Hochschild cohomology of the deformed weak category.

The d$(\mathbb{Z}/2)$g category $\mathsf{Fuk}_{\mathfrak{L}}$ over $\mathbb{C}_q$ is compact but not smooth. If the collection $\mathfrak{L}$ is chosen to be big enough, i.e. if it generates the full Fukaya category, then $\mathsf{Fuk}_{\mathfrak{L}}$ is the large volume limit of $\mathsf{Fuk}(X, \omega)$, i.e. the limit in which all disc instantons for ω are supressed.

Now the formalism of Section 2.5.3 should associate with $\mathsf{Fuk}_{\mathfrak{L}} = (A - \mathsf{mod})$ and $\boldsymbol{\gamma}$ a finite collection $\{\boldsymbol{c}_i\}$ of formal series in positive powers of q and a collection $\{\mathsf{Fuk}_i\}$ of non-trivial smooth and compact modifications of the Fukaya category whose Hochschild homologies are the regular singularity constitutents of the Hochschild homology of the q-family of Fukaya categories near the large volume limit. In this geometric context, we expect that the $\{\boldsymbol{c}_i\}$ are the eigenvalues of the quantum multiplication operator $c_1(T_X) *_q (\bullet)$ acting on $H^\bullet(X, \mathbb{C}) \otimes \mathbb{C}[[u]]$. Some evidence for this comes from the observation that when $c_1(T_X)$ vanishes in $H^2(X, \mathbb{Z})$, then the Fukaya category is $\mathbb{Z}$-graded and thus is a fixed point of the renormalization group. There is also a more explicit direct argument identifying the class $c_1(T_X)$ with the infnitesimal generator of the renormalization group, but we will not discuss it here.

The formalism of Section 2.5.3 now predicts that the periodic cyclic homology of the Fukaya category, which additively should be the same as the de Rham cohomology of X, should carry a natural **nc**-Hodge structure satisfying the degeneration conjecture from Section 2.2.4. This expectation is supported by ample evidence coming from mirror symmetry for Calabi-Yau complete intersections. Here we present further evidence by describing a natural **nc**-Hodge structure on the de Rham cohomology of a symplectic manifold and by showing that as ω approaches the large volume limit this structure fits in a natural variation of **nc**-Hodge structures.

Using the quantum product $*_q$ we will attach to (X, ω) a variation $((\mathscr{H}, \nabla), \mathscr{E}_B,$ **iso**) of **nc**-Hodge structures over a small disc $\{q \in \mathbb{C} | \, |q| < r\}$ in the q-plane. First we describe the **nc**-Hodge filtration $(\mathscr{H}, \nabla)$ and its variation in the q-direction:

- $\mathscr{H} := H^{\bullet}(X, \mathbb{C}) \otimes \mathbb{C}\{u, q\}$ and

$$\mathscr{H}^0 := \left(\bigoplus_{k=d \mod 2} H^k(X, \mathbb{C}) \right) \otimes \mathbb{C}\{u, q\}$$

$$\mathscr{H}^1 := \left(\bigoplus_{k=d+1 \mod 2} H^k(X, \mathbb{C}) \right) \otimes \mathbb{C}\{u, q\}$$

- ∇ is a meromorphic connection on $\mathscr{H}$ with poles along the coordinate axes $u = 0$ and $q = 0$, given by

$$\nabla_{\frac{\partial}{\partial u}} := \frac{\partial}{\partial u} + u^{-2} (\kappa_X *_q \bullet) + u^{-1}\mathbf{Gr},$$

$$\nabla_{\frac{\partial}{\partial q}} := \frac{\partial}{\partial q} - q^{-1}u^{-1} ([\omega] *_q \bullet),$$

where:
$\kappa_X \in H^2(X, \mathbb{Z})$ denotes the first Chern class of the cotangent bundle of X computed w.r.t. any ω-compatible almost complex structure, and

$\mathbf{Gr} : \mathscr{H} \to \mathscr{H}$ is the grading operator, defined to be $\mathbf{Gr}_{|H^k(X,\mathbb{C})} := \frac{k-d}{2} \mathrm{id}_{H^k(X,\mathbb{C})}$.

The data $(\mathscr{H}, \nabla)$ define a q-variation of (the de Rham part of) **nc**-Hodge structures. Defining the $\mathbb{Q}$-structure is much more delicate. To gain some insight into the shape of the rational local system $\mathscr{E}_B$ one can look at the monodromy in the q direction of the algebraic bundle with connection

$$(H, \nabla)_{|(\mathbb{A}^1 - \{0\}) \times \{q \in \mathbb{C} | \; |q| < R\}}, \qquad (H, \nabla) = \mathfrak{B}_{\text{along } u}((\mathscr{H}, \nabla)).$$

In some cases the facts that $\mathscr{E}_B$ should be preserved by ∇ and that the Stokes filtration is rational with respect to $\mathscr{E}_B$ are enough to determine $\mathscr{E}_B$ completely:

Proposition 3.1. *Let $X = \mathbb{CP}^{n-1}$ and let ω be the Fubini-Study form. Let (H, ∇) be the holomorphic bundle with meromorphic connection on $(\mathbb{A}^1 - \{0\}) \times \{q \in \mathbb{C} | \; |q| < R\}$ defined above. Let $\psi \in H$ be a holomorphic section which is covariantly constant with respect to ∇. Then*

(a) *For every $u \neq 0$, $\psi \neq 0$ the limit (in a sector of the q-plane)*

$$\psi_{\mathrm{cl}}(u) = \lim_{q \to 0} \left(\exp \left(-\frac{\log(q)}{u} ([\omega] \wedge (\bullet)) \right) \right) \psi$$

exists. Furthermore, ψ_{cl} satisfies the differential equation

$$\left(\frac{d}{du} + u^{-2}\kappa_X \wedge + u^{-1}\mathbf{Gr} \right) \psi_{\mathrm{cl}} = 0.$$

(b) *The vector*

$$\psi_{\mathrm{const}}(u) := \exp(\log(u)\mathbf{Gr}) \exp \left(\frac{\log(u)}{u} \kappa_X \wedge (\bullet) \right) \psi_{\mathrm{cl}} \in H^{\bullet}(X, \mathbb{C})$$

is independent of u.

(c) *Define the rational structure* $\mathscr{E}_B \subset H^\nabla$ *as the subsheaf of all covariantly constant sections* ψ *for which the vector* $\psi_{\mathrm{const}} \in H^\bullet(X, \mathbb{C})$ *belongs to the image of the map*

$$H^\bullet(X,\mathbb{Q}) \xrightarrow{\ \mathfrak{d}\ } H^\bullet(X,\mathbb{C}) \xrightarrow{\widehat{\Gamma}(X)\wedge(\bullet)} H^\bullet(X,\mathbb{C}),$$

where $\mathfrak{d} \in GL(H^\bullet(X,\mathbb{C}))$ *is the operator of multiplication by* $(2\pi i)^{k/2}$ *on* $H^k(X,\mathbb{C})$, *and* $\widehat{\Gamma}(X)$ *is a new characteristic class of* X *defined as*

$$\widehat{\Gamma}(X) := \exp\left(\boldsymbol{C}\,\mathrm{ch}_1(T_X) + \sum_{n\geq 2} \frac{\zeta(n)}{n}\mathrm{ch}_n(T_X)\right),$$

where

$$\boldsymbol{C} = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)\right)$$

is Euler's constant, and $\zeta(s)$ *is Riemann's zeta function.*

Then the inclusion $\mathscr{E}_B \subset H^\nabla$ *is compatible with Stokes data, i.e. the rational structure* $\mathscr{E}_B$ *satisfies* **($\mathbb{Q}$-structure axiom)**^{exp}.

The calculation presented below was known already to B.Dubrovin [**Dub98**, Section 4.2.1], where he also obtained a Taylor expansions of a power of a gamma function in quantum cohomology, although he did not identify it with a characteristic class.

Proof of Proposition 3.1. In the standard basis $\{1, h, h^2, \ldots, h^{n-1}\}$ of $H^\bullet(\mathbb{P}^{n-1}, \mathbb{C})$ the connection ∇ on H is given by

$$\nabla_{\frac{\partial}{\partial u}} = \frac{\partial}{\partial u} + u^{-2}\begin{pmatrix} 0 & & & nq \\ n & 0 & & \\ & \ddots & \ddots & \\ & & n & 0 \end{pmatrix} + u^{-1}\begin{pmatrix} \frac{1-n}{2} & & & 0 \\ & & \ddots & \\ & & \ddots & \\ 0 & & & \frac{n-1}{2} \end{pmatrix}$$

$$\nabla_{\frac{\partial}{\partial q}} = \frac{\partial}{\partial q} - q^{-1}u^{-1}\begin{pmatrix} 0 & & & q \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

If $\psi = \sum_{i=1}^{n} \psi_i h^{i-1}$ is a local section of H, a straightforward check shows that the condition on ψ to be ∇-horizontal is solved by the following ansatz:

$$\begin{aligned}
\psi_n &= u^{\frac{1-n}{2}} \int\limits_{\Gamma_{u,q}} \exp(\mathcal{F}) \prod_{i=1}^{n-1} \frac{dz_i}{z_i} \\
\psi_{n-1} &= \left(uq\frac{\partial}{\partial q} \right) \psi_n \\
\psi_{n-2} &= \left(uq\frac{\partial}{\partial q} \right)^2 \psi_n \\
&\dots \\
\psi_1 &= \left(uq\frac{\partial}{\partial q} \right)^{n-1} \psi_n.
\end{aligned}$$

Here $\mathcal{F}$ is the function on $(\mathbb{C}^\times)^{n-1}$ with coordinates $z_1, \dots, z_{n-1}$ depending on parameters $u, q \neq 0$ and given by

$$\mathcal{F}(z_1, z_2, \dots, z_{n-1}; u, q) := u^{-1} \left(z_1 + z_2 + \cdots + z_{n-1} + \frac{q}{z_1 z_2 \cdots z_{n-1}} \right).$$

The integral is taken over some fixed $(n-1)$-dimensional semi-algebraic non-compact cycle $\Gamma_{u,q}$ in $(\mathbb{C}^\times)^{n-1}$ (depending on the parameters u, q) which is going to infinity in directions where $\mathrm{Re}(\mathcal{F}) \to -\infty$.

More generally, the domain of integration $\Gamma_{u,q}$ used for defining ψ_n can be taken to be an $(n-1)$-dimensional rapid decay homology chain in $(\mathbb{C}^\times)^{n-1}$. The rapid decay homology cycles on smooth complex algebraic varieties are the natural domains of integration for periods of cohomology classes of irregular connections. The rapid decay homology was introduced and studied by Hien [**Hie07, Hie08**], following previous works of Sabbah [**Sab00**] and Bloch-Esnault [**BE04**]. In particular by recent work of Mochizuki [**Moc08a, Moc08b**] and Hien [**Hie08**] it follows that (after a birational base change) taking periods induces a perfect pairing between the de Rham cohomology of an irregular connection and the rapid decay homology. This powerful general theory is not really needed in our case where the manifold is the affine algebraic torus $(\mathbb{C}^\times)^{n-1}$, but it does provide a useful perspective.

Explicitly the non-compact cycles that we will use to generate horizontal sections of (H, ∇) will be the $(n-1)$-dimensional relative cycles for a pair $(\boldsymbol{\mathcal{X}}, Z)$ constructed as follows. Start with a smooth projective compactification $\mathfrak{X}$ of $(\mathbb{C}^\times)^{n-1}$ with a normal crossing boundary divisor D which is adapted to $\mathcal{F}$ in the sense that if u and q are nonzero, the divisors of zeroes and poles of $\mathcal{F}$ in $\mathfrak{X}$ do not intersect with each other, and locally at points of D the function $\mathcal{F}$ can be written as a product of an invertible holomorphic function and a monomial in the local coordinates. Let $\boldsymbol{\mathcal{X}}$ be the real oriented blow-up of $\mathfrak{X}$ along the divisor D. Now consider the real boundary $\partial\boldsymbol{\mathcal{X}}$ of $\boldsymbol{\mathcal{X}}$, i.e. the union of all the boundary divisors of the real oriented blow-up. The boundary $\partial\boldsymbol{\mathcal{X}}$ contains a natural open real semi-algebraic subset $Z \subset \partial\boldsymbol{\mathcal{X}}$ consisting of all points $b \in \partial\boldsymbol{\mathcal{X}}$, such that $|\mathcal{F}(z; u, q)| \to \infty$ when $z \to b$, and for points $z \in t(\mathbb{C}^\times)^{n-1}$ near b the argument of $\mathcal{F}(z; u, q)$ lies strictly in the left half-plane of $\mathbb{C}$. Note that the real blow-up X has the same homotopy type as $\mathfrak{X} - D = (\mathbb{C}^\times)^{n-1}$ and so relative cycles on $(\boldsymbol{\mathcal{X}}, Z)$ can be thought of as non-compact cycles on $(\mathbb{C}^\times)^{n-1}$. Moreover since Z is defined by our condition on the argument of $\mathcal{F}$, it follows that relative cycles with boundaries in Z give rise to well defined integrals of $\exp(\mathcal{F}) \prod z_i^{-1} dz_i$.

Next observe that the integrals over relative cycles with integral coefficients, i.e. elements in $H_{n-1}(\boldsymbol{\mathcal{X}}, Z; \mathbb{Z})$, give rise to a covariantly constant integral lattice in the bundle (H, ∇). Furthermore the Deligne-Malgrange-Stokes filtration is integral with respect to this lattice. Indeed if we fix a real number λ, then whenever $\mathrm{Re}\,(\mathcal{F}) < \lambda \cdot |u|^{-1}$, it follows that $|\exp(\mathcal{F})| < \exp(\lambda \cdot |u|^{-1})$ when $u \to 0$. Hence the steps of the Deligne-Malgrange-Stokes filtration of (H, ∇) are easy to describe in this language: they correspond to periods of $\exp(\mathcal{F}) \prod z_i^{-1} dz_i$ on relative cycles on $(\boldsymbol{\mathcal{X}}, Z)$ whose boundary is contained in half-planes of the form $\mathrm{Re}(\mathcal{F}) < \mathrm{const}$. The periods over cycles with integral coefficients and the same boundary property then give a full integral lattice in each such step.

Now to finish the proof of the proposition we just have to calculate the limiting lattice (which is independent of u and q) consisting of vectors $\psi_{\mathrm{const}} \in H^\bullet(X, \mathbb{C})$ defined in terms of ψ by the formula in part (b) of the statement of the proposition.

For a general ∇-horizontal local section $\psi = \sum_{i=1}^n \psi_i h^{i-1}$ in a sector at 0 in the q-plane (for given $u \neq 0$) one has an asymptotic expansion of ψ at $q \to 0$ given by:

$$\psi_n = \sum_{i=0}^{n-1} a_i(u)(\log q)^i + \boldsymbol{O}(q(\log q)^n) + \cdots \tag{3.1.2}$$

Then we have that the "classical limit" (at $q \to 0$ where the quantum multiplication becomes classical) is given by

$$\psi_{\mathrm{cl}}(u) = \begin{pmatrix} (n-1)! u^{n-1} a_{n-1}(u) \\ (n-2)! u^{n-2} a_{n-2}(u) \\ \vdots \\ 0! u^0 a_0(u) \end{pmatrix}.$$

Now we restrict to the case where all variables are real, $u < 0$, $q > 0$ and the contour of integration is being the positive octant $\{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, z_i > 0\ \forall i\}$.

The function $\psi_n = \psi_n(u, q)$ decays exponentially fast at $q \to +\infty$ for a given $u < 0$, hence one can extract its asymptotic expansion at $q \to 0$ through the Mellin transform:

$$\int_0^{+\infty} \psi_n q^s \frac{dq}{q} = \sum_{i=0}^{\infty} a_i(u) \frac{i!(-1)^i}{s^{i+1}} + \boldsymbol{O}(1), \quad s \to 0.$$

This integral can be calculated explicitly

$$\int_0^{+\infty} \psi_n q^s \frac{dq}{q} = u^{\frac{1-n}{2}} \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{n \text{ times}} \frac{dq}{q} \prod_{i=1}^{n-1} \frac{dz_i}{z_i}$$

$$\cdot \exp\left(u^{-1} \left(z_1 + z_2 + \cdots + z_{n-1} + \frac{q}{z_1 z_2 \cdots z_{n-1}} \right) \right) q^s$$

$$= u^{\frac{1-n}{2}} \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{n-1 \text{ times}} \prod_{i=1}^{n-1} \frac{dz_i}{z_i} \exp\left(u^{-1} \sum_i^{n-1} z_i \right)$$

$$\cdot \underbrace{\int_0^{+\infty} \exp\left(\frac{q}{u z_1 z_2 \cdots z_{n-1}} \right) q^s \frac{dq}{q}}_{\substack{\| \\ \Gamma(s)(-u z_1 z_2 \cdots z_{n-1})^s}}$$

$$= u^{\frac{1-n}{2}} (-u)^s \Gamma(s) \int_0^{+\infty} \cdots \int_0^{+\infty} \prod_{i=1}^{n-1} \left(\frac{dz_i}{z_i} z_i^s \exp \frac{z_i}{u} \right)$$

$$= u^{\frac{1-n}{2}} (-u)^s \Gamma(s) \left((-u)^s \Gamma(s) \right)^{n-1}$$

$$= u^{\frac{1-n}{2}} (-u)^{ns} \Gamma(s)^n.$$

The conclusion is that the chosen branch $\psi_{\mathrm{cl}}(u)$ is completely defined by the expansion

$$u^{\frac{1-n}{2}} (-u)^{ns} \Gamma(s)^n = \frac{\psi_{\mathrm{cl},n}(u)}{(-u)^0 s} + \frac{\psi_{\mathrm{cl},n-1}(u)}{(-u)^1 s^2} + \cdots + \frac{\psi_{\mathrm{cl},1}(u)}{(-u)^{n-1} s^n} + \boldsymbol{O}(1), \quad s \to 0$$

Furthermore, all the other branches can be obtained by acting on the branch we know by the monodromy transformations (around $q = 0$)

$$\frac{(2\pi\sqrt{-1})^i}{u^i} \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}^i,$$

for $i = 0, \ldots, n-1$.

Section ψ_{cl} satisfies the differential equation

$$\left(\frac{d}{du} + u^{-2} \kappa_X \wedge + u^{-1} \mathbf{Gr} \right) \psi_{\mathrm{cl}} = 0.$$

which is the classical limit (at $q \to 0$) of the equation

$$\nabla_{\frac{\partial}{\partial u}} (\psi) = 0$$

One can check that the operator $\frac{d}{du} + u^{-2}\kappa_X \wedge + u^{-1}\mathbf{Gr}$ can be written as

$$\exp\left(-\frac{\log(u)}{u}\kappa_X \wedge (\bullet)\right)\exp(-\log(u)\mathbf{Gr})\circ\frac{d}{du}\circ\exp(\log(u)\mathbf{Gr})\exp\left(\frac{\log(u)}{u}\kappa_X \wedge (\bullet)\right)$$

This follows from the commutation relation

$$[\kappa_X \wedge (\bullet), \mathbf{Gr}] = -\kappa_X \wedge (\bullet)$$

Finally, in the above formulas one can replace $\log(u)$ by $\log(-u)$ (and also $u^{\frac{1-n}{2}}$ by $(-u)^{\frac{1-n}{2}}$) with principal values at the domain $u < 0$. Having this modification in mind, we conclude that the vector

$$\psi_{\text{const}} = \psi_{\text{const}}(u) := \exp(\log(-u)\mathbf{Gr})\exp\left(\frac{\log(-u)}{u}\kappa_X \wedge (\bullet)\right)\psi_{\text{cl}} \in H^\bullet(X, \mathbb{C})$$

is independent of u, and in particular it coincides with $\psi_{\text{cl}}(-1)$, as for $u = -1$ the correction matrices relating $\psi_{\text{const}}(u)$ and $\psi_{\text{cl}}(u)$ are identity matrices. Therefore the vector ψ_{const} is given by Taylor coefficients

$$\psi_{\text{const},1}s^0 + \cdots + \psi_{\text{const},n}s^{n-1} = s^n\Gamma(s)^n + \boldsymbol{O}(s^n) = \Gamma(1+s)^n + \boldsymbol{O}(s^n)$$

We see that $\psi_{\text{const}} \in H^\bullet(X, \mathbb{C})$ (after rescaling by the operator $\mathfrak{d}$ from the Proposition) with the value of the multiplicative characteristic class associated to the series $\Gamma(1+s) = 1 + \boldsymbol{O}(s) \in \mathbb{C}[[s]]$ and the tangent bundle T_X, because $[T_X] = n[\mathcal{O}(1)] - [\mathcal{O}]$ for $X = \mathbb{CP}^n$, and by the classical expansion

$$\log(\Gamma(1+s)) = \boldsymbol{C}s + \sum_{k\geq 2}\frac{\zeta(k)}{k}s^k$$

The action of the monodromy corresponds (up to torsion) to the multiplication by $\kappa_X \in H^\bullet(X, \mathbb{Z})$. □

The previous proposition suggests the following general definition:

Definition 3.2. *The rational structure on (H, ∇) is the local subsystem $\mathscr{E}_B \subset H_{|\mathbb{A}^1 - \{0\}}$ of multivalued ∇-horizontal sections whose values at 1 belong to the image of*

$$H^\bullet(X, \mathbb{Q}) \xrightarrow{\mathfrak{d}} H^\bullet(X, \mathbb{C}) \xrightarrow{\widehat{\Gamma}(TX)\wedge(\bullet)} H^\bullet(X, \mathbb{C}),$$

where $\mathfrak{d} \in GL(H^\bullet(X, \mathbb{C}))$ is the operator of multiplication by $(2\pi i)^{k/2}$ on $H^k(X, \mathbb{C})$, and $\widehat{\Gamma}(TX)$ is a new characteristic class of X defined as

$$\widehat{\Gamma}(TX) := \prod_{i=1}^{d}\Gamma(1+\lambda_i),$$

where $\Gamma(s)$ is the classical gamma function and λ_i are the Chern roots of T_X computed in any ω-admissible almost complex structure.

Remark 3.3. Apart from the calculation in Proposition 3.1 there are a few other (loose) motivations for this definition:

- The class $\widehat{\Gamma}$ appears in the context of deformation quantization in the work of the second author [**Kon99**, Section 4.6].
- The number $\chi(X)\zeta(3)$ appears in the mirror formula for the quintic threefold.
- Golyshev's description [**Gol01, Gol07**] of the **nc**-motives associated with the Landau-Ginzburg mirror of a toric Fano involves similar hypergeometric series.

• The same class $\widehat{\Gamma}$ was derived and a definition similar to Definition 3.2 was proposed in the recent work of Iritani [**Iri07**] for the case of toric orbifolds by tracing out the mirror image of rational structure of the mirror Landau-Ginzburg model.

CONJECTURE 3.4. *The triple* $(H, \mathscr{E}_B, \mathsf{iso})$ *associated above with a symplectic manifold* (X, ω) *is a variation of* nc*-Hodge structures of exponential type.*

REMARK 3.5. **(i)** In general it is not clear if the **($\mathbb{Q}$-structure axiom)$^{\mathbf{exp}}$** holds in this case. It does hold trivially in the graded case, i.e. when X is a Calabi-Yau.
(ii) At the moment the "exponential type" part of the conjecture is not supported by any evidence beyond the graded case in which the nc-Hodge structure is regular. It is possible that for non-Kähler symplectic manifolds the nc-Hodge structure on the de Rham cohomology is not of exponential type.

3.2. B-model Hodge structures: holomorphic Landau-Ginzburg models. Suppose we have an algebraic map $\mathsf{w} : Y \to \mathbb{C}$, where Y is a smooth quasi-projective manifold and w has a compact critical locus $\mathrm{crit}(\mathsf{w}) \subset Y$. Let $S = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_m\} \subset \mathbb{C}$ denote the critical values of w.

A pair (Y, w) like that is called a ***holomorphic Landau-Ginzburg model*** and often arises (see e.g. [**HV00, HKK$^+$03**]) as the mirror of a symplectic manifold underlying a hypersurface, or a complete intersection in a toric variety. Remarkably the pair (Y, w) gives rise to a natural nc-space $\mathsf{nc}(Y, \mathsf{w})$. The category $C_{\mathsf{nc}(Y,\mathsf{w})}$ can be described in two equivalent ways (in fact these descriptions are valid even if the critical locus of w is not compact). First note that it is enough to define $\mathsf{Perf}_{C_{\mathsf{nc}(Y.\mathsf{w})}}$ since the category $C_{\mathsf{nc}(Y,\mathsf{w})}$ can be thought of as the homotopy colimit completion of $\mathsf{Perf}_{C_{\mathsf{nc}(Y.\mathsf{w})}}$. For the latter we have two models:

$\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$ **as a category of matrix factorizations:** This model was proposed originally by the second author as a mathematical description of the category of D-branes and was subsequently studied extensively in the physics and mathematics literature, see [**KL03, KL04**] and [**Orl04, Orl05b, Orl05a**].

A matrix factorization on (Y, w) is a pair $\left(E = E^0 \oplus E^1, d_E \in \mathrm{End}(E)^{\mathrm{opp}}\right)$, where

E is a $\mathbb{Z}/2$-graded algebraic vector bundle on Y, and
d_E is an odd endomorphism satisfying $d_E^2 = \mathsf{w} \cdot \mathrm{id}_E$.

In the case when Y is affine the $\mathbb{Z}/2$-graded complex $\underline{\mathrm{Hom}}((E, d_E), (F, d_F))$ of homomorphisms between two matrix factorizations is defined as $\underline{\mathrm{Hom}}((E, d_E), (F, d_F)) := (\mathrm{Hom}(E, F), d)$ where for a $\varphi : E \to F$ we have $d\varphi := \varphi \circ d_E - d_F \circ \varphi$. For general Y the same definition works if we replace $\mathrm{Hom}(E, F)$ by some acyclic model, e.g. if we use the Dolbeault resolution. The resulting category $\mathsf{MF}(Y, \mathsf{w})$ of matrix factorizations is a $\mathbb{C}$-linear $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ category. We define $\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$ to be the derived category $D^b(\mathsf{MF}(Y, \mathsf{w}))$ of the category of matrix factorizations.

To construct $D^b(\mathsf{MF}(Y, \mathsf{w}))$ one notes that in addition to being a $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ category $\mathsf{MF}(Y, \mathsf{w})$ can also be viewed as a curved $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ category with central curvature w (see e.g. [**PP05**] for the definition) or as a $\mathbb{Z}/2$-graded weak A_∞-category, i.e. an A_∞ category with an m_0-operation given by w

(see e.g. [**Sch03, LH03**] for the definition). In particular we can form the associated homotopy category (in the A_∞-sense) which by definition will be the derived category of matrix factorizations.

Alternatively, one can use the following two step construction proposed by Orlov. First we pass to the homotopy category of $\mathsf{MF}(Y, \mathsf{w})$, i.e. we consider the category whose objects are matrix factorizations and whose morphisms are given by the quotient of $\underline{\mathrm{Hom}}((E, d_E), (F, d_F))$ by homotopy equivalences. Next (following the standard wisdom) we need to quotient $\mathrm{Ho}(\mathsf{MF}(Y, \mathsf{w}))$ by the subcategory of acyclic factorizations. Since the matrix factorizations are not complexes, they do not have cohomology and so we can not define acyclicity in the usual way. But there is another point of view on acyclicity. If we have a short exact sequence of usual complexes, then the total complex of this diagram will be an acyclic complex. So we define ***acyclic*** matrix factorizations as the total matrix factorization of an exact sequence of factorizations. With this definition we get a thick subcategory in the homotopy category $\mathrm{Ho}(\mathsf{MF}(Y, \mathsf{w}))$ of matrix factorizations and then we can pass to the Serre quotient of $\mathrm{Ho}(\mathsf{MF}(Y, \mathsf{w}))$ by this thick subcategory. We set $D^b(\mathsf{MF}(Y, \mathsf{w}))$ to be this Serre quotient.

$\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}}$ **as a category of singularities:** This model was proposed originally by D. Orlov as an alternative to the matrix factorization description which is localized near the critical set of w. Orlov proved the equivalence of the two models, various versions of the localization theorem, and proved several duality statements relating derived categories of singularities to other familiar categories [**Orl04, Orl05b, Orl05a**].

Suppose Z is a quasi-projective complex scheme. The ***derived category*** $D^b_{\mathrm{Sing}}(Z)$ ***of singularities*** of Z is defined as the quotient

$$D^b_{\mathrm{Sing}}(Z) := D^b(\mathsf{Coh}(Z))/\,\mathsf{Perf}_Z$$

of the (dg enhancement of the) bounded derived category $D^b(\mathsf{Coh}(Z))$ of coherent sheaves on Z by the thick subcategory of perfect complexes on Z. The syzygy theorem implies that $D^b_{\mathrm{Sing}}(Z) = 0$ whenever Z is smooth and so $D^b(\mathsf{Coh}(Z))$ can be thought of as an invariant of the singularities of Z.

If now $\mathsf{w} : Y \to \mathbb{C}$ is a holomorphic Landau-Ginzburg model we write Y_c for the fiber $\mathsf{w}^{-1}(c)$ and set

$$\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}} := D^b_{\mathrm{Sing}}(Y_0).$$

Note that if $0 \in \mathbb{A}^1$ is not a critical value of w, then with this definition we will get $\mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w})}} = 0$. In order to get non-trivial categories we will use the critical values $S = \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ to shift the potential $\mathsf{w} \rightsquigarrow \mathsf{w} - \boldsymbol{c}_i$ and associate with $\mathsf{nc}(Y, \mathsf{w})$ honest categories $\mathsf{Perf}_i := \mathsf{Perf}_{C_{\mathsf{nc}(Y,\mathsf{w}-\boldsymbol{c}_i)}} = D^b_{\mathrm{Sing}}(Y_{\boldsymbol{c}_i})$. Conjecturally, these categories are smooth and compact.

Mirror symmetry suggests that the nc-space $\mathsf{nc}(Y, \mathsf{w})$ gives rise to the B-model geometric and Hodge theoretic data described in Section 2.5.2, and in particular that the periodic cyclic homology of $C_{\mathsf{nc}(Y,\mathsf{w})}$ carries a canonical nc-Hodge structure. In fact we have already described the geometric part of the data, namely the numbers

$\{c_i\}$ and the categories $\{\mathsf{Perf}_i\}$. These data of course fix the regular type (in the sense of Theorem 2.35) of the **nc**-Hodge structure but we are still missing the gluing data. Here we propose a construction of the Hodge structure on the periodic cyclic homology of $C_{\mathbf{nc}(Y,\mathsf{w})}$ but similarly to the A-model we have to rely on the actual geometry of (Y,w) in order to produce the gluing data. At present it is not clear if the gluing data can be reconstructed from the category $C_{\mathbf{nc}(Y,\mathsf{w})}$ or more generally from its one-parameter deformation.

First we discuss the appropriate cohomologies of the Landau-Ginzburg model. Let

$$\begin{aligned}\mathscr{H}^{\bullet}_{\text{for}} &:= H^{\bullet}_{\text{DR}}((Y,\mathsf{w});\mathbb{C})\\ &= \mathbb{H}^{\bullet \bmod 2}_{\text{Zar}}\left(Y, \left(\Omega^{\bullet}_Y[[u]], ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right)\end{aligned}$$

be the $\mathbb{Z}/2$-graded $\mathbb{C}[[u]]$-module of algebraic de Rham cohomology of the potential w. In the case when $\text{crit}(\mathsf{w})$ is compact, the $\mathbb{C}[[u]]$-module $\mathscr{H}^{\bullet}_{\text{for}}$ is known to be free by the work of Barannikov and the second author (unpublished), Sabbah [**Sab99**], or Ogus-Vologodsky [**OV05**]. This implies the following

LEMMA 3.6. *Assume that Y is quasi-projective and the critical locus of w is compact. Then we have:*

(i) *The fiber of $\mathscr{H}^{\bullet}_{\text{for}}$ at $u = 0$ is the algebraic Dolbeault cohomology*

$$\mathbb{H}^{\bullet}_{Zar}\left(Y, \left(\Omega^{\bullet}_Y, d\mathsf{w}\wedge\right)\right) \cong \mathbb{H}^{\bullet}_{an}\left(Y, \left(\Omega^{\bullet}_Y, d\mathsf{w}\wedge\right)\right)$$

of the potential w.

(ii) *There is a canonical isomorphism*

$$\mathbb{H}^{\bullet}_{Zar}\left(Y, \left(\Omega^{\bullet}_Y[[u]], ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right) \cong \mathbb{H}^{\bullet}_{an}\left(Y, \left(\Omega^{\bullet}_Y[[u]], ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right)$$

(iii) *If the map w is proper then $\mathscr{H}^{\bullet}_{\text{for}}$ is the formal germ at $u = 0$ of an algebraic vector bundle on the affine line*

$$\mathscr{H}^{\bullet}_{\text{alg}} := \mathbb{H}^{\bullet \, mod\, 2}_{Zar}\left(Y, \left(\Omega^{\bullet}_Y[u], ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right)$$

Proof. The cohomology sheaves of the complex $(\Omega^{\bullet}_Y, d\mathsf{w}\wedge)$ are supported on the critical locus of w and so, by our compactness assumption, must be coherent sheaves on Y both in the analytic and in the Zariski topology. The hypercohomology spectral sequence then implies that the hypercohomology of the complex $(\Omega^{\bullet}_Y, d\mathsf{w}\wedge)$ is finite dimensional and the spectral sequence associated with the filtration induced by multiplication by u implies that $\mathbb{H}^{\bullet}_{\text{Zar/an}}\left(Y, \left(\Omega^{\bullet}_Y[[u]], ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right)$ is a finite rank $\mathbb{C}[[u]]$-module. Furthermore, the same spectral sequence implies that

$$\dim_{\mathbb{C}((u))} \mathbb{H}^{\bullet}_{\text{Zar/an}}\left(Y, \left(\Omega^{\bullet}_Y((u)), ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right) \leq \dim_{\mathbb{C}} \mathbb{H}^{\bullet}_{\text{Zar/an}}\left(Y, \left(\Omega^{\bullet}_Y, d\mathsf{w}\wedge\right)\right).$$

The freeness statement of Barannikov and the second author (see e.g. [**Sab99**]) now gives that these two dimensions are equal and so $\mathbb{H}^{\bullet}_{\text{Zar}}\left(Y, \left(\Omega^{\bullet}_Y[[u]], ud_{\text{DR}} + d\mathsf{w}\wedge\right)\right)$ is a free finite rank module over $\mathbb{C}[[u]]$. This proves part **(i)** of the lemma.

For part **(ii)** we only need to notice that the two spaces in question are computed by spectral sequences associated with the filtrations by the powers of u and that these spectral sequences have E_2-levels whose entries are finite sums of copies of $\mathbb{H}^{\bullet}_{\text{Zar}}\left(Y, \left(\Omega^{\bullet}_Y, d\mathsf{w}\wedge\right)\right)$ and $\mathbb{H}^{\bullet}_{\text{an}}\left(Y, \left(\Omega^{\bullet}_Y, d\mathsf{w}\wedge\right)\right)$, respectively. Each of these can in turn be computed from the hypercohomology spectral sequence for the complex $(\Omega^{\bullet}_Y, d\mathsf{w}\wedge)$ of (Zariski or analytic) coherent sheaves. But the cohomology sheaves of this complex are supported on the zero locus of $d\mathsf{w}$ which by assumption is projective. Hence by GAGA the Zariski and analytic cohomologies of this complex are naturally isomorphic. This gives isomorphisms of the hypercohomology and filtration spectral sequences in the Zariski and the analytic setup, respectively, and so the two types of hypercohomologies are isomorphic.

Finally, part **(iii)** was also proven by Barannikov and the second author, and by Sabbah [**Sab99**]. □

REMARK 3.7. The isomorphism in part **(ii)** of the previous lemma is not convergent for $u \to 0$ in general. Indeed if $u \neq 0$ is a complex number, then the complex vector space $\mathbb{H}^{\bullet}_{\mathrm{an}}\left(Y, (\Omega^{\bullet}_Y, ud_{\mathrm{DR}} + d\mathsf{w}\wedge)\right)$ is the same as the usual de Rham cohomology $H^{\bullet}_{\mathrm{DR}}(Y, \mathbb{C})$ of Y. Indeed, for such a fixed $u \neq 0$, the complex $(\Omega^{\bullet}_Y, ud_{\mathrm{DR}} + d\mathsf{w}\wedge) \cong \left(\Omega^{\bullet}_Y, d_{\mathrm{DR}} + u^{-1}d\mathsf{w}\wedge\right)$ is the holomorphic de Rham complex of the local system $(\mathcal{O}_Y, d_{\mathrm{DR}} + u^{-1}d\mathsf{w})$. But the multiplication by $\exp(-u^{-1}\mathsf{w})$ is an analytic automorphism of the line bundle $\mathcal{O}_Y$ which gauge transforms the connection $d_{\mathrm{DR}} + u^{-1}d\mathsf{w}$ into the trivial connection d_{DR}. Hence $\exp(-u^{-1}\mathsf{w})$ identifies $(\Omega^{\bullet}_Y, ud_{\mathrm{DR}} + d\mathsf{w}\wedge)$ with the holomorphic de Rham complex $(\Omega^{\bullet}_Y, d_{\mathrm{DR}})$ and $\mathbb{H}^{\bullet}_{\mathrm{an}}\left(Y, (\Omega^{\bullet}_Y, ud_{\mathrm{DR}} + d\mathsf{w}\wedge)\right)$ with $H^{\bullet}_{\mathrm{DR}}(Y, \mathbb{C})$. On the other hand, the space $\mathbb{H}^{\bullet}_{\mathrm{Zar}}\left(Y, (\Omega^{\bullet}_Y, ud_{\mathrm{DR}} + d\mathsf{w}\wedge)\right)$ depends on the potential in an essential way. For instance, if $\mathsf{w} : Y \to \mathbb{A}^1$ is a Lefschetz fibration, then the complex $(\Omega^{\bullet}_Y, d\mathsf{w}\wedge)$ is just the Koszul complex associated with the regular section $d\mathsf{w} \in \Omega^1_Y$. In particular the space $\mathbb{H}^{\bullet}_{\mathrm{Zar}}(Y, (\Omega^{\bullet}_Y, d_{\mathrm{DR}} + d\mathsf{w}\wedge)) \cong \mathbb{H}^{\bullet}_{\mathrm{Zar}}(Y, (\Omega^{\bullet}_Y, d\mathsf{w}\wedge))$ has dimension equal to the number of critical points of w. More generally $\mathbb{H}^{\bullet}_{\mathrm{Zar}}(Y, (\Omega^{\bullet}_Y, d_{\mathrm{DR}} + d\mathsf{w}\wedge))$ can be identified (see e.g. [**Kap91**]) with the cohomology of the perverse sheaf of vanishing cycles of w.

REMARK 3.8. Under our assumptions, the algebraic de Rham and Dolbeault cohomologies $H^{\bullet}_{\mathrm{DR}}((Y, \mathsf{w}); \mathbb{C})$ and $H^{\bullet}_{\mathrm{Dol}}((Y, \mathsf{w}); \mathbb{C})$ of the potential w can be identified respectively with the periodic cyclic and Hochschild homologies $HP_{\bullet}(C_{\mathsf{nc}(Y,\mathsf{w})})$ and $HH_{\bullet}(C_{\mathsf{nc}(Y,\mathsf{w})})$ of the **nc**-space $C_{\mathsf{nc}(Y,\mathsf{w})}$ (more precisely, of the collection of categories Perf_i labeled by numbers $\{\boldsymbol{c}_i\}$). This can be done, e.g. by choosing strong generators $\mathcal{E}_i$ of Perf_i, and then identifying $HP_{\bullet}(C_{\mathsf{nc}(Y,\mathsf{w})})$ and $HH_{\bullet}(C_{\mathsf{nc}(Y,\mathsf{w})})$ with the periodic cyclic and Hochschild homologies of the curved d($\mathbb{Z}/2$)g algebra, which consists of the d($\mathbb{Z}/2$)g algebra $R\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ and a central curvature given by w. A detailed proof of the comparison theorem giving the identifications $H^{\bullet}_{\mathrm{DR}}((Y, \mathsf{w}); \mathbb{C}) \cong HP_{\bullet}(C_{\mathsf{nc}(Y,\mathsf{w})})$ and $H^{\bullet}_{\mathrm{Dol}}((Y, \mathsf{w}); \mathbb{C}) \cong HH_{\bullet}(C_{\mathsf{nc}(Y,\mathsf{w})})$ can be found in the recent work of Junwu Tu [**Tu08**].

We will construct an **nc**-Hodge structure on $H^{\bullet}_{\mathrm{DR}}((Y, \mathsf{w}); \mathbb{C})$ by using the dual description of **nc**-Hodge structures given in Theorem 2.35. Here we will assume that we choose an open subset (in the analytic topology) $Y' \subset Y$ such that

- $\mathrm{crit}(\mathsf{w}) \subset Y'$,
- $\mathsf{w}(Y')$ is an open disc in $\mathbb{C}$,
- the closure $\overline{Y}'$ of Y' is a manifold with corners,
- the restriction of w to the part of the boundary of $\overline{Y}'$ lying over $\mathsf{w}(Y')$ is a smooth fibration.

In the case when w is already proper one can choose Y' to be the pre-image under w of an open disc in $\mathbb{C}$ containing all the critical values $\boldsymbol{c}_i$.

Label the critical values of w: $S = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_n\}$, and let $\boldsymbol{c}_0 \in \mathsf{w}(Y') - S$. Choose a system of paths $\{a_i\}_{i=1}^n$ and discs $\{\boldsymbol{D}\}_{i=1}^n$ as in the proof of Theorem 2.35. Choose $\boldsymbol{c}_0$-based loops $\gamma_1, \dots, \gamma_n$, so that γ_i goes once around c_i in the counterclockwise direction, all γ_i intersect only at $\boldsymbol{c}_0$, and each γ_i encloses the path a_i and the disc $\boldsymbol{D}_i$ (see Figure 4). Let Γ_i denote the closed region in $\mathbb{C}$ enclosed by γ_i. Adjusting if necessary the choice of the γ_i we can ensure also that each Γ_i is convex. From now on we will always assume that this is the case.

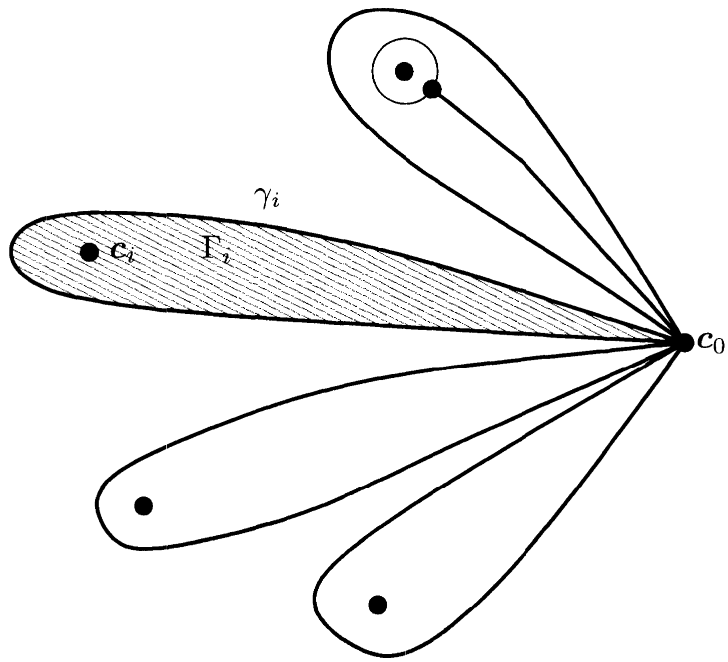


FIGURE 4. A system of thickened loops for $S \subset \mathbb{C}$.

For $i = 1, \dots, n$ set $Y_i := \mathsf{w}^{-1}(\Gamma_i) \cap Y'$ and consider the $\mathbb{Q}$-vector spaces of relative cohomology

$$U_i := H^\bullet(Y_i, Y_{\boldsymbol{c}_0}; \mathbb{Q}),$$

and

$$\begin{aligned} U &:= \oplus_{i=1}^n U_i \\ &= H^\bullet(\mathsf{w}^{-1}\left(\cup_{i=1}^n \Gamma_i\right), Y_{\boldsymbol{c}_0}; \mathbb{Q}) \\ &= H^\bullet(Y, Y_{\boldsymbol{c}_0}; \mathbb{Q}). \end{aligned}$$

Let $T_i : U \to U$ be the monodromy along γ_i. By definition T_i satisfies

$$(T_i - 1)_{|\oplus_{j \neq i} U_j} = 0$$

and so we get operators $T_{ji} : U_i \to U_j$, such that $T_{i|U_i} = \sum_{j=1}^n T_{ji}$. By construction the operator T_{ii} is the monodromy along γ_i of the local system on Γ_i of local relative cohomology, i.e. the local system of $\mathbb{Q}$-vector spaces whose fiber at $\boldsymbol{c} \in \Gamma_i$ is $H^\bullet(Y_i, Y_{\boldsymbol{c}}; \mathbb{Q})$. Hence T_{ii} is an isomorphism, and so the data $(S, \{U_i\}_{i=1}^n, \{T_{ij}\})$ are **nc**-Betti data of type **ncB(iii)**.

REMARK 3.9. (a) By Lemma 2.30 the data $(S, \{U_i\}_{i=1}^n, \{T_{ij}\})$ are the same thing as a constructible sheaf $\mathscr{F}$ of $\mathbb{Q}$-vector spaces on $\mathbb{C}$, satisfying $R\Gamma(\mathbb{C}, \mathscr{F}) = 0$.

The sheaf $\mathscr{F}$ can be described directly in terms of the geometry of (Y, w): for a $\boldsymbol{c} \in \mathbb{C}$ the stalk $\mathscr{F}_{\boldsymbol{c}}$ of $\mathscr{F}$ at $\boldsymbol{c}$ is the relative cohomology $H^{\bullet}(Y, Y_{\boldsymbol{c}}; \mathbb{Q})$.

(b) The geometric construction of $\mathscr{F}$ makes sense for every cohomology theory K. Indeed for every such K we can form a constructible sheaf of abelian groups ${}^{\mathsf{K}}\mathscr{F}$ whose stalk at $\boldsymbol{c} \in \mathbb{C}$ is $\mathsf{K}(Y, Y_{\boldsymbol{c}})$ and which again satisfies $R\Gamma(\mathbb{C}, {}^{\mathsf{K}}\mathscr{F}) = 0$. The vanishing of cohomology here is not obvious but can be proven as follows. Given a disk $\boldsymbol{D} \subset \mathsf{w}(Y') \subset \mathbb{C}$ such that $\partial \boldsymbol{D} \cap S = \varnothing$, and given any point $\boldsymbol{c} \in \partial \boldsymbol{D}$, consider the abelian group $A(\boldsymbol{D}, \boldsymbol{c}) := \mathsf{K}(\mathsf{w}^{-1}(\boldsymbol{D}), Y_{\boldsymbol{c}})$. The collection of abelian groups $A(\boldsymbol{D}, \boldsymbol{c})$ satisfies:

- $A(\boldsymbol{D}, \boldsymbol{c})$ are locally constant under small perturbations of $(\boldsymbol{D}, \boldsymbol{c})$, and
- for every decomposition $(\boldsymbol{D}, \boldsymbol{c}) = (\boldsymbol{D}_1, \boldsymbol{c}) \cup (\boldsymbol{D}_2, \boldsymbol{c})$ of $\boldsymbol{D}$ obtained by cutting D along a chord starting at c, we have $A(\boldsymbol{D}, \boldsymbol{c}) = A(\boldsymbol{D}_1, \boldsymbol{c}) \oplus A(\boldsymbol{D}_2, \boldsymbol{c})$.

This immediately gives us an equivalent description of ${}^{\mathsf{K}}\mathscr{F}$ via data of type **ncB(iii)**, which in turn yields the vanishing of cohomology of ${}^{\mathsf{K}}\mathscr{F}$.

Next, in order to complete the data **ncB(iii)** to a full-fledged **nc**-Hodge structure of exponential type, we need to construct:

- a collection $\{(\mathscr{R}_i, \nabla_i)\}_{i=1}^{m}$ of holomorphic bundles $\mathscr{R}_i$ over $\mathbb{C}\{u\}$ equipped with meromorphic connections ∇_i with at most second order poles and regular singularities, and
- for each $i = 1, \ldots, m$, an isomorphism $\mathfrak{f}_i$ between the local system on $\boldsymbol{S}^1$ induced from $(\mathscr{R}_i, \nabla_i)$ and the local system on $\boldsymbol{S}^1$ corresponding to the vector space $U_i \otimes \mathbb{C}$ and the monodromy operator T_{ii}.

As explained above the local system on the circle corresponding to the vector space $U_i \otimes \mathbb{C}$ and the monodromy operator T_{ii} can be described geometrically as the sheaf of complex vector spaces on the loop γ_i, whose stalk at $\boldsymbol{c} \in \gamma_i$ is $H^{\bullet}((Y_i, Y_{\boldsymbol{c}}); \mathbb{C})$. We will exploit this geometric picture to produce $(\mathscr{R}_i, \nabla_i)$ and the isomorphism $\mathfrak{f}_i$. The most convenient way to define the ∇_i is by using a Betti-to-de Rham cohomology isomorphism given by oscillating integrals.

Fix $i \in \{1, \ldots, m\}$ and let $Z := Y_i$, $\boldsymbol{\Delta} := \Gamma_i - \boldsymbol{c}_i \subset \mathbb{C}$, $\boldsymbol{f} := \mathsf{w} - \boldsymbol{c}_i$. By construction we have:

Z is a C^{∞}-manifold with boundary which is the closure of an open (in the classical topology) subset in the quasiprojective complex manifold Y.

$\boldsymbol{\Delta} \subset \mathbb{C}$ is a closed disc containing zero.

$\boldsymbol{f} : Z \to \boldsymbol{\Delta}$ is an analytic surjective map whose only critical value is zero and whose critical locus $\operatorname{crit}(\boldsymbol{f}) \subset Z$ is compact.

Consider now the $\mathbb{Z}/2$-graded $\mathbb{C}[[u]]$-module $H^{\bullet}_{\mathrm{DR}}((Z, \boldsymbol{f}); \mathbb{C})$ of de Rham cohomology of $(Z, \boldsymbol{f})$. By Lemma 3.6 we know that $H^{\bullet}_{\mathrm{DR}}((Z, \boldsymbol{f}); \mathbb{C})$ is a free $\mathbb{C}[[u]]$-module which can be computed as the cohomology of the complex $(\mathcal{A}^{\bullet}(Z)[[u]], d_{\mathrm{tot}})$, where $\mathcal{A}^{\bullet}(Z)[[u]]$ are the global C^{∞} complex-valued differential forms on Z, and

$d_{\text{tot}} := \bar{\partial} + u\partial + d\boldsymbol{f}\wedge$. The $\mathbb{C}[[u]]$-module $H^\bullet_{\text{DR}}((Z,\boldsymbol{f});\mathbb{C})$ carries a natural meromorphic connection ∇ differentiating in the u-direction and having a second order pole at $u=0$. This connection is induced from a connection $\boldsymbol{\nabla}$ on the $\mathbb{C}[[u]]$-module $\mathcal{A}^\bullet(Z)[[u]]$ which also has a second order pole and is defined by the formula

$$\boldsymbol{\nabla}_{u^2\frac{d}{du}} := u^2\tfrac{d}{du} - \boldsymbol{f}\cdot(\bullet) + u\mathsf{Gr} : \mathcal{A}^\bullet(Z)[[u]] \longrightarrow \mathcal{A}^\bullet(Z)[[u]],$$

where

$$\mathsf{Gr}_{|\mathcal{A}^{p,q}(Z)[[u]]} := \frac{q-p}{2}\cdot \mathrm{id}_{\mathcal{A}^{p,q}(Z)[[u]]}$$

is the grading operator coming from $\mathbf{nc}$-geometry (compare with 2.1.7).

With this definition we have

LEMMA 3.10. *The operator* $\boldsymbol{\nabla}_{u^2\frac{d}{du}}$ *satisfies:*

(a) $\left[\boldsymbol{\nabla}_{u^2\frac{d}{du}}, d_{\text{tot}}\right] = \frac{u}{2}\cdot d_{\text{tot}}$.

(b) $\boldsymbol{\nabla}_{u^2\frac{d}{du}}$ *preserves* $\ker(d_{\text{tot}})$ *and* $\operatorname{im}(d_{\text{tot}})$ *and so induces a meromorphic connection* ∇ *with a second order pole on the* $\mathbb{C}[[u]]$*-module* $H^\bullet_{\text{DR}}((Z,\boldsymbol{f});\mathbb{C})$.

Proof. We compute

$$\begin{aligned}\left[\boldsymbol{\nabla}_{u^2\frac{d}{du}}, d_{\text{tot}}\right] &= \left[u^2\frac{d}{du} - \boldsymbol{f} + u\mathsf{Gr}, \bar{\partial} + u\partial + d\boldsymbol{f}\wedge\right] \\ &= \left[u^2\frac{d}{du}, u\partial\right] - [\boldsymbol{f}, u\partial] + \left[u\mathsf{Gr}, \bar{\partial} + u\partial + d\boldsymbol{f}\wedge\right] \\ &= u^2\partial + ud\boldsymbol{f}\wedge + \frac{u\bar{\partial}}{2} - \frac{ud\boldsymbol{f}\wedge}{2} - \frac{u^2\partial}{2} \\ &= \frac{u}{2}\cdot d_{\text{tot}}.\end{aligned}$$

Part **(b)** follows immediately from **(a)** ☐

Suppose now that $\alpha = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + \cdots \in \mathcal{A}^\bullet(Z)[[u]]$, $\alpha_i = \sum \alpha_i^{p,q}$, $\alpha_i^{p,q} \in \mathcal{A}^{p,q}(Z)$ is a d_{tot}-cocycle. Then the differential $d + u^{-1}d\boldsymbol{f}\wedge = \bar{\partial} + \partial + u^{-1}d\boldsymbol{f}\wedge = u^{-1/2}u^{\mathsf{Gr}}d_{\text{tot}}u^{-\mathsf{Gr}}$ will kill the element

$$u^{\mathsf{Gr}}\alpha := \sum_{\substack{i\geq 0\\ 0\leq p,q\leq \dim Z}} \alpha_i^{p,q}u^{i+\frac{q-p}{2}} \in \mathcal{A}^\bullet(Z)((u^{1/2})).$$

Therefore the expression $e^{\frac{\boldsymbol{f}}{u}}u^{\mathsf{Gr}}\alpha$ satisfies formally

$$d\left(e^{\frac{\boldsymbol{f}}{u}}u^{\mathsf{Gr}}\alpha\right) = 0,$$

i.e. is d-closed. Moreover, the action of the operator $\boldsymbol{\nabla}_{u^2\frac{d}{du}}$ on α translates to the action of $u^2\frac{d}{du}$ on the above expression modulo formally exact forms.

Consider now a closed connected arc $\delta \subset \partial\boldsymbol{\Delta} = \gamma_i$ and let $\mathrm{Sec}(\delta) \subset \boldsymbol{\Delta}$ be the corresponding open sector (see Figure 5) with vertex at $0 \in \boldsymbol{\Delta}$, and boundary made out of the arc δ and the segments connecting 0 with the end points of δ. Note that

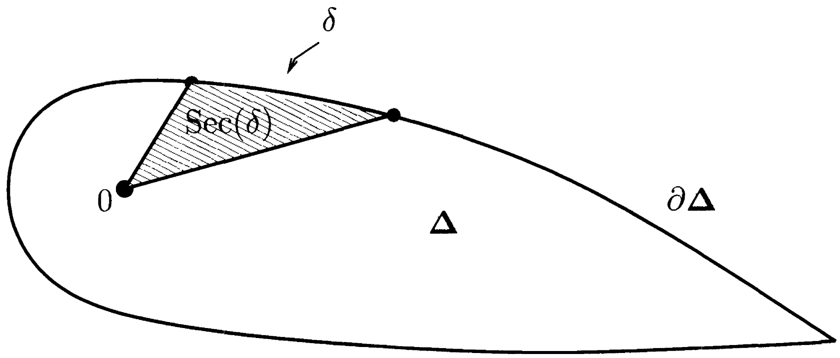


FIGURE 5. A sector in $\boldsymbol{\Delta}$.

the convexity of $\boldsymbol{\Delta}$ assures that $\mathrm{Sec}(\delta) \subset \boldsymbol{\Delta}$. Denote by $\mathrm{Sec}(\delta)^{\vee} \subset \mathbb{C}$ the dual angle sector consisting of $u \in \mathbb{C}$ such that $\mathrm{Re}(w/u) < 0$ for all $w \in \mathrm{Sec}(\delta)$.

Clearly, for each class in the relative integral homology $H_{\bullet}(Z, \boldsymbol{f}^{-1}(\delta); \mathbb{Z})$ we can choose a relative chain $\mathfrak{c}$ representing it, so that $\mathfrak{c}$ satisfies:

(†)
- $\mathfrak{c}$ is piece-wise real analytic;
- $\boldsymbol{f}(\mathrm{supp}(\mathfrak{c})) \subset \mathrm{Sec}(\delta)$;
- $\boldsymbol{f}(\mathrm{supp}(\partial\mathfrak{c})) \subset \delta$.

For every such relative chain $\mathfrak{c}$ we now have:

LEMMA 3.11. *For every d_{tot}-closed formal power series of forms $\alpha \in \mathcal{A}^{\bullet}_{Z}(Z)[[u]]$ and every relative chain $\mathfrak{c} \in C_{\bullet}(Z, \boldsymbol{f}^{-1}(\delta); \mathbb{Z})$ satisfying (†) the oscillating integral*

$$\int_{\mathfrak{c}} e^{\frac{\boldsymbol{f}}{u}} u^{\mathsf{Gr}} \alpha$$

is well defined as an asymptotic series in $u^{\mathbb{Q}}(\log u)^{\mathbb{N}}$ in the sector $\mathrm{Sec}(\delta)^{\vee}$.

Proof. Let $N \geq 0$ be a non-negative integer. Clearly the expression

$$e^{\boldsymbol{f}/u} u^{\mathsf{Gr}} \left(\sum_{0 \leq i \leq N} \alpha_i u^i \right)$$

is a well defined analytic function on $Z \times \mathrm{Sec}(\delta)^{\vee}$. Using the fact that $(d + u^{-1} d\boldsymbol{f} \wedge) u^{\mathsf{Gr}} \alpha = 0$ and the Malgrange-Sibuya theory of asymptotic sectorial solutions to analytic differential equations, we get that

$$\int_{\mathfrak{c}} e^{\boldsymbol{f}/u} u^{\mathsf{Gr}} \left(\sum_{0 \leq i \leq N} \alpha_i u^i \right) \simeq \sum_{j \in \mathbb{Q}, k \in \mathbb{N}} c_{j,k} u^j (\log(u))^k \tag{3.2.1}$$

is asymptotic to a series in $u^{\mathbb{Q}}(\log u)^{\mathbb{N}}$ in which the logarithms enter with bounded powers. Thus the limit of (3.2.1) as $N \to \infty$ is asymptotic to a series in $u^{\mathbb{Q}}(\log u)^{\mathbb{N}}$ on $\mathrm{Sec}(\delta)^{\vee}$. □

The previous lemma shows that the $\mathbb{C}[[u]]$-module with connection $(H^{\bullet}_{\mathrm{DR}}((Z, \boldsymbol{f}); \mathbb{C}), \nabla)$ is formally isomorphic to a meromorphic local system of the form $\boldsymbol{\mathcal{E}}^{\boldsymbol{f}/u} \otimes (\mathscr{R}_i, \nabla_i)$, where $\mathscr{R}_i$ is a free $\mathbb{C}[[u]]$-module, and ∇_i has regular singularities. Furthermore the lemma shows that the oscillating integrals above identify the local system on γ_i given by $(\boldsymbol{c} \in \gamma_i) \mapsto H^\bullet((Y_i, Y_{\boldsymbol{c}}), \mathbb{Q})$ with a rational structure on $(\mathscr{R}_i \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u)), \nabla_i)$. In particular the data $\{(\mathscr{R}_i, \nabla_i)\}_{i=1}^m$ and $(S, \{U_i\}, \{T_{ij}\})$ constitute the regular type and gluing data (in the sense of Theorem 2.35) of an **nc**-Hodge structure of exponential type.

Usually if one tries to make a Landau-Ginzburg model with proper map w from non-proper examples above, one gets new parasitic critical points. Choosing an appropriate domain $Y' \subset Y$ one can define the gluing data for the relevant critical points.

3.3. Mirror symmetry examples. Finally, in order to give a general idea of the mirror correspondence, we briefly discuss three examples of Landau-Ginzburg models mirror dual to symplectic manifolds of positive, vanishing, and negative anti-canonical class respectively.

- For $X = \mathbb{CP}^n$ one of the possible mirror dual Landau-Ginzburg models is given by $Y = (\mathbb{C}^\times)^n$ endowed with potential
$$\mathsf{w}(z_1, \dots, z_n) = z_1 + \cdots + z_n + \frac{q}{z_1 \cdots z_n}$$
where $q \in \mathbb{C}^\times$ is a parameter. In this model the map w is not proper. This can be repaired by compactifying the fibers of w to $(n-1)$-dimensional projective Calabi-Yau varieties. The compactification is not unique, it depends on combinatorial data, but the compactified space has the same critical points as Y. In general, for symplectic manifolds (X, ω) with ω representing the anticanonical class, one can combine equations for the connection in the q and u directions and get a beautiful variation of Hodge structures with strong arithmetic properties as predicted by our considerations in Section 3.1 (see also Golyshev's work [**Gol01, Gol07**]).
- For a smooth projective Calabi-Yau variety X one can take for Y the product $(X^\vee \times \mathbb{A}^{2N}, \mathsf{w})$ where $X^\vee$ is a Calabi-Yau variety mirror dual to X, $N \geq 1$ is an arbitrary integer and w is the pullback from $\mathbb{A}^{2N}$ of a non-degenerate quadratic form. In general, the complex dimension of the Landau-Ginzburg model is equal to half of the real dimension of X modulo 2.
- For X being a complex curve of genus $g \geq 2$ (considered as a symplectic manifold), the first author proposed several years ago a mirror Landau-Ginzburg model (Y, w) which is a complex algebraic 3-dimensional manifold with non-vanishing algebraic volume element, such that locally (in the analytic topology) near each point the pair (Y, w) is isomorphic to
$$\mathsf{w} : \mathbb{C}^3 \to \mathbb{C}, \qquad (x, y, z) \mapsto xyz$$
The set of critical points of w is the union of $3g - 3$ copies of $\mathbb{CP}^1$ glued along points $0, \infty$ meeting 3 curves at a point. The graph obtained by contracting each copy of $\mathbb{C}^\times$ to an edge is a connected 3-valent graph with g loops, representing a maximal degeneration point in the Deligne-Mumford moduli stack of stable genus g curves.

4. Generalized Tian-Todorov theorems and canonical coordinates

In this section we will examine more closely the other direction of the mirror symmetry correspondence, i.e. the situation in which symplectic Landau-Ginzburg models appear as mirrors of complex manifolds with a fixed anti-canonical section. In order to understand the Hodge theoretic implications of this process we first revisit a classical concept in the subject: the notion of canonical coordinates.

4.1. Canonical coordinates for Calabi-Yau variations of nc-Hodge structures.

4.1.1. ***Variations over supermanifolds.*** We begin with a reformulation of the definition of variations of **nc**-Hodge structures (Definition 2.7) to allow for bases that are supermanifolds:

DEFINITION 4.1. *For a complex analytic supermanifold S, a **variation of nc-Hodge structures over** S (respectively a **variation of nc-Hodge structures over** S **of exponential type**) is a triple $(H, \mathscr{E}_B, \mathsf{iso})$, where*

- *H is a holomorphic $\mathbb{Z}/2$-graded vector bundle on $\mathbb{A}^1 \times S$ which is algebraic in the $\mathbb{A}^1$-direction;*
- *$\mathscr{E}_B$ is a local system of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vector spaces on $(\mathbb{A}^1 - \{0\}) \times S$;*
- ***iso** is an analytic isomorphism of holomorphic vector bundles*

$$\mathsf{iso} : \mathscr{E}_B \otimes \mathcal{O}_{(\mathbb{A}^1-\{0\})\times S} \xrightarrow{\cong} H_{|(\mathbb{A}^1-\{0\})\times S};$$

so that:

- ◊ *the induced meromorphic connection ∇ on $H_{|(\mathbb{A}^1-\{0\})\times S}$ satisfies: locally on S, for every section ξ of T_S, the operators $\nabla_{u^2\frac{\partial}{\partial u}}$, $\nabla_{u\xi}$ extend to operators on $\mathbb{A}^1 \times S$, and*
- ◊ *the triple $(H, \mathscr{E}_B, \mathsf{iso})$ satisfies the* **($\mathbb{Q}$-structure axiom)** *and the* **(Opposedness axiom)** *(respectively (H, ∇) is of exponential type and $(H, \mathscr{E}_B$,* **iso**) *satisfies the* **($\mathbb{Q}$-structure axiom)$^{\mathsf{exp}}$** *and the* **(Opposedness axiom)$^{\mathsf{exp}}$**).

REMARK 4.2. From now on we will suppress the $\mathbb{Q}$-structure and the opposedness axioms since they will not play any special role in our analysis. At any given stage of the discussion they can be added without any harm or alteration to the arguments.

4.1.2. ***Calabi-Yau variations.*** Suppose now that $(H, \mathscr{E}_B, \mathsf{iso})$ is a variation of **nc**-Hodge structures over a supermanifold S. For any point $x \in S$ let $H_{0,x}$ denote the fiber of H at $(0, x) \in \mathbb{A}^1 \times S$. We get a canonical map

$$\mu_x : T_x S \to \mathrm{End}\,(H_{0,x}),$$

defined as follows: Extend the tangent vector $v \in T_x S$ to some analytic vector field ξ defined in a neighborhood of x. Consider the holomorphic first order differential operator $\nabla_{u\xi} : H \to H$. By construction this operator has symbol $(u\xi) \otimes \mathrm{id}_H$. In particular, the restriction of $\nabla_{u\xi}$ to the slice $\{0\} \times S \subset \mathbb{A}^1 \times S$ will have zero symbol, and so will be an $\mathcal{O}$-linear endomorphism of $H_{|\{0\}\times S}$. We define $\mu_x(v)$ to be the action of this $\mathcal{O}$-linear map on the fiber $H_{(0,x)}$. It is straightforward to check that this action is independent of the extension ξ and depends only on v.

DEFINITION 4.3. *Let S be a complex analytic supermanifold. We say that a variation $(H, \mathscr{E}_B, \mathsf{iso})$ of* **nc***-Hodge structures on S is of* ***Calabi-Yau type at a point*** *$x \in S$ if there exists an (even or odd) vector $h \in H_{(0,x)}$, so that the linear map*

$$\begin{aligned} T_x S &\longrightarrow H_{(0,x)} \\ v &\longmapsto (\mu_x(v))(h) \end{aligned}$$

is an isomorphism. Such a vector h will be called a ***generating vector for H at x****.*

It follows from the definition that if S is the base of a variation of **nc**-Hodge structures which is of Calabi-Yau type at a point $x \in S$, then the tangent space $T_x S$ is a unital commutative associative algebra acting on $H_{0,x}$ via the map μ_x and such that $H_{0,x}$ is a free module of rank one. The condition on a variation to have a Calabi-Yau type (even or odd) is an open condition on $x \in S$. Variations of **nc**-Hodge structures of Calabi-Yau type should arise naturally on the periodic cyclic homology of smooth and compact d($\mathbb{Z}/2$)g categories which are Calabi-Yau in the sense of [**KS06b**]. The basic geometric example of a Calabi-Yau variation is an extension of the setup we discussed in section 3.1:

EXAMPLE 4.4. Let (X, ω) be a compact symplectic manifold with $\dim_{\mathbb{R}} X = 2d$. Conjecturally there exists a non-empty open subset $S \subset H^{\bullet}(X, \mathbb{C})$ so that the big quantum product $*_x$ is absolutely convergent for all $x \in S$ (the product is given by a formula similar to one on page 78). The manifold S has a natural structure of a supermanifold, being an open subset in the affine superspace $H^{\bullet}(X, \mathbb{C})$. As in Section 3.1 we define a variation of **nc**-Hodge structures $(H, \mathscr{E}_B, \mathsf{iso})$ on S by taking H to be the trivial vector bundle on $\mathbb{A}^1 \times S$ with fiber $H^{\bullet}(X, \mathbb{C})$, and defining the connection ∇ on H by the formulas:

$$\nabla_{\frac{\partial}{\partial u}} := \frac{\partial}{\partial u} + u^{-2}\left(\kappa_X *_x \bullet\right) + u^{-1}\mathsf{Gr},$$

$$\nabla_{\frac{\partial}{\partial t^i}} := \frac{\partial}{\partial t^i} - q^{-1}u^{-1}\left(t_i *_x \bullet\right),$$

where the (t_i) form a basis on $H^{\bullet}(X, \mathbb{C})$, and (t^i) are the dual linear coordinates.

Clearly, if we restrict (H, ∇) to $S \cap H^2(X, \mathbb{C})$ we will get back the bundle with connection we defined in section 3.1. We now define the integral lattice $\mathscr{E}_B$ and isomorphism iso on S as the ∇-horizontal extensions of the integral lattice and isomorphism we had defined on $S \cap H^2(X, \mathbb{C})$. Finally, in order to match the framework of **nc**-geometry, we should change the parity of the bundle H in the case $d \equiv 1 \bmod 2$.

4.1.3. ***Decorated Calabi-Yau variations.*** The variations of **nc**-Hodge structures of Calabi-Yau type need to be decorated by a few additional pieces of data before we can extract canonical coordinates from them. To motivate our choice of such data we first recall the Deligne-Malgrange classification of logarithmic holomorphic extensions of regular connections.

Let S be a complex analytic supermanifold, let $\boldsymbol{D}$ be a one-dimensional complex disc, and let $\mathscr{E}$ be a complex local system on $(\boldsymbol{D} - \{\mathrm{pt}\}) \times S$ and let $(\mathscr{E}, \nabla)$ be the associated holomorphic bundle $E := \mathscr{E} \otimes \mathcal{O}_{(D-\{\mathrm{pt}\})\times S}$ on $(D - \{\mathrm{pt}\}) \times S$ with the induced flat connection ∇. Suppose that $\widetilde{E}$ is a holomorphic bundle on $\boldsymbol{D} \times S$ which extends E and on which ∇ has a logarithmic pole. The restriction $\widetilde{E}_{|\{\mathrm{pt}\}\times S}$ is a holomorphic bundle on S and ∇ induces: a holomorphic connection ${}^{\widetilde{E}}\nabla$ and an $\mathcal{O}_S$-linear residue endomorphism $\mathrm{Res}_{\widetilde{E}}(\nabla)$ on $\widetilde{E}_{|\{\mathrm{pt}\}\times S}$. Furthermore the integrability of ∇ on $(\boldsymbol{D}-\{\mathrm{pt}\})\times S$ implies that ${}^{\widetilde{E}}\nabla$ is also integrable and that the endomorphism $\mathrm{Res}_{\widetilde{E}}(\nabla)$ is covariantly constant with respect to ${}^{\widetilde{E}}\nabla$ [**Sab02**, Section 0.14b].

Recall next that by Deligne's extension theorem (see e.g. [**Del70**, Chapter II.5] or [**Sab02**, Corollary II.2.21]) meromorphic bundles with connections with regular singularities always admit functorial holomorphic extensions across the pole divisor. Deligne's extension procedure is not unique and depends on the choice of a set-theoretic section of the quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. We fix $\mathcal{V}$ to be the unique Deligne extension of E for which ∇ has a logarithmic pole at $\{\mathrm{pt}\} \times S$ and a residue with eigenvalues whose real parts are in the interval $(-1, 0]$. Now the classification theorem of Deligne-Malgrange [**Sab02**, Theorem III.1.1] asserts that there is a natural equivalence of categories

$$\begin{pmatrix}\text{Holomorphic extensions of } E \text{ to} \\ \boldsymbol{D}\times S \text{ for which } \nabla \text{ has a logarith-} \\ \text{mic singularity along } \{\mathrm{pt}\}\times S\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{Decreasing filtrations of } \mathscr{E} \text{ by } \mathbb{C}\text{-} \\ \text{local subsystems on } (\boldsymbol{D}-\{\mathrm{pt}\})\times \\ S\end{pmatrix}.$$

The equivalence depends on the chosen Deligne extension and is explicitly given as follows. Let t be a complex coordinate on $\boldsymbol{D}$ which vanishes at $\{\mathrm{pt}\} \in \boldsymbol{D}$. Consider the restriction $\mathcal{V}/t\mathcal{V}$ of $\mathcal{V}$ to $\{\mathrm{pt}\} \times S$. This is a holomorphic bundle on S equipped as above with the holomorphic connection ${}^{\mathcal{V}}\nabla$ and the covariantly constant residue endomorphism $\mathrm{Res}_{\mathcal{V}}(\nabla)$. Suppose now that $\widetilde{E}$ is another holomorphic bundle on $\boldsymbol{D} \times S$ which extends E and on which ∇ has a logarithmic pole. For any $k \in \mathbb{Z}$ we define a subbundle $(\mathcal{V}/t\mathcal{V})^k \subset \mathcal{V}/t\mathcal{V}$ by setting

$$(\mathcal{V}/t\mathcal{V})^k := \frac{\mathcal{V}\cap t^k\widetilde{E}}{t\mathcal{V}\cap t^k\widetilde{E}}$$

where $\mathcal{V}$ and $\widetilde{E}$ are viewed as subsheaves in the meromorphic bundle E.

By construction the subbundles $(\mathcal{V}/t\mathcal{V})^k$ are preserved both by ${}^{\mathcal{V}}\nabla$ and by the residue endomorphism $\mathrm{Res}_{\mathcal{V}}(\nabla)$ and so give rise to ∇-covariantly constant meromorphic subbundles of E, or equivalently to $\mathbb{C}$-local subsystems of $\mathscr{E}$.

Alternatively we can use a more intrinsic description of holomorphic extensions of $(\mathscr{E}, \nabla)$ which is beter adapted to our examples and in particular to Example 4.8. Namely, instead of relying on the Deligne extension and the induced filtration we can use decreasing filtrations $\mathscr{E}_{\leq\lambda}$ of $\mathscr{E}$ labeled by real numbers $\lambda \in \mathbb{R}$ and such that on the associated graded pieces the monodromy on $\boldsymbol{D} - \{\mathrm{pt}\}$ has eigenvalues in $\mathbb{R}_+ \times \exp(2\pi i\lambda)$.

We can now introduce the additional data that one needs for the canonical coordinates

Definition 4.5. *Let S be a complex supermanifold and let $(H, \mathscr{E}_B, \mathsf{iso})$ be a variation of* **nc**-*Hodge structures of Calabi-Yau type on S. A* ***decoration*** *on $(H, \mathscr{E}_B, \mathsf{iso})$ is a pair $(\widetilde{H}, \psi)$ where:*

$\widetilde{H}$ is an extension of H to a $(\mathbb{Z}/2)$-graded vector bundle on $\mathbb{P}^1 \times S$ so that ∇ has a regular singularity at $\{\infty\} \times S$.

ψ is a ${}^{\widetilde{H}}\nabla$-covariantly constant section of $\widetilde{H}_{\{\infty\}\times S}$.

A decoration is called ***rational*** *iff the $\mathbb{R}$-filtration on the local system $\mathscr{E}_B \otimes \mathbb{C}$ is compatible with the rational structure, and if the vector $\psi(x) \in \widetilde{H}_{\{\infty\}\times\{x\}} = \mathrm{gr}(\mathscr{E}_B \otimes \mathbb{C})_x$ is rational, i.e. if $\psi(x) \in \mathrm{gr}(\mathscr{E}_B)_x$.*

The previous discussion applied to the local system $\mathscr{E}_B \otimes \mathbb{C}$, the disc $\boldsymbol{D} = \{|u| > 1\} \cup \{\infty\}$ and the point $\mathrm{pt} = \infty$ shows that the data of a decoration are equivalent to the data $((\mathscr{E}_B \otimes \mathbb{C})_{\leq\bullet}, \psi)$, where $(\mathscr{E}_B \otimes \mathbb{C})_{\leq\bullet}$ is a decreasing filtration of $\mathscr{E}_B \otimes \mathbb{C}$ (labelled by real numbers) and ψ is a covariantly constant section (along S) of the corresponding logarithmic holomorphic extension of H. We will freely go back and forth between these two points of view.

Any decorated variation $(H, \mathscr{E}, \mathsf{iso}; \widetilde{H}, \psi)$ of nc-Hodge structures of Calabi-Yau type gives rise to a natural open domain $U \subset S$ defined by

$$U := \left\{ x \in S \;\middle|\; \begin{array}{l} \widetilde{H}_{\mathbb{P}^1\times\{x\}} \text{ is holomorphically trivial and if } s \in \Gamma\left(\mathbb{P}^1 \times \{x\}\right) \\ \text{is such that } s_x(\infty) = \psi(\infty, x), \text{ then } s_x(0) \text{ is a generating} \\ \text{vector for } (H, \mathscr{E}, \mathsf{iso}). \end{array} \right\}$$

Furthermore for every $x \in U$ we get a natural map $\mathsf{can}_x : T_xS \to \widetilde{H}_{\infty,x}$ defined as the composition

$$\begin{tikzcd} T_xS \arrow[r, "\mu_x(\bullet)(s_x(0))"] \arrow[rrr, bend right, "\mathsf{can}_x"'] & H_{0,x} \arrow[r, "\mathrm{ev}^{-1}_{(0,x)}"] & \Gamma\left(\widetilde{H}_{|\mathbb{P}^1\times\{x\}}\right) \arrow[r, "\mathrm{ev}_{(\infty,x)}"] & \widetilde{H}_{\infty,x}. \end{tikzcd}$$

where $\mathrm{ev}_{(t,x)} : \Gamma\left(\mathbb{P}^1, \widetilde{H}_{|\mathbb{P}^1\times\{x\}}\right) \to \widetilde{H}_{t,x}$ denotes the natural evaluation of sections, which is invertible by the triviality assumption on $\widetilde{H}_{|\mathbb{P}^1\times\{x\}}$.

The pullback of the flat connection ${}^{\widetilde{H}}\nabla$ by the map can induces a flat connection on $TS_{|U}$. The canonical coordinates on S come from the following easy claim whose proof we omit:

CLAIM 4.6. *The flat connection $\mathsf{can}^*\left({}^{\widetilde{H}}\nabla\right)$ on $TS_{|U}$ is torsion-free and so gives rise to a natural affine structure and affine coordinates on U. If the decoration is rational then the tangent bundle $TS_{|U}$ carries a natural rational structure.*

REMARK 4.7. **(i)** The canonical coordinates on U corresponding to a decorated nc-variation of Hodge structures are only affine coordinates and are defined only up to a translation.

(ii) For any $u \in \mathbb{A}^1 - \{0\}$ we can introduce another affine structure which is a **vector structure**. In fact, we get an analytic isomorphism between U and a domain in $H_{u,\bullet} = (\mathscr{E}_B)_{u,\bullet} \otimes \mathbb{C}$:

$$x \in U \mapsto \mathrm{ev}_{(u,x)}\, \mathrm{ev}^{-1}_{(\infty,x)}(\psi(x)) \in H_{(u,x)}.$$

One can use this to show that the local Torelli theorem holds for decorated Calabi-Yau variations of **nc**-Hodge structures.

EXAMPLE 4.8. The setup of Example 4.4 gives not only a variation of **nc**-Hodge structures but in fact gives a rationally decorated **nc**-Hodge structure of Calabi-Yau type. Indeed by definition the fibers of H are identified with $\mathbf{\Pi}^d H^{\bullet}(X, \mathbb{C})$. The monodromy of the connection around $\infty \in \mathbb{P}^1$ is the operator acting by $(-1)^{i+d} \exp(\kappa_X \wedge (\bullet))$ on $H^i(X, \mathbb{C})$. Consider the monodromy invariant filtration on $H^{\bullet}(X, \mathbb{C})$ whose component in degree $\frac{d-i}{2}$ is $H^{\geq i}(X, \mathbb{C})$. Let $\widetilde{H}$ be the corresponding logarithmic extension of H and let ψ be the section of $\widetilde{H}$ corresponding to the image of $1 \in H^0(X, \mathbb{C}) \subset H^{\bullet}(X, \mathbb{C})$. The bundle $\widetilde{H}_{|\{\infty\} \times S}$ is trivialized and $\nabla_{\frac{\partial}{\partial t^i}} = \frac{\partial}{\partial t^i}$ in this trivialization. This gives the desired decoration $(\widetilde{H}, \psi)$ and the associated canonical coordinates are the standard canonical coordinates in Gromov-Witten theory.

4.1.4. ***Generalized decorations.*** The notion of a decorated Calabi-Yau variation of **nc**-Hodge structures can be generalized in various ways. For instance, instead of specifying a covariantly constant filtration on H giving the extension $\widetilde{H}$ we can start with any holomorphic bundle H' defined on $\{u \in \mathbb{P}^1 | \, |u| \geq R\}$, and an identification of C^{∞}-bundles

$$p_1^* \left(H'_{|\{|u|=R\}} \right) \cong (\mathscr{E}_B \otimes \mathbb{C})_{\{|u|=R\} \times S},$$

where $p_1 : \{|u| = R\} \times S \to \{|u| = R\}$ is the projection on the first factor.

Furthermore (locally in S) the holomorphic bundle $p_1^* H'$ on $\{u \in \mathbb{P}^1 | \, |u| \geq R\} \times S$ carries a flat connection defined along S only. We can use the above identification to glue this together with H along $\{|u| = R\} \times S$ to get a bundle $\widetilde{H}$ on $\mathbb{P}^1 \times S$ equipped with a flat connection $\nabla_{/S}$ along S. This generalizes the first part of the decoration. For the second part we will take a $\nabla_{/S}$-covariantly constant section ψ of $\widetilde{H}_{|\{\infty\} \times S}$. Now the same definition of the set U and the canonical map can make sense in this context. The resulting connection on $TS_{|U}$ is again torsion-free.

4.1.5. ***Formal variations of Calabi-Yau type.*** The notion of a Calabi-Yau variation extends readily to the formal context. Suppose $S = \operatorname{Spf} \mathbb{C}[[x_1, \ldots, x_N, \xi_1, \ldots, \xi_M]]$ be a formal algebraic supermanifold, where x_i are even and ξ_j are odd formal variables. The de Rham part of a formal variation of **nc**-Hodge structures on S is a pair (H, ∇) where H is a $(\mathbb{Z}/2)$-graded algebraic vector bundle over $\mathbb{D} \times S$, where $\mathbb{D}$ is the one-dimensional formal disc $\mathbb{D} := \operatorname{Spf}(\mathbb{C}[[u]])$. Here ∇ is a meromorphic connection on H such that $\nabla_{u^2 \frac{\partial}{\partial u}}$, $\nabla_{u \frac{\partial}{\partial x_i}}$, $\nabla_{u \frac{\partial}{\partial \xi_j}}$ are regular differential operators on H.

We say that such a pair (H, ∇) has the Calabi-Yau property if we can find a vector $h \in H_{0,0}$, so that the natural linear map $T_0 S \to H_{0,0}$, $v \mapsto \mu_0(v)(h)$ is an isomorphism.

Finally a decoration of a formal Calabi-Yau de Rham data (H, ∇) is a pair $(\boldsymbol{e}, h)$, where $\boldsymbol{e}$ is a trivialization $\boldsymbol{e} : H_{|\mathbb{D} \times \{0\}} \to H_{0,0} \otimes \mathcal{O}_{\mathbb{D} \times \{0\}}$, and $h \in H_{0,0}$ is a generating vector for the Calabi-Yau property.

Again a decorated de Rham data of Calabi-Yau type gives an affine structure and canonical formal coordinates on S.

4.2. Algebraic framework: dg Batalin-Vilkovisky algebras. In this section we discuss the aspects of algebraic deformation theory relevant to the study of **nc**-Hodge structures. We will work over $\mathbb{C}$ but all algebraic considerations in this section make sense over any field of characteristic zero.

4.2.1. ***Preliminaries on L_∞ algebras.*** Our main objects of interest here will be differential $\mathbb{Z}/2$-graded algebras over $\mathbb{C}$ or more generally $\mathbb{Z}/2$-graded L_∞-algebras over $\mathbb{C}$. We begin with a definition:

DEFINITION 4.9. *A complex differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ (or a $\mathbb{Z}/2$-graded L_∞-algebra) is called* ***homotopy abelian*** *if it is L_∞ quasi-isomorphic to an abelian $d(\mathbb{Z}/2)g$ Lie algebra.*

REMARK 4.10. Homotopy abelian differential $\mathbb{Z}/2$-graded Lie algebras can be characterized in a variety of ways. In particular we have the following statements that follow readily from the definition:

- A differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is homotopy abelian if and only if all the higher operations m_n vanish on its L_∞ minimal model $\mathfrak{g}^{\min} = H^\bullet(\mathfrak{g}, d_{\mathfrak{g}})$, i.e. $m_n = 0$ for $n \geq 1$.
- A differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is homotopy abelian if and only if there exist d($\mathbb{Z}/2$)g Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and morphisms of d($\mathbb{Z}/2$)g Lie algebras:
$$\begin{tikzcd} & \mathfrak{g}_1 \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathfrak{g} & & \mathfrak{g}_2 \end{tikzcd}$$
so that $\mathfrak{g}_2$ is an abelian d($\mathbb{Z}/2$)g Lie algebra, and the morphisms $\mathfrak{g}_1 \to \mathfrak{g}$ and $\mathfrak{g}_1 \to \mathfrak{g}_2$ are quasi-isomorphisms.
- A differential $\mathbb{Z}/2$-graded Lie algebra $\mathfrak{g}$ is homotopy abelian if and only if the Lie algebra cohomology algebra $H^\bullet(\mathfrak{g}, \mathbb{C})$ is free, i.e. is isomorphic to the algebra of formal power series on some (possibly infinitely many) supervariables. Here the Lie algebra cohomology is defined as
$$H^\bullet(\mathfrak{g}, \mathbb{C}) := H^\bullet\left(\prod_{n\geq 0} \mathrm{Hom}_{(\mathbb{C}-\mathsf{Vect})}(\mathrm{Sym}^n \mathbf{\Pi}\mathfrak{g}, \mathbb{C})^\bullet, d\right)$$
where d is the cochain Cartan-Eilenberg differential.

After the pioneering work of Deligne and Drinfeld in the 80's, it is by now common wisdom (see e.g. [**Man99a**, Chapter III.9]) that dg Lie algebras give rise to solutions of moduli problems. In particular a homotopy abelian d($\mathbb{Z}/2$)g Lie algebra $\mathfrak{g}$ gives rise to a moduli space - the formal supermanifold $\mathbb{M}\mathsf{od}_{(\mathfrak{g}, d_{\mathfrak{g}})} := \mathrm{Spf}(H^\bullet(\mathfrak{g}, \mathbb{C}))$.

The property of being homotopy abelian is preserved by suitably non-degenerate deformations and various other natural operations:

PROPOSITION 4.11. **(i)** *Let $\mathfrak{g}$ be a flat family of $d(\mathbb{Z}/2)g$ Lie algebras (or $(\mathbb{Z}/2)$-graded L_∞ algebras) over $\mathbb{C}[[u]]$. That is, $\mathfrak{g}$ is a flat $(\mathbb{Z}/2)$-graded $\mathbb{C}[[u]]$-module, and the Lie bracket and differential on $\mathfrak{g}$ are $\mathbb{C}[[u]]$-linear. Assume further that*

(A) *$\mathfrak{g}_{\mathrm{gen}} := \mathfrak{g} \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u))$ is homotopy abelian over $\mathbb{C}((u))$, and*
(B) *$H^\bullet(\mathfrak{g}, d_\mathfrak{g})$ is a flat $\mathbb{C}[[u]]$-module.*

Then the special fiber $\mathfrak{g}_0 := \mathfrak{g}\widehat{\otimes}_{\mathbb{C}[[u]]}\mathbb{C}$ is also a homotopy abelian $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *Lie algebra over $\mathbb{C}$.*

(ii) *If $\mathfrak{g}$ is a homotopy abelian* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *Lie algebra over $\mathbb{C}$, and $\mathfrak{g}_1 \to \mathfrak{g}$ is a morphism of L_∞-algebras inducing a monomorphism $H^\bullet(\mathfrak{g}_1, d_{\mathfrak{g}_1}) \hookrightarrow H^\bullet(\mathfrak{g}, d_\mathfrak{g})$, then $\mathfrak{g}_1$ is homotopy abelian as well.*

(iii) *If $\mathfrak{g}$ is a homotopy abelian* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *Lie algebra over $\mathbb{C}$, and $\mathfrak{g} \to \mathfrak{g}_2$ is a morphism of L_∞-algebras inducing an epimorphism $H^\bullet(\mathfrak{g}, d_\mathfrak{g}) \twoheadrightarrow H^\bullet(\mathfrak{g}_2, d_{\mathfrak{g}_2})$, then $\mathfrak{g}_2$ is homotopy abelian as well.*

Proof. The proof is standard so we only mention some of the highlights of the argument. First note that parts **(ii)** and **(iii)** follow immediately by passing to minimal models. For part **(i)** we note first that the assumption *(B)* implies (and is in fact equivalent to) the existence of $\mathbb{C}[[u]]$-linear quasi-isomorphisms p_1, p_2 of complexes:

$$(H^\bullet(\mathfrak{g}_0, d_{\mathfrak{g}_0})[[u]], 0) \quad \cong \quad (H^\bullet(\mathfrak{g}, d_\mathfrak{g}), 0) \underset{p_2}{\overset{p_1}{\rightleftarrows}} (\mathfrak{g}, d_\mathfrak{g}),$$

and a $\mathbb{C}[[u]]$-linear homotopy h so that

$$\begin{aligned} p_2 \circ p_1 &= \mathrm{id} \\ p_1 \circ p_2 &= \mathrm{id} + [d_\mathfrak{g}, h]. \end{aligned}$$

Next note that the homological perturbation theory of [**KS01**] carries over verbatim to the L_∞-context and gives explicit expressions for the higher products m_n on $(H^\bullet(\mathfrak{g}_0, d_{\mathfrak{g}_0})[[u]], 0)$ as a polynomial expression in p_1, p_2 and h. In particular the operations m_n are all $\mathbb{C}[[u]]$-linear and are given by universal expressions. But by assumption *(A)* we know that the higher operations are zero after tensoring with $\otimes_{\mathbb{C}[[u]]}\mathbb{C}((u))$ and so $m_n = 0$ as formal power series in u for all $n \geq 1$. This implies that $m_{n|u=0} = 0$ for all $n \geq 1$ and so the proposition is proven. ☐

4.2.2. ***DG Batalin-Vilkovisky algebras.*** Recall [**Man99a**, Chapter III.10] the notion of a dg BV algebra:

DEFINITION 4.12. *A differential $\mathbb{Z}/2$-graded Batalin-Vilkovisky algebra over $\mathbb{C}$ is the data (A, d, Δ), where A is a $\mathbb{Z}/2$-graded supercommutative associative unital algebra, and $d : A \to A$, $\Delta : A \to A$ are odd $\mathbb{C}$-linear maps satisfying:*

- *$d(1) = \Delta(1) = 0$,*
- *d is a differential operator of order ≤ 1 on A,*
- *Δ is a differential operator of order ≤ 2 on A,*
- *$d^2 = \Delta^2 = d\Delta + \Delta d = 0$.*

Note that the first two properties in the definition imply that d is a derivation of A. Also $\mathfrak{g} := \mathbf{\Pi} A$ together with $[a,b] := \Delta(ab) - \Delta(a)b - (-1)^{\deg(a)} a\Delta(b)$ is a Lie superalgebra with two anti-commuting differentials d and Δ.

DEFINITION 4.13. *We will say that a* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *Batalin-Vilkovisky algebra* (A, d, Δ) *has the* ***degeneration property*** *if for every* $N \geq 1$ *we have that* $H^{\bullet}(A[u]/(u^N), d + u\Delta)$ *is a free* $\mathbb{C}[u]/(u^N)$*-module.*

Equivalently (A, d, Δ) has the degeneration property iff $H^{\bullet}(A[[u]], d + u\Delta)$ is a topologically free (flat) $\mathbb{C}[[u]]$-module. This in turn is equivalent to the existence of a (non-unique) isomorphism of topological $\mathbb{C}[[u]]$-modules:

$$T : \ H^{\bullet}(A[[u]], d + u\Delta) \xrightarrow{\cong} H^{\bullet}(A, d)[[u]]. \tag{4.2.1}$$

In this situation we will always normalize T so that $T_{|u=0} = \mathrm{id}_{H^{\bullet}(A,d)}$.

The degeneration property for dg Batalin-Vilkovisky algebras defined above is weaker than the $\partial\bar{\partial}$-lemma used by Barannikov and the second author in [**BK98**] and by Manin in [**Man99a, Man99b**]. In particular it has potentially a wider scope of applications – a feature that we will exploit next. We begin with a general smoothness result which was also proven by J.Terilla [**Ter07**].

THEOREM 4.14. *Suppose* (A, d, Δ) *is a* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *Batalin-Vilkovisky algebra which has the degeneration property. Let* $\mathfrak{g} := \mathbf{\Pi} A$ *be the associated super Lie algebra with anti-commuting differentials* d *and* Δ*. Then:*

(1) *The* $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ *Lie algebra* $(\mathfrak{g}, d)$ *is homotopy abelian, i.e. is quasi-isomorphic to* $H^{\bullet}(\mathfrak{g}, d)$ *endowed with the trivial bracket and the zero differential. In particular the associated moduli space* $\mathsf{Mod}_{(\mathfrak{g},d)}$ *is (non-canonically) isomorphic to a formal neighborhood of* 0 *in the superaffine space* $\mathbf{\Pi} H^{\bullet}(\mathfrak{g}, d)$*.*

(2) *Every choice of a normalized degeneration isomorphism* T *as in equation* (4.2.1) *gives an identification of formal manifolds*

$$\Phi_T : \ \mathsf{Mod}_{(\mathfrak{g},d)} \xrightarrow{\cong} \begin{pmatrix} \textit{formal neighborhood of } 0 \\ \textit{in } \mathbf{\Pi} H^{\bullet}(\mathfrak{g}, d) \end{pmatrix}$$

Proof. Part **(1)** of the theorem follows immediately from

LEMMA 4.15. *The* $d(\mathbb{Z}/2)g$ *Lie algebra* $(\mathfrak{g}((u)), d + u\Delta)$ *is homotopy abelian over* $\mathbb{C}((u))$*.*

Proof. Consider the formal completion at zero $\widehat{A}$ of the vector superspace underlying $A = \mathbf{\Pi}\mathfrak{g}$ as an algebraic supermanifold, and let as before $\mathbb{D} = \mathrm{Spf}(\mathbb{C}[[u]])$ be the formal one-dimensional disc. The $\mathrm{d}(\mathbb{Z}/2)\mathrm{g}$ Lie algebra structure on $\mathfrak{g}[[u]]$ is encoded in an odd vector field $\boldsymbol{\xi} \in \Gamma(\widehat{A} \times \mathbb{D}, T)$ on the supermanifold $\widehat{A} \times \mathbb{D}$, defined by

$$\dot{a} := \boldsymbol{\xi}(a) = da + u\Delta a + \frac{1}{2}[a, a].$$

There is a natural automorphism (i.e. a formal change of coordinates) $F : \widehat{A} \times \mathbb{D}^{\times} \to \widehat{A} \times \mathbb{D}^{\times}$ on the formal supermanifold $\widehat{A} \times \mathbb{D}^{\times}$ given by

$$F(a) := u\left(\exp\left(\frac{a}{u}\right) - 1\right) = a + \frac{1}{u}\frac{1}{2!}a^2 + \frac{1}{u^2}\frac{1}{3!}a^3 + \cdots,$$

and in the new coordinates $b = F(a)$ the vector field $\boldsymbol{\xi}$ is linear:

$$\begin{aligned}
\dot{b} &= \dot{a} \cdot \exp\left(\frac{a}{u}\right) \\
&= \left(da + u\Delta a + \frac{1}{2}[a,a]\right) \cdot \exp\left(\frac{a}{u}\right) \\
&= u \cdot \left(\frac{da}{u} + u\Delta\left(\frac{a}{u}\right) + u\frac{1}{2}\left[\frac{a}{u}, \frac{a}{u}\right]\right) \cdot \exp\left(\frac{a}{u}\right) \\
&= u \cdot (d + u\Delta)\exp\left(\frac{a}{u}\right) = (d+u\Delta)b.
\end{aligned}$$

So, in the b-coordinates, the vector field $\boldsymbol{\xi}$ depends only on the differential $d + u\Delta$ and does not depend on any higher operations. Passing to the minimal model we see that $(\mathfrak{g}((u)), d + u\Delta)$ is homotopy abelian, which proves the lemma. $\square$

The lemma implies that the hypothesis *(A)* of Proposition 4.11 **(i)** holds. On the other hand the hypothesis (B) holds by the degeneration assumption. Therefore by Proposition 4.11 **(i)** we conclude that $(\mathfrak{g}, d)$ is homotopy abelian. This proves part **(1)** of the theorem.

Next we construct the identification Φ_T. Given a formal path in $\mathbb{M}\mathsf{od}_{(\mathfrak{g},d)}$, i.e. a family of solutions (up to gauge equivalence)

$$\begin{aligned}
a(\varepsilon) &= a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots \in \varepsilon A[[\varepsilon]] \\
d(a(\varepsilon)) &+ \frac{1}{2}[a(\varepsilon), a(\varepsilon)] = 0
\end{aligned}$$

of the Maurer-Cartan equation in $(\mathfrak{g}, d)$, we have to construct the corresponding formal path through the origin in $H^\bullet(\mathfrak{g}, d)$.

As a first step choose a lift of the formal arc $a(\varepsilon)$ to a formal series in two variables $\tilde{a}(\varepsilon, u) \in \varepsilon A[[\varepsilon, u]]$ such that

$$\begin{aligned}
&(d{+}u\Delta)\tilde{a} + \frac{1}{2}[\tilde{a}, \tilde{a}] = 0, \\
&a(\varepsilon, 0) = a(\varepsilon).
\end{aligned}$$

Consider the reparameterization

$$\tilde{b} = F(\tilde{a}) = u\left(\exp\left(\frac{\tilde{a}}{u}\right) - 1\right) \in \varepsilon A((u))[[\varepsilon]].$$

Arguing as before we see that $\tilde{b}$ satisfies $(d + u\Delta)\tilde{b} = 0$. So if we expand

$$\tilde{b} = \tilde{b}_1\varepsilon + \tilde{b}_2\varepsilon^2 + \cdots, \qquad \text{where } \tilde{b}_n \in A((u)) \text{ satisfy } (d+u\Delta)\tilde{b}_n = 0,$$

we can define cohomology classes $\left[\tilde{b}_n\right] \in H^\bullet(A((u)), d + u\Delta)$. We can now apply the isomorphism $T \otimes_{\mathbb{C}[[u]]} \mathbb{C}((u))$ to the series

$$\sum_{n \geq 1} \left[\tilde{b}_n\right] \varepsilon^n \in \varepsilon H^\bullet(A((u)), d + u\Delta)[[\varepsilon]],$$

to obtain an element

$$T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\varepsilon^n\right)\in \varepsilon H^\bullet(A,d)((u))[[\varepsilon]].$$

In fact one has the following lemma whose proof we will skip since it is a somewhat tedious application of homological perturbation theory:

LEMMA 4.16. *There exists a lift $\tilde{a}(\varepsilon,u)$ of $a(\varepsilon)$ such that the associated class $T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\right)$ belongs to $\varepsilon H^\bullet(A,d)[[\varepsilon]]\subset \varepsilon H^\bullet(A,d)((u))[[\varepsilon]]$. Any such lift $\tilde{a}$ produces the same class $T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\right)$ and this class depends only on the gauge equivalence class of the original arc a, i.e. on the image $\underline{a}(\varepsilon)$ of $a(\varepsilon)$ in $\mathbb{M}\mathsf{od}_{(\mathfrak{g},d)}$.*

Now by definition the map Φ_T assigns the class $T\left(\sum_{n\geq 1}\left[\tilde{b}_n\right]\right)\subset \varepsilon H^\bullet(A,d)[[\varepsilon]]$ to the formal arc $\underline{a}(\varepsilon)$. $\square$

4.2.3. ***Geometric interpretation.*** The previous discussion can be repackaged geometrically as follows. A $(\mathbb{Z}/2)$-graded Batalin-Vilkovisky algebra (A,d,Δ) gives rise to a family $\mathscr{M}\to\mathbb{D}=\mathrm{Spf}(\mathbb{C}[[u]])$ of formal manifolds over the one-dimensional formal disc. The family $\mathscr{M}$ is the total space of the relative moduli space $\mathbb{M}\mathsf{od}_{(\mathfrak{g},d+u\Delta)}$ over $\mathbb{C}[[u]]$. If (A,d,Δ) has the degeneration property, then by Lemma 4.15 we have an affine structure on the generic fiber $\mathscr{M}^{\mathrm{gen}}:=\mathscr{M}\otimes_{\mathbb{C}[[u]]}\mathbb{C}((u))$ of the family (see Figure 6) given by the map F.

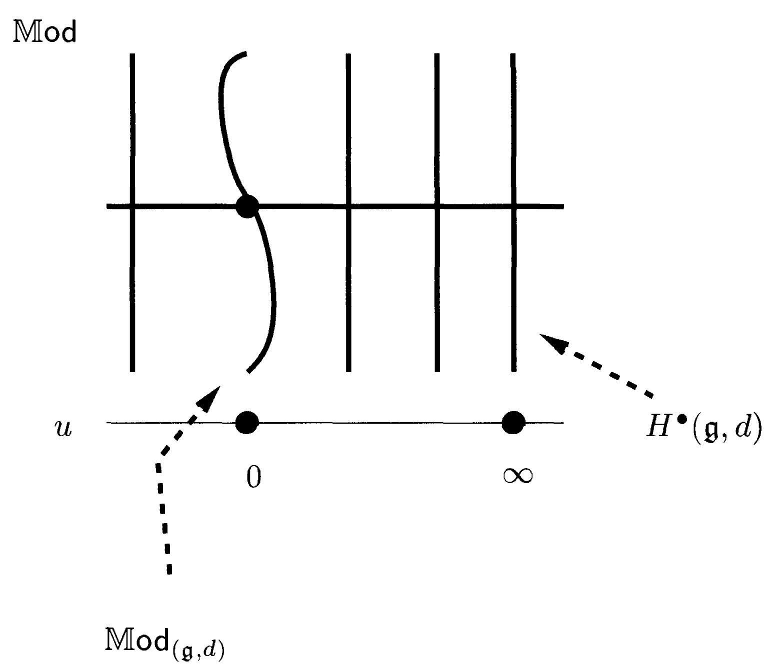


FIGURE 6. The relative moduli $\mathbb{M}\mathsf{od}\to\mathbb{P}^1$.

Furthermore the map T can be viewed as an extension of the affine bundle $\mathscr{M}^{\mathrm{gen}}\to\mathbb{D}^\times$ to a trivial bundle on $\mathbb{P}^1-\{0\}$ of formal superaffine spaces, where the fiber at ∞ is the superaffine space $H^\bullet(\mathfrak{g},d)$. This results in a family $\mathbb{M}\mathsf{od}\to\mathbb{P}^1$ of formal supermanifolds, which is a trivial vector bundle outside of zero but has a non-linear fiber at $0 \in \mathbb{P}^1$. Moreover by picking the closed point in each fiber we get a section of $\mathsf{Mod} \to \mathbb{P}^1$, which is just the zero section of the vector bundle $\mathsf{Mod}_{|\mathbb{P}^1 - \{0\}} \to \mathbb{P}^1 - \{0\}$. The normal bundle to this section in Mod is trivial (hence Mod is trivial as a non-linear bundle over $\mathbb{P}^1$), and the map Φ_T gives a (non-linear) trivialization of Mod over $\mathbb{P}^1$. This type of geometry was already discussed in [**CKS05**].

4.2.4. ***Relation to Calabi-Yau variations of* nc*-Hodge structures.*** Suppose (A, d, Δ) is a d($\mathbb{Z}/2$)g Batalin-Vilkovisky algebra which has the degeneration property. In this generality one does not expect to find a natural connection on $H^\bullet(A, d + u\Delta)$ along u, i.e. one does not expect to have a general formal analogue of an **nc**-Hodge structure.

However, a natural connection along the u-line may exist if we specify some additional data on (A, d, Δ). Following the analogy with the **nc**-Hodge structure associated with a symplectic manifold and the Gromov-Witten invariants, it is sufficient to specify:

- an even element $\kappa \in A$, with $d\kappa = 0$, and
- a grading operator $\mathbf{Gr} : A \to A$,

so that if we consider $\Gamma_{-1} := \mathbf{Gr} : A \to A$, and $\Gamma_{-2} : A \to A$ (the operator of multiplication by κ), then we have the commutation relations:

$$[\Gamma_{-1}, \Delta] = -\frac{1}{2}\Delta$$
$$[\Gamma_{-2}, d] = 0$$
$$d = [\Gamma_{-1}, d] + [\Gamma_{-2}, \Delta].$$

These commutation relations imply the identity

$$\left[u\frac{\partial}{\partial u} + u^{-1}\Gamma_{-2} + \Gamma_{-1}, d + u\Delta\right] = \frac{1}{2}(d + u\Delta),$$

which is consistent with the general formulas from Section 2.2.5. In particular, we can define a connection on $H^\bullet(A, d + u\Delta)$ along the u-line by setting

$$\nabla_{\frac{\partial}{\partial u}} := \frac{\partial}{\partial u} + u^{-2}\Gamma_{-2} + u^{-1}\Gamma_{-1}.$$

EXAMPLE 4.17. Let Y be a (possibly non-compact) d-dimensional Calabi-Yau manifold with a fixed holomorphic volume form $\mathbf{\Omega}_Y$. Let $\mathsf{w} : Y \to \mathbb{C}$ be a proper holomorphic function. This geometry gives rise to a natural dg Batalin-Vilkovisky algebra:

$$A := \Gamma_{C^\infty}\left(Y, \wedge^\bullet T_Y^{1,0} \otimes \wedge^\bullet A_Y^{0,1}\right),$$
$$d := \bar{\partial} + \iota_{d\mathsf{w}},$$
$$\Delta := \mathsf{div}_{\mathbf{\Omega}_Y} = \iota_{\mathbf{\Omega}_Y}^{-1} \circ \partial \circ \iota_{\mathbf{\Omega}_Y},$$

where $\iota_{\mathbf{\Omega}_Y} : \wedge^\bullet T_Y^{1,0} \to \wedge^{d-\bullet}\Omega_Y^{1,0}$ denotes the contraction with $\mathbf{\Omega}_Y$.

As discussed in Section 3.2 in this situation we get a connection along u which conjecturally defines an **nc**-Hodge structure. The connection is defined by the above formula with Γ_{-2} = the operator of multiplication by $-\mathsf{w}$, and $\Gamma_{-1} = \mathbf{Gr} : A \to A$, the grading operator which is equal to $\frac{q+p-d}{2} \cdot \mathrm{id}$ on $\Gamma_{C^\infty}\left(Y, \wedge^p T_Y^{1,0} \otimes \wedge^q A_Y^{0,1}\right)$.

We will elaborate on this geometric picture in the next section.

4.3. B-model framework: manifolds with anticanonical sections.

4.3.1. ***The classical Tian-Todorov theorem.*** Let X be a compact Kähler manifold. By Kodaira-Spencer theory we know that the deformations of X are controlled by the dg Lie algebra

$$\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right) := \left(\Gamma_{C^\infty}\left(X, T_X^{1,0} \otimes_{C_X^\infty} A_X^{0,\bullet}\right), \bar{\partial}\right).$$

The classical Tian-Todorov theorem [**Tia87, Tod89**] can be formulated as follows:

THEOREM 4.18. *If X is a compact Kähler manifold with $c_1(X) = 0 \in \mathrm{Pic}(X)$, then $\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right)$ is homotopy abelian. In particular the formal moduli space of X is smooth.*

Proof. Since $c_1(X) = 0 \in \mathrm{Pic}(X)$, up to scale we can find a unique holomorphic volume form $\mathbf{\Omega}_X$ on X. As in example 4.17 the pair $(X, \mathbf{\Omega}_X)$ gives rise to a dg Batalin-Vilkovisky algebra (A, d, Δ):

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_X^{1,0} \otimes \wedge^\bullet A_X^{0,1}\right) \\ d &:= \bar{\partial} \\ \Delta &:= \mathsf{div}_{\mathbf{\Omega}_X} = \iota_{\mathbf{\Omega}_X}^{-1} \circ \partial \circ \iota_{\mathbf{\Omega}_X}. \end{aligned}$$

Consider the associated dg Lie algebra $(\mathfrak{g}, d_{\mathfrak{g}}) := (\mathbf{\Pi} A, d)$. We have a natural inclusion of dg Lie algebras

$$\begin{array}{ccc} \left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right) & \hookrightarrow & (\mathfrak{g}, d_{\mathfrak{g}}) \\ \| & & \| \\ \left(\Gamma_{C^\infty}\left(X, T_X^{1,0} \otimes_{C_X^\infty} A_X^{0,\bullet}\right), \bar{\partial}\right) & \hookrightarrow & \Gamma_{C^\infty}\left(X, \wedge^\bullet T_X^{1,0} \otimes \wedge^\bullet A_X^{0,1}\right) \end{array}$$

which embeds $\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right)$ as a direct summand in $(\mathfrak{g}, d_{\mathfrak{g}})$, and so induces an embedding $H^\bullet\left(\mathfrak{g}^{(1)}, d_{\mathfrak{g}^{(1)}}\right) \subset H^\bullet(\mathfrak{g}, d_{\mathfrak{g}})$ in cohomology. So by Proposition 4.11 it suffices to check that $(\mathfrak{g}, d_{\mathfrak{g}})$ is homotopy abelian.

On the other hand the contraction map $\iota_{\mathbf{\Omega}_X}$ gives an isomorphism of bicomplexes between the dg Batalin-Vilkovisky algebra (A, d, Δ) and the Dolbeault bicomplex $(A^\bullet(X), \bar{\partial}, \partial)$. Since X is assumed compact and Kähler, the Hodge-to-de Rham spectral sequence degenerates on X, which is equivalent to the equality $\dim H^k_{\mathrm{dR}}(X, \mathbb{C}) = \dim(\oplus_{p+q=k} H^p(X, \Omega_X^q))$ which implies that the Dolbeault bicomplex $(A^\bullet(X), \bar{\partial}, \partial)$ has the degeneration property. Thus by Theorem 4.14 **(1)** it follows that $(\mathfrak{g}, d_{\mathfrak{g}})$ is homotopy abelian. The theorem is proven. $\square$

4.3.2. ***Canonical coordinates on the moduli of Calabi-Yau manifolds.*** Let X be a Calabi-Yau manifold, i.e. a d-dimensional compact Kähler manifold with $c_1(X) = 0$ in $\mathrm{Pic}(X)$. Let (A, d, Δ) be the dg Batalin-Vilkovisky algebra defined in Section 4.3.1. The contraction map $\iota_{\mathbf{\Omega}_X}$ identifies the $\mathbb{C}[[u]]$-module $H^\bullet(\mathfrak{g}[[u]], d + u\Delta)$ with the Rees module of the $\mathbf{nc}$-Hodge filtration on $H^\bullet_{\mathrm{dR}}(X, \mathbb{C})$

for which $H^{p,q}(X) \subset F^{\frac{p-q}{2}}$. Now choose one of the following equivalent pieces of data:

- a filtration $G^\bullet$ on $H^\bullet_{\mathrm{dR}}(X, \mathbb{C})$ which is opposed to the **nc**-Hodge filtration,
- a splitting of the **nc**-Hodge filtration,
- an extension of the associated **nc**-Hodge structure to a trivial bundle on $\mathbb{P}^1$ such that the connection has at most a first order pole at infinity.

Each such choice gives rise to an affine structure on $\mathbb{M}\mathsf{od}_{(\mathfrak{g},d_\mathfrak{g})}$. This affine structure is the same as the one described in Section 4.1.3 corresponding to the **nc**-Hodge structure above and the decoration ψ given by the class $[\boldsymbol{\Omega}_X]$ in the associated graded $\mathsf{gr}_{G^\bullet} H^\bullet_{\mathrm{dR}}(X, \mathbb{C})$.

In mirror symmetry considerations a choice of this type arises naturally when X is a Calabi-Yau manifold near a large complex structure limit point. Concretely, suppose $X = X_\mathfrak{z}$ is a member in a holomorphic family $\{X_z\}$ of compact d-dimensional Calabi-Yau manifolds parameterized by z in a polydisc $\prod_{i=1}^M \{z_i \in \mathbb{C} \,|\, 0 < |z_i| \ll 1\}$, and such that:

- $M = \dim_\mathbb{C} H^1(X_z, T_{X_z})$;
- for each $i = 1, \ldots, M$ the monodromy operator $t_i \in GL\left(H^1\left(X_\mathfrak{z}, T_{X_\mathfrak{z}}\right)\right)$ assigned to the circle (traced counterclockwise)
$$\gamma_i = \left\{ z \;\middle|\; \begin{array}{l} z_j = \mathfrak{z}_j,\ j \neq i, \\ |z_i| = |\mathfrak{z}_i| \end{array} \right\}$$
is unipotent of order d.

In this setup, the filtration $G^\bullet$ of $H^\bullet(X_\mathfrak{z}, \mathbb{C})$ invariant under all unipotent operators $\prod_{i=1}^M t_i^{a_i}$, $a_i \in \mathbb{Z}_{>0}$ will be opposed to the Hodge filtration and will thus give us canonical coordinates on the polydisc. This affine structure corresponds to a rational decoration of a Calabi-Yau variation of **nc**-Hodge structures.

4.3.3. ***Generalizations.*** Here we generalize the previous discussion to the case of varieties with divisors.

(i) Let X be a d-dimensional smooth projective variety over $\mathbb{C}$, and let $D \subset X$ be a normal crossings anti-canonical divisor, i.e $\mathcal{O}_X(D) = K_X^{-1} \in \operatorname{Pic}(X)$. Typically such an X will be a Fano or a quasi-Fano. If D is smooth, then by adjunction D will be a Calabi-Yau. Specifying such a divisor is equivalent to specifying a logarithmic volume form on X. This is up to scale a unique n-form $\boldsymbol{\Omega}_{X \log D} \in \Gamma\left(X, \Omega_X^d(\log D)\right)$ on X which has a first order pole along D and does not vanish anywhere on $X - D$.

Let $T_{X,D}$ be the subsheaf of T_X of holomorphic vector fields on X which at the points of D are tangent to D. This is a locally free subsheaf of T_X of rank d which controls the deformation theory of the pair (X, D). The relevant dg Batalin-Vilkovisky algebra (A, d, Δ) is an obvious generalization of the one in the absolute case:

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_{X,D} \otimes_{\mathcal{C}^\infty_X} \wedge^\bullet A_X^{0,1}\right) \\ d &:= \bar{\partial} \\ \Delta &:= \mathsf{div}_{\boldsymbol{\Omega}_{X \log D}} = \iota^{-1}_{\boldsymbol{\Omega}_{X \log D}} \circ \partial \circ \iota_{\boldsymbol{\Omega}_{X \log D}}, \end{aligned}$$

where $\iota_{\boldsymbol{\Omega}_{X \log D}} : \wedge^\bullet T_{X,D} \to \Omega_X^{d-\bullet}(\log D)$ is the isomorphism given by contraction with $\boldsymbol{\Omega}_{X \log D}$.

Again, the map $\iota_{\mathbf{\Omega}_{X \log D}}$ identifies (A, d, Δ) with the logarithmic Dolbeault bicomplex $\left(A^{\bullet,\bullet}(\log D), \bar{\partial}, \partial\right)$. In particular, for all $u \neq 0$ we get an identification of the cohomology of the complex $(A, d+u\Delta)$ with the cohomology of the total complex of the double complex $\left(\Omega_X^{\bullet,\bullet}(\log D), \bar{\partial}, \partial\right)$, which is equal [**Voi03**, Section 6.1] to the cohomology of the open variety $X - D$. In other words for all $u \neq 0$ we have an isomorphism

$$H^{\bullet}(A, d + u\Delta) \cong H^{\bullet}_{\mathrm{dR}}(X - D, \mathbb{C}). \tag{4.3.1}$$

Now mixed Hodge theory implies the following

LEMMA 4.19. *The logarithmic dg Batalin-Vilkovisky algebra (A, d, Δ) has the degeneration property. In particular the formal moduli space of the pair (X, D) is smooth.*

We will return to the proof of this lemma in section 4.3.4 but first we will discuss a couple of variants of this geometric setup.

(ii) Suppose X is a smooth projective d-dimensional Calabi-Yau manifold. Let as before $\mathbf{\Omega}_X$ be the holomorphic volume form on X. Let $D \subset X$ be a normal crossings divisor. Typically if D is smooth, it will be a variety of general type.

Consider the dg Batalin-Vilkovisky algebra (A, d, Δ) given by

$$\begin{aligned} A &:= \Gamma_{C^{\infty}}\left(X, \wedge^{\bullet} T_{X,D} \otimes_{C^{\infty}_X} \wedge^{\bullet} A^{0,1}_X\right) \\ d &:= \bar{\partial} \\ \Delta &:= \mathsf{div}_{\mathbf{\Omega}_X} = \iota^{-1}_{\mathbf{\Omega}_X} \circ \partial \circ \iota_{\mathbf{\Omega}_X}, \end{aligned}$$

The contraction $\iota_{\mathbf{\Omega}_X}$ identifies this algebra with the dg Batalin-Vilkovisky algebra

$$\left(\Gamma_{C^{\infty}}\left(X, \Omega^{\bullet}_X(\mathrm{rel}\, D) \otimes_{C^{\infty}_X} \wedge^{\bullet} A^{0,1}_X\right), \bar{\partial}, \partial\right),$$

where $\Omega^k_X(\mathrm{rel}\, D) \subset \Omega^k_X$ denotes the subsheaf of all holomorphic k-forms that restrict to $0 \in \Omega^k_{D-\mathrm{sing}(D)}$. The cohomology of the total complex associated with this double complex is the de Rham cohomology of the pair (X, D), and so again we get an identification

$$H^{\bullet}(A, d + u\Delta) \cong H^{\bullet}_{\mathrm{dR}}(X, D; \mathbb{C}) \tag{4.3.2}$$

valid for all fixed $u \neq 0$. Again using this identification and mixed Hodge theory one deduces the following

LEMMA 4.20. *The dg Batalin-Vilkovisky algebra (A, d, Δ) has the degeneration property and hence the formal moduli space of the pair (X, D) is smooth.*

(iii) The setups **(i)** and **(ii)** have a natural common generalization. Fix a smooth projective complex variety of dimension d, a normal crossings divisor $D = \cup_{i \in I} D_i \subset X$, and a collection of weights $\{a_i\}_{i \in I} \subset [0, 1] \cap \mathbb{Q}$, so that

$$\sum_{i \in I} a_i [D_i] = -K_X \in \mathrm{Pic}(X) \otimes \mathbb{Q}.$$

Represent the a_i's by reduced fractions, take $N \geq 1$ to be the least common multiple of the denominators of these fractions and such that

$$\sum_{i \in I} (N a_i)[D_i] = -N K_X \in \mathrm{Pic}(X),$$

and set $n_i := a_i N$. In particular up to scale we have a unique section $\widetilde{\mathbf{\Omega}}_X \in \Gamma\left(X, K_X^{\otimes(-N)}\right)$ whose divisor is $\sum_{i\in I} n_i D_i$. In this situation we can again promote the Dolbeault dg Lie algebra which computes the deformation theory of (X, D) to a dg Batalin-Vilkovisky algebra (A, d, Δ), where

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(X, \wedge^\bullet T_{X,D} \otimes_{\mathcal{C}_X^\infty} \wedge^\bullet A_X^{0,1}\right) \\ d &:= \bar\partial \\ \Delta &:= \mathsf{div}_{\widetilde{\mathbf{\Omega}}_X}. \end{aligned}$$

The divergence operator $\mathsf{div}_{\widetilde{\mathbf{\Omega}}_X}$ is defined as follows. Restricting the section $\widetilde{\mathbf{\Omega}}_X$ to $X - D$ we get a nowhere vanishing section of $K_{X-D}^{\otimes(-N)}$, i.e. a flat holomorphic connection on K_{X-D}. If $U \subset X - D$ is a simply connected open, then we can choose $\mathbf{\Omega}_U$ a holomorphic volume form on U which is covariantly constant for this flat connection, and define the associated divergence operator $\mathsf{div}_{\mathbf{\Omega}_U} := \iota_{\mathbf{\Omega}_U}^{-1} \circ \partial \circ \iota_{\mathbf{\Omega}_U}$. But by the flatness of the connection it follows that any other covariantly constant volume form on U will be proportional to $\mathbf{\Omega}_U$ with a constant proportionality coefficient. Since by definition $\mathsf{div}_{c\mathbf{\Omega}_U} = \mathsf{div}_{\mathbf{\Omega}_U}$ for any constant c we get a well defined divergence operator on $X - U$. Furthermore, locally this divergence operator is given by a holomorphic volume form which is a branch of $\left(\widetilde{\mathbf{\Omega}}_X\right)^{-1/N}$ and so by continuity it gives a well defined map of locally free sheaves $\mathsf{div}_{\widetilde{\mathbf{\Omega}}_X} : \wedge^i T_{X,D} \to \wedge^{i-1} T_{X,D}$.

Again, we claim that

LEMMA 4.21. *The dg Batalin-Vilkovisky algebra (A, d, Δ) has the degeneration property and the formal moduli space of the pair (X, D) is smooth.*

Proof. The proof of this lemma again reduces to mixed Hodge theory via a map similar to the isomorphisms (4.3.1) and (4.3.2). However constructing this map is a bit more involved than the arguments we used to construct (4.3.1) and (4.3.2).

Consider the root stack $Z = X\left\langle \left\{\frac{D_i}{N}\right\}_{i\in I}\right\rangle$ as defined in e.g. [**MO05, IS07**]. By construction Z is a smooth proper Deligne-Mumford stack, equipped with a finite and flat morphism $\pi : Z \to X$.

Conceptually the best way to define the stack Z is as a moduli stack classifying (special) log structures associated with X, the divisor D and the number N (see [**MO05**] for the details). Étale locally on X the stack Z can be described easily as a quotient stack. Indeed, choose étale locally an identification of X with a neighborhood of zero in $\mathbb{A}^d$ with coordinates $z_1, \ldots, z_d$, so that $D = D_1 \cup \cdots \cup D_r$ and D_i is identified with the hyperplane $z_i = 0$. Then the corresponding étale local patch in Z is canonically isomorphic to the stack quotient

$$\Big[\mathbb{A}^d / \underbrace{\boldsymbol{\mu}_N \times \cdots \times \boldsymbol{\mu}_N}_{r\text{-times}}\Big],$$

where $\boldsymbol{\mu}_N \subset \mathbb{C}^\times$ is the group of N-th roots of unity, and $(\zeta_1, \ldots, \zeta_r) \in \boldsymbol{\mu}_N^{\times r}$ acts as $(z_1, \ldots, z_r, z_{r+1}, \ldots, z_d) \mapsto (\zeta_1 z_1, \ldots, \zeta_r z_r, z_{r+1}, \ldots, z_d)$.

In particular, this description shows (see [**MO05**, Theorem 4.1]) that:

- The map π is an isomorphism over $X - D$ and in general identifies X with the coarse moduli space of Z;
- There is a strict normal crossings divisor $\widetilde{D} = \cup_{i\in I}\widetilde{D}_i \subset Z$, such that
$$\mathcal{O}_Z\left(-N\widetilde{D}_i\right) = \pi^*\mathcal{O}_X\left(-D_i\right)$$
as ideal subsheaves of $\mathcal{O}_Z$;
- For all j we have the Hurwitz formula $\Omega^j_Z(\log\widetilde{D}) = \pi^*\Omega^j_X(\log D)$.

In particular we have canonical isomorphisms
$$\pi^*K_X \cong \mathcal{O}_Z\left(-\sum_{i\in I} n_i\widetilde{D}_i\right)$$
$$\pi^*K_X \cong K_Z\otimes\mathcal{O}_Z\left((1-N)\sum_{i\in I}\widetilde{D}_i\right)$$
the first given by the section $\pi^*\widetilde{\mathbf{\Omega}}_X$ and the second coming from the Hurwitz formula.

There is a natural complex local system of rank one on $X - D$ with monodromy in $\boldsymbol{\mu}_N$ associated with the choices of N-th root of the section $\widetilde{\mathbf{\Omega}}_X$. It is easy to see that the pullback of this local system admits a canonical extension (as a local system) to Z, which we denote by Ξ. Moreover, we have a canonical meromorphic section $\mathbf{\Omega}_Z$ of $K_Z\otimes_{\mathbb{C}}\Xi$ with divisor $\sum_{i\in I}(N-1-n_i)\widetilde{D}_i$. It is easy to check locally by using the étale local description of Z as a quotient stack that the contraction $\iota_{\mathbf{\Omega}_Z}$ gives a well defined isomorphism of locally free sheaves:
$$\iota_{\mathbf{\Omega}_Z}:\quad \wedge^j T_{Z,\widetilde{D}} \xrightarrow{\cong} \Omega_Z^{d-j}\left(\log\widetilde{D}_{(1)}, \operatorname{rel}\widetilde{D}_{(0)}\right)\otimes_{\mathbb{C}}\Xi.$$
Here
$$\widetilde{D}_{(0)} := \cup_{i\in I_0}\widetilde{D}_i \qquad I_0 = \{i\in I | a_i = 0\}$$
$$\widetilde{D}_{(1)} := \cup_{i\in I_1}\widetilde{D}_i \qquad I_1 = \{i\in I | a_i = 1\}.$$
Now taking into account the Hurwitz isomorphism $\wedge^j T_{Z,\widetilde{D}} \cong \pi^*\wedge^j T_{X,D}$ and using adjunction, we can view $\iota_{\mathbf{\Omega}_Z}$ as an isomorphism

$$\wedge^j T_{X,D} \xrightarrow{\cong} \left(\pi_*\Omega_Z^{d-j}\left(\log\widetilde{D}_{(1)}, \operatorname{rel}\widetilde{D}_{(0)}\right)\otimes_{\mathbb{C}}\Xi\right) \tag{4.3.3}$$

It is immediate from the definition that the isomorphism (4.3.3) (taken for all j) identifies the dg Batalin-Vilkovisky algebra (A, d, Δ) with the Dolbeault bicomplex
$$\left(\Gamma_{C^\infty}\left(X, \left(\pi_*\Omega_Z^\bullet\left(\log\widetilde{D}_{(1)}, \operatorname{rel}\widetilde{D}_{(0)}\right)\otimes_{\mathbb{C}}\Xi\right)\otimes_{C^\infty_X} A_X^{0,\bullet}\right), \bar{\partial}, \partial\right).$$
But the above complex equipped with the differential $\partial+\bar{\partial}$ is the Dolbeault resolution of the complex of sheaves $\pi_*\left(\Omega_Z^\bullet\left(\log\widetilde{D}_{(1)}, \operatorname{rel}\widetilde{D}_{(0)}\right)\otimes_{\mathbb{C}}\Xi, \partial\right)$ which is equal to the derived direct image $R\pi_*\left(\Omega_Z^\bullet\left(\log\widetilde{D}_{(1)}, \operatorname{rel}\widetilde{D}_{(0)}\right)\otimes_{\mathbb{C}}\Xi, \partial\right)$ since π is finite. Now combined with the Leray spectral sequence for π this gives, for all $u\neq 0$, an isomorphism

$$H^\bullet(A, d+u\Delta) \cong H^\bullet_{\mathrm{dR}}\left(Z - \widetilde{D}_{(1)}, \widetilde{D}_{(0)} - \widetilde{D}_{(1)}; \Xi\right), \tag{4.3.4}$$

which specializes to both isomorphisms (4.3.1) and (4.3.2).

Now the fact that Z is a smooth and proper Deligne-Mumford stack and mixed Hodge theory (see 4.3.4) for $\left(Z - \widetilde{D}_{(1)}, \widetilde{D}_{(0)} - \widetilde{D}_{(1)}\right)$ endowed with the local system Ξ imply that (A, d, Δ) has the degeneration property. ☐

REMARK 4.22. The fact that the root stack in the previous proof can be viewed as the moduli stack of special log structures is very interesting. It suggests that the setup we just discussed may fit naturally in the recent approach of Gross-Siebert [**GS06, GS07**] to mirror symmetry and instanton corrections via log degenerations of toric Fano manifolds (see also [**KS06a, KS01**]). The relationship between these two setups is certainly worth studying and we plan to return to it in the future.

(iv) Yet another generalization of the previous picture arises when we take the variety X to be a normal-crossings Calabi-Yau. More precisely, assume that X is a strict normal crossings variety with irreducible components $X = \cup_{i\in I} X_i$ equipped with a holomorphic volume form $\boldsymbol{\Omega}_X$ on $X - X^{\text{sing}}$ such that the restriction of $\boldsymbol{\Omega}_X$ on each X_i has a logarithmic pole along $X_i \cap (\cup_{j\neq i} X_j)$ and the residues of these restricted forms cancel along each $X_i \cup X_j$. Taking a colimit along the projective system of all finite intersections of components of X we get again a dg Batalin-Vilkovisky algebra $A_{\text{tot}}(X) = \mathsf{colim}_{J\subset I} A\left(\cap_{i\in J} X_i\right)$ and again by using mixed Hodge theory we can check that this algebra has the degeneration property.

4.3.4. ***Mixed Hodge theory in a nutshell.*** In this section we briefly recall the basic arguments from Deligne's mixed Hodge theory [**Del74**] that are necessary for proving the degeneration property of the dg Batalin-Vilkovisky algebras in section 4.3.3 **(i)-(iv)**.

Suppose we are given:

- a finite ordered collection (X_α) of smooth complex projective varieties;
- for every α a choice of a $\mathbb{Z}\times\mathbb{Z}$-graded complex of sheaves of differential forms which are either C^∞ or are $C^{-\infty}$ (i.e. currents) and constrained so that their wave-front (singular support) is contained in a given conical Lagrangian in $T^\vee X_\alpha$ which is the conormal bundle to a normal crossing divisor in X_α;
- a collection of integers $n_\alpha \in \mathbb{Z}$.

Consider the complex $C^{\text{tot}} = \oplus_\alpha C^\bullet_\alpha[n_\alpha]$ equipped with three differentials ∂, $\bar{\partial}$, δ, where $\delta = \sum_{\alpha<\beta} \delta_{\alpha\beta}$, and the $\delta_{\alpha\beta}$ come from pullbacks and pushforwards for some maps $X_\beta \hookrightarrow X_\alpha$ or $X_\alpha \hookrightarrow X_\beta$. The statement we need now can be formulated as follows:

CLAIM 4.23. *For every $k \geq 1$ the cohomology*

$$H^\bullet\left(C^{\text{tot}}[u]/(u^k), \bar{\partial} + \delta + u\partial\right)$$

is a free $\mathbb{C}[u]/(u^k)$-module.

Proof. If X is smooth projective over $\mathbb{C}$ and if $\left(A^\bullet(X), \bar{\partial}\right)$ is the $\bar{\partial}$-complex of (either C^∞ or $C^{-\infty}$) differential forms on X, then the inclusion

$$\left(\ker \partial, \bar{\partial}\right) \hookrightarrow \left(A^\bullet(X), \bar{\partial}\right)$$

is a quasi-isomorphism.

This implies that the horizontal arrows in the diagram of complexes

$$\begin{array}{ccc} \left(\ker\partial[u]/(u^k), \bar{\partial}+\delta+u\partial\right) & \longrightarrow & \left(C^{\mathrm{tot}}[u]/(u^k), \bar{\partial}+\delta+u\partial\right) \\ \| & & \\ \left(\ker\partial[u]/(u^k), \bar{\partial}+\delta\right) & \longrightarrow & \left(C^{\mathrm{tot}}, \bar{\partial}+\delta\right)[u]/(u^k), \end{array}$$

are quasi-isomorphisms. Indeed, this follows by noticing that there are natural filtrations on both sides (by the powers of u and the index α) which give rise to convergent spectral sequences and induce the quasi-isomorphic inclusion $\left(\ker\partial, \bar{\partial}\right) \hookrightarrow \left(C^{\mathrm{tot}}, \bar{\partial}\right)$ on the associated graded. This proves the claim. □

REMARK 4.24. • Note that the same reasoning implies that the natural map

$$\left(\ker\partial, \bar{\partial}+\delta\right) \twoheadrightarrow \left(\ker\partial/\operatorname{im}\partial, \bar{\partial}+\delta\right) = \left(H^\bullet(X_\alpha), \delta\right),$$

is also a quasi-isomorphism, which reduces the problem of computing $H^\bullet\left(C^{\mathrm{tot}}[u]/(u^k), \bar{\partial}+\delta+u\partial\right)$ to a homological algebra question on a complex of finite dimensional vector spaces.

• There is a useful variant of the theory, also discussed in [**Del74**]: the previous discussion immediately generalizes to the case of cochain complexes of a collection of projective manifolds with coefficients in some unitary local systems.

Next we discuss a few examples and applications of the geometric setup from section 4.3.3.

4.3.5. ***The moduli stack of Fano varieties.*** As a consequence of section 4.3.3 **(iii)** we get a new proof and a refinement of the following result of Ran [**Ran92, Kaw92**]:

THEOREM 4.25. *Let X be a complex Fano manifold, that is, let X be a smooth proper $\mathbb{C}$-variety for which K_X^{-1} is ample. Then the versal deformations of X are unobstructed.*

Proof: Choose $N > 1$ so that $K_X^{\otimes(-N)}$ is very ample and all the higher cohomology groups $H^k\left(X, K_X^{\otimes(-N)}\right)$ vanish for $k \geq 1$. Choose a generic section $\widetilde{\boldsymbol{\Omega}}_X \in H^0\left(X, K_X^{\otimes(-N)}\right) = 0$ whose zero locus is a smooth and connected divisor $D \subset X$.

Consider now $\mathfrak{g} = \boldsymbol{\Pi} R\Gamma\left(X, \wedge^\bullet T_{X,D}\right)$ with the Schouten bracket. By Lemma 4.21 this d$(\mathbb{Z}/2)$g Lie algebra is homotopy abelian and so as in the proof of Theorem 4.18 we conclude that $\mathfrak{g}^{(1)} = R\Gamma\left(X, T_{X,D}\right)$ is homotopy abelian. Since this d$(\mathbb{Z}/2)$g Lie algebra governs the deformation theory of (X, D) as a variety with a divisor, it follows that the formal germ of the deformation space of the pair (X, D) is smooth. Next we will need the following simple

LEMMA 4.26. *Suppose (X', D') is a small deformation of (X, D) as a variety with divisor. Then X' is still a Fano with $K_{X'}^{\otimes(-N)}$ very ample and $D' \in \left|K_{X'}^{\otimes(-N)}\right|$.*

Proof: The condition of $K_X^{\otimes(-N)}$ being very ample is open in the moduli of X. Furthermore by definition $K_X^{\otimes(-N)} \otimes \mathcal{O}_X(-D) = \mathcal{O}_X$ and so by the small deformation hypothesis it follows that $K_{X'}^{\otimes(-N)} \otimes \mathcal{O}_{X'}(-D)$ is in the connected component of the identity of $\mathrm{Pic}(X')$. But X' is a Fano and so $\mathrm{Lie}(\mathrm{Pic}^0(X')) = H^1(X', \mathcal{O}_{X'}) = 0$. Hence $K_{X'}^{\otimes(-N)} \otimes \mathcal{O}_{X'}(-D) = \mathcal{O}_X$ as well. $\square$

The theorem now follows easily. The versal deformation space of smooth connected D's for a given X is smooth and isomorphic to a domain in $\mathbb{P}^{h^0\left(X, K_{X'}^{\otimes(-N)}\right)-1}$. Since the dimension of these projective spaces is locally constant in X by Riemann-Roch and vanishing of the higher cohomologies, it follows that the map from the versal deformation space of the pairs (X, D) to the versal deformation stack of X is smooth. In other words the versal deformation stack of X has a presentation in the smooth topology with a smooth atlas – the versal deformation space for (X, D). Hence the versal deformations of X form a smooth stack. $\square$

4.3.6. ***Algebras for the Landau-Ginzburg model.*** Consider again the setup of a holomorphic Landau-Ginzburg model. Suppose Y is smooth and quasiprojective over $\mathbb{C}$ and of dimension $\dim Y = d$. Suppose there exists a nowhere vanishing algebraic volume form $\boldsymbol{\Omega}_Y \in \Gamma(Y, K_Y)$, and let $\mathsf{w} : Y \to \mathbb{A}^1$ be a regular function with compact critical locus.

This data gives a dg Batalin-Vilkovisky algebra (A, d, Δ) where

$$\begin{aligned} A &:= \Gamma_{C^\infty}\left(Y, \wedge^\bullet T_Y^{1,0} \otimes_{C_Y^\infty} \wedge^\bullet A_Y^{0,1}\right) \\ d &:= \bar\partial + \iota_{d\mathsf{w}} \\ \Delta &:= \mathsf{div}_{\boldsymbol{\Omega}_Y}. \end{aligned}$$

Again the contraction $\iota_{\boldsymbol{\Omega}_Y}$ identifies (A, d, Δ) with the twisted Dolbeault bicomplex $\left(A^\bullet(Y), \bar\partial + d\mathsf{w}\wedge, \partial\right)$. The latter satisfies the degeneration property by the work of Barannikov and the second author, Sabbah [**Sab99**], or Ogus-Vologodsky [**OV05**].

REMARK 4.27. It will be interesting to combine the previous discussion with the discussion in section 4.3.3 **(iii)** or with the broken Calabi-Yau geometry from section 4.3.3 **(iv)**. Suppose we have a quasiprojective smooth complex Y, a regular function $\mathsf{w} : Y \to \mathbb{A}^1$ with compact critical locus, and suppose we are given a normal crossings divisor $D = \cup_{i\in I} D_i$ and a system of weights $\{a_i\}_{i\in I}$ as in Section 4.3.3 **(iii)**. Then we can write the w-twisted version of the dg Batalin-Vilkovisky algebra for (Y, D) which by general nonsense will compute the deformation theory of the data (Y, D, w). Similarly we can add a potential to Y which itself is a normal-crossings Calabi-Yau, as in Section 4.3.3 **(iv)**. We expect that the resulting algebras will again have the degeneration property, but we have not investigated this question.

4.4. Categorical framework: spherical functors.

In this section we briefly discuss some algebraic aspects of the deformation theory of $\mathbf{nc}$-spaces (see section 2.2.1). For simplicity we will discuss the $\mathbb{Z}$-graded case but in fact all definitions and statements readily generalize to the $\mathbb{Z}/2$ case.

4.4.1. ***Calabi-Yau $\mathbf{nc}$-spaces.*** Suppose $X = \mathbf{ncSpec}(A)$ is a graded $\mathbf{nc}$-affine $\mathbf{nc}$-space over $\mathbb{C}$. If X is smooth, then $A \in \mathsf{Perf}_{X\times X^{\mathrm{op}}} = \mathsf{Perf}\,(A \otimes A^{\mathrm{op}} - \mathsf{mod})$ and we define the ***smooth dual*** of A to be $A^! := \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(A, A \otimes A)$. Similarly if X is compact, then $A \in \mathsf{Perf}_{\mathrm{pt}}$ and we define the ***compact dual*** of A to be $A^* := \mathrm{Hom}_{\mathbb{C}}(A, \mathbb{C}) \in (A \otimes A^{\mathrm{op}} - \mathsf{mod})$.

If X is both a smooth and compact **nc**-space, then we have isomorphisms

$$A^! \otimes_A A^* \cong A^* \otimes_A A^! \cong A$$

in the category $(A \otimes A^{\mathrm{op}} - \mathsf{mod})$. The endofunctor $S_X : C_X \to C_X$ given by the A-bimodule A^* is called ***the Serre functor*** of X. It is an autoequivalence of C_X which is central (i.e. commutes with all autoequivalences). Moreover, for any two objects $\mathcal{E}, \mathcal{F} \in \mathsf{Perf}_X$ there is a functorial identification

$$\mathrm{Hom}_X(\mathcal{E}, \mathcal{F})^\vee \cong \mathrm{Hom}_X(\mathcal{F}, S_X\mathcal{E}).$$

With this notation we have the following definition (see also [**KS06b**]):

DEFINITION 4.28. *We say that a smooth graded* **nc**-*affine* **nc**-*space* $X = \mathbf{ncSpec}(A)$ *is a* ***Calabi-Yau of dimension*** $d \in \mathbb{Z}$ *if* $A^! \cong A[-d]$ *in* $(A \otimes A^{\mathrm{op}} - \mathsf{mod})$. *We say that a compact* **nc**-*affine* **nc**-*space* $X = \mathbf{ncSpec}(A)$ *is a* ***Calabi-Yau of dimension*** $d \in \mathbb{Z}$ *if* $A^* \cong A[d]$ *in* $(A \otimes A^{\mathrm{op}} - \mathsf{mod})$.

The definition works also in the $\mathbb{Z}/2$-graded case, where the dimension d is understood as an element of $\mathbb{Z}/2$.
For an **nc**-space which is both smooth and compact the two conditions are equivalent and are equivalent to having an isomorphism of endofunctors $S_X \cong [d]$.

REMARK 4.29. This definition of a Calabi-Yau structure on a smooth compact **nc**-space is somewhat simplistic and should be taken with a grain of salt. The true definition (see [**KS06b**]) implies the isomorphism of functors $S_X \cong [d]$ but also involves higher homotopical data which are encoded in a cyclic category structure on C_X. We will suppress the cyclic structure here in order to simplify the discussion.

We are interested in **nc**-space analogues of the Tian-Todorov theorem. The unobstructedness of graded smooth and compact **nc**-Calabi-Yau spaces was recently analyzed by Pandit [**Pan08**] via the T^1-lifting property of Ran [**Ran92**] and Kawamata [**Kaw92**]. Here we formulate the following general

THEOREM 4.30. *Suppose that X is a smooth and compact* **nc**-*Calabi-Yau space of dimension $d \in \mathbb{Z}$ (or of dimension $d \in \mathbb{Z}/2$ in the $\mathbb{Z}/2$-graded case). Assume that X satisfies the degeneration conjecture (see Section 2.2.4). Then:*

- *the Hochschild cochain algebra $C^\bullet(X)$ of X is a homotopy abelian L_∞ algebra;*
- *the formal moduli space $\mathbb{M}\mathsf{od}_X$ of X is a formal supermanifold, i.e.*

$$\mathbb{M}\mathsf{od}_X := \mathbb{M}\mathsf{od}_{C^\bullet(A,A)} \cong \mathrm{Spf}\, \mathbb{C}[[x_1, \ldots, x_N, \xi_1, \ldots, \xi_M]];$$

- *the negative cyclic homology of the universal family over $\mathbb{M}\mathsf{od}_X$ gives a vector bundle $H \to \mathbb{M}\mathsf{od}_X \times \mathbb{D}$ which is equipped with a flat meromorphic connection ∇ so that $\nabla_{u\partial/\partial x_i}$, $\nabla_{u\partial/\partial \xi_j}$, and $\nabla_{u^2\partial/\partial u}$ are regular;*
- *(H, ∇) is the de Rham part of a Calabi-Yau variation of* **nc**-*Hodge structures.*

We will only sketch some of the highlights of the proof of this theorem here since going into full detail would take us too far afield. The proof is based on a mildly generalized version of Deligne's conjecture (see e.g. [**KS00, Tam03**]) which states that the Hochschild cochain complex of an affine **nc**-space is also an algebra over the operad of chains of the little discs operad. The first step is to show that under the Calabi-Yau assumption the Hochschild cochain complex $C^\bullet(X)$ is also naturally an algebra over the cyclic operad of chains of the framed little discs operad (i.e. the operad of little discs with a marked point on the boundary). Next one shows that the validity of the degeneration conjecture for X implies that the induced S^1-action on the cochain complex is homotopically trivial. Finally by a topological argument one deduces from this the fact that all the higher L_∞ operations on $C^\bullet(X)$ must vanish.

REMARK 4.31. It seems certain[2] that from deformation quantization it follows that if X is a smooth and projective Calabi-Yau variety, then the data described in the above theorem are canonically isomorphic to the formal completion of the variation of **nc**-Hodge structures described in section 4.3.2.

For a general smooth and compact **nc**-Calabi-Yau space we expect that the formal variation of **nc**-de Rham data in Theorem 4.30 converges to give analytic de Rham data which contain a compatible **nc**-Betti data $\mathscr{E}_B$ and so extend to an honest variation of **nc**-Hodge structures.

4.4.2. ***Spherical functors.*** In this section we introduce a special version of the general notion of a spherical functor [**Ann07**] which is tailored to the Calabi-Yau condition. We begin with a definition:

DEFINITION 4.32. *Let X and Y be two graded* **nc***-spaces. A* ***morphism*** $f : X \to Y$ *is a triple of functors*

$$\begin{tikzcd} C_X \arrow[d, "f_*"] \\ C_Y \arrow[u, bend left, "f^*"] \arrow[u, bend right, "f^!"'] \end{tikzcd}$$

so that (f^*, f_*) and $(f_*, f^!)$ are (left, right) pairs of adjoint functors.

Suppose now X, Y are smooth and compact graded **nc**-spaces and let Y be an **nc**-Calabi-Yau of dimension d.

DEFINITION 4.33. *A morphism $f : X \to Y$ is* ***spherical*** *if:*

- **(a)** *the cone of the natural adjunction morphism $\mathrm{id}_{C_X} \to f^! \circ f_*$ is isomorphic to the shifted Serre functor of X:* $\mathrm{cone}\left(\mathrm{id}_{C_X} \to f^! \circ f_*\right) \cong S_X[1-d]$,
- **(b)** *the natural map $f^! \to S_X[1-d] \circ f^*$, induced from the isomorphism in* **(a)** *and the adjunction $f^! \to f^! \circ f_* \circ f^*$, is an isomorphism of functors.*

REMARK 4.34. **(a)** If f is spherical, then the associated ***reflection functor*** $\mathcal{R}_f := \mathrm{cone}\left(f_* \circ f^! \to \mathrm{id}_{C_Y}\right)$ is an auto-equivalence of C_Y [**Ann07**].
(b) Similarly to the definition of a Calabi-Yau structure the above notion of a spherical functor should be viewed as a weak preliminary version of a stronger more refined notion which has to involve higher homotopical data and has yet to be defined carefully.

EXAMPLE 4.35. **(i)** Let $X = \mathrm{pt}$ and Y be a d-dimensional smooth and compact **nc**-Calabi-Yau and let $\mathcal{E} \in C_Y$ be a spherical object, i.e. an object for which the complex of $\mathbb{C}$-vector spaces $\mathrm{Hom}_Y(\mathcal{E}, \mathcal{E})$ is quasi-isomorphic to $(H^\bullet(S^d, \mathbb{C}), 0)$. The

[2] We borrowed this delightful expression from [**Kap91**].

morphism of **nc**-spaces $f : \mathrm{pt} \to Y$ given by $f_*(V) = \mathcal{E} \otimes V$ for any $V \in C_{\mathrm{pt}} = (\mathsf{Vect}_{\mathbb{C}})$ is spherical.

(ii) Let X be smooth and projective of dimension $d+1$, and let $i : Y \hookrightarrow X$ be a smooth anti-canonical divisor in X. Then Y is a d-dimensional Calabi-Yau and we have a natural spherical **nc**-morphism $f : X \to Y$ given by $f_* := i^*$, $f^! := i_*$, etc.

(iii) Let Y be a smooth projective d-dimensional Calabi-Yau. Let $i : X \hookrightarrow Y$ be a smooth hypersurface. Then we have a natural spherical **nc**-morphism $f : X \to Y$ given by $f_* = i_*$, $f^! = i^!$, and $f^* = i^*$.

REMARK 4.36. The geometry of Example 4.35 **(ii)**, where X is taken to be a smooth projective Fano, and $i : Y \hookrightarrow X$ is a smooth anti-canonical divisor, can be encoded algebraically in the categories $C_X = D(\mathsf{Qcoh}(X))$, $C_Y = D(\mathsf{Qcoh}(Y))$, the functor $f_* = i^*$, and another natural triple of categories:

- the compact category $D_{\substack{\text{compact}\\\text{support}}}(\mathsf{Qcoh}(X - Y)) = \ker(f_*)$,
- the compact category $D_{\mathrm{supp}\ Y}(\mathsf{Qcoh}(X))$ = the subcategory in $D(\mathsf{Qcoh}(X))$ generated by $i_*D(\mathsf{Qcoh}(Y))$,
- the smooth category $D(\mathsf{Qcoh}(X - Y))$ = the quotient $D(\mathsf{Qcoh}(X))/D_{\mathrm{supp}\ Y}(\mathsf{Qcoh}(X))$.

There is a similar triple of categories for the setup in Example 4.35 **(iii)**. It will be very interesting to describe the categorical data that encode anti-canonical divisors with normal crossings or more generally the fractional anti-canonical divisor setup from section 4.3.3 **(iii)**. It seems likely that in this situation one gets a system of nested categories and functors with a "spherical" condition imposed on the whole system rather than on individual functors. This is a very interesting question that we plan to investigate in the future.

REMARK 4.37. It is clear from the examples above that spherical functors give a unifying framework for handling different types of geometric pairs.

Suppose that X and Y are smooth and compact **nc**-spaces, Y is an **nc**-Calabi-Yau, $f : X \to Y$ is a spherical map, and the degeneration conjecture holds for both X and Y. In this situation we expect that the deformation theory of $f : X \to Y$ is controlled by a homotopy abelian d($\mathbb{Z}/2$)g Lie algebra which is L_∞-quasi-isomorphic to

$$\operatorname{cone}\left(C_\bullet(Y) \xrightarrow{f^!} C_\bullet(X) \right)[1-d]. \tag{4.4.1}$$

Moreover, using f_* (or $f^!$) we can build a new **nc**-space Z by taking C_Z to be the semi-orthogonal extension $C_Z = \langle C_X, C_Y \rangle$, where we set

$$\mathrm{Hom}_Z\left(C_Y, C_X\right) := 0$$

$$\mathrm{Hom}_Z\left(\mathcal{E}, \mathcal{F}\right) := \mathrm{Hom}_Y\left(f_*\mathcal{E}, \mathcal{F}\right) \quad \text{for all } \mathcal{E} \in C_X, \mathcal{F} \in C_Y.$$

We expect that the deformation theory of $f : X \to Y$ is equivalent to the deformation theory of Z and in particular that the L_∞ algebra $C^\bullet(Z)$ is quasi-isomorphic to the algebra (4.4.1).

REMARK 4.38. We should point out that even though deformation quantization provides a conceptual bridge between the categorical framework and the geometric framework of the previous section, the actual connection between the two frameworks is tenuous at best. The source of the problem lies in the fact that the deformation quantization of general Poisson maps can be obstructed [**Wil07**].

4.5. *A*-model framework: symplectic Landau-Ginzburg models. We already noted in Examples 4.4 and 4.8 that there are natural canonical coordinates and a Calabi-Yau variation of **nc**-Hodge structures that one can attach to the A-model on a compact symplectic manifold. An interesting open problem is to find an algebraic description of these coordinates and variations in terms of some d($\mathbb{Z}/2$)g Batalin-Vilkovisky algebra that is naturally attached to the Fukaya category. This question is hard and we will not study it directly here. Instead we will look at the question of finding canonical coordinates and variations in another symplectic context, i.e. for symplectic Landau-Ginzburg models, and try to get an insight into a possible algebraic formulation in that case. It will be interesting to compare our formalism with the recent work of Fan-Jarvis-Ruan [**FJR07**] on the symplectic geometry of quasi-homogeneous Landau-Ginzburg potentials with isolated singularities but at the moment we do not see a direct relationship.

4.5.1. ***Symplectic geometry with potentials.*** The objects we would like to understand are triples (Y, w, ω), where

- Y is a C^∞-manifold and ω is a C^∞-symplectic form on Y.
- $\mathsf{w} : Y \to \mathbb{C}$ is a proper C^∞-map such that there exists an $R > 0$ so that over $\{z \in \mathbb{C} | \, |z| \geq R\}$ the map w is a smooth fibration with fibers symplectic submanifolds in (Y, ω).

Similarly to the case of compact symplectic manifolds one can associate Gromov-Witten invariants to such a geometry. Specifically, if we fix $n \geq 1$, $g \geq 0$, and $\beta \in H_2(Y, \mathbb{Z})$, then we can use stable pseudo-holomorphic pointed curves in Y to define a natural linear (correlator) map

$$I^{(1)}_{g,\beta,n-1} : \quad H^\bullet(Y, \mathbb{Q})^{\otimes(n-1)} \otimes H^\bullet\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right) \longrightarrow H^\bullet(Y, \mathbb{Q}).$$

Indeed, note that Poincaré duality gives an identification

$$H^\bullet(Y, \mathbb{Q}) \cong H_\bullet(Y, Y_R; \mathbb{Q})[-\dim Y],$$

where $R > 0$ is as above and $Y_R = \mathsf{w}^{-1}\left(\{z \in \mathbb{C} | \, |z| \geq R\}\right) \subset Y$. Combining this identification with the isomorphism $(H^\bullet)^\vee = H_\bullet$ we see that $I^{(1)}_{g,\beta,n-1}$ will be given by a class in $H_\bullet(Y, \mathbb{Q})^{\otimes(n-1)} \otimes H_\bullet(Y, Y_R; \mathbb{Q}) \otimes H_\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Next consider the usual moduli stack $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ of stable pseudo-holomorphic maps. Here it will be convenient to assume that an almost-complex structure on Y tamed by ω is chosen in such a way that $\mathsf{w}_{|Y_R}$ is holomorphic. The stack $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ is non-compact but near infinity it parameterizes only pseudo-holomorphic maps $\varphi : C \to Y$ such that $\mathsf{w} \circ \varphi : C \to \mathbb{C}$ is constant and $\mathsf{w} \circ \varphi(C) \in \mathbb{C}$ is close to infinity. Thus the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(Y, \beta)$ is well defined as a class in the relative homology

$$\begin{aligned} \left[\overline{\mathcal{M}}_{g,n}(Y, \beta)\right]_{\mathrm{vir}} &\in H_\bullet\left(Y^n \times \overline{\mathcal{M}}_{g,n}, Y^{n-1} \times Y_R \times \overline{\mathcal{M}}_{g,n}; \mathbb{Q}\right) \\ &= H_\bullet(Y, \mathbb{Q})^{\otimes(n-1)} \otimes H_\bullet(Y, Y_R; \mathbb{Q}) \otimes H_\bullet(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}). \end{aligned}$$

We define $I^{(1)}_{g,\beta,n-1}$ to be the map given by the relative virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(Y,\beta)]_{\text{vir}}$.

This collection of correlators satisfies analogues of the usual axioms of a cohomological field theory [**KM94**] but we will not discuss them here. Consider now a cohomology class

$$x = (x_2, x_{\neq 2}) \in H^{\bullet}(Y,\mathbb{C}) = H^2(Y,\mathbb{C}) \oplus H^{\neq 2}(Y,\mathbb{C}),$$

where $H^{\bullet}(Y,\mathbb{C})$ is viewed as a supermanifold over $\mathbb{C}$.

Now for every such x we define a quantum product

$$\bullet *_x \bullet : \quad H^{\bullet}(Y,\mathbb{C}) \otimes H^{\bullet}(Y,\mathbb{C}) \longrightarrow H^{\bullet}(Y,\mathbb{C})$$

by the formula

$$\begin{aligned} \alpha_1 *_x \alpha_2 \quad := & \sum_{m\geq 0} \sum_{\beta \in H_2(Y,\mathbb{Z})} \exp\left(\langle \beta, x_2\rangle\right) \\ & \cdot \frac{1}{m!} I^{(1)}_{g,\beta,m+1}\big((\alpha_1 \otimes \alpha_2 \otimes \underbrace{x_{\neq 2} \otimes \cdots \otimes x_{\neq 2}}_{m \text{ times}}) \otimes 1_{\overline{\mathcal{M}}_{0,m+1}}\big). \end{aligned}$$

Now this quantum multiplication together with the usual formulas (see Examples 4.4 and 4.8) can be used to define a decorated variation of **nc**-Hodge structures over the (conjecturally non-empty) domain in $H^{\bullet}(Y,\mathbb{C})$ where the series defining $*_x$ is absolutely convergent.

REMARK 4.39. There are some interesting variants of this construction. For instance we can take a symplectic manifold (Y,ω) with no potential and a pseudoconvex boundary. In this situation $\overline{\mathscr{M}}_{g,n}(Y,\beta)$ is already compact, as long as $\beta \neq 0$. Also, in a symplectic Landau-Ginzburg model (Y,ω,w) we can allow for w to be non-proper and instead require that its fibers have pseudoconvex boundary. Finally one can consider a symplectic Y equipped with a proper map $Y \to \mathbb{C}^k$, holomorphic at infinity and with $k \geq 2$.

4.5.2. ***Categories of branes.*** Let (Y,ω,w) be a symplectic geometry with a proper potential. There are two natural categories that we can attach to this geometry: the Fukaya category of the general fiber of w, and the Fukaya-Seidel category of w. Understanding the structural properties of these categories or even defining them properly is a difficult task which requires a lot of effort and hard work. We will not explain any of these intricate details but will rather use the Fukaya and Fukaya-Seidel categories as conceptual entities. For details of the definitions and a rigorous development of the theory we refer the reader to the main sources [**FOOO07, FO01**], [**Sei07b, Sei07a**]. The categories that we are interested in are:

(1) The Fukaya-Seidel category $\mathsf{FS}(Y,\omega,\mathsf{w})$ of the potential w has objects which are unitary local systems $\mathbb{V}$ on (graded) ω-Lagrangian submanifolds $L \subset Y$ such that:

- $\mathsf{w}(L) \subset (\text{compact}) \cup \mathbb{R}_{\leq 0}$;
- The restriction of L over the ray $\mathbb{R}_{\leq 0}$ is a fibration on $\mathbb{R}_{\leq -R}$ and when $z \in \mathbb{R}_{\leq 0}$, and $z \to -\infty$, we have that the fiber $L_z \subset Y_z$ is a Lagrangian submanifold in the symplectic manifold $(Y_z, \omega_{|Y_z})$.

The morphisms between two objects $(L_1, \mathbb{V}_1)$ and $(L_2, \mathbb{V}_2)$ are defined as homomorphisms between the fibers of the local systems at the intersection points of the two Lagrangians. As usual, to make this work one has to perturb one of the Lagrangians, say L_2, by a Hamiltonian isotopy to ensure transversality of the intersection. A new feature of this setup (compared to the situation of symplectic manifolds with no potential) is that the allowable isotopies are tightly controlled – they correspond to small wiggling, see Figure 7, of the tail of the tadpole-like image of the Lagrangian in $\mathbb{C}$ and a lift of this wiggling to Y given by a non-linear symplectic connection identifying the fibers of Y.

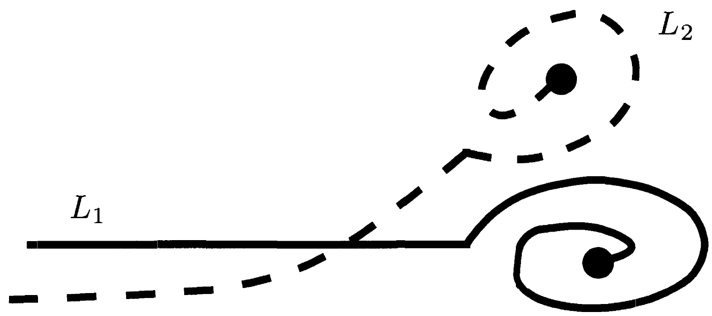


FIGURE 7. Tadpole-like w-images of two Lagrangian submanifolds.

The compositions of morphisms are given by correlators counting pseudo-holomorphic discs whose boundary is contained in the given Lagrangian submanifolds.

(2) The Fukaya category $\mathsf{Fuk}(Y_z)$ of a fiber $\left(Y_z, \omega_{|Y_z}\right)$ over a point $z \in \mathbb{C}$ which is not a critical value for w. The objects in this category are again pairs consisting of (graded) Lagrangian submanifolds in Y_z equipped with unitary local systems, and morphisms and compositions are defined again by maps between the fibers of the local systems at the intersection points and by counting discs. The parallel transport w.r.t. a non-linear symplectic connection on $\mathsf{w} : Y \to \mathbb{C}$ identifies symplectically all fibers $\left(Y_z, \omega_{|Y_z}\right)$ over points $z \in \mathbb{R}_{\leq 0}$ when $z \to -\infty$. We will denote any one such fiber by $(Y_{-\infty}, \omega_{-\infty})$.

Now observe that by intersecting a Lagrangian $L \subset Y$ with the fiber $Y_{-\infty}$ we get an assignment $L \mapsto L_{-\infty} := L \cap Y_{-\infty}$. We expect that this assignment can be promoted to a spherical functor (see also [**Sei07a**] for a similar discussion)

$$F: \quad \mathsf{FS}(Y, \mathsf{w}, \omega) \longrightarrow \mathsf{Fuk}(Y_{-\infty}, \omega_{-\infty})$$

so that the associated spherical twist $\mathcal{R}_F : \mathsf{Fuk}(Y_{-\infty}, \omega_{-\infty}) \to \mathsf{Fuk}(Y_{-\infty}, \omega_{-\infty})$ categorifies the monodromy around the circle $\{z \in \mathbb{C} | \, |z| = R\}$.

In this situation one can also define relative Gromov-Witten invariants

$$J^{(1)}_{g,\beta,n-2} : \quad H^\bullet(Y, Y_{-\infty}; \mathbb{Q}) \otimes H^\bullet(Y, \mathbb{Q})^{\otimes(n-2)} \otimes H^\bullet\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right) \longrightarrow H^\bullet(Y, Y_{-\infty}; \mathbb{Q}).$$

For this we again use the duality $(H^\bullet)^\vee \cong H_\bullet$ and the Poincaré duality $H^\bullet(Y, Y_{-\infty}; \mathbb{Q}) \cong H_\bullet(Y, Y_{+\infty}; \mathbb{Q})$ to rewrite $J^{(1)}_{g,\beta,n-2}$ as a class in

$$H_\bullet(Y, Y_{-\infty}; \mathbb{Q}) \otimes H_\bullet(Y, Y_{+\infty}; \mathbb{Q}) \otimes H_\bullet(Y, \mathbb{Q})^{\otimes(n-2)} \otimes H_\bullet\left(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}\right).$$

This class can again be defined as a virtual fundamental class space $\overline{\mathcal{M}}_{g,n}(Y,\beta)$ of stable pseudo-holomorphic maps. Again we can interpret the virtual class as a relative homology class:

$$\begin{aligned}\left[\overline{\mathcal{M}}_{g,n}(Y,\beta)\right]_{\text{vir}} \in H_\bullet \left(Y^n \times \overline{\mathcal{M}}_{g,n},\, Y^{n-2} \times \left((Y^-_{R,\varepsilon} \times Y) \cup (Y \times Y^+_{R,\varepsilon})\right) \times \overline{\mathcal{M}}_{g,n}; \mathbb{Q}\right) \\ = H_\bullet(Y, Y^-_{R,\varepsilon}; \mathbb{Q}) \otimes H_\bullet(Y, Y^+_{R,\varepsilon}; \mathbb{Q}) \otimes H_\bullet(Y,\mathbb{Q})^{\otimes(n-2)} \otimes H_\bullet\left(\overline{\mathcal{M}}_{g,n},\mathbb{Q}\right) \\ = H_\bullet(Y, Y_{-\infty}; \mathbb{Q}) \otimes H_\bullet(Y, Y_{+\infty}; \mathbb{Q}) \otimes H_\bullet(Y,\mathbb{Q})^{\otimes(n-2)} \otimes H_\bullet\left(\overline{\mathcal{M}}_{g,n},\mathbb{Q}\right),\end{aligned}$$

and so it gives the desired map $J^{(1)}_{g,\beta,n-2}$.

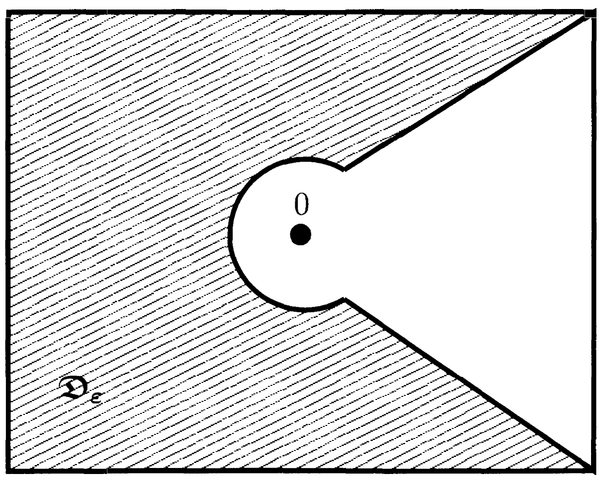


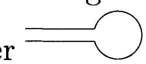
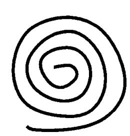
FIGURE 8. The domain $\mathfrak{D}_\varepsilon$.

Here $1 \gg \varepsilon > 0$, and $Y^{\pm}_{R,\varepsilon} = \mathsf{w}^{-1}(\pm\mathfrak{D}_\varepsilon)$, where $\mathfrak{D}_\varepsilon \subset \mathbb{C}$ is the domain given by (see Figure 8)

$$\mathfrak{D}_\varepsilon := \left\{ z \in \mathbb{C} \;\middle|\; |z| \geq R \text{ and } \operatorname{Arg} z \in \left(\frac{\pi}{2} - \varepsilon, \frac{3\pi}{2} + \varepsilon\right) \right\}.$$

Again the relative invariants $J^{(1)}_{g,\beta,n-2}$ give rise to a quantum multiplication and through the usual formulas from Examples 4.4 and 4.8 we again get a decorated variation of **nc**-Hodge structures over a (conjecturally non-empty) domain in $H^\bullet(Y,\mathbb{C})$ with fiber $H^\bullet(Y, Y_{-\infty})$.

4.5.3. ***Mirror symmetry.*** In conclusion we systematize (see also [**Aur07**]) all the objects introduced above in a mirror table describing the corresponding A and B-model entities in parallel:

invariants	A-model	B-model
geometry	a triple (Y, w, ω) where: $\mathsf{w} : Y \to \mathbb{C}$ is a proper C^∞-map, and (Y, ω) is symplectic with $c_1(T_Y) = 0$	a pair $Z \subset X$ where: X is smooth projective, and $Z \subset X$ is a smooth anticanonical divisor
cohomology	$\left.\begin{array}{l} H^\bullet(Y, \mathbb{C}) \\ H^\bullet(Y, Y_{-\infty}; \mathbb{C}) \\ H^\bullet(Y_{-\infty}, \mathbb{C}) \end{array}\right\}$ variations of **nc**HS over a domain in $H^\bullet(Y, \mathbb{C})$	$\left.\begin{array}{l} H^\bullet(X-Z, \mathbb{C}) \\ H^\bullet(X, \mathbb{C}) \\ H^\bullet(Z, \mathbb{C}) \end{array}\right\}$ variations of **nc**HS over a domain in $H^\bullet(X-Z, \mathbb{C})$
categories	$\left.\begin{array}{c} \mathsf{Fuk}(Y_{-\infty}) \\ \Big\uparrow{\scriptstyle F} \\ \mathsf{FS}(Y) \end{array}\right\}$ $\mathsf{Fuk}(Y_{-\infty})$ is a CY category and F is a spherical functor	$\left.\begin{array}{c} D(Z) \\ \Big\uparrow{\scriptstyle F} \\ D(X) \end{array}\right\}$ $D(Z)$ is a CY category and F is a spherical functor
	The part of $\mathsf{FS}(Y)$ consisting of Lagrangians fibered over  where the circle is of radius $R \gg 0$.	$D_{\mathrm{supp}\, Z}(X) \left\{ \begin{array}{l} \text{a full compact} \\ \text{(non smooth)} \\ \text{subcategory in} \\ D(X) \end{array}\right.$
	The part of $\mathsf{FS}(Y)$ consisting of compact Lagrangian sumanifolds in Y.	$D_{\substack{\text{compact} \\ \text{support}}}(X - Z) \left\{ \begin{array}{l} \text{a full compact (non} \\ \text{smooth) subcategory in} \\ D(X) \end{array}\right.$
	The wrapped FS category: the Hom space between $(L_1, \mathbb{V}_1)$ and $(L_2, \mathbb{V}_2)$ is the sum of $\mathrm{Hom}(\mathbb{V}_1, \mathbb{V}_2)_x$, $x \in L_1 \cap L_2$, and L_2 is deformed so that $\mathsf{w}(L_2)$ becomes a spiral:  wrapped infinitely many times	$D(X - Z) \left\{ \begin{array}{l} \text{a smooth (non} \\ \text{compact) category} \end{array}\right.$

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