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To whom it may concern

Dear Colleagues,

Attached is my review of the dissertation of Prof. **Ludmil Katzarkov**. It was a pleasure to assist you in this matter.

Sincerely yours,

Yuri Tschinkel
Professor of Mathematics

PEER REVIEW

on the Thesis Submitted for Awarding the Scientific Degree "Doctor of Sciences" in the Professional Field 4.5. Mathematics Scientific Specialty "Geometry and Topology"

Author of the thesis: Ludmil Katzarkov, Ph.D.

Title of the thesis: "Symplectic Topology, Non-commutative Geometry, and Mirror Symmetry"

Member of the Scientific Jury: Yuri Tschinkel, Ph.D., Professor of Mathematics, Courant Institute, NYU, member Leopoldina, German National Academy of Sciences

The thesis covers a wide range of topics at the interface of symplectic topology and algebraic geometry. These are established and mature fields that have experienced tremendous growth in the last several years, inspired by developments in a branch of mathematical physics, string theory. These developments suggested a mysterious duality, the so called *Mirror symmetry*, between symplectic and algebraic invariants of geometric objects. The initial formulation of the duality has undergone several enhancements; the author has been one of the principal architects of this theory. The main contributions of this thesis are: an explicit description of this duality in many situations and applications of the new structures to a classical problem in algebraic geometry, the rationality problem.

The thesis is written in English on 343 pages. It contains a short introduction on 2 pages, two chapters with three, respectively, two sections with main results and constructions, a conclusion with a description of future directions, and 16 pages of references, including publications of the author. Throughout the text, the author formulates many questions and conjectures.

1. Timeliness of the problems studied in the thesis. The theory of Gromov-Witten invariants and its incarnations in the language of derived categories are central subjects in modern geometry, intensely studied by cohorts of researchers worldwide. The initial discovery of Mirror Symmetry, making unexpected predictions about rational curves on Calabi-Yau varieties via hypergeometric functions computed from the periods of the variety, has stimulated intense efforts which led to the creation of a completely new framework describing basic geometries. This framework is now known as *Homological Mirror Symmetry* (HMS).

The main thrust of this thesis lies in applications of HMS to classical problems in

algebraic geometry concerning rationality of algebraic varieties, in particular, of low-degree hypersurfaces in projective space. The technical core of this work is the development of enhancements of Hodge theory. These enhancements reflect simultaneously symplectic and algebraic aspects, and a substantial part of the thesis is devoted to explorations of mutual dependencies between the respective structures. One of the key new definitions is that of a quantum spectrum. A crucial result for applications is the theorem that the spectral decomposition of quantum cohomology is a birational invariant.

The range of methods used is enormous and reflects encyclopedic knowledge of the author, as well as technical strength.

2. Scientific contributions.

The results presented in this thesis are at the cutting edge of modern symplectic and algebraic geometry. They are not only fundamental for our understanding of deep new connections between these fields uncovered by mathematical physicists, but have also the potential to revolutionize the field of birational geometry, leading to proofs of long-standing conjectures, such as the nonrationality of a general cubic fourfold.

The first part is devoted to constructive proofs of various versions of Homological Mirror Symmetry for Fano varieties, e.g., Del Pezzo surfaces, or hypersurfaces in toric varieties, or complete intersections. Generally, HMS predicts an equivalence of certain triangulated categories: the bounded derived catetory of coherent sheaves on the algebraic side and the Fukaya category (based on Lagrangian Floer theory), on the symplectic side. A crucial insight by the author was that in the cases he studies, the mirror manifold is given as a *Landau-Ginzburg model*, i.e., a fibration over the affine line. The corresponding structures are beautifully explicit. On the algebraic side of the mirror correspondence, the author gives several descriptions of the derived category of coherent sheaves (via exceptional sequences, or as derived categories of DG-algebras); on the symplectic side, the author observes that the Fukaya category (which is not yet defined in full generality) has a much simpler (but still, combinatorially highly nontrivial!) shape in the cases at hand, namely as the derived category of Lagrangian vanishing cycles of the Landau-Ginzburg fibration. This first part contains beautiful examples, an overwhelming amount of explicit computations, e.g., with theta-functions, asymptotic analysis, homological algebra...

Armed with a plethora of examples, the author proceeds in the second part with general categorical investigations. He develops the abstract theory of noncommutative Hodge structures and analyzes various filtrations on them. He shows the appearance of such structures on both the algebraic and symplectic sides of HMS. He studies global and local deformations of these structures in the context of different geometries of interest, and establishes analogs of the Tian-Todorov theorem. There is an all-encompassing summary and comparison in Section

4.5.3., passing from geometry to a matching of categories. The thesis ends with the study of *spectra*, giving rise to remarkable filtrations of derived categories. These filtrations are labelled by numbers arising as asymptotics of limiting stability conditions, or equivalently, asymptotics of a certain PDE on quantum cohomology. The main theorem that this filtration is a birational invariant seems miraculous to me. Even more astonishing are other related spectra, e.g., the spectrum of the Landau-Ginzburg model, noncommutative spectrum, etc. The rationality results derived from these considerations, Theorems 2.14 and 2.15, state nonrationality of smooth hypersurfaces in projective space of small degree. In my opinion, this is the biggest breakthrough in birational geometry since the work of Clemens-Griffiths, Manin-Iskovskikh, and Artin-Mumford in the 1970s, who proved the existence of unirational but nonrational threefolds.

3. Analysis of the publications on the thesis. The list of publications forming the basis of this thesis has 10 items. All of these papers have appeared in the very top international journals: *Inventiones Math.*, *Publications IHES*, *Annals of Math.*, *Journal of the American Math. Society*, *Topology*. These papers have exceptionally high citation numbers, and have been very influential. An absolute gem is the joint paper with Kontsevich and Pantev, on Hodge-theoretic aspects of mirror symmetry, cited 132 times. Overall, there are 475 citation to these 10 papers, an impressive accomplishment. The author has joint papers with Fields medalists (Kontsevich, Donaldson), as well as young and less established mathematicians (Ballard, Favero, Efimov). According to *Mathscinet*, the author has published joint papers with 73 mathematicians, which is ample proof of his leadership.

4. Approbation of the results. The results of this thesis have been presented at more than 50 conferences and schools, in many countries. I have personally attended about 20 talks by the author and have followed with great interest the evolution of his ideas.

5. Recommendations and critical remarks. While there are many immediate targets in this research project, the author might consider expanding this text to a monograph, or even a series of volumes, explaining in more detail his ideas and results. As is, the thesis is inspiring for experts, but might seem formidable for outsiders.

With such a scope of work it is perhaps inevitable that there are small issues with spelling and typesetting here and there, misplaced citations, formatting errors in diagrams.

6. The abstract of the thesis and the abstract of the contributions are precise and concise. It briefly describes the historical background, mostly from physics, and the integration of physical ideas into new mathematical theories. it summarizes well the main contributions of the author, from proofs of HMS for rational surfaces and open Calabi-Yau varieties, to new structural results concerning noncommutative enhancements of Hodge theory, leading to the proof of birational invariants of quantum spectra. It concludes with a persuasive list of future directions.

Conclusion.

I strongly recommend to the Scientific Jury to award Professor Ph.D. Ludmil Katzarkov the scientific degree *Doctor of Sciences* in the Professional Direction 4.5., Mathematics.