R E P O R T

by Prof. Gueo Grantcharov
Florida International University
on the Dissertation Thesis
entitled
"Symplectic topology, non-commutative geometry, and mirror symmetry"
by Prof. Ludmil Katzarkov
submitted for obtaining the scientific degree "Doctor of Science"
in the Area of Higher Education: 4. Natural Sciences, Mathematics and Informatics,
Professional Field: 4.5 Mathematics

I am a member of the Scientific Jury for the defense of this dissertation according to the report on Order 550/27.11.2023 of the Director of the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences.

The report is prepared in accordance with the requirements of the Act on Development of the Academic Staff in the Republic of Bulgaria, the Regulations for its application, and the Rules for the conditions and regulations for acquiring scientific degrees and occupying academic positions in the *Bulgarian Academy of Sciences.* From the required documents and papers submitted by *Ludmil Katzarkov,* I have made sure that they and the applicant meet the requirements of Chapter 2 of the Act on Development of the Academic Staff in the Republic of Bulgaria. *Ludmil Katzarkov* holds a PhD degree from University of Pennsylvania, recognized with order No. 000077, issued on October 28, 2019 by the Bulgarian Academy of Sciences, the minimum national requirements for obtaining the scientific degree "Doctor of Science" in the professional field 4.5. Mathematics in accordance with the Regulations for the Application of the Act on Development of the Academic Staff in the Republic of Bulgaria are fulfilled.

Overview

The dissertation has 343 pages and 256 references. It contains an Introduction, two parts with three Chapters in the first and two Chapters in the second part, an Index and List of names. The numbering is separate for each chapter. The dissertation is devoted to the homological mirror symmetry – a correspondence relation between symplectic and algebraic geometry, and its application to noncommutative invariants. The dissertation is theoretical in nature and meets the requirements for the scientific degree "Doctor of Science".

Description of the results

The dissertation contains 2 parts with 3 Chapters in the first part and 2 chapters in the second. The first 2 chapters of Part 1 are devoted to the proofs of the homological mirror symmetry for 3 important examples of Fano manifolds – the weighted projective planes, the Hirzebruch surfaces and the Dell Pezzo surfaces. The homological mirror symmetry (HMS for short) conjecture is formulated by M. Kontsevich following ideas in physics and suggests in general an equivalence between derived category of coherent sheaves of a complex variety and the derived Fukaya category of its symplectic mirror. The mirror for Fano varieties is a Landau-Ginzburg model – and affine symplectic manifold X with a superpotential W – a complex-valued function whose level sets are symplectic. The category of coherent sheaves is well-known in complex analysis and algebraic geometry, while the derived Fukaya category of a Landay-Ginzburg model has as objects vanishing Lagrangian cycles associated to the critical points of W.

The first chapter in part 1 is devoted to proving the homological mirror symmetry conjecture for weighted projective planes and Hirzebruch surfaces, as well as to some of their noncommutative deformations. The main result is Theorem 1.2 which proves the HMS for weighted projective planes $\mathbf{CP^2}(a,b,c)$ which are Fano surfaces with singularities.

In case of $\mathbf{CP^2}(a,b,c)$ the mirror X is the affine hypersurface $\{x^a y^b z^c = 1\}$ in $(C^*)3$ with an exact symplectic form and $W=x+y+z$. The HMS for the Hirzebruch surfaces is reduced to the case $\mathbf{CP^2}(1,1,n)$ in Theorem 2.29 on the complex side and Proposition 5.5 for the symplectic side. The results of this chapter appeared in An*nals of Mathematics.*

In the second chapter of Part 1 the HMS conjecture is proven for the Dell Pezzo surfaces. In case of blow-up of $\mathbf{CP^2}$ at k points ($k<9$), the mirror Landau-Ginzburg model is an elliptic fibration of **C** with k+3 nodal singular fibers. The main result is Theorem 1.5, which shows that the HMS conjecture is valid for some choice of a (complexifed) symplectic form. Then Theorem 1.6 extends the HMS to any noncommutative deformation of the Dell Pezzo surfaces. The results of this chapter are published in *Inventiones Mathematicae.*

In the last chapter of Part 1, the mirror symmetry is approached via Strominger-Yau-Zaslow (SYZ) conjecture. Using ideas from HMS a proposition for SYZ-type mirror pairs for varieties of general type is presented. The original SYZ relates Lagrangian torus fibrations of Calabi-Yau manifolds, which do not exists on manifolds of general type. The suggested correspondence is between A-side and B-side of Landau-Ginzburg models which are defined in Definition 1.2. To formulate the main result of this Section, let V be a toric variety and H is a tropical hypersurface in it. Define X to be the blow-up of $V\times C$ along $H\times 0$. Then the main result is in Theorem 1.5 which states that under some positivity type intersection assumption, for a given such X, there is an explicitly constructed B-side Landau-Ginzburg model which is SYZ-mirror of X. Then Theorem 1.6 extends the result to existence of such B-side Landau-Ginzburg model for H. The B-side model is an open manifold, and in

Theorem 1.7 its usual SYZ-mirror is found, which is an open Calabi-Yau manifold. Results of this Chapter are published in *Publications Mathematiques de l'Institut des Hautes Etudes Scientifiques.*

The second Part of the dissertation develops some aspects of non-commutative geometry. Although seemingly unrelated, it uses many ideas of mirror symmetry. The first Chapter in it is building foundations of non-commutative Hodge theory. It is relatively long and technical, but with a lot of further applications, as it is seen from the list of citations to the original paper by the author and his collaborators. The Chapter starts with the definition on non-commutative Hodge structures (nc-Hodge structures). It incorporates many other notions, and is related through its variations to the usual variations of Hodge structures. The definition is based on a generalization of a correspondence between bundles with meromorphic connections on the punctured affine line A-{0} and particular type of modules over Laurant series with only pole at 0. In particular, there is a faithful functor from the category of usual variations of Hodge structures and the variations of the nc-Hodge structures (Lemma 2.9). The main result of this Chapter is Theorem 2.35 about the gluing of nc-Hodge structures. The second part of this Chapter shows how these structures are related to mirror symmetry. The variations of nc-Hodge structures are considered in depth in the third part of this Chapter, where the properties analogous to the Bogomolov-Tian-Todorov Lemma are provided. The sub-Section 4.5.3 contains a nice summery of the results in the Chapter. The results are published in in *Proceedings of Symposia in Pure Mathematics, vol. 78.*

The last Chapter of the dissertation, which is the second Chapter of the second Part, is based on a new research projects and many of the results are part of different works in progress. It connects various notions of spectrum via categorical approach. The spectrum here is a spectrum of an operator acting on various spaces and categories attached to an algebraic variety – quantum cohomology, Fukaya-Seidel category and others related to topology. In one of the works in progress, it was proven that the spectral decomposition corresponding to the quantum spectrum is birational invariant. In the Chapter, one of the main results is an extension of this result in Theorem 2.14, where a new obstructions for rationality of a Fano variety based on noncommutative spectra are provided. They are tested on some known non-rational examples. The relations with topological invariants, like Alexander polynomials for knots, Seiberg-Witten and Gromov-Witten symplectic invariants for elliptic surfaces, Witten-Reshetikhin-Turaev invariants of 3-manifolds and others, are mentioned in the next sections. The results are published in *Proceedings in Mathematics and Statistics, vol. 409.*

In general, the dissertation shows that the author is a leading expert in an active area of research like HMS.

Approbation of the results:

According to the information submitted by the applicant for the fulfilment of the minimum national requirements in accordance with the Regulations for the Application of the Act on Development of the Academic Staff in the Republic of Bulgaria in the professional field 4.5. Mathematics, in the current procedure for obtaining the scientific degree "Doctor of Science" Ludmil Katzarkov has applied with 10 published papers with 476 citations. Seven of the papers are in Q1 journals and five of them are of highest rank, two are in a conference proceedings.

The dissertation contains the material of papers with numbers 3, 4, 5, 9, 10 in the "List of publications on the dissertation", which are cited 325 times. All papers are in collaboration with different mathematicians, some of them of highest level (recipients of the Fields medal), and I assume that the candidate's contributions are equal to his collaborators.

Critical notes and recommendations

Due to its length of 343 pages the dissertation contains some technical errors, for example:

1. The HMS conjecture is repeated as Conjecture 1.1 in both the first and second chapters of part 1.

2. Some of the references are not updated ([42], [161]) others are incomplete [30], [111],[226].

Additionally, I would note that the last chapter contains main results which are not published or officially announced yet. In connection with this I would recommend providing short proofs.

Quality of the Summary

The Summary contains 14 pages and correctly represents the content of the dissertation, as well as the relevant motivation and future directions of study.

Conclusions

The review of the proposed dissertation thesis, the summary, and the related scientific papers and documents shows that Ludmil Vasilev Katzarkov is a world-leading expert in the field of algebraic geometry with significant personal contributions to the topic of the dissertation. I evaluate the work of the applicant and his dissertation most positively. I strongly recommend to the Scientific Jury to award Ludmil Katzarkov the scientific degree "Doctor of Sciences" in the Professional field 4.5 Mathematics.

January 24, 2023 Member of the Scientific Jury:

(Prof. Gueo Grantcharov)