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SCHUR PROPERTY OF GENERALIZED ORLICZ-LORENTZ SEQUENCE SPACES

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Abstract: In this article we give some sufficient conditions for Musielak-Orlicz function $\varphi = (\varphi_n)$ such that the generalized Orlicz-Lorentz sequence space λ_φ has the Schur property.

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1. Introduction

In this paper all the notations follow that of [1]. We use notations $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ for sets of real, nonnegative real and natural numbers, respectively. Let σ denotes a bijection map on $\mathbb{N}$. If $(X\,, \|.\|)$ be a norm space, the closed unit ball of X is $\{x \in X : \|x\| \leq 1\}$. By $(\mathbb{N}\,, 2^{\mathbb{N}}\,, m)$, we denote the counting measure space. Let $\ell_0 = \ell_0(m)$ be the set of all real sequence. Let $p \in [1, +\infty)$. ℓ_p denotes all $x = (x_n) \in \ell_0$ satisfying $\sum\limits_{n=1}^{\infty} |x_n|^p < \infty$, endowed with the norm $\|x\|_p = \left(\sum\limits_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$. If $x = (x_n) \in \ell_p$ and for every $n \in \mathbb{N}$, $x_n \geq 0$, then $x \in \ell_p^+$. The space ℓ_∞ denotes all $x = (x_n) \in \ell_0$ that x is bounded and endowed with the norm $\|x\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$. For every $x = (x_n) \in \ell_0$ we define its distribution function $\mu_x : [0, \infty) \to \{0, \infty\} \cup \mathbb{N}$ by $\mu_x(\lambda) = m\{n \in \mathbb{N} : |x_n| > \lambda\}$

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and rearrangement $x^* = (x_n^*)$ with $x_n^* = \inf\{\lambda > 0 : \mu_x(\lambda) < n\}$. The sequences $x, y \in \ell_0$ is called equimeasurable if $\mu_x = \mu_y$ on $\mathbb{R}^+$. A sequential Banach space $X = (X, \|.\|)$ is said to be a rearrangement invariant space if $x \in X$, $y \in \ell_0$ and x, y equimeasurable, then $y \in X$ and $\|x\| = \|y\|$. If X is rearrangement invariant sequence space, then $\ell_1 \subseteq X \subseteq \ell_\infty$ (see [3]). The unit ball of each rearrangement invariant sequence space contain a sequence (x_n) if and only if contain $(x_\sigma(n))$, for every bijection map σ.

A function $\varphi : [-\infty, +\infty] \to [0, +\infty]$ is said to be Orlicz function, if φ is a nonzero function that is convex, even, vanishing at zero and left continuous on $(0, \infty)$ and continuous at zero. A sequence $\varphi = (\varphi_n)$ of Orlicz functions is called a Musielak-Orlicz function. Let $b_\varphi^1 = sup\{u \geq 0 : \varphi_n(u) < \infty\}$. We say that a Musielak-Orlicz function $\varphi = (\varphi_n)$ satisfies condition (L_1) if $p_n \geq p_{n+1}$ for all $n \in \mathbb{N}$ and $u \in [0, b_\varphi^1)$, where p_n denotes the right derivative of φ_n.

The sequence space λ_φ, generated by a Musielak-Orlicz function $\varphi = (\varphi_n)$, is the set of all $x = (x_n)$ such that $\varrho_\varphi(x) = \sup\limits_\sigma \sum\limits_{n=1}^{\infty} \varphi_n(\beta x_{\sigma(n)}) < \infty$ for some $\beta > 0$, Which becomes a normed space under the Luxemberg norm $\|x\| = \inf\{\beta > 0 : \varrho_\varphi(\frac{x}{\beta}) \leq 1\}$. λ_φ is a Banach rearrangement space (see [1]).

If $\varrho_\varphi = \sum\limits_{n=1}^{\infty} \varphi_n(x_n)$ for any $x \in \lambda_\varphi$, then λ_φ will be called Musielak-Orlicz space. We say that a Banach space X has the Schur property if every weakly-convergent sequence is norm-convergent. ℓ_1 space has this property (see [1]). In [4] has given conditions for Musielak-Orlicz function φ and Musielak-Orlicz sequence space generated by φ, that this space has Schur property. If $\varrho_\varphi(x) = \sum\limits_{n=1}^{\infty} \varphi_n(x_n^*)$ for any $x \in \lambda_\varphi$, then λ_φ will be called the generalized Orliz-Lorentz space. Some geometric and topological properties of this space have been published in the papers [1] and [5]. Necessary and sufficient condition for λ_φ will be the generalized Orliz-Lorentz space is φ satisfies condition (L_1) (see [1]). In the remain part we will assume that the Musielak-Orlicz φ has this condition.

Let $p \in [1, +\infty)$. if there exist bounded sequence $b = (b_n)$ and $c = (c_n) \in \ell_1^+$ such that $b_n\varphi_n(u) + c_n \geq u^p$, for every $n \in \mathbb{N}$ and $u > 0$, then space λ_φ is isomorphic to a subspace of ℓ_p. Because if $x = (x_n) \in \lambda_\varphi$, then we have $b_n\varphi_n(x_{\sigma(n)}) + c_n \geq \left|x_{\sigma(n)}\right|^p$. Assume that the upper bound of b is M. Thus $\sum\limits_{n=1}^{\infty} \left|\beta x_{\sigma(n)}\right| \leq \sum\limits_{n=1}^{\infty} \left\{M\varphi_n(\beta x_{\sigma(n)}) + c_n\right\} < \infty$. Therefore $x \in \ell_p$. In particular if $p = 1$, then $\lambda_\varphi = \ell_1$.

2. Main Results

Let $\varphi = (\varphi_n)$ be a Musielak-Orlicz function. The main results of this note are as follows.

Theorem 1. *If one of the following condition is satisfied:*

1. *There is $x_1 , x_2 > 0$ such that $x_1 \neq x_2$ and $\varphi_n(x_1) = x_1, \varphi_n(x_2) = x_2$ for every $n \in \mathbb{N}$,*
2. *φ_n is differentiable in point zero and $\varphi'_n(0) \geq 1$ for every $n \in \mathbb{N}$,*

then $\lambda_\varphi = \ell_1$.

To prove these result, we need the following easily verified lemmas.

Lemma 2. *If a real function f is convex on (a , b) and $a < s < t < u < b$, then*

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Lemma 3. *Let a Banach space X be endowed with two norms, denote $X_1 = (X , \|, \|_1)$ and $X_2 = (X , \|.\|_2)$. If the identity map $Id : X_1 \to X_2$ defined by $Id(x) = x$ is continuous, then Id is an isomorphism.*

3. Proofs

Proof of Lemma 3. Since X_1 is Hausdorff, the set $\{(x , x) : x \in X\}$ is closed. Thus linear operator Id has a closed graph. By using closed graph theorem and open mapping theorem, Id is an open map. Also Id is a bijection. Therefore Id is an isomorphism between the spaces X_1 and X_2.

Proof of Theorem 1. In the two cases, we claim that $\varphi_n(x) \geq x$ for any $x > 0$ and $n \in \mathbb{N}$.

Let condition (1) hold. Without loss of generality, we can assume $x_1 < x_2$. Assume that there exists $x_3 > 0$ and $n \in \mathbb{N}$ such that $\varphi_n(x_3) < x_3$. If $x_3 > x_1 > 0$ then there is $\lambda \in (0 , 1)$ as $x_1 = \lambda x_3$. Hence

$$\lambda x_3 = x_1 = \varphi_n(x_1) = \varphi_n(\lambda x_3) \leq \lambda\varphi_n(x_3) < \lambda x_3,$$

which is contradiction. And if $x_3 < x_1$ then by Lemma 2, we would have

$$\frac{\varphi_n(x_1) - \varphi_n(x_3)}{x_1 - x_3} \leq \frac{\varphi_n(x_2) - \varphi_n(x_1)}{x_2 - x_1} = 1.$$

Therefore $\varphi_n(x_3) \geq x_3$, which is impossible.

Now assume condition (2) holds. If $x = (x_n) \in \lambda_\varphi$ then there is $\beta > 0$ that $\varrho_\varphi(\beta x) < \infty$. Assume there is $u_0 > 0$ such that $\varphi_n(\beta u_0) < \beta u_0$. We have

$$\sup_{0<\delta} \inf_{0<\beta u<\delta} \frac{\varphi_n(\beta u)}{\beta u} = \lim_{\beta u \to 0} \frac{\varphi_n(\beta u)}{\beta u} = \varphi_n'(0) \geq 1 > \frac{\varphi_n(\beta u_0)}{\beta u_0}.$$

Then we found $\delta > 0$ that $\frac{\varphi_n(\beta u)}{\beta u} > \frac{\varphi_n(\beta u_0)}{\beta u_0}$ for every $0 < u < \frac{\delta}{\beta}$. So we get

$$\frac{\varphi_n(\beta u) - \varphi_n(\beta u_0)}{\beta u - \beta u_0} > \frac{\varphi_n(\beta u_0)}{\beta u_0}. \tag{1}$$

Define the function $f(u) = \varphi_n(u) - u$ on $\mathbb{R}$. The function f is continuous and $f(\beta u) > 0$, $f(\beta u_0) < 0$. Consequently it has at least one root u_1 in interval $(\beta u\,, \beta u_0)$. We get

$$\frac{\varphi_n(\beta u_0)}{\beta u_0} \geq \frac{\varphi_n(\beta u_0) - \varphi_n(u_1)}{\beta u_0 - u_1}.$$

So by equation (1) we have

$$\frac{\varphi_n(\beta u) - \varphi_n(\beta u_0)}{\beta u - \beta u_0} > \frac{\varphi_n(\beta u_0) - \varphi_n(u_1)}{\beta u_0 - u_1},$$

which is a contradiction with Lemma 2.

So in two cases, we can assume that $\varphi_n(x) \geq x$ for every $x > 0$ and $n \in \mathbb{N}$. Now let $x = (x_n) \in \lambda_\varphi$. Then there is $\beta > 0$ such that $\varrho_\varphi(\beta x) < \infty$. We obtain

$$\sum_{n=1}^{\infty} \beta \left| x_{\sigma(n)} \right| \leq \sum_{n=1}^{\infty} \varphi_n(\beta x_{\sigma(n)}) \leq \sup_\sigma \sum_{n=1}^{\infty} \varphi_n(\beta x_{\sigma(n)}) < \infty.$$

Then $(x_n) \in \ell_1$. Therefore $\lambda_\varphi = \ell_1$.

We prove the identity map $Id : \ell_1 \to \lambda_\varphi$ is continuous. Let $\varepsilon > 0$ be arbitrary. Choose $x \in \lambda_\varphi$ such that $\|x\| < \varepsilon$. By definition, there is $0 < \beta < \varepsilon$ such that $\varrho_\varphi(\frac{x}{\beta}) \leq 1$. We have

$$\sum_{n=1}^{\infty} \left| \frac{x_n}{\beta} \right| \leq \sum_{n=1}^{\infty} \varphi_n \left(\frac{x_n}{\beta} \right) \leq \sup_\sigma \sum_{n=1}^{\infty} \varphi_n \left(\frac{x_{\sigma(n)}}{\beta} \right) = \varrho_\varphi \left(\frac{x}{\beta} \right) \leq 1.$$

Thus $\sum\limits_{n=1}^{\infty} |x_n| \leq \beta < \varepsilon$, that is $\|x\|_1 < \varepsilon$. Therefore Id is continuous.

By using Lemma 3, ℓ_1 and λ_φ are isomorphic.

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