On the p-Rank of the Incidence Matrix of a Projective Hjelmslev Plane

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Abstract. In this paper we estimate the p-rank of the points-by-lines incidence matrix of a projective Hjelmslev plane over a chain ring with 4 or 9 elements. The proof uses a characterization of all divisible arcs in the corresponding projective planes. Furthermore, we prove lower and upper bounds on the p-rank of the incidence matrix of the projective Hjelmslev plane over an arbitrary finite chain ring of nilpotency index 2.

Keywords: projective Hjelmslev plane · finite chain ring · incidence matrix · p-rank

1 Preliminaries

In this paper we shall use the basic definitions and notations from [1,2,4,5].

A set of points X in $\mathrm{PG}(2,q)$ or $\mathrm{PHG}(2,R)$, $R/\mathrm{Rad}\,R \cong \mathbb{F}_q$, is said to be linearly independent if there exists an arc $\mathcal{K}$ with support $\mathrm{Supp}\,\mathcal{K} \subseteq X$ such that every line has multiplicity $0 \mod p$, i.e. for every line L it holds $\mathcal{K}(L) \equiv 0 \pmod p$. If X is a linearly independent set of points in a finite plane ($\mathrm{PG}(2,q)$ or $\mathrm{PHG}(2,R)$), and A is the incidence matrix of that plane then we have the following inequality

$$\mathrm{rk}\,A \geq |X|. \tag{1}$$

It is known that the rank of the points-by-lines incidence matrix of $\mathrm{PG}(2,q)$, $q = p^h$, is

$$\left(\frac{p(p+1)}{2}\right)^h + 1.$$

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The incidence matrix of $\mathrm{AG}(2,q)$ has rank

$$\left(\frac{p(p+1)}{2}\right)^h.$$

In the case $q=p$, the rank of $\mathrm{PG}(2,p)$ (resp. $\mathrm{AG}(2,p)$) is $\binom{p+1}{2}+1$, resp. $\binom{p+1}{2}$. It is known that the neighbor classes of points in $\mathrm{PHG}(2,R)$, are affine planes of order q [2]. This implies the following lemma.

Lemma 1. *Let R be a finite chain ring with, $|R|=q^2$ and $R/\operatorname{Rad}R\cong\mathbb{F}_q$. If X is a linearly independent set of points in $\mathrm{PHG}(2,R)$ then for every neighbor class of points $[P]$, it holds*

$$|X\cap[P]|\le\left(\frac{p(p+1)}{2}\right)^h.$$

Corollary 2. *(i) If $|R|=4$ then $|X\cap[P]|\le 3$.*
(ii) If $|R|=9$ then $|X\cap[P]|\le 6$.

Theorem 3. *Let A be the incidence matrix points-by-lines of the projective plane $\mathrm{PHG}(2,R)$, where $R/\operatorname{Rad}R\cong\mathbb{F}_q$. Then*

$$\mathrm{rk}\,(A)\le\left(\frac{p(p+1)}{2}\right)^h(q^2+q+1).$$

Proof. The theorem follows immediately by Lemma 1.

An arc in which every line has multiplicity $c \mod p$ is called a $(c \mod p)$-arc.

Lemma 4. *Let Y be linearly independent set of points in $\mathrm{PHG}(2,R)$, and let $\mathcal{K}$ be a $(0 \mod p)$-arc with $\operatorname{Supp}\mathcal{K}\subseteq Y$. Then there exist a constant c such that for every neighbor class of points $[P]$, it holds*

$$\mathcal{K}[P]\equiv c \pmod p.$$

Proof. Let $[L]$ be a neighbor class of points, and let $[P_i]$, $i=0,\dots,q$, be the neighbor classes of points incident with this class of lines. Let $L_1\in[L]$ and denote by $L_1,\dots,L_q$ all lines that contain the line segment $L_1\cap P_0$.

Set

$$x_{ij}=\mathcal{K}([P_i]\cap L_j), i=0,\dots,q, i=1,\dots,q.$$

Counting the multilicities of the points through the segment $[P_0]\cap L_1$, one gets

$$\begin{aligned}
x_{01}+x_{11}+x_{21}+\dots+x_{q1}&\equiv 0 \pmod p\\
x_{01}+x_{12}+x_{22}+\dots+x_{q2}&\equiv 0 \pmod p\\
&\ \vdots\\
x_{01}+x_{1q}+x_{2q}+\dots+x_{qq}&\equiv 0 \pmod p
\end{aligned}$$

This implies

$$\mathcal{K}[P_1] + \mathcal{K}[P_2] + \cdots + \mathcal{K}[P_q] \equiv 0 \pmod{p}.$$

Thus we have that

$$\mathcal{K}[L] - \mathcal{K}[P_j] \equiv 0 \pmod{p}, \tag{2}$$

for every $j = 0, 1, \ldots, q$. This in turn implies

$$\mathcal{K}[P_0] \equiv \mathcal{K}[P_1] \equiv \cdots \equiv \mathcal{K}[P_q] \pmod{p}.$$

Theorem 5. *Let R be a finite chain ring with $|R| = q^2$, $R/\mathrm{Rad}\, R \cong \mathbb{F}_q$, $q = p^h$, where $p \geq 3$. Denote by A the points-by-lines incidence matrix of* $\mathrm{PHG}(2, R)$. *Then*

$$\mathrm{rk}_p\, A \geq \binom{p+1}{2}^h (q+1) + 2q^2 - 1.$$

Proof. Define the pointset X as follows: select a line class $[L]$, i.e. a line in the factor geometry, and sets of $\binom{p+1}{2}^h$ points in each point class on $[L]$. These points in each of the classes on $[L]$ should form an independent set (i.e. an independent set in $\mathrm{AG}(2, q)$). Further select a point in the point class $[P_0] \notin [L]$ and two points in each of the remaining point classes not on $[L]$ or different from $[P_0]$. There are no restrictions on the point in $[P_0]$ and the remaining $2(q^2 - 1)$ points are selected in the following way.

Let $[L']$ be any line class through $[P_0]$. Denote by $[P_i]$, $i = 0, \ldots, q-1$, the point classes on $[L']$. Consider a line segment in $[L] \cap [L']$ that has the direction of L' and denote by L'_i, $i = 1, \ldots, q$, the lines through this segment. Now the two points of X in each of the point classes $[P_i]$, $i = 1, \ldots, q-1$, are selected so that $[P_i]$ contains points incident with L'_i and L'_{i+1}. These two points from $X \cap [[P_i]$ are denoted by P'_i and P''_i. The same selection is made for the points in all line classes through $[P_0]$ (see Fig. 1).

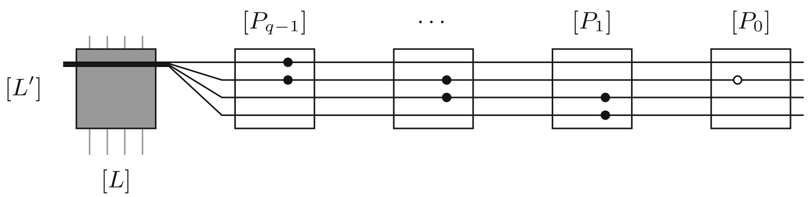

Fig. 1. A neighborclass of lines in $\mathrm{PG}(2, 2)$

Hence by construction we have that

$$|X| = \binom{p+1}{2}^h (q+1) + 2q^2 - 1.$$

We are going to prove that X is an independent set. It is enough to demonstrate that every $(0 \mod p)$-arc with support contained on X is the trivial zero-arc.

Let $\mathcal{K}$ be a $(0 \mod p)$-arc in $\mathrm{PHG}(2,R)$ with $\operatorname{Supp}\mathcal{K} \subseteq X$. By Lemma 4 every point class has the same multiplicity modulo p. Since $p+12 \equiv 0 \pmod p$ we get that every point class has multiplicity $0 \pmod p$. This implies in particular that $\mathcal{K}([P_0]) = 0$.

Furthermore, since $\mathcal{K}(L_i') \equiv 0$, $i = 1, \ldots, q$, we get that

$$\mathcal{K}(P_i') \equiv a \pmod p, \mathcal{K}(P_i'') \equiv b \pmod p,$$

for some constants $a, b \in \{0, \ldots, p-1\}$. Moreover we have

$$a \equiv a + b \equiv b \pmod p.$$

This implies that $a \equiv b \equiv 0 \pmod p$, i.e. $a = b = 0$. We have obtained so far that for all points P in $X \setminus [L]$ we have $\mathcal{K}(P) = 0$.

Let $[L]$ contain the points $[Q_0]$, $[Q_1], \ldots, [Q_q]$. Consider one of them, $[Q_0]$ say. Clearly $\mathcal{K}([Q_0]) \equiv 0 \pmod p$. All line segments in $[Q_0]$, except for those contained in one parallel class (the one with the direction of L), have multiplicity $0 \pmod P$. Now an easy counting gives that also the segment in this parallel class have multiplicity $0 \pmod P$. Hence $\mathcal{K}|_{[Q_0]}$ is a $(0 \mod p)$-arc. By the fact that $X \cap [Q_0]$ is an independent set, $\mathcal{K}(P) = 0$ for all points $P \in [Q_0]$. Similarly, $\mathcal{K}(P) = 0$ for all points $P \in [Q_i]$ for all $i = 1, \ldots, q$. Thus $\mathcal{K}$ is the trivial zero arc on X. This implies that X is an independent set and

$$\operatorname{rk}_p A \geq |X| = \binom{p+1}{2}^h (q+1) + 2q^2 - 1.$$

This theorem can be improved slightly by taking suitably a line class with a maximal number of independent points, a line in the affine part containing one point in each neighbour class, and sets of three points (suitably chosen) in each of the remaining point classes.

Theorem 6. *Let R be a finite chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$, $q = p^h$, where $p \geq 3$. Denote by A the points-by-lines incidence matrix of* $\mathrm{PHG}(2,R)$*. Then*

$$\operatorname{rk}_p A \geq \binom{p+1}{2}^h (q+1) + 3q^2 - 2q.$$

2 The Case $|R| = 4$

In this case the point multiplicities are contained in $\{0, 1, \ldots, p-1\} = \{0, 1\}$. If X is a set that supports a $(0 \mod p)$-arc by Lemma 4, all point classes contain even or odd number of points. Therefore we have for all points P either $|X \cap [P]| \in \{0, 2, 4\}$, or else $|X \cap [P]| \in \{1, 3\}$.

We can also make the following observation. If X is a $(0 \mod 2)$-arc and if we replace the intersection of this arc with some point class by its complement in the point class, the result is again a $(0 \mod 2)$-arc. In other words, if X is a $(0 \mod 2)$-arc then

$$(X \setminus (X \cap [P])) \cup ([P] \setminus (X \cap [P]))$$

is again a $(0 \mod 2)$-arc.

Let A_R, where $R = \mathbb{Z}_4$ or $\mathbb{F}_2[u]/(u^2)$, be the rank of the points-by-lines incidence matrix of the plane $\mathrm{PHG}(2, R)$. In both cases, we have (by Theorem 5)

$$\mathrm{rk}\, A_r \geq 12.$$

Denote by V the vector space of all $(0 \mod 2)$-arcs in $\mathrm{PHG}(2, R)$, $|R| = 4$. This vector space can be viewed as a subspace of $\mathbb{F}_2^{28}$. More generally, the vector space of all $(0 \mod p)$-arcs in $\mathrm{PHG}(2, R)$, $|R| = q^2, q = p^h$, can be viewed as a subspace of $\mathbb{F}_p^{q^2(q^2+q+1)}$ by identifying each arc with its characteristic vector $\boldsymbol{x}$.

If $\boldsymbol{x}$ is the characteristic vector of a $(0 \mod 2)$-arc then $\boldsymbol{x}A_R = \mathbf{0}$ and $\boldsymbol{x}$ is a solution to a homogeneous system of linear equations with a coefficient matrix A_R. Hence

$$\mathrm{rk}_p\, A_R + \dim V = 28.$$

In the general case, we have

$$\mathrm{rk}_p\, A_R + \dim V = q^2(q^2 + q + 1).$$

Note that all $(0 \mod 2)$-arcs that have an even number of points in each point class form a subspace V_0 of V. Now we are going to construct all $(0 \mod 2)$ arcs from V_0.

(1) Every neighbor class contains 0 or 4 points. The number of such arcs is 2^7.
(2) Every neighbor class with 2 points, where the two points in every class determine all possible directions (i.e. they determine a all lines in the factor geometry $\mathrm{PG}(2, q)$). Two points P, Q are said to determine the line class $[L]$ if $\langle P, Q \rangle \in [L]$. If B is the incidence matrix of $\mathrm{PG}(2, 2)$ then the number of such arcs is $\mathrm{per}(B) \cdot 2^7$, i.e. this number is $24 \cdot 2^7$. Here $\mathrm{per}(B)$ is the permanent of the of B.
(3) Four classes have 0 or four points and three classes with two points. The classes with two point should form a triangle. The two points in each of these classes should determine directions that point at the nucleus. Since the number of triangles is $\dfrac{7 \cdot 6 \cdot 4}{3!} = 28$, the total number of such arcs is $28 \cdot 2^7$ (see Fig. 2).
(4) Three classes have 0 or four points and the remaining four classes have two points. The classes with 2-points form a hyperoval and the 0/4-classes are collinear. The pairs of points in each of the 2-point classes determine a line which points at the same point class on the line of 0/4-points (see Fig. 3). Altogether we have 7 choices for the 0/4 line and 3 possibilities for the point on it. So, altogether we have $21 \cdot 2^7$ such arcs.

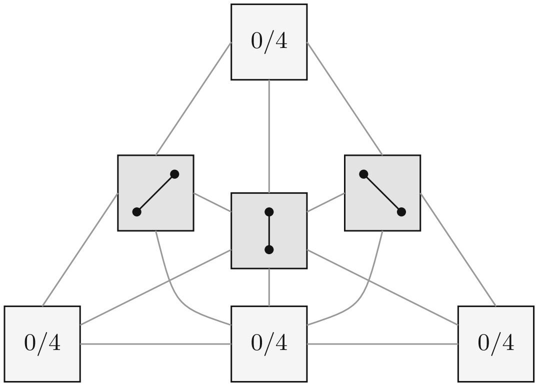

Fig. 2. An arc of type (3) in PG(2, 2)

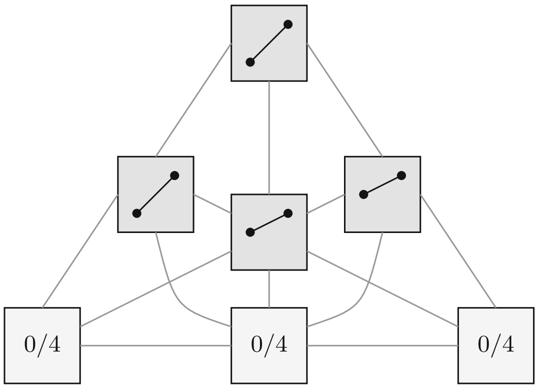

Fig. 3. An arc of type (4) in PG(2, 2)

(5) Two classes have 0 or 4 points, and the remaining five classes have 2 points. The two points in the class which is the third point on the line $[L]$ defined by the 0/4-point classes define a line in the class $[L]$. In the remaining four 2-point classes the directions are as introduced on Fig. 4. The point classes $[Q_1]$ and $[Q_2]$ can be selected in $\binom{7}{2} = 21$ ways. Furthermore, the two point classes with two points that define a line pointing at $[Q_i]$, $i = 1, 2$, can be selected in four ways. Hence the number of such arcs is equal to $84 \cdot 2^7$.

(6) One class with 0/4 points (see Fig. 5).
Total number of arcs: $98 \cdot 2^7$

Summing up, the total number of $(0 \mod 2)$ arcs with an even number of points in each point class is:

$$2^7 + 24 \cdot 2^7 + 28 \cdot 2^7 + 21 \cdot 2^7 + 84 \cdot 2^7 + 98 \cdot 2^7 = 256 \cdot 2^7 = 2^{15}.$$

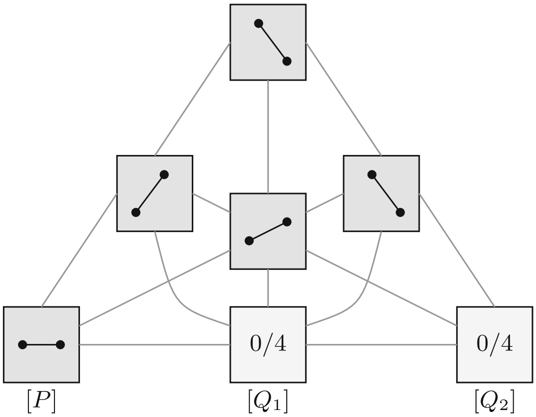

Fig. 4. An arc of type (5) in PG(2, 2)

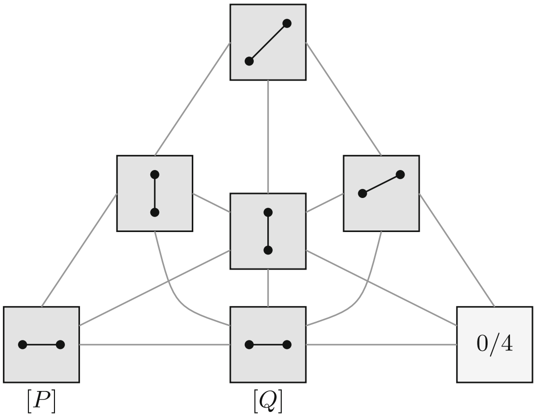

Fig. 5. An arc of type (6) in PG(2, 2)

Hence $\dim V_0 = 15$.

Now it remains to count the number of all $(0 \mod 2)$-arcs with an odd number of points in each neighbor class of points.

In the case when $R = \mathbb{F}_2[u]/(u^2)$ there exist no hyperovals and hence no $(0 \mod 2)$-arcs with an odd number of points in each class. Hence $\dim V = 15$.

In the case $R = \mathbb{Z}_4$ there exist $(0 \mod 2)$-arcs that have an odd number of points in each point class [3]. If we select arbitrarily four points in general position in the point classes $[U]$, $[U_0]$, $[U_1]$, $[U_2]$, where

$$U = (1, 1, 1), U_0 = (1, 0, 0), U_1 = (0, 1, 0), U_2 = (0, 0, 1),$$

then they can be extended uniquely to a hyperoval. Therefore the number of the $(0 \mod 2)$-arcs with an odd number of points in each neighbor class is $4^4 \cdot 2^7 = 2^{15}$. Thus in the case $R = \mathbb{Z}_4$, we get that $\dim V = 16$.

Thus we have proved the following theorem.

Theorem 7.

$$\mathrm{rk}_2\,(A_R) = \begin{cases} 12 & \textit{if } R = \mathbb{Z}_4, \\ 13 & \textit{if } R = \mathbb{F}_2[u]/(u^2). \end{cases}$$

3 The Case $|R| = 9$

Since $\mathrm{rk}_3\, A_{\mathrm{AG}(2,3)} = 6$, we have the upper bound

$$\mathrm{rk}_3\, A_{\mathrm{PHG}(2,R)} \leq 6 \cdot 13 = 78.$$

On the other hand, by Theorem 6 we get also a lower bound, which gives altogether

$$45 \leq \mathrm{rk}_3\, A_{\mathrm{PHG}(2,R)} \leq 76.$$

Lemma 8. *Let R be a chain ring with $|R| = 9$. Consider two lines $[L_1]$ and $[L_2]$ in the factor geometry of* $\mathrm{PHG}(2, R)$*. Let X be an arbitrary point set containing six points in each of the point classes in $[L_1]$ and $[L_2]$ but not in the point class $[L_1] \cap [L_2]$. Then X is a linearly dependent set.*

Proof. Denote the nonempty point classes on $[L_1]$ by $[P_1]$, $[P_2]$, and $[P_3]$, and the nonempty point classes on $[L_2]$ by $[Q_1]$, $[Q_2]$, and $[Q_3]$. Without loss of generality $[P_i] \cap X$ and $[Q_j] \cap X$ are independent sets and hence a triangle. Otherwise there is nothing to prove.

Consider any of these six point classes. We can prescribe multiplicities to the points in these sets in such way that all lines in the same direction have the same multiplicity, and the multiplicities in the four directions are either $(1, 1, 1, 0)$, or $(2, 2, 2, 0)$. This can be done in two ways. The two possibilities are presented on Fig. 6.

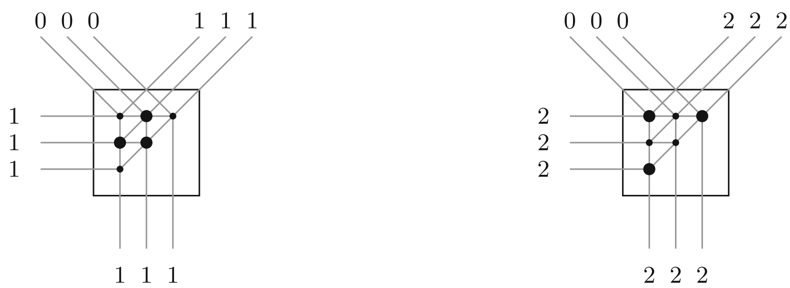

Fig. 6. A neighbor class of points in PG(2, 3)

The same multiplicities are obtained from two line segments with points of multiplicity 1 or 2 (see Fig. 7).

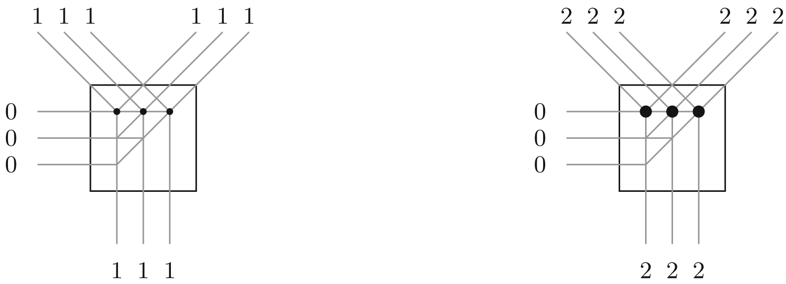

Fig. 7. A neighbor class of points in $\mathrm{PG}(2,3)$ with six non-zero points

Each of the classes $[P_i]$, $[Q_j]$ contains a triangle whose sides (the line segments with three points) determine three directions. We are going to prove that no matter how the directions of the sides of the triangle are selected we can prescribe multiplicities to the points from X in such way that the obtained arc is a (0 mod 3)-arc.

Assume that some class $[P_i]$ (or, $[Q_i]$) has a triple of collinear points that determine the line $\langle [P_i], [R] \rangle$ (or $\langle [Q_j], [R] \rangle$). Then we prescribe to these three points multiplicity 1 if they are in some $[P_i]$, and multiplicity 2 if they are in some $[Q_j]$.

If in some $[P_i]$ (resp., $[Q_j]$) the sides of the triangle determine all directions different from $\langle [P_i], [R] \rangle$ (resp., $\langle [Q_j], [R] \rangle$) then we select the multiplicities in such way that the lines in the direction of the point $[R]$ have multiplicities 0 mod 3 and in all other directions multiplicity $1 \mod 3$ (resp., $2 \mod 3$). Now it is easily checked that all lines have multiplicity $0 \mod 3$ (Fig. 8).

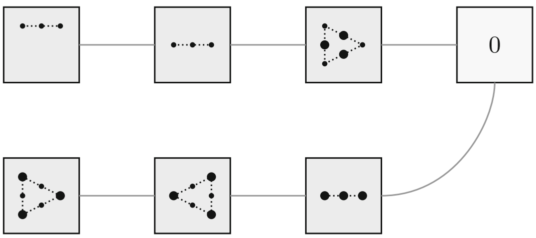

Fig. 8. Two neighbor classes of lines in $\mathrm{PG}(2,3)$ with three non-zero points

Corollary 9. *Let X be a linearly independent set of points. Then at most seven of the point classes can have six points. Consequently for every independent set X, it holds $|X| \leq 72$.*

Proof. If there are eight neighbor classes with six points, we necessarily have two line classes satisfying the conditions of Lemma 8 and hence the set X is dependent. Therefore the number of neighbor classes with 6 points is at most seven and hence

$$|X| \leq 7 \cdot 6 + 6 \cdot 5 = 72.$$

Now by Theorem 6 and Corollary 9

$$45 \leq \mathrm{rk}_3 \, A_{\mathrm{PHG}(2,R)} \leq 72.$$

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