Far East Journal of Mathematical Sciences (FJMS)
Volume 49, Number 1, 2011, Pages 7-20
Published Online: February 25, 2011
This paper is available online at http://pphmj.com/journals/fjms.htm

SOME PROPERTIES OF THE CESÁRO-MUSIELAK-ORLICZ SEQUENCE SPACES

MARYAM BAJALAN and DARYOUSH BEHMARDI

Department of Mathematics
Alzahra University
Vanak, Tehran, Iran
e-mail: behmardi@alzahra.ac.ir

Abstract

In this paper, we define a subspace $\Lambda_p(\varphi)$ of the Cesáro-Musielak-Orlicz sequence space $ces_p(\varphi)$ and show that $\Lambda_p(\varphi)$ is the rearrangement invariant Banach space. Also, we show that $ces_p(\varphi)$ has the property (H), whenever the Musielak-Orlicz function φ satisfies the Δ_2-condition. It is also proved that $ces_p(\varphi)$ has the Fatou-Levy property. Finally, we give the necessary condition such that $ces_p(\varphi)$ is the separable and reflexive space.

0. Preliminaries

For all notations and terms, we refer to [3], [5] and [13]. We denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ for the sets of the natural, real and nonnegative real numbers, respectively. A bijection map σ on $\mathbb{N}$ is called a *permutation*. If $(X, \|\cdot\|)$ is a norm space, then the set $B_X = \{x \in X : \|x\| \leq 1\}$ denotes the unit ball of $(X, \|\cdot\|)$ and the set

2010 Mathematics Subject Classification: 46E30, 46B20, 46A45, 46A80, 46B42, 15A60.

Keywords and phrases: Cesáro-Musielak-Orlicz space, rearrangement invariant Banach space, property (H), Fatou-Levy property.

The second author is supported by Alzahra University.

Received December 7, 2010; Revised January 26, 2011

$S_X = \{x \in X : \| x \| = 1\}$ denotes the unit sphere of $(X, \| \cdot \|)$. By $(\mathbb{N}, 2^{\mathbb{N}}, m)$, we denote the counting measure space. Let ℓ_0 be the space of all real sequences. For every $x = (x_n) \in \ell_0$, we write $| x | = (| x_n |)$. Also we write $| x | \leq | y |$, if $| x_n | \leq | y_n |$ for all $n \in \mathbb{N}$ and define distribution function $\mu_x : [0, \infty) \to \mathbb{N} \cup \{0, \infty\}$ by $\mu_x(\lambda) = m\{n \in N : | x_n | > \lambda\}$ and define decreasing rearrangement $x^* = (x_n^*)$ with $x_n^* = \inf\{\lambda > 0 : \mu_x(\lambda) < n\}$. We refer to [5] to see $x_n^* = \inf_{m(J)<n} \sup_{i \in \mathbb{N} \setminus J} | x_i |$. The sequences $x, y \in \ell_0$ is called *equimeasurable*, if $\mu_x = \mu_y$ on $\mathbb{R}^+$. Let $(X, \| \cdot \|)$ denote a sequential Banach space. The space $(X, \| \cdot \|)$ is said *symmetric*, if for any $x \in X$ and for any arbitrary permutation σ, $x \circ \sigma \in X$. The unit ball of each symmetric space contains x if and only if contains $x \circ \sigma$, for any arbitrary permutation σ. If X is a symmetric space, then $\ell_1 \subseteq X \subseteq \ell_\infty$ (see [6]). The space $(X, \| \cdot \|)$ is called *Banach lattice*, if it satisfies the following two conditions:

(1) If $x \in X$, $y \in \ell_0$ and $|y| \leq |x|$, then $y \in X$ and $\| y \| \leq \| x \|$.

(2) There is $x \in X$ such that $x_n > 0$, for all $n \in \mathbb{N}$.

Also the space $(X, \| \cdot \|)$ is called *rearrangement invariant Banach space*, if it satisfies the following condition:

(1) If $x \in X$, $y \in \ell_0$ and $\mu_y = \mu_x$, then $y \in X$ and $\| y \| = \| x \|$.

It is clear that, $(X, \| \cdot \|)$ is a rearrangement invariant Banach lattice if and only if it satisfies the following condition:

(1) If $x \in X$, $y \in \ell_0$ and $y^* \leq x^*$, then $y \in X$ and $\| y \| \leq \| x \|$.

Every rearrangement invariant sequence space is the symmetric space. If E is a subset of the rearrangement invariant Banach lattice X, then $\overline{E}^X$ is also the rearrangement invariant Banach lattice (see [9, Lemma 4.4]). The rearrangement invariant Banach lattice is useful in study the Interpolation theory (see [1, 9]).

The space $(X, \| \cdot \|)$ is said to have the property (H) (or kadec norm), if weak and norm convergence coincide, for any sequence on the unit sphere X. If $(X, \| \cdot \|)$

has property (H), then the Identity map $Id : (X, \sigma(X, X^*)) \to (X, \|\cdot\|)$ is continuous. Also B_X is weakly closed (see [12, Proposition 4]).

The space $(X, \|\cdot\|)$ has the Fatou-Levy property, if (x_m) is a sequence in X such that $\sup_m \| x_m \| < \infty$ and $0 \le x_m \uparrow x,$ then $x \in X$ and $\| x_m \| \to \| x \|.$

Let $p \in [1, \infty).$ For any $x = (x_n) \in \ell_0,$ we denote $S_n^p(x) = \left(\frac{1}{n}\sum_{i=1}^{n} | x_i |\right)^p.$ A vector space $ces_p,$ defined by

$$ces_p = \left\{x \in \ell_0 : \sum_{n=1}^{\infty} S_n^p(x) < \infty\right\}$$

and equipped with the norm $\| x \|_{ces_p} = \left(\sum_{n=1}^{\infty} S_n^p(x)\right)^{\frac{1}{p}},$ is called the *Cesáro sequence space*. It is known $ces_1 = \{0\}.$ Also it is known ces_p is reflexive and separable Banach space and it contains ℓ_p space, for any $p \in (1, \infty)$ (see [7, 10]). This space has property (H), for any $p \in [1, \infty)$ (see [13]). The Cesáro sequence space is useful in study the Matrix theory (see [8]).

Let X be the real vector space. Then a function $\varrho : X \to R^+$ is called the *convex modular* if it satisfies the following condition:

(1) $\varrho(0) = 0.$

(2) $\varrho(x) = \varrho(-x),$ for any $x \in X.$

(3) $\varrho(\alpha x + \beta y) \le \alpha\varrho(x) + \beta\varrho(x),$ for any $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha + \beta = 1$ and for any $x, y \in X.$

A vector space X_ϱ defined by $X_\varrho = \{x \in X : \varrho(\beta x) < \infty, \text{ for some } \beta > 0\},$ is called the *Modular space generated* by $\varrho.$ The space X_ϱ equipped with the Luxemburg norm

$$\| x \| = \inf\left\{\beta > 0 : \varrho\left(\frac{x}{\beta}\right) \le 1\right\},$$

is the Banach space (see [11]).

A function $\varphi : [-\infty, +\infty] \to [0, +\infty]$ is said to be *Orlicz function* if φ is a nonzero function that is convex, even, vanishing at zero, left continuous on $(0, \infty)$ and continuous at zero. A sequence $\varphi = (\varphi_n)$ of the Orlicz functions is called a *Musielak-Orlicz function*. We suppose that $\varphi = (\varphi_n)$ is the Musielak-Orlicz function. We say φ satisfies the condition (L_2), if $\sum_{n=1}^{\infty} \varphi_n(u) = \infty$, for all $u > 0$. Also we say φ satisfies the Δ_2-condition, if there is $k > 0$ such that $\varphi_n(2u) \leq k\varphi_n(u)$, for any $u \geq 0$ and for all $n \in \mathbb{N}$.

From now on we let $p \in [1, \infty)$ and the symbol φ will denote the Musielak-Orlicz function (φ_n).

The space $ces_p(\varphi) = \{x \in \ell_0 : \rho_\varphi(\beta x) < \infty, \text{ for some } \beta > 0\}$, where $\rho_\varphi(x)$ is the convex modular defined by $\rho_\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(S_n^p(x))$, is called the *Cesáro-Musielak-Orlicz sequence space*. This space endows with the Luxemburg norm $\| x \| = \inf\left\{\beta > 0 : \rho_\varphi\left(\frac{x}{\beta}\right) \leq 1\right\}$. Banach lattice $ces_p(\varphi)$ is not always rearrangement invariant Banach space. We define one closed subspace of $ces_p(\varphi)$ as follows

$$ces_p^0(\varphi) = \{x \in \ell_0 : \rho_\varphi(\beta x) < \infty, \text{ for all } \beta > 0\}.$$

We define the symmetric space $\Lambda_p(\varphi)$ by

$$\Lambda_p(\varphi) = \{x \in \ell_0 : \varrho_\varphi(\beta x) < \infty, \text{ for some } \beta > 0\},$$

where ϱ_φ is the convex modular defined by $\varrho_\varphi(x) = \sup_\sigma \sum_{n=1}^{\infty} \varphi_n(S_n^p(x \circ \sigma))$. We endow this space with the Luxemburg norm

$$\| x \| = \inf\left\{\beta > 0 : \varrho_\varphi\left(\frac{x}{\beta}\right) \leq 1\right\}.$$

It is easy to check that the modular space $\Lambda_p(\varphi)$ is the Banach lattice. Also we define one closed subspace of $\Lambda_p(\varphi)$ as follows

$$\Lambda_p^0(\varphi) = \{x \in \ell_0 : \varrho_\varphi(\beta x) < \infty, \text{ for all } \beta > 0\}.$$

First, we show that if φ satisfies the condition (L_2), then $ces_p(\varphi)$ contains isometric copy of ℓ_∞. Also we establish that $\Lambda_p(\varphi)$ is the rearrangement invariant space. Then property (H) of the space $ces_p(\varphi)$ considered, if φ satisfies the Δ_2-condition. Also it is proved that $ces_p(\varphi)$ has the Fatou-Levy property. Finally, we will give criteria which $ces_p(\varphi)$ be the separable and reflexive space.

1. Results

Lemma 1.1. *The following assertions are equivalent*:

(1) $ces_p(\varphi) \subseteq c_0$.

(2) $ces_p(\varphi) \subseteq \ell_\infty$ *and* φ *satisfies the condition* (L_2).

Proof. Assume that φ does not satisfy the condition (L_2). Hence there exists $u > 0$ such that $\sum_{n=1}^{\infty} \varphi_n(u) < \infty$. Put $x = \left(u^{\frac{1}{p}}, u^{\frac{1}{p}}, \cdots\right)$. We have $x \in ces_p(\varphi)$. Then $u = 0$, a contradiction.

Assume that $x = (x_n) \in ces_p(\varphi)\backslash c_0$. We have $x^* \in \ell_\infty$. Then the sequence $(S_n^p(x^*))$ has the upper bound $M > 0$. Also there is $n_0 \in \mathbb{N}$ such that $\varphi_{n_0}(M) > 0$. We claim that there is $\beta > 0$ such that $\rho_\varphi(\beta x^*) < \infty$. At first, we suppose that $\varphi_{n_0}(M) = \infty$. In this case, there is $\beta > 0$ that $\varphi_{n_0}(\beta S_{n_0}^p(x^*)) < \varphi_{n_0}(M)$. If $\beta \geq 1$, then we have $\rho_\varphi(x^*) < \sum_{n=1}^{\infty} \varphi_n(M) = \infty$ and if $\beta < 1$, then we have $\rho_\varphi(\beta x^*) < \sum_{n=1}^{\infty} \varphi_n(M) = \infty$. Now we suppose that

$\varphi_{n_0}(M) > 0.$ Then there is $a, b \in \mathbb{R}^+$ such that $M \in [a, b]$ and the function φ_{n_0} is strictly increasing in the interval $[a, b]$. Therefore, $\varphi_{n_0}(\beta S_{n_0}^p(x^*)) < \varphi_{n_0}(\beta M)$. Then $\rho_\varphi(\beta S_n^p(x^*)) < \infty$. We know $x^* \notin c_0$. Thus there are $\varepsilon > 0$ and subsequence $(x_{n_k}^*)$ such that $x_{n_k}^* \geq \varepsilon$, for all $k \in \mathbb{N}$. Therefore, $x_n^* \geq \varepsilon$, for all $n \in \mathbb{N}$. Then we have $\sum_{n=1}^{\infty} \varphi_n(\beta\varepsilon) \leq \rho_\varphi(\beta x^*) < \infty$, a contradiction. □

Similar to Lemma 1.1, we can prove Lemma 1.2.

Lemma 1.2. *The following assertions are equivalent*:

(1) $\Lambda_p(\varphi) \subseteq c_0$.

(2) φ *satisfies the condition* (L_2).

In Lemma 1.3 and Theorem 1.7, we will assume that $ces_p(\varphi) \subseteq \ell_\infty$.

Lemma 1.3. φ *satisfies the condition* (L_2) *if and only if* $\inf_n \varphi_n(u) > 0$, *for all* $u > 0$.

Proof. If there is $u > 0$ such that $\inf_n \varphi_n(u) = 0$, then $\inf_n \varphi_n(t_i) = 0$, for all $t_i \leq u$ such that $(t_i) \notin c_0$. So for any $i \in \mathbb{N}$, there is $n_i \in \mathbb{N}$ such that $\varphi_{n_i}(t_i) < \frac{1}{2^i}$. We define the sequence $x = (x_n)$ such that if $n \neq n_i, n_{i+1}, x_n = 0$ and if $n = n_i$, $x_n = nt_i$ and if $n = n_{i+1}$, $x_n = -nt_i$. We have $S_n^1(x) = \begin{cases} 0 & n \neq n_i, \\ t_i & n = n_i. \end{cases}$ So $\sum_{n=1}^{\infty} \varphi_n(S_n^1(x)) < \infty$. Therefore, $x \in ces_1(\varphi)$. Then $x \in c_0$, a contradiction.

The inverse is clear. □

Lemma 1.4. *The following assertions are equivalent*:

(1) *The spaces* $\Lambda_p(\varphi)$ *and* ℓ_∞ *are isomorphic*.

(2) φ *does not satisfy the condition* (L_2).

Proof. Assume there is $x \in \ell_\infty \backslash \Lambda_p(\varphi)$. By assertion (2), there exists $u > 0$ such that $\sum_{n=1}^{\infty} \varphi_n(u) < \infty$. If M is the upper bound of x, then we get $\varrho_\varphi\left(\frac{u}{M} x\right) < \infty$, a contradiction. Therefore, $\ell_\infty = \Lambda_p(\varphi)$.

Now assume $\varepsilon > 0$ is fixed and $\| x \|_\infty < \varepsilon$. Thus $\varrho_\varphi\left(\frac{x}{\varepsilon}\right) \leq \varrho_\varphi(1)$. If $\varrho_\varphi(1) \leq 1$, then $\| x \| < 2\varepsilon$. If $\varrho_\varphi(1) > 1$, then put $c = \max\{1, \varrho_\varphi(1)\}$. So $\| x \| < c\varepsilon$. Therefore, the Identity map $Id : (\Lambda_p(\varphi), \| \cdot \|) \to (\ell_\infty, \| \cdot \|_\infty)$ is continuous. By the Open Mapping theorem, *Id* is an isomorphism.

By Lemma 1.2, the inverse is clear. □

Lemma 1.5. *Suppose that* ϱ *is a convex modular on* X_ϱ, $x \in X_\varrho$ *and* (x_m) *is a sequence in* X_ϱ. *Then* $\| x_m - x \| \to 0$ *if and only if* $\varrho(\lambda(x_m - x)) \to 0$, *for all* $\lambda > 0$.

Proof. See [11, Theorems 1-6]. □

Lemma 1.6. *If* $x = (x_n) \in \ell_0$ *and* $| x_m | \geq \inf_k \sup_{n \geq k} | x_n |$, *for all* $m \in \mathbb{N}$, *then there are* $N_0 \subseteq \mathbb{N}$ *and the bijection map* $\delta : N_0 \to \mathbb{N}$ *such that* $x^* = | x | \circ \delta$.

Proof. See [4]. □

Theorem 1.7. $\Lambda_p(\varphi)$ *is the rearrangement invariant Banach space*.

Proof. Let $x \in \Lambda_p(\varphi)$ and $\mu_x = \mu_y$. Assume that φ does not satisfy the condition (L_2). Because ℓ_∞ is the rearrangement invariant space, by our assumption, we get $y \in \ell_\infty$. We know there are $c_1, c_2 > 0$ such that

$$c_1 \| x \|_\infty \leq \| x \|, \| y \| \leq c_2 \| x \|_\infty.$$

We have $\varrho_\varphi(\lambda(\| y \| - \| x \|)) = 0$, for any $\lambda > 0$. So $\| x \| = \| y \|$.

Now assume φ satisfies the condition (L_2). Then there is $n \in \mathbb{N}$ such that the equality $\varphi_n(u) = 0$ implies $u = 0$. Therefore, there exists $a \in (0, \infty)$ such that φ_n

is monotone increasing on $[0, a]$. Since $y \in c_0$, it has the upper bound M. We can choose $\beta > 0$ such that $\beta M^p < a$. We get $\sup_{\sigma} \varphi_n(\beta S_n^p(y \circ \sigma)) < \varphi_n(a)$. We obtain $\varrho_\varphi(\beta y) < \sum_{n=1}^{\infty} \varphi_n(a) = \infty$. Then $y \in \Lambda_p(p)$. Now we proof $\| x \| = \| y \|$. We have $x, y \in c_0$. Then there are $N_1, N_2 \subseteq \mathbb{N}$ and the bijection map $\delta_1 : N_1 \to \mathbb{N}$ and $\delta_2 : N_2 \to \mathbb{N}$ such that $| x | \circ \delta_1 = | y | \circ \delta_2$. If we have $| x_m | < x_n^*$ for any $n \in \mathbb{N}$, then $| x_m | = 0$. Because if $| x_m | > 0$, then there is $t_0 \in \mathbb{N}$ such that $| x_t | < | x_m |$, for any $t \geq t_0$. So there is $n_1 \in \mathbb{N}_1$ such that $x_{n_1}^* < x_n^*$, for all $n \in \mathbb{N}$, a contradiction. Similarly, if we have $| y_m | < y_n^*$, for all $m \in \mathbb{N}$, then $| y_m | = 0$. Therefore, $\varrho_\varphi(x) = \varrho_\varphi(y)$ and this completes the proof. □

Lemma 1.8. *Let* $x, y \in ces_p(\varphi)$ *and* (x_m) *be the sequence in* $ces_p(\varphi)$. *Then the following assertions are true*:

(1) *If* $0 < a < 1$, *then* $a\rho_\varphi\left(\frac{x}{a}\right) \leq \rho_\varphi(x)$.

(2) *If* $a \geq 1$, *then* $\frac{1}{a}\rho_\varphi(x) \leq \rho_\varphi\left(\frac{x}{a}\right)$.

(3) $\rho_\varphi(x + y) \leq \rho_\varphi(x) + \rho_\varphi(y)$.

(4) *If* $0 < a < 1$, *then* $\| x \| > a$ *implies* $\rho_\varphi(x) > a$.

(5) *If* $a \geq 1$, *then* $\| x \| < a$ *implies* $\rho_\varphi(x) < a$.

(6) *If* $\lim_{m\to\infty} \| x_m \| = 1$, *then* $\lim_{m\to\infty} \rho_\varphi(x_m) = 1$.

(7) *If* $\lim_{m\to\infty} \rho_\varphi(x_m) = 0$, *then* $\lim_{m\to\infty} \| x_m \| = 0$.

Proof. We define the function $f(\beta) = \rho_\varphi\left(\frac{| x |}{\beta}\right)$ on $\mathbb{R}^+$. If $a\rho_\varphi\left(\frac{x}{a}\right) > \rho_\varphi(x)$, then $a f(a) > f$ (1). Also we know $f(1) \geq f(a)$. So $a f(a) > f(a)$, a contradiction. (2) follows similarly. (3) follows by (1). If $\| x \| > a$, then $\rho_\varphi\left(\frac{x}{a}\right) > 1$. So (4) follows from (1). (5) is similar to (4). Suppose that $\varepsilon \in (0, 1)$ is arbitrary. Then there is $m_0 \in \mathbb{N}$ such that $1 - \varepsilon < \| x_m \| < 1 + \varepsilon$, for any $m \geq m_0$. Then $1 - \varepsilon < \rho_\varphi(x_m) < 1 + \varepsilon$, that is $\lim_{m \to \infty} \rho_\varphi(x_m) = 1$. (7) follows similarly (6). □

Lemma 1.9. *If* φ *satisfies the* Δ_2*-condition*, *then the following assertions are true*:

(1) $ces_p(\varphi) = ces_p^0(\varphi)$.

(2) *For any* $x \in ces_p(\varphi)$, *we have* $\| x \| = 1$ *if and only if* $\rho_\varphi(x) = 1$.

Proof. (1) Suppose $x \in ces_p(\varphi)$. Then there is $\beta > 0$ which $\rho_\varphi(\beta x) < \infty$. We give an arbitrary real number $\mu > 0$. If $\mu \leq \beta$, then $\rho_\varphi(\mu x) < \infty$. If $\mu > \beta$, then there exists $r > 0$ such that $\mu \leq 2^r \beta$. Let k be as in the definition of the Δ_2-condition. We have $\rho_\varphi(\mu x) \leq k^r \rho_\varphi(\beta x) < \infty$. So $x \in ces_p^0(\varphi)$.

(2) We need only to show that $\| x \| = 1$ implies $\rho_\varphi(x) = 1$, because the opposite implication holds in any modular space. Suppose that $\rho_\varphi(x) < 1$. We define the function $f(\beta) = \rho_\varphi(\beta x)$ on $\mathbb{R}^+$. The function f is infinite and convex. So it is continuous. Note that there is $\beta_0 > 1$ such that $\rho_\varphi(\beta_0 x) > 1$. Then we have $f(1) < 1 < f(\beta_0)$. So there is $\lambda \in (1, \beta_0)$ that $\rho_\varphi(\lambda x) = 1$. Therefore, $\| x \| < 1$, a contradiction. □

Lemma 1.10. *Let* φ *satisfy the* Δ_2*-condition*, x *be a point of the unit sphere* $ces_p(\varphi)$ *and* (x_m) *is a sequence in the unit sphere* $ces_p(\varphi)$ *such that* $x_m \to x$ *coordinatewise*. *If* $\lim_{m \to \infty} \rho_\varphi(x_m) = \rho_\varphi(x)$, *then* $\lim_{m \to \infty} x_m = x$.

Proof. Let $\varepsilon > 0$, $t \in \mathbb{N}$ be an arbitrary number and k be as in the definition of Δ_2-condition. We have $S_n^p(x_m - x) \to 0$. Then there is $M_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{t} \varphi_n(S_n^p(x_m - x)) < \frac{\varepsilon}{2}.$$

Since $\rho_\varphi(x_m) \to \rho_\varphi(x)$, there is $N_0 \in \mathbb{N}$ such that

$$\sum_{n=1}^{t} \varphi_n(S_n^p(x)) - \sum_{n=1}^{t} \varphi_n(S_n^p(x_m)) < \frac{\varepsilon}{8k^p}.$$

Also there is $P_0 \in \mathbb{N}$ such that $\rho_\varphi(x_m) \leq \rho_\varphi(x) + \frac{\varepsilon}{8k^p}$, for any $m \geq P_0$. Also we can find $t_0 \in \mathbb{N}$ such that $\sum_{n=t_0+1}^{\infty} \varphi_n(S_n^p(x)) < \frac{\varepsilon}{8k^p}$. Put $m_0 = \max\{M_0, N_0, P_0\}$. So for any $m \geq m_0$, we obtain

$$\begin{aligned}
\rho_\varphi(x_m - x) &< \frac{\varepsilon}{2} + \sum_{n=t_0+1}^{\infty} \varphi_n(|\, S_n^p(x_m - x)\,|) \\
&\leq \frac{\varepsilon}{2} + \sum_{n=t_0+1}^{\infty} \varphi_n(2^p[|\, S_n^p(x_m)\,| + |\, S_n^p(x)\,|]) \\
&\leq \frac{\varepsilon}{2} + k^p\left[\sum_{n=t_0+1}^{\infty} \varphi_n(S_n^p(x_m)) + \sum_{n=t_0+1}^{\infty} \varphi_n(S_n^p(x_m))\right] \\
&< \frac{\varepsilon}{2} + k^p\left[\rho_\varphi(x) + \frac{\varepsilon}{8k^p} - \sum_{n=1}^{t_0} \varphi_n(S_n^p(x_m)) + \sum_{n=t_0+1}^{\infty} \varphi_n(S_n^p(x))\right] \\
&< \frac{\varepsilon}{2} + k^p\left[\rho_\varphi(x) + \frac{\varepsilon}{4k^p} - \sum_{n=1}^{t_0} \varphi_n(S_n^p(x)) + \sum_{n=t_0+1}^{\infty} \varphi_n(S_n^p(x))\right] \\
&< \frac{\varepsilon}{2} + k^p\left[\frac{\varepsilon}{4k^p} + \frac{\varepsilon}{4k^p}\right] = \varepsilon.
\end{aligned}$$

So we have $\rho_\varphi(x_m - x) \to 0$. Therefore, $x_m - x \to 0$ □

Theorem 1.11. *If* φ *satisfies the* Δ_2*-condition, then space* $ces_p(\varphi)$ *has the property* (*H*).

Proof. Assume that *x* is a point of the unit sphere $ces_p(\varphi)$ and (x_m) is a sequence in the unit sphere $ces_p(\varphi)$ such that (x_m) is weak convergence to *x*. By

$x_m \overset{w}{\to} x,$ we get $x_m(n) \to x(n)$ as $m \to \infty,$ for all $n \in \mathbb{N}.$ Now, by Lemma 1.10, we have $\| x_m - x \| \to 0.$

Corollary 1.12. *If* φ *satisfies the* Δ_2*-condition, then the unit ball* $ces_p(\varphi)$ *is weakly closed.*

Theorem 1.13. *The space* $ces_p(\varphi)$ *has the Fatou-Levy property.*

Proof. Suppose that $x \in \ell_0$ and (x_m) is a sequence in $ces_p(\varphi)$ such that $0 \le x_m \uparrow x$ and $\sup_m \| x_m \| < \infty.$ Put $A = \sup_m \| x_m \|.$ Let m be fixed. Then we have $S_n^p\left(\frac{x_m}{A}\right) \le S_n^p\left(\frac{x_m}{\| x_m \|}\right),$ for any $n \in \mathbb{N}.$ So

$$\sum_{n=1}^{\infty} \varphi_n\left(S_n^p\left(\frac{x_m}{A}\right)\right) \le \sum_{n=1}^{\infty} \varphi_n\left(S_n^p\left(\frac{x}{\| x_m \|}\right)\right).$$

Therefore,

$$\sum_{n=1}^{\infty} \varphi_n\left(S_n^p\left(\frac{x_m}{A}\right)\right) \le 1, \quad \forall m \in \mathbb{N}. \tag{1.1}$$

Since $x_m \to x,$ we have $S_n^p\left(\frac{x_m}{A}\right) \uparrow S_n^p\left(\frac{x}{A}\right).$ Hence $\lim_{m\to\infty} \varphi_n\left(S_n^p\left(\frac{x_m}{A}\right)\right) = \varphi_n\left(S_n^p\left(\frac{x}{A}\right)\right).$ By using Monotone convergence theorem, we get

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \varphi_n\left(S_n^p\left(\frac{x_m}{A}\right)\right) = \sum_{n=1}^{\infty} \varphi_n\left(S_n^p\left(\frac{x}{A}\right)\right).$$

Now, by equation (1.1), we have $x \in ces_p(\varphi).$

We suppose $\| x_m \|$ does not convergent to $\| x \|.$ Then there are $\varepsilon > 0$ and subsequence $(x_{m_k})_{k=1}^{\infty}$ such that $|\, \| x_{m_k} \| - \| x \| \,| > \varepsilon,$ for all $n \in \mathbb{N}.$ So we have

$$\varphi_n\left(S_n^p\left(\frac{x_{m_k}}{\| x \| - \varepsilon}\right)\right) \le \varphi_n\left(S_n^p\left(\frac{x_{m_k}}{\| x_{m_k} \|}\right)\right) \le 1, \tag{1.2}$$

for any $k \in \mathbb{N}$. Also, by Monotone convergence theorem, we have

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}\varphi_n\left(S_n^p\left(\frac{x_{m_k}}{\|x\|-\varepsilon}\right)\right) = \sum_{n=1}^{\infty}\varphi_n\left(S_n^p\left(\frac{x}{\|x\|-\varepsilon}\right)\right).$$

Then, by equation (1.2), we get $\|x\| \leq \|x\| - \varepsilon$, a contradiction. □

Theorem 1.14. *If one of the following conditions satisfies*:

(1) *there is* $x_1, x_2 > 0$ *such that* $x_1 \neq x_2$ *and* $\varphi_n(x_1) = x_1$, $\varphi_n(x_2) = x_2$, *for all* $n \in \mathbb{N}$,

(2) φ_n *is differentiable in the point zero and* $\varphi_n'(0) \geq 1$, *for all* $n \in \mathbb{N}$,

then the space $ces_p(\varphi)$ *is separable and reflexive*.

Proof. In two cases we claim that $\varphi_n(x) \geq x$, for any $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. If $x \in [-\infty, 0)$, then it holds. So we suppose that $x \in [0, \infty]$. Let condition (1) hold. Without loss of generality, we can assume that $x_1 < x_2$. We claim that $\varphi_n(x) \geq x$, for any $x \geq 0$ and for all $n \in \mathbb{N}$. Assume that there are $x_3 > 0$ and $n \in \mathbb{N}$ such that $\varphi_n(x_3) < x_3$. If $x_3 > x_1 > 0$, then there is $\lambda \in (0, 1)$ as $x_1 = \lambda x_3$. Hence, $\lambda x_3 = \varphi_n(x_1) \leq \lambda\varphi_n(x_3) < \lambda x_3$, a contradiction. And if $x_3 < x_1$, then by convexity φ_n, we have

$$\frac{\varphi_n(x_1) - \varphi_n(x_3)}{x_1 - x_3} \leq \frac{\varphi_n(x_2) - \varphi_n(x_1)}{x_2 - x_1} = 1.$$

Therefore, $\varphi_n(x_3) \geq x_3$ which is impossible. Now let condition (2) hold. Assume that there are $u_0 > 0$ and $n \in \mathbb{N}$ such that $\varphi_n(u_0) < u_0$. We have

$$\sup_{\delta>0}\inf_{0<u<\delta}\frac{\varphi_n(u)}{u} = \varphi_n'(0) \geq 1 > \frac{\varphi_n(u_0)}{u_0}.$$

Then we can find $\delta > 0$ that for any $0 < u < \delta$,

$$\frac{\varphi_n(u) - \varphi_n(u_0)}{u - u_0} > \frac{\varphi_n(u_0)}{u_0}. \tag{1.3}$$

Define the function $f(u) = \varphi_n(u) - u$ on $\mathbb{R}$. This function has at least one root u_1 in interval (u, u_0). We get $\frac{\varphi_n(u_0)}{u_0} \geq \frac{\varphi_n(u_0) - \varphi_n(u_1)}{u_0 - u_1}$. So by equation (1.3), we have

$$\frac{\varphi_n(u) - \varphi_n(u_0)}{u - u_0} > \frac{\varphi_n(u_0) - \varphi_n(u_1)}{u_0 - u_1},$$

a contradiction.

So in two cases, we have $\sum_{n=1}^{\infty} \beta(S_n^p x) \leq \rho_\varphi(\rho_x)$, for any $x \in ces_p(\varphi)$ and for any $\beta > 0$. Therefore, $ces_p(\varphi) \subseteq ces_p$. Thus $ces_p(\varphi)$ is the separable and reflexive space. □

Lemma 1.15. *Suppose that X is the Banach lattice. Then the following assertions are equivalent*:

(1) *X is the reflexive space*.

(2) *X has the Fatou-Levy property and on* B_X, *pointwise convergence topology and weak topology are coincide*.

Proof. See [2, Lemma 2]. □

Corollary 1.16. *If two conditions of the before theorem hold*, *then pointwise convergence topology and weak topology are coincide on the unit ball* $ces_p(\varphi)$.

Remark 1.17. We can prove similarly all theorems and lemmas for the space $\Lambda_p(\varphi)$.

References

[1] C. Bennett and R. Sharpley, Interpolation of Operators, Academic Press, New York, 1988.

[2] J. Cerda, H. Hdzik, A. Kaminska and M. Mastoylo, Geometry properties of symmetric spaces with application to Orlicz-Lorentz spaces, Positivity 2 (1998), 311-337.

[3] M. Fabian, P. Habala, P. Hajek, V. M. Santalucia, J. Pelant and V. Zizler, Functional Analysis and Infinite-dimensional Geometry, Springer-Verlag, New York, 2001.

[4] P. Forlewski, H. Hudzik and L. Szymaszkiewics, Local rotundity structure of generalized Orlicz-Lorentz sequence spaces, Nonlinear Anal. 68 (2008), 2709-2718.

[5] P. Foralewski, H. Hudzik and L. Szymaszkiewicz, On some geometric and topological properties of generalized Orlicz-Lorentz sequence spaces, Math. Nachar. 281 (2008), 181-198.

[6] D. J. H. Garling, Inequalities: A Journey into Linear Analysis, Cambridge University, 2007.

[7] A. A. Jagers, A note on the Cesáro sequence spaces, Nieuw Arch. Wiskd. 22 (1974), 113-124.

[8] F. M. Khan and M. A. Khan, Matrix transformations between Cesáro-sequence Spaces, Indian J. Pure Appl. Math. 25 (1994), 641-645.

[9] S. G. Krein, J. I. Petunin and E. M. Seminov, Interpolation of linear operator, Proc. Amer. Math. Soc., 1982.

[10] G. M. Leibowitz, A note on the Cesáro sequence spaces, Tamkang J. Math. 2 (1971), 151-157.

[11] J. Musielak, Orlicz spaces and modular spaces, Lecture Notes in Mathematics, Springer-Verlag, 1983.

[12] M. Raja, Kadec norm and Borel sets in a Banach space, Studia Math. 136 (1999), 1-16.

[13] S. Suantai, On some convexity properties of generalized Cesáro sequence spaces, Georgian Math. J. 10 (2003), 193-200.