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Global stabilizability of an anaerobic biodegradation process via piecewise constant feedback

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Summary

We consider a nonlinear model of an anaerobic digestion process with methane production. We propose a stabilizing control law involving piecewise constant feedback and study the asymptotic stability of the obtained closed-loop system. An extremum seeking algorithm is applied to maximize the output. Numerical simulation results are also presented to illustrate the theoretical investigations.

KEYWORDS

asymptotic stabilization, continuous bioreactor, dynamical nonlinear model, piecewise constant feedback control

1 | INTRODUCTION

We consider a well-known mathematical model of biological anaerobic treatment (methane fermentation) of organic wastes in a continuously stirred tank bioreactor (see Reference 1 and the references therein). The model is described by two nonlinear ordinary differential equations

$$\begin{aligned} \frac{ds(t)}{dt} &= -k_1\mu(s(t))x(t) + u(s_{\text{in}} - s(t)) \\ \frac{dx(t)}{dt} &= (\mu(s(t)) - \alpha u)x(t) \end{aligned} \tag{1}$$

and one algebraic equation, presenting the gaseous output (methane)

$$Q(s(t), x(t)) = k_2\mu(s(t))x(t). \tag{2}$$

The meaning of the state variables $x = x(t)$, $s = s(t)$ and of the model parameters is given below:

- x is biomass concentration (g/L)
- s is substrate concentration (g/L)
- u is dilution rate (1/day)
- s_{in} is influent organic pollutant concentration (g/L)
- k_1 is yield coefficient
- k_2 is coefficient
- Q is methane (biogas) flow rate (liter biogas for liter of the medium per day).

The coefficients k_1 and k_2 are positive. The parameter $\alpha \in (0, 1)$ represents the proportion of bacteria that are affected by the dilution; $\alpha = 0$ and $\alpha = 1$ correspond to an ideal fixed bed reactor and to an ideal continuous stirred tank reactor, respectively.[2] The influent substrate concentration s_{in} is assumed to be constant.

The dilution rate u is considered as a control variable.

The model function $\mu(s)$ [1/day] represents the specific growth rate of the methane producing bacteria x. We make the following general assumption on $\mu(s)$.

Assumption 1. The function $\mu(s)$ is defined for $s \in [0, +\infty)$, $\mu(0) = 0$ and $\mu(s) > 0$ whenever $s > 0$; $\mu(s)$ is continuously differentiable and bounded for all $s > 0$.

The above model (1) can be considered as the most simple input-output model of the methane fermentation or as a simplified form of more complex nonlinear models and is very appropriate to check different control strategies. The practical applicability of the model is demonstrated.[1,3]

It is well known that the feedback control of bioreactor (chemostat) models provides many advantages in operating a plant and is used to increase its efficiency and sustainable long-term energy production. We consider here a feedback control law of the form $u \equiv \kappa(s, x) = \beta k_2 \mu(s) x$, where β is an auxiliary positive and properly chosen parameter. Obviously, this feedback is based on the output (ie, on the biogas flow rate, see (2)) measurements, which are always on-line available. This feedback is shown to stabilize globally the dynamics (1).[4] Now we modify the feedback $\kappa(\cdot)$ by introducing a sampling period delay $\tau > 0$. The design and analysis of sampled-data control systems have been of continuing interest for several decades[5-7] and the references therein. The main objective of the present article is to achieve global asymptotic stabilization of the so obtained sample-and-hold control system.

Introducing a sampled-data control law converts the model (1) into a system of ordinary differential equations with piecewise (with respect to time) constant arguments. This class of equations combines the properties of differential and differential-difference equations. A recently developed mathematical approach for proving global attractivity of solutions of equations with piecewise constant arguments[8] is based on constructing a suitable Lyapunov function and we apply this idea in our study.

The article is organized in the following way. Section 2 is devoted to stability analysis of the model involving a piecewise constant feedback control law. First, we show in Theorem 1 that under suitable assumptions and for any sampling period $\tau \geq 0$ the trajectories of the closed-loop system approach in finite time the set $\{(s, x) : 0 < s^- \leq s \leq s^+ < s_{\text{in}}, x > 0\}$, where the substrate concentrations s^- and s^+ are defined in a proper way. Then the global stability of the closed-loop system is established in Theorem 2 for sufficiently small values of τ. Based on these results, a numerical extremum seeking algorithm (ESA) is applied to stabilize the system toward the maximum methane flow rate. The ESA is shortly described in Section 3. Section 4 contains results from computer simulations. The stabilizability of the dynamics is illustrated in subsection 4.1 for different values of the sampling period $\tau > 0$, whereas subsection 4.2 demonstrates and visualizes the numerical outputs of ESA. Detailed proofs of Theorems 1 and 2 are given in the Appendix.

2 | STABILITY ANALYSIS OF THE MODEL

The control variable in the model (1) is the dilution rate u. In practice, the dilution rate is proportional to the speed of the pumping mechanism that feeds the bioreactor, thus u is always lower- and upper-bounded, that is, there exist $u_{\min} > 0$ and $u_{\max}$ such that $u_{\min} \leq u \leq u_{\max}$. In the model-based studies it is reasonable to assume that $u_{\max} < \frac{1}{\alpha}\mu(s_{\text{in}})$ to avoid wash-out of the bacteria.[2]

Define the function

$$\kappa(s, x) = \beta k_2 \mu(s) x, \tag{3}$$

where β is a positive parameter, which varies in given bounds. To determine these bounds, we need the following assumption.

Assumption 2. Lower bounds $s_{\text{in}}^- > 0$ and $k_2^- > 0$ for the values of s_{in} and k_2, respectively, and an upper bound $k_1^+ > 0$ for the value of k_1 are known.

Assumption 2 is not restrictive in the sense that very often in practice we do not know exact values for the model parameters but only some bounds for them. This is especially valid for fermentation processes which are known to be highly uncertain.

Define

$$\beta^- := \frac{k_1^+}{k_2^- s_{\text{in}}^-} > 0.$$

The value β^- will serve as a lower bound for the parameter β, involved in the expression for $\kappa(s,x)$ from (3).

Let us fix an arbitrary $\beta \in (\beta^-, +\infty)$ and set

$$\bar{x} := \frac{1}{\alpha\beta k_2}, \quad \bar{s} := s_{\text{in}} - \alpha k_1 \bar{x}, \quad \overline{p}_\beta = (\bar{s}, \bar{x}). \tag{4}$$

Obviously, $\bar{x}$ and $\bar{s}$ depend on the parameter β; according to the choice of β we have that $0 < \bar{s} < s_{\text{in}}$. If β is too large, then $\bar{x}$ becomes too small, and $\bar{s}$ becomes very close to s_{in}. It is straightforward to see that $\overline{p}_\beta$ is an equilibrium point of the closed-loop system using $\kappa(s,x)$ instead of u.

To design our feedback, we choose values u^- and u^+, $u^- < u^+$, such that $[u^-, u^+] \subset (u_{\min}, u_{\max})$. Then each point from the interval $[u^-, u^+]$ is an admissible value for the control function u.

Assumption 3. The following conditions hold true:

(i) there exist values s^- and s^+ of the substrate concentration s such that $0 < s^- < \bar{s} < s^+ < s_{\text{in}}$ and $\mu(s^-) = \alpha u^-$, $\mu(s^+) = \alpha u^+$;

(ii) the function μ is strictly increasing on the interval (s^-, s^+);

(iii) the following inequality holds true

$$\mu(s^-) > \mu(s) \quad \text{for each } s \in [0, s^-).$$

(iv) there exist $\hat{\varepsilon} \in (0, s_{\text{in}} - s^+)$ and $\eta > 0$ such that the following inequality holds true:

$$\mu(s) \geq \mu(s^+) + \eta \quad \text{for each } s \in [s^+ + \hat{\varepsilon}, s_{\text{in}}).$$

Assumption 3 is technical and is used in the theoretical studies of the model. It is always fulfilled when the function $\mu(s)$ is strictly increasing (like the Monod specific growth rate). If the function $\mu(s)$ is not increasing (like, eg, the Haldane law) then the points $\bar{s}$ (ie, the value for β), s^- and s^+ have to be chosen in a proper way to satisfy Assumption 3. For details see the numerical experiments in Section 4.

It follows from Assumption 3(ii) that the following inequalities hold true

$$\mu(s^-) < \mu(s) < \mu(s^+) \quad \text{for each } s \in (s^-, s^+).$$

Using the same value for β as in (4), we set

$$\bar{u} := \kappa(\bar{s}, \bar{x}) = \frac{\mu(\bar{s})}{\alpha}; \tag{5}$$

obviously the inclusion $\bar{u} \in (u^-, u^+)$ is fulfilled. The equality (5) is called regulability in Reference 2. It means that there should be a constant value $\bar{u}$ of the dilution rate, corresponding to the nontrivial equilibrium $\overline{p}_\beta = (\bar{s}, \bar{x})$.

Let $\tau > 0$. We set $s(0) = s^0 > 0$, $x(0) = x^0 > 0$ and consider the following closed-loop system Σ

$$\frac{ds(t)}{dt} = -k_1 \mu(s(t)) x(t) + \chi(t)(s_{\text{in}} - s(t)) \tag{6}$$

$$\frac{dx(t)}{dt} = (\mu(s(t)) - \alpha\chi(t))\, x(t), \tag{7}$$

obtained from the original system (1) by substituting the dilution rate u by the following piecewise continuous feedback control law

$$\chi(t) := \psi(s(\theta(t)), x(\theta(t)))$$

within

$$\psi(s(t), x(t)) := \begin{cases} u^-, & \text{if } \kappa(s(t), x(t)) \le u^-, \\ \kappa(s(t), x(t)), & \text{if } u^- \le \kappa(s(t), x(t)) \le u^+, \\ u^+, & \text{if } \kappa(s(t), x(t)) \ge u^+ \end{cases} \tag{8}$$

for each $t \in [\theta(t), \theta(t) + \tau)$ with $\theta(t) := \lfloor t/\tau \rfloor \tau$, where $\lfloor t/\tau \rfloor$ denotes the largest nonnegative integer k satisfying the inequality $k\tau \le t$.

In fact,

$$\chi(t) = \psi(s(k\tau), x(k\tau)) \quad \text{for each } t \in [k\tau, (k+1)\tau), \ k = 0, 1, 2, \dots,$$

that is, the value of the feedback $\chi(\cdot)$ at the time t is equal to $\psi(s(\cdot), x(\cdot))$ at the sampling time $k\tau$, and holds on the interval $[k\tau, (k+1)\tau)$, $k = 0, 1, 2, \dots$.[5,6]

It is straightforward to see that the point $\overline{p}_\beta = (\overline{s}, \overline{x})$ from (4) is an equilibrium point of Σ, that is, of (6) to (7). Moreover, all equilibrium points of Σ lie on the straight line $s + \alpha k_1 x = s_{\text{in}}$ (see Figures 2 to 4).

We shall prove under suitable assumptions that the feedback law $\chi(\cdot)$ stabilizes asymptotically the closed-loop system Σ to the equilibrium point $\overline{p}_\beta$.

Denote

$$\Omega := \{\zeta = (s, x) : s > 0, \ x > 0\}$$

and let $\zeta^0 = (s^0, x^0) \in \Omega$ be an arbitrary point such that $s(0) = s^0 > 0$ and $x(0) = x^0 > 0$. Denote by $\phi(\cdot, \zeta^0) = (s(\cdot), x(\cdot))$, the corresponding solution of Σ starting from ζ^0. Important properties of $\phi(\cdot, \zeta^0)$ are established in the next two lemmas.

Lemma 1. *For each point $\zeta^0 = (s^0, x^0) \in \Omega$ the corresponding solution $\phi(t, \zeta^0) = (s(t), x(t))$ is defined for each $t > 0$. Moreover, for each $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for each $t > T_\varepsilon$ the following inequalities hold true:*

$$s_{\text{in}} - \varepsilon < s(t) + k_1 x(t) < \frac{s_{\text{in}}}{\alpha} + \varepsilon. \tag{9}$$

Lemma 2. *For each point $\zeta^0 = (s^0, x^0) \in \Omega$ there exist $\varepsilon > 0$ and $T > 0$ so that for each $t > T$ the following inequalities hold true*

$$s(t) < s_{\text{in}} \quad \text{and} \quad x(t) \ge \frac{\varepsilon}{k_1} =: x_{\min} > 0, \tag{10}$$

where $\phi(t, \zeta^0) = (s(t), x(t))$ with $t \ge 0$ is the corresponding trajectory of Σ.

Lemmas 1 and 2 extend assertions that can be found in References 2,9, and 10 for different chemostat models. We present their proofs in the Appendix for our model (6) to (7), for completeness and reader's convenience.

Remark 1. Lemmas 1 and 2 imply that for each point $\zeta^0 = (s^0, x^0) \in \Omega$ the corresponding solution $(s(t), x(t))$ with $(s(0), x(0)) = (s^0, x^0)$ is defined for each $t > 0$, takes only positive values and is bounded. Moreover, from these lemmas we obtain the existence of positive constants L_s, L_x, B_s, and B_x, such that for each point $\zeta^0 = (s^0, x^0) \in \Omega$ and for each positive reals t_2 and t_1 the following relations hold true

$$|\mu(s(t_2)) - \mu(s(t_1))| = \frac{d}{ds}\mu(s(\xi_s))\frac{d}{dt}(s(\xi_s))|t_2 - t_1| \le L_s|t_2 - t_1|,$$
$$|x(t_2) - x(t_1)| = \frac{d}{dt}x(\xi_x)|t_2 - t_1| \le L_x|t_2 - t_1|,$$

where ξ_s and ξ_x belong to the open interval determined by t_1 and t_2; further let

$$\mu(s(t)) \leq B_s \quad \text{and} \quad x(t) \leq B_x \quad \text{for all } t \geq 0.$$

Moreover, the proofs of Lemmas 1 and 2 imply that the constants L_s, L_x, B_s, and B_x do not depend on the particular choice of the parameter β (ie, on the choice of the equilibrium point $\overline{p}_\beta$). Also, the lower bound $x_{\min}$ (see (10)) does not depend on β.

The first result is the following theorem, which proof is given in the Appendix.

Theorem 1. *Let the Assumptions 1, 2, and 3 be fulfilled. Then for each sampling period $\tau > 0$ and for each point $\zeta^0 = (s^0, x^0) \in \Omega$ the corresponding solution $\phi(t, \zeta^0)$ of Σ with $(s(0), x(0)) = \zeta^0$ exists on $[t, +\infty)$ and has the following property: there exists $\hat{t} > 0$ such that*

$$\phi(\hat{t}, \zeta^0) \in \Omega_s := \{(s, x) : s \in (s^-, s^+), x > 0\}.$$

The main result of the article is given in the next Theorem 2.

Theorem 2. *Let the Assumptions 1, 2, and 3 be fulfilled. Then there exists a sampling period $\tau^* > 0$ so that for each $\tau \in (0, \tau^*)$ and for each point $\zeta^0 = (s^0, x^0) \in \Omega$ the corresponding solution $\phi(t, \zeta^0)$ of Σ converges asymptotically toward $\overline{p}_\beta = (\overline{s}, \overline{x})$.*

A detailed proof of Theorem 2 can be found in the Appendix.

Remark 2. It follows from the proof of Theorem 2 that there exist bounds (estimates) for τ, for which the global stabilizability of the dynamics is achieved. These bounds are theoretical and a priori. To estimate the values of τ guaranteeing global asymptotic stability of the dynamics one has to make a number of computer simulations and/or real experiments corresponding to the concrete fermentation process.

3 | OUTPUT OPTIMIZATION VIA EXTREMUM SEEKING

Consider Equation (2) describing the process output, that is, the methane production. We shall apply a numerical extremum seeking algorithm (ESA),[4,11] to steer and stabilize the dynamics (6) to (7) toward an equilibrium point, where the maximum methane flow rate $Q_{\max}$ is achieved. For that purpose, we first compute Q on the set of all equilibrium points $\{\overline{p}_\beta\}$, parameterized with respect to $\beta \in (\beta^-, +\infty)$. Denote the so obtained function by $Q(\beta)$. The function $Q(\beta)$ is called input-output static characteristic of the model. Assume that $Q(\beta)$, $\beta \in (\beta^-, +\infty)$, is strongly unimodal, that is, there exists a unique point $\beta_{\max} \in (\beta^-, +\infty)$, where $Q(\beta)$ takes a maximum $Q_{\max} = Q(\beta_{\max})$, $Q(\beta)$ strongly increases in the interval $(\beta^-, \beta_{\max})$ and strongly decreases in $(\beta_{\max}, +\infty)$.

Denote by

$$\overline{p}_{\beta_{\max}} = (\overline{s}_{\max}, \overline{x}_{\max})$$

the equilibrium point where $Q_{\max}$ is achieved.

Our goal is to stabilize in real time the system (6) to (7) toward this (unknown) equilibrium point $\overline{p}_{\beta_{\max}}$ and therefore to the maximum methane flow rate $Q_{\max}$. This is realized by applying a numerical model-based ESA.

The ESA exploits the fact that we have the freedom in choosing values of the parameter $\beta > \beta^-$ taking into account the conditions of Theorems 1 and 2. Hence, we construct a sequence of points $\beta^1, \beta^2, \ldots, \beta^n, \ldots$, which tends to $\beta_{\max}$. Theorem 2 guarantees that the dynamics is globally asymptotically stabilizable to each point $\overline{p}_{\beta^j}$ and thus to $\overline{p}_{\beta_{\max}}$ for reasonable values of the sampling period $\tau > 0$ (see Remark 2). Then by computing and comparing the values $Q(\beta^1), Q(\beta^2), \ldots, Q(\beta^n), \ldots$, the desired equilibrium point $\overline{p}_{\beta_{\max}}$ and thus $Q_{\max}$ are achieved.

In the computer simulation ESA is carried out in two stages. In the first stage, "rough" intervals $[\beta]$ and $[Q]$ are found which enclose $\beta_{\max}$ and $Q_{\max}$, respectively; in the second stage, the interval $[\beta]$ is refined using an elimination procedure based on the golden mean value (or Fibonacci search) strategy. The second stage produces the final intervals $[\beta_{\max}] = [\beta^-_{\max}, \beta^+_{\max}]$ and $[Q_{\max}]$ such that $\beta_{\max} \in [\beta_{\max}]$, $Q_{\max} \in [Q_{\max}]$, and $\beta^+_{\max} - \beta^-_{\max} \leq \epsilon$, where $\epsilon > 0$ is user specified tolerance.

ESA is presented in more details in Reference 11 for the same model using an adaptive feedback control. Now the algorithm is adopted to system (6) to (7). The numerical simulations are carried out in the SmoWeb Computational Platform; more information can be found on http://platform.sysmoltd.com/.

4 | NUMERICAL EXPERIMENTS

In the computer simulations, we consider the Haldane model function for the specific growth rate

$$\mu(s) = \frac{m_1 s}{k_s + s + s^2 k_i},$$

where m_1 is the maximum specific growth rate of the microorganisms [1/day], k_s and k_i are the saturation and inhibition constants, respectively.

We use the following values for the model coefficients, obtained by laboratory experiments and parameter estimation:[1]

$$s_{\text{in}} = 2, \quad k_1 = 3, \quad m_1 = 0.35, \quad k_s = 0.7, \quad k_i = 0.6, \quad \alpha = 0.5, \quad k_2 = 5.6. \tag{11}$$

With $k_1^+ = 3.1$ and $s_{\text{in}}^- = 1.95$, $k_2^- = 5.59$, we find $\beta^- = \frac{k_1^+}{s_{\text{in}}^- k_2^-} \approx 0.2844$. Thus, the parameter β is assumed to vary in the interval $(0.2844, +\infty)$.

The function $\mu(s)$ achieves its maximum at the point $s_\mu = \sqrt{k_s k_i} \approx 0.6481$ (see Figure 1 (left)). Hence, $\mu(s)$ is monotone increasing for $s \in (0, s_\mu)$ and monotone decreasing for $s > s_\mu$. The equilibrium component $\bar{s}$ depends on the parameter β, that is, $\bar{s} = \bar{s}(\beta)$ (see (4)). Solving the equation $\bar{s}(\beta) = s_\mu$ with respect to β implies $\beta = \beta_\mu \approx 0.3963$. Since $\bar{s}(\beta)$ is an increasing function of β, it suffices to consider $\beta \in (\beta^-, \beta_\mu)$ to have Assumption 3(ii) satisfied.

For the simulations, we also choose the following numerical values

$$\begin{array}{ll} s^- = \bar{s}(\beta^-) \approx 0.1163, & u^- \approx 0.097032, \\ s^+ = 0.43 < s_\mu, & u^+ \approx 0.2093. \end{array}$$

4.1 | Simulation of the global behavior of the dynamics

First, we shall demonstrate numerically the stabilizability of the closed-loop system (6) to (7) toward the equilibrium (*target*) point $\bar{p}_\beta$ for different values of the sampling period τ.

We choose and fix $\beta = 0.33 \in \left(\beta^-, \beta_\mu\right)$. Then the target point is

$$\bar{p}_\beta = (\bar{s}, \bar{x}) \approx (0.37662,\ 1.08225).$$

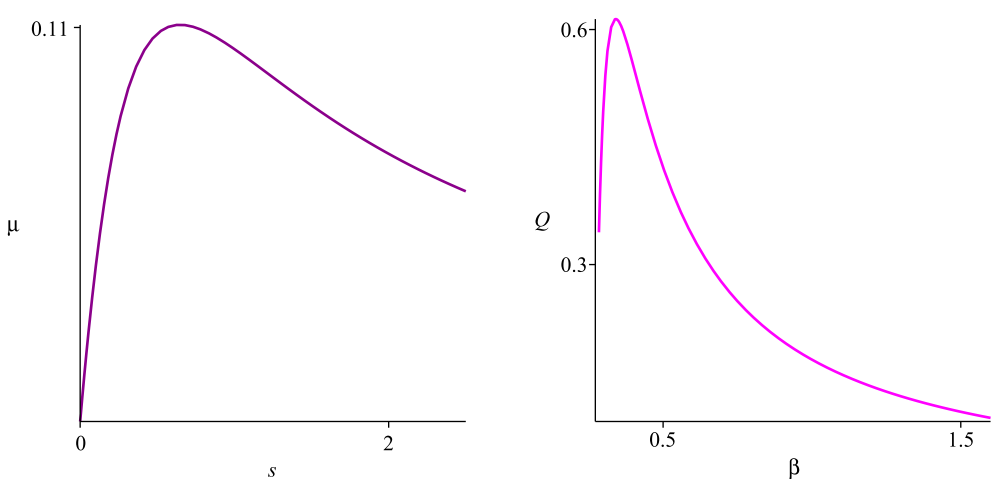

FIGURE 1 Graphs of $\mu(s)$ (left) and of $Q(\beta)$ (right) [Color figure can be viewed at wileyonlinelibrary.com]

Theorem 2 suggests a theoretical upper bound $\tau^* > 0$ such that the control system (6) to (7) is globally stabilizable for any $\tau < \tau^*$. On the other hand, it follows from Proposition 1 (see Appendix) that the model dynamics (1) is globally stabilizable for any value of $u \in [u^-, u^+]$. Computer simulations using the parameter values (11) show that the trajectories of (1) approach the corresponding equilibrium point within time $t^* = 65$ days. Since in practice stabilization with constant dilution rate is the most simple and usual approach, the value t^* will serve us as a time measure for achieving global stabilizability of the sampled-data control system Σ. We shall determine numerically the values of the sampling period τ for which the corresponding trajectories of the closed-loop system Σ reach the target point $\overline{p}_\beta$ within the same reasonable time $t^* = 65$ days.

Figures 2 and 3 show that this is possible for $\tau < 5$ days. For larger values of the sampling period, for example, $\tau = 10$ days, the target point $\overline{p}_\beta$ cannot be reached within $t = 65$ days; the trajectory stops at some end point, which is relatively away from the target point $\overline{p}_\beta$ (see Figure 4).

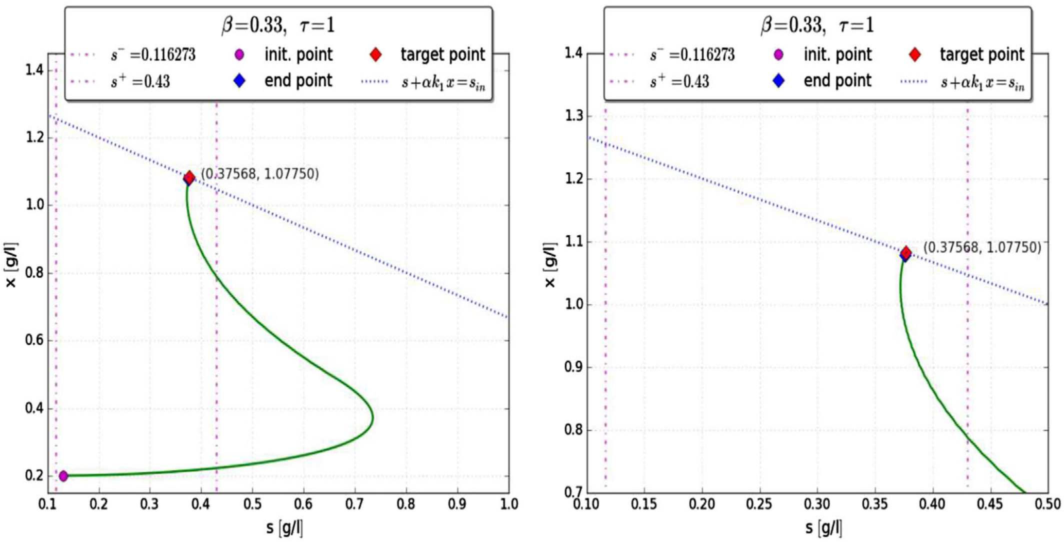

FIGURE 2 $\beta = 0.33$, $\tau = 1$: A trajectory of Σ in the (s, x) phase plane (left); enlarged fragment of the trajectory in a neighborhood of the target point $\overline{p}_\beta$ (right). The vertical dash-dot lines pass through the points s^- and s^+ [Color figure can be viewed at wileyonlinelibrary.com]

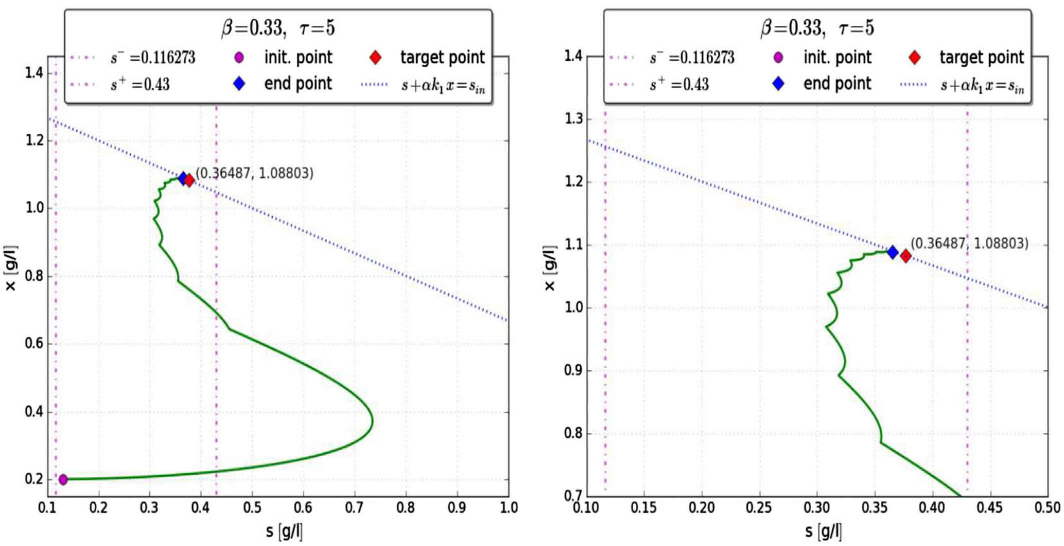

FIGURE 3 $\beta = 0.33$, $\tau = 5$: A trajectory of Σ in the (s, x) phase plane (left); enlarged fragment of the trajectory in a neighborhood of the target point $\overline{p}_\beta$ (right). The vertical dash-dot lines pass through the points s^- and s^+ [Color figure can be viewed at wileyonlinelibrary.com]

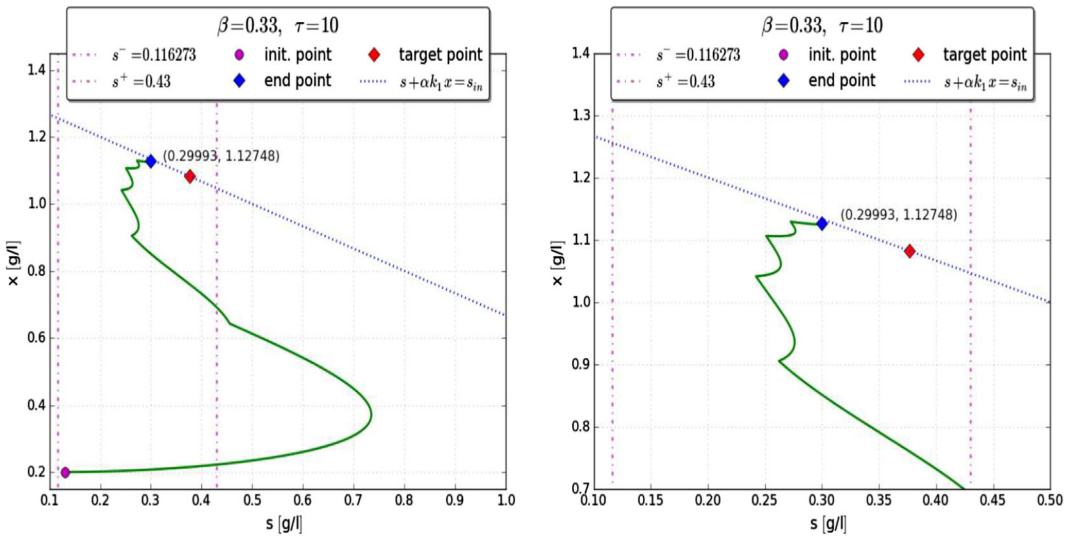

FIGURE 4 $\beta = 0.33$, $\tau = 10$: A trajectory of Σ in the (s, x) phase plane (left); enlarged fragment of the trajectory in a neighborhood of the target point $\overline{p}_\beta$ (right). The vertical dash-dot lines pass through the points s^- and s^+ [Color figure can be viewed at wileyonlinelibrary.com]

4.2 | Simulation results from ESA

Using the above values (11) of the model coefficients, the input-output static characteristic $Q(\beta)$ is unimodal and takes its maximum for $\beta = \beta_{\max} \approx 0.34115 < \beta_\mu$ (see Figure 1 (right)). Then

$$Q_{\max} = Q(\overline{p}_{\beta_{\max}}) \approx 0.61338, \quad \overline{s}_{\max} \approx 0.42971, \quad \overline{x}_{\max} \approx 1.04687.$$

The results from the ESA are presented in Figures 5 and 6 for values of the sampling period $\tau = 1$ and $\tau = 3$, respectively. The "jumps" in the left-hand side graphs correspond to the different choices of the points β^j, $j = 1, 2, \ldots$.

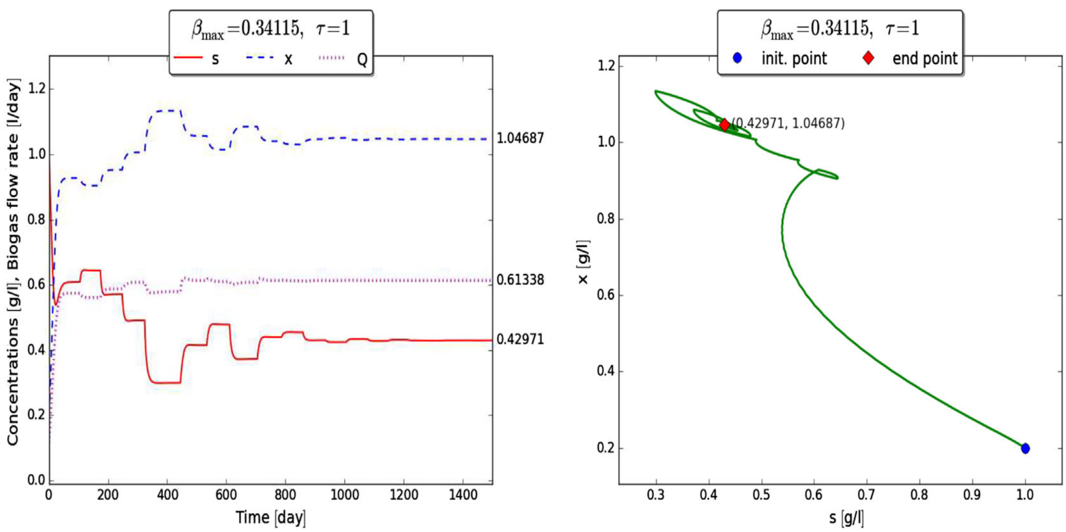

FIGURE 5 $\tau = 1$: Visualization of the numerical results from the ESA: left plot—time evolution of s, x, and Q; right plot—the trajectory in the phase plane (s, x). [Color figure can be viewed at wileyonlinelibrary.com]

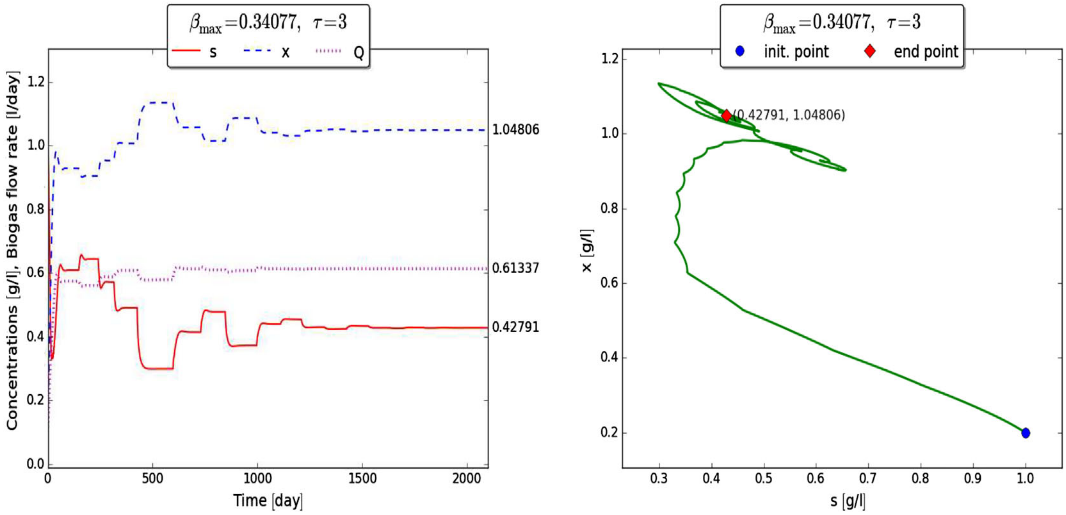

FIGURE 6 $\tau = 3$: Visualization of the numerical results from the ESA: left plot—time evolution of s, x, and Q; right plot—the trajectory in the phase plane (s, x). [Color figure can be viewed at wileyonlinelibrary.com]

5 | CONCLUSION

The article presents results on global stabilizability of a two-dimensional model of an anaerobic bioreactor with methane (biogas) production. The global stabilization is achieved by means of a bounded piecewise constant feedback control law. The feedback is related to the gaseous output only and involves an auxiliary positive parameter β. This parameter is lower bounded and its bound can be computed knowing respective bounds for the model parameters. A nontrivial equilibrium point $\overline{p}_\beta = (\overline{s}, \overline{x})$ of the closed-loop system is determined as a function of β. Then Theorem 1 shows that all trajectories of the closed loop system Σ enter the set $\{(s, x) : s^- \le s \le s^+, x > 0\}$ in finite time and for any value of the sampling period $\tau > 0$. We prove further in Theorem 2, the global stabilizability of Σ toward the previously chosen equilibrium (target) point $\overline{p}_\beta = (\overline{s}, \overline{x})$. The proof proposes an upper bound τ^* of the sampling period τ so that the global convergence of the solutions is achieved for any $\tau \in (0, \tau^*)$. This bound is theoretical and is supposed to be sufficiently small. The numerical simulations in Section 4 demonstrate that global stability can be achieved for practically reasonable values of the sampling period τ. For values of τ, for which the closed-loop system is globally stable, a numerical ESA is applied to stabilize the dynamics toward the equilibrium point where maximum methane flow rate is achieved. Numerical simulation results demonstrate the facilities of ESA as well.

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APPENDIX

The next lemma (known as Barbalat's Lemma, cf. [12]) is used in the proofs of Lemma 2 and Theorem 1.

Barbălat's Lemma. If $f : (0, \infty) \to R$ is Riemann integrable and uniformly continuous, then $\lim_{t\to\infty} f(t) = 0$.

The next Proposition 1 is a simple corollary of theorem 3 from Reference 4, concerning the model (1). Proposition 1 is used in the proof of Theorem 2.

Proposition 1. *Let Assumption* 1 *be fulfilled. Choose some* $\tilde{u} \in \left(0, \frac{1}{\alpha}\mu(s_{\text{in}})\right)$ *and let* $\tilde{p} = \left(\tilde{s}, \frac{s_{\text{in}}-\tilde{s}}{\alpha k_1}\right)$ *be the equilibrium point of the model* (1) *with* $\tilde{s}$ *satisfying the equality* $\tilde{u} = \frac{1}{\alpha}\mu(\tilde{s})$. *Assume that* μ *satisfies the following inequalities:* $\mu(s) < \mu(\tilde{s})$ *for each* $s \in (0, \tilde{s})$ *and* $\mu(s) > \mu(\tilde{s})$ *for each* $s \in (\tilde{s}, s_{\text{in}})$. *Then the solution* $(\tilde{s}(t), \tilde{x}(t))$ *of* (1) *starting from an arbitrary point* $(s(0), x(0)) \in \Omega$ *and corresponding to* $\tilde{u}$ *converges asymptotically toward* $\tilde{p}$.

Proof of Lemma 1. We fix an arbitrary point $\zeta^0 \in \Omega$. Clearly, $s(t) > 0$ and $x(t) > 0$ for each $t > 0$, where the solution is defined. We set

$$q_1(t) := s(t) + k_1 x(t) - \frac{s_{\text{in}}}{\alpha} \quad \text{and} \quad q_2(t) := s(t) + k_1 x(t) - s_{\text{in}}.$$

One can directly check that

$$\dot{q}_1(t) = \chi(t)(s_{\text{in}} - s(t)) - \alpha\chi(t)k_1x(t) \le \chi(t)\left(s_{\text{in}} - \alpha\left(s(t) + k_1x(t)\right)\right) = -\alpha\chi(t)q_1(t),$$

and hence

$$q_1(t) \le q_1(0) \cdot e^{-\alpha\int_0^t \chi(\sigma)\, d\sigma} \le q_1(0) \cdot e^{-\alpha t u^-}. \tag{A1}$$

Analogously one can obtain that

$$q_2(t) \ge q_2(0) \cdot e^{-\int_0^t \chi(\sigma)\, d\sigma} \ge q_2(0) \cdot e^{-t u^+}. \tag{A2}$$

The definitions of $q_1(\cdot)$ and $q_2(\cdot)$, and the estimates (A1) and (A2) imply that for each ε there exists $T_\varepsilon > 0$ such that for each $t \geq T_\varepsilon$ the inequalities (9) hold true.

Since $s(t)$ and $x(t)$ are positive, it follows from (9) that $s(t)$ and $x(t)$ are bounded. Thus, the trajectory $\phi(t,\zeta^0) = (s(t), x(t))$ is well defined and bounded for each $t \geq 0$. ■

Proof of Lemma 2. Suppose that $s(t) \geq s_{\text{in}}$ for each $t \geq 0$. Then we have from (6) that

$$\dot{s}(t) = \chi(t)(s_{\text{in}} - s(t)) - k_1\mu(s(t))x(t) < 0.$$

According to Barbălat's Lemma we obtain

$$0 = \lim_{t\to\infty} \dot{s}(t) = \lim_{t\to\infty} [\chi(t)(s_{\text{in}} - s(t)) - k_1\mu(s(t))x(t)].$$

The latter equalities imply that $\chi(t)(s(t) - s_{\text{in}}) \to 0$ and $x(t) \to 0$ as $t \to \infty$. Because $\chi(t) \geq u^- > 0$ for each $t > 0$, we obtain that $\lim_{t\to\infty} s(t) = s_{\text{in}}$.

Since $s(t)$ is bounded and $x(t) \to 0$ as $t \to \infty$, there exists $\tilde{T}_1$ so that $\chi(t) = u^-$ for each $t \geq \tilde{T}_1$. According to Assumptions 3(ii) and (iv), there exists $\eta > 0$ such that $\mu(s_{\text{in}}) > \mu(s^+) + \eta > \mu(s^-) + \eta$. Taking into account that $u^- = \mu(s^-)/\alpha$, we obtain that $\mu(s_{\text{in}}) > \alpha u^- + \eta$. The continuity of the function μ implies the existence of $\delta > 0$ such the $\mu(s) > \mu(s_{\text{in}}) - \eta/2$ whenever $|s - s_{\text{in}}| < \delta$. Then the equality $\lim_{t\to\infty} s(t) = s_{\text{in}}$ implies the existence of $\tilde{T}_2$ such that $|s(t) - s_{\text{in}}| < \delta$ for each $t \geq \tilde{T}_2$, and hence $\mu(s(t)) > \mu(s_{\text{in}}) - \eta/2$ for each $t \geq \tilde{T}_2$. Thus

$$\mu(s(t)) > \mu(s_{\text{in}}) - \frac{\eta}{2} > \mu(s^-) + \frac{\eta}{2} = \alpha u^- + \frac{\eta}{2} = \alpha\chi(t) + \frac{\eta}{2}$$

for each $t \geq \max(\tilde{T}_1, \tilde{T}_2)$. However, then we have that

$$\dot{x}(t) = (\mu(s(t)) - \alpha\chi(t))x(t) \geq \frac{\eta}{2}x(t) > 0$$

for each $t \geq \max(\tilde{T}_1, \tilde{T}_2)$. It follows from here that $x(t)$ cannot tend to zero as t tends to $+\infty$. This contradiction shows that there exists a sufficiently large $T > 0$ with $s(T) < s_{\text{in}}$. Moreover, if the equality $s(\tilde{t}) = s_{\text{in}}$ holds true for some $\tilde{t} \geq T$, then we have

$$\dot{s}(\tilde{t}) = \chi(\tilde{t})(s_{\text{in}} - s(\tilde{t})) - k_1\mu(s(\tilde{t}))x(\tilde{t}) = -k_1\mu(s(\tilde{t}))x(\tilde{t}) < 0.$$

The last inequality shows that $s(t) < s_{\text{in}}$ for each $t > T$.Let us choose $\varepsilon \in (0, (s_{\text{in}} - s^+ - \hat{\varepsilon})/2)$ (cf Assumption 3(iv)). According to Lemma 1, there exists $T_\varepsilon > 0$ so that for each $t \geq T_\varepsilon$ the following inequality is fulfilled

$$s_{\text{in}} - \varepsilon \leq s(t) + k_1x(t) \leq \frac{s_{\text{in}}}{\alpha} + \varepsilon. \tag{A3}$$

Let us assume that $x(t) \leq \varepsilon/k_1$ for some $t \geq T_\varepsilon$. Then we obtain from (A3) that

$$s(t) \geq s_{\text{in}} - \varepsilon - k_1x(t) \geq s_{\text{in}} - 2\varepsilon \geq s^+ + \hat{\varepsilon}.$$

It follows from Assumption 3(iv) that

$$\dot{x}(t) = (\mu(s(t)) - \alpha\chi(t))x(t) \geq (\mu(s^+) + \eta - \alpha\chi(t))x(t) = (\eta + \alpha u^+ - \alpha\chi(t))x(t) \geq \eta x(t). \tag{A4}$$

Hence, $x(t)$ will be a strictly increasing function for each $t \geq T_\varepsilon$, for which $x(t) \leq \varepsilon/k_1$. If the equality $x(\tilde{t}) = \varepsilon/k_1$ holds for some $\tilde{t} \geq T_\varepsilon$, then we obtain from (A4) that $\dot{x}(\tilde{t}) > 0$. This completes the proof. ■

Proof of Theorem 1. Let us fix an arbitrary $\tau > 0$. If $s(t) \in (s^-, s^+)$ for some $t > 0$, we are done.

Let us assume that $s(t) \leq s^-$ for each $t \geq 0$. Then Assumption 3(iii) implies that

$$\mu(s(t)) \leq \mu(s^-) = \alpha u^-.$$

It follows from the definition of $\chi(\cdot)$ that $\chi(t) \geq u^-$ for each $t \geq 0$. Then

$$\mu(s(t)) - \alpha\chi(t) \leq \mu(s(t)) - \alpha u^- \leq 0$$

for each $t \geq 0$, and hence $\dot{x}(t) = (\mu(s(t)) - \alpha\chi(t))x(t) \leq 0$. Hence, $x(t)$, $t \geq 0$, is nonincreasing, and thus there exists $x^* := \lim_{t\to\infty} x(t)$. Applying Barbălat's Lemma, we obtain that

$$\dot{x}(t) = (\mu(s(t)) - \alpha\chi(t))x(t) \to 0 \ \text{ as } \ t \to +\infty.$$

Since $x(t)$ has a positive lower bound $x_{\min}$ (see (10)), we obtain from $\dot{x}(t) = (\mu(s(t)) - \alpha\chi(t))x(t)$ that

$$(\mu(s(t)) - \alpha u^-) + \alpha(u^- - \chi(t)) \to 0 \quad \text{as } \ t \to +\infty.$$

Because both addends are nonpositive, the last relation leads to

$$\mu(s(t)) \to \alpha u^- \quad \text{and} \quad \chi(t) \to u^- \ \text{ as } \ t \to +\infty.$$

According to Assumption 3(iii) we obtain that $s(t) \to s^-$ as $t \to +\infty$. Barbălat's Lemma implies

$$\dot{s}(t) = \chi(t)(s_{\text{in}} - s(t)) - k_1\mu(s(t))x(t) \to 0 \quad \text{as } \ t \to \infty,$$

and hence there exists x^* so that

$$u^-(s_{\text{in}} - s^-) - k_1\mu(s^-)x^* = 0, \ \text{ ie, } u^-(s_{\text{in}} - s^-) - \alpha k_1 u^- x^* = 0.$$

Therefore,

$$s_{\text{in}} = s^- + \alpha k_1 x^*. \tag{A5}$$

Remind that

$$\chi(t) = \begin{cases} u^-, & \text{if } \beta k_2\mu(s(\theta(t)))x(\theta(t)) \leq u^-, \\ \beta k_2\mu(s(\theta(t)))x(\theta(t)), & \text{if } u^- \leq \beta k_2\mu(s(\theta(t)))x(\theta(t)) \leq u^+, \\ u^+, & \text{if } \beta k_2\mu(s(\theta(t)))x(\theta(t)) \geq u^+ \end{cases} \tag{A6}$$

for each $t \in [\theta(t), \theta(t) + \tau)$. Also, we have that $\chi(t) \to u^-$ as $t \to +\infty$. This is possible iff for each $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that $\beta k_2\mu(s(\theta(t)))x(\theta(t)) < u^- + \varepsilon$ for each $t > T_\varepsilon$ for which $\beta k_2\mu(s(\theta(t)))x(\theta(t)) > u^-$ (if $\beta k_2\mu(s(\theta(t)))x(\theta(t)) \leq u^-$, then $\chi(t) = u^-$). Then for each $t > T_\varepsilon$ we have that

$$\beta k_2\mu(s(\theta(t)))x(\theta(t)) = \frac{\mu(s(\theta(t)))x(\theta(t))}{\alpha\bar{x}} < u^- + \varepsilon.$$

Taking a limit in this inequality, we obtain that

$$\frac{\mu(s^-)x^*}{\alpha\bar{x}} \leq u^-, \qquad \text{ie, } \ \frac{\alpha u^- x^*}{\alpha\bar{x}} \leq u^- \ \text{ or } \ x^* \leq \bar{x}.$$

However, this is impossible because (see (A5))

$$x^* = \frac{s_{\text{in}} - s^-}{\alpha k_1} > \frac{s_{\text{in}} - \bar{s}}{\alpha k_1} = \bar{x}.$$

Assuming that $s(t) \geq s^+$ for each $t \geq 0$, we obtain a contradiction in a similar way. Therefore, there exists a time moment $\hat{t}$ such that $s(\hat{t})$ belongs to the interval (s^-, s^+). This completes the proof. ■

Proof of Theorem 2. We divide the proof in three steps for better readability and clarity.

Claim 1. There exist a positive real δ and a time moment $T_\delta > 0$ so that $s(t) \in [\bar{s} - \delta, \bar{s} + \delta]$ for each $t \geq T_\delta$.

Proof of Claim 1. According to Theorem 1, there exists $\hat{t} > 0$ with $s(\hat{t}) \in (s^-, s^+)$. Then $\chi(\hat{t}) = \beta k_2 \mu(s(\theta(\hat{t})))x(\theta(\hat{t}))$, and

$$\begin{aligned}
\dot{s}(\hat{t}) &= \beta k_2 \mu(s(\theta(\hat{t})))x(\theta(\hat{t}))(s_{\text{in}} - s(\hat{t})) - k_1 \mu(s(\hat{t}))x(\hat{t}) \\
&= \beta k_2 \mu(s(\theta(\hat{t})))x(\theta(\hat{t})) \left(s_{\text{in}} - s(\hat{t}) - \frac{\alpha k_1}{\alpha \beta k_2} \right) + k_1 \left(\mu(s(\theta(\hat{t})))x(\theta(\hat{t})) - \mu(s(\hat{t}))x(\hat{t}) \right) \\
&= \beta k_2 \mu(s(\theta(\hat{t})))x(\theta(\hat{t})) \left(s_{\text{in}} - s(\hat{t}) - \alpha k_1 \bar{x} \right) + k_1 \left(\mu(s(\theta(\hat{t})))x(\theta(\hat{t})) - \mu(s(\hat{t}))x(\hat{t}) \right) \\
&= -\beta k_2 \mu(s(\theta(\hat{t})))x(\theta(\hat{t})) \left(s(\hat{t}) - \bar{s} \right) + k_1 \left(\mu(s(\theta(\hat{t})))x(\theta(\hat{t})) - \mu(s(\hat{t}))x(\hat{t}) \right).
\end{aligned} \tag{A7}$$

Let us fix an arbitrary $\delta_0 \in (0, \min(\bar{s} - s^-, s^+ - \bar{s}))$ and set

$$K^\pm := \frac{\alpha u^\pm}{\mu(\bar{s} \pm \delta_0)}. \tag{A8}$$

Clearly, $K^+ > 1$ and $K^- \in (0, 1)$.

Denote by $L_\mu > 0$ a Lipschitz constant of the function μ, by $\hat{C}$ an upper bound of the set of real numbers $\{\mu'(s) : s \in [s^-, s^+]\}$, and set

$$A^+ := \frac{\mu(s^+)B_x}{\alpha x_{\min}} + k_1 B_x \hat{C}, \quad A^- := \frac{\mu(s^+)B_x}{\alpha x_{\min}} + 2k_1 B_x L_\mu,$$

where B_x is defined in Remark 1.

Choose a real number

$$\delta \in (0, \min(\delta_0, \omega)), \tag{A9}$$

where

$$\omega := \min\left(\frac{(K^+ - 1)\mu^2(s^-)K^+ x_{\min}}{2\left(\hat{C}A^+ + \mu(s^+)B_x L_\mu (1 + K^+)\right)}, \frac{(1 - K^-)\mu^2(s^-)x_{\min}^2}{2B_x \left(\hat{C}A^- + \mu(s^+)B_x L_\mu (1 + K^-)\right)} \right).$$

We set

$$\tau_1 := \frac{u^- \delta}{2\bar{k}_1(B_s L_x + L_s B_x)} \quad \text{with } \bar{k}_1 := \max(1, k_1). \tag{A10}$$

We assume that the sampling period $\tau \in (0, \tau_1)$. Taking into account Remark 1, we obtain according to (A10) that for each $t \in [\theta(\hat{t}), \theta(\hat{t}) + \tau)$

$$\begin{aligned}
&|\mu(s(t))x(t) - \mu(s(\theta(t)))x(\theta(t))| \\
&\quad \leq \mu(s(t))|x(t) - x(\theta(t))| + |\mu(s(t)) - \mu(s(\theta(t)))|x(\theta(t)) \\
&\quad \leq B_s L_x |t - \theta(t)| + L_s B_x |t - \theta(t)| \\
&\quad \leq (B_s L_x + L_s B_x)\tau.
\end{aligned} \tag{A11}$$

First, we assume that $s(\hat{t}) \in (s^-, \bar{s} - \delta]$. We have according to (A10) and the estimate (A11) that for each $t \in [\theta(\hat{t}), \theta(\hat{t}) + \tau)$

$$|\mu(s(t))x(t) - \mu(s(\theta(t)))x(\theta(t))| \leq \frac{u^- \delta}{2\bar{k}_1},$$

and hence from (A7)

$$\dot{s}(\hat{t}) \geq u^- \left(\bar{s} - s(\hat{t}) \right) - \frac{k_1 u^- \delta}{2\bar{k}_1} \geq u^- \delta - \frac{u^- \delta}{2} = \frac{u^- \delta}{2} > 0.$$

Next, we assume that $s(\hat{t}) \in [\bar{s} + \delta, s^+)$. Using the estimate (A11), we obtain from (A10) and the obvious inequality $u^- < u^+$ that

$$|\mu(s(t))x(t) - \mu(s(\theta(t)))x(\theta(t))| \leq \frac{u^+\delta}{2\bar{k}_1},$$

and therefore

$$\dot{s}(\hat{t}) \leq -u^+ \left(s(\hat{t}) - \bar{s}\right) + \frac{k_1 u^+ \delta}{2\bar{k}_1} \leq -u^+\delta + \frac{u^+\delta}{2} = -\frac{u^+\delta}{2} < 0.$$

These two cases show that there exists $T_\delta > \hat{t} > 0$ such that the following inclusion holds true

$$s(t) \in [\bar{s} - \delta, \bar{s} + \delta] \quad \text{for all } t \geq T_\delta. \tag{A12}$$

Claim 2. There exist time $\overline{T} > T_\delta > 0$ so that $\kappa(s(t), x(t)) \in (u^-, u^+)$ for each $t \geq \overline{T}$.

Proof of Claim 2. Let us assume that $\kappa(s(t), x(t)) \leq u^-$ for each $t \geq T_\delta$. Then applying Proposition 1 with $\tilde{u} = u^-$, we obtain that $(s(t), x(t)) \to (s^-, x^-)$ as $t \to \infty$ with $\mu(s^-) = \alpha u^-$ and

$$x^- := (s_{\text{in}} - s^-)/(\alpha k_1). \tag{A13}$$

According to the definition of the feedback (8), we have that

$$\kappa(s(t), x(t)) = \beta k_2 \mu(s(t))x(t) \leq u^- \quad \text{for all} \;\; t \geq T_\delta.$$

Taking a limit as $t \to +\infty$, we obtain

$$\beta k_2 \mu(s^-)x^- \leq u^-,$$

and from (4) and Assumption 3(i) it follows that

$$\frac{\mu(s^-)x^-}{\alpha\bar{x}} \leq \frac{\mu(s^-)}{\alpha}, \quad \text{ie,} \;\; x^- \leq \bar{x}.$$

Hence, using (4) and (A13), we obtain that

$$s_{\text{in}} = s^- + \alpha k_1 x^- = \bar{s} + \alpha k_1 \bar{x}.$$

Because $s^- < \bar{s}$, the last equality implies that $x^- > \bar{x}$. The obtained contradiction shows that the assumption $\kappa(s(t), x(t)) \leq u^-$ for each $t \geq T_\delta$ is wrong. Analogously, one can prove that the assumption $\kappa(s(t), x(t)) \geq u^+$ for each $t \geq T_\delta$ is also impossible.

Therefore there exist $\hat{t} \geq T_\delta$ such that

$$s(\hat{t}) \in [\bar{s} - \delta, \bar{s} + \delta] \quad \text{and} \quad \kappa(s(\hat{t}), x(\hat{t})) \in (u^-, u^+).$$

Let us choose an arbitrary time moment $\bar{t} \geq \hat{t}$. We have

$$\begin{aligned} \frac{d}{dt}\kappa(s(\bar{t}), x(\bar{t})) &= \frac{1}{\alpha\bar{x}}\left[\mu'(s(\bar{t}))\dot{s}(\bar{t}) + \mu(s(\bar{t}))\dot{x}(\bar{t})\right] \\ &= \frac{1}{\alpha\bar{x}}\left[\mu'(s(\bar{t}))\left(\frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(s_{\text{in}} - s(\bar{t})) - k_1\mu(s(\bar{t}))x(\bar{t})\right)\right. \\ &\quad \left. + \mu(s(\bar{t}))x(\bar{t})\left(\mu(s(\bar{t})) - \alpha\frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}\right)\right]. \end{aligned} \tag{A14}$$

We remind that $K^+ := \alpha u^+/\mu(\bar{s}+\delta_0)$ and $K^+ > 1$ (see (A8)). First, we assume that $\bar{x} \le x(\bar{t}) < K^+\bar{x}$. Assumption 3(ii) and the definition of the function $\kappa(\cdot)$ imply that

$$\kappa(s(\bar{t}), x(\bar{t})) = \frac{\mu(s(\bar{t}))x(\bar{t})}{\alpha\bar{x}} < \frac{\mu(\bar{s}+\delta)K^+\bar{x}}{\alpha\bar{x}} = \frac{\mu(\bar{s}+\delta)\alpha u^+\bar{x}}{\mu(\bar{s}+\delta_0)\alpha\bar{x}} < u^+.$$

Next we assume that $x(\bar{t}) \ge K^+\bar{x}$. Without loss of generality (see (A12)) we may think that $s(\theta(\bar{t})) \in [\bar{s}-\delta, \bar{s}+\delta] \subset (s^-, s^+)$. Then we obtain from (A14)

$$\begin{aligned}
&\frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(s_{\text{in}} - s(\bar{t})) - k_1\mu(s(\bar{t}))x(\bar{t}) \\
&= \frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(\bar{s} - s(\bar{t}) + \alpha k_1\bar{x}) - k_1\mu(s(\bar{t}))x(\bar{t}) \\
&= \frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(\bar{s} - s(\bar{t})) + k_1\mu(s(\theta(\bar{t})))x(\theta(\bar{t})) - k_1\mu(s(\bar{t}))x(\bar{t}) \\
&< \frac{\mu(\bar{s}+\delta)B_x\delta}{\alpha\bar{x}} + k_1\left[\mu(s(\theta(\bar{t})))x(\theta(\bar{t})) - \mu(s(\bar{t}))x(\bar{t})\right] \\
&\le \frac{\mu(\bar{s}+\delta)B_x\delta}{\alpha\bar{x}} + k_1\left[\mu(\bar{s}+\delta)(x(\bar{t}) + L_x\tau)) - \mu(s(\bar{t}))x(\bar{t})\right] \\
&= \frac{\mu(\bar{s}+\delta)B_x\delta}{\alpha\bar{x}} + k_1\left[\mu(\bar{s}+\delta)L_x\tau + x(\bar{t})(\mu(\bar{s}+\delta) - \mu(s(\bar{t})))\right] \\
&< \frac{\mu(s^+)B_x}{\alpha x_{\min}}\delta + k_1\left[\mu(s^+)L_x\tau + B_x\hat{C}\delta\right] = \left[\frac{\mu(s^+)B_x}{\alpha x_{\min}} + k_1B_x\hat{C}\right]\delta + k_1\mu(s^+)L_x\tau \\
&= A^+\delta + B\tau,
\end{aligned}$$

where $B := k_1\mu(s^+)L_x$. In the same way we obtain that

$$\begin{aligned}
\mu(s(\bar{t})) - \frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\bar{x}} &< \mu(\bar{s}+\delta) - \frac{\mu(\bar{s}-\delta)x(\theta(\bar{t}))}{\bar{x}} \\
&= \mu(\bar{s}+\delta) - \frac{\mu(\bar{s}-\delta)(x(\theta(\bar{t})) - x(\bar{t}))}{\bar{x}} - \frac{\mu(\bar{s}-\delta)x(\bar{t})}{\bar{x}} \\
&< \mu(\bar{s}+\delta) + \frac{\mu(s^-)L_x\tau}{\bar{x}} - K^+\mu(\bar{s}-\delta) \\
&= \mu(\bar{s})(1-K^+) + (\mu(\bar{s}+\delta) - \mu(\bar{s})) + K^+(\mu(\bar{s}) - \mu(\bar{s}-\delta)) + \frac{\mu(s^-)L_x\tau}{\bar{x}} \\
&< \mu(s^-)(1-K^+) + L_\mu(1+K^+)\delta + \frac{\mu(s^-)L_x\tau}{\bar{x}}.
\end{aligned}$$

Let us remind that $\hat{C}$ is an upper bound for the set of real numbers $\{\mu'(s) : s \in [s^-, s^+]\}$ and that $x(\bar{t}) \ge K^+\bar{x}$ holds true. Then the presentation (A14) and the above written estimates imply

$$\begin{aligned}
\frac{d}{dt}\kappa(s(\bar{t}), x(\bar{t})) &< \frac{1}{\alpha\bar{x}}\left[\hat{C}(A^+\delta + B\tau) + \mu(s(\bar{t}))x(\bar{t})\left(\mu(s^-)(1-K^+) + L_\mu(1+K^+)\delta + \frac{\mu(s^-)L_x\tau}{\bar{x}}\right)\right] \\
&< -\frac{(K^+-1)\mu^2(s^-)K^+}{\alpha} + \frac{1}{\alpha\bar{x}}\left[\hat{C}(A^+\delta + B\tau) + \mu(s(\bar{t}))x(\bar{t})\left(L_\mu(1+K^+)\delta + \frac{\mu(s^-)L_x\tau}{\bar{x}}\right)\right] \\
&< -\frac{(K^+-1)\mu^2(s^-)K^+}{\alpha} + \frac{\hat{C}(A^+\delta + B\tau)}{\alpha x_{\min}} + \frac{\mu(s^+)B_x\left(L_\mu(1+K^+)\delta + \frac{\mu(s^-)L_x\tau}{x_{\min}}\right)}{\alpha x_{\min}} \\
&= -\frac{(K^+-1)\mu^2(s^-)K^+}{\alpha} + \frac{\hat{C}A^+ + \mu(s^+)B_xL_\mu(1+K^+)}{\alpha x_{\min}}\delta + \left(\frac{\hat{C}B}{\alpha x_{\min}} + \frac{\mu(s^-)\mu(s^+)L_xB_x}{\alpha x_{\min}^2}\right)\tau \\
&< -\frac{(K^+-1)\mu^2(s^-)K^+}{4\alpha} =: \gamma^- < 0.
\end{aligned}$$

The last inequality is obtained according to the choice of δ (see (A9)) and for each $\tau \in (0, \min(\tau_1, \tau_2))$, where

$$\tau_2 := \frac{(K^+ - 1)\mu^2(s^-)K^+ x_{\min}^2}{4\left(\hat{C}Bx_{\min} + \mu(s^-)\mu(s^+)L_x B_x\right)}.$$

This inequality means that $\frac{d}{dt}\kappa(s(\bar{t}), x(\bar{t})) < \gamma^- < 0$ at each point $\bar{t} \geq T_\delta$ for which $\kappa(s(\bar{t}), x(\bar{t})) < u^+$ and $x(\bar{t}) \geq K^+\bar{x}$.

We remind that $K^- := \alpha u^- / \mu(\bar{s} - \delta_0)$ and $K^- \in (0, 1)$ (see (A8)). We assume first that $\bar{x} \geq x(\bar{t}) > K^-\bar{x}$. The definition of the function $\kappa(\cdot)$ and the choice of δ (see (A9)) imply

$$\kappa(s(\bar{t}), x(\bar{t})) = \frac{\mu(s(\bar{t}))x(\bar{t})}{\alpha\bar{x}} > \frac{\mu(\bar{s} - \delta)K^-\bar{x}}{\alpha\bar{x}} = \frac{\mu(\bar{s} - \delta)\alpha u^-\bar{x}}{\mu(\bar{s} - \delta_0)\alpha\bar{x}} > u^-.$$

Next we assume that $x(\bar{t}) \leq K^-\bar{x}$. Without loss of generality we may think that $s(\theta(\bar{t})) \in [\bar{s} - \delta, \bar{s} + \delta] \subset (s^-, s^+)$. Then using (A14) we obtain

$$\begin{aligned}
\frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(s_{\text{in}} - s(\bar{t})) - k_1\mu(s(\bar{t}))x(\bar{t}) &= \frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(\bar{s} - s(\bar{t}) + \alpha k_1\bar{x}) - k_1\mu(s(\bar{t}))x(\bar{t}) \\
&= \frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\alpha\bar{x}}(\bar{s} - s(\bar{t})) + k_1\left[\mu(s(\theta(\bar{t})))x(\theta(\bar{t})) - \mu(s(\bar{t}))x(\bar{t})\right] \\
&\geq -\frac{\mu(\bar{s} + \delta)B_x}{\alpha\bar{x}}\delta + k_1\left[\mu(\bar{s} - \delta)(x(\bar{t}) - L_x\tau) - \mu(\bar{s} + \delta)x(\bar{t})\right] \\
&> -\frac{\mu(s^+)B_x}{\alpha\bar{x}}\delta - k_1\mu(s^+)L_x\tau + k_1 x(\bar{t})\left(\mu(\bar{s} - \delta) - \mu(\bar{s} + \delta)\right) \\
&> -\frac{\mu(s^+)B_x}{\alpha x_{\min}}\delta - k_1\mu(s^+)L_x\tau - 2k_1 B_x L_\mu\delta = -\left(\frac{\mu(s^+)B_x}{\alpha x_{\min}} + 2k_1 B_x L_\mu\right)\delta - k_1\mu(s^+)L_x\tau \\
&= -A^-\delta - B\tau,
\end{aligned}$$

where $B := k_1\mu(s^+)L_x$. In the same way we obtain according to (A14) that

$$\begin{aligned}
\mu(s(\bar{t})) - \frac{\mu(s(\theta(\bar{t})))x(\theta(\bar{t}))}{\bar{x}} &\geq \mu(\bar{s} - \delta) - \frac{\mu(\bar{s} + \delta)x(\theta(\bar{t}))}{\bar{x}} \\
&= \mu(\bar{s} - \delta) - \frac{\mu(\bar{s} + \delta)(x(\theta(\bar{t})) - x(\bar{t}))}{\bar{x}} - \frac{\mu(\bar{s} + \delta)x(\bar{t})}{\bar{x}} \\
&> \mu(\bar{s} - \delta) - \frac{\mu(s^+)L_x}{\bar{x}}\tau - K^-\mu(\bar{s} + \delta) \\
&= \mu(\bar{s})(1 - K^-) + (\mu(\bar{s} - \delta) - \mu(\bar{s})) + K^-(\mu(\bar{s}) - \mu(\bar{s} + \delta)) - \frac{\mu(s^+)L_x\tau}{\bar{x}} \\
&> \mu(s^-)(1 - K^-) - L_\mu(1 + K^-)\delta - \frac{\mu(s^+)L_x\tau}{\bar{x}}.
\end{aligned}$$

Let us remind that $\hat{C}$ is an upper bound for the set $\{\mu'(s) : s \in [s^-, s^+]\}$ and that $x(\bar{t}) \leq K^-\bar{x}$. Then using (A14), the above written estimates and Lemma 2 we obtain

$$\begin{aligned}
\frac{d}{dt}\kappa(s(\bar{t}), x(\bar{t})) &> \frac{1}{\alpha\bar{x}}\left[-\hat{C}(A^-\delta + B\tau) + \mu(s(\bar{t}))x(\bar{t})\left(\mu(s^-)(1 - K^-) - L_\mu(1 + K^-)\delta - \frac{\mu(s^+)L_x\tau}{\bar{x}}\right)\right] \\
&> \frac{(1 - K^-)\mu^2(s^-)x_{\min}}{\alpha\bar{x}} - \frac{\hat{C}(A^-\delta + B\tau) + L_\mu(1 + K^-)\delta\mu(s(\bar{t}))x(\bar{t})}{\alpha\bar{x}} - \frac{\mu(s(\bar{t}))x(\bar{t})\mu(s^+)L_x\tau}{\alpha\bar{x}^2} \\
&> \frac{(1 - K^-)\mu^2(s^-)x_{\min}}{\alpha\bar{x}} - \frac{\hat{C}(A^-\delta + B\tau) + L_\mu(1 + K^-)\delta\mu(s^+)B_x}{\alpha\bar{x}} - \frac{\mu^2(s^+)B_x L_x\tau}{\alpha\bar{x}^2} \\
&> \frac{(1 - K^-)\mu^2(s^-)x_{\min}}{\alpha B_x} - \frac{\hat{C}A^- + \mu(s^+)B_x L_\mu(1 + K^-)}{\alpha x_{\min}}\delta - \frac{\hat{C}Bx_{\min} + \mu^2(s^+)B_x L_x}{\alpha x_{\min}^2}\tau \\
&> \frac{(1 - K^-)\mu^2(s^-)x_{\min}}{4\alpha B_x} =: \gamma^+ > 0.
\end{aligned}$$

The last inequality is satisfied according to the choice of δ from (A9) and for each $\tau \in (0, \min(\tau_1, \tau_2, \tau_3))$, where

$$\tau_3 := \frac{(1-K^-)\mu^2(s^-)x_{\min}^3}{4B_x\left(\hat{C}Bx_{\min} + \mu^2(s^+)B_xL_x\right)}.$$

The above inequality means that $\frac{d}{dt}\kappa(s(\bar{t}), x(\bar{t})) > \gamma^+ > 0$ at each point $\bar{t} \geq \hat{t}$ for which $\kappa(s(\bar{t}), x(\bar{t})) > u^-$ and $x(\bar{t}) \leq K^-\overline{x}$.

In conclusion, we have shown that for each $\tau \in (0, \tau_3)$, the inclusion $\kappa(s(\bar{t}), x(\bar{t})) \in (u^-, u^+)$ holds true whenever $x(\bar{t}) \in (K^-\overline{x}, K^+\overline{x})$, and

$$\frac{d}{dt}\kappa(s(\bar{t}), x(\bar{t})) \begin{cases} < \gamma^- < 0 \text{ for } x(\bar{t}) \geq K^+\overline{x} > \overline{x}, \\ > \gamma^+ > 0 \text{ for } x(\bar{t}) \leq K^-\overline{x} < \overline{x}. \end{cases}$$

From here and because $\kappa(s(\hat{t}), x(\hat{t})) \in (u^-, u^+)$, it follows that $\kappa(s(\bar{t}), x(\bar{t})) \in (u^-, u^+)$ for each $\bar{t} \geq \hat{t} \geq T_\delta$. This proves Claim 2.

It follows from Claim 1 and Claim 2 that there exist $\overline{T} > T_\delta$ such that

$$s(t) \in [\overline{s} - \delta, \overline{s} + \delta] \quad \text{and} \quad \kappa(s(t), x(t)) \in (u^-, u^+) \text{ for each } t \geq \overline{T}. \tag{A15}$$

Finally, we shall prove the main part of the theorem, namely

Claim 3. There exists a sampling period $\tau^* > 0$ so that for each $\tau \in (0, \tau^*)$ the solution $\phi(t, \zeta^0)$ of Σ converges asymptotically toward the equilibrium point $\overline{p}_\beta = (\overline{s}, \overline{x})$.

Proof of Claim 3. Let us fix an arbitrary $\bar{t} > \overline{T}$.

We set $y(t) := (s(t), x(t))$ and $f(y(t), y(\theta(t)) := \frac{d}{dt}\phi(t, \zeta^0) = \frac{d}{dt}y(t) = \dot{y}(t)$ for each $t \in [\theta(\bar{t}), \theta(\bar{t}) + \tau]$. Then

$$f(y(t), y(\theta(t))) = \begin{pmatrix} -k_1\mu(s(t))x(t) + \psi(s(\theta(t)), x(\theta(t)))(s_{\text{in}} - s(t)) \\ (\mu(s(t)) - \alpha\psi(s(\theta(t)), x(\theta(t))))x(t) \end{pmatrix},$$

and hence

$$\begin{aligned} f(y(t), y(t)) &= \begin{pmatrix} -k_1\mu(s(t))x(t) + \psi(s(t), x(t))(s_{\text{in}} - s(t)) \\ (\mu(s(t)) - \alpha\psi(s(t), x(t)))x(t) \end{pmatrix} \\ &= \begin{pmatrix} \beta k_2\mu(s(t))x(t)(s_{\text{in}} - \alpha k_1/(\alpha\beta k_2) - s(t)) \\ \alpha\beta k_2\mu(s(t))x(t)(1/(\alpha\beta k_2) - x(t)) \end{pmatrix} \\ &= \begin{pmatrix} -\beta k_2\mu(s(t))x(t)(s(t) - \overline{s}) \\ -\alpha\beta k_2\mu(s(t))x(t)(x(t) - \overline{x}) \end{pmatrix}. \end{aligned}$$

In particular, we have that

$$f(\overline{p}_\beta, \overline{p}_\beta) = \begin{pmatrix} -\beta k_2\mu(\overline{s})\overline{x}(\overline{s} - \overline{s}) \\ -\alpha\beta k_2\mu(\overline{s})\overline{x}(\overline{x} - \overline{x}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The next estimates are proved following an idea from Reference 8. For each $t \in [\theta(\bar{t}), \theta(\bar{t}) + \tau)$, the following presentation holds true

$$\begin{aligned} y(t) &= y(\theta(\bar{t})) + \int_{\theta(\bar{t})}^{t} \dot{y}(\sigma)\, d\sigma = y(\theta(\bar{t})) + \int_{\theta(\bar{t})}^{t} f(y(\sigma), y(\theta(\bar{t})))\, d\sigma \\ &= y(\theta(\bar{t})) + \int_{\theta(\bar{t})}^{t} (f(y(\sigma), y(\theta(\bar{t}))) - f(\overline{p}_\beta, \overline{p}_\beta))\, d\sigma, \end{aligned}$$

and thus

$$\|y(t) - \overline{p}_\beta\| \leq \|y(\theta(\bar{t})) - \overline{p}_\beta\| + \int_{\theta(\bar{t})}^{t} \left(L_y\|y(\sigma) - \overline{p}_\beta\| + L_\theta\|y(\theta(\bar{t})) - \overline{p}_\beta\|\right) d\sigma,$$

where L_y and L_θ are upper bounds of the partial derivatives of f with respect to the first and to the second variable, respectively.

Denote $\overline{\tau}_3 := \min(\tau_1, \tau_2, \tau_3)$. Applying the Gronwall inequality, we obtain the first estimate we need further:

$$\|y(t) - \overline{p}_\beta\| \leq C_y \|y(\theta(\bar{t})) - \overline{p}_\beta\| \ \text{ with } \ C_y := (1 + L_\theta \overline{\tau}_3) e^{L_y \overline{\tau}_3} \quad \text{for each } \ t \in [\theta(\bar{t}), \theta(\bar{t}) + \overline{\tau}_3]. \tag{A16}$$

Using the presentation

$$y(\theta(\bar{t})) = y(t) - \int_{\theta(\bar{t})}^{t} \dot{y}(\sigma)\, d\sigma$$

we obtain in the same way as above that

$$\|y(\theta(\bar{t})) - \overline{p}_\beta\| \leq \|y(t) - \overline{p}_\beta\| + \int_{\theta(\bar{t})}^{t} \left(L_y \|y(\sigma) - \overline{p}_\beta\| + L_\theta \|y(\theta(\bar{t})) - \overline{p}_\beta\| \right) d\sigma$$

and, according to the estimate (A16) it follows that

$$\|y(\theta(\bar{t})) - \overline{p}_\beta\| \leq \|y(t) - \overline{p}_\beta\| + \int_{\theta(\bar{t})}^{t} \left(L_y (1 + L_\theta \overline{\tau}_3) e^{L_y \overline{\tau}_3} \|y(\theta(\bar{t})) - \overline{p}_\beta\| + L_\theta \|y(\theta(\bar{t})) - \overline{p}_\beta\| \right) d\sigma.$$

We set

$$\tau_4 := \frac{1}{2L_y (1 + L_\theta \overline{\tau}_3) e^{L_y \overline{\tau}_3} + L_\theta}$$

and denote by $\overline{\tau}_4 = \min(\overline{\tau}_3, \tau_4)$. Then for each $\tau \in (0, \min(\tau_1, \tau_2, \tau_3, \tau_4))$ and for each $t \in [\theta(\bar{t}), \theta(\bar{t}) + \tau)$, the following estimate holds true

$$\|y(\theta(\bar{t})) - \overline{p}_\beta\| \leq C_\theta \|y(t) - \overline{p}_\beta\| \ \text{ with } \ C_\theta := \frac{1}{1 - \overline{\tau}_4 \left(L_y (1 + L_\theta \overline{\tau}_3) e^{L_y \overline{\tau}_3} + L_\theta \right)}. \tag{A17}$$

Define the function $V(y(t)) = V(s(t), x(t)) = \frac{1}{2} \left((s(t) - \overline{s})^2 + (x(t) - \overline{x})^2 \right)$; then we have

$$V'(y(t)) f(y(t), y(t)) = -\kappa(s(t), x(t)) \left((s(t) - \overline{s})^2 + \alpha (x(t) - \overline{x})^2 \right). \tag{A18}$$

In the following we shall use the obvious presentation

$$\begin{aligned}
\frac{d}{dt} V(y(t)) &= V'(y(t)) \frac{d}{dt} y(t) = V'(y(t)) f(y(t), y(\theta(\bar{t}))) \\
&= V'(y(t)) f(y(t), y(t)) + V'(y(t)) \Big(f(y(t), y(\theta(\bar{t}))) - f(y(t), y(t)) \Big)
\end{aligned}$$

as well as the inequality

$$\begin{aligned}
\mu(s(t)) x(t) &= \mu(s(\theta(t))) x(\theta(t)) + (\mu(s(t)) x(t) - \mu(s(\theta(t))) x(\theta(t))) \\
&\geq \mu(s(\theta(t))) x(\theta(t)) - |\mu(s(t)) x(t) - \mu(s(\theta(t))) x(\theta(t))|.
\end{aligned}$$

Then, using the inequality $\kappa(s(t), x(t)) > u^-$ (see (A15)) we obtain from (A18)

$$\begin{aligned}
V'(y(t)) f(y(t), y(t)) &= -\kappa(s(t), x(t)) \left((s(t) - \overline{s})^2 + \alpha (x(t) - \overline{x})^2 \right) \\
&< -\alpha u^- \left((s(t) - \overline{s})^2 + (x(t) - \overline{x})^2 \right).
\end{aligned} \tag{A19}$$

Also, it follows from (A16) that

$$\begin{aligned}
&\|f(y(t),y(\theta(\bar{t})))-f(y(t),y(t))\| \le L_\theta\|y(t)-y(\theta(\bar{t}))\| \le L_\theta\left\|\int_{\theta(\bar{t})}^{t}\dot{y}(\sigma)\,d\sigma\right\|\\
&=L_\theta\left\|\int_{\theta(\bar{t})}^{t} f(y(\sigma),y(\theta(\bar{t})))\,d\sigma\right\| = L_\theta\left\|\int_{\theta(\bar{t})}^{t}(f(y(\sigma),y(\theta(\bar{t})))-f(\overline{p}_\beta,\overline{p}_\beta))\,d\sigma\right\|\\
&\le L_\theta\int_{\theta(\bar{t})}^{t}\left(L_y\|y(\sigma)-\overline{p}_\beta\|+L_\theta\|y(\theta(\bar{t}))-\overline{p}_\beta\|\right)\,d\sigma \le \tau L_\theta(L_yC_y+L_\theta)\|y(\theta(\bar{t}))-\overline{p}_\beta\|\\
&\le \tau C_\theta L_\theta(L_yC_y+L_\theta)\|y(t)-\overline{p}_\beta\|.
\end{aligned}$$

The last inequality and (A19) imply

$$\begin{aligned}
\frac{d}{dt}V(y(t)) &= \frac{1}{2}\frac{d}{dt}\|y(t)-\overline{p}_\beta\|^2 = V'(y(t))\left[f(y(t),y(t))+f(y(t),y(\theta(\bar{t})))-f(y(t),y(t))\right]\\
&\le -\alpha u^-\left((s(t)-\overline{s})^2+(x(t)-\overline{x})^2\right)+\tau C_\theta L_\theta(L_yC_y+L_\theta)\|y(t)-\overline{p}_\beta\|^2\\
&= -\Gamma_\tau\left((s(t)-\overline{s})^2+(x(t)-\overline{x})^2\right) \text{ with } \Gamma_\tau := \alpha u^- - \tau C_\theta L_\theta(L_yC_y+L_\theta).
\end{aligned}$$

Finally, we set

$$\tau_5 := \frac{\alpha u^-}{2C_\theta L_\theta(L_yC_y+L_\theta)} \quad \text{and} \quad \tau^* = \min(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5).$$

Then for each $\tau \in (0,\tau^*)$ the following inequality holds true

$$\frac{d}{dt}V(y(t)) = \frac{1}{2}\frac{d}{dt}((s(t)-\overline{s})^2+(x(t)-\overline{x})^2) \le -\frac{1}{2}\alpha u^-\left((s(t)-\overline{s})^2+(x(t)-\overline{x})^2\right). \tag{A20}$$

Note that the upper bound τ^* of τ depends only on $x_{\min}$, L_s, L_x, B_s, and B_x, as well as on the function μ and the model parameters, but it does not depend on the choice of the equilibrium point $\overline{p}_\beta$.

It follows from (A20) that the distance between the points $(s(t),x(t))$ and $(\overline{s},\overline{x})$ decreases as the time t increases, and this is valid on the whole interval $[\theta(\bar{t}),\theta(\bar{t})+\tau)$. In the same way, one can prove that (A20) holds true on the intervals $[\theta(\bar{t})+\tau,\theta(\bar{t})+2\tau)$, $[\theta(\bar{t})+2\tau,\theta(\bar{t})+3\tau)$, and so on. Hence, this inequality is fulfilled for each $t \ge \theta(\bar{t})$. From here we obtain that $\phi(t,\zeta^0)$, $t \ge 0$, tends to $\overline{p}_\beta$ as t tends to $+\infty$ for any starting point $\zeta^0 \in \Omega$. The Lyapunov stability of the closed-loop system Σ follows from the existence of the Lyapunov function V. This completes the proof of Claim 3 and of Theorem 2. ■