

Model-based stabilization of a fermentation process using output feedback with discrete time delay

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Abstract. The present study is devoted to the stabilization of a bioreactor model, describing an anaerobic fermentation process for biological degradation of organic wastes with methane production. The stabilization is realized by means of a feedback control law related to the model output and involving a discrete time delay. We determine a nontrivial equilibrium point of the closed-loop system and investigate its asymptotic stability as well as the appearance of bifurcations with respect to the delay parameter. We establish the existence of an attracting and invariant region around the equilibrium such that all trajectories enter this region in finite time for some values of the delay and remain there. An iterative numerical extremum seeking algorithm is applied to the closed-loop system aimed to maximize the methane flow rate in real time. Simulation results are presented to illustrate the theoretical studies.

1 Introduction

Delayed mathematical models of bioprocesses have been extensively studied in recent years in order to explain the appearance of different phenomena in the real process, cf. [8], [9] and the references therein. In the same time, feedback control of bioreactor models provides many advantages in operating a plant and is used to increase its efficiency. In the present paper we combine the above mentioned approaches in studying a bioprocess mathematical model, namely we use a feedback control related to the (on-line measurable) process output for the dynamics stabilization, and introduce a discrete time delay. The time delay is involved in the feedback, because there is always a delay between the output measurements and the system’s response (cf. [4]).

We consider one of the bench-mark mathematical models of the continuous methane fermentation, the so-called “single biomass/single substrate” model. It

is described by two nonlinear ordinary differential equations

$$\begin{aligned}\frac{ds}{dt} &= -k_1\mu(s)x + u(s_{in} - s) \\ \frac{dx}{dt} &= (\mu(s) - \alpha u)x\end{aligned}\tag{1}$$

and one algebraic equation for the gaseous output

$$Q(s, x) = k_2\mu(s)x.\tag{2}$$

The state variables $x = x(t)$ and $s = s(t)$ represent biomass concentration [mg/dm³] and substrate concentration [mg/dm³] respectively, s_{in} is influent substrate concentration [mg/dm³], u is dilution rate [day⁻¹], k_1 is yield coefficient, k_2 is coefficient [(dm³)²/mg], and Q is methane gas flow rate [dm³/day]. The parameter $\alpha \in (0, 1)$ accounts for the biomass retention. The model function $\mu(s)$ presents the specific growth rate of the biomass.

The paper is organized as follows. The next Section 2 contains the assumptions imposed on the model. Section 3 is devoted to stability and bifurcation analysis of the equilibrium points with respect to the delay parameter. In Section 4 we proof the existence of an attracting and invariant set in the phase plane, such that all trajectories enter it in finite time and remain there for sufficiently small values of the delay. The last Section 5 demonstrates the applicability of the model-based extremum seeking algorithm using a numerical example.

2 Assumptions on the model

The theoretical studies of the model (1) are carried out under several assumptions presented below.

Assumption 1. The function μ is defined for $s \in [0, +\infty)$, $\mu(0) = 0$, $\mu(s) > 0$ for $s > 0$, and $\mu(s)$ is continuously differentiable for all $s \geq 0$.

Assumption 2. Lower bounds s_{in}^- and k_2^- for the values of s_{in} and k_2 respectively, and an upper bound k_1^+ for the value of k_1 are known.

Denote $\beta^- = \frac{k_1^+}{k_2^- s_{in}^-}$ and consider the feedback control law

$$\kappa(s(t), x(t)) = \beta k_2 \mu(s(t)) x(t) \quad \text{with} \quad \beta \in (\beta^-, +\infty).\tag{3}$$

Obviously, $\kappa(\cdot) = \beta Q(\cdot)$ holds true. Replacing in the model (1) the dilution rate u by the feedback $\kappa(s(t-\tau), x(t-\tau))$, where $\tau > 0$ is a discrete delay, we obtain

$$\frac{ds}{dt} = -k_1\mu(s(t))x(t) + \kappa(s(t-\tau), x(t-\tau))(s_{in} - s(t))\tag{4}$$

$$\frac{dx}{dt} = \mu(s(t))x(t) - \alpha\kappa(s(t-\tau), x(t-\tau))x(t).\tag{5}$$

Choose some $\beta \in (\beta^-, +\infty)$ and define

$$\bar{s} = s_{in} - \frac{k_1}{k_2\beta}, \quad \bar{x} = \frac{1}{\alpha\beta k_2}, \quad \bar{p}_\beta = (\bar{s}, \bar{x}).\tag{6}$$

It is straightforward to see that \bar{p}_β is an equilibrium point for (4)–(5), and \bar{s} belongs to the interval $(0, s_{in})$.

Assumption 3. There exist points s^- and s^+ such that $s^- < \bar{s} < s^+ < s_{in}$ and

- (i) the function μ is strictly increasing on the interval $(s^-, s^+]$;
- (ii) $\mu(s) < \mu(s^-) < \mu(s_{in})$ for each $s \in [0, s^-)$;
- (iii) there exists $\varepsilon > 0$ such that $\mu(s^+) < \mu(s)$ for each $s \in (s^+, s^+ + \varepsilon)$.

Denote

$$u^- = \mu(s^-)/\alpha, \quad u^+ = \mu(s^+)/\alpha. \quad (7)$$

Assumption 3 implies that $u^- < u^+$.

Assumption 4. Each point from the interval $[u^-, u^+]$ is an admissible value for the control function u .

Denote further $\bar{u} = \kappa(\bar{s}, \bar{x}) = \mu(\bar{s})/\alpha$; obviously $\bar{u} \in (u^-, u^+)$ holds true.

3 Stability and bifurcations of the equilibrium point

We shall investigate the local asymptotic stability of the equilibrium point \bar{p}_β from (6) with respect to the parameters of the system (4)–(5).

The characteristic equation of system (4)–(5) evaluated at the equilibrium point \bar{p}_β has the form (cf. [5], [7])

$$\lambda^2 + a\lambda + b + (c\lambda + d)e^{-\lambda\tau} = 0, \quad (8)$$

where λ is a complex number, and

$$\begin{aligned} a &= a(\beta) = k_1 \bar{x} \mu'(\bar{s}) + \frac{1}{\alpha} \mu(\bar{s}), & b &= b(\beta) = k_1 \bar{x} \mu(\bar{s}) \mu'(\bar{s}), \\ c &= c(\beta) = \mu(\bar{s}) - k_1 \bar{x} \mu'(\bar{s}), & d &= d(\beta) = \mu(\bar{s}) \left(\frac{1}{\alpha} \mu(\bar{s}) - k_1 \bar{x} \mu'(\bar{s}) \right). \end{aligned}$$

Theorem 1. *Let Assumptions 1, 2 and 3 be satisfied. (i) If $b \geq d$ then the equilibrium point \bar{p}_β is locally asymptotically stable for any value of the delay $\tau \geq 0$. (ii) If $b < d$ then there exists $\tau_0 > 0$ such that the equilibrium point \bar{p}_β is locally asymptotically stable for all values τ such that $0 < \tau < \tau_0$; the equilibrium is locally unstable if $\tau \geq \tau_0$, and a Hopf bifurcation occurs for $\tau = \tau_0$.*

Proof. First we shall show that if $\tau = 0$ then the characteristic equation does not possess roots λ with nonnegative real part. For $\tau = 0$ equation (8) takes the form

$$\lambda^2 + (a + c)\lambda + b + d = 0. \quad (9)$$

Assumption 3(i) implies $a > 0$ and $b > 0$. Since $a + c = (1 + \frac{1}{\alpha}) \mu(\bar{s}) > 0$ and $b + d = \frac{1}{\alpha} \mu^2(\bar{s}) > 0$ it follows that the roots of the quadratic equation (9) have negative real parts.

Let $\tau > 0$. We are looking for purely imaginary roots $\lambda = \pm i\omega$ of (8) with $\omega > 0$. We obtain consecutively:

$$\begin{aligned} -\omega^2 + ai\omega + b + (ci\omega + d)e^{-i\omega\tau} &= 0, \\ -\omega^2 + ai\omega + b + (ci\omega + d)(\cos(\omega\tau) - i\sin(\omega\tau)) &= 0. \end{aligned}$$

Separating the real and the imaginary parts of the last equation leads to

$$\begin{aligned} -\omega^2 + b &= -c\omega \sin(\omega\tau) - d \cos(\omega\tau) \\ a\omega &= -c\omega \cos(\omega\tau) + d \sin(\omega\tau). \end{aligned} \quad (10)$$

Squaring both sides of the equations (10) and adding them together implies

$$\omega^4 - (c^2 - a^2 + 2b)\omega^2 + b^2 - d^2 = 0.$$

With $v := \omega^2$ we obtain the quadratic equation

$$v^2 - (c^2 - a^2 + 2b)v + b^2 - d^2 = 0. \quad (11)$$

It is straightforward to see that the discriminant $\Delta = (c^2 - a^2)(c^2 - a^2 + 4b) + 4d^2$ of (11) is strongly positive, i. e. the quadratic equation (11) possesses two real roots v_1 and v_2 , say $v_1 < v_2$, satisfying the relations $v_1 + v_2 = c^2 - a^2 + 2b < 0$, $v_1 v_2 = b^2 - d^2 = (b - d)(b + d)$.

Case 1: $0 < b = d$. In this case $v_1 = c^2 - a^2 + 2b < 0$ and $v_2 = 0$, thus the characteristic equation (8) does not possess purely imaginary roots for any $\tau > 0$. However, $\lambda = 0$ is not a root of (8) since $b + d = 2b > 0$ holds. Hence, there is no stability switch of the equilibrium point \bar{p}_β for any $\tau > 0$.

Case 2: $b > d$. Now the two real roots v_1 and v_2 are strongly negative, so the characteristic equation (8) does not have purely imaginary roots for any $\tau > 0$. The equilibrium point \bar{p}_β is locally asymptotically stable for any $\tau > 0$.

Case 3: $0 < b < d$. Equation (11) has one negative and one positive root; the positive root is $v_2 = \frac{1}{2}(c^2 - a^2 + 2b + \sqrt{\Delta})$, i. e. the characteristic equation (8) possesses a purely imaginary root when τ takes certain values. Denoting $\omega_+ = \sqrt{v_2}$, these values of τ can be determined from system (10):

$$\sin(\omega_+\tau) = \frac{\omega_+(c\omega_+ + ad - bc)}{c^2\omega_+^2 + d^2}, \quad \cos(\omega_+\tau) = \frac{(d - ac)\omega_+^2 - bd}{c^2\omega_+^2 + d^2}. \quad (12)$$

If $c < 0$ then $ad - bc > 0$; if $c > 0$ then $ad - bc > d(a - c) > 0$ holds. Therefore, we have $\sin(\omega_+\tau) > 0$. Denote $\theta = \omega_+\tau$, $0 < \theta < \pi$. If $\cos(\theta) > 0$, then we take $0 < \theta < \pi/2$, otherwise we take $\pi/2 < \theta < \pi$. Hence,

$$\theta = \operatorname{arccot} \frac{(d - ac)\omega_+^2 - bd}{\omega_+(c\omega_+^2 + ad - bc)}.$$

Denote $\tau_0 = \frac{\theta}{\omega_+} > 0$. We shall see whether a Hopf bifurcation occurs at $\tau = \tau_0$.

To check the transversality condition for a Hopf bifurcation (cf. [5], [7]), we need to determine the sign of the derivative of $\operatorname{Re}\lambda(\tau)$ at the point where $\lambda(\tau)$ is purely imaginary. Differentiating implicitly (8) with respect to τ we obtain

$$\frac{d\lambda}{d\tau} = \frac{(c\lambda + d)\lambda e^{-\lambda\tau}}{2\lambda + a + (c - (c\lambda + d)\tau)e^{-\lambda\tau}}.$$

For convenience, we shall study the sign of $\left(\frac{d\lambda}{d\tau}\right)^{-1}$. We have

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1} &= \frac{2\lambda + a + (c - (c\lambda + d)\tau)e^{-\lambda\tau}}{(c\lambda + d)\lambda e^{-\lambda\tau}} = \frac{(2\lambda + a)e^{\lambda\tau} + c}{\lambda(c\lambda + d)} - \frac{\tau}{\lambda} \\ &= -\frac{2\lambda + a}{\lambda(\lambda^2 + a\lambda + b)} + \frac{c}{\lambda(c\lambda + d)} - \frac{\tau}{\lambda},\end{aligned}$$

and further

$$\begin{aligned}\text{sign} \left(\frac{d(\text{Re } \lambda)}{d\tau} \right)_{\lambda=i\omega_+} &= \text{sign} \left\{ \text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \right\}_{\lambda=i\omega_+} \\ &= \text{sign} \left\{ \text{Re} \left(-\frac{2\lambda + a}{\lambda(\lambda^2 + a\lambda + b)} \right) + \text{Re} \left(\frac{c}{\lambda(c\lambda + d)} \right) \right\}_{\lambda=i\omega_+} \\ &= \text{sign} \left\{ \text{Re} \left(-\frac{(2i\omega_+ + a)(-a\omega_+^2 - i\omega_+(b - \omega_+^2))}{\omega_+^2(a^2\omega_+^2 + (b - \omega_+^2)^2)} \right) + \text{Re} \left(\frac{c\omega_+(-c\omega_+ - id)}{\omega_+^2(c^2\omega_+^2 + d^2)} \right) \right\} \\ &= \text{sign} \left\{ \frac{a^2 - 2(b - \omega_+^2)}{a^2\omega_+^2 + (b - \omega_+^2)^2} - \frac{c^2}{c^2\omega_+^2 + d^2} \right\} = \text{sign} \{ 2\omega_+^2 - (c^2 - a^2 + 2b) \} = +1.\end{aligned}$$

The last result means that all roots that cross the imaginary axis at $i\omega_+$, cross this axis from left to right as τ increases. The proof is completed.

4 Asymptotic stabilization of the model solutions

In practice, the dilution rate u is proportional to the speed of the input mechanism which feeds the bioreactor with substrate. Thus u is always lower- and upper-bounded [3]. Let u^- and u^+ be determined according to (7).

Define the set

$$\Omega = \{ \zeta = (s, x) : s > 0, x > 0 \}.$$

Let $\tau > 0$ and $\zeta^0 = (s^0, x^0) \in \Omega$ be an arbitrary point such that $s(t) = s^0 > 0$, $x(t) = x^0 > 0$ for each $t \in [-\tau, 0]$. Consider the following closed-loop system Σ

$$\dot{s}(t) = -k_1\mu(s(t))x(t) + \chi(t)(s_{in} - s(t)) \quad (13)$$

$$\dot{x}(t) = (\mu(s(t)) - \alpha\chi(t))x(t), \quad (14)$$

where $\chi(t)$ is defined in the following way:

$$\chi(t) = \begin{cases} u^-, & \text{if } \kappa(s(t - \tau), x(t - \tau)) \leq u^-, \\ \kappa(s(t - \tau), x(t - \tau)), & \text{if } u^- \leq \kappa(s(t - \tau), x(t - \tau)) \leq u^+, \\ u^+, & \text{if } \kappa(s(t - \tau), x(t - \tau)) \geq u^+. \end{cases} \quad (15)$$

Obviously, $\bar{p}_\beta = (\bar{s}, \bar{x})$ is an equilibrium point of Σ , i. e. of (13)–(14). Denote by $\varphi(\cdot, \zeta^0) = (s(\cdot), x(\cdot))$ the solution of Σ starting from ζ^0 . Important properties of $\varphi(\cdot, \zeta^0)$ are given in the next Lemma 1. Similar assertions can be found e. g. in [3], [8], [9] for various bioreactor models.

Lemma 1. *For each point $\zeta^0 = (s^0, x^0) \in \Omega$ the solution $\varphi(t, \zeta^0) = (s(t), x(t))$ of Σ is defined for each $t > 0$, and*

(i) for each $\varepsilon_1 > 0$ there exists $T_1 > 0$ such that for each $t > T_1$ the inequalities $s_{in} - \varepsilon_1 < s(t) + k_1 x(t) < s_{in}/\alpha + \varepsilon_1$ hold true.

(ii) there exist $\varepsilon_2 > 0$ and $T_2 > 0$ such that for each $t > T_2$ the estimates $s(t) < s_{in}$ and $x(t) \geq \varepsilon_2/k_1 =: x_{\min} > 0$ hold true.

For the proof of the next theorem we need the following lemma.

Barbălat's Lemma (cf. [2]). *If $f : (0, \infty) \rightarrow \mathbb{R}$ is Riemann integrable and uniformly continuous, then $\lim_{t \rightarrow \infty} f(t) = 0$.*

Theorem 2. *Let Assumptions 1, 2, 3 and 4 be fulfilled. Then there exists $\bar{\tau} > 0$ such that for each $\tau \in (0, \bar{\tau})$ and for each point $\zeta^0 = (s^0, x^0) \in \Omega$ the solution $\varphi(t, \zeta^0)$ of Σ has the following property: there exists $T > 0$ such that for each $t > T$,*

$$\varphi(t, \zeta^0) \in \Omega_{s^-, s^+} := \{(s, x) : s \in [s^-, s^+], x > 0\}.$$

Proof. Let us fix an arbitrary $\tau > 0$ and assume that $s(t) \leq s^-$ for each $t \geq 0$. Then Assumption 3(ii) and (7) imply that $\mu(s(t)) \leq \mu(s^-) = \alpha u^-$. The definition of $\chi(\cdot)$ implies $\chi(t) \geq u^-$ for each $t \geq 0$. Then $\mu(s(t)) - \alpha \chi(t) \leq \mu(s(t)) - \alpha u^- \leq 0$ for each $t \geq 0$, and hence $\dot{x}(t) = (\mu(s(t)) - \alpha \chi(t))x(t) \leq 0$. Thus the function $x(\cdot)$ is non increasing and there exists $\tilde{x} = \lim_{t \rightarrow \infty} x(t)$. According to Barbălat's Lemma, we obtain that $\dot{x}(t) \rightarrow 0$ as $t \rightarrow +\infty$. Since $x(t) \geq x_{\min} > 0$ (see Lemma 1(ii)), equation (14) implies that $(\mu(s(t)) - \alpha u^-) + \alpha(u^- - \chi(t)) \rightarrow 0$ as $t \rightarrow +\infty$. The last relation leads to

$$\mu(s(t)) \rightarrow \alpha u^- \text{ and } \chi(t) \rightarrow u^- \text{ as } t \rightarrow +\infty.$$

Applying again Assumption 3(ii) we obtain that $s(t) \rightarrow s^-$ as $t \rightarrow +\infty$. It follows from Barbălat's Lemma that $\dot{s}(t) = \chi(t)(s_{in} - s(t)) - k_1 \mu(s^-)x(t) \rightarrow 0$ as $t \rightarrow \infty$, and hence

$$u^-(s_{in} - s^-) - k_1 \mu(s^-)\tilde{x} = 0, \text{ i. e. } u^-(s_{in} - s^-) - \alpha k_1 u^- \tilde{x} = 0.$$

Therefore, $s_{in} = s^- + \alpha k_1 \tilde{x}$. We also have $\chi(t) \rightarrow u^-$ as $t \rightarrow +\infty$. This is possible iff for each $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that $\kappa(s(t - \tau), x(t - \tau)) < u^- + \varepsilon$ for each $t > T_\varepsilon$ for which $\kappa(s(t - \tau), x(t - \tau)) > u^-$ (if $\kappa(s(t - \tau), x(t - \tau)) \leq u^-$, then $\chi(t) = u^-$). Then for each $t > T_\varepsilon$ we have

$$u^- + \varepsilon > \kappa(s(t - \tau), x(t - \tau)) = \beta k_2 \mu(s(t - \tau))x(t - \tau) = \frac{\mu(s(t - \tau))x(t - \tau)}{\alpha \bar{x}}.$$

Taking a limit in the latter inequality we obtain $\frac{\mu(s^-)\tilde{x}}{\alpha \bar{x}} \leq u^-$, i. e. $\frac{\alpha u^- \tilde{x}}{\alpha \bar{x}} \leq u^-$ or $\tilde{x} \leq \bar{x}$. But this is impossible because $\tilde{x} = \frac{s_{in} - s^-}{\alpha k_1} > \frac{s_{in} - \bar{s}}{\alpha k_1} = \bar{x}$.

Assuming that $s(t) \geq s^+$ for each $t \geq 0$, we obtain a contradiction in a similar way.

If $\tau > 0$ is sufficiently small, one can easily apply the Lyapunov functions approach to prove that $(s(t), x(t))$ tends to (\bar{s}, \bar{x}) as $t \rightarrow \infty$. This completes the proof of Theorem 2.

Remark 1. It follows from Theorem 2 that the feedback (15) ensures attractivity and invariance of the region Ω_{s^-, s^+} for some values of the time delay $\tau \geq 0$. If s^- and s^+ are sufficiently close to each other and $\bar{p}_\beta = (\bar{s}, \bar{x})$ is locally asymptotically stable, then the trajectories remain close to \bar{p}_β because $\bar{s} \in (s^-, s^+)$ holds true. The existence of Ω_{s^-, s^+} is important for the practical applications (cf. [3]) and we shall exploit it in the next section.

5 Numerical extremum seeking

Consider the Haldane model function for the specific growth rate (cf. [6])

$$\mu(s) = \frac{m_1 s}{k_s + s + s^2/k_i},$$

where m_1 is the maximum specific growth rate of the microorganisms [1/day], k_s and k_i are the saturation and inhibition constants respectively. We use the following values for the model coefficients (cf. [6]):

$$k_1 = 3, s_{in} = 2, m_1 = 0.35, k_s = 0.7, k_i = 0.6, \alpha = 0.5, k_2 = 5.6.$$

With $k_1^+ = 3.1$, $s_{in}^- = 1.95$ and $k_2^- = 5.59$ we obtain $\beta^- \approx 0.2844$.

The function $\mu(s)$ achieves its maximum at the point $s_{\mu_{\max}} = \sqrt{k_s k_i} \approx 0.6481$ and $\mu(s)$ is strongly increasing for $s \in (0, s_{\mu_{\max}})$. Solving the equation $\bar{s} = s_{\mu_{\max}}$ with respect to β implies $\beta = \beta_{\mu_{\max}} \approx 0.3963$. Since \bar{s} is an increasing function of β , it suffice to consider $\beta \in (\beta^-, \beta_{\mu_{\max}})$ in order to have Assumption 3 satisfied.

Consider equation (2) describing the process output, and evaluate the function Q on the set of all equilibrium points \bar{p}_β , parameterized with respect to β . The so obtained function $Q(\beta)$ is called input-output static characteristic of the model. It is straightforward to see that $Q(\beta)$ is strongly unimodal, i. e. there exists a unique point $\beta_{\max} \approx 0.3411 < \beta_{\mu_{\max}}$, where $Q(\beta)$ takes a maximum, $Q_{\max} = Q(\beta_{\max})$, the function strongly increases in the interval (β^-, β_{\max}) and strongly decreases in $(\beta_{\max}, \beta_{\mu_{\max}})$. Denote by $p_{\beta_{\max}} = (s_{\max}, x_{\max})$ the steady state where Q_{\max} is achieved. We have $s_{\max} \approx 0.4294$, $x_{\max} \approx 1.047$, and $Q_{\max} = Q(p_{\beta_{\max}}) \approx 0.6134$. Our goal is to stabilize in real time the system (13)–(14) towards this (unknown) equilibrium point $p_{\beta_{\max}}$ and therefore to the maximum methane flow rate. This is realized by applying a numerical model-based extremum seeking algorithm (ESA). The ESA is described in details in [1] for the same model (1) with another feedback and without delay. Now ESA is adopted to Σ and implemented in the programme language *Python*.

In the simulation process we consider $\beta \in (0.29, 0.39)$ and take $s^- = 0.1527$, $u^- = 0.1199$, $s^+ = 0.6264 < s_{\mu_{\max}}$, $u^+ = 0.2214$; obviously $s_{\max} \in (s^-, s^+)$. We choose $\beta = 0.37$. According to Theorem 1(ii) a stability switch of \bar{p}_β may occur at $\tau_0 = 121$ [days], and \bar{p}_β is locally asymptotically stable if $\tau < \tau_0$. The delay

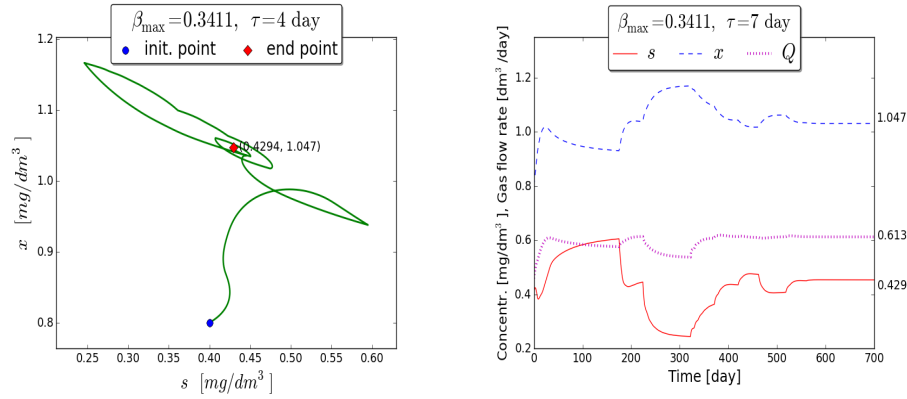


Fig. 1. A trajectory in the (s, x) phase plane for $\tau = 4$ (left); time evolution of $s(t)$, $x(t)$, $Q(t)$ towards s_{\max} , x_{\max} , Q_{\max} respectively for $\tau = 7$ (right)

$\tau_0 = 121$ [days] is however rather large and not feasible in actual practice. The numerical results from ESA are visualized in Figure 1 for $\tau = 4$ and $\tau = 7$.

Acknowledgements. The work of the first and the second author has been partially supported by the Bulgarian Academy of Sciences, the Program for Support of Young Scientists and Scholars, grant No. DFP-17-25/25.07.2017. The work of the third author has been partially supported by the Sofia University “St. Kl. Ohridski” under contract No. 80-10-133/25.04.2018.

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