Functional differential model of an anaerobic biodegradation process*

Milen K. Borisov[1], Neli S. Dimitrova[1] and Mikhail I. Krastanov[1,2]

[1]Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl. 8, 1113 Sofia, Bulgaria,
milen_kb@math.bas.bg; nelid@math.bas.bg
[2]Faculty of of Mathematics and Informatics, Sofia University
5 James Bourchier Blvd., 1164 Sofia, Bulgaria, krastanov@fmi.uni-sofia.bg

Abstract. In this paper we study a nonlinear functional differential model of a biological digestion process, involving two microbial populations and two substrates. We establish the global asymptotic stability of the model solutions towards a previously chosen equilibrium point and in the presence of two different discrete delays. Numerical simulation results are also included.

1 Introduction

We consider a well-known anaerobic digestion model for biological treatment of wastewater in a continuously stirred tank bioreactor (cf. for example [2], [3]). Here we include discrete time delays in the equations to model the delay in the conversion of nutrient consumed by the viable biomass. For more detailed motivation see [13], [14] and the references therein. The model is described by the following nonlinear differential equations:

$$\begin{aligned}
\frac{d}{dt}s_1(t) &= (s_1^i - s_1(t))u - k_1\mu_1(s_1(t))x_1(t) \\
\frac{d}{dt}x_1(t) &= \mu_1(s_1(t-\tau_1))x_1(t-\tau_1) - \alpha u x_1(t) \\
\frac{d}{dt}s_2(t) &= (s_2^i - s_2(t))u + k_2\mu_1(s_1(t))x_1(t) - k_3\mu_2(s_2(t))x_2(t) \\
\frac{d}{dt}x_2(t) &= \mu_2(s_2(t-\tau_2))x_2(t-\tau_2) - \alpha u x_2(t).
\end{aligned} \tag{1}$$

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively: s_1 is the organic substrate, characterized by its chemical oxygen demand (COD), s_2 denotes the volatile fatty acids (VFA), x_1 and x_2 are the acidogenic and methanogenic bacteria respectively; s_1^i and s_2^i are the input substrate concentrations. The constants $\tau_i \geq 0$, $i = 1, 2$, stand for the time delay

* This research has been partially supported by the Sofia University "St Kl. Ohridski" under contract No. 08/26.03.2015

in conversion of the corresponding substrate to viable biomass for the ith bacterial population. The parameter $\alpha \in (0,1)$ represents the proportion of bacteria that are affected by the dilution rate u. The constants k_1, k_2 and k_3 are yield coefficients related to COD degradation, VFA production and VFA consumption respectively. For biological evidence, s_1^i and s_2^i as well as all parameters in (1) are assumed to be positive.

The functions $\mu_1(s_1)$ and $\mu_2(s_2)$ model the specific growth rates of the bacteria. Following [9] we impose the following assumption on μ_1 and μ_2:

Assumption A1: For each $j = 1, 2$ the function $\mu_j(s_j)$ is defined for $s_j \in [0, +\infty)$, $\mu_j(0) = 0$, and $\mu_j(s_j) > 0$ for each $s_j > 0$; the function $\mu_j(s_j)$ is bounded and Lipschitz continuous for all $s_j \in [0, +\infty)$.

The equations (1) with $\tau_1 = \tau_2 = 0$ have been already investigated by the authors; thereby, global stabilizability via feedback control is proposed in [4], whereas [5] considers the case of global stabilization of the solutions using constant dilution rate u. This second approach is now extended to model (1) involving discrete delays $\tau_j > 0$, $j = 1, 2$. More precisely, in this paper we define a suitable positive constant u_b and prove that for any (admissible) value of the dilution rate $u \in (0, u_b)$ there exists an equilibrium point which is globally asymptotically stable for system (1). To our knowledge, such investigations have not been carried out for this model.

2 Global asymptotic stabilizability of the model

We set $u_b = \dfrac{1}{\alpha} \min\left\{\mu_1(s_1^i),\ \mu_2(s_2^i)\right\}$ and make the following

Assumption A2. For each point $\bar{u} \in (0, u_b)$ there exist points $s_1(\bar{u}) = \bar{s}_1 \in \left(0, s_1^i\right)$ and $s_2(\bar{u}) = \bar{s}_2 \in \left(0, s_2^i\right)$, such that the following equalities hold true

$$\bar{u} = \frac{1}{\alpha}\mu_1(\bar{s}_1) = \frac{1}{\alpha}\mu_2\left(\bar{s}_2\right).$$

Assumption A2 is called in [7] regulability of the system.

Let $\bar{s}_1$ and $\bar{s}_2$ be determined according to Assumption A2. Compute further

$$x_1(\bar{u}) = \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad x_2(\bar{u}) = \bar{x}_2 = \frac{s_2^i - \bar{s}_2 + \alpha k_2 \bar{x}_1}{\alpha k_3}. \tag{2}$$

Then the point $p(\bar{u}) = \bar{p} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ is a nontrivial (positive) equilibrium point for system (1).

Assumption A3. There exist positive numbers ν_1 and ν_2 such that the following inequalities hold true

$$\mu_1(s_1^-) < \mu_1(\bar{s}_1) < \mu_1(s_1^+), \quad \mu_2(s_2^-) < \mu_2(\bar{s}_2) < \mu_2(s_2^+)$$

for each

$$s_1^- \in (0, \bar{s}_1), s_1^+ \in (\bar{s}_1, s_1^i + \nu_1], s_2^- \in (0, \bar{s}_2) \text{ and } s_2^+ \in (\bar{s}_2, s_2^i + \nu_2].$$

Assumption A3 is always fulfilled when the functions $\mu_j(\cdot)$, $j = 1, 2$, are monotone increasing (like the Monod specific growth rate). If at least one function $\mu_j(\cdot)$ is not monotone increasing (like the Haldane law) then the points $\bar{s}_j$ have to be chosen sufficiently small in order to satisfy Assumption A3.

Denote by R^+ the set of all positive real numbers and by C^+_τ – the nonnegative cone of continuous functions $\varphi : [-\tau, 0] \to R^+$, where $\tau = \max\{\tau_1, \tau_2\}$, and set $C^4_\tau := \{\varphi = (\varphi_{s_1}, \varphi_{x_1}, \varphi_{s_2}, \varphi_{x_2}) \in C^+_\tau \times C^+_\tau \times C^+_\tau \times C^+_\tau\}$.

Let $\bar{u} \in (0, u_b)$ be chosen in such a way that Assumptions A2 and A3 are satisfied. Denote by Σ the system obtained from (1) by substituting the parameter u by $\bar{u}$. Using the Schauder fixed-point theorem it is easy to prove that for each $\varphi \in C^4_\tau$ there exists $\varrho > 0$ and a unique solution $\Phi(t, \varphi) = (s_1(t, \varphi), x_1(t, \varphi), s_2(t, \varphi), x_2(t, \varphi))$ of (1) defined on $[-\tau, \varrho)$ such that $\Phi(t, \varphi) = \varphi(t)$ for each $t \in [-\tau, 0]$ (cf. Theorem 2.1 in [8]).

We shall prove below that the equilibrium point $\bar{p}$ is globally asymptotically stable for system Σ.

Theorem 1. *Let the Assumptions A1, A2 and A3 be fulfilled and let φ_0 be an arbitrary element of C^4_τ. Then the corresponding solution $\Phi(t, \varphi_0)$ is well defined on $[-\tau, +\infty)$ and converges asymptotically towards $\bar{p}$.*

Proof. We fix an arbitrary $\varphi_0 \in C^4_\tau$. Then there exists $\varrho > 0$ such that the corresponding solution $\Phi(t, \varphi_0)$ of Σ (denoted by $\Phi(t) := (s_1(t), x_1(t), s_2(t), x_2(t))$ for simplicity) is defined on $[-\tau, \varrho)$. The proof uses some ideas from [13] and [14]. For the reader's convenience we subdivide the proof in five claims.

Claim 1. The components of $\Phi(t)$ take positive values for each $t \in [-\tau, \varrho)$.

Proof of Claim 1. If $s_1(t) = 0$ for some $t \in [0, \varrho)$, then $\dot{s}_1(t) > 0$. This implies that $s_1(t) > 0$ for each $t \in [-\tau, \varrho)$. Analogously one can obtain that $s_2(t) > 0$ for each $t \in [-\tau, \varrho)$. Since

$$x_j(t) = \varphi_{x_j}(0)e^{-\alpha\bar{u}t} + \int_0^t e^{-\alpha\bar{u}(t-\sigma)}\mu_j(s_j(\sigma - \tau_j))x_j(\sigma - \tau_j)d\sigma,\ j = 1, 2,$$

then $x_j(t) > 0$ for each $t \in [-\tau, \varrho)$. This completes the proof of Claim 1. ♢

Claim 2. The solution $\Phi(t)$ of Σ is defined for each $t \in [-\tau, +\infty)$ and is bounded.

Proof of Claim 2. Denote $s(t) := k_2 s_1(t) + k_1 s_2(t)$ and $s^i = k_2 s^i_1 + k_1 s^i_2$. Then $s(t)$ satisfies the differential equation $\dot{s}(t) = \bar{u}(s^i - s(t)) - k_1 k_3 \mu_2(s_2(t))x_2(t)$. We set $q_1(t) := s(t) + k_1 k_3 x_2(t + \tau_2) - s^i/\alpha$ and $q_2(t) := s(t) + k_1 k_3 x_2(t + \tau_2) - s^i$. Then

$$\dot{q}_1(t) = \bar{u}[s^i - s(t) - \alpha k_1 k_3 x_2(t+\tau_2)] \le \bar{u}[s^i - \alpha\,(s(t) + k_1 k_3 x_2(t + \tau_2))] = -\alpha\bar{u}q_1(t),$$

and hence

$$q_1(t) \le q_1(0) \cdot e^{-\alpha t\,\bar{u}}. \tag{3}$$

The latter inequality shows that $q_1(t)$ is bounded. Using the fact that the values of $s_1(t)$, $s_2(t)$ and $x_2(t)$ are positive, it follows that $s_1(t)$, $s_2(t)$ and $x_2(t)$ are bounded as well. Analogously one can obtain that

$$q_2(t) \ge q_2(0) \cdot e^{-t\,\bar{u}}. \tag{4}$$

The estimates (3), (4) and the definition of $s(\cdot)$ imply that for each $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for each $t \geq T_\varepsilon$ the following inequalities hold true

$$s^i - \varepsilon < k_2 s_1(t) + k_1 s_2(t) + k_1 k_3 x_2(t + \tau_2) < \frac{s^i}{\alpha} + \varepsilon. \tag{5}$$

It is easy to see (in the same way as the estimates (5)) that for each $\varepsilon > 0$ there exists a finite time $T_\varepsilon > 0$ such that for all $t \geq T_\varepsilon$ the following inequalities hold

$$s_1^i - \varepsilon < s_1(t) + k_1 x_1(t + \tau_1) < \frac{s_1^i}{\alpha} + \varepsilon. \tag{6}$$

The inequalities (6) imply that $x_1(t)$ is also bounded. Thus the trajectory $\Phi(t)$ of Σ is well defined and bounded for all $t \geq -\tau$ (cf. also Theorem 3.1 of [8]). This completes the proof of Claim 2. $\diamondsuit$

Claim 3. There exists $T_0 > 0$ such that $s_1(t) < s_1^i$ and $s_2(t) < s_2^i + k_2 s_1^i / k_1$ for each $t \geq T_0$.

Proof of Claim 3. First let us assume that there exists $\bar{t} > 0$ such that $s_1(t) \geq s_1^i$ for all $t \geq \bar{t}$. Then we have

$$\dot{s}_1(t) = \bar{u}(s_1^i - s_1(t)) - k_1 \mu_1(s_1(t)) x_1(t) < 0.$$

Since $s_1(\cdot)$ and $x_1(\cdot)$ are bounded differentiable functions defined on $[-\tau, +\infty)$, then $\dot{s}_1(\cdot)$ is an uniformly continuous function. Barbălat's Lemma (cf. [6]) leads to

$$0 = \lim_{t \to \infty} \dot{s}_1(t) = \lim_{t \to \infty} [\bar{u}(s_1^i - s_1(t)) - k_1 \mu_1(s_1(t)) x_1(t)].$$

Because $s_1^i - s_1(t) \leq 0$ and $x_1(t) > 0$, the above equalities imply that $s_1(t) \downarrow s_1^i$ and $x_1(t) \downarrow 0$ as $t \uparrow \infty$. On the other hand, if we set (cf. Lemma 2.2 of [14])

$$z_1(t) := x_1(t) + \int_{t-\tau_1}^{t} \mu_1(s_1(\sigma)) x_1(\sigma) d\sigma,$$

we obtain according to Assumption 3 that $\dot{z}_1(t) = x_1(t)(\mu_1(s_1(t)) - \alpha \bar{u}) > 0$ for all $t \geq \bar{t}$, and so $z_1(t) \uparrow z_1^* > 0$ as $t \uparrow \infty$. But this is impossible according to the definition of $z_1(\cdot)$ and because we have already shown that $x_1(t) \downarrow 0$ as $t \uparrow \infty$.

Hence, there exists a sufficiently large $T_0 > 0$ with $s_1(T_0) \leq s_1^i$. Moreover, if the equality $s_1(\bar{t}) = s_1^i$ holds true for some $\bar{t} \geq T_0$, then we have

$$\dot{s}_1(\bar{t}) = \bar{u}(s_1^i - s_1(\bar{t})) - k_1 \mu_1(s_1(\bar{t})) x_1(\bar{t}) = -k_1 \mu_1(s_1(\bar{t})) x_1(\bar{t}) < 0.$$

The last inequality shows that $s_1(t) < s_1^i$ for each $t > T_0$.

Further with $s(t) = k_2 s_1(t) + k_1 s_2(t)$ and $s^i = k_2 s_1^i + k_1 s_2^i$ we obtain

$$\dot{s}(t) = \bar{u}(s^i - s(t)) - k_1 k_3 \mu_2(s_2(t)) x_2(t).$$

One can show in the same way as above that $s(t) < s^i$ for each $t \geq T_0$ (if necessary T_0 can be enlarged). This establishes Claim 3. $\diamondsuit$

Claim 4. Denote

$$\gamma_j := \limsup_{t\uparrow\infty} x_j(t), \quad \delta_j := \liminf_{t\uparrow\infty} x_j(t), \quad j = 1, 2$$

$$v_1(t) := s_1(t) + k_1 x_1(t + \tau_1), \quad v_2(t) := k_2 s_1(t) + k_1 s_2(t) + k_1 k_3 x_2(t + \tau_2),$$

$$\alpha_j := \limsup_{t\uparrow\infty} v_j(t), \quad \beta_j := \liminf_{t\uparrow\infty} v_j(t), \quad j = 1, 2.$$

Then the following relations hold true: $\delta_1 > 0$, $\alpha_1 = \beta_1$ and $\gamma_1 = \delta_1$, $\alpha_2 = \beta_2$ and $\gamma_2 = \delta_2$.

Proof of Claim 4. Let us assume that $\delta_1 = 0$. Choose an arbitrary $\varepsilon \in (0, (s_1^i - \bar{s}_1)/(1 + k_1))$. According to Claim 2 (see (6)) there exists $T_\varepsilon > 0$ such that for all $t \geq T_\varepsilon$ the following inequalities hold true

$$s_1^i - \varepsilon < s_1(t - \tau_1) + k_1 x_1(t) < \frac{s_1^i}{\alpha} + \varepsilon. \tag{7}$$

Since $\delta_1 = 0$ there exists $t_0 > \max(T_\varepsilon, T_0)$ such that $x_1(t_0) < \varepsilon$. We set (cf. Lemma 3.5 of [14])

$$\sigma := \min\{x_1(t) : \; t \in [t_0 - \tau_1, t_0]\}$$

$$\bar{t} := \sup\{t \geq t_0 - \tau_1 : \; x_1(\tau) \geq \sigma \text{ for all } \tau \in [t_0 - \tau_1, t]\}.$$

Clearly $\sigma \in (0, \varepsilon]$, $\bar{t} \in [t_0 - \tau_1, +\infty)$, $x_1(t) \geq \sigma$ for all $t \in [t_0 - \tau_1, \bar{t}]$ and

$$x_1(\bar{t}) = \sigma, \quad \dot{x}_1(\bar{t}) \leq 0. \tag{8}$$

Taking into account (7) and the choice of ε, we obtain consecutively

$$s_1^i > s_1(\bar{t} - \tau_1) \geq s_1^i - k_1 x_1(\bar{t}) - \varepsilon \geq s_1^i - (1 + k_1)\varepsilon > \bar{s}_1,$$

$$\dot{x}_1(\bar{t}) = \mu_1(s_1(\bar{t} - \tau_1)) x_1(\bar{t} - \tau_1) - \alpha\bar{u} x_1(\bar{t}) > \alpha\bar{u}\sigma - \alpha\bar{u}\sigma = 0.$$

The last inequality contradicts (8), which means that $\delta_1 > 0$.

The proof of the equalities $\alpha_j = \beta_j$ and $\gamma_j = \delta_j$, $j = 1, 2$, is based on similar ideas used in the proofs of Lemma 4.3 of [14] and Theorem 3.1 of [13], so we omit it here due to the limited paper length. ♢

Claim 5. The equilibrium point $\bar{p}$ is locally asymptotically stable for all values of the delays $\tau_1 \geq 0$ and $\tau_2 \geq 0$.

Proof of Claim 5. Denote for simplicity $a = k_1 \mu_1'(\bar{s}_1)\bar{x}_1$ and $b = k_3 \mu_2'(\bar{s}_2)\bar{x}_2$. It follows from Assumption A3 that $a > 0$ and $b > 0$ hold true. The characteristic equation of Σ corresponding to the equilibrium point $\bar{p}$ has the form

$$0 = P(\lambda; \tau_1, \tau_2) = P_1(\lambda; \tau_1) \times P_2(\lambda; \tau_2),$$

where λ is a complex number and

$$P_1(\lambda; \tau_1) = \lambda^2 + (\bar{u} + a + \alpha\bar{u})\lambda + \alpha\bar{u}(\bar{u} + a) - \alpha\bar{u}(\bar{u} + \lambda)e^{-\lambda\tau_1},$$
$$P_2(\lambda; \tau_2) = \lambda^2 + (\bar{u} + b + \alpha\bar{u})\lambda + \alpha\bar{u}(\bar{u} + b) - \alpha\bar{u}(\bar{u} + \lambda)e^{-\lambda\tau_2}.$$

First it is straightforward to see that if $\tau_1 = \tau_2 = 0$ then there exist no roots λ of $P(\lambda; \tau_1, \tau_2) = 0$ with $Re(\lambda) \geq 0$. Let $\tau_1 > 0$ and $\tau_2 > 0$. We are looking for purely imaginary roots $\lambda = i\omega$ of $P_j(\lambda; \tau_j) = 0$ with $\omega > 0$, $j = 1, 2$. For $P_1(i\omega; \tau_1) = 0$ we obtain

$$-\omega^2 + (\bar{u} + a + \alpha\bar{u})i\omega + \alpha\bar{u}(\bar{u} + a) - \alpha\bar{u}(\bar{u} + i\omega)e^{-i\omega\tau_1} = 0,$$
$$-\omega^2 + (\bar{u} + a + \alpha\bar{u})i\omega + \alpha\bar{u}(\bar{u} + a) - \alpha\bar{u}(\bar{u} + i\omega)(\cos(\tau_1\omega) - i\sin(\tau_1\omega)) = 0.$$

Separating the real and the imaginary parts of the last equation implies

$$\begin{aligned} -\omega^2 + \alpha\bar{u}(\bar{u} + a) &= \alpha\bar{u}^2\cos(\tau_1\omega) + \alpha\bar{u}\omega\sin(\omega\tau_1) \\ (\bar{u} + a + \alpha\bar{u})\omega &= -\alpha\bar{u}^2\sin(\tau_1\omega) + \alpha\bar{u}\omega\cos(\omega\tau_1). \end{aligned} \tag{9}$$

Squaring both sides of the equations (9) and adding leads to

$$\omega^4 + (\bar{u} + a)^2\omega^2 + \alpha^2\bar{u}^2 a(2\bar{u} + a) = 0.$$

Obviously, the latter equation does not possess positive real roots since $a > 0$. The same conclusion holds true for $P_2(i\omega; \tau_2) = 0$. Therefore, $P(\lambda; \tau_1, \tau_2) = 0$ does not have purely imaginary roots for any $\tau_1 > 0$ and $\tau_2 > 0$. Applying Theorem 3 and Corollary 4 from [10] (see also [11], [12] for similar results) to the exponential polynomial $P(\lambda; \tau_1, \tau_2)$ we obtain that the characteristic equation does not have roots with nonnegative real parts. This means that for any $\tau_1 \geq 0$ and $\tau_2 \geq 0$ the equilibrium $\bar{p}$ is locally asymptotically stable. ♢

The local asymptotic stability of the equilibrium $\bar{p}$ together with the convergence of the solution $\Phi(t)$ and the attractivity of $\bar{p}$, proved above throughout Claims 1 to 4, imply that $\bar{p}$ is globally asymptotically stable.

The proof of Theorem 1 is completed. ♦

3 Computer simulation

Consider the following specific growth rate functions in the model (1), taken from [1], [2] and [3]:

$$\mu_1(s_1) = \frac{m_1 s_1}{k_{s_1} + s_1} \text{ (Monod law)}, \quad \mu_2(s_2) = \frac{m_2 s_2}{k_{s_2} + s_2 + (s_2/k_I)^2} \text{ (Haldane law)}.$$

In the simulation process we shall use the following numerical values for the model coefficients, which are obtained by real experiments and given in [1]:

$k_1 = 10.53$	$k_2 = 28.6$	$k_3 = 1074$	$s_1^i = 7.5$	$s_2^i = 75$	$\alpha = 0.5$
$m_1 = 1.2$	$k_{s_1} = 7.1$	$m_2 = 0.74$	$k_{s_2} = 9.28$	$k_I = 16$	

Within the above coefficient values we compute the admissible upper bound $u_b = 1.044$ for u, thus $u \in (0, 1.044)$.

As an example let us take $\bar{u} = 0.85$. Then the corresponding internal equilibrium is $\bar{p} = (3.893548387, 0.6849860613, 13.47803015, 0.1328068353)$. Using the initial conditions $\varphi_{s_1}(t) = 2$, $\varphi_{x_1}(t) = 0.1$ for $t \in [-\tau_1, 0]$, and $\varphi_{s_2}(t) = 10$, $\varphi_{x_2}(t) = 0.05$ for $t \in [-\tau_2, 0]$, we consider different values for the delays $\tau_1 \geq 0$ and $\tau_2 \geq 0$. The numerical outputs are visualized in the next Figure 1.

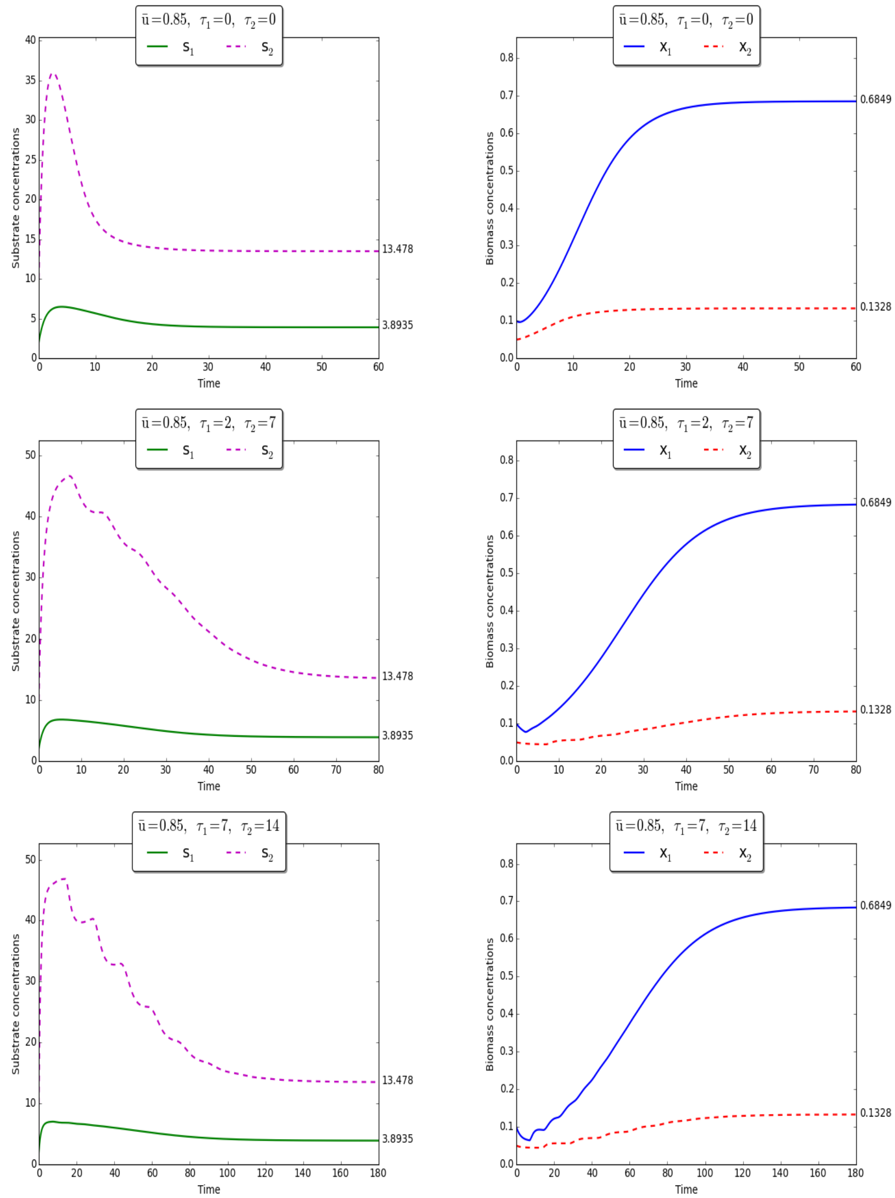

Fig. 1. Time evolution of $s_1(t)$, $s_2(t)$ (left) and $x_1(t)$, $x_2(t)$ (right)

4 Conclusion

In this paper we investigate a bioreactor model for wastewater treatment by anaerobic digestion. The model equations (1) involve discrete delays, describing the time delay in nutrient conversion to viable biomass. Using a properly chosen admissible value for the dilution rate $\bar{u}$ we prove the global convergence of the solutions towards an equilibrium point, corresponding to $\bar{u}$. To authors' knowledge, such kind of investigations have not been yet fulfilled for this delay bioreactor model. Numerical simulation is included to confirm the theoretical results.

References

1. Alcaraz-González, V., Harmand, J., Rapaport, A., Steyer, J.-P., González–Alvarez, V., Pelayo-Ortiz, C.: Software sensors for highly uncertain WWTPs: a new apprach based on interval observers. Water Research **36**, 2515–2524 (2002).
2. Bernard, O., Hadj-Sadok, Z., Dochain, D.: Advanced monitoring and control of anaerobic wastewater treatment plants: dynamic model develop- ment and identication. In: Proceedings of Fifth IWA Inter. Symp. WATERMATEX, Gent, Belgium, 3.57–3.64 (2000).
3. Bernard, O., Hadj-Sadok, Z., Dochain, D., Genovesi, A., Steyer, J.-P.: Dynamical model development and parameter identication for an anaerobic wastewater treatment process. Biotechnology and Bioengineering **75**, 424–438 (2001).
4. Dimitrova, N. S., Krastanov, M. I.: On the Asymptotic Stabilization of an Uncertain Bioprocess Model. In: Large-Scale Scientific Computing (LSSC'2011), I. Lirkov, S. Margenov, J. Wa/'sniewski (eds.): Lecture Notes in Computer Sciences 7116, Springer, 115–122 (2012).
5. Dimitrova, N. S., Krastanov, M. I.: Model-based optimization of biogas production in an anaerobic biodegradation process. Computers and Mathematics with Applications **68**, 986–993 (2014).
6. Gopalsamy, K.: Stability and oscillations in delay differential equations of population dynamics. Kluwer Academic Publishers Dordrect, 1992.
7. Grognard F., Bernard, O.: Stability analysis of a wastewater treatment plant with saturated control. Water Science Technology **53**, 149–157 (2006).
8. Hale, J. K.: Theory of functional differential equations. Springer, 1977.
9. Maillert, L., Bernard, O., Steyer, J.-P.: Robust regulation of anaerobic digestion processes. Water Science and Technology **48** (6), 87–94 (2003).
10. Ruan, S.: On nonlinear dynamics of predator-prey models with discrete delay. Math. Model. Nat. Phenom. **4** (2), 140–188 (2009).
11. Ruan, S., Wei, J.: On the zeroes of transcendental functions with applications to stability of delay differential equations. Dynam. Contin. Impuls. Syst. **10**, 863–874 (2003).
12. Smith, H.: An introduction to delay differential equations with applications to the life sciences. Springer, 2011.
13. Wang, L., Wolkowicz, G.: A delayed chemostat model with general nonmonotone response functions and differential removal rates, J. Math. Anal. Appl. **321**, 452-468 (2006).
14. Wolkowicz, G., Xia, H.: Global asymptotic behavior of a chemostat model with discrete delays, SIAM J. Appl. Math. **57** (4), 1019–1043 (1997).