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On the socles of fully invariant subgroups of Abelian p-groups

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Abstract. The classification of the fully invariant subgroups of a reduced Abelian p-group is a difficult long-standing problem when one moves outside of the class of fully transitive groups. In this work we restrict attention to the socles of fully invariant subgroups and introduce a new class of groups which we term *socle-regular groups*; this class is shown to be large and strictly contains the class of fully transitive groups. The basic properties of such groups are investigated but it is shown that the classification of even this simplified class of groups, seems extremely difficult.

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0. Introduction. The classification of all the fully invariant subgroups of a reduced Abelian p-group is a difficult and long-standing problem, not withstanding the progress made by Kaplansky in the 1950s utilizing the notion of a fully transitive group, see §18 in [8]. Further progress was made for the special class of so-called *large* subgroups by Pierce in [10, Theorem 2.7]. A somewhat less ambitious programme is to try to characterize the socles of fully invariant subgroups and this is the subject of our discussions here. Despite the seeming simplification engendered by restricting attention to socles, the situation is still complicated once one moves away from fully transitive groups. We will show by means of examples that full transitivity is not the real core of the problem. We remark at the outset that the consideration of reduced groups only, is not a serious restriction; see the Note after Lemma 1.1 below. *Hence in the sequel we shall assume that our groups are always reduced p-groups for some prime p.*

Our notation is standard and follows [5, 8], an exception being that maps are written on the right. Finally we recall the notion of a U-sequence from [8]: a

U-sequence relative to a p-group G is a monotone increasing sequence of ordinals $\{\alpha_i\}(i \geq 0)$ (each less than the length of the group G) except that it is permitted that the sequence be ∞ from some point on but that if a gap occurs between α_n and α_{n+1}, the α_n^{th} Ulm invariant of G is non-zero.

We introduce two additional concepts, the first of which shall be the primary focus our interest:

(i) A group G is said to be *socle-regular* if for all fully invariant subgroups F of G, there exists an ordinal α (depending on F) such that $F[p] = (p^\alpha G)[p]$.

(ii) Suppose that H is an arbitrary subgroup of the group G. Set $\alpha = \min\{h_G(y) : y \in H[p]\}$ and write $\alpha = \min(H[p])$; clearly $H[p] \leq (p^\alpha G)[p]$.

If K is also a subgroup of G containing H, then of course there may be two different values of min associated to H, depending on where the heights of elements are calculated. We will distinguish these if necessary by writing $\min^G(H[p])$ and $\min^K(H[p])$; note that if K is an isotype subgroup of G then the respective values of min coincide. However if K is not an isotype subgroup of G then all that one can say is that $\min^K(H[p]) \leq \min^G(H[p])$. Our first result collects some elementary facts about the function min.

Proposition 0.1. (i) *If F is a subgroup of the group G and $(p^n G)[p] \leq F[p]$ for some integer n, then* $\min(F[p])$ *is finite.*

(ii) *If F is a fully invariant subgroup of the group G and* $\min(F[p]) = n$*, a finite integer, then $F[p] = (p^n G)[p]$.*

Proof. (i) Suppose that $\alpha = \min(F[p])$, so that $\alpha \leq \min\{h_G(x) : x \in (p^n G)[p]\}$. Now if $\alpha \geq \omega$, then $(p^n G)[p] \leq p^\omega G = p^\omega(p^n G)$, so that writing $X = p^n G$, one has $X[p] \leq p^\omega X$, which forces X to be divisible contrary to the assumption that G is reduced provided $p^n G$ is non-zero. Hence $\min(F[p])$ is finite as required.

(ii) As observed above, one inclusion holds always. Conversely, suppose that $x \in F[p]$ and $h_G(x) = n$. Then $x = p^n y$ and the subgroup generated by y is a direct summand of G, see, e.g., Corollary 27.2 in [5]. Thus $G = \langle y \rangle \oplus G_1$ for some subgroup G_1. Now if $0 \neq z$ is an arbitrary element of $(p^n G)[p]$, then $z = p^n w$ for some $w \in G$. Since the elements y, w are both of order p^{n+1} we may define a homomorphism $\phi : G \to G$ by sending $y \mapsto w$ and mapping G_1 to zero; note that $x\phi = z$. Since $F[p]$ is fully invariant in G, it follows that $z \in F[p]$ and so $(p^n G)[p] \leq F[p]$. □

Corollary 0.2. *If G is a separable group, then G is socle-regular.*

Proof. This is immediate since the hypothesis of separability implies that for any fully invariant subgroup F of G, $\min(F[p])$ is finite. □

Corollary 0.2 could have been deduced directly from our next result but we preferred to give the more elementary proof as an introduction to the type of arguments needed.

Theorem 0.3. *If G is a fully transitive group, then G is socle-regular.*

Proof. Since G is, by hypothesis, fully transitive, one may make use of Kaplansky's classification of fully invariant subgroups, see Theorem 25 in [8]. Thus the fully invariant subgroup F has the form $F = \{x \in G : U_G(x) \geq U\}$, where $U = \{\alpha_i\}$ is a U-sequence relative to G. Now if $x \in F[p]$, then $U_G(x) = \{\beta, \infty, \dots\}$ for some ordinal $\beta \geq \alpha_0$. Clearly $x \in (p^{\alpha_0}G)[p]$ and so $F[p] \leq (p^{\alpha_0}G)[p]$.

Conversely if $y \in (p^{\alpha_0}G)[p]$, then $U_G(y) = \{\gamma, \infty, \dots\}$ where $\gamma \geq \alpha_0$. But now it is immediate that $y \in \{x \in G : U_G(x) \geq U\} = F$, so that $(p^{\alpha_0}G)[p] \leq F[p]$. This completes the proof. $\square$

It follows, of course, that the class of socle-regular groups is large since the class of fully transitive groups is known to contain the λ-separable groups for all limit ordinals λ, the totally projective groups and Crawley's generalized torsion-complete groups; for further details of the latter see [6]. It is perhaps worth remarking that, as observed in [6], for $p \neq 2$, the concept of full transitivity coincides with Krylov's notion of transitivity, i.e. there exists an endomorphism mapping any element of the group to any other element which has the same Ulm sequence.

1. The class of socle-regular groups. In this section we explore some of the properties of the class of socle-regular groups. We shall have need of the following result which is a slight variation of a well-known result.

Lemma 1.1. *Suppose that $A = \bigoplus_{i \in I} G_i$ and that F is fully invariant in A. Then*

1. $F = \bigoplus_{i \in I} (G_i \cap F)$
2. *each $G_i \cap F$ is fully invariant in G_i.*

Proof. Let $\pi_i : A \twoheadrightarrow G_i$ denote the canonical projections onto G_i. It is easy to see that $F = \bigoplus_{i \in I} F\pi_i$. Since F is fully invariant in A, $F\pi_i \leq F$ and it follows easily that $F\pi_i = G_i \cap F$, establishing (i). Suppose now that ϕ_i is an arbitrary endomorphism of G_i. Then $(G_i \cap F)\phi_i = F\pi_i\phi_i \leq F$ since F is fully invariant in A and $\pi_i\phi_i$ can be identified with an endomorphism of A. Since $(G_i \cap F)\phi_i \leq G_i$ also, the result follows. $\square$

Note. This Lemma allows one to justify the restriction of consideration to reduced groups. For if $G = D \oplus R$ is a group with maximal divisible subgroup D, then for any fully invariant subgroup F of G, one has $F = (F \cap D) \oplus (F \cap R)$ and $F \cap D$, $F \cap R$ are fully invariant in D, R respectively. However it is well known that the socle $(F \cap D)[p]$ must be either 0 or $D[p]$ and so the determination of $F[p]$ reduces to the determination of the socle of the fully invariant subgroup $F \cap R$ of the reduced group R.

Given that the class of fully transitive groups is closed under the addition of separable summands—see e.g. [1, Proposition 2.6]—it is reasonable to ask whether the class of socle-regular groups has a similar property. A strong positive answer is given by:

Theorem 1.2. *Suppose that $A = G \oplus H$ where H is separable, then A is socle-regular if, and only if, G is socle-regular.*

Proof. Suppose that G is socle-regular and that F is fully invariant in A, so that by Lemma 1.1 $F = (F \cap G) \oplus (F \cap H)$ and $(F \cap G), (F \cap H)$ are fully invariant in G, H respectively. If $F \cap H \neq 0$ then, since H is separable, it follows that $\min^H((F \cap H)[p])$ is finite. But $F[p] = (F \cap G)[p] \oplus (F \cap H)[p]$ and so

$$\min{}^A(F[p]) \leq \min{}^A((F \cap H)[p]) = \min{}^H((F \cap H)[p]),$$

the last equality following since H is pure in A. Thus it follows that $\min^A(F[p])$ is also finite, and so by Proposition 0.1, $F[p] = (p^n A)[p]$ for some integer n.

If $F \cap H = 0$, then F is a fully invariant subgroup of the socle-regular group G. Hence $F[p] = (p^\alpha G)[p]$ for some ordinal α. If $\alpha \geq \omega$, then $p^\alpha A = p^\alpha G$ since H is separable and so $F[p] = (p^\alpha A)[p]$. Otherwise $F[p] = (p^n G)[p]$ and F is a fully invariant subgroup of G. It follows from Proposition 0.1(i) that $\min^G(F[p])$ is finite, and since G is pure in A, we also have that $\min^A(F[p])$ is finite. Now an appeal to Proposition 0.1(ii) yields the desired result.

Conversely suppose that A is socle-regular and assume for a contradiction that G is not. Then there exists a fully invariant subgroup K of G such that $K[p] \neq (p^\alpha G)[p]$ for any α. Note that $\min(K[p])$ must be infinite, for if it were finite, then by Proposition 0.1(ii), $K[p] = (p^n G)[p]$ for some finite n—contradiction. So $\min(K[p])$ is infinite and thus $K[p] \leq p^\omega G$. Furthermore $K[p]$ is fully invariant in G since K is. It follows from Lemma 1.3 below that $K[p]$ is fully invariant in the socle-regular group A. Thus $K[p] = (p^\alpha A)[p]$ for some α. Since $K[p] \leq p^\omega G$, α must be infinite. But then $p^\alpha H = 0$ and so $K[p] = (p^\alpha G)[p] \oplus (p^\alpha H)[p] = (p^\alpha G)[p]$—contradiction. Thus G is socle-regular as required. □

We remark that the last possibility examined in the proof above never actually occurs: $\min^G(F[p])$ finite implies that there is an $x \in F[p]$ which can be embedded in a cyclic summand of G and then this element x can be mapped outside of F contrary to full invariance of F.

The proof of Theorem 1.2 is completed by the following:

Lemma 1.3. *A subgroup F of G is fully invariant in $A = G \oplus H$, where H is separable, if F is fully invariant in G and $F \leq p^\omega G$.*

Proof. Suppose that $F \leq p^\omega G$ and that F is fully invariant in G. Let $\Phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ be any endomorphism of A. Then $(F \oplus 0)\Phi \leq (F\alpha \oplus F\gamma) \leq (F \oplus F\gamma)$ since F is fully invariant in G. Moreover, γ is a homomorphism : $G \to H$, and since H is separable, $p^\omega G$ must be mapped to zero by γ. Since $F \leq p^\omega G$, one must have that $F\gamma = 0$ and so $(F \oplus 0)\Phi \leq (F \oplus 0)$ and F is fully invariant in A as required. $\square$

We can also show that direct powers of a single socle-regular group are again socle-regular. In fact we have the stronger:

Theorem 1.4. *The group G is socle-regular if, and only if, the direct sum $G^{(\kappa)}$ is socle-regular for any cardinal κ.*

Proof. Suppose that F is fully invariant in $G^{(\kappa)}$, so that in view of Lemma 1.1, $F = \bigoplus_{i<\kappa}(G_i \cap F)$ where each G_i is isomorphic to G. Then the socle $F[p] = \bigoplus_{i<\kappa}(G_i \cap F)[p]$ and each $G_i \cap F$ is fully invariant in G_i. Since G is socle-regular, each $(G_i \cap F)[p]$ can be expressed as $(p^{\alpha_i}G_i)[p]$ for ordinals α_i. However if the α_i are not all equal, the subgroup $\bigoplus_{i<\kappa}(p^{\alpha_i}G_i)[p]$ is not fully invariant. It follows immediately that $F[p] = (p^\alpha G^{(\kappa)})[p]$, where α is the common value of the α_i, as required.

Conversely suppose that $G^{(\kappa)}$is socle-regular and that F is an arbitrary fully invariant subgroup of G. Since the endomorphism ring of $G^{(\kappa)}$ may be construed as the ring of row-finite matrices over $\mathrm{End}(G)$, it is easy to see that the subgroup $F^{(\kappa)}$ is fully invariant in $G^{(\kappa)}$. Since the latter is socle-regular, we have $(F^{(\kappa)})[p] = (p^\alpha G^{(\kappa)})[p]$ for some ordinal α. It follows immediately that $F[p] = (p^\alpha G)[p]$ and thus G is socle-regular. $\square$

Recall that a fully invariant subgroup L of a group G is said to be large if $G = L + B$ for every basic subgroup B of G. Our next result shows that socle-regularity is inherited by large subgroups.

Proposition 1.5. *If A is a socle-regular group and L is a fully invariant subgroup of A such that $p^\omega L = p^\omega A$, then L is socle-regular. In particular, large subgroups of socle-regular groups are again socle-regular.*

Proof. Let F be a fully invariant subgroup of L. Then F is also fully invariant in A and hence, as A is socle-regular, $F[p] = (p^\alpha A)[p]$ for some ordinal α. Since $p^\omega A = p^\omega L$ by hypothesis, it follows from a simple transfinite induction argument that $p^\alpha A = p^\alpha L$ for all ordinals $\alpha \geq \omega$. Thus, if $\alpha \geq \omega$, $F[p] = (p^\alpha A)[p] = (p^\alpha L)[p]$. If α is finite, then $F[p] = (p^n A)[p] \geq (p^n L)[p]$ and so it follows from Proposition 0.1(i) that $\min^L(F[p])$ is finite. Applying the second part of the same Proposition gives that $F[p] = (p^m L)[p]$ for some integer m. The final claim in relation to large subgroups follows from the fact that if L is a large subgroup of A, then $p^\omega A = p^\omega L$, see, e.g., §46.1 in [11]. $\square$

Once we drop the hypothesis of full transitivity, it is possible to exhibit groups of varying levels of complexity which are not socle-regular. Our first result shows that this failure can happen at the next stage beyond separability. We give two examples, the first based on the well-known realization theorem of Corner in [2] while the second is essentially due to Megibben [9].

Theorem 1.6. *There exist groups of length $\omega + 1$ which are not socle-regular.*

Proof. For the first class of examples let $H = \langle a \rangle \oplus \langle b \rangle$ where a, b are of order p and set $K = \langle a \rangle$ and $L = \langle b \rangle$. The endomorphism ring of H contains a subring Φ consisting of the diagonal matrices with entries from $\mathrm{End}(K)$ and $\mathrm{End}(L)$. Now apply Corner's realization result Theorem 6.1 in [2] to obtain a group G such that $p^\omega G = H$ and $\mathrm{End}(G) \restriction H = \Phi$. (Note that G is neither transitive nor fully transitive since K, L are both fully invariant subgroups of G but the elements a, b have the same Ulm sequence $(\omega, \infty, \dots)$.)

In particular K is fully invariant in G and $K[p] = K$. However $(p^\omega G)[p] = K \oplus L$, $p^{\omega+1}G = 0$ and $p^n G$ is unbounded for all positive integers n, so that $K[p] \neq (p^\alpha G)[p]$ for any α. Hence G is not socle-regular as desired.

For the second class of examples let $A = G \oplus H$, where $p^\omega G \cong p^\omega H \cong \mathbb{Z}(p)$, $G/p^\omega G$ is a direct sum of cyclic groups and $H/p^\omega H$ is torsion-complete. It follows easily – e.g., see Theorem 2.4 in [9] – that $p^\omega H$ is fully invariant in A. We claim that A is not socle-regular. If it were, then there is an ordinal $\alpha \geq 0$ such that $p^\omega H = (p^\omega H)[p] = (p^\alpha A)[p] = (p^\alpha G)[p] \oplus (p^\alpha H)[p]$. Therefore, $(p^\alpha G)[p] = 0$, i.e., $p^\alpha G = 0$ and hence $\alpha = \omega + 1$. Thus, $p^\omega H = (p^{\omega+1} H)[p] = 0$, a contradiction. $\square$

Note. (i) The first class of examples shows that elongations of socle-regular groups by socle-regular groups need not be socle-regular: $p^\omega G$ and $G/p^\omega G$ are clearly both socle-regular while G is not. Notice, however, that it is easy to show that for any ordinal α and any socle-regular group A, the subgroup $p^\alpha A$ is always socle-regular.

(ii) These same examples show that Kaplansky's classification of fully invariant subgroups fails if we drop the full transitivity hypothesis: the subgroup K above is fully invariant but it cannot have the form $M(\{\alpha_i\})$ for any U-sequence $\{\alpha_i\}$. To see this observe that $U_G(a) = (\omega, \infty, \dots)$ and so if $K = M(\{\alpha_i\})$ for some U-sequence $\{\alpha_i\}$, then $\alpha_0 \leq \omega$. But it follows immediately that b, which has Ulm sequence $U_G(b) = (\omega, \infty, \dots)$, must also belong to $M(\{\alpha_i\})$, implying that $b \in K$ – contradiction. A similar observation has been made by Megibben in [9].

(iii) The second class of examples shows that one cannot drop the separability condition from Theorem 1.2: since $p^\omega G \cong p^\omega H \cong \mathbb{Z}(p)$, it is easy to see that G, H are both fully transitive and hence socle-regular by Theorem 0.3. However $A = G \oplus H$ is not socle-regular and so direct sums of socle-regular groups need not be socle-regular.

As noted above, elongations of socle-regular groups by socle-regular groups need not be socle-regular. We can however obtain some additional information in the special situation where the quotient $G/p^\omega G$ is a direct sum of cyclic groups.

Theorem 1.7. *Let G be a group such that $G/p^\omega G$ is a direct sum of cyclic groups. Then G is socle-regular if, and only if, $p^\omega G$ is socle-regular.*

Proof. We have already noted that G socle-regular implies that $p^\alpha G$ is socle-regular for any ordinal α, so it suffices to handle the sufficiency. Let F be an arbitrary fully invariant subgroup of G. Consider the socle $F[p]$. If $F[p] \nleq (p^\omega G)[p]$, then $\min(F[p])$ is finite and it follows from Proposition 0.1 that $F[p] = (p^n G)[p]$ for some finite integer n. If, however, $F[p] \leq (p^\omega G)[p]$ we claim that $F[p]$ is fully invariant in $p^\omega G$. Assuming that this is true, it then follows immediately that $F[p] = (p^\alpha(p^\omega G))[p]$ since $p^\omega G$ is socle-regular by hypothesis. Thus $F[p] = (p^{\omega+\alpha}G)[p]$ and we are finished. Thus it remains to show that $F[p]$ is fully invariant in $p^\omega G$.

If ϕ is an arbitrary endomorphism of $p^\omega G$, then it follows from Hill's work on totally projective groups – see Theorem 2 in [7] – that every endomorphism of $p^\omega G$ is induced from an endomorphism of G in this situation. The desired result follows immediately. $\square$

We have seen in Theorem 1.4 that direct powers of socle-regular groups must be socle-regular, but we have been unable to determine whether or not summands of socle-regular groups are, in general, socle-regular. The best we can offer is the rather weak:

Proposition 1.8. *Let $G = A \oplus B$ be a socle-regular group such that every homomorphism from A to B is small, then A is socle-regular.*

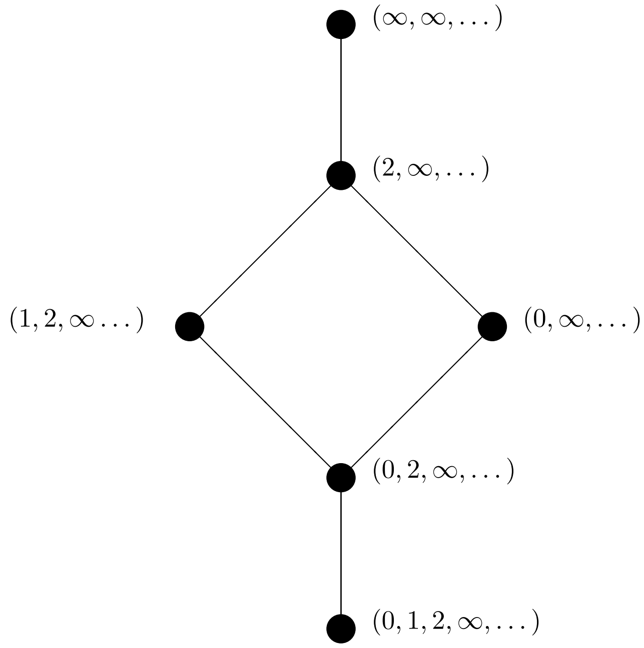
Proof. Let F be a fully invariant subgroup of A. If $\min(F[p])$ is finite then it follows from Proposition 0.1 that $F[p] = (p^n A)[p]$ for some finite integer n. If $\min(F[p])$ is infinite, then $F[p] \leq p^\omega A$. Claim that $F[p] \oplus 0$ is fully invariant in G: the argument is similar to that used in Lemma 1.3 with smallness replacing the argument using separability. If Φ is any endomorphism of G then Φ may be written as a matrix $\left(\begin{smallmatrix}\alpha & \gamma\\ \delta & \beta\end{smallmatrix}\right)$, where $\gamma \in \mathrm{Hom}(A, B)$, so that γ is small. Then $(F[p] \oplus 0)\Phi \leq (F[p]\alpha \oplus F[p]\gamma)$. However $F[p] \leq p^\omega A$ implies that $F[p]\gamma = 0$ as γ is small. So $F[p] \oplus 0$ is fully invariant in G and hence $F[p] \oplus 0 = (p^\nu G)[p]$ for some ordinal ν. Hence $F[p] = (p^\nu A)[p]$ as required. $\square$

It is, however, possible to construct a group which is not fully transitive but is transitive (and hence is a 2-group) and has the property that it is socle-regular.

Example. Let G be the transitive, non fully transitive 2-group constructed by Corner in [3]. The group G has the property that $2^\omega G = H$, $\mathrm{Aut}(G) \restriction 2^\omega G = \mathrm{Aut}(H)$, $\mathrm{End}(G) \restriction 2^\omega G = \Phi$, where Φ is the subring of $\mathrm{End}(H)$ generated by $\mathrm{Aut}(H)$ and the group $H = \langle a \rangle \oplus \langle b \rangle$, where a has order 2 and b has order 8. Note that H has six different associated Ulm sequences:

$$(\infty, \infty, \dots); (2, \infty, \dots); (0, \infty, \dots); (1, 2, \infty, \dots); (0, 2, \infty, \dots); (0, 1, 2, \infty, \dots).$$

Fuller details of this group, relevant for our present purposes, may be found in [6, Example 3.16]. In particular, the associated lattice has just one pair of incomparable Ulm types and it is easy to check, using the calculations and discussions of Example 3.16 in [6], that the only fully invariant subgroups of G contained in $2^\omega G$ are $F_1 = \{0, 4b, a-2b, a+2b\}, F_2 = \{0, a, 4b, a+4b\}$ and $F_3 = \{0, 4b\}$. (This is essentially because it is possible to map from any vertex of the lattice, other than the vertex labelled $(0, 2, \infty, \dots)$, to any other one above it.)



Now if F is an arbitrary fully invariant subgroup of G and $\min(F[2])$ is finite, then $F[2] = (2^n G)[2]$ for $n = \min(F[2])$ by Proposition 0.1. If $\min(F[2]) \geq \omega$ then $F[2]$ is one of $F_i[2]$, $i = 1, 2, 3$. However, a simple check shows that $F_1[2] = (2^{\omega+1}G)[2]$, $F_2[2] = (2^\omega G)[2]$ while $F_3[2] = (2^{\omega+2}G)[2]$. Thus the socle of each fully invariant subgroup of G is of the form $(2^\alpha G)[2]$ for some α and G is socle-regular.

Note. It is now rather easy to show that neither transitivity nor full transitivity is the core concept in determining whether or not a group is socle-regular. For if G is the group in the example above, it follows from Theorem 1.4 that $A = G \oplus G$ is socle-regular. However A is neither transitive nor fully transitive; it cannot be fully transitive since direct summands of such groups are again fully transitive [1, Theorem 3.4] and it cannot be transitive since if it were, it would follow from [4, Corollary 3] that G was fully transitive which it is not.

We finish off our discussion by posing three questions:

(1) Does there exist a transitive group which is not socle-regular? Such a group would, of course, necessarily be a 2-group.

(2) Does Theorem 1.7 generalize to arbitrary infinite ordinals α, if $G/p^{\alpha}G$ is assumed to be totally projective?

(3) Is a summand of a socle-regular group again socle-regular?

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