Houston Journal of Mathematics
© 2012 University of Houston
Volume 38, No. 4, 2012

ON n-SIMPLY PRESENTED PRIMARY ABELIAN GROUPS

PATRICK W. KEEF AND PETER V. DANCHEV

Communicated by Manfred Dugas

Abstract. An abelian p-group G is defined to be *(strongly) n-simply presented* if it has a (nice) p^n-bounded subgroup P such that G/P is simply presented. These notions combine and generalize both the theories of simply presented groups and $p^{\omega+n}$-projective groups. The summands of the (strongly) n-simply presented groups are described, which expand the concept of balanced projective groups. Finally, important results of Nunke on totally projective groups and Crawley-Hales on simply presented groups are generalized to this new framework.

1. Introduction

Throughout, by the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration of the paper. Our terminology and notation will be based upon [5]. For example, if α is an ordinal, then a group G will be said to be *p^α-projective* if $p^\alpha \mathrm{Ext}(G,X) = \{0\}$ for all groups X. We will denote the height of an element $x \in G$ by $|x|_G$. We will say G is *Σ-cyclic* if it is isomorphic to a direct sum of cyclic groups.

The *totally projective* groups have a central position in the study of abelian p-groups (see Chapter XII of [5] or Chapter VI of [9]). One reason for their importance is the number of different ways they can be characterized; recall that a group G is totally projective if any one of the following equivalent conditions is satisfied:

(1) G is *simply presented*;
(2) G is *balanced projective*, i.e., $\mathrm{Bext}(G,X) = \{0\}$ for all groups X;
(3) $G/p^\alpha G$ is p^α-projective for every ordinal α;

2000 *Mathematics Subject Classification.* 20K10.

Key words and phrases. Simply presented, balanced projective, nice subgroup.

(4) G has a *nice system*;

(5) G has a *nice composition series*.

It is worth pointing out that, unlike the treatment in [5], we do not require a simply presented group to be reduced.

In a somewhat different direction, if n is a non-negative integer (that will be fixed for the remainder of this paper), then the group G is $p^{\omega+n}$-projective iff there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (see, e.g., [13]). So, a group is p^{ω}-projective iff it is Σ-cyclic. It follows easily that the class of $p^{\omega+n}$-projectives is closed under arbitrary subgroups. In addition, if G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric (i.e., there exists an isomorphism that preserves the height functions on the two subgroups; see [7]).

A number of papers have been written over the years that combine elements of these two important components of the study of abelian p-groups (see, for example, [8] and [11]). In this and a subsequent paper, we will consider several other interesting ways to combine them. Generalizing (1), a group G will be said to be *n-simply presented* if there is a subgroup $P \subseteq G[p^n]$ such that G/P is simply presented. Such a subgroup will be called *n-simply representing*. It follows, therefore, that the class of n-simply presented groups includes both the simply presented groups and the $p^{\omega+n}$-projective groups.

We say a short exact sequence $0 \to X \to Y \to G \to 0$ is *n-balanced exact* if it represents an element of $p^n\mathrm{Bext}(G, X)$. Generalizing (2), we say G is *n-balanced projective* if every such n-balanced exact sequence splits. We show that G is n-balanced projective iff it is a summand of a group that is n-simply presented, and that there are enough n-balanced projectives (Theorem 2.1). We also show that a separable group G is n-simply presented iff it is n-balanced projective iff it is $p^{\omega+n}$-projective (Proposition 2.2).

If G is $p^{\omega+n}$-projective and P is a subgroup of $G[p^n]$ such that G/P is Σ-cyclic, then P will, in fact, be *nice* in G (i.e., every coset $x+P$ will contain an element of maximal height). This leads to a further generalization of (1): We say the group G is *strongly n-simply presented* if it has an n-simply representing subgroup which is nice.

We say a short exact sequence $0 \to X \to Y \xrightarrow{\phi} G \to 0$ is *strongly n-balanced exact* if it is balanced and there is a height-preserving homomorphism $\nu : G[p^n] \to Y[p^n]$ such that $\phi \circ \nu$ is the identity on $G[p^n]$ (note that if $n \geq 1$, then the latter condition already implies that the sequence is balanced - see, for example, Proposition 80.2 of [5]). In other words, we are requiring that the induced exact sequence, $0 \to X[p^n] \to Y[p^n] \to G[p^n] \to 0$, is split in the category of valuated groups. We can, therefore, consider the class of *strongly n-balanced projectives.*

In parallel with the above, we next show that a group G is strongly n-balanced projective iff it is a summand of a group that is strongly n-simply presented, and that there are enough strongly n-balanced projectives (Theorem 2.4). We also show that a $p^{\omega+n}$-bounded group G is strongly n-simply presented iff it is strongly n-balanced projective iff it is $p^{\omega+n}$-projective (Proposition 2.5).

One of the most useful and important results in the study of totally projective groups is a theorem of Nunke [14] which states that if λ is an ordinal, then a group G is totally projective iff $p^\lambda G$ and $G/p^\lambda G$ are both totally projective (see, for example, Theorem 74 of [9]). The same property was independently proved by Crawley-Hales for simply presented groups (see [2] and [3]). It is not hard to see that if G is (strongly) n-simply presented or (strongly) n-balanced projective, then $p^\lambda G$ and $G/p^\lambda G$ must share the corresponding property (Theorem 3.4(a) and Proposition 3.5(a)). The converse is more complicated. We show that if $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are strongly n-simply presented or strongly n-balanced projective, then so is G (Theorem 3.4(b) and Proposition 3.5(b)). On the other hand, for ordinals not of the form $\lambda+n$ (e.g., limit ordinals), we show that this can fail for strongly n-simply presented groups (Example 3.1).

The fourth section of the paper is devoted to showing that for an arbitrary ordinal λ, if $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented or n-balanced projective, then the same can be said of G (Theorem 4.4 and Corollary 4.6). This surprisingly difficult proof requires a detailed examination of the behavior of bounded subgroups P of G for which G/P is simply presented.

These properties allow us to conclude that for any group G of length strictly less than ω^2, that G is (strongly) n-simply presented iff it is (strongly) n-balanced projective (Corollaries 3.6 and 4.7). In other words, the (strongly) n-simply presented groups of length less than ω^2 are closed under taking direct summands. In section 5 we establish some further statements of this sort.

2. Definitions and Foundational Results

A group G is $p^{\omega+n}$-projective iff there is a Σ-cyclic group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$ (see, e.g., [7]). The proof of this property depends solely on the fact that T is Σ-cyclic iff p^nT is Σ-cyclic. Similarly, we say G is *n-co-simply presented* if there is a simply presented group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$. Since T is also simply presented iff p^nT is simply presented, the same proof shows that G is n-simply presented iff it is n-co-simply presented.

We begin by describing the summands of the n-simply presented groups.

Theorem 2.1. *The group G is n-balanced projective iff it is a summand of a group that is n-simply presented. There are enough n-balanced projectives.*

Proof. Suppose first that G is n-simply presented, and hence n-cosimply presented, i.e. there is a simply presented (and hence balanced projective) group T and a subgroup $Q \subseteq T[p^n]$ such that $T/Q \cong G$. For any group A we have an exact sequence

$$\to \mathrm{Hom}(T,A) \to \mathrm{Hom}(Q,A) \xrightarrow{\phi} \mathrm{Ext}(G,A) \xrightarrow{\mu} \mathrm{Ext}(T,A).$$

It follows that $\mu(\mathrm{Bext}(G,A)) \subseteq \mathrm{Bext}(T,A) = \{0\}$, so that $\mathrm{Bext}(G,A) \subseteq \phi(\mathrm{Hom}(Q,A))$. Since $p^n\mathrm{Hom}(Q,A) = \{0\}$, we can conclude that $p^n\mathrm{Bext}(G,A) = \{0\}$, so that G is n-balanced projective. Therefore, any direct summand of a group that is n-simply presented is also n-balanced projective.

We now show the converse, and at the same time we show that there are enough n-balanced projectives. Let $0 \to X \to Y \to G \to 0$ be a balanced projective resolution of G (i.e., it is balanced and Y is simply presented). Consider the pull-back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G[p^n] & = & G[p^n] & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & X & \longrightarrow & Z & \longrightarrow & G & \to & 0 \\
 & & \| & & \downarrow \gamma & & \downarrow p^n & & \\
0 & \to & X & \longrightarrow & Y & \longrightarrow & G & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G/p^nG & = & G/p^nG & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Obviously, the upper short exact row is n-balanced exact. We claim that Z is n-simply presented: Note that Y is simply presented and $p^n(Y/\gamma(Z)) = \{0\}$. It follows from general properties of simply presented groups, therefore, that $\gamma(Z)$ is simply presented (or see Lemma 3.2(a) below). Since the middle column determines an isomorphism $Z/G[p^n] \cong \gamma(Z)$, we can infer that $G[p^n]$ is an n-simply representing subgroup of Z, i.e., Z is n-simply presented, as claimed.

By the first part of the proof, we can deduce that Z is n-balanced projective; and since $0 \to X \to Z \to G \to 0$ is n-balanced exact, there are enough n-balanced projectives.

Finally, if G is n-balanced projective, then there is a splitting $Z \cong G \oplus X$, so that G is a summand of a group which is n-simply presented, as required. $\square$

Proposition 2.2. *If G is a separable (i.e., p^ω-bounded) group, then the following are equivalent:*

(a) G is n-simply presented;

(b) G is n-balanced projective;

(c) G is $p^{\omega+n}$-projective.

Proof. We begin by showing that (a) and (c) are equivalent. Observe first that if G is $p^{\omega+n}$-projective, then there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic, and hence totally projective. It follows immediately that G must be n-simply presented (this argument does not use the separability of G). Conversely, suppose P is an n-simply representing subgroup of G. If $\overline{P}$ is the closure of P in the p-adic topology of G, then $\overline{P} \subseteq G[p^n]$ and $\overline{P}/P = p^\omega(G/P)$. Therefore, $G/\overline{P} \cong (G/P)/p^\omega(G/P)$ will also be Σ-cyclic, so that G is $p^{\omega+n}$-projective.

Next, observe that since the collection of $p^{\omega+n}$-projective groups is closed under direct summands, by Theorem 2.1, the equivalence of (a) and (b) follows from the equivalence of (a) and (c). $\square$

We will extensively employ concepts related to *valuated groups* and *valuated vector spaces* which can be found, for example, in [15] and [6], and which we briefly review. Let $\mathcal{O}$ be the class of ordinals and $\mathcal{O}_\infty = \mathcal{O} \cup \{\infty\}$, where we agree that $\alpha < \infty$ for all $\alpha \in \mathcal{O}_\infty$. A *valuation* on a group V is a function which assigns to every $x \in V$ an element $|x|_V \in \mathcal{O}_\infty$ such that for every $x, y \in V$, $|x \pm y|_V \geq \min\{|x|_V, |y|_V\}$ and $|px|_V > |x|_V$. As a result, for all $\alpha \in \mathcal{O}_\infty$, $V(\alpha) = \{x \in V : |x|_V \geq \alpha\}$ is a subgroup of V with $pV(\alpha) \subseteq V(\alpha+1)$.

A homomorphism between two valuated groups will be said to be *valuated* if it does not decrease values and an *isometry* if it is bijective and preserves values. If V_i, $i \in I$, is a collection of valuated groups, then the usual direct sum $V = \bigoplus_{i\in I} V_i$ has a natural valuation, where $V(\alpha) = \bigoplus_{i\in I} V_i(\alpha)$ for every $\alpha \in \mathcal{O}_\infty$.

If W is any subgroup of V, then restricting $|\ |_V$ to W turns W into a valuated group with $W(\alpha) = W \cap V(\alpha)$ for all $\alpha \in \mathcal{O}_\infty$. A *valuated vector space* W is a p-bounded valuated group, so each $W(\alpha)$ will be a subspace of W; further, we say W is *free* if it is isometric to a valuated direct sum of cyclic groups (of order p). If V is a valuated group, then its socle $V[p]$ is a valuated vector space. Clearly, any group is a valuated group, using the height function as its valuation.

If V is a valuated group, then in [15] a functorial group $\mathcal{G}(V)$ was defined such that

(a) V is a nice subgroup of $\mathcal{G}(V)$;
(b) the valuation on V agrees with the height valuation on $\mathcal{G}(V)$;
(c) $\mathcal{G}(V)/V$ is simply presented;
(d) $V(\alpha) = \{0\}$ iff $p^\alpha \mathcal{G}(V) = \{0\}$.

It follows that if V is p^n-bounded, then $\mathcal{G}(V)$ is strongly n-simply presented.

We extend this construction in the following way: If G is a group and $n \geq 1$, then let $H(G) = \mathcal{G}(G[p^n])$.

Lemma 2.3. *Suppose G is a group and $n \geq 1$.*

(a) The identity map $G[p^n] \to G[p^n]$ extends to a homomorphism $\pi : H(G) \to G$;

(b) If $K(G)$ is the kernel of π, then $0 \to K(G) \to H(G) \to G \to 0$ is strongly n-balanced exact.

Proof. (a) This follows from the fact that $G[p^n]$ is nice in $H(G)$, $H(G)/G[p^n]$ is simply presented and the identity map clearly does not decrease heights (see, for example, Corollary 81.4 of [5]).

(b) Observe first that if $x \in G[p]$, then $|x|_G = |x|_{H(G)} \leq |x|_{\pi(H(G))} \leq |x|_G$, so that $|x|_{\pi(H(G))} = |x|_G$. It follows that $\pi(H(G))$ is an isotype subgroup of G containing $G[p]$, so that $\pi(H(G)) = G$ and π is surjective.

Next, the identity map $G[p^n] \to G[p^n]$ induces a valuated splitting

$$H(G)[p^n] \cong K(G)[p^n] \oplus G[p^n],$$

and (b) follows. $\square$

We have the following analogue of Theorem 2.1.

Theorem 2.4. *The group G is strongly n-balanced projective iff it is a summand of a group that is strongly n-simply presented. There are enough strongly n-balanced projectives.*

Proof. Note that if $n = 0$, the result is well known, so assume $n \geq 1$. If G is strongly n-simply presented, then it has a nice n-simply representing subgroup N. Suppose $E : 0 \to X \to Y \to G \to 0$ is strongly n-balanced exact and ϕ is the right homomorphism; by definition, then, there is a valuated splitting $\nu : G[p^n] \to Y[p^n]$. Again referring to Corollary 81.4 of [5], the restriction of ν to $N \to Y[p^n]$ extends to a homomorphism $h : G \to Y$ such that $\phi \circ h$ is the identity on N. It follows that $1_G - \phi \circ h$ is zero on N, so it induces a homomorphism $G/N \to G$. However, since G/N is simply presented and E is balanced, there is a homomorphism $G/N \to Y$ such that if h' is the composition $G \to G/N \to Y$, then $1_G - \pi \circ h = \pi \circ h'$. Since $1_G = \pi \circ (h + h')$, it follows that E splits, so that G is strongly n-balanced projective.

Therefore, any summand of a strongly n-simply presented group is strongly n-balanced projective, and by Lemma 2.3(b), there are enough strongly n-balanced projectives.

Conversely, if G is strongly n-balanced projectives, then it must be a summand of $H(G)$, which is strongly n-simply presented. □

The following is an analogue of Proposition 2.2:

Proposition 2.5. *If G is a $p^{\omega+n}$-bounded group, then the following are equivalent:*

(a) G is strongly n-simply presented;

(b) G is strongly n-balanced projective;

(c) G is $p^{\omega+n}$-projective.

Proof. Note that if $n = 0$, all these statements simply say that G is Σ-cyclic, so assume $n \geq 1$. As in Proposition 2.2, the result will follow once we show the equivalence of (a) and (c). Note if (c) holds, then there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic. If follows that this P will necessarily be nice in G, so that G is strongly n-simply presented.

Conversely, suppose P is a nice n-simply presenting subgroup of G and $A = G/P$. Note $p^\omega G \subseteq G[p^n]$, so that $P' = p^\omega G + P$ is p^n-bounded. In addition,

$$G/P' = G/[p^\omega G + P] \cong (G/P)/[p^\omega G + P]/P = A/p^\omega A$$

is Σ-cyclic, and it follows that G is $p^{\omega+n}$-projective, as required. □

We say G is *strongly n-co-simply presented* if there is a simply presented group T and a nice subgroup $Q \subseteq T[p^n]$ such that $G \cong T/Q$. Though a group is n-simply presented iff it is n-cosimply presented, we make the following observation:

Example 2.1. *There is a group G which is strongly 1-co-simply presented, which is not strongly 1-balanced projective (and so not strongly 1-simply presented).*

Proof. Let M be some separable non-Σ-group with basic subgroup B, and let $0 \to Y \to X \to M \to 0$ be a pure-projective resolution of M, where we assume $Y \subseteq X$ and $X/Y = M$. Let $P = Y[p]$, so that $Z = X/P$ is a separable $p^{\omega+1}$-projective group which is not Σ-cyclic. Next, let D be the subgroup of X containing Y such that $D/Y = B$. There is a splitting, $D \cong Y \oplus B$, hence $E = D/P \cong (Y/P) \oplus B \cong pY \oplus B$ will also be Σ-cyclic. Note that D is pure and dense in X, so that E is pure and dense in Z.

Let C be a group such that $p^\omega C$ is a direct sum of cyclic groups of order p for which $C/p^\omega C$ can be identified with X such that $D = [D' + p^\omega C]/p^\omega C$ for some high subgroup D' of C. [For example, if $K = X/D$, we may let $C = \{(x,k) \in X \oplus K : x + D = pk\}$.] If $P' \subseteq D'$ satisfies $P = [P' + p^\omega C]/p^\omega C$, then we let $G = C/P'$.

The following can now be checked:

(a) C is a dsc-group of length $\omega + 1$ (since $C/p^\omega C = X$ is Σ-cyclic).

(b) P' is nice in C (since $P' \cap p^\omega C = \{0\}$ and $[P' + p^\omega C]/p^\omega C = P$ is nice in $X = C/p^\omega C$).

(c) G is strongly 1-co-simply presented (by (a) and (b)).

(d) $E' = D'/P'$ is a high subgroup of G (since D' is a maximal subgroup of C containing P' intersecting $p^\omega C$ trivially, we have E' is a maximal subgroup of G intersecting $p^\omega G = [p^\omega C + P']/P'$ trivially).

(e) $G[p]$ is a free valuated vector space ($E' = D'/P' \cong D/P = E$ is Σ-cyclic and there is an isometry $G[p] = E'[p] \oplus p^\omega G$ where both terms are, in fact, free as valuated vector spaces).

(f) G is not strongly 1-balanced projective: If G were strongly 1-balanced projective, then by Proposition 2.5, it would be $p^{\omega+1}$-projective. In this case, (e) would imply that G is totally projective (since $G[p]$ will be isometric to the socle of a totally projective group and $p^{\omega+1}$-projective groups with isometric socles are, in fact, isomorphic). However, this contradicts the fact that

$$\begin{aligned} G/p^\omega G &= [C/P']/[(p^\omega C + P')/P'] \\ &\cong C/(p^\omega C + P') \\ &\cong [C/p^\omega C]/[(p^\omega C + P')/p^\omega C] = X/P = Z \end{aligned}$$

is not Σ-cyclic. □

Since the group in this example is strongly 1-co-simply presented, it is also 1-co-simply presented and hence 1-simply presented and 1-balanced projective, i.e., the classes of 1-simply presented and 1-balanced projective groups properly contain the classes of strongly 1-simply presented and strongly 1-balanced projective groups, respectively. In other words, though "0-simply presented" = "strongly 0-simply presented" = "0-balanced projective" = "strongly 0-balanced projective" = "simply presented," for $n \geq 1$ the containments "strongly n-simply presented"

$\subset$ "n-simply presented" and "strongly n-balanced projective" $\subset$ "n-balanced projective" are proper. This also implies that for $n \geq 1$, there are strongly n-balanced short exact sequences that are not n-balanced short exact.

3. Nunke-esque Results

We collect a number of routine observations in the following:

Lemma 3.1. *Suppose λ is an ordinal, G is a group with a subgroup $P \subseteq G[p^n]$, $A = G/P$ and X is the subgroup of G containing P such that $X/P = p^{\lambda+n}A$. Then*

(a) $p^{\lambda+n}G + P \subseteq X \subseteq p^{\lambda}G + P$;

(b) there is a short exact sequence

$$0 \to p^{\lambda+n}G/[p^{\lambda+n}G \cap P] \to p^{\lambda+n}A \to B_1 \to 0$$

where B_1 is bounded;

(c) there is a short exact sequence

$$0 \to B_2 \to A/p^{\lambda+n}A \to G/[p^{\lambda}G + P] \to 0$$

where $B_2 \subseteq p^{\lambda}(A/p^{\lambda+n}A)$ is bounded.

Proof. (a) Clearly $[p^{\lambda+n}G + P]/P \subseteq p^{\lambda+n}A$, so that $p^{\lambda+n}G + P \subseteq X$. On the other hand, there is a short exact sequence

$$0 \to [p^{\lambda}G + P]/p^{\lambda}G \to G/p^{\lambda}G \to G/[p^{\lambda}G + P] \to 0.$$

Since $p^{\lambda}(G/p^{\lambda}G) = \{0\}$ and $p^n([p^{\lambda}G + P]/p^{\lambda}G) = \{0\}$, we have $p^{\lambda+n}(G/[p^{\lambda}G + P]) = \{0\}$.

Now, there is another short exact sequence

$$0 \to [p^{\lambda}G + P]/P \to A \to G/[p^{\lambda}G + P] \to 0.$$

It therefore follows that $p^{\lambda+n}A \subseteq [p^{\lambda}G + P]/P$, giving $X \subseteq p^{\lambda}G + P$, as required.

(b) There is a short exact sequence

$$0 \to [p^{\lambda+n}G + P]/P \to X/P \to X/[p^{\lambda+n}G + P] \to 0.$$

Clearly, the first two terms agree with those in (b), and we let B_1 be the third term. By (a), $p^nX \subseteq p^n(p^{\lambda}G + P) \subseteq p^{\lambda+n}G + P$, showing that B_1 is bounded.

(c) There is a short exact sequence

$$0 \to [p^{\lambda}G + P]/X \to G/X \to G/[p^{\lambda}G + P] \to 0.$$

There is also an isomorphism $G/X \cong (G/P)/(X/P) = A/p^{\lambda+n}A$. Finally,

$$B_2 = [p^\lambda G + P]/X = [p^\lambda G + X]/X \subseteq p^\lambda(G/X) \cong p^\lambda(A/p^{\lambda+n}A)$$

is p^n-bounded. □

The following well-known technicality is even more straightforward.

Lemma 3.2. *Let λ be an ordinal and Z be a group.*

(a) Suppose Y is a subgroup of Z such that Z/Y is bounded. Then Z is simply presented iff Y is simply presented.

(b) Suppose Y is a bounded subgroup of $p^\lambda Z$. Then $p^\lambda Z$ is bounded and Z is simply presented iff $p^\lambda(Z/Y)$ is bounded and Z/Y is simply presented.

Proof. (a) Suppose Z is simply presented. Then $p^\omega Y = p^\omega Z$ is simply presented, and $Y/p^\omega Y$ embeds in $Z/p^\omega Z$, which is Σ-cyclic. Therefore, $Y/p^\omega Y$ is Σ-cyclic, and Y is simply presented.

Conversely, if Y is simply presented, then for some k, $p^k Z \subseteq Y$, so that by the first part of the proof, $p^k Z$ is simply presented, implying that Z is simply presented, as well.

(b) Note that $p^\lambda Z$ is bounded iff $p^\lambda(Z/Y) = p^\lambda Z/Y$ is bounded. In this case, it follows that Z is simply presented iff $Z/p^\lambda Z$ is simply presented iff $(Z/Y)/p^\lambda(Z/Y)$ is simply presented iff Z/Y is simply presented. □

We put these together in the following:

Lemma 3.3. *Suppose λ is an ordinal, G is a group with a subgroup P such that $p^n P = \{0\}$ and $A = G/P$. Then*

(a) If $m < \omega$, then P is an n-simply representing subgroup of G iff $p^m G \cap P$ is an n-simply representing subgroup of $p^m G$;

(b) A is simply presented iff both $p^\lambda G/[p^\lambda G \cap P]$ and $G/[p^\lambda G + P]$ are simply presented.

Proof. In can be checked that (a) is a consequence of Lemma 3.2(a). As to (b), by Lemma 3.1(b) and Lemma 3.2(a), it follows that $p^{\lambda+n}A$ is simply presented iff $p^{\lambda+n}G/[p^{\lambda+n}G \cap P]$ is simply presented, and by (a) this is equivalent to $p^\lambda G/[p^\lambda G \cap P]$ being simply presented. Again, by Lemma 3.1(c) and Lemma 3.2(b), $A/p^{\lambda+n}A$ is simply presented iff $G/[p^\lambda G + P]$ is simply presented. Since A is simply presented iff $p^{\lambda+n}A$ and $A/p^{\lambda+n}A$ are simply presented, the result follows. □

Theorem 3.4. *Suppose λ is an ordinal and G is a group.*

(a) If G is (strongly) n-simply presented, then both $p^\lambda G$ and $G/p^\lambda G$ are (strongly) n-simply presented.

(b) If both $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are (strongly) n-simply presented, then G is (strongly) n-simply presented.

Proof. Suppose first that G is n-simply presented. Let P be an n-simply representing subgroup of G. By Lemma 3.3(b),

$$(G/p^\lambda G)/([p^\lambda G + P]/p^\lambda G) \cong G/[p^\lambda G + P]$$

and $p^\lambda G/[p^\lambda G \cap P]$ are simply presented. In addition, since

$$p^n[p^\lambda G \cap P] = \{0\} = p^n \left([p^\lambda G + P]/p^\lambda G\right),$$

we can conclude that $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.

Observe that if G were actually *strongly* n-simply presented, then we could assume P is nice in G, and it would follow that $P \cap p^\lambda G$ is nice in $p^\lambda G$ and $[p^\lambda G + P]/p^\lambda G$ is nice in $G/p^\lambda G$, so that these two groups are, in fact, strongly n-simply presented, as well.

Turning to (b), suppose that P_1 is a subgroup of G containing $p^{\lambda+n}G$ for which $P_1/p^{\lambda+n}G$ is an n-simply representing subgroup of $G/p^{\lambda+n}G$. Let Y be a maximal p^n-bounded summand of $p^\lambda G$, so that there is a decomposition $p^\lambda G = X \oplus Y$. Let H be a $p^{\lambda+n}$-high subgroup of G containing Y (i.e., H is maximal with respect to intersecting $p^{\lambda+n}G$ trivially).

We next claim that $(G/p^{\lambda+n}G)[p^n] = (X \oplus H[p^n])/p^{\lambda+n}G$: Note that $X[p] = (p^{\lambda+n}G)[p]$, so that $X \cap H = \{0\}$; this means that $X \oplus H[p^n]$ really is an internal direct sum. Since $p^n X \subseteq p^{\lambda+n}G$ and $p^n H[p^n] = \{0\}$, inclusion in one direction is clear. So assume $z \in G$ and $p^n z \in p^{\lambda+n}G$; we need to show that $z \in X \oplus H[p^n]$. If $x \in X$ is chosen so that $p^n x = p^n z$, then replacing z by $z - x$, we may assume $p^n z = 0$. Next, since $G[p] = (p^{\lambda+n}G)[p] \oplus H[p]$, H is pure in G and $(p^{\lambda+n}G) = p^n X$, it follows that $G[p^n] = X[p^n] \oplus H[p^n]$. Therefore, $z = x' + h$, where $x' \in X[p^n] \subseteq X$ and $h \in H[p^n]$, as required.

It follows from the last paragraph that $P_1 \subseteq X \oplus H[p^n]$. Let

$$P_2 = (X + P_1) \cap H[p^n] \subseteq G[p^n].$$

Clearly, $P_2 \subseteq H$ implies that $P_2 \cap p^{\lambda+n}G = \{0\}$. In addition, $P_1 \subseteq X \oplus H[p^n]$ also implies that

$$X + P_1 = X + [(X + P_1) \cap H[p^n]] = X + P_2.$$

We can therefore conclude that $p^\lambda G + P_1 = p^\lambda G + P_2$.

Next, let P_3 be an n-simply representing subgroup of $(p^{\lambda+n}G)[p^n]$. We then let $P = P_2 + P_3$, so that $P \subseteq G[p^n]$. We clearly have $p^{\lambda+n}G \cap P = P_3$, so that $p^{\lambda+n}G/[p^{\lambda+n}G \cap P]$ is simply presented. By Lemma 3.3(a), we also can conclude that $p^\lambda G/[p^\lambda G \cap P]$ is simply presented. In addition, $p^\lambda G + P = p^\lambda G + P_2 = p^\lambda G + P_1$.

Note that $G/P_1 \cong (G/p^{\lambda+n}G)/(P_1/p^{\lambda+n}G)$ is simply presented; and since $p^n(P_1/p^{\lambda+n}G) = \{0\}$, it follows that $p^\lambda(G/P_1)$ is bounded (by p^{2n}). Therefore, $[p^\lambda G + P_1]/P_1$ is a bounded subgroup of $p^\lambda(G/P_1)$. Applying Lemma 3.2(b) to G/P_1, we can deduce that

$$G/[p^\lambda G + P] = G/[p^\lambda G + P_1] \cong (G/P_1)/([p^\lambda G + P_1]/P_1)$$

is simply presented. Therefore, by Lemma 3.3(b), we have that G/P is simply presented, as desired.

Suppose $p^{\lambda+n}G$ and $G/p^{\lambda+n}G$ are actually *strongly* n-simply presented. In this case, we can choose $P_3 = p^{\lambda+n}G \cap P$ to be nice in $p^{\lambda+n}G$ and $P_1/p^{\lambda+n}G$ to be nice in $G/p^{\lambda+n}G$. If, as above, $P = P_2 + P_3$, then $P \cap p^{\lambda+n}G = P_3$ being nice in $p^{\lambda+n}G$ and $p^\lambda G/p^{\lambda+n}G$ being bounded readily imply that $P \cap p^\lambda G$ is nice in $p^\lambda G$. In addition,

$$\begin{aligned}[p^\lambda G + P]/p^\lambda G &= [p^\lambda G + P_1]/p^\lambda G \\ &\cong [(p^\lambda G/p^{\lambda+n}G) + (P_1/p^{\lambda+n}G)]/(p^\lambda G/p^{\lambda+n}G)\end{aligned}$$

is nice in $G/p^\lambda G \cong (G/p^{\lambda+n}G)/p^\lambda(G/p^{\lambda+n}G)$. Together, these assure that P is nice in G, hence G is strongly n-simply presented. □

We can easily extend the last result to summands.

Proposition 3.5. *Suppose C is a group and λ is an ordinal.*

(a) If C is (strongly) n-balanced projective, then both $p^\lambda C$ and $C/p^\lambda C$ are (strongly) n-balanced projective.

(b) If $p^{\lambda+n}C$ and $C/p^{\lambda+n}C$ are (strongly) n-balanced projective, then C is (strongly) n-balanced projective.

Proof. For (a), note that if C is (strongly) n-balanced projective, then C is a summand of a (strongly) n-simply presented group G. It follows that $p^\lambda C$ and $C/p^\lambda C$ are summands of $p^\lambda G$ and $G/p^\lambda G$, respectively, and since the latter two groups are (strongly) n-simply presented, it follows that $p^\lambda C$ and $C/p^\lambda C$ are (strongly) n-balanced projectives.

Turning to (b), suppose $p^{\lambda+n}C$ and $C/p^{\lambda+n}C$ are (strongly) n-balanced projective. Observe first that there are groups Z and Y such that $p^{\lambda+n}C \oplus Z$ and $(C/p^{\lambda+n}C) \oplus Y$ are (strongly) n-simply presented. It follows that $p^{\lambda+n}Y \cong$

$p^{\lambda+n}((C/p^{\lambda+n}C) \oplus Y)$ is (strongly) n-simply presented. Construct a group X such that $p^{\lambda+n}X \cong Z$ and $X/p^{\lambda+n}X$ is simply presented (see, for instance, [9]).

The proof will be complete if we can show $C \oplus X \oplus Y$ is (strongly) n-simply presented. Note that $p^{\lambda+n}(C \oplus X \oplus Y) \cong (p^{\lambda+n}C \oplus Z) \oplus p^{\lambda+n}Y$ is (strongly) n-simply presented. In addition, since $X/p^{\lambda+n}X$ is simply presented,

$$\begin{aligned}&(C \oplus X \oplus Y)/p^{\lambda+n}(C \oplus X \oplus Y) \cong \\ &\quad ((C/p^{\lambda+n}C) \oplus Y)/p^{\lambda+n}((C/p^{\lambda+n}C) \oplus Y) \oplus (X/p^{\lambda+n}X)\end{aligned}$$

is (strongly) n-simply presented. Therefore, by Theorem 3.4, $C \oplus X \oplus Y$ is (strongly) n-simply presented, as required. $\square$

Corollary 3.6. *Suppose G is a group of length strictly less than ω^2. Then the following conditions are equivalent:*

(a) G is strongly n-simply presented;

(b) G is strongly n-balanced projective;

(c) For every non-negative integer m, the factor group $p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$ is $p^{\omega+n}$-projective.

Proof. Note that there are only a finite number of non-zero factor groups $p^{\omega\cdot m+n}G/p^{\omega\cdot(m+1)+n}G$, and it follows from Theorem 3.4 that G is strongly n-simply presented (respectively, strongly n-balanced projective) iff each of these factor groups share that property. Finally, by Proposition 2.5, these two conditions are equivalent on each of these factors, and further, they are equivalent to condition (c). $\square$

We now illustrate that the full Nunke property does *not* hold for *strongly n-simply presented* groups.

Example 3.1. *There is a group G for which $p^\omega G$ and $G/p^\omega G$ are strongly 1-simply presented, but G itself is not strongly 1-simply presented (or even strongly 1-balanced projective).*

Proof. Consider the group G of Example 2.1. In discussing this example, it was noted that $G' = G/p^\omega G$ is $p^{\omega+1}$-projective, and hence strongly 1-simply presented. Since G is $p^{\omega+1}$-bounded, we also can conclude that $p^\omega G$ is Σ-cyclic, and hence strongly 1-simply presented. We know, however, that G is not strongly 1-balanced projective. $\square$

In fact, in the last example, we really could be more general. If G is any group of length $\omega+1$ which is not $p^{\omega+1}$-projective, but for which $G/p^\omega G$ is $p^{\omega+1}$-projective, then G will satisfy our requirements.

4. The Nunke Property for n-Simply Presented Groups

The purpose of this section is to verify that, as opposed to the case of strongly n-simply presented groups, the full Nunke property holds for the larger class of n-simply presented groups. Before doing so, however, we first take a fairly extended detour into the realm of valuated vector spaces. A valuated vector space is said to be *subfree* if it embeds as an isometric subspace of a free valuated vector space. The following property is well known.

Lemma 4.1. *If H is a totally projective group, then $H[p]$ is subfree.*

Proof. We verify this by induction on the length of H, which we denote by γ. If γ is a limit, then H is isomorphic to a direct sum $\bigoplus_{\alpha<\gamma} H_\alpha$, where $p^\alpha H_\alpha = \{0\}$. By induction, each $(H_\alpha)[p]$ is subfree, hence the same holds for H. Assuming $\gamma = \beta + 1$ is isolated, if H' is a p^β-high subgroup of H, then there is an isometry $H[p] = H'[p] \oplus p^\beta H$. Clearly $p^\beta H$ is free. In addition, $H'[p]$ embeds isometrically in $(H/p^\beta H)[p]$. Since $H/p^\beta H$ is totally projective of length β, by induction, its socle is subfree. Hence $H'[p]$ is also subfree, so that $H[p]$ is subfree, establishing the result. □

By a *graded vector space*, we will mean a collection of vector spaces indexed by the ordinals, $U = [U_\alpha]_{\alpha<\infty}$, such that there is an ordinal λ with $U_\alpha = \{0\}$ for all $\alpha \geq \lambda$; the smallest such ordinal λ we call the *length* of U. The definition of a graded homomorphism or isomorphism follows naturally and the resulting category of graded vector spaces clearly has direct sums. We say $x \in U$ if there is an α such that $x \in U_\alpha$ and if $x \neq 0$ we write $|x|_U = \alpha$. We say U is *admissible* if its *Ulm function* $f_U(\alpha) = r(U_\alpha)$ is admissible in the usual sense. Let $R(U) = \sum_{\alpha<\infty} r(U_\alpha)$, and if β is an ordinal, let $R_\beta(U) = \sum_{\beta\leq\alpha<\beta+\omega} r(U_\alpha)$.

Our motivating example is where V is a valuated vector space (e.g., the socle of some group) and $U(V)$ is the graded vector space $[U_\alpha(V)]_{\alpha<\infty} = [V(\alpha)/V(\alpha+1)]_{\alpha<\infty}$. We let $R(V) = R(U(V))$ and $R_\beta(V) = R_\beta(U(V))$. If $\mathcal{L}$ is a subset of a valuated vector space V, then for each ordinal α we let $\mathcal{L}_\alpha = \{x \in \mathcal{L} : |x|_V = \alpha\}$ and we let $\operatorname{span}(\mathcal{L})$ be the graded vector space $[\operatorname{span}(\mathcal{L}_\alpha)]_{\alpha<\infty}$ (where we are identifying each element of $\mathcal{L}_\alpha$ with its image in $U_\alpha(V)$). We say $\mathcal{L}$ is *linearly independent* if $\mathcal{L}_\alpha$ is linearly independent in $U_\alpha(V)$ for every α, and a *basis* if, in addition, $U(V) = \operatorname{span}(\mathcal{L})$. If $\mathcal{L}$ is linearly independent, let $R(\mathcal{L}) = |\mathcal{L}| = R(\operatorname{span}(\mathcal{L}))$, and if β is an ordinal, let $R_\beta(\mathcal{L}) = |\{x \in \mathcal{L} : \beta \leq |x|_V < \beta + \omega\}| = R_\beta(\operatorname{span}(\mathcal{L}))$. We say $\mathcal{L}$ is *admissible* if the function $f_{\mathcal{L}}(\alpha) = |\mathcal{L}_\alpha|$ is an admissible function.

Lemma 4.2. *If V is a subfree valuated vector space and $V(\infty) = \{0\}$, then $R(V) = r(V)$.*

Proof. Suppose V is a valuated subspace of the free valuated vector space W and λ is the length of V. We induct on λ, so assume the result holds for all subfree valuated vector spaces of smaller length. Replacing W by $W/W(\lambda)$, we may assume $W(\lambda) = \{0\}$.

Fix a decomposition $W = \oplus_{\alpha<\lambda} B_\alpha$, where $|x|_W = \alpha$ for all $0 \neq x \in B_\alpha$. For $\gamma \leq \lambda$, let $W_\gamma = \oplus_{\alpha<\gamma} B_\alpha \subseteq W$ and $V_\gamma = V \cap W_\gamma$.

Case 1 - λ is a limit ordinal:

Using the induction hypothesis, it can be seen that

$$r(V) = \sup\{r(V_\gamma) : \gamma < \lambda\} = \sup\{R(V_\gamma) : \gamma < \lambda\} = R(V).$$

Case 2 - $\lambda = \gamma + 1$ is an isolated ordinal:

Again, there is a valuated decomposition $V \cong V(\gamma) \oplus (V/V(\gamma))$, where the first term is already free and the second term is sub-free of smaller length. Applying the induction hypothesis to the second term gives the result. $\square$

We now verify an important technical observation.

Lemma 4.3. *Suppose U is a graded vector space whose length is a limit ordinal λ, $\kappa \geq |\lambda|$ is a cardinal and $R_\beta(U) = \kappa$ for all $\beta < \lambda$ (so that $R(U) = \kappa$ as well, and U is admissible). Let I be a set of cardinality κ, and for each $i \in I$, let W_i be a graded subspace of U with $R(W_i) = \kappa$. Then U is an (internal) direct sum, $\bigoplus_{i\in I} V_i$, where each V_i is admissible of length $\lambda_i < \lambda$, and $V_i \cap W_i \neq \{0\}$.*

Proof. Identify I with κ. Suppose we have defined s_i and t_i for all $i < \ell < \kappa$ satisfying

(a1) $s_i \in W_i$ for all $i < \ell$;

(b1) if $\mathcal{L}_\ell = \{s_i : i < \ell\}$ and $\mathcal{M}_\ell = \{t_i : i < \ell\}$, then $\mathcal{L}_\ell \cup \mathcal{M}_\ell$ is linearly independent;

(c1) for each $i < \ell$, $|s_i|_U \leq |t_i|_U < |s_i|_U + \omega$.

To define s_ℓ, note that $R(W_\ell) = \kappa$ and $|\mathcal{L}_\ell \cup \mathcal{M}_\ell| < \kappa$, so we can find an $s_\ell \in W_\ell$ which is not in $\mathrm{span}(\mathcal{L}_\ell \cup \mathcal{M}_\ell)$. Since $R_{|s_\ell|_U}(U) = \kappa$, we can therefore find a t_ℓ so that (b1) and (c1) are valid for $\ell' = \ell + 1$. Therefore, by induction, we can define these elements so that (a1), (b1) and (c1) hold for all $i < \ell = \kappa$, and we let $\mathcal{L} = \{s_i : i < \kappa\}$ and $\mathcal{M} = \{t_i : i < \kappa\}$.

Note that if $\beta < \lambda$ is an ordinal, then (c1) implies that $R_\beta(\mathcal{L}) \leq R_\beta(\mathcal{M})$. Expand $\mathcal{M}$ to a set $\mathcal{P}$ such that $\mathcal{P} \cap \mathcal{L} = \emptyset$ and $\mathcal{L} \cup \mathcal{P}$ is a basis for U. Observe that (c1) implies that for all $\beta < \lambda$, $R_\beta(\mathcal{L}) \leq R_\beta(\mathcal{P})$; and $R_\beta(U) = \kappa$ implies that $R_\beta(\mathcal{P}) = \kappa$. This means that we can decompose $\mathcal{P}$ into the disjoint union of admissible subsets $\mathcal{P}_i$, for $i < \kappa$, of length λ_i, where $|s_i|_U \leq \lambda_i < \lambda$ (just construct them such that $R_\beta(\mathcal{P}_i) = \kappa$ for all $\beta < \lambda_i$). Letting $W_i = \mathrm{span}(\mathcal{P}_i \cup \{s_i\})$ proves the result. □

This brings us to the objective of this section.

Theorem 4.4. *Suppose G is a group and λ is any ordinal. Then G is n-simply presented iff $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented.*

Proof. By Theorem 3.4(a), if G is n-simply presented, then $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented. On the other hand, if $p^\lambda G$ and $G/p^\lambda G$ are n-simply presented, then clearly $p^{\lambda+n}G = p^n(p^\lambda G)$ is n-simply presented. If we let $G' = G/p^{\lambda+n}G$, it follows from Theorem 3.4(b) that G is n-simply presented iff G' has that property. If $\lambda = \mu + m$, where μ is a limit ordinal and $m < \omega$, then

$$G'/p^\mu G' = (G/p^{\lambda+n}G)/p^\mu(G/p^{\lambda+n}G) \cong (G/p^\lambda G)/p^\mu(G/p^\lambda G)$$

is n-simply presented. In addition, $p^\mu G' = p^\mu G/p^{\mu+m+n}G$ is bounded. Our result, therefore, can be reduced to the following special case. Because of its importance, we formulate it separately. □

Theorem 4.5. *If G is a group and λ is a limit ordinal such that $p^\lambda G$ is bounded and $G/p^\lambda G$ is n-simply presented, then G is n-simply presented.*

Proof. We begin with some simplifying assumptions. Consider a subgroup Q of G containing $p^\lambda G$ such that $Q/p^\lambda G$ is an n-simply representing subgroup of $G/p^\lambda G$. As in the proof of Proposition 2.2, it is easily checked that if $\overline{Q}$ is the closure of Q in the p^λ-topology of G (which uses the subgroups $p^\alpha G$ for $\alpha < \lambda$ as a neighborhood basis of 0), then $\overline{Q}/p^\lambda G$ is also an n-simply representing subgroup of $G/p^\lambda G$. We may therefore assume that Q is closed, so that $p^\lambda(G/Q) = \{0\}$.

Next, observe that if $\alpha < \lambda$ is an ordinal, then by Theorem 3.4,

$$G/p^{\alpha+n}G \cong (G/p^\lambda G)/p^{\alpha+n}(G/p^\lambda G)$$

is n-simply presented, and so G is n-simply presented iff $p^{\alpha+n}G$ is n-simply presented. This means that we may replace G by $p^{\alpha+n}G$, if necessary. For example, the λ-*final rank* of G is the infimum of the set $\{r(p^\alpha G) : \alpha < \lambda\}$, and we may clearly assume that G has rank and λ-final rank equal to some infinite cardinal κ. Note also that if $\kappa = \aleph_0$, then G is countable, and hence trivially n-simply presented. We may therefore assume that κ is uncountable.

In fact, we can refine these conditions.

Assumption: If $\pi_Q : G \to G/Q$ is the natural homomorphism, then for some positive integer k we have

$$r(\pi_Q(G[p^{k-1}])) < \kappa \text{ and } r(\pi_Q((p^\alpha G)[p^k])) = \kappa$$

for all $\alpha < \lambda$.

To verify that we can make this Assumption, for any ordinal α and integer $k \geq 1$, let $\rho(\alpha, k) = r(\pi_Q((p^\alpha G)[p^k]))$. Since $p^\lambda G$ and $Q/p^\lambda G$ are bounded, there is an $m < \omega$ such that $p^m Q = \{0\}$. For $k > m$, the fact that the rank and λ-final rank of G both equal κ implies that $\rho(\alpha, k) = \kappa$ for all $\alpha < \lambda$. For each $k \leq m$, we can find an $\alpha_k < \lambda$ such that $\rho(\alpha, k)$ is constant for all $\alpha_k < \alpha < \lambda$. If $\beta = \max\{\alpha_k : k \leq m\}$, we then replace G by $p^{\beta+n}G$ and Q by $p^{\beta+n}G \cap Q$ and we can let k be the smallest integer such that $\rho(\beta + n, k) = \kappa$.

For the rest of the proof, Q and k will be defined as in the Assumption. The next definition is the key concept in verifying the full Nunke property for n-simply presented groups. A subgroup P of G containing $p^\lambda G$ is an (n, λ, κ)-*subgroup* if $P/p^\lambda G$ is an n-simply representing subgroup of $G/p^\lambda G$, and if $\pi_P : G \to G/P$ is the usual homomorphism, then there is a decomposition

$$H_P = G/P = \bigoplus_{i \in I} Y_i$$

where $|I| = \kappa$, such that

(a2) Y_i has length strictly less than λ;

(b2) If $K \subseteq I$ with $|K| < \kappa$, and $\alpha < \lambda$, then there is an $x \in (p^\alpha G)[p] - P$ such that $\pi_P(x) = y + z$, where $0 \neq y \in \bigoplus_{i \in I-K} Y_i$, $z \in \bigoplus_{i \in K} Y_i$ and $|y|_{H_P} \leq |z|_{H_P}$.

Intuitively, an (n, λ, κ) subgroup is one where, for all $\alpha < \lambda$, $H_P[p]$ has "enough" elements of the form $x_P = \pi_P(x)$, where $x \in (p^\alpha G)[p]$; we are essentially demanding that they are "spread widely" among the summands of some decomposition. The next statement is a refinement of a construction that appeared in [10].

Claim A: G has an (n, λ, κ)-subgroup.

Clearly, $|\lambda| \leq \kappa$, so if I is a set of cardinality κ there is a function $\phi : I \to \lambda$ such that for all $\alpha < \lambda$, the set of $i \in I$ such that $\phi(i) = \alpha$ also has cardinality κ. Denote $\phi(i)$ by α_i. Consider the graded vector space $U = U(H_Q[p])$, where for each $i \in I$ we let

$$W_i = U(H_Q[p] \cap \pi_Q((p^{\alpha_i} G)[p^k])) \subseteq U.$$

By Lemma 4.1, U, and hence W_i, is subfree. So, by Lemma 4.2 and our Assumption, we know that

$$R(W_i) = r(H_Q[p] \cap \pi_Q((p^{\alpha_i}G)[p^k])) = \kappa.$$

Consequently, in view of Lemma 4.3, we can conclude that there is a decomposition $U = \bigoplus_{i \in I} V_i$ where each V_i is an admissible graded space of length $\lambda_i < \lambda$, and an element $s_i \in (p^{\alpha_i}G)[p^k]$ such that $s_{Q,i} = \pi_Q(s_i)$ represents a non-zero element of V_i.

Let T_i be a simply presented group such that $U(T_i)$ is isomorphic to V_i. If $T = \bigoplus_{i \in I} T_i$, then the usual approach to the classification of totally projective groups (see, for example, Lemma 77.1 of [5]) implies these isomorphisms are induced by a group isomorphism $T \to H_Q$ which we interpret as an equality. All of this work has the following consequence:

(a3) If $i \in I$ and we consider $s_i \in (p^{\alpha_i}G)[p^k] - Q$, then $s_{Q,i} \in H_Q[p]$ is a non-zero element whose i^{th} coordinate has minimal value in our decomposition $H_Q = \bigoplus_{i \in I} T_i$.

If $J \subseteq I$, we let $\Sigma_J = \bigoplus_{i \in J} T_i$. By our Assumption, $\pi_Q(G[p^{k-1}]))$ has rank strictly less than κ, so there is a subset $J \subseteq I$ such that $|J| < \kappa$ and $\pi_Q(G[p^{k-1}])) \subseteq \Sigma_J$. We let R be the subgroup of G containing Q such that $R/Q = (\Sigma_J)[p^{k-1}]$; clearly $G[p^{k-1}] \subseteq R$. The proof of Claim A will be completed by the next observation.

Claim B: $P = p^{k-1}R$ is an (n, λ, κ)-subgroup of G.

We break this into a sequence of statements.

Subclaim B1: $p^n P = p^{n+k-1}R \subseteq p^\lambda G$.
Note that $p^{k-1}R \subseteq Q$, so that $p^{n+k-1}R \subseteq p^n Q \subseteq p^\lambda G$, as required.

Subclaim B2: $p^{k-1}G/P \cong (\bigoplus_{i \in I-J} T_i) \oplus (\bigoplus_{j \in J} p^{k-1}T_j)$.

We have a sequence of isomorphisms,

$$\begin{aligned} p^{k-1}G/p^{k-1}R &\cong (G/G[p^{k-1}])/(R/G[p^{k-1}]) \cong G/R \cong (G/Q)/(R/Q) \\ &\cong (\Sigma_I)/(\Sigma_J)[p^{k-1}] \cong (\Sigma_{I-J}) \oplus (p^{k-1}\Sigma_J), \end{aligned}$$

which clearly gives the Subclaim.

Note that $p^{k-1}(G/P) = p^{k-1}G/P$, so that G/P is simply presented. It follows that for all $i \in I$, we can construct a group Y_i such that

(a4) there is an isomorphism $\bigoplus_{i\in I} Y_i \cong G/P$ such that
(b4) it restricts to an isomorphism of $p^{k-1}Y_j$ and $p^{k-1}T_j$ whenever $j \in J$;
(c4) it restricts to an isomorphism of $p^{k-1}Y_i$ and T_i whenever $i \in I - J$.

Again, interpret this isomorphism as an equality. Since (a2) apparently holds, the following completes the proof of Claims A and B.

Subclaim B3: (b2) holds.

Given $K \subseteq I$ with $|K| < \kappa$ and $\alpha < \kappa$, find an $\ell \in I - (J \cup K)$ such that $\alpha_\ell = \alpha$. Let $x = p^{k-1}s_\ell \in p^\alpha G[p]$. Note that in the isomorphism of Subclaim B2 we have

$$\begin{array}{ccccccc} G/Q & \to & G/R & \cong & p^{k-1}G/P & \subseteq & G/P \\ | & & | & & | & & | \\ \bigoplus_{i\in I} T_i & \to & (\bigoplus_{i\in I-J} T_i) \oplus (\bigoplus_{j\in J} p^{k-1}T_j) & \cong & \bigoplus_{i\in I} p^{k-1}Y_i & \subseteq & \bigoplus_{i\in I} Y_i \end{array}$$

It follows from (a3) that the ℓ^{th} coordinate of $s_{Q,\ell} = s_\ell + Q$ has the minimum height in $H_Q = \bigoplus_{i\in I} T_i$. This shows that the ℓ^{th} coordinate of $x_P = x + P$ has minimum height in $H_P = \bigoplus_{i\in I} Y_i$. Therefore, (b2) must hold for this x.

All of the above work was intended to establish the following, from which we can conclude that Theorem 4.5 holds by inducting on n.

Claim C: There is a subgroup $S \subseteq G[p]$ such that if $G' = G/S$, then $G'/p^\lambda G'$ is $n-1$-simply presented.

By Claim A, there is an (n, λ, κ)-subgroup P of G, and we continue to use the notation given there; so, for example, if $x \in G$, we let $x_P = x + P \in H_P$. In addition, if $J \subseteq I$, we now let $\Sigma_J = \bigoplus_{i\in J} Y_i \subseteq H_P$. Let $P_1 = \{w \in P : pw \in p^\lambda G\}$ and $(w_\gamma, \alpha_\gamma)$ for $\gamma < \kappa$ be an enumeration of $P_1 \times \lambda$ (where we just repeat terms if $|P_1 \times \lambda| < \kappa$). We inductively define elements $x_\gamma \in (p^{\alpha_\gamma}G)[p] - P$ and $u_\gamma \in p^{\alpha_\gamma}G$ with the following properties:

(a5) If $K_\gamma \subseteq I$ is the union of the supports of $x_{P,\delta} = x_\delta + P$ and $u_{P,\delta} = u_\delta + P$ for all $\delta < \gamma$, then $x_{P,\gamma} = y_\gamma + z_\gamma$, where $0 \neq y_\gamma \in \Sigma_{I-K_\gamma}$ and $z_\gamma \in \Sigma_{K_\gamma}$, and $|y_\gamma|_{H_P} \leq |z_\gamma|_{H_P}$;
(b5) $pu_\gamma = pw_\gamma \in p^\lambda G$;
(c5) $|u_\gamma|_G > |x_{P,\gamma}|_{H_P}$;
(d5) $\mathrm{supp}(u_{P,\gamma}) \cap \mathrm{supp}(x_{P,\gamma}) = \emptyset$.

The existence of a x_γ that satisfies (a5) follows from (b2). Having chosen x_γ, let $\beta < \lambda$ be chosen large enough so that $p^\beta Y_i = \{0\}$ for any $i \in \mathrm{supp}(x_{P,\gamma})$. If we then choose $u_\gamma \in G$ satisfying (b5) such that $|u_\gamma|_G > \beta$, then it is easy to check that (c5) and (d5) will hold, as well.

For $\gamma < \kappa$, let $r_\gamma = x_\gamma - u_\gamma + w_\gamma$. Note that $pr_\gamma = px_\gamma - p(u_\gamma - w_\gamma) = 0$, so that if $S = \langle r_\gamma : \gamma < \kappa \rangle$, then $S \subseteq G[p]$.

Claim D: $[P_1 + S]/S \subseteq p^\lambda(G/S) = p^\lambda G'$.

Let $w \in P_1$, so that $w + S \in G/S = G'$. For any $\alpha < \lambda$, let $\gamma < \kappa$ be chosen such that $w = w_\gamma$, $\alpha = \alpha_\gamma$. Then

$$w + S = u_\gamma - x_\gamma + S \in [p^\alpha G + S]/S \subseteq p^\alpha(G/S) = p^\alpha G'.$$

Since this holds for all $\alpha < \lambda$, we can conclude that $w + S \in p^\lambda G'$.

Observe that Claim D implies that $p^{n-1}[P + S]/S \subseteq [P_1 + S]/S \subseteq p^\lambda G'$. Therefore, Claim C, and hence the entire result, will follow once we establish our next statement.

Claim E: $G'/([P + S]/S) \cong G/[P + S]$ is simply presented.

Note first that $G/[P + S] \cong (G/P)/([P + S]/P)$. For each $\gamma \leq \kappa$, let $K_\gamma \subseteq I$ again denote the union of the supports of $x_{P,\delta}$ and $u_{P,\delta}$ for all $\delta < \gamma$, so that $[P + S]/P \subseteq \Sigma_{K_\kappa}$. Next, define

$$S_\gamma = \langle r_{P,\delta} : \delta < \gamma \rangle = \langle x_{P,\delta} - u_{P,\delta} : \delta < \gamma < \lambda \rangle \subseteq \Sigma_{K_\gamma} \subseteq \Sigma_I = H_P = G/P.$$

If $H_\gamma = \Sigma_{K_\gamma}/S_\gamma$, then $G/[P + S] \cong H_\kappa \oplus \Sigma_{I-K_\kappa}$. Since H_κ is the direct limit of the H_γ, for $\gamma < \kappa$, Claim E, and hence the entire result, will once again follow from our next statement.

Claim F: For every $\gamma < \kappa$, there is a split short exact sequence

$$0 \to H_\gamma \to H_{\gamma+1} \to L_\gamma \to 0$$

where

$$L_\gamma = \Sigma_{(K_{\gamma+1} - K_\gamma)}/\langle y_\gamma \rangle$$

is a p^λ-bounded simply presented group.

To verify this, note that in moving from γ to $\gamma + 1$, by (a5) and (d5), we have added two types of summands Y_i; those corresponding to elements $i \in \mathrm{supp}(u_{P,\gamma}) - K_\gamma$, and those corresponding to elements $i \in \mathrm{supp}(y_\gamma)$. Also, in going from S_γ to $S_{\gamma+1}$ we included exactly one more relator,

$$r_{P,\gamma} = x_{P,\gamma} - u_{P,\gamma} = y_\gamma + z_\gamma - u_{P,\gamma}.$$

Note that including $r_{P,\gamma}$ has the effect of identifying $y_\gamma \in \Sigma_{\mathrm{supp}(y_\gamma)}$ with

$$u_{P,\gamma} - z_\gamma \in \Sigma_{K_\gamma \cup \mathrm{supp}(u_{P,\gamma})}.$$

In more detail, observe that $|y_\gamma|_{H_P} = |x_{P,\gamma}|_{H_P} \leq |z_\gamma - u_{P,\gamma}|_{H_P}$. Since the subgroup $\langle y_\gamma \rangle$ of $\Sigma_{\mathrm{supp}(y_\gamma)}$ has order p, it is nice and $\Sigma_{\mathrm{supp}(y_\gamma)}/\langle y_\gamma \rangle$ is simply presented. Therefore, the assignment $y_\gamma \mapsto z_\gamma - u_\gamma$ extends to a homomorphism $\phi : \Sigma_{\mathrm{supp}(y_\gamma)} \to \Sigma_{K_\gamma \cup \mathrm{supp}(u_{P,\gamma})}$. It follows that

$$(a, b) \mapsto (a + \phi(b), b)$$

is an automorphism of

$$\Sigma_{K_{\gamma+1}} \cong \Sigma_{K_\gamma \cup \mathrm{supp}(u_{P,\gamma})} \oplus \Sigma_{\mathrm{supp}(y_\gamma)}$$

(where $\phi^{-1}(a, b) = (a - \phi(b), b)$), which takes $S_\gamma \oplus \langle y_\gamma \rangle$ to $S_{\gamma+1}$. Therefore,

$$\Sigma_{K_{\gamma+1}}/S_{\gamma+1} \cong (\Sigma_{K_\gamma}/S_\gamma) \oplus (\Sigma_{(K_{\gamma+1}-K_\gamma)}/\langle y_\gamma \rangle).$$

This proves Claim F, and hence the entire result. □

Our next statement follows as in the proof of Proposition 3.5.

Corollary 4.6. *If C is a group and λ is an ordinal, then C is n-balanced projective iff both $p^\lambda C$ and $C/p^\lambda C$ are n-balanced projective.*

And as in the proof of Corollary 3.6 we have

Corollary 4.7. *Suppose G is a group of length strictly less than ω^2. Then the following conditions are equivalent:*

(a) G is n-simply presented;

(b) G is n-balanced projective;

(c) For every non-negative integer m, the Ulm factor $p^{\omega \cdot m}G/p^{\omega \cdot (m+1)}G$ is $p^{\omega+n}$-projective.

5. Summands of n-Simply Presented Groups

We begin with the following quite natural question:

Problem 1. Is a group G (strongly) n-simply presented iff it is (strongly) n-balanced projective?

It seems plausible that this is true, at least for groups of countable length. It is also plausible that it holds for one class of groups, but not for the other. It is worthwhile restating that, according to Theorems 2.1 and 2.4, Problem 1 is tantamount to asking whether the (strongly) n-simply presented groups are closed under summands.

The following generalizes Corollaries 3.6 and 4.7.

Proposition 5.1. *Suppose G is a group such that $p^\lambda G$ is (strongly) n-simply presented for some $\lambda < \omega^2$. Then G is (strongly) n-balanced projective iff it is (strongly) n-simply presented.*

Proof. One direction being an immediate consequence of Theorems 2.1 and 2.4, we consider the converse. If $p^\lambda G$ is (strongly) n-simply presented, then by Theorem 3.4(a), $p^{\lambda+n}G$ is (strongly) n-simply presented. If, in addition, G is (strongly) n-balanced projective, then by Proposition 3.5(a) we can conclude $G/p^{\lambda+n}G$ is (strongly) n-balanced projective. Since $\lambda+n < \omega^2$, by Proposition 3.6 and Corollary 4.7, $G/p^{\lambda+n}G$ is (strongly) n-simply presented. An appeal to Theorem 3.4(b) completes the argument. $\square$

A homomorphism $f : G \to A$ is said to be ω_1-*bijective* if its kernel and cokernel are countable. This condition has proven useful in a number of contexts (see, for example, [1], [4], and [12]). The following applies this idea to our investigation.

Proposition 5.2. *Suppose $f : G \to A$ is an ω_1-bijective homomorphism.*
(a) If G is n-simply presented, then A is n-simply presented.
(b) If G is n-balanced projective, then A is n-balanced projective.

Proof. (a) Suppose K is the kernel of f, P is an n-simply representing subgroup of G and $Q = f(P) \subseteq A[p^n]$. If $f' : G/P \to A/Q$ is the induced homomorphism, then the kernel of f' is $[P + K]/P$, which is countable. In addition, the cokernels of f and f' are isomorphic, and hence they are both countable. Therefore, f' is also an ω_1-bijection. It follows from Theorem 2.4 of [4] that A/Q is simply presented, so that Q is an n-simply representing subgroup of A.

(b) If X is a group such that $G \oplus X$ is n-simply presented, then the induced homomorphism $f \oplus 1_X$ shows that $A \oplus X$ is also n-simply presented, so that A is n-balanced projective. $\square$

In Example 2.3 of [4], a group G which is *not* (0-)simply presented was constructed with a countable (and, in fact, pure) subgroup K such that G/K is (0-)simply presented. It follows that the converses of both parts of Proposition 5.2 fail. On the other hand, we do have the following partial result in this direction.

Proposition 5.3. *Suppose G is a group, $p^\omega G$ is countable, and K is a countable and nice subgroup of G. If G/K is n-simply presented, then G is n-simply presented.*

Proof. The niceness of K in G implies that there is a short exact sequence

$$0 \to K/[p^\omega G \cap K] \to G/p^\omega G \to (G/K)/p^\omega(G/K) \to 0.$$

Since G/K is n-simply presented, so is $(G/K)/p^{\omega}(G/K)$; and since it is separable, we can infer from Proposition 2.2 that it is $p^{\omega+n}$-projective. Since the left-hand group is countable, by Theorem 4.2 of [4], we can conclude that $G/p^{\omega}G$ is $p^{\omega+n}$-projective, i.e., n-simply presented. Since $p^{\omega}G$ is clearly n-simply presented, by Theorem 4.4, so is G. □

Corollary 5.4. *Suppose K is a countable subgroup of G. If G/K is separable and n-simply presented, then G is n-simply presented.*

Proof. Since K is nice in G and $p^{\omega}G \subseteq K$ is countable, the conclusion follows directly from Proposition 5.3. □

The analogue of the last corollary fails for strongly n-simply presented groups. As noted previously, there are many $p^{\omega+n}$-bounded groups G for which $K = p^{\omega}G$ is countable, $G/p^{\omega}G$ is $p^{\omega+n}$-projective (and so strongly n-simply presented), such that G is *not* $p^{\omega+n}$-projective (and hence not strongly n-simply presented).

In parallel with the above, a homomorphism $f : G \to A$ is *ω-bijective* if its kernel and cokernel are finite. It is easy to check that in this case, G is simply presented *iff* A is simply presented. The proof of Proposition 5.2(a) then shows that G is n-simply presented *iff* A is n-simply presented. Since finite subgroups are always nice, that argument also shows that if G is strongly n-simply presented, then the same is true of A. On the other hand, in the examples mentioned in the last paragraph, $p^{\omega}G$ can easily be chosen to be finite, showing that the converse of this statement fails.

Proposition 5.5. *If $A = B \oplus C$ is n-simply presented, where $p^{\lambda}C$ is countable for some $\lambda < \omega^2$, then B is n-simply presented.*

Proof. Since $p^{\lambda}A$ is n-simply presented, $p^{\lambda}C$ is countable and $p^{\gamma}B \cong p^{\lambda}A/p^{\lambda}C$, it follows from Proposition 5.2(a) that $p^{\lambda}B$ is n-simply presented. Since B is clearly n-balanced projective, the result follows from Proposition 5.1. □

A similar argument gives our last observation.

Proposition 5.6. *If $A = B \oplus C$ is strongly n-simply presented, where $p^{\lambda}C = \{0\}$ for some $\lambda < \omega^2$, then B is strongly n-simply presented.*

We close the present work with the following special case of Problem 1, which is parallel to Proposition 5.5:

Problem 2 If $A = B \oplus C$ is strongly n-simply presented and C is countable, does it follow that B is also strongly n-simply presented?

References

[1] B. Balof and P. Keef, *Invariants on primary abelian groups and a problem of Nunke*, Note di Matematica, **28**(2): 83–115, 2008.

[2] P. Crawley and A. W. Hales, *The structure of abelian p-groups given by certain presentations*, J. Algebra, **12**(1): 10–23, 1969.

[3] P. Crawley and A. W. Hales, *The structure of abelian p-groups given by certain presentations. II*, J. Algebra, **18**(2): 264–268, 1971.

[4] P. Danchev and P. Keef, *Generalized Wallace theorems*, Math. Scand., **104**(1), 33–50, 2009.

[5] L. Fuchs, *Infinite Abelian Groups, Volumes I & II*, Academic Press, New York, 1970 and 1973.

[6] L. Fuchs, *Vector spaces with valuations*, J. Algebra, **35**: 23–38, 1975.

[7] L. Fuchs, *On $p^{\omega+n}$-projective abelian p-groups*, Publ. Math. Debrecen, **23**(3-4): 309–313, 1976.

[8] L. Fuchs and J. Irwin, *On elongations of totally projective p-groups by $p^{\omega+n}$-projective p-groups*, Czechoslavak Math. J., **32**(4): 511–515, 1982.

[9] P. Griffith, *Infinite Abelian Group Theory*, The University of Chicago Press, Chicago and London, 1970.

[10] J. Irwin and P. Keef, *Primary abelian groups and direct sums of cyclics*, J. Algebra, **159**(2): 387–399, 1993.

[11] P. Keef, *Elongations of totally projective groups and $p^{\omega+n}$-projective groups*, Commun. Algebra, **12**(18): 4377–4385, 1990.

[12] P. Keef, *Partially Decomposable Primary abelian Groups and The Generalized Core Class property*, in Models, Modules and Abelian Groups, pages 289–299, Walter de Gruyter, Berlin, 2008.

[13] R. Nunke, *Purity and subfunctors of the identity*, in Topics in Abelian Groups, pages 121–171, Scott, Foresman and Co., 1962.

[14] R. Nunke, *Homology and direct sums of countable abelian groups*, Math. Z., **101**(3): 182–212, 1967.

[15] F. Richman and E. Walker, *Valuated groups*, J. Algebra, **56**(1): 145–167, 1979.

Received March 12, 2010

Department of Mathematics, Whitman College, Walla Walla, WA 99362
E-mail address: keef@whitman.edu

Department of Mathematics, Plovdiv University "P. Hilendarski", Plovdiv 4000, Bulgaria
E-mail address: pvdanchev@yahoo.com