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An Application of Set Theory to $\omega + n$-Totally $p^{\omega+n}$-Projective Primary Abelian Groups

Peter V. Danchev and Patrick W. Keef*

Abstract. Several equivalent descriptions are given of the class of primary abelian groups whose separable subgroups are all direct sums of cyclic groups; such groups are called *ω-totally Σ-cyclic*. This establishes the converse of a theorem due to Megibben. For $n < \omega$, this is generalized to a consideration of the class of primary abelian groups whose $p^{\omega+n}$-bounded subgroups are all $p^{\omega+n}$-projective. The question of whether there are such groups that are *proper* in the sense that they are neither $p^{\omega+n}$-projective nor ω-totally Σ-cyclic is shown to be logically equivalent to a natural question about the structure of valuated vector spaces. Finally, it is shown that both of these statements are independent of ZFC.

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1. Introduction and Terminology

By the term "group" we will mean an abelian p-group, where p is a prime fixed for the duration. Our group theoretic terminology and notation will generally follow that found in [7]. In particular, $p^\omega G$ denotes the first Ulm subgroup of a group G consisting of all elements of infinite height, and $p^{\omega+n}G = p^n(p^\omega G)$. The cyclic group of order p^k will be denoted by $\mathbb{Z}_{p^k}$ and the infinite cocyclic group will be denoted by $\mathbb{Z}_{p^\infty}$. We will say a group G is Σ-*cyclic* if it is isomorphic to a direct sum of cyclic groups. A group G is a *dsc-group* if it is isomorphic to a direct sum of countable groups. In particular, we are

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not assuming that our dsc-groups are necessarily reduced; in fact, they are a direct sum of a divisible group and a reduced group where the second summand is dsc-group in the sense of [7]. Following [11] and [13], a group G is said to be a *Σ-group* if one (and hence every) *high* subgroup of G is Σ-cyclic (where a subgroup X of G is high if it is maximal with respect to the property $X \cap p^\omega G = \{0\}$).

It was asked in [11] and [13] whether or not subgroups of Σ-groups are again Σ-groups. In general, a subgroup of a Σ-group is not necessarily a Σ-group (see Example 2 of [14]). We will say G is a *totally Σ-group* if every subgroup of G is also a Σ-group. Our first objective is to give several different characterizations of this class (Theorem 2.6). For example, G is a totally Σ-group iff it is the direct sum of a countable group and a Σ-cyclic group. Alternatively, we will say that G is *ω-totally Σ-cyclic* if every separable subgroup S of G is Σ-cyclic. It is elementary that G is a totally Σ-group iff it is ω-totally Σ-cyclic (Proposition 2.1).

The class of ω-totally Σ-cyclic groups can be described in other ways. For example, it coincides with the class of *ω-totally pure-complete groups*, i.e., those groups all of whose separable subgroups are pure-complete (where a group X is pure-complete if for every subgroup $S \subseteq X[p]$ there is a pure subgroup $P \subseteq X$ such that $P[p] = S$). It also coincides with the class of *$\omega+n$-totally dsc-groups*, i.e., those groups all of whose $p^{\omega+n}$-bounded subgroups are dsc-groups.

Expanding slightly on the example of Megibben in [14], if H is any group (e.g., a torsion-complete group), then there is a group G such that $p^\omega G = H$ and $G/p^\omega G$ is Σ-cyclic. Since for any high subgroup Z of G there is an embedding $Z \to G/p^\omega G$, Z must be Σ-cyclic, so that G will be a Σ-group containing H. On the other hand, if H is not countable, then G will not be a totally Σ-group. We sharpen this observation by showing that any separable group S can be embedded as a subgroup in a group G of length $\omega + 1$ which is a Σ-group (but not a totally Σ-group - Proposition 2.9).

More generally, if $\mathbf{C}$ is a class of groups and α is an ordinal, we will say that G is *α-totally* $\mathbf{C}$ if every p^α-bounded subgroup of G is a member of $\mathbf{C}$. Again, it is elementary that G is α-totally $\mathbf{C}$ iff every subgroup of G has the property that all of its *p^α-high* subgroups are in $\mathbf{C}$ (where a subgroup X of a group Y is p^α-high iff it is maximal with respect to the property that $X \cap p^\alpha Y = \{0\}$). In fact, we will mainly be concerned with the case where $n < \omega$, $\alpha = \omega + n$ and $\mathbf{C}$ is the class of *$p^{\omega+n}$-projective groups*; recall that G is $p^{\omega+n}$-projective if $p^{\omega+n}\mathrm{Ext}(G, X) = 0$ for all X, or equivalently, if there is a subgroup $P \subseteq G[p^n]$ such that G/P is Σ-cyclic (see, e.g., [16]). So, a group is p^ω-projective iff it is Σ-cyclic. It follows easily that the class of $p^{\omega+n}$-projectives is closed under arbitrary subgroups. In addition, if G_1 and G_2 are $p^{\omega+n}$-projectives, then G_1 and G_2 are isomorphic iff $G_1[p^n]$ and $G_2[p^n]$ are isometric (i.e., there is an isomorphism that preserves the height functions on the two groups; see [9]). So, if $\mathbf{C}$ is the class of $p^{\omega+n}$-projective groups and $\alpha = \omega + n$, we have that a group G is $\omega + n$-totally $p^{\omega+n}$-projective iff every $p^{\omega+n}$-bounded subgroup X of G is $p^{\omega+n}$-projective. And since a group is p^{ω}-projective iff it is Σ-cyclic, a group is ω-totally p^{ω}-projective iff it is ω-totally Σ-cyclic.

Note that if $p^{\omega+n}G = \{0\}$, then G is $\omega+n$-totally $p^{\omega+n}$-projective iff it is $p^{\omega+n}$-projective. It is also straightforward to verify that the class of $\omega+n$-totally $p^{\omega+n}$-projectives contains the class of ω-totally Σ-cyclic groups (Corollary 2.8). We will say an $\omega+n$-totally $p^{\omega+n}$-projective group G is *proper* if it does not belong to either of these two classes; i.e., iff it is not $p^{\omega+n}$-projective and not ω-totally Σ-cyclic. In particular, there are no proper ω-totally p^{ω}-projectives. For $0 < n < \omega$ we study the question of whether there are, in fact, any proper $\omega+n$-totally $p^{\omega+n}$-projective groups. In fact, we show that this question is equivalent to a natural construction expressible using *valuated vector spaces* (see, for example, [17] and [8]).

If V is a group, then a *valuation* on V is a function $v : V \to \mathbf{O}_{\infty}$ (where $\mathbf{O}_{\infty}$ is the class of all ordinals plus the symbol ∞), such that for all $x, y \in V$, $v(x \pm y) \geq \min\{v(x), v(y)\}$ and $v(px) > v(x)$. It follows that for every $\alpha \in \mathbf{O}_{\infty}$, $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$ is a subgroup of V. If V and W are valuated groups, then a homomorphism $\phi : V \to W$ will be said to be *valuated* if $v(x) \leq v(\phi(x))$ for all $x \in V$, and an *isometry* if it is bijective and preserves all values. Note that if G is any group and H is a subgroup of G, then the height function on G restricts to a valuation on H. The category of valuated groups clearly has direct sums.

Naturally, a valuated group V is a *valuated vector space* if $pV = \{0\}$. In particular, the socle of a group will always be a valuated vector space. The valuated vector space V will be said to be *separable* if $V(\omega) = \{x \in V : v(x) \geq \omega\} = \{0\}$ and *free* if it is isometric to the valuated direct sum of valuated vector spaces of rank one. If W is a subspace of V, then the *corank* of W is the dimension of V/W. A subspace E of V will be called *cofree* if there is a valuated decomposition $V = E \oplus F$, where F is free [in other words, V is algebraically the internal direct sum of E and F, and $v(x+y) = \min\{v(x), v(y)\}$ for all $x \in E$ and $y \in F$].

If κ is an infinite cardinal, then a valuated vector space V will be said to be *κ-coseparable* if it is separable and every subspace W of corank strictly less than κ contains a subspace $E \subseteq W$ that is cofree in V. We will really only be concerned with the cases where $\kappa = \aleph_0$ or $\aleph_1$. A κ-coseparable valuated vector space will be said to be *proper* if it is not free. In [6] the existence of a proper $\aleph_1$-coseparable valuated vector space was shown to be equivalent to a question involving the structure of abelian groups, and to be independent of ZFC. We conclude this paper by showing that for $0 < n < \omega$, the existence of a proper $\aleph_0$-coseparable valuated vector space is equivalent to the existence of a proper $\omega+n$-totally $p^{\omega+n}$-projective group, and we prove that both of these propositions are independent of ZFC (Theorem 3.11).

2. $\omega+n$-Totally $p^{\omega+n}$-Projective Groups

We begin with the following elementary assertion:

Proposition 2.1. *If G is a group, α is an ordinal and $\mathbf{C}$ is a class of groups, then G is α-totally $\mathbf{C}$ iff every subgroup $T \subseteq G$ has the property that every p^α-high subgroup of T is a member of $\mathbf{C}$.*

Proof. Suppose G is α-totally $\mathbf{C}$ and T is an arbitrary subgroup of G. If S is a p^α-high subgroup of T, then S is p^α-bounded, so by hypothesis, S is in $\mathbf{C}$. So, one direction has been established.

Conversely, suppose every p^α-high subgroup of a subgroup of G is in $\mathbf{C}$. If S is any p^α-bounded subgroup of G, then S is a p^α-high subgroup of itself, so it must be in $\mathbf{C}$, so that G is α-totally $\mathbf{C}$. □

The following is a special case of a general result on extending homomorphisms on nice subgroups.

Lemma 2.2. *Suppose G and H are groups, H has infinite cardinality κ, P is a subgroup of H such that H/P is Σ-cyclic and there is an injective homomorphism $\phi : P \to G$ such that (1) for every $x \in P$, $ht_G(\phi(x)) \geq ht_H(x)$; and (2) for every $m < \omega$, $(p^mG)[p]/(p^mG \cap \phi(P))[p]$ has cardinality at least κ. Then ϕ extends to an injective homomorphism $\Phi : H \to G$.*

Proof. By adding a Σ-cyclic summand if needed, there is clearly no loss of generality in assuming that H/P has cardinality κ. Suppose $H/P \cong \bigoplus_{i<\kappa} \langle x_i + P\rangle$, and for $\alpha < \kappa$ let $H_\alpha = P + \langle x_i : i < \alpha\rangle$; thus $H = \cup_{\alpha<\kappa} H_\alpha$. We inductively extend ϕ to an injection $\phi_\alpha : H_\alpha \to G$, so that $\beta < \alpha$ implies that ϕ_α agrees with ϕ_β on H_β. Assume we have constructed ϕ_β for all $\beta < \alpha$. If α is a limit, then we clearly need just take unions. On the other hand, suppose α is isolated and $x_{\alpha-1} + P$ has order p^m in H/P. It follows that $p^m x_{\alpha-1} \in P$, and so $\phi(p^m x_{\alpha-1})$ is defined. By condition (1), we have $ht_G(\phi(p^m x_{\alpha-1})) \geq m$; let $u \in G$ satisfy $p^m u = \phi(p^m x_{\alpha-1})$. Clearly $[\langle u\rangle + \phi_{\alpha-1}(H_{\alpha-1})]/\phi(P)$ has rank $|\alpha| < \kappa$. By condition (2) there is an element

$$w \in (p^{m-1}G)[p] - (\langle u\rangle + \phi_{\alpha-1}(H_{\alpha-1}))\,. \qquad (*)$$

Choose $z \in G$ such that $p^{m-1}z = w$. We let ϕ_α agree with $\phi_{\alpha-1}$ on $H_{\alpha-1}$ and $\phi_\alpha(x_{\alpha-1}) = u+z$. To show ϕ_α is an injection, suppose $y \in H_\alpha$ and $\phi_\alpha(y) = 0$. Let $y = a + kx_{\alpha-1}$, where $a \in H_{\alpha-1}$ and k is an integer. We first claim that $p^m|k$: If this failed, then for some integer ℓ we would have $\ell k \equiv p^{m-1}$ modulo the order of z. Therefore,

$$0 = \phi_\alpha(\ell y) = \phi_\alpha(\ell a + \ell k x_{\alpha-1}) = \phi_{\alpha-1}(\ell a) + \ell k u + \ell k z = \phi_{\alpha-1}(\ell a) + \ell k u + w.$$

This implies that $w = -\ell k u - \phi_{\alpha-1}(\ell a)$, which contradicts $(*)$. We can, therefore, conclude that $p^m|k$, so that $kx_{\alpha-1} \in P$, and hence, $y \in H_{\alpha-1}$. Since $\phi_{\alpha-1}$ is injective, we have $y = 0$, as required.

Letting $\Phi = \cup_{\alpha<\kappa}\phi_\alpha$ completes the proof. □

Recall that G is ω-totally $p^{\omega+n}$-projective means that every separable subgroup of G is $p^{\omega+n}$-projective.

Proposition 2.3. *If $n < \omega$ and G is ω-totally $p^{\omega+n}$-projective, then $p^{\omega+n}G$ is countable.*

Proof. Suppose on the contrary that $p^{\omega+n}G$ is uncountable. Let H be a separable group of cardinality $\aleph_1$ which is $p^{\omega+n+1}$-projective, but not $p^{\omega+n}$-projective. [To construct such a group, let A be a separable group of cardinality $\aleph_1$ with a countable basic subgroup, so that A is not p^{α}-projective for any ordinal α. If C is a Σ-group of rank and final rank $\aleph_1$, then by Theorem 8 of [2], $A \oplus C$ has a subgroup H of the required form.]

Let P be a subgroup of H such that $p^{\omega+n+1}P = \{0\}$ and H/P is Σ-cyclic. Since $p^{\omega+n}G$ is uncountable, there is a subgroup P' of $p^{\omega}G$ which is isomorphic to P such that $(p^{\omega}G)[p]/P'[p]$ is uncountable. By Lemma 2.2, the isomorphism of P and P' extends to an embedding $H \to G$. Since H is separable and not $p^{\omega+n}$-projective, we can conclude that G is not ω-totally $p^{\omega+n}$-projective, contrary to our assumption. □

Since an $\omega{+}n$-totally $p^{\omega+n}$-projective group is ω-totally $p^{\omega+n}$-projective, we have the following:

Corollary 2.4. *If $n < \omega$ and G is $\omega + n$-totally $p^{\omega+n}$-projective, then $p^{\omega+n}G$ is countable.*

Corollary 2.5. *If $n < \omega$ and G is ω-totally $p^{\omega+n}$-projective, then G is a dsc-group iff $G/p^{\omega}G$ is Σ-cyclic.*

Proof. By Lemma 78.1 of [7], if G is a dsc-group, then $G/p^{\omega}G$ is Σ-cyclic. Conversely, Proposition 2.3 ensures that $p^{\omega+n}G$ is countable, so that $p^{n}(p^{\omega}G)$ is a dsc-group, and hence, so is $p^{\omega}G$. If, in addition, $G/p^{\omega}G$ is Σ-cyclic, then G must be a dsc-group. □

The following characterizes the class of groups that are ω-totally Σ-cyclic ($= \omega$-totally p^{ω}-projective).

Theorem 2.6. *If G is a group, then the following are equivalent:*

(a) *G is a totally Σ-group;*
(b) *G is ω-totally Σ-cyclic;*
(c) *G is a Σ-group and $p^{\omega}G$ is countable;*
(d) *$G/p^{\omega}G$ is Σ-cyclic and $p^{\omega}G$ is countable;*
(e) *$G \cong C \oplus M$, where C is countable and M is Σ-cyclic;*
(f) *G is ω-totally pure-complete;*
(g) *For all $n < \omega$, G is an $\omega + n$-totally dsc-group;*
(h) *For some $n < \omega$, G is an $\omega + n$-totally dsc-group.*

Proof. By Proposition 2.1, (a) and (b) are equivalent, and we begin by verifying that these imply (c); so suppose that G is a totally Σ-group. Clearly, if G is a totally Σ-group, then it is a Σ-group. By Proposition 2.3, $p^{\omega}G$ is countable.

Next assume (c), so that G is a Σ-group with countable $p^{\omega}G$, and we show that $G/p^{\omega}G$ is Σ-cyclic, as required in (d). Suppose Z is a high subgroup of G, so that Z is Σ-cyclic. Since $p^{\omega}G$ embeds as an essential subgroup of

G/Z, it follows that G/Z is countable. Since there is a surjection $G/Z \to G/[Z + p^\omega G]$, it follows that latter group is also countable. However, since there exists a short exact sequence

$$0 \to Z \to G/p^\omega G \to G/[Z + p^\omega G] \to 0$$

it follows that $G/p^\omega G$ is Σ-cyclic (see, for example, Corollary 3.1 of [5]).

The equivalence of (d) and (e) is another elementary exercise in the theory of totally projective groups (again, see Chapter XII of [7]). So, suppose G satisfies (d) and (e), and we verify that (b) holds as well. If S is any separable subgroup of G, then $S/(S \cap p^\omega G)$ embeds in $G/p^\omega G$, and since $G/p^\omega G$ is Σ-cyclic, it follows that $S/(S \cap p^\omega G)$ is Σ-cyclic. Since $S \cap p^\omega G$ is countable, it follows that S is Σ-cyclic, as required (see, for example, Theorem 4.2 of [5]).

To establish the equivalence of (b) and (f), note that if G is ω-totally Σ-cyclic and X is a separable subgroup of G, then X must be Σ-cyclic. Since any Σ-cyclic group is pure-complete, it follows that G is ω-totally pure-complete. Conversely, suppose that G is not ω-totally Σ-cyclic, so it has a separable subgroup S which is not Σ-cyclic. By virtue of the "core class property" from [1], one may infer that S contains a subgroup X which is $p^{\omega+1}$-projective but not Σ-cyclic. But by Theorem 2 of [12] (or see [4]), a pure-complete $p^{\omega+1}$-projective group must be Σ-cyclic, so that S is not pure-complete. It follows that G is not ω-totally pure-complete, proving the result.

Finally, turning to the equivalence of (g) and (h) with the other conditions, suppose first that G is ω-totally Σ-cyclic. It follows that every subgroup of G must also be ω-totally Σ-cyclic, and hence a dsc-group. In particular, for every $n < \omega$, every $p^{\omega+n}$-bounded subgroup of G is a dsc-group, i.e., G is an $\omega + n$-totally dsc-group and (g) follows.

Clearly (g) implies (h), so assume (h) hold for some positive integer n. It follows that every separable subgroup of G is a separable dsc-group, i.e., every separable subgroup of G is Σ-cyclic. This shows that (h) implies (b), completing the proof. □

Remark 2.7. In [14] (Theorem 7), Megibben noted that, in our terminology, a group which is isomorphic to a direct sum of a countable group and a Σ-cyclic group is a totally Σ-group; Theorem 2.6 gives the converse of this observation.

Corollary 2.8. *If $n < \omega$ and G is ω-totally Σ-cyclic, then G is $\omega + n$-totally $p^{\omega+n}$-projective.*

Proof. If G is ω-totally Σ-cyclic, then it is an $\omega + n$-totally dsc-group, and since a $p^{\omega+n}$-bounded dsc-group is $p^{\omega+n}$-projective, G must be $\omega + n$-totally $p^{\omega+n}$-projective. □

We now observe that every separable group can be embedded in a Σ-group of minimal length.

Proposition 2.9. *If S is any separable group, then there is a Σ-group G of length $\omega + 1$ containing S as a subgroup.*

Proof. Suppose T is any dsc-group of length $\omega + 1$ for which there is an isomorphism $\phi : p^\omega T \to S[p]$. Let

$$X = \{(x, \phi(x)) : x \in p^\omega T\} \subseteq T \oplus S, \text{ and } G = [T \oplus S]/X.$$

Since $T \cong ([T \oplus \{0\}] + X)/X \subseteq G$ and $S \cong ([\{0\} \oplus S] + X)/X \subseteq G$, we may identify S and T with subgroups of G so that $G = S + T$ and $p^\omega T = S[p] = T \cap S$. Since $p^\omega T \subseteq p^\omega G$ and

$$\begin{aligned} G/p^\omega T &\cong [T + S]/[T \cap S] \\ &\cong (T/[T \cap S]) \oplus (S/[T \cap S]) \\ &= (T/p^\omega T) \oplus (S/S[p]) \\ &\cong (T/p^\omega T) \oplus pS \end{aligned}$$

is separable, it follows that $p^\omega G = p^\omega T$, so that G has length $\omega + 1$.

If Z is a high subgroup of G, then $Z \cap S[p] = Z \cap p^\omega T = \{0\}$, so that $Z \cap S = \{0\}$, as well. Since $Z \cap S = \{0\}$ is the kernel of the composite homomorphism

$$Z \hookrightarrow G = T + S \to (T + S)/S \cong T/(S \cap T) = T/p^\omega T,$$

it follows that this is an embedding. However, since $T/p^\omega T$ is Σ-cyclic, we have that Z is also Σ-cyclic, so that G is a Σ-group. $\square$

Note that in Proposition 2.9, if S is not Σ-cyclic, then G is a Σ-group which is not a totally Σ-group.

The following property of valuated vector spaces is well-known: If $\phi : V \to F$ is a valuated vector space homomorphism and F is separable and free, then the kernel of ϕ is cofree in V. [See, for example, Lemma 1 of [12]. If W is this kernel, then the separability of F implies that W is nice in V, that is, every coset has an element of maximal value, and the quotient valuated vector space V/W is separable. Since F is the union of bounded subspaces B_k, for $k < \omega$, it follows that V/W will be the union of bounded subspaces $\phi^{-1}(B_k)/W$, again for $k < \omega$. This means that V/W is also free, so that V is isometric to the valuated direct sum $W \oplus (V/W)$.]

We now introduce two useful functors. If G is a group, we let $K(G) = (G/p^\omega G)[p]$ and $K_0(G) = \{(G[p] + p^\omega G)/p^\omega G\} \subseteq K(G)$. Note that $K_0(G)$ is dense in $K(G)$ in the induced p-adic topology. [If $x + p^\omega G \in K(G)$ and $m < \omega$, then $px \in p^\omega G$, so there is a $y \in G$ such that $p^{m+1} y = px$. It follows that $x + p^\omega G = (x - p^m y + p^\omega G) + (p^m y + G)$ so that $K(G) = K_0(G) + K(G)(m)$.] Another way to interpret this notion is to check that the map $x + p^\omega G \mapsto px + p^{\omega+1} G$ is a well-defined surjective homomorphism $K(G) \to p^\omega G/p^{\omega+1} G$, and that $K_0(G)$ is the kernel of this map; thus $K(G)/K_0(G) \cong p^\omega G/p^{\omega+1} G$.

Lemma 2.10. *Suppose G is a group such that $G/p^\omega G$ is $p^{\omega+1}$-projective. Then the following are equivalent:*

- (a) *There is a group decomposition $G = H \oplus M$ where H is separable and $M/p^\omega M$ is Σ-cyclic;*
- (b) *$G/p^{\omega+1} G$ is $p^{\omega+1}$-projective;*

(c) $K_0(G) \subseteq K(G)$ *contains a cofree subspace of* $K(G)$.

Proof. We first show (a) implies (c). If $G \cong H \oplus M$ is as described, then clearly $H[p]$ maps to a subspace of $K_0(G)$, and $K(G)$ is isometric to $H[p] \oplus (M/p^\omega M)[p]$ where the latter summand is free. This proves (c).

Suppose now that (c) holds, and we will prove (b) does, as well. Let $K(G)$ be the valuated direct sum $E \oplus F$, where $E \subseteq K_0(G)$ and F is free. Since $G_1 = G/p^\omega G$ is $p^{\omega+1}$-projective, there is a subgroup $P \subseteq K(G)$ such that G_1/P is Σ-cyclic. If $Q = P \cap E$, then Q is the kernel of the valuated homomorphism $P \subseteq K(G) \to F$, so that it follows that there is a valuated decomposition $P = Q \oplus F'$, where F' is free. Let C be a Σ-cyclic group such that there is an isometry $\phi : F' \to C[p]$. Letting $\phi(Q) = 0$ then gives a valuated homomorphism $P \to C[p]$, and since G_1/P is Σ-cyclic, this extends to a homomorphism $\phi : G_1 \to C$ such that $P \cap ker(\phi) = Q$. It therefore follows that the map $G_1 \to (G_1/P) \oplus C$ given by $g \mapsto (g + P, \phi(g))$ has Q as its kernel, so that G_1/Q is also Σ-cyclic. Replacing P by Q, then, we may assume that $P \subseteq E \subseteq K_0(G)$. This implies there is a subgroup $P_0 \subseteq G[p]$ such that $P_0 \cap p^\omega G = \{0\}$ and $P = [P_0 \oplus p^\omega G]/p^\omega G$. We then let

$$P_1 = ([P_0 \oplus p^{\omega+1}G]/p^{\omega+1}G) \oplus (p^\omega G/p^{\omega+1}G) \subseteq G/p^{\omega+1}G.$$

It follows that $pP_1 = \{0\}$ and $(G/p^{\omega+1}G)/P_1 \cong G_1/P$ is Σ-cyclic, so $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective, as required.

Finally, we assume that (b) holds and prove (a). Note that there is a decomposition:

$$G/p^{\omega+1}G = H \oplus Y$$

where H is separable and $p^{\omega+1}$-projective, and Y is a dsc-group (see, e.g., [10]). We define $L, M \subseteq G$ by the conditions $p^{\omega+1}G = L \cap M$, $L/p^{\omega+1}G = H$ and $M/p^{\omega+1}G = Y$. Note that $G/M \cong H$ is separable, so that $p^\omega G \subseteq M$. This implies that for every $x \in p^{\omega+1}G$, there is a $y \in p^\omega G \subseteq M$, such that $py = x$. We now prove by induction on m that $p^\omega G \subseteq p^m M$, which we have just observed holds for $m = 0$. Suppose next that it holds for m and $z \in p^\omega G$. Considering $G/p^{\omega+1}G \cong (L/p^{\omega+1}G) \oplus (M/p^{\omega+1}G)$, there is a $w \in M$ such that $x_1 = p^{m+1}w - z \in p^{\omega+1}G$. This means that $x_1 = py_1$ for some $y_1 \in p^\omega G \subseteq p^m M$. Therefore, $y_1 = p^m u$ for some $u \in M$, so that $z = p^{m+1}w - x_1 = p^{m+1}w - py_1 = p^{m+1}(w - u) \in p^{m+1}M$, as required. We can conclude that $p^\omega G \subseteq p^\omega M \subseteq p^\omega G$, so that $p^\omega G = p^\omega M$, and hence $p^{\omega+1}M = p^{\omega+1}G$.

We therefore have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & p^{\omega+1}M & \to & L & \to & H & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M & \to & G & \to & H & \to & 0
\end{array}$$

By Proposition 56.1 of [7], it follows that the bottom row is $p^{\omega+1}$-pure, and since H is $p^{\omega+1}$-projective, we have $G \cong H \oplus M$. Finally, $M/p^\omega M \cong Y/p^\omega Y$ is Σ-cyclic. □

As a consequence, we have the following:

Corollary 2.11. *Suppose that G is a group and $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.*

(i) *If $p^{\omega}G$ is countable, then G is the direct sum of a separable $p^{\omega+1}$-projective group and a countable group.*

(ii) *If $p^{\omega+1}G$ is countable, then G is the direct sum of a $p^{\omega+1}$-projective group and a countable group.*

Proof. Since $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective, it easily follows that $G/p^{\omega}G$ is $p^{\omega+1}$-projective. Applying Lemma 2.10(a), one may write $G = H \oplus M$ where H is a separable $p^{\omega+1}$-projective group and M is a group with the property that $M/p^{\omega}M$ is Σ-cyclic. Regarding (i), since $p^{\omega}M$ is countable, it follows that M can be written as a direct sum of a Σ-cyclic group and a countable group. So, (i) is sustained.

As for (ii), it is easy to see that $M/p^{\omega+1}M$ is a dsc-group and $p^{\omega+1}M$ is countable. Therefore, M is itself a dsc-group which, because of the countability of $p^{\omega+1}M$, can be decomposed as a direct sum of a dsc-group of length $\omega + 1$, which is certainly $p^{\omega+1}$-projective, and a countable group. □

The following example illustrates that neither statement in Corollary 2.11 holds for $n \geq 2$.

Example. There is a group G such that $G/p^{\omega+2}G$ is $p^{\omega+2}$-projective and $p^{\omega}G$ is countable which is not the direct sum of a $p^{\omega+2}$-projective group and a countable group.

Proof. Suppose A is an unbounded separable $p^{\omega+2}$-projective group with the property that every summand of A which is Σ-cyclic must be bounded (an example of which was constructed by Cutler and Missel in [3]). Since any unbounded $p^{\omega+1}$-projective group has unbounded Σ-cyclic summands, it follows that A is not $p^{\omega+1}$-projective. Let $P \subseteq A[p^2]$ be a subgroup such that A/P is Σ-cyclic.

We claim that $(p^m A)[p]$ is not contained in P for any $m < \omega$: Assume this fails for some m. If we let $P_0 = (p^m A \cap P)/(p^m A)[p]$, it follows that $pP_0 = \{0\}$. In addition, $(p^m A/(p^m A)[p])/P_0 \cong p^m A/(p^m A \cap P)$ embeds in A/P, so in particular, it is Σ-cyclic. This implies that $p^{m+1}A \cong p^m A/(p^m A)[p]$ is $p^{\omega+1}$-projective; which in turn would imply that A is $p^{\omega+1}$-projective, which is not the case.

This last claim implies that we can construct a dense subsocle $D \subseteq A[p]$ containing $P[p]$ such that $A[p]/D$ has rank 1. Let L be a subgroup of A containing P that is maximal with respect to $L[p] = D$. It follows that L is pure and dense in A and there is an isomorphism $\varphi : A/L \cong \mathbb{Z}_{p^\infty}$. Let

$$G = \{(a, z) : a \in A, z \in \mathbb{Z}_{p^\infty} \text{ and } \varphi(a) = p^3 z\}.$$

It readily follows that $p^{\omega}G = \{0\} \oplus \mathbb{Z}_{p^\infty}[p^3]$, which we denote by J, and that $G/J \cong A$. In addition, let $P' = P \oplus \{0\} \subseteq G$.

Note that $J \cap P' = \{0\}$, and so $P' \oplus J$ can also be viewed as a p^3-bounded subgroup of G containing $p^{\omega+2}G = p^2 J$. Since

$$(G/p^{\omega+2}G)/[(P' \oplus J)/p^{\omega+2}G] \cong G/[P' \oplus J] \cong A/P$$

is Σ-cyclic and $p^2[(P' \oplus J)/p^{\omega+2}G] = \{0\}$, $G/p^{\omega+2}G$ is $p^{\omega+2}$-projective.

On the other hand, if $G = C \oplus G'$, where C is countable and G' is $p^{\omega+2}$-projective, then $p^{\omega+2}G' = \{0\}$, so that $p^{\omega+2}C = p^{\omega+2}G \neq \{0\}$. In particular, $p^m C \neq \{0\}$ for all $m < \omega$. However,

$$A \cong G/J = G/p^\omega G \cong (C/p^\omega C) \oplus (G'/p^\omega G'),$$

where $C/p^\omega C$ is an unbounded Σ-cyclic, which contradicts the fact that A has no unbounded Σ-cyclic summands. □

We can now extend Theorem 2.6(b) $\Leftrightarrow$ (e) for $n = 1$ in the following way:

Proposition 2.12. *An $\omega+1$-totally $p^{\omega+1}$-projective group G is a direct sum of a $p^{\omega+1}$-projective group and a countable group iff $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.*

Proof. If G is the direct sum of a $p^{\omega+1}$-projective group and a countable group, say $H \oplus C$, then it plainly follows that $G/p^{\omega+1}G \cong H \oplus (C/p^{\omega+1}C)$ is $p^{\omega+1}$-projective, as well.

Conversely, suppose G is a $\omega + 1$-totally $p^{\omega+1}$-projective group such that $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective. Employing Proposition 2.3 and Corollary 2.11(ii), we deduce the desired decomposition of G. □

3. Proper $\omega + n$-Totally $p^{\omega+n}$-Projective Groups

Though this section contains a discussion of the structure of proper $\omega + n$-totally $p^{\omega+n}$-projective groups, we pause for a few general observations on κ-coseparable valuated vector spaces. It can be easily verified that the class of κ-coseparable valuated vector spaces is closed under valuated direct sums and summands, and that it contains all the separable free valuated vector spaces. In particular, if the separable valuated vector space V is the valuated direct sum $W \oplus F$, where F is free, then V is κ-coseparable iff W is κ-coseparable. In addition, a separable valuated vector space V is $\aleph_0$-coseparable iff every subspace $W \subseteq V$ of corank one contains a cofree subspace (this follows since the intersection of a finite collection of cofree subspaces is also cofree).

The following result is our main tool in analyzing proper $\omega + n$-totally $p^{\omega+n}$-projective groups. Since non-free separable valuated vector spaces are usually not $\aleph_0$-coseparable, it puts a serious limitation on the structure of proper $\omega + n$-totally $p^{\omega+n}$-projectives, showing that they are relatively rare phenomena.

Theorem 3.1. *Suppose $n < \omega$ and G is a proper $\omega+n$-totally $p^{\omega+n}$-projective group. If V is a separable valuated vector space for which there is an injective valuated homomorphism $V \to G[p]$, then V is $\aleph_0$-coseparable.*

Proof. We may clearly assume V is unbounded and our valuated injection $V \to G[p]$ is an inclusion such that for all $x \in V$, $v(x) \leq ht_G(x)$. If necessary, we may replace G by $p^m G$ and V by $V(m)$, so that there is no loss of generality in assuming that the rank and final rank of G is some cardinal κ, and that the rank of V is at most κ. Since $p^{\omega+n}G \neq \{0\}$, we can find some non-zero $x \in (p^{\omega+n}G)[p]$. Observe first that if $x \in V$, then $\langle x\rangle$ is a valuated summand of V, and if $V = \langle x\rangle \oplus V_0$, then V is $\aleph_0$-coseparable iff V_0 is. Replacing V by V_0, we may therefore assume that $x \notin V$. Find $y \in p^\omega G$ such that $p^n y = x$, so that $\langle y\rangle \cap V = \{0\}$.

Let Y be a high subgroup of G, so that Y is $p^{\omega+n}$-projective and there is a (ht_G-)valuated decomposition $G[p] = Y[p] \oplus (p^\omega G)[p]$. It follows from Corollary 26 of [12] that $Y[p]$ is isometric to $Q \oplus F$, where F is a free valuated vector space of final rank κ. Consider the valuated composition $V \to G[p] \to Y[p] = Q \oplus F \to F$ whose kernel is $V_1 = V \cap (Q + p^\omega G[p])$. We can conclude that V is isometric to $V_1 \oplus F'$ where F' is free; therefore, V is $\aleph_0$-coseparable iff V_1 is. Replacing V by V_1, we may assume $V \subseteq Q + (p^\omega G)[p]$, so that $F \cap V = \{0\}$. This means that if $m < \omega$, that $(p^m G)[p]/(p^m G \cap V)[p]$ has cardinality κ, since it contains a copy of $F(m)$.

Let D be a subspace of V of corank one; we need to exhibit a subspace of D which is cofree in V. If D is not dense in V, then D will be a valuated summand of V, so it will be cofree. We may therefore assume that D is dense in V. Suppose $z \in V - D$, and let $P = D \oplus \langle z + y\rangle \subseteq G$. Note that there is a surjective homomorphism $\rho : P \to V$ which is the identity on D and maps $z + y$ to z; the kernel of this homomorphism is clearly $\langle py\rangle \subseteq P$. We define a valuation v_P on P as follows: Suppose $u \in P$; if $u = 0$, then let $v_P(u) = \infty$; otherwise, if $\rho(u) \neq 0$, then let $v_P(u) = v(\rho(x))$; finally, if $\rho(u) = 0$, then $u = p^k q(py)$, where $(p, q) = 1$ and $k < n$, and we let $v_P(u) = \omega + k$. It is straightforward to check that v_P is a valuation, and if $v_P(u)$ is infinite and $\beta < v_P(u)$, then there is a $w \in P$ such that $pw = u$ and $\beta \leq v_P(w)$. By a variation on a construction in [17], there is a group H of rank at most κ containing P as a subgroup such that

(1) the height function on H agrees with v_P on P;
(2) H/P is Σ-cyclic of rank at most κ.

[Let H be generated by P and a set of elements x_u, for $u \in P - P(\omega)$, subject to the relations $p^{v_P(u)}x_u = u$.]

It follows that $p^\omega H = P(\omega) = \langle py\rangle$, so G is $p^{\omega+n}$-bounded. It is easy to verify that for all $u \in P$, $v_P(u) \leq ht_G(u)$, so by Lemma 2.2, the inclusion $P \subseteq G$ extends to an embedding $H \to G$.

Since G is $\omega + n$-totally $p^{\omega+n}$-projective, we can conclude that H is $p^{\omega+n}$-projective. Therefore, there is a subgroup $R \subseteq H[p^n]$ such that H/R is Σ-cyclic. Note that $P(\omega) \subseteq R \cap P \subseteq P[p^n] = D \oplus \langle py\rangle$, so that if $E = \rho(R \cap P)$, then $E \subseteq D$. In addition, E is the kernel of the valuated composition: $V \cong P/P(\omega) \to H/P(\omega) \to H/R$. Since $(H/R)[p]$ is free, it follows that $E \subseteq D$ is cofree in V, as required. □

A separable valuated vector space V is *efi* (for *essentially finitely indecomposable*) iff it does not have a valuated summand which is an unbounded free valuated vector space. In particular, an unbounded efi valuated vector space cannot be $\aleph_0$-coseparable. Therefore, we have the following direct consequence of Theorem 3.1.

Corollary 3.2. *Suppose G is a proper $\omega+n$-totally $p^{\omega+n}$-projective and V is an unbounded valuated vector space that is efi. Then there does not exist a valuated injection $V \to G[p]$.*

We have seen by Theorem 2.6 that if G is ω-totally Σ-cyclic, then $p^\omega G$ is countable. More generally, by Corollary 2.4, if $n > 0$ and G is $\omega+n$-totally $p^{\omega+n}$-projective, then $p^{\omega+n}G$ must be countable, but $p^\omega G$ does not have to be countable: for example, if G is $p^{\omega+n}$-projective (such as a $p^{\omega+n}$-bounded dsc-group), $p^\omega G$ can be made as large as we please. We now investigate the question of the countability of $p^\omega G$ for *proper* $\omega+n$-totally $p^{\omega+n}$-projective groups.

Let σ be the smallest cardinal such that there is a separable valuated vector space of cardinality σ which is *not* $\aleph_0$-coseparable. Since any countable separable valuated vector space is free, and hence $\aleph_0$-coseparable, we can conclude that $\sigma \geq \aleph_1$.

Corollary 3.3. *If G is a proper $\omega+n$-totally $p^{\omega+n}$-projective, then $r(p^\omega G) < \sigma$.*

Proof. Let V be a separable valuated vector space of cardinality σ which is not $\aleph_0$-coseparable. If $r(p^\omega G) \geq \sigma$, then there is an injective group homomorphism $V \to (p^\omega G)[p] \subseteq G[p]$, which certainly does not decrease values, contradicting Theorem 3.1. □

It is clear that the class of ω-totally Σ-cyclic groups is closed under countable direct sums. On the other hand, this property does not generalize to integers $0 < n < \omega$. However, for any natural number n, arbitrary direct sums of $p^{\omega+n}$-projective groups are again $p^{\omega+n}$-projective.

Corollary 3.4. *If $0 < n < \omega$, then the class of $\omega+n$-totally $p^{\omega+n}$-projective groups is not closed under (finite) direct sums.*

Proof. Let A be a $p^{\omega+1}$-bounded dsc-group such that $r(p^\omega A) \geq \sigma$. Then A is $p^{\omega+1}$-projective, and hence $p^{\omega+n}$-projective, and hence $\omega+n$-totally $p^{\omega+n}$-projective. Next, let M be a countable reduced group such that $p^{\omega+n}M \neq 0$. Then M is ω-totally Σ-cyclic, and hence $\omega+n$-totally $p^{\omega+n}$-projective.

Note that if $G = A \oplus M$ were $\omega+n$-totally $p^{\omega+n}$-projective, then since G is not $p^{\omega+n}$-projective and $p^\omega G$ is not countable, it would have to be proper. Since $r(p^\omega G) \geq \sigma$, however, this would contradict Corollary 3.3. □

Corollary 3.3 implies that we would like to know whether $\sigma = \aleph_1$. To investigate this question, we extend our brief detour into the theory of valuated vector spaces. If λ is a cardinal number, let D_λ be a valuated vector space of dimension λ such that $v(x) = \omega$ for all non-zero $x \in D_\lambda$. Let $\phi_\lambda : F_\lambda \to D_\lambda$ be a surjective homomorphism, where F_λ is a free separable valuated vector space of cardinality $\lambda \cdot \aleph_0$ such that if M_λ is the kernel of ϕ_λ, then $\omega = \max\{v(x+y) : y \in M_\lambda\}$ for every $x \in F_\lambda - M_\lambda$ (i.e., M_λ is a dense subspace of F_λ of corank λ). If V and W are valuated vector spaces, let $\mathrm{Hom}_v(V, W)$ denote the collection of all valuated homomorphisms $f : V \to W$.

Lemma 3.5. *Suppose κ is an infinite cardinal and V is a separable valuated vector space. Then V is κ-coseparable iff for every cardinal $\lambda < \kappa$,*

$$\mathrm{Hom}_v(V, F_\lambda) \to \mathrm{Hom}_v(V, D_\lambda)$$

is surjective, i.e., for every homomorphism $f : V \to D_\lambda$ (which is automatically valuated) there is a valuated homomorphism $g : V \to F_\lambda$ such that $f = \phi_\lambda \circ g$. If $\kappa = \aleph_0$, then this need only be true for $\lambda = 1$.

Proof. We will concentrate on the case where $\kappa = \aleph_0$ and $\lambda = 1$, which is the only one we will use in the rest of the paper. (The general case follows in an almost identical way.) Suppose V is $\aleph_0$-coseparable and $f : V \to D_1$ is a homomorphism. If W is the kernel of f, then it follows that V/W has rank at most one. Since V is $\aleph_0$-coseparable, it follows that $V = E \oplus F$, where $E \subseteq W$ and F is free. Since F is free, there is a valuated homomorphism $g : F \to F_1$ such that $f|_F = \phi_1 \circ g$. If we then define $g(E) = 0$, then it follows that $f = \phi_1 \circ g$.

Conversely, suppose V satisfies this homological condition and W is a subspace of V of corank one. Then there is a valuated composite homomorphism $f : V \to V/W \to D_1$ with kernel W. If $g : V \to F_1$ is the valuated homomorphism satisfying $f = \phi_1 \circ g$, then letting E be the kernel of g, it follows that $E \subseteq W$. Since F_1 is separable and free, it follows that E is cofree in V, as required. □

The following gives a great deal of information about the size of σ.

Proposition 3.6. *The following relations hold:*

(a) $\sigma \leq c = 2^{\aleph_0}$;
(b) *If* $2^{\aleph_0} < 2^{\aleph_1}$, *then* $\sigma = \aleph_1$.

Proof. Regarding (a), let B be a countable separable unbounded free valuated vector space. If $V = \overline{B}$ is the p-adic completion of B, then V has cardinality c. Since $\overline{B}$ is efi, it follows that it is not $\aleph_0$-coseparable, so that $\sigma \leq c$.

Turning to (b), again let B be a countable separable unbounded free valuated vector space, but this time, let V be a subspace of $\overline{B}$ containing B of cardinality $\aleph_1$. We claim that V is not $\aleph_0$-coseparable. To that end, consider the valuated sequence

$$0 \to M_1 \to F_1 \to D_1 \to 0$$

from Lemma 3.5. Note that any valuated homomorphism $g : V \to F_1$ is determined by its restriction to B. It follows that $\mathrm{Hom}_v(V, F_1)$ has cardinality at most $2^{\aleph_0}$. On the other hand, since any homomorphism $f : V \to D_1$ is valuated, the cardinality of $\mathrm{Hom}_v(V, D_1)$ is $2^{\aleph_1}$. It follows that $\mathrm{Hom}_v(V, F_1) \to \mathrm{Hom}_v(V, D_1)$ is not surjective, so that V is not $\aleph_0$-coseparable. This implies that $\sigma = \aleph_1$, as required. □

Combining the Corollary 3.3 and Proposition 3.6, we derive:

Corollary 3.7. *If $2^{\aleph_0} < 2^{\aleph_1}$ (e.g., in any set-theoretic environment in which CH is valid) and G is a proper $\omega+n$-totally $p^{\omega+n}$-projective group, then $p^\omega G$ must be countable.*

A simple combination of Theorem 2.6 and Corollary 3.7 leads us to the following supplement to Corollary 2.5:

Corollary 3.8. *If $2^{\aleph_0} < 2^{\aleph_1}$ and G is a proper $\omega + n$-totally $p^{\omega+n}$-projective group, then G is not a dsc-group.*

We will have use for the following technical observation:

Lemma 3.9. *Suppose κ is an infinite cardinal, V is a κ-coseparable valuated vector space, W is a separable valuated vector space and $\phi : W \to V$ is a valuated vector space homomorphism with finite kernel J. Then W is also κ-coseparable.*

Proof. Since J is finite, there is a valuated decomposition $W = J \oplus W'$. It follows that W is κ-coseparable iff W' is κ-coseparable, so, without loss of generality, we may assume $J = \{0\}$, $W = W'$ and ϕ is injective (note that ϕ may increase values computed in W and V). Considering Lemma 3.5, if $\lambda < \kappa$ is a cardinal and $f_W : W \to D_\lambda$ is a homomorphism, then there is a homomorphism $f_V : V \to D_\lambda$ such that $f_W = f_V \circ \phi$. Since V is κ-coseparable, there is a valuated homomorphism $g_V : V \to F_\lambda$ such $f_V = \phi_\lambda \circ g_V$. If $g_W = g_V \circ \phi$, it follows that $f_W = f_V \circ \phi = \phi_\lambda \circ g_V \circ \phi = \phi_\lambda \circ g_W$, so that W is κ-coseparable, as required. □

A group G will be said to be *special* if it is isomorphic to a direct sum $H \oplus M$, where:

(a) H is a separable $p^{\omega+1}$-projective group and $H[p]$ is an $\aleph_0$-coseparable valuated vector space;
(b) M is a dsc-group and $p^\omega M$ is finite.

Clearly, a special group is reduced, and, in fact, $p^{\omega+n}G = \{0\}$ for some $n < \omega$. Since M can be decomposed as a direct sum of a Σ-cyclic group and a countable group, we may assume M is countable.

Theorem 3.10. *The following hold:*

(a) *A group G is special iff $p^\omega G$ is finite, $G/p^\omega G$ is $p^{\omega+1}$-projective and $K(G)$ is $\aleph_0$-coseparable.*
(b) *The class of special groups is closed under arbitrary subgroups.*
(c) *Any special group is $\omega + n$-totally $p^{\omega+n}$-projective for all $0 < n < \omega$.*

Proof. Regarding (a), if $G \cong H \oplus M$ is special, then clearly $p^\omega G \cong p^\omega M$ is finite, and $G/p^\omega G \cong H \oplus (M/p^\omega M)$ is $p^{\omega+1}$-projective. Note that $K(G)$ is isometric to the valuated sum $H[p] \oplus K(M)$, and since the first summand is $\aleph_0$-coseparable and the second summand is separable and free, it follows that $K(G)$ is also $\aleph_0$-coseparable.

Suppose now that G satisfies the conditions listed in the last half of (a). Since $K(G)$ is $\aleph_0$-coseparable and $K(G)/K_0(G) \cong p^\omega G/p^{\omega+1}G$ is finite, it follows from Lemma 2.10 that $G \cong H \oplus M$, where H is a separable $p^{\omega+1}$-projective and M is such that $p^\omega M$ is finite and $M/p^\omega M$ is Σ-cyclic, thus a dsc-group, so that (a) follows.

Turning to (b), suppose G is special and A is some subgroup of G. Since $p^\omega A \subseteq p^\omega G$ and the latter is finite, it follow that $p^\omega A$ is finite, as well. Next note that there is an induced homomorphism $\phi : A/p^\omega A \to G/p^\omega G$ which restricts to a homomorphism $K(A) \to K(G)$. The kernel of ϕ is $[A \cap p^\omega G]/p^\omega A$ which is finite (so that it embeds in a finite summand of $A/p^\omega A$), and it follows easily that $A/p^\omega A$ is $p^{\omega+1}$-projective. Finally, since $K(G)$ is $\aleph_0$-coseparable and $K(A) \to K(G)$ has finite kernel, it follows from Lemma 3.9 that $K(A)$ is $\aleph_0$-coseparable. This proves that A is special and concludes the proof of (b).

Finally, to show (c), if $0 < n < \omega$, G is special, and A is a $p^{\omega+n}$-bounded subgroup of G, then in view of (b) we have that A is also special. If follows that $A \cong H' \oplus M'$, where H' is $p^{\omega+1}$-projective, and M' is a countable group with $p^{\omega+n}M' = \{0\}$. Since H' and M' are $p^{\omega+n}$-projective, we can conclude that A is $p^{\omega+n}$-projective and hence G is $\omega+n$-totally $p^{\omega+n}$-projective. $\square$

We come now to our main theorem on proper $\omega+n$-totally $p^{\omega+n}$-projectives.

Theorem 3.11. *The equivalence of the following three statements is a theorem in ZFC:*

(a) *There is a proper $\omega+n$-totally $p^{\omega+n}$-projective group for some $0 < n < \omega$.*
(b) *There is a proper $\aleph_0$-coseparable valuated vector space.*
(c) *There is a separable $p^{\omega+1}$-projective group A which is not Σ-cyclic such that whenever G is a group with $p^\omega G \cong \mathbb{Z}_p$ and $G/p^\omega G \cong A$, then G must also be $p^{\omega+1}$-projective.*

On the other hand, all three are undecidable in ZFC; in particular, they all hold in a model of MA+$\neg$CH, whereas they all fail in a model of V=L.

Proof. We begin by showing that (b) implies (a); to that end, suppose V is a non-free $\aleph_0$-coseparable valuated vector space. Then there is a group H containing $V \subseteq H[p]$ such that the valuation on V agrees with the height function on H, and for which H/V is Σ-cyclic. Note that such an H will be separable and $p^{\omega+1}$-projective. As we have observed several times in the past, $H[p]$ is isometric to $V \oplus F$, where F is a free valuated vector space. It therefore follows that $H[p]$ is also a proper $\aleph_0$-coseparable valuated vector space. If M is any countable group such that $p^\omega M$ is finite and $p^{\omega+n}M \neq \{0\}$, then $G = H \oplus M$ will be special, and hence $\omega+n$-totally $p^{\omega+n}$-projective, by Theorem 3.10(c). Since M is not $p^{\omega+n}$-bounded and H is not Σ-cyclic, G is necessarily proper, thus proving (a).

We next verify that (a) implies (b), so suppose G is a proper $\omega+n$-totally $p^{\omega+n}$-projective group. Suppose first that $p^\omega G$ is uncountable. In this case, Corollary 3.3 implies that $\sigma > \aleph_1$. However, if we let V be any separable valuated vector space of rank $\aleph_1$ with a countable basic subspace, then V is clearly not free, but since $r(V) < \sigma$, V must be a proper $\aleph_0$-coseparable valuated vector space, proving (b) in this case.

On the other hand, assume that $p^\omega G$ is countable. Let H be a high subgroup of G. Note that if H is Σ-cyclic, then G must be a Σ-group. However, since $p^\omega G$ is countable, Theorem 2.6(c) $\Rightarrow$ (b) would imply that G is ω-totally Σ-cyclic, contrary to assumption. It follows that $H[p]$ is not free. Since there is obviously a valuated injection $H[p] \to G[p]$, it follows from Theorem 3.1 that $H[p]$ is $\aleph_0$-coseparable, which establishes (b).

Assume now that (b) holds, and we will prove (c). Let V be a proper $\aleph_0$-coseparable valuated vector space. It follows that there is a separable $p^{\omega+1}$-projective group A containing V as a subgroup where the height function on A coincides with the valuation on V, and such that A/V is Σ-cyclic. If G is any group with $p^\omega G \cong \mathbb{Z}_p$ and $G/p^\omega G \cong A$, it follows from Theorem 3.10(a) that G is special. Therefore, $G \cong H \oplus M$, where H is a separable $p^{\omega+1}$-projective group and M is countable. Since G is $p^{\omega+1}$-bounded, so is M, so that G is necessarily $p^{\omega+1}$-projective.

Conversely, suppose that (c) holds, and we establish (b). Let $V = A[p]$; since A is not Σ-cyclic, V is not free. Suppose D is a subspace of V of corank one. If there is an $m < \omega$ such that $V(m) \subseteq D$, then D is already cofree, so assume D is dense in V. If L is a pure subgroup of A with $L[p] = D$, then there is a surjective homomorphism $\phi : A \to \mathbb{Z}_{p^\infty}$ with kernel L. Let

$$G = \{(a, z) : a \in A, z \in \mathbb{Z}_{p^\infty} \text{ and } \phi(a) = pz\}.$$

It follows that $G/p^\omega G \cong A$, $p^\omega G$ is cyclic of order p and $D \cong K_0(G)$. Since G must be $p^{\omega+1}$-projective, by Lemma 2.10(c), $D = K_0(G)$ contains a cofree subspace of $K(G)$, so that $V = A[p] = K(G)$ must be a proper $\aleph_0$-coseparable valuated vector space.

We next show that all of them are valid in a model of MA+$\neg$CH. In this set-theoretic context, by Theorem 3.4(a) and 3.3 of [6], there is a proper $\aleph_1$-coseparable valuated vector space. Since an $\aleph_1$-coseparable valuated vector space is also $\aleph_0$-coseparable, we have that (b) holds.

Finally, arguing as in [6], we show that (c) does not hold in a model of V=L. Suppose, therefore, that A satisfies (c). Note that if G is some group such that $p^\omega G \cong \mathbb{Z}_p$ and $G/p^\omega G \cong A$, then G is $p^{\omega+1}$-projective, so that $G \cong C \oplus S$, where C is a dsc-group and S is separable. If $H_{\omega+1}$ is the generalized Prüfer group of length $\omega + 1$, there is clearly a homomorphism $G \to C \to H_{\omega+1}$ which is non-zero on $p^\omega G \cong p^\omega C$. In the presence of V=L, by Theorem 2.2 of [15], A must be Σ-cyclic, contrary to assumption. $\square$

The last proof actually shows the following:

Corollary 3.12. *If there is a proper $\omega + n$-totally $p^{\omega+n}$-projective for some $0 < n < \omega$, then there is a proper $\omega + n$-totally $p^{\omega+n}$-projective for all $0 < n < \omega$.*

Corollary 3.13. *In V=L, if $n < m < \omega$ and G is $\omega+n$-totally $p^{\omega+n}$-projective, then it is $\omega + m$-totally $p^{\omega+m}$-projective.*

Proof. Since by Theorem 3.11 the group G cannot be proper, it must either be $p^{\omega+n}$-projective or ω-totally Σ-cyclic. In either case it will be $\omega+m$-totally $p^{\omega+m}$-projective. □

There are still unanswered questions that pertain to the structure of proper $\omega+n$-totally $p^{\omega+n}$-groups, at least in those set-theoretic environments in which they exist. For example, we have the following:

Problem 1: In ZFC, does $\sigma = \aleph_1$?

Problem 2: In ZFC, if G is a proper $\omega + n$-totally $p^{\omega+n}$-projective, does it follow that $p^\omega G$ is necessarily countable?

By Corollary 3.3, an affirmative answer to Problem 1 implies an affirmative answer to Problem 2.

Problem 3: In ZFC, if $n < m < \omega$ and G is $\omega + n$-totally $p^{\omega+n}$-projective, must it also be $\omega + m$-totally $p^{\omega+m}$-projective?

Problem 4: If $n < \omega$, describe the class of ω-totally $p^{\omega+n}$-projectives (which contains the class of $\omega + n$-totally $p^{\omega+n}$-projectives).

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