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ON ABELIAN GROUPS HAVING ALL PROPER FULLY INVARIANT SUBGROUPS ISOMORPHIC

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We introduce two classes of abelian groups which have either only trivial fully invariant subgroups or all their nontrivial (respectively nonzero) fully invariant subgroups are isomorphic, called IFI-groups and strongly IFI-groups, such that every strongly IFI-group is an IFI-group, respectively. Moreover, these classes coincide when the groups are torsion-free, but are different when the groups are torsion as well as, surprisingly, mixed groups cannot be IFI-groups. We also study their important properties as our results somewhat contrast with those from [13] and [14].

Key Words: Abelian groups; Characteristic subgroups; Fully invariant subgroups; Projection invariant subgroups.

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1. INTRODUCTION AND MAIN DEFINITIONS

Throughout the present article, let all groups into consideration be *additively* written and *abelian*. Our notations and terminology from group theory are mainly standard and follow those from [9] and [16]. For instance, if p is a prime integer and G is an arbitrary group, $p^nG = \{p^ng \mid g \in G\}$ denotes the p^n*th power subgroup* of G consisting of all elements of p-height greater than or equal to $n \in \mathbb{N}$, $G[p^n] = \{g \in G \mid p^ng = 0, n \in \mathbb{N}\}$ denotes the p^n-*socle* of G, and $G_p = \cup_{n<\omega} G[p^n]$ denotes the *p-component* of the *torsion part* $tG = \oplus_p G_p$ of G.

On the other hand, if G is a torsion-free group and $a \in G$, then let $\chi_G(a)$ denote the *characteristic* and let $\tau_G(a)$ denote the *type* of a, respectively. Specifically, the class of equivalence in the set of all characteristics is just called *type*, and we write τ. If $\chi_G(a) \in \tau$, then we write $\tau_G(a) = \tau$, and so $\tau(G) = \{\tau_G(a) \mid 0 \neq a \in G\}$ is the set of types of all nonzero elements of G. The set $G(\tau) = \{g \in G \mid \tau(g) \geqslant \tau\}$ forms a pure fully invariant subgroup of the torsion-free group G. Recall that a torsion-free group G is called *homogeneous* if all its nonzero elements have the same type.

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Concerning ring theory, suppose that all rings which we consider are *associative* with *identity* element. For any ring R, the letter R^+ will denote its *additive group*. To simplify the notation and to avoid a risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism ring of G and $\mathrm{End}(G) = \mathrm{E}(G)^+$ for the endomorphism group of G.

As usual, a subgroup F of a group G is called *fully invariant* if $\phi(F) \subseteq F$ for any $\phi \in \mathrm{E}(G)$. In addition, if ϕ is an invertible endomorphism (i.e., an automorphism), then F is called *a characteristic* subgroup, while if ϕ is an idempotent endomorphism (i.e., a projection), then F is called *a projection invariant* subgroup.

Classical examples of important fully invariant subgroups of an arbitrary group G are the subgroups defined above p^nG and $G[p^n]$ for any natural n as well as tG and the maximal divisible subgroup dG of G; actually dG is a fully invariant direct summand of G (see, for instance, [9]).

We shall say that a group G has only *trivial fully invariant subgroups* if $\{0\}$ and G are the only ones. Same appears for characteristic and projection invariant subgroups, respectively.

The following notions are our major tools.

Definition 1. A nonzero group G is said to be an *IFI-group* if either it has only trivial fully invariant subgroups, or all its nontrivial fully invariant subgroups are isomorphic otherwise.

Definition 2. A nonzero group G is said to be an *IC-group* if either it has only trivial characteristic subgroups, or all its nontrivial characteristic subgroups are isomorphic otherwise.

Definition 3. A nonzero group G is said to be an *IPI-group* if either it has only trivial projection invariant subgroups, or all its nontrivial projection invariant subgroups are isomorphic otherwise.

Note that Definition 3 implies Definition 1 and Definition 2 implies Definition 1. In other words, any IPI-group is an IFI-group and any IC-group is an IFI-group; in fact, every fully invariant subgroup is both characteristic and projection invariant.

Definition 4. A nonzero group G is called a *strongly IFI-group* if either it has only trivial fully invariant subgroups, or all its nonzero fully invariant subgroups are isomorphic otherwise.

Definition 5. A nonzero group G is called a *strongly IC-group* if either it has only trivial characteristic subgroups, or all its nonzero characteristic subgroups are isomorphic otherwise.

Definition 6. A nonzero group G is called a *strongly IPI-group* if either it has only trivial projection invariant subgroups, or all its nonzero projection invariant subgroups are isomorphic otherwise.

Notice that Definition 6 implies Definition 4 and Definition 5 implies Definition 4.

On the other hand, it is obvious that Definition 4 implies Definition 1, whereas the converse fails as the next example shows: In fact, construct the group $G \cong \mathbb{Z}(p) \oplus \oplus_{\aleph_o} \mathbb{Z}(p^2)$. Since it is fairly clear that $G \neq pG$, $G \neq G[p]$ and $G = G[p^2]$, we deduce that $pG \cong \oplus_{\aleph_0} \mathbb{Z}(p) \cong G[p]$ that are the only proper fully invariant subgroups of G. However, $G \not\cong G[p]$, as required. Thus there exists a p-primary IFI-group which is not a strongly IFI-group, as asserted.

However, in the torsion-free case, Definitions 1 and 4 are tantamount (see Proposition 2.2 below).

Moreover, each subgroup of an indecomposable group is projection invariant, so that an indecomposable group is an IPI-group if and only if it is either a cyclic group of order p for some prime p, or is isomorphic to the additive group of integers $\mathbb{Z}$.

It is worthwhile noticing in the current context that in [13] and [14] were studied p-groups which are isomorphic to their fixed proper fully invariant subgroup as well as in [1] were examined the so-called *IP-groups* that are isomorphic to their fixed pure subgroup. On the other vein, in [10] and [11] the classes of *minimal* and *quasi-minimal groups*, having some specific properties of subgroups, were investigated as well.

Our purpose in this article is to explore some crucial properties of the defined above new classes of groups. The chief results are stated and proved in the next section.

2. BASIC RESULTS

As usual, $\oplus_m G = G^{(m)}$ will denote the *external* direct sum of m copies of the group G, where m is some ordinal (finite or infinite). The following statement asserts that in a special case the three classes from Definitions 1, 2, and 3 do coincide.

Theorem 2.1. *Let G be a p-group, and let $m \geqslant 2$ be an ordinal. Then $G^{(m)}$ is an IFI-group if and only if G is an IC-group if and only if G is an IPI-group.*

Proof. The statement follows directly by results from [7] and [8], where it is shown that in this case characteristic and projection invariant subgroups are fully invariant. □

Remark 1. In [3] and [4] some other special properties of projection invariant subgroups were considered and, in addition, when they are fully invariant (see [19] too). The results established there can also be applied successfully to the proof of Theorem 2.1.

Proposition 2.2. *Let G be a torsion-free group. Then G is an IFI-group if and only if G is a strongly IFI-group.*

Proof. One direction being elementary, we assume now that G is a torsion-free IFI-group containing all nontrivial fully invariant subgroups isomorphic. So, for all primes p, we have that $G[p] = \{0\}$, and consequently, $G \cong G/\{0\} = G/G[p] \cong pG \neq \{0\}$. If $G = pG$ for any prime p, it follows from [9] that G is a torsion-free divisible group, whence it does not contain proper fully invariant subgroups, and so we are finished.

Suppose now that $G \neq pG$ for some prime p. Since $pG \neq \{0\}$ is fully invariant in G for every p, it follows by definition that each other nontrivial fully invariant subgroup of G must be isomorphic to pG, and hence to G. So, all nonzero fully invariant subgroups of G (including the full group G) must be mutually isomorphic, i.e., G is a strongly IFI-group, as claimed. □

For torsion (strongly) IFI-groups, we can obtain a complete description; however, the torsion-free case is rather more complicated. We first need a series of technical claims.

The next technicality is quite easy, but we provide a proof only for the sake of completeness and for the readers' convenience.

Lemma 2.3.

(a) *A fully invariant subgroup of an IFI-group is an IFI-group.*
(b) *A fully invariant subgroup of a strongly IFI-group is a strongly IFI-group.*

Proof. (a) Let G be an IFI-group with a fully invariant subgroup F. If either $F = \{0\}$ or $F = G$, we are done. Suppose now that K and L are two different proper fully invariant subgroups of F. Since they are obviously proper fully invariant in G, we deduce that $K \cong L$, as required.

(b) The same idea as that in the preceding point successfully works to get the claim. □

Before proceeding by proving our main characterization theorem, we need one more useful observation.

Proposition 2.4. *A nonzero IFI-group is either divisible or reduced.*

Proof. If $dG = \{0\}$ or $dG = G$, we are finished. If now $G \neq dG \neq \{0\}$, we have that $G[p] = \{0\}$ or that $G[p] = G$ for any p, because otherwise $dG \cong G[p]$ assures that $dG = \{0\}$, a contradiction. In the latter case, $dG = \{0\}$, again a contradiction. So, let $G[p] = \{0\}$ for all p. But $G \neq pG$ for some p; if not G should be divisible — contrary to our assumption. Therefore, $G \cong G/\{0\} = G/G[p] \cong pG$ and thus $dG \cong pG \cong G$, which is false since G is not divisible. □

In accordance to the last statement, since divisible groups are well-classified (cf. [9]), we will henceforth consider only *reduced* groups.

Theorem 2.5. *The following two points hold*:

(i) *A nonzero group G is an IFI-group if and only if one of the following holds*:

- *For some prime p either $pG = \{0\}$, or $p^2G = \{0\}$ with $r(G) = r(pG)$*;
- *G is a homogeneous torsion-free IFI-group of an idempotent type.*

(ii) *A nonzero torsion group G is a strongly IFI-group if and only if it is an elementary p-group for some prime p.*

Proof. (i) Suppose first that G is torsion, that is, $G = tG$. If $G = G[p]$, the assertion follows. So, assume now that $G \neq G[p]$. We next claim that $G = G[p^2]$ or, equivalently, $p^2G = \{0\}$. If $G \neq G[p^2]$, then $G[p^2] \cong G[p]$ which is untrue, so that the claim is sustained. But moreover $G[p]$ and $(pG)[p] = pG$ are both non-trivial fully invariant in G, whence they should be isomorphic. In addition, appealing to [9], one may derive that $r(G) = r(G[p]) = r((pG)[p]) = r(pG)$, as stated.

Reciprocally, if G is an elementary p-group, it contains only trivial fully invariant subgroups, and thus we are done. So, let G be p^2-bounded. It is well known in this case that the only proper fully invariant subgroups of G are $G[p]$ and $pG = (pG)[p]$. Now, the rank condition allows us to infer that they are isomorphic, as required. This completes the proof of the torsion case.

Assume now that G is torsion-free, i.e., $G \neq tG = \{0\}$. Since $G(\tau) \cong G$, we can write $G = G(\tau)$. But $G(\chi)$ is also fully invariant in G for a characteristic $\chi \in \tau$ and, because $G \cong G(\chi)$, we conclude that the type τ must be an idempotent, that is, $\tau^2 = \tau$, as claimed. This completes the torsion-free case.

Finally, we will show that an IFI-group cannot be mixed. In fact, applying Lemma 2.3, tG is an IFI-group. By what we have shown above, tG has to be a p^2-bounded p-group for some prime p. This means that G splits, that is, $G = tG \oplus R$ where R is torsion-free (see, for instance, [9]). Since both $tG \neq \{0\}$ and $p^2R = p^2G \neq \{0\}$ are obviously nontrivially fully invariant in G (this is because $G \neq tG$ and $p^2R = G = R \oplus tG$ ensures that $tG = \{0\}$ which is against our assumption), they must be mutually isomorphic. But this is manifestly wrong, because p^2R remains torsion-free while tG is torsion, which gives the desired contradiction. This completes the proof of the mixed case.

(ii) If G possesses only two trivial fully invariant subgroups, we are done. Suppose now that $G_p \neq \{0\}$ for some prime p. Since both $G \neq \{0\}$ and $G[p] \neq \{0\}$ are fully invariant in G, they should be isomorphic, so that G must be an elementary p-group, as asserted.

Conversely, it is apparent that each elementary p-group G, where p is a prime, is a strongly IFI-group because it has only two fully invariant subgroups, namely $\{0\}$ and G. □

In conjunction with the last statement, we will hereafter be interested only in torsion-free groups.

It is self-evident that any rank one torsion-free group of an idempotent type is an IFI-group; these groups are realized as subgroups of the additive group of rational numbers $\mathbb{Q}$. Thus a question related to torsion-free IFI-groups, which immediately arises, is the following one: Is it true that all homogeneous torsion-free groups of an idempotent type are IFI-groups? Unfortunately, this problem has a negative resolution; especially, there is a homogeneous torsion-free group of an idempotent type with arbitrary rank greater than 1 which is not an IFI-group. In fact, the following concrete example is true.

Example 2.6. There exists a homogeneous torsion-free group of an idempotent type with arbitrary large rank > 1 that is not an IFI-group.

Proof. Letting $\mathbb{Q}_p$ be the ring of all rational numbers with denominator q such that $(q, p) = 1$, we employ [9, Paragraph 110, Exercize 7] or [6] to find that there is a reduced indecomposable torsion-free group G of rank 2 with endomorphism ring $\mathrm{E}(G) = \mathbb{Q}_p$. Therefore, G is a $\mathbb{Q}_p$-module, whence G is a homogeneous torsion-free group of an idempotent type with the property that $\mathrm{E}(G)a \cong \mathbb{Q}_p^+$ for any $0 \neq a \in G$, but $\mathrm{E}(G)a \ncong G$. However, one may see that G is not an IFI-group.

Even more, the following generalized construction holds: Suppose that $L = \mathbb{Q}_p^{(n)} = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ (n-times), where n is a natural. So, using the construction of G demonstrated above, there exists a reduced indecomposable torsion-free group G_n of rank $2n$ with endomorphism ring $\mathrm{E}(G) = L$, and hence $G_1 = G$. Consequently, $\mathrm{rank}(\mathrm{E}(G_n)a) \leqslant n$ for any $0 \neq a \in G_n$ and thus $\mathrm{E}(G_n)a \ncong G_n$. So, for each n, we have constructed a homogeneous group of an idempotent type and rank $2n$ which is not an IFI-group. Set $A = G^{(\kappa)} \oplus \mathbb{Q}_p^+$, where κ is an arbitrary cardinal. It is not too hard to see that the group A is endocyclic, that is, $A = \mathrm{E}(A)a$ for some $a \in A$, although G is obviously not endocyclic. Furthermore, $\mathrm{Hom}(G, \mathbb{Q}_p^+) = 0$, because otherwise G will have a direct summand isomorphic to $\mathbb{Q}_p^+$, which will contradict the fact that G is indecomposable. Thus, $\mathrm{Hom}(G^{(\kappa)}, \mathbb{Q}_p^+) = 0$, i.e., $G^{(\kappa)}$ is a fully invariant subgroup in A. But it is clear that $G^{(\kappa)} \ncong A$, as wanted. Therefore, if $\kappa = l$ is a natural, for each l we have constructed a homogeneous group of an idempotent type and rank $2l + 1$ which is not an IFI-group. For an infinite ordinal κ such a group has exactly rank κ.

Note also that, if $2 \leqslant \kappa \leqslant 2^{\aleph_0}$ and B is a pure subgroup of rank κ of the group $\mathbb{J}_p$ of p-adic integers, then $B^{(m)} \oplus \mathbb{Q}_p^+$ is also not an IFI-group for any cardinal m. □

It follows directly from the proof of Theorem 2.5 that the following proposition is true.

Proposition 2.7. *Suppose G is a divisible group. Then G is an IFI-group if and only if it is a torsion-free group.*

As an explicit example to this fact, it is worthwhile noticing that $\mathbb{Q}$ is a torsion-free divisible group of rank 1, whence it is an IFI-group.

Since the divisible part is always a fully invariant subgroup of the whole group, then the (torsion-free) IFI-group is either divisible or reduced. That is why we may hereafter assume that all groups are *reduced*.

Observe also that Theorem 2.5 gives a chance to describe some partial classes of IFI-groups. So, the following corollary holds.

Corollary 2.8.

(1) *A coperiodical group is an IFI-group if and only if it is either an elementary p-group, or is a torsion-free p-adic algebraically compact group, for some single prime p.*
(2) *A vector torsion-free group is an IFI-group if and only if it is a direct product of groups of rank* 1 *with the same idempotent type.*

Proof. (1) Applying Theorem 2.5, such a group should be either an elementary p-group or a torsion-free group. In the latter case, in accordance with [9, Corollary 54.5], a torsion-free coperiodical group is algebraically compact (for more details, the interested reader can see cf. [9] too). Since each its nonzero p-adic component is fully invariant, we are done.

(2) Owing to [9, Lemma 96.4], such a group should be homogeneous and separable, whence it is an IFI-group. □

As already illustrated in the proof of point (2) of Corollary 2.8, since any nonzero fully invariant subgroup of a group G is of the form nG, where $n \in \mathbb{N}$, it easily follows that every separable homogeneous torsion-free group of idempotent type is an IFI-group.

Furthermore, recall that a torsion-free group A is called *fully transitive* if, for each two elements $0 \neq a, b \in A$ with $\chi_A(a) \leqslant \chi_A(b)$, there exists $f \in \mathrm{E}(A)$ such that $f(a) = b$. This class of groups is quite large and, for instance, it contains algebraically compact torsion-free groups and homogeneous separable groups (see, for example, [17]). Using this definition, the last claim about separable homogeneous torsion-free groups stated above can be somewhat extended thus:

Proposition 2.9. *Every homogeneous fully transitive torsion-free group of an idempotent type is an IFI-group.*

Proof. In Paragraph 25, Exercise 11 of [17] it was proved that every fully invariant subgroup of a torsion-free group G has the form nG for some integer $n \geqslant 0$ if and only if G is a homogeneous torsion-free fully transitive group of an idempotent type. And since nG is isomorphic to G, all nontrivial fully invariant subgroups are mutually isomorphic, so that the assertion follows. □

On the other hand, if an almost completely decomposable group (for the definition, we refer the reader to [18]) is an IFI-group, then by virtue of Theorem 2.5 it is homogeneous of an idempotent type. Likewise, excepting the case where it is isomorphic to its regulator, an almost completely decomposable IFI-group A should be a completely decomposable homogeneous group with the property that each its fully invariant subgroup has the form nA.

As a consequence to Proposition 2.9, we obtain the following (see also Problem 1 below).

Corollary 2.10. *A direct summand of a fully transitive torsion-free IFI-group is again a fully transitive IFI-group.*

Proof. In view of Theorem 2.5, the group G should be homogeneous of an idempotent type. Moreover, it follows from [17] that any direct summand of a fully transitive torsion-free group is again a torsion-free fully transitive group. But it must also be homogeneous of an idempotent type, so that Proposition 2.9 is applicable to get the claim. □

In [12], a group G is called *an H-group* if any its fully invariant subgroup F has the form $F = \{a \in G \mid \mathbf{H}(a) \geqslant M\}$, where $\mathbf{H}(a)$ is the *height matrix* of the element a and M is some $\omega \times \omega$-matrix with ordinal numbers and symbol ∞ for entries. Likewise, it is shown there that every H-group is a fully transitive group and that a p-group is a H-group if and only if it is fully transitive. However, there are fully transitive torsion-free groups that are not H-groups. Nevertheless, torsion-free homogeneous fully transitive groups are necessarily H-groups.

The next assertions shed some light about the relationships between IFI-groups and H-groups (compare also with Theorem 2.19 below).

Proposition 2.11. *Suppose that G is a torsion-free* H*-group. Then G is an IFI-group if and only if G is a homogeneous group of an idempotent type.*

Proof. The necessity follows directly from Theorem 2.5. Since as observed above H-groups are fully transitive, the sufficiency follows directly from Proposition 2.9. □

Mimicking [20], a ring R with identity is said to be an *E-ring* if $\mathrm{Hom}_{\mathbb{Z}}(R, R) = \mathrm{Hom}_R(R, R)$, where $\mathbb{Z}$ is the ring of integers. Note that every E-ring is necessarily commutative. The additive groups of E-rings are just called *E-groups*. Notice also that the group A is an E-group if and only if $A \cong \mathrm{End}(A)$ and the ring $\mathrm{E}(A)$ is commutative. Furthermore, if R is a commutative ring, then the right R-module A is said to be an *E-module* if $\mathrm{Hom}_{\mathbb{Z}}(R, A) = \mathrm{Hom}_R(R, A)$.

We also recall that the commutative ring R with identity is called *a principal ideal ring* if each its ideal is principal, that is, it is of the form xR for some element $x \in R$.

Theorem 2.12. *Suppose $A \neq 0$ is a torsion-free group whose nonzero endomorphisms are monomorphisms. Then A is an IFI-group if and only if A is an E-group and $E(A)$ is a principal ideal ring.*

Proof. "**Necessity**." Set $R = \mathrm{E}(A)$. For each $0 \neq a \in A$ the map of R^+ onto Ra, defined by $R^+ \ni \beta \mapsto \beta a$, gives the group isomorphism $R^+ \cong Ra$. Thus $A \cong Ra \cong R^+$. In fact, let $f\colon R^+ \to A$ be an isomorphism. Now, the map $\psi\colon ra \mapsto f(ra) - r(f(a))$ defines for each fixed $0 \neq a \in A$ a group homomorphism $Ra \to A$ with nonzero kernel. Therefore, $\psi = 0$ which forces an isomorphism $Ra \cong A$. So, $f(ra) = r(f(a))$ and, hence, f is an R-modular isomorphism. But $R^+ \cong A$ implies the equality $\mathrm{Hom}_{\mathbb{Z}}(R, R) = \mathrm{Hom}_R(R, R)$, that is, the ring R is an E-ring. Every ideal I of R as a submodule of an E-module R_R is an E-module as well. Consequently, the isomorphism $I^+ \cong Ra$ is an R-modular isomorphism and so the ideal I is principal, i.e., R is a principal ideal ring.

"**Sufficiency**." Since $A \cong \mathrm{End}(A)$, then we can determine on A the structure of the ring $\mathrm{E}(A)$, so that all non-zero fully invariant subgroups of A can be considered as the ideals of the ring $\mathrm{E}(A)$. According to the condition on additive groups, such ideals are obviously isomorphic to $\mathrm{End}(A)$, as required. □

Lemma 2.13. *If A is a torsion-free IFI-group, then all its nonzero central endomorphisms are monomorphisms.*

Proof. If α is a central endomorphism and $\ker\alpha \neq 0$, then $\ker\alpha \cong A$, and therefore, there exists a monomorphism $f \in \mathrm{E}(A)$ such that $\alpha f = 0$. But $\alpha f = f\alpha$, whence $\alpha = 0$ as needed. □

It follows from [2, Lemma 1.3] that in any quasi-homogeneous torsion-free fully transitive group all nonzero central endomorphisms are monomorphisms. Besides, in [5] were found some necessary and sufficient conditions for groups to be torsion-free fully transitive, provided that their endomorphism ring is commutative.

A group is said to be *irreducible* if it does not have proper pure fully invariant subgroups. So, elementary p-groups can be considered as irreducible. If now A is a torsion-free IFI-group of finite rank, then any its pure fully invariant subgroups coincides with the full group A; in particular, the group A is irreducible.

The following sheds some light on the endomorphism ring structure of such groups.

Proposition 2.14. *If A is a torsion-free IFI-group of finite rank, then the following conditions are equivalent*:

(1) *All nonzero endomorphisms of A are monomorphisms*;
(2) *A is a strongly indecomposable group*;
(3) *$E(A)$ is a commutative ring.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, while the implication (1) $\Rightarrow$ (3) follows from Theorem 2.12. The validity of the implication (3) $\Rightarrow$ (1) was noted above. Now, we will show that (2) $\Rightarrow$ (1) is true. In fact, in a strongly indecomposable torsion-free *IFI*-group of finite rank any pure fully invariant subgroup coincides with the whole group. Consequently, according to Corollary 5.14 from [17], all its nonzero endomorphisms are monomorphisms which guarantees the wanted implication. □

A combination of Theorem 2.12 and Proposition 2.14 gives the following corollary.

Corollary 2.15. *Suppose $A \neq 0$ is a strongly indecomposable torsion-free IFI-group of finite rank. Then A is an E-group and* $\mathrm{E}(A)$ *is a principal ideal ring.*

Homogeneous fully transitive torsion-free groups A of an idempotent type are *endocyclic groups*, that is, $A = \mathrm{E}(A)a$ for a certain element $a \in A$; in conjunction with Proposition 2.9, they are also IFI-groups. All fully invariant subgroups of such a group A are submodules of the R-module ${}_RA$, where $R = \mathrm{E}(A)$. If in the determination of the torsion-free IFI-group we require an R-module isomorphism, then under the validity of the isomorphism $A \cong Ra$, where $0 \neq a \in A$, the group A is endocyclic. Moreover, a more general class form the so-called *endofinite groups* that are groups considered as finitely generated modules over their endomorphism rings.

Theorem 2.16. *Suppose A is an irreducible endofinite torsion-free group, the center C of $E(A)$ is a principal ideal domain, and the module ${}_CA$ has rank $\leqslant \aleph_0$. Then A is an*

IFI-group. Besides, if the group A is decomposable, then it is both an IC-group and an IP-group.

Proof. According to [15] (see also [17, Corollary 8.6]), one sees that A is a free C-module. If now H is a fully invariant subgroup of A, then H is a submodule of the module ${}_CA$. Since C is a principal ideal domain, then H is also a free C-module (same rank as ${}_CA$ under the truthfulness of the fully invariance of the subgroup H). Consequently, the module ${}_CH$ is isomorphic to ${}_CA$, and hence we have the group isomorphism $H \cong A$, as desired.

The second part is immediate. □

We shall say that R is *a ring with property* $(*)$ if R^+ is a torsion-free group and the factor-ring R/pR is a domain for any prime number p such that $pR \neq R$. With [17, Lemma 44.6] at hand, it will follow that in such a ring R the equality $\chi(ab) = \chi(a) + \chi(b)$ holds for any $a, b \in R$.

The following technicality is pivotal.

Lemma 2.17. *Let R be an E-ring with property $(*)$. Then the following conditions are equivalent*:

(1) *R^+ is irreducible*;
(2) *any element of R is an integer multiplied by invertible*;
(3) *R^+ is a homogeneous fully transitive group.*

Proof. "(1) ⇒ (2)." Since R^+ is irreducible, then it is homogeneous, and since $\chi(1)$ is the least characteristic, then the type of R^+ is an idempotent. Supposing that $I = xR$ is a main ideal, we write $x = nx_0$ where $\chi(x_0) = \chi(1)$ and $J = x_0R$. If $y = x_0z \in J$, where $z \in R$, then $\chi(y) = \chi(x_0) + \chi(z) = \chi(z)$ because $\chi(x_0)$ is a characteristic consisting only of 0 and ∞. So, if $p^kt = y$, then $z \in p^kR$ and $y \in p^kJ$. Equivalently, J^+ is a pure fully invariant subgroup in R^+ because R is an E-ring. Consequently, $J = R$ ensures that the element x_0 is invertible, as required.

"(2) ⇒ (3)." Since any element of R is an integer multiplied by invertible, the group R^+ is homogeneous of an idempotent type. Let $0 \neq a, b \in R^+$ and $\chi(a) \leqslant \chi(b)$. Assuming $na_0 = a$, where a_0 is invertible, we obtain that $nb_0 = b$. Therefore, $b_0a_0^{-1}a = b$, as wanted.

"(3) ⇒ (1)." It is obvious. □

It is worthwhile noticing that, since the multiplication of elements of a ring by its invertible elements is an automorphism, all conditions of Lemma 2.17 are also equivalent to the fact that R^+ is a homogeneous transitive group. Besides, note that the ring R from Lemma 2.17 is a principal ideal domain.

Proposition 2.18. *Any countable irreducible and endofinite torsion-free group, for which the center of its endomorphism ring is a principal ideal domain with property $(*)$, is both a fully transitive and transitive group.*

Proof. Let A be such a group, and let C be the center of $\mathrm{E}(A)$. In accordance with [17, Theorem 8.7], C is an E-ring. With [17, Corollary 8.6] at hand, A is a free C-module. However, as a direct summand of A, the group C^+ is irreducible. Thus the proof goes on by virtue of Lemma 2.17 and [17, Corollary 40.5]. □

The next statement also describes certain cases of IFI-groups (compare with Proposition 2.14 above).

Theorem 2.19. *For a torsion-free group G of finite rank, for which the center C of $E(G)$ is a ring satisfying property $(*)$, the following four conditions are equivalent*:

(1) *G is an IFI-group*;
(2) *G is an irreducible endofinite group and C is a principal ideal E-ring*;
(3) *$G \cong (C^+)^{(n)}$, where n is some natural number and C^+ is a strongly indecomposable E-group of finite rank*;
(4) *G is a homogeneous fully transitive group of idempotent type.*

Proof. "(1) $\Rightarrow$ (2)" and "(1) $\Rightarrow$ (3)." We have noted before the statement of Proposition 2.14 that G is irreducible and that we have proved in Lemma 2.13 that all nonzero endomorphisms in C are monomorphisms. Since $G \cong \mathrm{E}(G)a$ for any fixed $0 \neq a \in G$, the subgroup $\mathrm{E}(G)a$ has finite index in G because of the finite rank of G. So G is an endofinite group. Invoking [17, Corollary 8.8] and Corollary 2.15, C is a principal ideal E-ring, and G is quasi-isomorphic to $(C^+)^{(n)}$ for some n. As being a quasi-summand of G, the group C^+ is irreducible, so referring to Lemma 2.17 and [17, Corollary 8.6], we obtain that $G \cong (C^+)^{(n)}$ that substantiates the proof of these two implications.

"(2) $\Rightarrow$ (3)." It follows from [17, Corollary 11.5].

"(3) $\Rightarrow$ (4)." By virtue of [17, Corollary 8.10], the group $\mathrm{End}(C^+)$ is an irreducible group, and hence in view of Lemma 2.17 the group $C^+ \cong \mathrm{End}(C^+)$ is a fully transitive group, so G as being isomorphic to the direct sum of copies of a fixed fully transitive group is again fully transitive, as required.

"(4) $\Rightarrow$ (1)." It follows directly from Proposition 2.9. □

Remark 2. Note that Theorem 2.5 guarantees the validity only of a part of implication (1) $\Rightarrow$ (4), namely that G is a homogeneous group of an idempotent type.

Now we will consider the question of when an arbitrary direct sum of IFI-groups is again an IFI-group. Before doing this, it is worthy of noticing that any IFI-group G has no nontrivial fully invariant direct summand (i.e., a fully invariant direct summand $\neq 0, G$). To that goal, Theorem 2.5 settles this when G is a torsion group. Letting now G be torsion-free, we write in a way of contradiction that $G = A \oplus B$, where $A \neq 0$ is fully invariant in G. Then $A \cong G$, so one can infer that $A = A_1 \oplus B_1$, where $A_1 \cong A$ and $B_1 \cong B$. But this allows us to conclude that $\mathrm{Hom}(A, B) \neq 0$, and so the desired claim follows.

Proposition 2.20. *Suppose A_i $(i \in I)$ is a system of nonzero torsion-free IFI-groups. Then $G = \bigoplus_{i\in I} A_i$ is an IFI-group if and only if at most one of the following two conditions is valid*:

(1) *For any pair $i, j \in I$ and for each $0 \neq a \in A_i$, there exists $\varphi \in Hom(A_i, A_j)$ with the property $\varphi(a) \neq 0$;*
(2) *$\bigoplus_{j\in J_K} A_j \cong G$ for each $K \subseteq I$ $(K \neq \varnothing, I)$, where $J_K = \{j \in I \mid \bigcap_{f\in \mathrm{Hom}(A_j,A_k),\ k\in K} \ker f \neq 0\}$.*

Proof. "**Necessity**." Assume that $J_K \neq \varnothing$ for some $\varnothing \neq K \subsetneqq I$. Thus $G = B \oplus C$, where $B = \bigoplus_{j\in J_K} A_j$ and $C = \bigoplus_{i\in I\setminus J_K} A_i$. Set

$$H_j = \bigcap_{f\in \mathrm{Hom}(A_j,A_k),\ k\in K} \ker f,$$

where $j \in J_K$. It is clear that $H = \bigoplus_{j\in J_K} H_j$ is a fully invariant subgroup in B. But if $i \in I\setminus J_K$, then for any $0 \neq a \in A_i$ there exist $k \in K$ and $f \in \mathrm{Hom}(A_i, A_k)$ with the property $f(a) \neq 0$. So H is a fully invariant subgroup of G and H_j is a fully invariant subgroup of A_j for each $j \in J_K$, respectively. Therefore, $H \cong G$ and hence $B \cong H \cong G$, as required.

"**Sufficiency**." If H is a fully invariant subgroup of G, then it is well known that $H = \bigoplus_{i\in I}(H \cap A_i)$, where every $H \cap A_i$ is a fully invariant subgroup of A_i. If now condition (1) holds, then $H \cap A_i \neq 0$ for any $i \in I$. However, $H \cap A_i \cong A_i$, which assures that $H \cong G$, as needed.

If we set $K = \{k \in I \mid H \cap A_k = 0\} \neq \varnothing$ and $J = \{j \in I \mid H \cap A_j \neq 0\}$, then one sees that $J \cup K = I$ and $J \cap K = \varnothing$, so that

$$J = J_K = \left\{ s \in I \mid \bigcap_{f\in \mathrm{Hom}(A_s,A_k),\ k\in K} \ker f \neq 0 \right\}.$$

Next, in the presence of condition (2), we conclude that $H \cap A_j \cong A_j$ for each $j \in J$ and consequently $H \cong G$, as required. □

We notice the obvious fact that condition (1) in Proposition 2.20 is not equivalent to $\mathrm{Hom}(A_i, A_j) \neq 0$ for any $i, j \in I$; in fact, it is weaker than that inequality because in (1) the homomorphism φ depends on the choice of the element a.

The next assertion, however, shows that under some additional circumstances on the family $\{A_i\}_{i\in I}$, the last statement can be somewhat reversed.

Proposition 2.21. *Let A_i $(i \in I)$ be a system of nonzero irreducible IFI-groups. Then $G = \bigoplus_{i\in I} A_i$ is an IFI-group if and only if $Hom(A_i, A_j) \neq 0$ for any $i, j \in I$.*

Proof. "**Necessity**." Assume that $\varphi(a) = 0$ for some $0 \neq a \in A_i$ and for each $\varphi \in \mathrm{Hom}(A_i, A_j)$. If we set $B = \bigoplus_{k\in I\setminus\{j\}} A_k$ and $C = A_j$, then $a \in H = \bigcap_{f\in \mathrm{Hom}(B,C)} \ker f$, where it is readily checked that H is a pure fully invariant subgroup of G. So, it follows that $H = \bigoplus_{i\in I}(H \cap A_i)$, where $H \cap A_i$ are fully invariant pure subgroups of A_i and thus $H \cap A_i = A_i$ if $H \cap A_i \neq 0$. Consequently, H is a nonzero fully invariant direct summand of G that contradicts the remark listed before Proposition 2.20. In particular, $\mathrm{Hom}(A_i, A_j) \neq 0$ for any $i, j \in I$, as desired.

"**Sufficiency**." By hypothesis, it follows that either all A_i are elementary p-groups for a fixed prime natural p and hence G is an elementary p-group, or all A_i are torsion-free groups. In the second case, these A_i are irreducible groups. So, for any $0 \neq a, b \in A_i$, we find $f \in \mathrm{E}(A_i)$ with the property that $f(a) = kb$ for some natural number k. Thus, if H is a fully invariant subgroup of G, then $H \cap A_i \neq 0$ for every $i \in I$. As in the proof of Proposition 2.20, we deduce that $H \cong G$, whence G is an IFI-group, as claimed. □

As an immediate consequence to Proposition 2.20, we also derive:

Corollary 2.22. *If G is an IFI-group, then $G^{(m)}$ is also an IFI-group for any ordinal m.*

It was proved in [12, Corollary 3.24] that if G is a homogeneous fully transitive torsion-free group and $\mathbf{K}$ is an arbitrary ideal of the Boolean algebra of all subsets of a certain set of indices I, then the $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} G$ remains a fully transitive group. If, additionally, the type of G is an idempotent, then $\bigoplus_{\mathbf{K}} G$ will also be homogeneous as a pure subgroup of the homogeneous group G^I (see, for instance, [9, Lemma 96.4]).

We thus deduce the following statement.

Proposition 2.23. *If G is a fully transitive torsion-free IFI-group, then any $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} G$ is an IFI-group.*

Recall that a torsion-free group is called *strongly irreducible* if any of its nonzero fully invariant subgroup has bounded index. Utilizing [9, Proposition 92.1], we directly obtain the following proposition.

Proposition 2.24. *Any strongly irreducible group G, satisfying the condition $|G/pG| \leqslant p$ for each prime p, is an IFI-group.*

3. LEFT-OPEN PROBLEMS

We close the work with some results of interest.

Problem 1. Is a direct summand of an IFI-group again an IFI-group?

Problem 2. Do there exist IFI-groups that are not fully transitive (in particular, that are not H-groups)?

Problem 3. Do there exist nonirreducible and nonendocyclic torsion-free IFI-groups?

Problem 4. Does there exist a strongly irreducible endocyclic group which is not an IFI-group?

Problem 5. If possible, construct an IFI-group that is neither an IC-group nor an IPI-group.

Problem 6. Let A_i ($i \in I$) be a system of reduced torsion-free IFI-groups, and let $\mathbf{K}$ be the ideal of the Boolean algebra of all subsets of I. Find a necessary and/or sufficient condition when the $\mathbf{K}$-direct sum $\bigoplus_{\mathbf{K}} A_i$ (in particular, the direct product $\prod_{i\in I} A_i$) is an IFI-group.

Problem 7. Can torsion-free IFI-groups be characterized by certain numerical invariants?

Problem 8. Determine when $\mathrm{E}(A)$ is a clean ring, provided that A is a torsion-free IFI-group.

Problem 9. Characterize those groups that have all (proper or nonzero, respectively) large subgroups isomorphic, if they eventually exist. We call such groups *IL-groups* or *strongly IL-groups*, respectively.

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REFERENCES

[1] Beaumont, R. A., Pierce, R. S. (1964). Isomorphic direct summands of abelian groups. *Math. Ann.* 153:21–37.
[2] Chekhlov, A. R. (2001). Fully transitive torsion-free groups of finite p-rank. *Algebra and Logic* 40:391–400.
[3] Chekhlov, A. R. (2009). Separable and vector groups whose projectively invariant subgroups are fully invariant. *Siberian Math. J.* 50:748–756.
[4] Chekhlov, A. R. (2010). On projective invariant subgroups of abelian groups. *J. Math. Sci.* 164:143–147.
[5] Chekhlov, A. R. (2013). Torsion-free weakly transitive E-engel abelian groups. *Math. Notes* 94:583–589.
[6] Corner, A. L. S. (1963). Every countable reduced torsion-free ring is an endomorphism ring. *Proc. London Math. Soc.* 13:687–710.
[7] Danchev, P., Goldsmith, B. (2010). On the socles of characteristic subgroup of abelian p-groups. *J. Algebra* 323:3020–3028.
[8] Danchev, P., Goldsmith, B. (2012). On projection-invariant subgroups of abelian p-groups. *Contemp. Math.* 576:31–40.
[9] Fuchs, L. (1970/1973). *Infinite Abelian Groups*. Vol. I and Vol. II. New York and London: Academic Press.
[10] Goldsmith, B., Óhógáin, S. (2002). On torsion and mixed minimal abelian groups. *Rocky Mount. J. Math.* 32:1451–1465.

[11] Goldsmith, B., Óhógáin, S., Wallutis, S. (2004). Quasi-minimal abelian groups. *Proc. Amer. Math. Soc.* 132:2185–2195.

[12] Grinshpon, S. Ya. (2002). Fully invariant subgroups of abelian groups and full transitivity. *Fundam. Prikl. Mat.* 8:407–473 (In Russian).

[13] Grinshpon, S. Ya., Nikolskaya (Savinkova), M. M. (2011). Fully invariant subgroups of abelian p-groups with finite Ulm-Kaplansky invariants. *Commun. Algebra* 39: 4273–4282.

[14] Grinshpon, S. Ya., Nikolskaya (Savinkova), M. M. (2011-2012/2014). Torsion IF-groups. *Fundam. Prikl. Mat.* 17:47–58; translated in *J. Math. Sci.* 197:614–622.

[15] Hausen, J. (1987). On strongly irreducible torsion-free abelian groups. *Abelian Group Theory*. New York: Gordon and Breach, pp. 351–358.

[16] Kaplansky, I. (1954/1969). *Infinite Abelian Groups*. University of Michigan Press, Ann Arbor.

[17] Krylov, P. A., Mikhalev, A. V., Tuganbaev, A. A. (2003). *Endomorphism Rings of Abelian Groups*. Dordrecht, Boston and London: Kluwer Acad. Publ.

[18] Mader, A. (1995). Almost completely decomposable torsion-free abelian groups. In: *Abelian Groups and Modules, Proc. Padova Conf., Padova, Italy, 1994*. Vol. 343. Kluwer Acad. Publ., pp. 343–366.

[19] Megibben, C. (1977). Projection-invariant subgroups of abelian groups. *Tamkang J. Math.* 8:177–182.

[20] Schultz, P. (1973). The endomorphism ring of the additive group of a ring. *J. Austral. Math. Soc.* 15:60–69.