Results. Math. 63 (2013), 1109–1130

1422-6383/13/031109-22
published online May 17, 2012
DOI 10.1007/s00025-012-0256-8

Results in Mathematics

On Projectively Fully Transitive Abelian p-Groups

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Abstract. There are two natural questions which arise in connection with the endomorphism ring of an Abelian group: when is the ring generated by its idempotents and when is the ring generated additively by its idempotents? The present work investigates these two questions for Abelian p-groups. This leads in a natural way to consideration of two strengthened versions of Kaplansky's notion of full transitivity, which we call *projective full transitivity* and *strong projective full transitivity*. We establish, *inter alia*, that these concepts are strictly stronger than the classical concept of full transitivity but there are nonetheless many strong parallels between the notions.

Mathematics Subject Classification. Primary 20K10, 20K12.

Keywords. Abelian p-groups, projection-invariant subgroups, fully invariant subgroups, transitive and fully transitive groups, projectively fully transitive groups.

1. Introduction

In 1952, Kaplansky [15] began his investigations into the fully invariant and characteristic subgroups of an Abelian p-group. He followed this up in his now famous "*little red book*", Infinite Abelian Groups [16], and introduced the notions of transitive and fully transitive p-groups in a natural way arising from his investigations in [15]; these notions have been of interest in Abelian group theory ever since. There is another notion, closely related to full invariance, which has also been studied: projection invariance. Recall that a subgroup H of the group G is said to be *projection-invariant in* G if $\pi(H) \leq H$ for all idempotent endomorphisms π of G. Significant work on this topic was produced by Hausen [12] and Megibben [18], concentrating in the main in establishing when projection-invariant subgroups are actually fully invariant; the socles of such subgroups have been investigated by the present authors in [8]. In this paper we follow a somewhat different path and explore a new notion of transitivity which we shall call projective full transitivity. Recall that a group G is said to be fully transitive if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there is an endomorphism ϕ of G with $\phi(x) = y$. Our modification is to say that G is *projectively fully transitive* if the endomorphism ϕ can be chosen to be in the subring of the full endomorphism ring generated by the idempotent endomorphisms; clearly a projectively fully transitive group is always fully transitive.

We shall establish a number of basic properties of projectively fully transitive groups; in particular we shall show that this class of groups is properly contained in the class of fully transitive groups. Moreover, the class is large but is not closed under the taking of direct summands, unlike the situation which pertains for fully transitive groups. Recent work on various types of transitivity—see, for e.g., [7]—has revealed the role played by 'squares' of a group in this connection and similar properties re-appear here.

To simplify the notation and to avoid risk of confusion, we shall write $\mathrm{E}(G)$ for the endomorphism *ring* of G and $\mathrm{End}(G)$ for the endomorphism *group* of G. We shall denote by $\mathrm{Proj}(G)$ the *subring* of $\mathrm{E}(G)$ generated by the idempotents of $\mathrm{E}(G)$; thus an element $\phi \in \mathrm{Proj}(G)$ will have the form $\phi = \sum_{\text{finite}} \pm\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$, where each π_{i_j} is an idempotent in $\mathrm{E}(G)$.

In the final section of the paper we shall examine briefly an apparently stronger notion. Following Hausen [13], we let $\Pi(G)$ denote the *subgroup* of the endomorphism group $\mathrm{End}(G)$ generated by the idempotent endomorphisms; so $\phi \in \Pi(G)$ has the form $\phi = \sum_{i=1}^{n} \pm\pi_i$ for some finite n, where each π_i is an idempotent endomorphism. Then a group G is said to be *strongly projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \Pi(G)$ with $\phi(x) = y$; clearly a strongly projectively fully transitive group is projectively fully transitive. Our results here are somewhat sketchier.

Throughout the word group will denote an additively written Abelian p-group. In this context our notation is standard and follows Fuchs [10] and Kaplansky [16]; mappings are written on the left.

2. Elementary Results

Since it is clear that a fully transitive group G is projectively fully transitive if $\mathrm{E}(G) = \mathrm{Proj}(G)$ (and similarly it is strongly projectively fully transitive if $\mathrm{End}(G) = \Pi(G)$), we consider firstly this situation. To simplify our terminology we shall say that a group G is an *idempotent-generated* group (or IG-group) if $\mathrm{E}(G) = \mathrm{Proj}(G)$; we say that G is an *idempotent-sum* group (or IS-group) if $\mathrm{End}(G) = \Pi(G)$. If $\mathrm{E}(G)$ is commutative, then it is obvious that $\mathrm{Proj}(G) = \Pi(G)$ so that the IG-groups are then precisely the IS-groups; in general an IS-group is always an IG-group. However, this situation is rather rare for a primary group: it follows from results of Szele and Szendrei—see

Exercise 6, p. 227 in [10]—that groups with commutative endomorphism ring are precisely subgroups of $\mathbb{Z}(p^\infty)$ and it is easy to see that any cyclic group is an IS-group, while the quasi-cyclic group $\mathbb{Z}(p^\infty)$ is not even an IG-group.

We begin with an elementary but useful observation:

Proposition 2.1. *If $G = A_1 \oplus \cdots \oplus A_n$ where the A_i are IG (resp. IS)-groups, then G is an IG (resp. IS)-group. In particular,*

(i) *if A is an IG (resp. IS)-group, then so also is $A^{(n)}$ for each finite n;*
(ii) *if F is a finite group, then it is an IS-group.*

Proof. By induction it suffices to show the result for the direct sum of two groups, so suppose that $G = A \oplus B$ where each A, B is an IG (resp. IS)-group. If $\phi \in \mathrm{E}(G)$, then we can write ϕ in the form $\phi = \left(\begin{smallmatrix} \alpha & \gamma \\ \delta & \beta \end{smallmatrix}\right)$ where $\alpha \in \mathrm{E}(A), \beta \in \mathrm{E}(B), \gamma \in \mathrm{Hom}(B, A)$ and $\delta \in \mathrm{Hom}(A, B)$. But then we have

$$\phi = \left(\begin{smallmatrix} \alpha - 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & \beta - 1_B \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1_A & \gamma \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1_B \end{smallmatrix}\right).$$

The latter two matrices represent idempotent endomorphisms of G. Since $\alpha - 1_A \in \mathrm{E}(A), \beta - 1_B \in \mathrm{E}(B)$ and A, B are IG (resp. IS)-groups, the endomorphisms $\alpha - 1_A, \beta - 1_B$ may be expressed as sums of products of idempotents (resp. sums of idempotents) and hence the matrices

$$\left(\begin{smallmatrix} \alpha - 1_A & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & \beta - 1_B \end{smallmatrix}\right)$$

have the same properties since the embeddings of $\mathrm{E}(A), \mathrm{E}(B)$ into $\mathrm{E}(G)$ are ring homomorphisms. The particular cases are immediate. $\square$

Corollary 2.2. *If $G = A \oplus B$, where A is a fully invariant subgroup of G, then G is an IG (resp. IS)-group if, and only if, both A, B are IG (resp. IS)-groups. In particular, if $G = D \oplus R$, where D is divisible and R is reduced, then G is an IG (resp. IS)-group if, and only if, both D, R are IG (resp. IS)-groups.*

Proof. If A, B are IG (resp. IS)-groups, then it follows immediately from Proposition 2.1 that G is an IG (resp. IS)-group; the full invariance of A is not needed here.

Conversely suppose that G ia an IG (resp. IS)-group and that A is fully invariant in G. Let χ denote the restriction map $\chi : \mathrm{E}(G) \to \mathrm{E}(A)$ with $\phi \mapsto \phi \restriction A$ for each $\phi \in \mathrm{E}(G)$; the full invariance of A ensures that χ is a ring homomorphism $\mathrm{E}(G) \twoheadrightarrow \mathrm{E}(A)$. Consequently, $\chi(\mathrm{Proj}(G)) \leq \mathrm{Proj}(A)$, $\chi(\Pi(G)) \leq \Pi(A)$ and hence $\mathrm{E}(A) = \chi(\mathrm{E}(G)) \leq \mathrm{Proj}(A) \leq \mathrm{E}(A)$ if G is an IG-group. Similarly, $\mathrm{E}(A) = \Pi(A)$ if G is an IS-group. Thus if G is an IG (resp. IS)-group, then so also is A. $\square$

It follows immediately from Corollary 2.2 that the study of IG (resp. IS)-groups may be reduced to the separate study of divisible and reduced IG (resp. IS)-groups.

In fact we can say a great deal more about IG-groups, thus generalizing Proposition 2.1 (i).

Proposition 2.3. *If A is an arbitrary group and $\kappa \geq 2$ is any cardinal, then $G = A^{(\kappa)}$ is an IG-group.*

Proof. If κ is finite then the result is immediate from Proposition 2.1 in [8]. If κ is infinite, then write $G = X \oplus X$, where $X \cong A^{(\kappa)}$, so that $\mathrm{E}(G)$ is isomorphic to the ring of 2×2 matrices over $S = \mathrm{E}(X)$. However, it follows again from Proposition 2.1 in [8] that $\mathrm{E}(G) = \mathrm{Proj}(G)$, so that G is an IG-group. □

Notice that it follows from Proposition 2.3 that a summand of an IG-group need not be an IG-group: in fact, if G is any group which is not an IG-group (for e.g., $\mathbb{Z}(p^\infty)$), then its square is an IG-group.

We also have the simple consequence:

Corollary 2.4. *A divisible group D is an IG-group if, and only if,* $\mathrm{rk}(D) \geq 2$.

Proof. The sufficiency follows immediately from Proposition 2.3. However, if $\mathrm{rk}(D) = 1$, then $\mathrm{E}(D) \cong J_p$, the ring of p-adic integers which has only 0 and 1 as idempotents, and consequently the endomorphism ring of D is not generated by idempotents. □

Note also that Proposition 2.3 does *not* generalize to IS-groups as the next example shows.

Example 2.5. The group $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ is not an IS-group.

Proof. The endomorphism ring of G is, of course, isomorphic to the ring of 2×2 matrices over J_p. We claim that the trace of an idempotent 2×2 p-adic matrix is one of $\{0, 1, 2\}$. To see this suppose that $\Delta = \left(\begin{smallmatrix} x & a \\ b & y \end{smallmatrix}\right)$ is an idempotent p-adic matrix. Direct calculation gives

(i) $xa + ay = a$ and $bx + yb = b$, so that $a(x + y - 1) = 0 = b(x + y - 1)$.
(ii) $x^2 + ab = x$ and $y^2 + ab = y$, so that $(x - y)(x + y - 1) = 0$.

Since J_p is a domain, we have that either $x + y - 1 = 0$ or $x = y$. In the first case, the trace of Δ is 1, so we may restrict our attention to the case where $x + y - 1 \neq 0$ and $x = y$. From the observation in (i) above, we conclude that $a = 0 = b$ and this in turn forces $x^2 = x, y^2 = y$. Thus x, y take the values 0 or 1 and hence the trace of Δ is either 0, 1 or 2, as required. In particular, we see that the trace of any idempotent matrix must be a non-zero integer in this situation.

Suppose now, for a contradiction, that $\mathrm{End}(G) = \Pi(G)$. Choose a p-adic integer u which is not a rational integer and consider the matrix $\phi = \left(\begin{smallmatrix} u & 0 \\ 0 & 0 \end{smallmatrix}\right)$. By assumption then ϕ is a linear combination of idempotent matrices and hence the trace of ϕ, u, is a finite sum of terms from the set $\{0, 1, 2\}$; in particular it is a rational integer—contradiction. □

Returning to the reduced situation, it seems rather difficult to give a satisfactory description of the class of IG (resp. IS)-groups. Since a bounded group can be expressed in the form $G = A \oplus F$ where F is finite and each homocyclic component of A is of rank at least 2, it follows easily from Proposition

2.1 and Proposition 2.3 that a bounded group is an IG-group. In fact Hausen has shown, [12, Corollary 6], that every bounded group is even an IS-group. On the other hand, it is relatively easy to exhibit reduced groups which are not IG-groups. A simple, but useful, observation here is that if the ring $\mathrm{E}(G)$ is generated as a ring (resp. additively) by idempotents, then the same is true of $\mathrm{E}(G)/I$ for any two-sided ideal I. This, combined with Corner's realization theorems [2,3], gives the following:

Proposition 2.6. *If A is a ring whose additive group is the completion of a free p-adic module of countable rank and A is not generated as a ring (resp. additively) by its idempotents, then there exists an unbounded separable group G_A which is not an IG (resp. IS)-group.*

Proof. From Theorem 1.1 in [2], we conclude that there is an unbounded separable group G_A with $\mathrm{E}(G_A) = A \oplus E_s(G_A)$, where $E_s(G_A)$ is the ideal of small endomorphisms of G_A. Since $\mathrm{E}(G_A)/E_s(G_A) \cong A$ is not generated as a ring (resp. additively) by its idempotents, we conclude that G_A is not an IG (resp. IS)-group. □

Rings of the type required for Proposition 2.6 are easy to construct: for e.g., the ring A which is the completion (in the p-adic topology) of the polynomial ring $J_p[X]$ has the property and so also does the ring direct product $A = J_p \times \cdots \times J_p = J_p^{(n)}$ for a finite n. In fact if A is a ring whose additive group is the completion of a free p-adic module of countable rank and A is commutative, then A is not generated as a ring by its idempotents: if it were, it would follow from Bergman's lemma [10, Lemma 97.2] that the additive group of A would be free—contradiction.

Corollary 2.7. *If G is an unbounded essentially indecomposable group, then G is not an IG-group.*

Proof. If G is an unbounded essentially indecomposable p-group, then it follows from a result of Monk [20, Corollary to Theorem 1] that the only idempotents in $\mathrm{E}(G)/E_s(G)$ are 0 and 1 but these cannot generate this quotient as a ring since the quotient must always contain a copy of the centre of $\mathrm{E}(G)$ which is isomorphic to J_p as G is unbounded. □

It is also easy to construct non-separable groups which are not IG (resp. IS)-groups; to do this we make use of Corner's second realization theorem [3, Theorem 10.2]. Hence we have:

Proposition 2.8. *If A is a reduced separable group with basic subgroups of rank ≥ 2 and of cardinal $< 2^{\aleph_0}$, then for any infinite ordinal $\alpha < \omega^2$, there is a group G with $p^\alpha G = A$ and G is not an IG-group.*

Proof. Using Corner's theorem we construct a group G with $p^\alpha G = A$ and $\mathrm{E}(G)_A = \{\phi \restriction A : \phi \in \mathrm{E}(G)\} = \Phi$, where Φ is any complete separable p-adic subalgebra of $\mathrm{E}(A)$. Consider firstly the case where A is unbounded. Then the choice $\Phi = J_p$ is possible and so we have $\mathrm{E}(G)_A = J_p$. Since J_p is not generated as a ring by its idempotents, the ring $\mathrm{E}(G)$ cannot be generated by its idempotents since $\mathrm{E}(G)_A$ is a ring homomorphic image of $\mathrm{E}(G)$.

Now suppose that A is bounded and write $A = B \oplus C$, where $B = \mathbb{Z}(p^{n_1}) \oplus \mathbb{Z}(p^{n_2})$ and $n_1 \leq n_2$; this is possible since by assumption the rank of a basic subgroup of A is at least 2. Let Φ be the set of matrices of the form $\left(\begin{smallmatrix} \phi & 0 \\ 0 & \psi \end{smallmatrix}\right)$, where $\phi \in \Phi_1, \psi = r1_C$ $(0 \leq r < p^{e(C)})$ and Φ_1 is a subring of $\mathrm{E}(B)$, which we will define. Since $\mathrm{E}(B)$ is finite, Φ will be complete and separable, so we may apply Corner's theorem to find a group G with $p^\alpha G = A$ and $\mathrm{E}(G)_A = \Phi$. Now choose Φ_1 to be the subring of $\mathrm{E}(B)$ consisting of lower triangular matrices of the form $\left(\begin{smallmatrix} r & 0 \\ s & r \end{smallmatrix}\right)$, where $0 \leq r < p^{n_2}$ and $s \in \mathrm{Hom}(\mathbb{Z}(p^{n_1}), \mathbb{Z}(p^{n_2}))$.

We claim that Φ is not generated as a ring by its idempotents - notice that this suffices to show that G is not an IG-group. For if it were, then the same would be true of any ring homomorphic image of Φ, in particular Φ_1 would be generated as a ring by its idempotents. However by direct calculation we can see that any idempotent matrix in Φ_1 must have entries satisfying $r^2 = r, 2rs = s$. Hence $r = 0$ or 1, which in turn implies that $s = 0$. Hence, the only idempotents in Φ_1 are $0, 1$ and these clearly do not generate all of Φ_1. This completes the proof. $\square$

We have already seen that a summand of an IG-group need not be an IG-group, however we do have:

Proposition 2.9. *If G is an IG (resp. IS)-group then so also is $p^n G$ for any finite n.*

Proof. The mapping $\chi : \mathrm{E}(G) \to \mathrm{E}(p^n G)$ given by $\chi(\phi) = \phi \restriction p^n G$ is a ring homomorphism. However, if $\theta \in \mathrm{E}(p^n G)$ then it follows easily from Proposition 113.3 in [10] that there is an endomorphism $\phi \in \mathrm{E}(G)$ with $\phi \restriction p^n G = \theta$. Hence the mapping χ is onto and $\mathrm{E}(p^n G)$ is a ring epimorphic image of $\mathrm{E}(G)$. The result follows now immediately. $\square$

This assertion may be extended to subgroups $p^\alpha G$ provided the quotient $G/p^\alpha G$ is totally projective: the proof is essentially identical but the ontoness property of the mapping χ now comes from Hill's result on totally projective groups [13]. So, we have:

Proposition 2.10. *If G is an IG (resp. IS)-group and the quotient $G/p^\alpha G$ is totally projective, then $p^\alpha G$ is an IG (resp. IS)-group.*

We close this section with the following question:

Problem 1. If A is an IS-group and $\kappa \geq 1$ is any cardinal, does it follow that $A^{(\kappa)}$ is also an IS-group?

3. Projectively Fully Transitive Groups

In the classical theory of transitive and fully transitive groups, it is usual to restrict consideration to reduced groups. However, it is not difficult to extend the theory to non-reduced groups. This is normally achieved by modifying the definition of an Ulm sequence for an element of a divisible group—see [16, p.57]—so that if D is divisible and $x \in D$, then $U_D(x) = (0, \ldots 0, \infty, \ldots)$ where the symbol ∞ occurs at precisely the $(n+1)^{st}$ place if x has order p^n; with this understanding it is easy to show that divisible groups are fully transitive—see, for e.g., [16, Exercise 71] or [1, Proposition 2.1]. In fact, we can show even that any divisible group is necessarily a projectively fully transitive group. Recall once again from the introduction that a group G is said to be *projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$; clearly a projectively fully transitive group is fully transitive.

Theorem 3.1. *If D is a divisible group, then D is a projectively fully transitive group.*

Proof. Since a divisible group is fully transitive and any divisible group of rank ≥ 2 is an IG-group (by Proposition 2.3), we deduce immediately that the result is true provided $\mathrm{rk}(D) \geq 2$. Suppose then that D is divisible with $\mathrm{rk}(D) = 1$. Note that, in this situation, we have for $x, y \in D$ that $o(x) \geq o(y)$ if, and only if, $U_D(x) \leq U_D(y)$.

Clearly, if $o(x) = p^n \geq o(y) = p^m$ we can write $x = ua, y = p^{n-m}va$ where a is the generator of the $\mathbb{Z}(p^\infty)[p^n]$ and $(u, p) = 1 = (v, p)$. Let $\lambda p^n + \mu u = 1$ so that $a = \mu ua$ and then observe that the mapping $\phi = \mu v p^{n-m} : x \mapsto vp^{n-m}\mu ua = y$. Since ϕ is an integer multiple of the identity map, it certainly belongs to $\mathrm{Proj}(\mathbb{Z}(p^\infty))$. This completes the proof. □

Recall [9, Definition 1] that the groups G_1, G_2 form a *fully transitive pair* if, for every $x \in G_i, y \in G_j (i, j \in \{1, 2\})$ with $U_G(x) \leq U_G(y)$, there exists $\alpha \in \mathrm{Hom}(G_i, G_j)$ with $\alpha(x) = y$. Note that $\{G_1, G_2\}$ is a fully transitive pair if, and only if, $G_1 \oplus G_2$ is fully transitive—see [9, Proposition 1].

Proposition 3.2. *If A, B are projectively fully transitive groups and $\{A, B\}$ is a fully transitive pair, then $A \oplus B$ is a projectively fully transitive group.*

Proof. Let $G = A \oplus B$ and suppose that $x, y \in G$ with $U_G(x) \leq U_G(y)$. We proceed by induction on the order of y. In fact, we claim that if there is always an endomorphism $\phi \in \mathrm{Proj}(G)$ mapping x to y when $o(y) = p$, then the proposition follows: for suppose we have shown the result for $o(y) = p^n$ and consider the situation where $o(y) = p^{n+1}$. Then, arguing exactly as in [11, Lemma 2.2], $U_G(px) \leq U_G(py)$ and $o(py) \leq p^n$. So there is a $\theta \in \mathrm{Proj}(G)$ with $\theta(px) = py$. Set $y' = y - \theta(x)$ so that $y' \in G[p]$ and clearly $U_G(x) \leq U_G(y')$. Hence there is $\varphi \in \mathrm{Proj}(G)$ with $\varphi(x) = y'$. But then $\theta + \varphi \in \mathrm{Proj}(G)$ and

$(\theta + \varphi)(x) = \theta(x) + (y - \theta(x)) = y$. So the claim is established and it remains only to verify the result when $o(y) = p$.

Let $x = (a, b), y = (a_1, b_1) \in A \oplus B$ with $py = 0$. By re-labelling if necessary, we may assume $ht_G(x) = ht_A(a)$. Now we have $U_A(a) \leq U_G(y) \leq U_B(b_1)$. In particular, since $\{A, B\}$ is a fully transitive pair, there is a homomorphism $\gamma : A \to B$ with $\gamma(a) = b_1$. However, $U_A(a) \leq U_A((a_1 - a)$ since $p(a_1 - a) = -pa$. Since by hypothesis A is a projectively fully transitive group, there is an endomorphism $\varphi \in \mathrm{Proj}(A)$ with $\varphi(a) = a_1 - a$. Now if $\Delta = \left(\begin{smallmatrix} \varphi & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\Gamma = \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 0 \end{smallmatrix}\right)$, then Δ and Γ are endomorphisms of G and $\Delta + \Gamma$ maps (a, b) to (a_1, b_1) as required. It remains to show that $\Delta, \Gamma \in \mathrm{Proj}(G)$. Since the embedding of $\mathrm{E}(A)$ into $\mathrm{E}(G)$ is a ring homomorphism, it is clear that $\Delta \in \mathrm{Proj}(G)$, but a direct calculation shows that Γ is idempotent also and so we have the result. $\square$

We note for later use that the proof of Proposition 3.2 carries over to the situation where the groups A, B are strongly projectively fully transitive groups.

We have a partial converse in the situation where one of the groups is divisible.

Proposition 3.3. *Suppose that D is a divisible group and R is reduced. If $G = D \oplus R$ is projectively fully transitive, then so also is R.*

Proof. Suppose that $x, y \in R$ and that $U_R(x) \leq U_R(y)$. Then $U_G((0, x)) \leq U_G((0, y))$ and so there is a $\varphi \in \mathrm{Proj}(G)$ with $\varphi(0, x) = (0, y)$, say $\varphi = \left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$ is the matrix representation. Now an idempotent matrix in $\mathrm{E}(G)$ necessarily has diagonal entries which are idempotents in $\mathrm{E}(D)$ and $\mathrm{E}(R)$ respectively. Consequently any product of idempotent matrices must also have idempotent diagonal entries. Since φ can be expressed as a sum (or difference) of such products, it follows that its diagonal entry β has the same property, i.e. $\beta \in \mathrm{Proj}(R)$. Since $\beta(x) = y$, R is projectively fully transitive. $\square$

Summarizing these results we have:

Theorem 3.4. *If $G = D \oplus R$, where D is divisible and R is reduced, then G is projectively fully transitive if, and only if, R is projectively fully transitive.*

Proof. The necessity follows from Proposition 3.3 and the sufficiency follows from Proposition 3.2 since R, being projectively fully transitive, is certainly fully transitive and then the divisibility of D ensures that $\{D, R\}$ is a fully transitive pair. $\square$

It follows immediately that we may restrict our attention to reduced groups when we are considering projective full transitivity. Hence for the remainder of this section we shall assume that all groups discussed are *reduced.*

Until this point we have not given an example of a fully transitive group which is not projectively fully transitive. We remedy this in the next result:

Proposition 3.5. *The class of projectively fully transitive groups is strictly contained in the class of fully transitive groups.*

Proof. In fact we will exhibit three examples of groups with this phenomenon; the first having an infinite elementary first Ulm subgroup, the second a finite elementary first Ulm subgroup and the third a finite non-elementary first Ulm subgroup.

(i) Suppose that G is a group in which $p^{\omega}G$ is elementary of infinite rank and G is fully transitive but not transitive—such groups exist, for example, the groups constructed by Corner in §3 of [4]. Then, as shown in Lemma 1.6 of [8], every subgroup of $p^{\omega}G$ is a projection-invariant subgroup of G. We claim that G is not projectively fully transitive: to show this choose basis elements a, b of $p^{\omega}G$ and note that $U_G(a) = (\omega, \infty, \dots) = U_G(b)$. However, if $\phi \in \mathrm{Proj}(G)$, then $\phi(\langle a\rangle) \leq \langle a\rangle$ since $\langle a\rangle$ is projection invariant in G; in particular there cannot be a $\phi \in \mathrm{Proj}(G)$ with $\phi(a) = b$, so that G is not projectively fully transitive, as claimed.

Our construction of the second and third examples is based on Corner's Theorem 6.1 in [3] and we make use of Lemma 3.12 below.

(ii) Let $H = \mathbb{Z}(p) \oplus \mathbb{Z}(p) = \langle a\rangle \oplus \langle b\rangle$ and define the endomorphism ϕ by $\phi : a \mapsto b,\ b \mapsto a + b$; note that $\phi^2 = I + \phi$, where I is the identity on H. Let Φ be the subring of $\mathrm{E}(G)$ generated by I, ϕ. Then Φ consists of the elements $\{rI + s\phi : 0 \leq r, s \leq p - 1\}$.

Suppose that $p \neq 2$ and we make the additional assumption that p is a prime of the form $p = 5n + 2$; note that it follows from Dirichlet's theorem on primes in arithmetic progression that there are infinitely many primes of this form. Consider an idempotent $rI + s\phi \in \Phi$. Then it follows immediately that $(r^2 + s^2 - r)I + (2rs + s^2 - s)\phi = 0$. Applying this expression to the element a, we deduce that

$$r^2 + s^2 - r \equiv 0 \mod p \tag{1}$$

and that $2rs + s^2 - s \equiv 0 \mod p$.

Consider now the situation where $s \neq 0$; the last congruence may then be simplified to

$$2r + s - 1 \equiv 0 \mod p. \tag{2}$$

Now multiply the relation (1) by 4 and substitute for $2r$, to obtain $(1 - s)^2 + 4s^2 - 2(1 - s) \equiv 0 \mod p$. Simplifying, we get

$$5s^2 \equiv 1 \mod p. \tag{3}$$

Since $s \neq 0$, the congruence (3) has a solution if, and only if, 5 is a quadratic residue mod p, i.e., employing the standard Legendre symbol notation, if, and only if, $(\frac{5}{p}) = -1$. Now it follows from the Quadratic Reciprocity theorem that $(\frac{5}{p})(\frac{p}{5}) = (-1)^{(5-1)(p-1)/4} = 1$ and hence we conclude that $(\frac{5}{p}) = (\frac{p}{5})$, since the Legendre symbol is ± 1 in each case.

We claim that $(\frac{p}{5}) = -1$; for suppose not, then we have $p \equiv x^2 \mod 5$ for some x and from this it follows that $x^2 \equiv 2 \mod 5$. This latter is impossible since the only squares mod 5 are $0, 1, 4$. Hence the only idempotents in Φ must have $s = 0$ and it follows from a straightforward calculation that then $r = 0, 1$ and so the only idempotents in Φ are $0, I$.

Note that as a consequence of $(\frac{5}{p}) = -1$, we have that the expression $t^2 + t - 1 \not\equiv 0 \mod p$ for any $0 \leq t \leq p-1$: for if $t^2 + t - 1 \equiv 0 \mod p$ then $4t^2 + 4t - 4 = (2t+1)^2 - 5 \equiv 0 \mod p$, contradicting $(\frac{5}{p}) = -1$.

Now construct, using Corner's theorem, a group G such that $p^\omega G = H$ and $\mathrm{E}(G) \restriction H = \Phi$. Since Proj$(\Phi)$ consists only of the multiples of the identity, it is clear that $\mathrm{E}(G)$ does not act projectively fully transitively on $p^\omega G$ and so G is certainly not projectively fully transitive by Lemma 3.12 below. However, Φ acts fully transitively on $p^\omega G$: to see this observe that the Ulm sequences of H are only of two types, viz., $(\infty, \infty, \dots)$ and $(0, \infty, \dots)$ and these correspond respectively to the sets of elements $\{0\}, \{ra + sb : 0 \leq r, s \leq p-1; r, s \text{ not both } 0\}$. Since it is trivial to find a map in Φ taking a to $ra + sb$, $(0 \leq r, s \leq p-1)$, it will suffice to show that for an arbitrary element $ra + sb$, with not both of $r, s = 0$, that there is a mapping in $yI + z\phi$ taking $ra + sb$ to a.

We consider a number of cases:

(a) if $s = 0$, choose $y = r^{-1}$ and $x = 0$;

(b) if $s \neq 0$, let $t = rs^{-1}$ and note that multiplication by s^{-1} maps $ra + sb \mapsto ta + b$. Thus it will suffice to show that we can map an arbitrary element of the form $(ta + b)$ to a. Applying the map $yI + x\phi$ to $(ta + b)$ we get $(yt + x)a + (y + xt + x)b$, so we need to choose y, x in order that $(yt + x) \equiv 1 \mod p$ and simultaneously that $(y + xt + x) \equiv 0 \mod p$. If we set $x = 1 - yt$ then certainly the first congruence is satisfied. Substituting we see that the second congruence reduces to $y(1 - t - t^2) + (1 + t) \equiv 0 \mod p$. As noted above, our choice of p ensures that $(1 - t - t^2) \not\equiv 0 \mod p$ and hence the choice $y = (1+t)/(t^2+t-1), x = 1 - yt$ guarantees that $yI + x\phi$ maps $(ta + b) \mapsto a$, as required.

It follows from Lemma 2.1 in [4] that G is fully transitive.

(iii) The proof of the final part is similar to that of (ii). Let $H = \mathbb{Z}(2) \oplus \mathbb{Z}(4) = \langle a \rangle \oplus \langle b \rangle$ and define the endomorphism ϕ by $\phi : a \mapsto a + 2b$, $b \mapsto a + b$; note that $\phi^2 = 3I$ and $2\phi = 2I$, where I is the identity on H. Let Φ be the subring of $\mathrm{E}(G)$ generated by I, ϕ. A routine check using the identities noted above shows that Φ has order 8 and consists of the elements $\{0, I, 2I, 3I, \phi, I + \phi, 2I + \phi, 3I + \phi\}$; observe that the only idempotents in Φ are $0, I$. Now construct, utilizing Corner's theorem, a group G such that $2^\omega G = H$ and $\mathrm{E}(G) \restriction H = \Phi$. Since Proj$(\Phi)$ consists only of the multiples of the identity, it is clear that

E(G) does not act projectively fully transitively on $2^{\omega}G$ and so G is certainly not projectively fully transitive by Lemma 3.12 below. However, Φ acts fully transitively on $2^{\omega}G$: to see this observe that the Ulm sequences of H form a chain with four nodes consisting of the elements with Ulm sequences $(0, 1, \infty, \dots), (0, \infty, \dots), (1, \infty, \dots)$ and $(\infty, \infty, \dots)$. The four types consist respectively of the sets of elements $\{b, 3b, a{+}b, a{+}3b\}, \{a, a{+}2b\}, \{2b\}, \{0\}$. A straightforward check shows that $3I$ interchanges $b, 3b$ and also $a + 3b, a + b$ while $\phi : b \mapsto a + b, 2I + \phi : a + b \mapsto b$; thus the elements of Ulm sequence $(0, 1, \infty, \dots)$ lie in a single orbit under the action of Φ. Since $2I + \phi : b \mapsto a$, $3I + \phi : a \mapsto 2b$ and $2I : 2b \mapsto 0$, we can establish that Φ acts fully transitively on H if we can show that $a, a + 2b$ are in the same orbit of Φ; but this is immediate since a simple calculation assures that ϕ interchanges $a, a + 2b$. It follows from Lemma 2.1 in [4] that G is fully transitive. □

Remark 3.6. The choice of the prime $p = 5n + 2$ in Proposition 3.5 (ii) was made purely to simplify the calculations; it is not a necessary condition. For example, it is straightforward to demonstrate that in the case $p = 2$, the only idempotents in the subring Φ are $0, I$ and that Φ acts fully transitively on H.

Our next result illustrates the close connection between the various types of transitivity that we have discussed:

Theorem 3.7. *Suppose $\kappa > 1$. Then the following are equivalent:*

(i) *G is fully transitive;*
(ii) *$G^{(\kappa)}$ is fully transitive;*
(iii) *$G^{(\kappa)}$ is transitive;*
(iv) *$G^{(\kappa)}$ is projectively fully transitive.*

Proof. The equivalence of (i) and (ii) follows from Corollary 1 in [9], while the equivalent statement of (ii) and (iii) follows from Corollary 4 of the same paper. We show the equivalence of (ii) and (iv). Since projectively fully transitive groups are always fully transitive, it is immediate that (iv) $\Rightarrow$ (ii). Conversely, if $G^{(\kappa)}$ is fully transitive, then, since we are assuming that $\kappa > 1$, it follows from Proposition 2.3 that $G^{(\kappa)}$ is a fully transitive IG-group and so necessarily is a projectively fully transitive group. □

Corollary 3.8. *If G is projectively fully transitive, then for every cardinal κ, $G^{(\kappa)}$ is projectively fully transitive.*

Corollary 3.9. *A direct summand of a projectively fully transitive group is not necessarily a projectively fully transitive group.*

Proof. Choose a group G as in Proposition 3.5 above, so that G is fully transitive but not projectively fully transitive. If $H = G \oplus G$, then it follows from Theorem 3.7 that H is projectively fully transitive while its summand G is not. □

Remark 3.10. The role of transitivity in this connection is not clear. In Theorem 3.7 it is not possible to replace condition (i) with the statement "G is projectively fully transitive": for if G is chosen as in the proof of Corollary 3.9, then G is not projectively fully transitive but its square $G \oplus G$ is transitive, since G is fully transitive—see Corollary 3 in [9]. It would be interesting to know if a transitive, fully transitive group is necessarily projectively fully transitive. In fact, it was shown in [8] that C_λ-groups of length λ are both transitive and fully transitive; however whether or not they are projectively fully transitive is not obvious (compare also with Corollary 3.19 (ii)). Moreover, note that it is well known that there exist transitive 2-groups which are not fully transitive (see, for example, §4 in [4]) and hence, *a fortiori*, not projectively fully transitive.

In the classical notions of transitivity a key observation due to Corner [4] is that the transitivity property depends on the action of the endomorphism ring on the first Ulm subgroup. A similar phenomenon occurs here; let us say that a subring Φ of $\mathrm{E}(G)$ acts *projectively fully transitively* on a subgroup X of G if, given $x, y \in X$ with $U_G(x) \leq U_G(y)$, there is an endomorphism $\phi \in \Phi$ such that $\phi(x) = y$ and ϕ belongs to the subring of Φ generated by the idempotents in Φ.

The following extremely simple assertion has been used previously by both Hausen [12] and Megibben [18]; we include the short proof for completeness.

Lemma 3.11. *If $G = A \oplus H$ and $\psi \in \mathrm{End}(G)$ with $\psi(A) \leq H, \psi(H) = 0$, then $\psi \in \Pi(G)$.*

Proof. The standard matrix representation for ψ is $\psi = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1_H \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1_H \end{smallmatrix}\right)$. Since there last two matrices are clearly idempotent, $\psi \in \Pi(G)$. □

Our next result is a careful re-working of Lemma 2.1 in [3]. We were faced here with the dilemma of whether to simply tell the reader that the necessary changes can be made to Corner's original proof or of reworking the proof in detail. We have chosen the latter since it enables us to point out the more substantial changes needed to extend the proof to strongly projectively fully transitive groups, which we will discuss in the next section.

Lemma 3.12. *A group G is projectively fully transitive if, and only if,* $\mathrm{E}(G)$ *acts projectively fully transitively on $p^\omega G$.*

Proof. The sufficiency is trivial; so assume that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$ and consider $x, y \in G$ with $U_G(x) \leq U_G(y)$. Let r, s be the least natural numbers such that $p^r x, p^s y \in p^\omega G$; if $r = 0$, then both $x, y \in p^\omega G$ and the result follows immediately, so we may assume that $r \neq 0$. Note that $s \leq r$, so that in any case $p^r y \in p^\omega G$ and $U_G(p^r x) \leq U_G(p^r y)$. By hypothesis there is an endomorphism $\phi_0 \in \mathrm{Proj}(G)$ with $\phi_0(p^r x) = p^r y$. Since $r \neq 0$, we may choose an integer $m > \max\{ht_G(p^{r-1}x), ht_G(p^{s-1}y)\}$—if $s-1$ is negative we simply omit the final term $ht_G(p^{s-1}y)$.

So we can choose $x_0 \in G$ with $p^r x = p^{r+m}x_0$; then $p^r y = \phi_0(p^r x) = p^{r+m}y_0$ where $y_0 = \phi_0(x_0)$. Thus $x = x_1 + p^m x_0, y = y_1 + p^m y_0$ where $p^r x_1 = p^r y_1 = 0$. Note that $o(x_1) = p^r$ for otherwise $p^t x = p^{t+m}x_0$ for some $t < r$, contradicting the choice of m. Also $ht_G(p^{r-1}x_1) = ht_G(p^{r-1}x)$ since $x_1 = x - p^m x_0$ and so $ht_G(p^{r-1}x_1) < m$. Thus $\langle x_1 \rangle \cap p^m G = 0$ and so there is a $p^m G$-high subgroup A such that $x_1 \in A$. It follows that A is necessarily a bounded pure subgroup and so we may write $G = A \oplus H$ for some complement H; note that $p^m G \leq H$. Let π denote the projection of G onto H with kernel A.

Let $y_1 = a_1 + h_1$, where $a_1 \in A, h_1 \in H$. Since the decomposition is direct, $U_G(y_1) = U_G(a_1) \wedge U_G(h_1)$ and so $U_G(x_1) \leq U_G(a_1)$ and $U_G(x_1) \leq U_G(h_1)$. Now $x_1, a_1 \in A$, a bounded group, and $U_G(x_1) = U_A(x_1)$ and $U_G(a_1) = U_A(a_1)$. Since bounded groups are fully transitive there is an endomorphism θ of A with $\theta(x_1) = a_1$. But, as observed earlier, a bounded group is an IG-group and so we can extend θ to an endomorphism θ_1 of G with $\theta_1 \in \mathrm{Proj}(G)$. (It is even possible to choose a suitable $\theta_1 \in \Pi(G)$ by using Corollary 6 in [12]).

Since A is a bounded summand we can certainly find an endomorphism ϕ' of G with $\phi'(x_1) = h_1$. Set $\psi = \pi\phi'(1-\pi)$ and observe that $\psi(x_1) = h_1$. Moreover, $\psi(A) \leq \pi(G) = H$ while $\psi(H) = 0$, so by Lemma 3.11, ψ is in $\Pi(G)$; in particular $\psi \in \mathrm{Proj}(G)$. Set $\phi_1 = \theta_1 + \psi$ and note that $\phi_1(x_1) = \theta_1(x_1) + \psi(x_1) = a_1 + h_1 = y_1$; also observe that $\phi_1 \in \mathrm{Proj}(G)$. (In fact it is even in $\Pi(G)$).

Finally set $\phi = \phi_0\pi + \phi_1(1-\pi)$ so that $\phi \in \mathrm{Proj}(G)$. (Note that this construction is not possible if one wants to stay within $\Pi(G)$.) Now $\phi(x) = \phi_0\pi(x) + \phi_1(1-\pi)(x)$ and, since $x = x_1 + p^m x_0$, we have $\phi_0\pi(x) = \phi_0\pi(p^m x_0)$ as $x_1 \in A$. So $\phi_0\pi(x) = \phi_0(p^m x_0)$, since $p^m G \leq H$, as noted above. Thus $\phi_0\pi(x) = p^m\phi_0(x_0) = p^m y_0$. We also have $\phi_1(1-\pi)(x) = \phi_1(1-\pi)(x_1 + p^m x_0) = \phi_1(1-\pi)(x_1)$ because $p^m x_0 \in H$, and this gives $\phi_1(1-\pi)(x) = \phi_1(x_1) = y_1$. Therefore $\phi(x) = y_1 + p^m y_0 = y$ with $\phi \in \mathrm{Proj}(G)$, as required. $\square$

Corollary 3.13. (i) *A separable group is projectively fully transitive;*
(ii) *if $p^\omega G \cong \mathbb{Z}(p^n)$ for some finite n, then G is projectively fully transitive;*
(iii) *if A is projectively fully transitive and B is separable, then $A \oplus B$ is projectively fully transitive.*

Proof. Part (i) follows immediately from Lemma 3.12. For (ii) observe that if $x, y \in p^\omega G$ with $U_G(x) \leq U_G(y)$, then it is easy to see that an integer multiple of the identity (and hence an endomorphism which is even in $\Pi(G)$) maps x to y; the result then follows from Lemma 3.12. For the final part, note that both A, B are fully transitive and the direct sum $A \oplus B$ is also fully transitive by [1, Proposition 2.6]. Hence $\{A, B\}$ is a fully transitive pair and the result follows from Proposition 3.2. $\square$

Note that it is not possible to extend part (ii) of Corollary 3.13 even to the situation where $p^\omega G = \mathbb{Z}(p) \oplus \mathbb{Z}(p)$; indeed, Megibben [19] has constructed an example of such a group which is not even fully transitive.

Recall that the notions of socle-regularity, strong socle-regularity have been introduced in [5,6]; these concepts were generalizations of full transitivity and transitivity respectively. In [8], a group G was said to be *projectively socle-regular* if for all projection-invariant subgroups P of G, there is an ordinal α (depending on P) such that $P[p] = (p^\alpha G)[p]$. Our next result shows that projective socle-regularity is likewise a generalization of projective full transitivity.

Proposition 3.14. *If G is projectively fully transitive, then G is projectively socle-regular; if $p \neq 2$, then G is strongly socle-regular.*

Proof. Suppose P is an arbitrary projection-invariant subgroup of G and $\alpha = \min\{ht_G(z) : z \in P[p]\}$, so that $P[p] \leq (p^\alpha G)[p]$. Choose $x \in P[p]$ of height exactly α so that $U_G(x) = (\alpha, \infty, \dots)$. Let $y \in (p^\alpha G)[p]$ be arbitrary; then $U_G(y) = (\beta, \infty, \dots)$ where $\beta \geq \alpha$. Since G is projectively fully transitive there is a $\phi \in \mathrm{Proj}(G)$ such that $\phi(x) = y$. But because ϕ is a linear combination of products of idempotents and P is projection invariant, we have that $y = \phi(x) \in P[p]$. Since y was arbitrary in $(p^\alpha G)[p]$, we deduce that $(p^\alpha G)[p] \leq P[p]$ and hence we have the desired equality. The final conclusion follows immediately from Proposition 1.5 in [8] once we have that G is projectively socle-regular. $\square$

We now consider subgroups of projectively fully transitive groups. We begin with the elementary:

Proposition 3.15. *If G is projectively fully transitive, then $p^\beta G$ is projectively fully transitive for all ordinals β.*

Proof. Let $H = p^\beta G$ and observe that if $x, y \in H$ with $U_H(x) \leq U_H(y)$, then $U_G(x) \leq U_G(y)$. So there is a $\phi \in \mathrm{Proj}(G)$ with $\phi(x) = y$. However, as H is fully invariant in G, it is easy to see that if $\phi \in \mathrm{Proj}(G)$, then $\phi \restriction H \in \mathrm{Proj}(H)$. $\square$

For *finite* ordinals β it is easy to establish the converse:

Proposition 3.16. *If $p^n G$ is projectively fully transitive for some finite n, then G is projectively fully transitive.*

Proof. By induction it suffices to establish the result for pG, so let $H = pG$. Furthermore, by Lemma 3.12 it suffices to show that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$. So let $x, y \in p^\omega G$ with $U_G(x) \leq U_G(y)$. Note that $x, y \in p^\omega G = p^\omega H$ since $p^\omega H = p^{1+\omega} G = p^\omega G$. Consider $U_H(x) = (\alpha_0, \alpha_1, \dots)$, say. Since $x \in p^\omega H$, each $\alpha_i \geq \omega$ and then $p_i^\alpha H = p_i^\alpha G$, so that $U_H(x) \leq U_H(y)$. Since H is, by assumption, projectively fully transitive there is a $\phi \in \mathrm{Proj}(H)$ with $\phi(x) = y$. It follows from Theorem 1.11 in [8] that every idempotent in $\mathrm{E}(H)$ lifts to an idempotent in $\mathrm{E}(G)$ and so every element of $\mathrm{Proj}(H)$ lifts to an element of $\mathrm{Proj}(G)$. In particular, ϕ lifts to an element $\psi \in \mathrm{Proj}(G)$ with $\psi(x) = y$. □

If we wish to extend Proposition 3.16 to ordinals $\beta \geq \omega$, it seems inevitable that we must introduce some restriction on the quotient $G/p^\beta G$: we know from the proof of Proposition 3.5 that there is a group G such that $p^\omega G$ is an elementary group of infinite rank (and hence projectively fully transitive) but G is not projectively fully transitive. An obvious restriction is to assume that the quotient $G/p^\beta G$ is totally projective. We begin by examining the situation when $\beta = \omega$.

Lemma 3.17. *If $G/p^\omega G$ is a direct sum of cyclic groups and $p^\omega G$ is projectively fully transitive, then G is projectively fully transitive.*

Proof. We show that $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$. Let $x, y \in p^\omega G$ with $U_G(x) \leq U_G(y)$. Since for any $g \in p^\omega G$, $ht_G(g) = \omega + ht_{p^\omega G}(g)$, we have $U_{p^\omega G}(x) \leq U_{p^\omega G}(y)$. By assumption there is a $\phi \in \mathrm{Proj}(p^\omega G)$ with $\phi(x) = y$. It follows from Theorem 11 in [14] that every idempotent in $\mathrm{E}(p^\omega G)$ lifts to an idempotent in $\mathrm{E}(G)$, so the mapping ϕ lifts to a mapping $\psi \in \mathrm{Proj}(G)$ and $\psi(x) = y$. Thus $\mathrm{E}(G)$ acts projectively fully transitively on $p^\omega G$ and, in virtue of Lemma 3.12, G is projectively fully transitive as required. □

Theorem 3.18. *Suppose that α is an ordinal strictly less than ω^2 and $G/p^\alpha G$ is totally projective. If $p^\alpha G$ is projectively fully transitive, then so also is G.*

Proof. The proof is by induction; if $\alpha \leq \omega$ we have already established the result in Proposition 3.16 and Proposition 3.17. So suppose that the result is true for all ordinals $< \alpha$. There are two possibilities: either α is a limit of cofinality ω or $\alpha = \beta + 1$ for some β.

Consider firstly the case $\alpha = \beta + 1$ for some β. Set $X = p^\beta G$ and note that $pX = p^\alpha G$ is projectively fully transitive. It follows from Proposition 3.16 that X is projectively fully transitive. Moreover, $G/p^\beta G \cong (G/p^\alpha G)/(p^\beta G/p^\alpha G) \cong (G/p^\alpha G)/p^\beta(G/p^\alpha G)$ and hence $G/p^\beta G$ is totally projective by a well-known result of Nunke—see, e.g., [10, Exercise 82.3]. So by our induction hypothesis we conclude that G is projectively fully transitive.

In the limit case $\alpha = \beta + \omega$ for some β. Set $X = p^\beta G$ so that $p^\omega X = p^\alpha G$ is projectively fully transitive. Now $X/p^\omega X \cong p^\beta G/p^\alpha G$ is totally projective again by the aforementioned Nunke's result. It follows from Proposition 3.17 that $X = p^\beta G$ is projectively fully transitive. Since $G/p^\beta G$ is totally projective and $\beta < \alpha$, the induction hypothesis gives us that G is projectively fully transitive. □

Corollary 3.19. (i) *If G is totally projective of length $\leq \omega^2$, then G is projectively fully transitive;*

(ii) *if λ is cofinal with ω and G is a C_λ-group of length $\lambda \leq \omega^2$, then G is projectively fully transitive.*

Proof. (i) If G is totally projective of length $< \omega^2$, then the result follows immediately from Theorem 3.18 above. If G has length ω^2, then G is actually a direct sum of totally projective groups of length $< \omega^2$, say $G = \bigoplus_{i \in I} G_i$ where $l(G_i) < \omega^2$ for each $i \in I$. If $x, y \in G$ and $U_G(x) \leq U_G(y)$, then there is a finite set $\{i_1, \ldots, i_n\} \subseteq I$ such that $x, y \in H = \bigoplus_{j=1}^{n} G_{i_n}$; moreover, $U_H(x) = U_G(x) \leq U_G(y) = U_H(y)$. If we can show that H is projectively fully transitive, then we have a mapping $\phi \in \mathrm{Proj}(H)$ with $\phi(x) = y$. If $G = H \oplus K$ and we set $\psi = \phi \oplus 0_K$, then it is easy to see that $\psi \in \mathrm{Proj}(G)$ and $\psi(x) = y$. Thus to establish part (i) it suffices to show that H is projectively fully transitive.

Now each G_{i_j} is totally projective of length $< \omega^2$, so each G_{i_j} is projectively fully transitive. Moreover, given any i_1, i_2, the sum $G_{i_1} \oplus G_{i_2}$ is totally projective and hence fully transitive, i.e. $\{G_{i_1}, G_{i_2}\}$ is a fully transitive pair and hence it follows from Proposition 3.2 that $G_{i_1} \oplus G_{i_2}$ is projectively fully transitive. A simple induction now yields the desired result that H is projectively fully transitive. (This argument is presented in a more formalized way in Corollary 4.7 below).

(ii) If G is a C_λ-group of length λ cofinal with ω and $x, y \in G$ with $U_G(x) \leq U_G(y)$, let $H = \langle x, y \rangle$. Since H is a finite group and λ is a limit ordinal, there is an ordinal $\alpha < \lambda$ such that $H \cap p^\alpha G = \{0\}$. Then it follows from [17, Proposition 4] that G decomposes as $G = A \oplus K$ where A is totally projective of length $< \lambda$ and $x, y \in A$. Since $U_A(x) = U_G(x) \leq U_G(y) = U_A(y)$ and A is projectively fully transitive by part (i), we have an endomorphism $\phi \in \mathrm{Proj}(A)$ with $\phi(x) = y$. But then an identical argument to that in the proof of part (i) gives a mapping $\psi \in \mathrm{Proj}(G)$ with $\psi(x) = y$. Thus G is projectively fully transitive as required. □

4. Strongly Projectively Fully Transitive Groups

Recall from the introduction that a group G is said to be *strongly projectively fully transitive* if, given $x, y \in G$ with $U_G(x) \leq U_G(y)$, there exists $\phi \in \Pi(G)$ with $\phi(x) = y$; clearly a strongly projectively fully transitive group is projectively fully transitive. We pointed out in the final paragraph of the proof of Lemma 3.12 the difficulty in extending that result to strongly projectively fully transitive groups. We can, however, obtain the corresponding result by taking a little more care. The notion of acting strongly projectively fully transitively on $p^\omega G$ is analogous to that acting projectively fully transitively: specifically, a subgroup Φ of $\mathrm{End}(G)$ acts *strongly projectively fully transitively* on a subgroup X of G if, given $x, y \in X$ with $U_G(x) \leq U_G(y)$, there is an endomorphism $\phi \in \Phi$ such that $\phi(x) = y$ and ϕ belongs to the subgroup of Φ additively generated by the idempotents in Φ.

Lemma 4.1. *A reduced group G is strongly projectively fully transitive if, and only if,* End(G) *acts strongly projectively fully transitively on $p^\omega G$.*

Proof. The sufficiency is trivial, so assume that End(G) acts strongly projectively fully transitively on $p^\omega G$. Our arguments follow exactly those described in the proof of Lemma 3.12 and we use the same notation as in that lemma. Observe firstly that the hypothesis that End(G) acts strongly projectively fully transitively on $p^\omega G$ means that ϕ_0 can be chosen to be in $\Pi(G)$. Now consider the endomorphism θ of the bounded group A. As noted in the proof of Lemma 3.12, we may use Hausen's result [12, Corollary 6] to choose $\theta \in \Pi(A)$. We now extend θ to an endomorphism of G taking a little more care than in the previous proof.

If ε is an idempotent endomorphism of the direct summand A, where $G = A \oplus H$, then we can extend ε to an endomorphism ε^* by setting $\varepsilon^* = \varepsilon \oplus 0_H$. Note that ε^* is then an idempotent endomorphism of G and, if π is the canonical projection of G onto H along A, we have $\varepsilon^*(1-\pi)(H) = 0 = \varepsilon^*(H)$, while $(\varepsilon^*(1-\pi))(a) = \varepsilon^*(a)$ for all $a \in A$. Consequently $\varepsilon^*(1-\pi) = \varepsilon^*$. Applying this method of extension to the map $\theta \in \Pi(A)$ we get an endomorphism $\theta_1 \in \Pi(G)$ and $\theta_1(1-\pi) = \theta_1$.

Returning to the proof of Lemma 3.12, we note that the mapping ψ, where $\psi(x_1) = h_1$, belongs to $\Pi(G)$ and by construction it satisfies $\psi = \psi(1-\pi)$. Therefore the map $\phi_1 = \theta_1 + \psi$ also belongs to $\Pi(G)$ and satisfies $\phi_1 = \phi_1(1-\pi)$.

In the final paragraph of the proof of Lemma 3.12 it is shown that the map $\phi = \phi_0\pi + \phi_1(1-\pi)$ has the desired property that $\phi(x) = y$. However, if we now define a new map $\phi^* = \phi_0 + \phi_1$, then certainly $\phi^* \in \Pi(G)$ since both $\phi_0, \phi_1 \in \Pi(G)$. But $G = A \oplus H$ and A is bounded, so $p^\omega G \leq H$ and hence $\phi_0\pi(x) = \phi_0(x)$ as $x \in p^\omega G$.

Moreover, as we noted above, $\phi_1 = \phi_1(1-\pi)$ and so $\phi^*(x) = \phi(x) = y$, as required. □

Corollary 4.2. (i) *If B is a separable group, then B is strongly projectively fully transitive;*

(ii) *if A is strongly projectively fully transitive and B is separable, then $A \oplus B$ is strongly projectively fully transitive;*

(iii) *if $p^\omega G \cong \mathbb{Z}(p^n)$ for some finite n, then G is strongly projectively fully transitive.*

Proof. Point (i) is immediate from the previous result, and (ii) follows immediately from part (i) and Proposition 3.2—recall our observation at the end of the proof of Proposition 3.2. The final part follows by an identical argument to that used in Corollary 3.13 (ii). □

We remark that it is possible to prove directly (i.e. without invoking Lemma 4.1) that a separable group is strongly projectively fully transitive: the argument utilizes Lemma 65.5 in [10].

Although a separable group is necessarily strongly projectively fully transitive, it does not follow that it is an IS-group; recall from Corollary 2.7 that a separable essentially indecomposable group need not be even an IG-group.

Since Proposition 3.2 and Proposition 3.3 carry over unchanged to strongly projectively fully transitive groups, we see that a group $G = D \oplus R$, with D divisible and R reduced, is strongly projectively fully transitive if, and only if, D, R are both strongly projectively fully transitive.

In fact we derive:

Theorem 4.3. *A group $G = D \oplus R$, where D is divisible and R is reduced, is strongly projectively fully transitive if, and only if, R is strongly projectively fully transitive.*

Proof. By the preceding observation it is clearly enough to show that any divisible group is strongly projectively fully transitive. If D is of rank one then the result follows from the proof of Theorem 3.1: just observe that the mapping used to send the element x to y was an integer multiple of the identity. If D is of finite rank then the result follows from Proposition 2.1 (ii) and the fact that a divisible group is always fully transitive. Finally, if D has infinite rank, the result follows from Proposition 4.6 below. □

Corollary 4.4. *A divisible group is strongly projectively fully transitive.*

The following somewhat combines Corollaries 4.2 (iii) and 4.4 into a more general case.

Proposition 4.5. *Let G be a group such that $p^\omega G$ is the direct sum of a divisible group and a cyclic group of order p^n for some $n \in \mathbb{N}$. Then G is strongly projectively fully transitive.*

Proof. One may decompose $G = D \oplus C$ where D is divisible and $p^\omega C \cong \mathbb{Z}(p^n)$. In fact, $p^{\omega+n}G$ is the maximal divisible part in $p^\omega G$, so that $p^\omega G = p^{\omega+n}G \oplus R$ where $R \cong \mathbb{Z}(p^n)$. But $G = p^{\omega+n}G \oplus C$ for some group C, and hence $C \cong G/p^{\omega+n}G$ and $p^\omega C \cong p^\omega(G/p^{\omega+n}G) = p^\omega G/p^{\omega+n}G \cong R$. This substantiates our claim. Furthermore, we apply a combination of Theorem 4.3 and Corollary 4.2 (iii) to deduce that G is strongly projectively fully transitive, as asserted. □

Proposition 4.6. *If the group $G^{(n)}$ is strongly projectively fully transitive for every finite n, then $H = G^{(\kappa)}$ is strongly projectively fully transitive for any infinite cardinal κ.*

Proof. If $x, y \in H$ with $U_H(x) \leq U_H(y)$, then there exists a finite integer m such that $x, y \in H_m = G^{(m)}$. Now $U_{H_m}(x) = U_H(x)$ and similarly for y, so there is a mapping $\phi \in \Pi(H_m)$ with $\phi(x) = y$. However ϕ can be expressed as a linear combination of idempotents in $\mathrm{E}(H_m)$ and each of these may be extended trivially to an idempotent of H by acting as the zero map on the canonical complement. The resulting sum is a map $\psi \in \Pi(H)$ with $\psi(x) = y$. Thus H is strongly projectively fully transitive, as required. □

In fact the argument in Proposition 4.6 easily generalizes to give:

Corollary 4.7. *If* $G_i (i \in I)$ *is a collection of groups with the property that* $\bigoplus_{i\in J} G_i$ *is strongly projectively fully transitive (respectively projectively fully transitive) for every finite subset* $J \subseteq I$*, then we have that* $\bigoplus_{i\in I} G_i$ *is strongly projectively fully transitive (respectively projectively fully transitive).*

Next, we record some crucial properties of strongly projectively fully transitive groups; the proofs of these results follow by identical arguments to those used for the corresponding results on projectively fully transitive groups; the proof of part (v) follows from part (iv) and Proposition 4.6.

Theorem 4.8. (i) *If G is strongly projectively fully transitive, then $p^\beta G$ is strongly projectively fully transitive for all ordinals β;*

(ii) *if $p^n G$ is strongly projectively fully transitive for some finite n, then G is strongly projectively fully transitive;*

(iii) *if α is an ordinal strictly less than ω^2 and $G/p^\alpha G$ is totally projective, then if $p^\alpha G$ is strongly projectively fully transitive, so also is G;*

(iv) *if A, B are strongly projectively fully transitive and $\{A, B\}$ is a fully transitive pair, then $A \oplus B$ is strongly projectively fully transitive;*

(v) *if G is strongly projectively fully transitive, then $G^{(\kappa)}$ is strongly projectively fully transitive for any cardinal κ;*

(vi) *if G is totally projective of length $\leq \omega^2$, then G is strongly projectively fully transitive;*

(vii) *if λ is cofinal with ω and G is a C_λ-group of length $\lambda \leq \omega^2$, then G is strongly projectively fully transitive.*

An easy consequence of Corollary 4.7 is the following result which generalizes Proposition 3.2:

Proposition 4.9. *If G is a fully transitive group which is an arbitrary direct sum of (strongly) projectively fully transitive groups, then G is (strongly) projectively fully transitive.*

In light of Theorem 3.7, one might expect a similar result with strongly projectively fully transitive groups replacing projectively fully transitive groups. This seems to be difficult and the best we can offer is the following.

Proposition 4.10. *If $p^\omega G$ is an elementary group, then G is fully transitive if, and only if, $G \oplus G$ is strongly projectively fully transitive.*

Proof. Sufficiency is immediate since summands of fully transitive groups are fully transitive; in fact there is no need for the additional hypothesis on $p^\omega G$ for this argument. Conversely suppose that G is fully transitive and $p^\omega G$ is elementary. Let $H = G \oplus G$ and consider any elements $(a, b), (c, d)$ in $p^\omega H$. Consider firstly the situation where $a, b \neq 0$. Since all the elements of $p^\omega G$ have the same Ulm sequence $(\omega, \infty, \dots)$ in G, there are endomorphisms $\gamma : b \mapsto c$ and $\delta : a \mapsto d$. The matrix $\Delta = \left(\begin{smallmatrix} 0 & \gamma \\ \delta & 0 \end{smallmatrix}\right)$ represents an endomorphism of H which maps (a, b) to (c, d), but $\Delta = \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and each of these matrices is idempotent, so that $\Delta \in \Pi(H)$.

If one of $a, b = 0$ (the situation where both are zero is trivial), then we may assume without loss that $a \neq 0, b = 0$. As before, we have the endomorphisms of G that are $\alpha : a \mapsto c$, $\delta : a \mapsto d$. Now the matrix $\Delta = \left(\begin{smallmatrix} \alpha & \alpha \\ \delta & 1-\alpha \end{smallmatrix}\right)$ represents an endomorphism of H and maps $(a, 0)$ to (c, d). However, $\Delta = \left(\begin{smallmatrix} \alpha & \alpha \\ 1-\alpha & 1-\alpha \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 0 \\ \delta+\alpha-1 & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and direct calculation gives that each of these matrices is idempotent. Thus $\Delta \in \Pi(H)$.

It follows immediately from Lemma 4.1 that $H = G \oplus G$ is strongly projectively fully transitive. □

Remark 4.11. *The condition that $p^\omega G$ be elementary in Proposition 4.10 is far from necessary. For instance, if C is a bounded group and G is a group with $p^\omega G = C$ constructed via Corner's Theorem 6.1 in [3], with* $\mathrm{E}(G)$ *acting on $p^\omega G$ in the same manner as the full endomorphism ring* $\mathrm{E}(C)$*, then G is certainly fully transitive and $H = G \oplus G$ is strongly projectively fully transitive. To see the latter, observe that if $(x, y), (u, v) \in p^\omega H$ with $U_H((x, y)) \leq U_H((u, v))$, then we can assume without loss that $U_G(x) \leq U_G(y)$ and $U_G(u) \leq U_G(v)$. By the full transitivity of G we have endomorphisms $\gamma : x \mapsto u$, $\delta : x \mapsto v$ and $\gamma \restriction p^\omega G \in$* $\mathrm{E}(C)$*. However, $\left(\begin{smallmatrix} 0 & 0 \\ \delta & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are both idempotents and so the sum $\Delta_1 = \left(\begin{smallmatrix} 0 & 0 \\ \delta & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Pi(H)$. Since $\gamma \restriction p^\omega G$ can be written as a sum of idempotents in* $\mathrm{E}(C)$*, say $\gamma \restriction p^\omega G = \pi_1 + \dots + \pi_n$, then we obtain from Corner's construction that there are idempotents $e_1, \dots, e_n \in$* $\mathrm{E}(G)$ *with $e_i \restriction p^\omega G = \pi_i$ and $\gamma = e_1 + \dots + e_n$. The matrices $\left(\begin{smallmatrix} e_i & 0 \\ 0 & 0 \end{smallmatrix}\right)$ are again idempotents in* $\mathrm{E}(H)$ *and if $\Delta_2 = \left(\begin{smallmatrix} e_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \dots + \left(\begin{smallmatrix} e_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$, then it is immediate that $\Delta = \Delta_1 + \Delta_2 \in \Pi(H)$ and Δ maps $(x, y) \mapsto (u, v)$, as required.*

Finally, we note that, similar to the situation for projectively fully transitive groups, a summand of a strongly projectively fully transitive group need not be strongly projectively fully transitive, even when the first Ulm subgroup is elementary.

Proposition 4.12. *There is a non-strongly projectively fully transitive group G, with elementary first Ulm subgroup, such that $G \oplus G$ is strongly projectively fully transitive.*

Proof. Let G be a fully transitive group as constructed in either part (i) or part (ii) of Proposition 3.5 above; note that in either case the first Ulm subgroup of G is an elementary group. It follows immediately from Proposition 4.10 that $G \oplus G$ is strongly projectively fully transitive. However, as pointed out in the proof of Proposition 3.5, neither group G is even projectively fully transitive. □

The reader will have noted that we have not shown that the classes of projectively fully transitive and strongly projectively fully transitive groups are distinct. This seems to be reasonably difficult, so we pose:

Problem 2. Find a projectively fully transitive group which is not strongly projectively fully transitive.

We finish the paper with a further question; we believe that an answer to this question will shed further light on the nature of projective and strong projective full transitivity.

Problem 3. Are reduced totally projective groups (in particular, reduced countable groups) necessarily (strongly) projectively fully transitive?

In closing we also state the following specification:

Remark. Question 2.2 from [7] has obviously a negative solution. In fact, every Krylov transitive group G such that all elements of $p^{\omega}G$ have comparable Ulm sequences, is fully transitive. To show that, we apply subsequently the first part of Theorem 2.13 and Corollary 2.8 again from [7]. Thus $G \oplus G$ has to be fully transitive, whence so is G as being a direct summand, as asserted.

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Received: August 29, 2011.
Accepted: April 24, 2012.