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On ω_1-n-simply presented abelian p-groups

Peter Danchev
Department of Mathematics, Plovdiv University
"P. Hilendarski", Plovdiv 4000, Bulgaria
pvdanchev@yahoo.com

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We define the class of ω_1-n-simply presented abelian p-groups which class properly contains the subclasses of n-simply presented groups and ω_1-$p^{\omega+n}$-projective groups, respectively. Some comprehensive results in this new way are also established. This substantially strengthens achievements due to both Keef–Danchev [On n-simply presented primary abelian groups, *Houston J. Math.* **38**(4) (2012) 1027–1050] and Keef [On ω_1-$p^{\omega+n}$-projective primary abelian groups, *J. Algebra Number Theory Acad.* **1**(1) (2010) 41–75].

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1. Introduction and Main Definitions

Throughout this paper, let all groups into examination be p-torsion abelian written additively as is the custom when discussing such groups. Also, let $n \geq 0$ be a non-negative integer. Most of the used notions and notations are standard and can be seen in the classical sources [9–11]. For the more specific terminology the interested reader can read [14, 15]. For instance, we will abbreviate G as a *dsc* group if it is a *direct sum of countable* groups. Besides, imitating [14] (see [15] too) a group G is called *n-simply presented* if there is $P \leq G[p^n]$ such that G/P is simply presented. When P is nice in G, such groups are said to be *strongly n-simply presented* or *nicely n-simply presented*. The last is a common generalization of the well-known concept of $p^{\omega+n}$-projectivity due to Nunke where G is *$p^{\omega+n}$-projective* whenever there exists a p^n-bounded subgroup $P \leq G$ such that G/P is *Σ-cyclic* (= a direct sum of cyclics). Later on, Keef enlarged in [13] that notion to the so-called *ω_1-$p^{\omega+n}$-projective* groups that are groups G for which there exist countable (nice) subgroups C such that G/C are $p^{\omega+n}$-projective.

This paper is an extension of n-simply presented groups in the spirit of (the previous generalizations of) ω_1-$p^{\omega+n}$-projective groups. It is organized as follows: In this section, we put the main definitions. In Sec. 2, we prove some useful preliminary assertions and state some background material, and in Sec. 3 we state with proofs the major results in the subject. Next, in Sec. 4, we prove a series of statements concerning the important Nunke-esque property, and we close in Sec. 5 with some unsettled challenging questions.

Definition 1. The group G is called *ω_1-n-simply presented* if there is a countable subgroup K of G such that G/K is n-simply presented. In addition, if K is finite, G is said to be *ω-n-simply presented*.

When $n = 0$, and as a result G/K is simply presented, we will just say that G is *ω_1-simply presented*. But if K is nice in G, G is just simply presented (see [1] or [8]). Likewise, ω-n-simply presented groups are precisely n-simply presented.

When K is a priory chosen nice in G, one may state the following.

Definition 2. The group G is called *nicely ω_1-n-simply presented* if there is a countable nice subgroup N of G such that G/N is n-simply presented.

When $n = 0$, and hence G/N is simply presented, we observe with the aid of [1] or [8] that G must be simply presented, too.

Definition 3. The group G is said to be *strongly ω_1-n-simply presented* if there exists a countable subgroup C of G such that G/C is strongly n-simply presented. In addition, if C is finite, we will say that G is *strongly ω-n-simply presented*.

In case that C is taken a priory nice in G, one can state the following.

Definition 4. The group G is said to be *strongly nice ω_1-n-simply presented* if there exists a countable nice subgroup M of G such that G/M is strongly n-simply presented.

Apparently, because $p^{\omega+n}$-projective groups are strongly n-simply presented, the ω_1-$p^{\omega+n}$-projectives, defined as in [13], are themselves strongly nice ω_1-n-simply presented. Moreover, strongly ω-n-simply presented groups are strongly nice ω_1-n-simply presented, because finite subgroups are always nice. As indicated in [14], strongly ω-n-simply presented groups need not be strongly n-simply presented.

Also, it is clear that Definition 4 yields Definition 2 and Definition 3 implies Definition 1.

Likewise, some enlargements of this kind for the n-totally projective groups from [15] can be given as well.

Hill and Megibben gave in [12] the definition of *a c.c. group* as a group G such that $p^\omega(G/C)$ is countable whenever $C \leq G$ is a countable subgroup. For our applicable purposes we shall now enlarge this concept to the so-called *α-countably* *groups* where α is an arbitrary ordinal. This is necessary because the approach used in [14] does not work here because $p^{\alpha+n}(G/C)$ is not always contained in $(p^\alpha G + C)/C$ if $p^n C \neq \{0\}$.

Definition 5. We will say that the group G is *α-countably* if for any of its countable subgroup C the factor-group $p^\alpha(G/C)/(p^\alpha G + C)/C$ is always countable.

Note also that if either C is a nice subgroup or $\alpha \in \mathbb{N}$, the factor-group $p^\alpha(G/C)/(p^\alpha G + C)/C$ equals to zero, and so Definition 5 is satisfied in both situations.

When $\alpha = \omega$, the posed condition is equivalent to the countability of the quotient $[\bigcap_{i<\omega}(p^i G + C)]/[\bigcap_{i<\omega} p^i G]$ which in turn is tantamount to the countability of the quotient $[\bigcap_{i<\omega}(p^i G + C)]/[(\bigcap_{i<\omega} p^i G) + C]$. Apparently, c.c. groups are always ω-countably. To treat the converse relationship, one sees that if $p^\omega G$ is countable, then every ω-countably group is a c.c. group, and thus these two notions do coincide. In particular, weakly ω_1-separable groups (which are of necessity separable), are ω-countably as well as ω-countably separable groups are weakly ω_1-separable.

Definition 6. We will say that the group G is *α-boundary* if for any of its countable subgroup C the factor-group $p^\alpha(G/C)/[(p^\alpha G + C)/C]$ is always bounded.

In particular, there is a natural number m such that the inclusion $p^{\alpha+m}(G/C) \subseteq (p^\alpha G + C)/C$ holds.

Note also that if either C is a nice subgroup or $\alpha \in \mathbb{N}$ then the quotient $p^\alpha(G/C)/[(p^\alpha G + C)/C]$ equals to zero, as well as if $p^m C = \{0\}$ then the inclusion $p^{\alpha+m}(G/C) \subseteq (p^\alpha G + C)/C$ holds appealing to [14, Lemma 3.1], and thus in all cases Definition 6 is fulfilled.

2. Preliminaries and Backgrounds

The following two technicalities possess a central role.

Lemma 2.1. *Suppose that α is an ordinal, and that G and F are groups where F is finite. Then the following formula is fulfilled*:

$$p^\alpha(G+F) = p^\alpha G + [F \cap p^\alpha(G+F)] \subseteq p^\alpha G + F.$$

Proof. We will use a transfinite induction on α. First, if $\alpha - 1$ exists, we have

$$\begin{aligned} p^\alpha(G+F) &= p(p^{\alpha-1}(G+F)) = p(p^{\alpha-1}G + [F \cap p^{\alpha-1}(G+F)]) \\ &= p(p^{\alpha-1}G) + p(F \cap p^{\alpha-1}(G+F)) \subseteq p^\alpha G + [F \cap p(p^{\alpha-1}(G+F))] \\ &= p^\alpha G + [F \cap p^\alpha(G+F)]. \end{aligned}$$

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha - 1$ does not exist, we have that $p^\alpha(G+F) = \bigcap_{\beta<\alpha}(p^\beta(G+F)) \subseteq \bigcap_{\beta<\alpha}(p^\beta G + F) = \bigcap_{\beta<\alpha} p^\beta G + F = p^\alpha G + F$. In fact, the second sign "$=$" follows like this: Given $x \in \bigcap_{\beta<\alpha}(p^\beta G + F)$, we write that $x = g_{\beta_1} + f_1 = \cdots = g_{\beta_s} + f_s = \cdots$ where $f_1, \ldots, f_s \in F$ are all the elements of F; $g_{\beta_1} \in p^{\beta_1}G, \ldots, g_{\beta_s} \in p^{\beta_s}G$ with $\beta_1 < \cdots < \beta_s < \cdots$.

Since F is finite, while the number of equalities is infinite due to the infinite cardinality of α, we infer that $g_{\beta_s} \in p^\beta G$ for any ordinal $\beta < \alpha$ which means that $g_{\beta_s} \in \bigcap_{\beta<\alpha} p^\beta G = p^\alpha G$. Thus $x \in \bigcap_{\beta<\alpha} p^\beta G + F = p^\alpha G + F$, as claimed. Furthermore, $p^\alpha(G+F) \subseteq (p^\alpha G + F) \cap p^\alpha(G+F) = p^\alpha G + [F \cap p^\alpha(G+F)]$ which is obviously equivalent to an equality. □

Lemma 2.2. *Let N be a nice subgroup of a group G. Then*:

(i) *$N + R$ is nice in G for every finite subgroup $R \leq G$;*
(ii) *N is nice in $G + F$ for each finite group F.*

Proof. (i) For any limit ordinal γ, we deduce that $\bigcap_{\delta<\gamma}(N + R + p^\delta G) \subseteq R + \bigcap_{\delta<\gamma}(N + p^\delta G) = R + N + p^\gamma G$, as required. Indeed, the relation "$\subseteq$" follows like this: Given $x \in \bigcap_{\delta<\gamma}(N+R+p^\delta G)$, we write $x = a_1 + r_1 + g_1 = \cdots = a_s + r_s + g_s = \cdots = a_k + r_1 + g_k = \cdots$, where $a_1, \ldots, a_k \in N$; $r_1, \ldots, r_k \in R$; $g_1 \in p^{\delta_1}G, \ldots, g_k \in p^{\delta_k}G$ with $\delta_1 < \cdots < \delta_k$. So $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \bigcap_{\delta<\gamma}(N + p^\delta G)$ and hence $x \in R + \bigcap_{\delta<\gamma}(N + p^\delta G)$, as requested.

(ii) Since N is nice in G, we may write $\bigcap_{\delta<\gamma}[N + p^\delta G] = N + p^\gamma G$ for every limit ordinal γ. Furthermore, with Lemma 2.1 at hand, we subsequently deduce that

$$\begin{aligned}\bigcap_{\delta<\gamma}[N + p^\delta(G+F)] &= \bigcap_{\delta<\gamma}[N + p^\delta G + (F \cap p^\delta(G+F))] \\ &\subseteq \bigcap_{\delta<\gamma}(N + p^\delta G) + [F \cap p^\gamma(G+F)] \\ &= N + p^\gamma G + [F \cap p^\gamma(G+F)] = N + p^\gamma(G+F).\end{aligned}$$

In fact, the inclusion "$\subseteq$" follows thus: Given $x \in \bigcap_{\delta<\gamma}[N + p^\delta G + (F \bigcap p^\delta(G+F))]$, we write $x = a_1 + g_1 + f_1 = \cdots = a_s + g_s + f_s = \cdots = a_k + g_k + f_1 = \cdots$, where $a_1, \ldots, a_k \in N$; $g_1 \in p^{\delta_1}G, \ldots, g_k \in p^{\delta_k}G$; $f_1 \in F \bigcap p^{\delta_1}(G+F), \ldots, f_k \in F \bigcap p^{\delta_k}(G+F)$ with $\delta_1 < \cdots < \delta_k$. Hence $a_1 + g_1 = \cdots = a_k + g_k = \cdots \in \bigcap_{\delta<\gamma}(N + p^\delta G)$ and because the number of the f_i's $(1 \leq i \leq k)$ is finite whereas the number of equalities is not, we can deduce that $f_1 \in \bigcap_{\delta<\gamma}(F \bigcap p^\delta(G+F)) = F \bigcap p^\gamma(G+F)$, as needed. □

Before continue, we pause for the following mere observations:

It was pointed out in [14] that G is (strongly) n-simply presented if and only if $p^i G$ is (strongly) n-simply presented whenever $i \in \mathbb{N}$ — see also [15, Lemma 1.3]. Moreover, it was proved in [14, Proposition 5.2] that if $f : G \to A$ is an ω_1-bijection, that is a (bijective) homomorphism whose kernels and co-kernels are both countable, then G being n-simply presented implies the same for A. In particular, if G

is n-simply presented, then G/H is n-simply presented whenever H is a countable subgroup. However, the converse does not hold; nevertheless it could be true provided in addition that H is nice in G — compare with Corollary 3.3 (for more details about that type of results the reader can see [8] too). When $n = 0$, i.e. in the case of simply presented groups, this is fulfilled (see [6, 8]); however for any $n \geq 1$ we have doubts about its validity; thus will exist a nicely ω_1-n-simply presented group (with uncountable first Ulm subgroup) which is not n-simply presented — compare with Theorem 3.2 stated in the sequel.

We proceed in this way with the following particular case of the aforementioned result from [14]; nevertheless we give a more conceptual and easy proof needed for applicable purposes.

Lemma 2.3. *If T is n-simply presented and G/T is countable, then G is n-simply presented.*

Proof. Write $G = T + K$ where K is countable. With [14] at hand, there exists $P \leq T[p^n]$ such that T/P is simply presented. Furthermore, $G/P = (T/P) + (K + P)/P$ where $(K + P)/P \cong K/(K \cap P)$ is countable. Thus [6, Theorem 2.4] can be successfully employed to show that G/P is simply presented, as required. □

Likewise, it was only pointed out in [14] without a proof that if $\varphi : G \to A$ is an ω-bijective homomorphism, that is, a homomorphism whose kernel and co-kernel are finite, then G is n-simply presented if and only if A is n-simply presented — we shall give a suitable confirmation to this fact below. Besides, if G is strongly n-simply presented, then A is strongly n-simply presented, whereas the converse is not true as some concrete examples from [14] show; compare also with the comments given below.

However, the defined above new group classes inherit ω-bijections as we will prove in the sequel (for another treatment see also [2]). First, one more preliminary claim is needed (see also [15, Lemma 1.3] for a more general treatment considering bounded factors).

Lemma 2.4. *If S is a subgroup of a group G such that G/S is finite, then G is (strongly) n-simply presented if and only if S is (strongly) n-simply presented.*

Proof. Write $G = S + F$ where $F \leq G$ is finite.

Suppose first that S is strongly n-simply presented. With [14] in hand, there is $Z \leq S[p^n]$ which is nice in S such that S/Z is simply presented. We therefore have that $G/Z = [S/Z] + [(F + Z)/Z]$ where $(F + Z)/Z \cong F/(F \cap Z)$ is finite. By what we have remarked above, G/Z should be simply presented. But Z is nice in G utilizing Lemma 2.2(ii), as required.

Reciprocally, let G be strongly n-simply presented. Since $p^tG = p^tS$ for some $t \in \mathbb{N}$ and in [14] was established that any group A is strongly n-simply presented if and only if so is p^tA, one may derive that S is strongly n-simply presented.

Actually, this idea also provides a new verification of sufficiency considered above.

The same method works for n-simply presented groups as well. $\square$

Continuing this approach, we can state that if $G = S + B$, where $B \leq G$ is bounded, then G is (strongly) n-simply presented if and only if S is (strongly) n-simply presented. In addition, it seems that the same procedure does not work for G/S being bounded.

As a final comment we note that [13, Lemma 1.9] asserts that the following two conditions are equivalent for any class $\mathbb{K}$ of abelian groups:

(∗) Whenever S is a subgroup of a group G with G/S countable, then $G \in \mathbb{K}$ if and only if $S \in \mathbb{K}$.

(∗∗) Whenever C is a countable subgroup of G, then $G \in \mathbb{K}$ if and only if $G/C \in \mathbb{K}$.

In addition they are equivalent for $\mathbb{K}$ to be closed under taking ω_1-bijections.

However, there is no equivalence if "countable" is replaced by "finite". Indeed, suppose $\mathbb{K}$ coincides with the class of strongly n-simply presented groups. It was proved in Lemma 2.4 that if G/S is finite, then G is strongly n-simply presented if and only if S is, so that condition $(*)$ is satisfied. Nevertheless, the same cannot be said of $(**)$; namely if G is strongly n-simply presented, then so is G/F for any finite subgroup F, but G/F being strongly n-simply presented does not imply the same for G — in fact, as noticed in [14], taking $p^\omega G \cong \mathbb{Z}(p)$ and $G/p^\omega G$ to be $p^{\omega+n}$-projective, and hence strongly n-simply presented, it follows from [13, Example 2.3] that G is ω-$p^{\omega+n}$-projective but not $p^{\omega+n}$-projective.

So, it will follow that the class of such groups G is closed under the formation of ω-bijections exactly when point $(*)$ is fulfilled for finite subgroups. It should be better if points $(*)$ and $(**)$ are tantamount with the word "finite" in the claims; in other words they must hold together. It is noteworthy that $(**)$ always implies $(*)$; in fact if $G = S + F$ for some finite subgroup $F \leq G$, then $G/F = (S+F)/F \cong S/(S \cap F)$ where $S \cap F$ is finite, thus sustaining our affirmation.

So, we are now able to give the promised above proof of the following statement which is no longer true for *strongly* n-simply presented groups as indicated above — compare with Lemma 2.4 too.

Lemma 2.5. *A group G is n-simply presented if and only if G/F is n-simply presented for some finite subgroup F of G.*

Proof. As aforementioned, the "only if" direction was proved in [14, Proposition 5.2].

To treat the "if" one, write $(G/F)/(A/F) \cong G/A$ is simply presented for some $A \leq G$ such that $p^n A \subseteq F \subseteq A$. Since $p^n A$ is finite, it is a routine technical exercise to check that $A = L + A[p^n]$ for some finite $L \leq A$. Furthermore, $G/A \cong$

$(G/A[p^n])/(A/A[p^n])$ being simply presented with finite $A/A[p^n] \cong L/L[p^n]$ implies with the help of [1] or [6] or [8] that $G/A[p^n]$ is simply presented, as required. $\square$

3. Some Basic Results

This work is mainly inspired by [14, Proposition 5.3] and it is a significant generalization of the stated above concept in [13]. So, in this light, we begin this section with some different characterizations of ω_1-n-simply presented groups.

Theorem 3.1. *The following points are equivalent*:

(i) G *is* ω_1-n-*simply presented*;

(ii) $G/(C \oplus L)$ *is simply presented where* C *is a countable subgroup of* G *and* L *is a* p^n-*bounded subgroup of* G;

(iii) G/L *is* ω_1-*simply presented for some* $L \leq G[p^n]$.

Proof. (i) $\Leftrightarrow$ (ii) Foremost, letting (i) be fulfilled, given G/K is n-simply presented for some countable subgroup $K \leq G$. Thus there is A/K with $A \leq G$ and $p^nA \subseteq K$ such that G/A is simply presented. But it is well known that $A = C \oplus L$, and hence (ii) holds.

Conversely, assume that (ii) is true. Thus $G/(C \oplus L) \cong [G/C]/[(C \oplus L)/C]$ is simply presented, where $(C \oplus L)/C \cong L$ is p^n-bounded. Therefore, G/C is n-simply presented, as required.

(ii) $\Leftrightarrow$ (iii) First, assuming that (ii) is valid, we see that $G/(C \oplus L) \cong [G/L]/[(C \oplus L)/L]$ is simply presented where $(C \oplus L)/L \cong C$ is countable. So, G/L is ω_1-simply presented.

Reciprocally, let (iii) be true, so given G/L is ω_1-simply presented for some p^n-bounded subgroup L. Hence there is a countable subgroup B/L with $B \leq G$ such that $(G/L)/(B/L) \cong G/B$ is simply presented. Besides, $B = L + K$ for some countable $K \leq B$. Since $p^nL = \{0\}$, we write $L = L_1 \oplus L_2$ where L_2 is countable and $L \cap K \subseteq L_2$. Observe that $B = L_1 + (K + L_2)$ where L_1 is p^n-bounded and $K + L_2$ is countable. Moreover, $L_1 \cap (K + L_2) = \{0\}$; indeed take $a = b + c$ where $a \in L_1$, $b \in K$ and $c \in L_2$. Furthermore, $a - c \in L \cap K \subseteq L_2$, whence $a \in L_1 \cap L_2 = \{0\}$ and so $a = 0$. Finally, $B = L_1 \oplus (K + L_2)$ and thus $G/(C \oplus M)$ is simply presented for the countable $C = K + L_2$ and the p^n-bounded $M = L_1$, as stated. $\square$

Remark 1. Unfortunately there is no absolute analogy with the corresponding result from [13]. In fact, the reciprocal conditions $G \cong S/(C \oplus L)$ where S is simply presented, C is countable, L is p^n-bounded and $G \cong T/K$ where T is n-simply presented and K is countable, are obviously equivalent to the fact that G is n-simply presented by utilizing [14]. Similarly $G \cong V/Z$ for some ω_1-simply presented group V and its p^n-bounded subgroup Z is tantamount again to the fact that G is n-simply presented.

The next main result shows how nicely ω_1-n-simply presented groups differ from the n-simply presented ones. Specifically, the next statement somewhat extends both Corollary 4.7 and Proposition 5.3 of [14]. As we will see below, it actually shows that p^λ-bounded n-simply presented groups are closed under $\aleph_0$-nice elongations whenever $\lambda < \omega^2$.

Theorem 3.2. *Nicely ω_1-n-simply presented groups of length $< \omega^2$ are n-simply presented.*

Proof. Suppose that G/K is n-simply presented for some countable nice subgroup $K \leq G$. We will transfinite induction on length$(G) = \lambda < \omega^2$.

First, assume that $\lambda = \omega$. Hence Proposition 3.4 applies to derive that G is $p^{\omega+n}$-projective and hence n-simply presented.

Let us assume that the claim is true for groups of length $\leq \omega \cdot t$ for some $t \in \mathbb{N}$ and we shall prove it for groups G of length $\leq \omega \cdot (t+1) = \omega \cdot t + \omega$. Thus

$$(G/K)/p^{\omega \cdot t}(G/K) \cong G/(p^{\omega \cdot t}G + K) \cong [G/p^{\omega \cdot t}G]/[(p^{\omega \cdot t}G + K)/p^{\omega \cdot t}G]$$

is n-simply presented with the aid of [14, Theorem 3.4(a)]. Since $(p^{\omega \cdot t}G + K)/p^{\omega \cdot t}G \cong K/(K \cap p^{\omega \cdot t}G)$ is countable and nice in $G/p^{\omega \cdot t}G$, the induction hypothesis yields that $G/p^{\omega \cdot t}G$ is n-simply presented. In addition, $p^{\omega \cdot t}(G/K) \cong p^{\omega \cdot t}G/(p^{\omega \cdot t}G \cap K)$ is also n-simply presented owing again to [14, Theorem 3.4(a)]. However, $p^\omega(p^{\omega \cdot t}G) = p^{\omega \cdot t+\omega}G = \{0\}$ and $p^{\omega \cdot t}G \cap K$ is countable and nice in $p^{\omega \cdot t}G$. Consequently, $p^{\omega \cdot t}G/(p^{\omega \cdot t}G \cap K)$ is separable $p^{\omega+n}$-projective, whence $p^{\omega \cdot t}G$ is separable $p^{\omega+n}$-projective in accordance with step one given above. Finally, [14, Theorem 4.4] ensures that G is n-simply presented, as asserted.

Let us now consider the general case when length$(G) = \omega \cdot t + l$ with $t, l \in \mathbb{N} \cup \{0\}$. As above

$$(G/K)/p^{\omega \cdot t}(G/K) \cong [G/p^{\omega \cdot t}G]/[(p^{\omega \cdot t}G + K)/p^{\omega \cdot t}G]$$

is n-simply presented of length at most $\omega \cdot t$, where the quotient $(p^{\omega \cdot t}G + K)/p^{\omega \cdot t}G \cong K/(p^{\omega \cdot t}G \cap K)$ is a countable nice subgroup of $G/p^{\omega \cdot t}G$. By what we have shown above $G/p^{\omega \cdot t}G$ is n-simply presented of length not exceeding $\omega \cdot t$. Besides, $p^{\omega \cdot t}G$ is obviously bounded by p^l, so that we appeal to [14, Theorem 4.5] to infer after all that G is n-simply presented, as stated. $\square$

For groups of length beyond or equal to ω^2 the validity of the last theorem remains left-open. We however conjecture that there exists a group G of length ω^2 which is nicely ω_1-n-simply presented but not n-simply presented; in fact, $p^\omega G$ should be uncountable.

As an immediate consequence, we yield the following.

Corollary 3.3. *Suppose K is a countable nice subgroup of a group G such that length$(G) < \omega^2$. Then G is n-simply presented if and only if G/K is n-simply presented.*

Proof. The necessity follows directly from [14, Proposition 5.2(a)], as the sufficiency follows directly from Theorem 3.2. $\square$

Again, as remarked in [14] and proved in Lemma 2.5, G is n-simply presented if and only if G/F is n-simply presented whenever $F \leq G$ is finite. Likewise, as indicated in [14] and commented above, the same type affirmation does not hold for strongly n-simply presented groups and thus for strongly nice ω_1-n-simply presented groups. In fact, there are too many groups G of length $\omega + n$ such that $p^\omega G$ is countable (and even finite) and $G/p^\omega G$ is $p^{\omega+n}$-projective but G is not $p^{\omega+n}$-projective — see [13] — such a group G is actually ω_1-$p^{\omega+n}$-projective (or even ω-$p^{\omega+n}$-projective).

We continue with some other structural affirmations.

Proposition 3.4. *If G is nicely ω_1-n-simply presented, then $G/p^\omega G$ is $p^{\omega+n}$-projective. In particular, separable nicely ω_1-n-simply presented groups are $p^{\omega+n}$-projective.*

Proof. According to [14, Proposition 2.2], $(G/N)/p^\omega(G/N) \cong G/(p^\omega G + N)$ is $p^{\omega+n}$-projective, where N is a countable nice subgroup of G. But $G/(p^\omega G + N) \cong [G/p^\omega G]/[(p^\omega G + N)/p^\omega G]$, where it is obvious that $(p^\omega G + N)/p^\omega G \cong N/(N \cap p^\omega G)$ is countable. Henceforth, we apply [6, Theorem 4.2] to get the first claim. The second part is its trivial consequence. □

Corollary 3.5. *Suppose G is a group for which $p^\omega G$ is countable. Then G is nicely ω_1-n-simply presented if and only if it is ω_1-$p^{\omega+n}$-projective.*

Proof. In view of Proposition 3.4, $G/p^\omega G$ is $p^{\omega+n}$-projective. Hence, in virtue of [13], G is ω_1-$p^{\omega+n}$-projective, as expected.

The reverse implication is obvious. □

Remark 2. Another proof might be like this: Utilizing Proposition 5.3 from [14], G must be n-simply presented. Therefore, [13] applies to conclude that G has to be ω_1-$p^{\omega+n}$-projective, as wanted.

Corollary 3.6. *Suppose that G is a group whose $p^{\omega+n}G$ is countable. Then G is strongly nice ω_1-n-simply presented if and only if it is ω_1-$p^{\omega+n}$-projective.*

Proof. Let G be strongly nice ω_1-n-simply presented. Owing to [14, Proposition 2.5], we have that the factor-group $(G/M)/p^{\omega+n}(G/M) \cong G/(p^{\omega+n}G + M) \cong [G/p^{\omega+n}G]/[(p^{\omega+n}G + M)/p^{\omega+n}G]$ is $p^{\omega+n}$-projective. However, $(p^{\omega+n}G + M)/p^{\omega+n}G \cong M/(M \cap p^{\omega+n}G)$ is obviously countable whence by the utilization of [13] we obtain that $G/p^{\omega+n}G$ is ω_1-$p^{\omega+n}$-projective. Thus again [13] is applicable to conclude that G is ω_1-$p^{\omega+n}$-projective, too, as asserted.

The converse part is immediate as mentioned in Sec. 1. □

Remark 3. The above statements generalize the facts that every n-simply presented group is ω_1-$p^{\omega+n}$-projective provided its first Ulm subgroup is countable, and that any strongly n-simply presented group G is ω_1-$p^{\omega+n}$-projective provided that $p^{\omega+n}G$ is countable.

Moreover, in this aspect, is it true that ω_1-n-simply presented groups G with countable $p^\omega G$, as well as strongly ω_1-n-simply presented groups G with countable $p^{\omega+n}G$, are ω_1-$p^{\omega+n}$-projective?

This can be partially settled like the following.

Proposition 3.7. *If G is an ω_1-n-simply presented and ω-countably group, then $G/p^\omega G$ is $p^{\omega+n}$-projective.*

Proof. Let G/K be n-simply presented where K is a countable subgroup of G. Therefore, we apply [14] to show that $(G/K)/p^\omega(G/K) \cong G/\bigcap_{i<\omega}(p^iG+K) \cong [G/p^\omega G]/[\bigcap_{i<\omega}(p^iG+K)/p^\omega G]$ is $p^{\omega+n}$-projective. Since $\cap_{i<\omega}(p^iG+K)/p^\omega G$ is countable, [6] applies to get that $G/p^\omega G$ remains $p^{\omega+n}$-projective, as desired. □

As two immediate consequences, we deduce the following.

Corollary 3.8. *Suppose G is a c.c. group. Then G is ω_1-n-simply presented if and only if G is ω_1-$p^{\omega+n}$-projective.*

Proof. The sufficiency being elementary, we deal with the necessity. Since c.c. groups are obviously countably separable with countable first Ulm subgroup, Proposition 3.7 allows us to conclude with the help of [13] that G is ω_1-$p^{\omega+n}$-projective, as stated. □

So, we directly obtain the following.

Corollary 3.9. *Suppose G is a weakly ω_1-separable group. Then G is ω_1-n-simply presented if and only if G is $p^{\omega+n}$-projective.*

Furthermore, we come to the following.

Proposition 3.10. *Suppose that A is a group with a countable subgroup L. Then A is ω_1-n-simply presented if and only if A/L is ω_1-n-simply presented.*

Proof. First, let us assume that A be ω_1-n-simply presented, hence A/K is n-simply presented for some countable $K \leq A$. But

$$[A/L]/[(L+K)/L] \cong A/(L+K) \cong [A/K]/[(L+K)/K],$$

where the last factor-group $[A/K]/[(L+K)/K]$ is n-simply presented by [14, Proposition 5.2(a)] since $(L+K)/K$ is countable. Therefore, $[A/L]/[(L+K)/L]$ is n-simply presented with countable $(L+K)/L \cong K/(K\cap L)$, as wanted.

Reciprocally, let A/L be ω_1-n-simply presented, and so let C/L be a countable subgroup of A/L for some $C \leq A$ such that $(A/L)/(C/L) \cong A/C$ is n-simply presented. Observing that C is of necessity countable, we deduce via Definition 1 that A is ω_1-n-simply presented, as formulated. □

As an easy consequence, we deduce the following.

Corollary 3.11. *Suppose A is a group such that $p^\alpha A$ is countable for some ordinal α. Then A is ω_1-n-simply presented if and only if $A/p^\alpha A$ is ω_1-n-simply presented.*

Proposition 3.12. *Let A be a group with a subgroup G such that A/G is countable. Then A is ω_1-n-simply presented if and only if G is ω_1-n-simply presented.*

Proof. Write $A = G + C$ where C is countable and assume that G is ω_1-n-simply presented. Now Definition 1 ensures that there is a countable subgroup K such that G/K is n-simply presented. Consequently, $A/K = (G/K)+(C+K)/K$. Employing Lemma 2.3, A/K is n-simply presented since $(C + K)/K$ is obviously countable. This gives that A is ω_1-n-simply presented, as desired.

Conversely, let us assume that A is ω_1-n-simply presented. Now, Proposition 3.10 guarantees that $(G + C)/C \cong G/(G \cap C)$ is ω_1-n-simply presented. But $G \cap C$ is countable and again Proposition 3.10 will work to get that G is ω_1-n-simply presented, as wanted. $\square$

Remark 4. Actually Propositions 3.10 and 3.12 are equivalent and can be deduced from one of the other. The above arguments also give a simpler verification to [13, Lemma 1.9].

We are now ready to prove the following central result.

Theorem 3.13. *The class of ω_1-n-simply presented groups is closed under the formation of ω_1-bijections, and is the smallest class containing n-simply presented groups with this property.*

In other words, if $f : G \to A$ is an ω_1-bijective homomorphism and G is an ω_1-n-simply presented group, then A is an ω_1-n-simply presented group, and ω_1-n-simply presented groups form the minimal class of groups possessing that property.

Proof. The first part follows by [13, Lemma 1.9], plus Propositions 3.10 and 3.12.

That is the minimal class possessing that property follows using [13, Proposition 1.10] and Theorem 3.1. $\square$

Proposition 3.14. *Suppose A is a group with a finite subgroup F. Then A is*

(a) *nicely ω_1-n-simply presented*;
(b) *strongly ω_1-n-simply presented*;
(c) *strongly nice ω_1-n-simply presented*,

if and only if A/F is.

Proof. (a) Assume first that A is nicely ω_1-n-simply presented, i.e. there is a countable nice subgroup N such that A/N is n-simply presented. Observing as above that

$$[A/F]/[(F+N)/F] \cong A/(F+N) \cong [A/N]/[(F+N)/N],$$

and that $[A/N]/[(F+N)/N]$ is n-simply presented, it follows that A/F is nicely ω_1-n-simply presented because $(F+N)/F$ is countable and nice in A/F in accordance with Lemma 2.2(i).

Conversely, given that A/F is nicely ω_1-n-simply presented, so there exists a countable nice subgroup C/F of A/F with $C \leq A$ such that $(A/F)/(C/F) \cong A/C$ is n-simply presented. Since F is nice in A, one can see that C is countable and nice in A (see [9]), whence Definition 2 gives the claim.

(b) In view of the aforementioned results from [14] concerning strongly n-simply presented groups and with Lemma 2.2(i) in hand, the assertion follows using the tricks in the previous point (a).

(c) Follows using the arguments in the preceding two points (a) and (b). □

Proposition 3.15. *Let A be a group with a subgroup G such that A/G is finite. Then A belongs to* (a), (b) *or* (c) *of Proposition* 3.14 *if and only if G belongs to one of them.*

Proof. Repeating the same method as in Proposition 3.10 combined with Proposition 3.14, we complete the arguments. □

We are now ready to establish the following main result.

Theorem 3.16. *The classes of strongly ω_1-n-simply presented groups, nicely ω_1-n-simply presented groups and strongly nice ω_1-n-simply presented groups are closed under taking ω-bijections. Moreover, the class of strongly nice ω_1-n-simply presented groups is the smallest* (*minimal*) *class containing strongly n-simply presented groups possessing that property.*

Proof. The first part follows by our discussion in Sec. 2 (see again [13, Lemma 1.9]) along with Propositions 3.14 and 3.15. The second one follows from [13, Proposition 1.10] and some similar arguments to that of Theorem 3.1. □

Analyzing the proof of Proposition 3.14, we now detect that this proposition can be somewhat (considerably) extended to proper $\aleph_0$-nice elongations. Specifically, the following is fulfilled.

Proposition 3.17. *Let A be a group with a countable nice subgroup N. If A/N is either*

(a) *nicely ω_1-n-simply presented*;
(b) *strongly ω_1-n-simply presented*;
(c) *strongly nice ω_1-n-simply presented*,

then so is A.

We finish off here with the intersection between some well-known group classes. To that goal, the definition of an n-summable group can be seen in [7].

Theorem 3.18. *If G is both a strongly nice ω_1-n-simply presented group and an n-summable group, then G is a C_{ω_1}-group.*

Proof. By definition, there is a countable nice subgroup M such that G/M is strongly n-simply presented.

On another vein, [8, Corollary 1.3] guarantees that G/M is also n-summable. We consequently apply [15, Corollary 4.7] to get that G/M is a C_{ω_1}-group. Finally, [8, Corollary 2.4] gives the desired fact that G is a C_{ω_1}-group, too. □

As a valuable consequence, we derive the following.

Corollary 3.19. *Suppose that* $\mathrm{length}(G) < \omega_1$. *Then G is strongly nice ω_1-n-simply presented and n-summable if and only if G is a dsc group.*

Proof. The necessity follows immediately from Theorem 3.18 because C_{ω_1}-groups of countable length are dsc groups.

The sufficiency is obvious. □

4. Nunke-Like Theorems

We will now prove some partial Nunke-esque results for the new group classes defined in Sec. 1.

Proposition 4.1. *Suppose λ is an ordinal such that G is a λ-boundary group. If G is ω_1-n-simply presented, then so are $p^\lambda G$ and $G/p^\lambda G$.*

Proof. Let G/K be n-simply presented for some countable $K \leq G$. Consequently, [14, Theorem 3.4(a)] gives that $p^\lambda(G/K)$ is also n-simply presented. But by assumption $p^\lambda(G/K)/(p^\lambda G+K)/K$ is bounded, which means that $(p^\lambda G+K)/K \cong p^\lambda G/(p^\lambda G \cap K)$ is n-simply presented as well. And since $p^\lambda G \cap K$ is countable, this substantiates that $p^\lambda G$ is ω_1-n-simply presented.

To prove the second statement, again [14, Theorem 3.4(a)] is applied to ensure that $(G/K)/p^\lambda(G/K) \cong [(G/K)/(p^\lambda G + K)/K]/[p^\lambda(G/K)/(p^\lambda G + K)/K] = [(G/K)/(p^\lambda G+K)/K]/p^\lambda((G/K)/(p^\lambda G+K)/K)$ is also n-simply presented. But $p^\lambda((G/K)/(p^\lambda G+K)/K) = p^\lambda(G/K)/(p^\lambda G+K)/K$ is bounded, and therefore [14, Theorem 4.5] is in use to get that $(G/K)/(p^\lambda G + K)/K \cong G/(p^\lambda G + K) \cong [G/p^\lambda G]/(p^\lambda G + K)/p^\lambda G$ is n-simply presented. This means that $G/p^\lambda G$ is ω_1-n-simply presented, because $(p^\lambda G + K)/p^\lambda G \cong K/(K \cap p^\lambda G)$ is countable, as required. □

Theorem 4.2. *Suppose that G is a λ-boundary group for which $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is ω_1-n-simply presented if and only if $p^\lambda G$ is ω_1-n-simply presented.*

Proof. "$\Rightarrow$". It follows directly from Proposition 4.1.

"$\Leftarrow$" Let $p^\lambda G/T = p^\lambda(G/T)$ be n-simply presented for some countable subgroup T. Observe that

$$(G/T)/p^\lambda(G/T) = (G/T)/(p^\lambda G/T) \cong G/p^\lambda G$$

is n-simply presented. We therefore apply [14, Theorem 4.4] to get that G/T is n-simply presented, as desired. $\square$

Theorem 4.3. *Suppose G is a λ-boundary group for some ordinal λ such that $p^\lambda G$ is n-simply presented. Then G is ω_1-n-simply presented if and only if $G/p^\lambda G$ is ω_1-n-simply presented.*

Proof. Necessity. It follows utilizing directly Proposition 4.1.

Sufficiency. Given $(G/p^\lambda G)/(C/p^\lambda G) \cong G/C$ is n-simply presented for some countable $C/p^\lambda G$ with $C \leq G$. Write $C = p^\lambda G + K$ for some countable subgroup K. That is why $G/(p^\lambda G + K) \cong [G/K]/[(p^\lambda G + K)/K]$ is n-simply presented too. Now [14, Theorem 3.4(a)] can be used to derive that

$$\begin{aligned}&[(G/K)/(p^\lambda G + K)/K]/p^\lambda((G/K)/(p^\lambda G + K)/K)\\ &\quad = [(G/K)/(p^\lambda G + K)/K]/[p^\lambda(G/K)/(p^\lambda G + K)/K] \cong (G/K)/p^\lambda(G/K)\end{aligned}$$

is n-simply presented. But on the other vein $(p^\lambda G + K)/K \cong p^\lambda G/(p^\lambda G \cap K)$, which leads with the help of [14, Proposition 5.2(a)] that they are n-simply presented because $p^\lambda G \cap K$ is countable. We therefore may employ [15, Lemma 1.3] to infer that $p^\lambda(G/K)$ is n-simply presented. Next, the usage of [14, Theorem 4.4] guarantees that G/K is n-simply presented, as needed. $\square$

Theorem 4.4. *Suppose G is a λ-countably group for some ordinal λ such that $p^\lambda G$ is n-simply presented. Then G is ω_1-n-simply presented if and only if $G/p^\lambda G$ is ω_1-n-simply presented.*

Proof. "$\Rightarrow$". Given G/K is an n-simply presented group for some countable $K \leq G$. Consequently, [14, Theorem 3.4(a)] forces that

$$\begin{aligned}(G/K)/p^\lambda(G/K) &\cong [(G/K)/(p^\lambda G + K)]/p^\lambda(G/K)/(p^\lambda G + K)/K\\ &= [(G/K)/(p^\lambda G + K)/K]/p^\lambda((G/K)/(p^\lambda G + K)/K)\end{aligned}$$

is also n-simply presented. Because of the countability of $p^\lambda(G/K)/(p^\lambda G+K)/K = p^\lambda((G/K)/(p^\lambda G + K)/K)$, a simple appeal to [14, Theorem 4.4] leads to n-simply presentness of

$$[G/K]/[(p^\lambda G + K)/K] \cong G/(p^\lambda G + K) \cong [G/p^\lambda G]/[(p^\lambda G + K)/p^\lambda G].$$

And since $(p^\lambda G + K)/p^\lambda G \cong K/(K \cap p^\lambda G)$ is countable, we are done.

"$\Leftarrow$" Let $(G/p^\lambda G)/(C/p^\lambda G) \cong G/C$ be n-simply presented for some countable $C/p^\lambda G$ with $C \leq G$. Write $C = p^\lambda G + K$ for some countable subgroup

K. So $G/(p^\lambda G + K) \cong [G/K]/[(p^\lambda G + K)/K]$ is n-simply presented; note that this gives with the aid of [14, Theorem 3.4(a)] that $p^\lambda((G/K)/(p^\lambda G + K)/K) = p^\lambda(G/K)/(p^\lambda G + K)/K$ is n-simply presented — however we have by assumption that this quotient is countable. Moreover, again [14, Theorem 3.4(a) or Proposition 5.2(a)] applies to conclude that

$$[(G/K)/(p^\lambda G + K)/K]/p^\lambda((G/K)/(p^\lambda G + K)/K)$$
$$\cong [(G/K)/(p^\lambda G + K)/K]/p^\lambda(G/K)/(p^\lambda G + K)/K \cong (G/K)/p^\lambda(G/K)$$

is n-simply presented. But on the other hand $(p^\lambda G + K)/K \cong p^\lambda G/(p^\lambda G \cap K)$, which means by [14, Proposition 5.2(a)] that the second term, and hence the first one, are n-simply presented because $p^\lambda G \cap K$ is countable. We therefore may apply Lemma 2.3 to derive that $p^\lambda(G/K)$ is n-simply presented. Finally, utilizing [14, Theorem 4.4] to get after all that G/K is n-simply presented, as expected. □

Notice that we have not used in the necessity the condition that $p^\lambda G$ is n-simply presented, so that what immediately arises is whether or not this limitation can be dropped off in the formulation of the theorem.

Our next result generalizes one way Corollary 3.11.

Proposition 4.5. *Suppose that G is a group such that $p^\lambda G$ is a dsc groups for some ordinal λ. If $G/p^\lambda G$ is ω_1-n-simply presented, then G is ω_1-n-simply presented.*

Proof. Let $(G/p^\lambda G)/(V/p^\lambda G) \cong G/V$ is n-simply presented for some countable factor $V/p^\lambda G$ with $V \leq G$. Thus using [6], V is a dsc group as well. Assume that V' is a direct summand of V such that V/V' is countable. Hence $G/V \cong (G/V')/(V/V')$ being n-simply presented forces by Definition 1 that G/V' is ω_1-n-simply presented. Furthermore, suppose as before V'' is a direct summand of V' such that V'/V'' is countable. That is why, referring to Proposition 3.10, we conclude that G/V'' remains ω_1-n-simply presented because $G/V' \cong (G/V'')/(V'/V'')$, etc. repeating the same procedure after a finite or infinite number of steps, we will obtain a countable subgroup $C \leq G$ such that G/C is ω_1-n-simply presented. A final employment of Proposition 3.10 assures the statement that G is ω_1-n-simply presented, after all. □

Proposition 4.6. *Let G be a group and α an ordinal. If G is nicely ω_1-n-simply presented, then $p^\alpha G$ and $G/p^\alpha G$ are nicely ω_1-n-simply presented.*

Proof. Let G/N be n-simply presented for some countable nice subgroup N of G. Hence, using [14], $p^\alpha(G/N) = (p^\alpha G + N)/N \cong p^\alpha G/(p^\alpha G \cap N)$ is n-simply presented, where $p^\alpha G \cap N$ is countable and nice in $p^\alpha G$ (cf. [9]).

Moreover, $(G/N)/p^\alpha(G/N) \cong G/(p^\alpha G + N) \cong [G/p^\alpha G]/[(p^\alpha G + N)/p^\alpha G]$ is n-simply presented where $(p^\alpha G + N)/p^\alpha G \cong N/(p^\alpha G \cap N)$ is countable and $(p^\alpha G + N)/p^\alpha G$ is nice in $G/p^\alpha G$, because $N + p^\alpha G$ is so in G. □

An interesting consequence is the following one, extending Proposition 3.17 in some aspect.

Corollary 4.7. *Suppose that G is a group such that $p^\lambda G$ is a Σ-cyclic group for some ordinal λ. Then G is nicely ω_1-n-simply presented if and only if $G/p^\lambda G$ is nicely ω_1-n-simply presented.*

Proof. The "necessity" follows immediately from Proposition 4.6.

To treat the "sufficiency", let $(G/p^\lambda G)/(N/p^\lambda G) \cong G/N$ be n-simply presented for some countable nice subgroup $N/p^\lambda G$ with $N \leq G$. Thus $N = p^\lambda G + K$ is nice in G for some countable $K \leq N$. Since $p^\lambda G$ is a Σ-cyclic group, appealing to [6] we have that N is a direct sum of a countable group and a Σ-cyclic group; in fact writing $p^\lambda G = C_1 \oplus C_2$ where C_1 is Σ-cyclic, and C_2 is countable with $p^\lambda G \cap K \subseteq C_2$, one plainly follows that $N = C_1 \oplus (C_2 + K)$. Henceforth, $G/N \cong (G/C_1)/(N/C_1)$ being n-simply presented, where $N/C_1 \cong C_2 + K$ is countable and nice in G/C_1, yields by Definition 2 that G/C_1 is nicely ω_1-n-simply presented. Note that C_1 remains nice in G because it is nice in $p^\lambda G$. Referring to Proposition 3.14, G/C_1' remains nicely ω_1-n-simply presented, where $G/C_1 \cong (G/C_1')/(C_1/C_1')$ and C_1/C_1' is finite such that $C_1 \cong C_1' \oplus (C_1/C_1')$, etc. repeating the same procedure, after a finite or infinite number of steps, we will obtain a finite subgroup $F \leq G$ such that G/F is nicely ω_1-n-simply presented. A final application of Proposition 3.14 ensures the assertion that G is nicely ω_1-n-simply presented, after all. $\square$

Proposition 4.8. *Suppose that $G/p^\lambda G$ is n-simply presented for some ordinal λ. Then G is nicely ω_1-n-simply presented if and only if $p^\lambda G$ is nicely ω_1-n-simply presented.*

Proof. The "only if" part follows by a direct application of Proposition 4.6.

As for the "if" part, let $p^\lambda G/Y = p^\lambda(G/Y)$ be n-simply presented for some nice countable subgroup Y. Hence Y is also nice in G (see, e.g. [9]), and besides $G/p^\lambda G \cong (G/Y)/p^\lambda(G/Y)$ is n-simply presented by assumption. Now the application of [14, Theorem 4.4] leads us to G/Y is n-simply presented, as wanted. $\square$

Remark 5. As a final notice, we emphasize that the same type of results concerning Ulm subgroups and Ulm factors can be formulated and proved also for strongly (nice) ω_1-n-simply presented groups and strongly ω-n-simply presented groups, but we omit their representations in order to avoid the similarity of the considerations.

5. Open Problems

In closing, we shall state some left-open problems that still elude us (see also [3–5] for an analogous problematic).

Problem 1. *Does it follow that if G is an ω_1-n-simply presented group with $p^\omega G = \{0\}$, then it is $p^{\omega+n}$-projective?*

Problem 2. *Let α be an ordinal. Does it follow that G is (nicely) ω_1-n-simply presented if and only if both $p^\alpha G$ and $G/p^\alpha G$ are (nicely) ω_1-n-simply presented?*

Problem 3. *Let α be an ordinal. Does it follow that G is strongly (nice) ω_1-n-simply presented if and only if $p^{\alpha+n}G$ and $G/p^{\alpha+n}G$ are both strongly (nice) ω_1-n-simply presented?*

Problem 4. *If α is an ordinal such that $p^\alpha G$ is countable, what is the structure of $p^\alpha(G/C)$ where $C \leq G$ is a countable subgroup? Does it follow that it is countable as well, or is simply presented, or something else?*

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