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# On exchange $\pi$-UU unital rings

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**Abstract.** We prove that a ring $R$ is exchange 2-UU if, and only if, $J(R)$ is nil and $R/J(R) \cong B \times C$, where $B$ is a Boolean ring and $C$ is a ring with $C \subseteq \prod_{\mu} \mathbb{Z}_3$ for some ordinal $\mu$. We thus somewhat improve on a result due to Abdolyousefi-Chen (J. Algebra Appl., 2018) by showing that it is a simple consequence of already well-known results of Danchev-Lam (Publ. Math. Debrecen, 2016) and Danchev (Commun. Korean Math. Soc., 2017).

## 1. Introduction and Background

Everywhere in the text of the current article, all our rings are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are mainly in agreement with the stated in [7]. For instance, for an arbitrary ring $R$, $U(R)$ will always denote the unit group with $n$-th power $U^n(R) = \{u^n \mid u \in U(R)\}$ where $n \in \mathbb{N}$, $J(R)$ the Jacobson radical, and $Nil(R)$ the set of all nilpotents. Recall also that a ring $R$ is said to be *tripotent* provided that the equality $x^3 = x$ holds for all $x \in R$.

We also need some other fundamentals as follows:

**Definition 1.1.** *([6]) A ring $R$ is said to be UU if $U(R) = 1 + Nil(R)$.*

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**Definition 1.2.** *A ring $R$ is said to be exchange if, for each $r \in R$, there is an idempotent $e \in rR$ such that $1 - e \in (1 - r)R$.*

It was proved in [6] that a ring $R$ is an exchange UU ring if, and only if, $J(R)$ is nil and $R/J(R)$ is Boolean.

Before proceed by proving our chief result, we need a few more technicalities, mainly developed by the current author in the papers cited in the reference list. And so, generalizing Definition 1.1, one can state the following.

**Definition 1.3.** *Let $n \in \mathbb{N}$ be fixed. A ring $R$ is called $n$-UU if, for any $u \in U(R)$, $u^n \in 1 + Nil(R)$, that is, the inclusion $U^n(R) \subseteq 1 + Nil(R)$ holds. If $n$ is the minimal natural with this property, $R$ is just said to be strongly $n$-UU.*

Clearly, UU rings just coincide with (strongly) 1-UU rings.

This can be freely expanded to the following:

**Definition 1.4.** *A ring $R$ is called $\pi$-UU if, for any $u \in U(R)$, there exists $i \in \mathbb{N}$ depending on $u$ such that $u^i \in 1 + Nil(R)$.*

The leitmotif of the present paper is to study exchange $n$-UU rings in the cases $n = 2$ and $n = 3$. Our results will considerably strengthen those from [1] and will also provide the interested reader with new simpler proofs. In closing we state a question which remains unanswered.

## 2. Main Results

The next statement considerably supersedes [1, Lemma 4.4] by dropping off the unnecessary limitation on the ring to be "exchange". The used technique was developed in [4] and [5].

**Proposition 2.1.** *Let $R$ be a 2-UU ring. Then $J(R)$ is nil.*

**Proof.** Given $x \in J(R)$, it follows that $(1+x)^2 = 1+2x+x^2 \in 1+Nil(R)$ which amounts to $2x + x^2 \in Nil(R)$. Similarly, replacing $x$ by $-x$, we derive that $-2x + x^2 \in Nil(R)$. Since these two sums commute, it follows immediately that $2x^2 \in Nil(R)$. Finally, using the above trick for $x^2$, we deduce that $2x^2 + x^4 \in Nil(R)$. Since $2x^2 \in Nil(R)$, we conclude that $x^4 \in Nil(R)$, i.e., $x \in Nil(R)$, as required. □

**Corollary 2.2.** *A ring $R$ is 2-UU if, and only if, $J(R)$ is nil and $R/J(R)$ is 2-UU.*

**Proof.** According to Proposition 2.1, the argument follows in the same manner as [6, Theorem 2.4 (2)]. □

**Lemma 2.3.** *Let $R$ be a ring. Then the following two points hold:*

*(i) If $R$ is $n$-UU for some $n \in \mathbb{N}$, then $eRe$ is also $n$-UU for any $e \in Id(R)$.*

*(ii) If $R$ is $\pi$-UU, then $eRe$ is also $\pi$-UU for any $e \in Id(R)$.*

**Proof.** We shall show the validity only of (ii). The proof of (i) is analogous and so it will be omitted. As in [6], letting $w \in U(eRe)$ with inverse $v$, it follows that $w + 1 - e \in U(R)$ with inverse $v + 1 - e$. Therefore, there exists $i \in \mathbb{N}$ such that $(w + 1 - e)^i = w^i + 1 - e \in 1 + Nil(R)$, that is, $w^i - e = q \in Nil(R)$. But $q \in Nil(R) \cap (eRe) = Nil(eRe)$ which leads to $w^i = e + q \in 1_{eRe} + Nil(eRe)$, as expected. □

**Lemma 2.4.** *For any $n \in \mathbb{N}$ and any non-zero ring $R$ the full matrix ring $\mathbb{M}_n(R)$ is not 2-UU.*

**Proof.** Since $\mathbb{M}_2(R)$ is isomorphic to a corner ring of $\mathbb{M}_n(R)$ for $n \geq 2$, in view of Lemma 2.3 it suffices to establish the claim for $n = 2$. To that goal, as in [6], let us consider the invertible matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, we infer that $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ which is the same invertible element with the inverse $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, and thus it is certainly not a nilpotent, as wanted. □

We shall now restate and reproof the main result from [1] by giving a more convenient form and more transparent proof arising from well-known recent results in [6] and [5], respectively. Actually, a new substantial achievement, including new points with more strategic estimations, arises as follows:

**Theorem 2.5.** *Suppose that $R$ is a ring. Then the following five items are equivalent:*

*(a) $R$ is exchange 2-UU.*

*(b) $J(R)$ is nil and $R/J(R)$ is commutative invo-clean.*

*(c) $J(R)$ is nil and $R/J(R) \cong B \times C$, where $B \subseteq \prod_\lambda \mathbb{Z}_2$ and $C \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinals $\lambda$ and $\mu$.*

*(d) $J(R)$ is nil and $R/J(R)$ is tripotent.*

*(e) $J(R)$ is nil and $R/J(R) \subseteq \prod_\lambda \mathbb{Z}_2 \times \prod_\mu \mathbb{Z}_3$ for some ordinals $\lambda$ and $\mu$.*

**Proof.** The equivalence (b) $\iff$ (c) is exactly [5, Corollary 2.17], whereas the equivalence (d) $\iff$ (e) is obvious.

We shall show that (a) $\iff$ (b) is valid. To prove the left-to-right implication, we first consider the semi-primitive case when $J(R) = \{0\}$. Imitating the basic idea from the proof of [6, Theorem 4.1], we arrive at the case when $eRe \cong \mathbb{M}_2(T)$ for some idempotent $e \in R$ and some non-zero ring $T$ depending on $R$, provided $Nil(R) \neq \{0\}$. However, with Lemma 2.3 at hand we deduce that $eRe$ is 2-UU, while with the aid of Lemma 2.4 this property does not hold for $\mathbb{M}_2(T)$. This contradiction substantiates that $R$ is reduced, i.e., $Nil(R) = \{0\}$ and thus abelian. Hence $R$ is clean with $U^2(R) = \{1\}$ which allows us to conclude with an appeal to [5] that $R$ is abelian invo-clean and so commutative invo-clean. Suppose now that $J(R) \neq \{0\}$. The fact that $J(R)$ is nil follows directly from Proposition 2.1. Owing to [8] and Corollary 2.2, one sees that $R/J(R)$ is exchange 2-UU, and so by what we have just already shown so far, the factor-ring $R/J(R)$ has to be commutative invo-clean, as asserted.

As for the right-to-left implication, it follows immediately by virtue of [8] that $R$ is an exchange ring. That $R$ is a 2-UU ring follows like this: Using the isomorphisms $U(R)/(1+J(R)) \cong U(R/J(R)) \cong U(B) \times \prod_\mu U(\mathbb{Z}_3)$, we so have $U^2(R/J(R)) = \{1\}$. Furthermore, for any $u \in U(R)$ it must be that $u+J(R) \in U(R/J(R))$ and hence $(u+J(R))^2 = u^2+J(R) = 1+J(R)$ which means that $u^2 - 1 \in J(R) \subseteq Nil(R)$, as required.

The implication (c) $\Rightarrow$ (d) is elementary. What remains to illustrate is the truthfulness of the implication (d) $\Rightarrow$ (a). Since tripotent rings are always exchange, the application of [8] shows that $R$ is exchange. On the other side, since $U(R/J(R)) \cong U(R)/(1+J(R))$ and $U^2(R/J(R)) = \{1\}$, as shown above it follows that $R$ is a 2-UU ring, thus completing the proof after all. $\square$

The next construction manifestly demonstrates that the theorem is no longer true for $n$-UU rings when $n > 2$.

**Example 2.6.** Consider the full matrix $2 \times 2$ ring $R = \mathbb{M}_2(\mathbb{Z}_2)$. It was proved in [2] that $R$ is nil-clean and hence exchange. Moreover, $R$ is a 3-UU ring. However, it is easily checked that $J(R) = \{0\}$ and that $R$ is even not tripotent (whence it is not boolean). In fact, $U(R)$ has 6 elements satisfying the following identities:

- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore, in all cases, $U^3(R) - 1 \subseteq Nil(R)$ and besides there are $u \in U(R)$ such that $u^3 = 1 \neq u$. This concludes our claim and thus the example ends.

We finish off our work with the following question of some interest and importance. Recall that a ring $R$ is termed *$\pi$-Boolean* if, for any $r \in R$, there is $i \in \mathbb{N}$, which depends on $r$, with $r^i = r^{2i}$.

**Problem 2.7.** *Does it follow that $R$ is an exchange $\pi$-UU ring if, and only if, $J(R)$ is nil and $R/J(R)$ is $\pi$-Boolean?*

If yes, this will resolve the basic problem from [3] in the affirmative.

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