Commun. Korean Math. Soc. **32** (2017), No. 1, pp. 19–27
https://doi.org/10.4134/CKMS.c160054
pISSN: 1225-1763 / eISSN: 2234-3024

INVO-CLEAN UNITAL RINGS

Peter V. Danchev

Abstract. We define and completely describe the structure of invo-clean rings having identity. We show that these rings are clean but not (weakly) nil-clean and thus they possess independent properties than these obtained by Diesl in [7] and by Breaz-Danchev-Zhou in [2].

1. Introduction and background

Throughout the present paper, all rings R considered shall be assumed to be associative and unital containing the identity element 1 which differs from the zero element 0. As usual, $U(R)$ denotes the set of all units of R, $Inv(R)$ the subset of $U(R)$ consisting of all involutions of R, $Id(R)$ the set of all idempotents of R and $Nil(R)$ the set of all nilpotents of R. Traditionally, $J(R)$ stands for the Jacobson radical of R. All other notions and notations, not explicitly stated herein, are at most standard.

The following concept appeared in [9].

Definition 1.1. A ring R is called *clean* if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique, then R is said to be *uniquely clean*.

A clean ring R with $ue = eu$ is said to be *strongly clean*. If again the existing idempotent e is unique, the ring is called *uniquely strongly clean*.

It is well known that uniquely clean rings being abelian clean rings are strongly clean. The converse, however, does not hold in general.

In particular, in [7] was introduced the following concept:

Definition 1.2. A ring R is called *nil-clean* if each $r \in R$ can be written as $r = q + e$, where $q \in Nil(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique, then R is said to be *uniquely nil-clean*.

Received March 15, 2016; Revised June 27, 2016.
2010 *Mathematics Subject Classification.* 16D60, 16U60.
Key words and phrases. clean rings, nil-clean rings, idempotents, nilpotents, units, involutions, invo-clean rings.

A nil-clean ring R with $qe = eq$ is said to be *strongly nil-clean*. If again the existing idempotent e is unique, the ring is called *uniquely strongly nil-clean*.

It is well known that uniquely nil-clean rings being abelian nil-clean rings are strongly nil-clean. Also, commutative nil-clean rings are always uniquely nil-clean (compare with [6]), and even it was proved in [7, Corollary 3.8] that strongly nil-clean rings and uniquely strongly nil-clean rings do coincide in general.

On the other hand, the latter concept of nil-cleanness was extended in [6] and [2] respectively by defining the notion of *weak nil-cleanness* as follows:

Definition 1.3. A ring R is called *weakly nil-clean* if every $r \in R$ can be presented as either $r = q + e$ or $r = q - e$, where $q \in Nil(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique in the sense that there exists a unique idempotent e such that exactly one of $r = q + e$ or $r = q - e$ holds, then R is said to be *uniquely weakly nil-clean*.

A weakly nil-clean ring with $qe = eq$ is said to be *weakly nil-clean with the strong property*. If again the existing idempotent e is unique, the ring is called *uniquely weakly nil-clean with the strong property*.

It was established in [2] and [6] that weakly nil-clean rings are themselves clean. Likewise, in [2] was established a complete characterization of abelian weakly nil-clean rings as those abelian rings R for which $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring B, or to $\mathbb{Z}_3$, or to $B \times \mathbb{Z}_3$. (See also [4] for the general case as well as [11] for a slightly different characterization.) We notice also that uniquely weakly nil-clean rings were classified in [3] as the abelian weakly nil-clean rings.

Definition 1.4. A ring R is said to be *invo-clean* if every $r \in R$ can be written as $r = v + e$, where $v \in Inv(R)$ and $e \in Id(R)$. If, in addition, the existing idempotent e is unique, then R is called *uniquely invo-clean*.

An invo-clean ring with $ve = ev$ is said to be *strongly invo-clean*. If again the existing idempotent e is unique, the ring is called *uniquely strongly invo-clean*.

Interestingly, any idempotent is an invo-clean element due to the record $e = (2e - 1) + (1 - e)$, because $(2e - 1)^2 = 1$ and $(1 - e)^2 = 1 - e$.

Moreover, simple examples of invo-clean rings that could be plainly verified are these: $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_8$. Oppositely, both $\mathbb{Z}_5$ and $\mathbb{Z}_7$ are not invo-clean but however they are clean being finite.

The objective of this article is to explore (strongly, uniquely) invo-clean rings by giving a complete description of their algebraic structure. As an application, we will characterize some related classes of rings. Our scientific work is organized in the next section, and in closing we end with two challenging questions.

2. Invo-clean rings

We begin with two useful technicalities.

Lemma 2.1. *Homomorphic images of invo-clean rings are again invo-clean.*

Proof. Since homomorphic images of involutions and idempotents are again involutions and idempotents, the assertion follows easily. □

Lemma 2.2. *If R is an invo-clean ring, then $24 = 0$. In particular, $6 \in Nil(R)$.*

Proof. Write $3 = v + e$, where v is an involution and e is an idempotent. Thus $(3-v)^2 = 3-v$ implies that $5v = 7$, whence $24 = 0$ by squaring both sides of the equality. In addition, $6^3 = 216 = 24.9 = 0$, hence $6 \in Nil(R)$, as asserted. □

Remark 2.3. If R is a nil-clean ring it was proved in [7] that $2 \in Nil(R)$, whereas if R is a weakly nil-clean ring it was established in [6] and [2] that $6 \in Nil(R)$.

The following technicality is our critical tool (see [4], too).

Lemma 2.4. *Suppose R is a ring with $u \in U(R)$ and $e \in Id(R)$ such that $u^2e = eu^2 = e$ and $u = e + q$, where $q \in Nil(R)$. Then $e = 1$.*

Proof. Letting $u = e+q$ for some $e \in Id(R)$ and $q \in Nil(R)$ with $q^t = 0, t \in \mathbb{N}$ say, we obtain that $u^2 = e+eq+qe+q^2$ and hence $u^2e = e = e+eqe+qe+q^2e$ which forces that $(q+q^2)e = -eqe$. Similarly, $eu^2 = e$ insures that $e(q+q^2) = -eqe$. Thus e commutes with the nilpotent $(q+q^2)^n = [q(1+q)]^n = q^n(1+q)^n$ for all $n \in \mathbb{N}$, and therefore the same is valid for u. Furthermore, $u-(q+q^2) = e-q^2$ with $u-(q+q^2) = u_2 = e-q^2$ being a unit, one sees that $u_2-(2q^3+q^4) = e-(q^2+2q^3+q^4) = e-(q+q^2)^2$. Putting $u_3 = u_2+(q+q^2)^2$, we observe that u_3 is a unit since u_2 commutes with $(q+q^2)^2$ and that $u_3 = e+q^3(2+q)$. In the same manner $u_4 = u_3-2(q+q^2)^3 = e-q^4(5+6q+2q^2)$, $u_5 = u_4+5(q+q^2)^4 = e+q^5(14+28q+20q^2+5q^3)$ and $u_6 = u_5-14(q+q^2)^5 = e-q^6c$, where $c = f(q)$ is a function (polynomial) of q. Repeating the same procedure t-times, we will find a unit u_t such that $u_t = e + q^t \cdot a = e$ for some element a depending on q; $a = -1 = -q^0$ provided $t = 2$. This yields that $e = 1$, which exhausts our claim. □

Remark 2.5. Notice that the method used in [11] cannot be applied in the proof of Lemma 2.4, because $u^2(1-e) = u^2 - u^2e = u^2 - e = 1-e$ holds, provided a priory that $u^2 = 1$ only. But this is not deducible at once, namely the fact that u has to be an involution (and hence a unipotent) will follow after certain additional arguments.

For applicable purposes we detect that the following is true (compare with [11] as well):

Corollary 2.6. *If R is a ring with $u \in Inv(R)$ such that $u = e+q$ for $e \in Id(R)$ and $q \in Nil(R)$, then $e = 1$.*

Proof. Just take $u \in Inv(R)$ in Lemma 2.4 and this allows us to infer that $e = 1$, as promised. □

Proposition 2.7. *If R is an invo-clean ring with $2 \in U(R)$, then $Nil(R) = J(R) = \{0\}$.*

Proof. If $q \in Nil(R)$, write $q = v + e$ where $v \in Inv(R)$ and $e \in Id(R)$. Thus $-v = -q + e$, where $-v \in Inv(R)$ and $-q \in Nil(R)$. Appealing to Corollary 2.6, we conclude that $e = 1$. Therefore, $q = v + 1$ and hence $q^2 = 2 + 2v = 2(1 + v) = 2q$. This leads to $q(2 - q) = 0$. Since $2 - q \in U(R)$, we finally infer that $q = 0$, as expected.

Concerning the second part, given $z \in J(R)$ we have $z = v + e$ for v, e as above. Consequently, $z - v = e \in U(R) \cap Id(R) = \{1\}$ means that $z = v + 1$ and since $2 - z \in U(R)$ the same trick as above works to get that $z = 0$, as promised. □

Proposition 2.8. *If R is an invo-clean ring with $Id(R) = \{0, 1\}$ and $2 \in U(R)$, then $R \cong \mathbb{Z}_3$.*

Proof. Each element r of R can be written as either $r = v + 1$ or $r = v$, where $v \in Inv(R)$. However, $\frac{1-v}{2}$ is always an idempotent, whence $\frac{1-v}{2} = 0$ or $\frac{1-v}{2} = 1$. In the first case $v = 1$, while in the second one $v = -1$. Consequently, all the elements of R are $\{0, -1, 1, 2\}$. But it must be that $2 = -1$, because only $2.(-1) = 1$ or $2.2 = 1$ is possible. So, $3 = 0$ and $R = \{0, 1, 2\}$, as needed. □

Proposition 2.9. *If R is an invo-clean ring with $2 \in Nil(R)$, then R is nil-clean with bounded index of nilpotence. In particular, an invo-clean ring is nil-clean if and only if 2 is a nilpotent.*

Proof. Given $r \in R$, we write $r = v + e$, where $v^2 = 1$ and $e^2 = e$. But $(1+v)^2 = 2+2v = 2(1+v)$ and hence $(1+v)^3 = 2(1+v)^2 = 2^2(1+v)$, etc. by induction we derive that $(1+v)^{n+1} = 2^n(1+v)$ for all $n \in \mathbb{N}$. Thus $(1+v)^t = 0$ for some appropriate natural t, that is, $1 + v \in Nil(R)$. Furthermore, one may write that $r = (v + 1) - (1 - e)$, whence R is nil-clean, as claimed.

For the second part, given $q \in Nil(R)$, we write that $q = i + e$ for some $i \in Inv(R)$ and $e \in Id(R)$. Thus $-i = (-q) + e$ and, since $-i \in Inv(R)$ and $-q \in Nil(R)$, Corollary 2.6 is applicable to infer that $q = i + 1$. Furthermore, one verifies that $q^2 = 2q$ and hence, by induction, $q^{n+1} = 2^n q$ for all $n \in \mathbb{N}$. Thus $q^k = 0$ for some fixed $k \in \mathbb{N}$, as required. □

Under certain additional circumstances the converse is true; even more a criterion when a nil-clean ring is invo-clean is deducible.

Proposition 2.10. *Suppose that R is a nil-clean ring. Then R is invo-clean if and only if any $q \in Nil(R)$ satisfies the equation $q^2 + 2q = 0$.*

Proof. "$\Rightarrow$" As in Proposition 2.9, we derive that $q^2 = 2q$. Substituting q by $-q$, we are set.

"$\Leftarrow$" Writing $r = q + e = (1 + q) - (1 - e)$ for any $r \in R$ with $q \in Nil(R)$ and $e \in Id(R)$, one checks that $(1+q)^2 = 1+2q+q^2 = 1$ and $(1-e)^2 = 1-e$, as required. $\square$

As an interesting consequence, we obtain the following one.

Corollary 2.11. *Suppose R is a nil-clean ring of characteristic 2. Then R is invo-clean if and only if the index of nilpotence of R is 2.*

Remark 2.12. In regard to the above statement, it is worth noticing that $\mathbb{Z}_8$ is both invo-clean and nil-clean containing the element 2 of nilpotence index 3. However, it is readily seen that 2 satisfies the equality $q^2 + 2q = 0$ because $2^2 + 2 \cdot 2 = 8 = 0$.

Likewise, $\mathbb{Z}_{16} = \mathbb{Z}_{2^4}$ is a nil-clean ring which is not necessarily invo-clean (compare with Corollary 2.11 above). In fact, $\mathbb{Z}_{16}$ is indecomposable, that is, the only idempotents are 0 and 1 as well as all involutions are 1, 7, 9 and 15. So, the unit 5 cannot be represented as a sum of an involution and an idempotent, as expected.

Proposition 2.13. *The (finite or infinite) direct product of invo-clean rings is again invo-clean.*

Proof. This fact has routinely technical check, so we leave it to the reader. $\square$

So, consulting with [6, Proposition 1.9(ii)], we come to the following:

Example 2.14. The ring $\mathbb{Z}_3 \times \mathbb{Z}_3$ is invo-clean but not weakly nil-clean. Also, referring to [2], $\mathbb{Z}_9$ is a weakly nil-clean ring but an easy computation shows that it is not invo-clean. Thereby these two notions are independent each to other.

We now come to our main result in which we give a complete description of invo-clean rings.

Theorem 2.15. *A ring R is invo-clean if and only if $R \cong R_1 \times R_2$, where R_1 is an invo-clean ring of characteristic at most 8 which is nil-clean, and R_2 is either $\{0\}$ or a commutative semiprimitive (and hence reduced) invo-clean ring of characteristic 3 such that each its element is the sum of two idempotents (respectively, of two involutions). In addition, R_2 can be embedded as an isomorphic copy in the direct product of copies of $\mathbb{Z}_3$.*

Proof. Treating the necessity, by virtue of Lemma 2.2 we know that $6^n = 0$ for some $n \in \mathbb{N}$. Since $(2^n, 3^n) = 1$, i.e., there exist non-zero integers k, l such that $2^n k + 3^n l = 1$, it follows that $R = 2^n R \oplus 3^n R$ because $2^n R \cap 3^n R = \{0\}$. In fact, to show that this intersection is zero, given $x = 2^n a = 3^n b$ for some $a, b \in R$, we have $2^n ak = 3^n bk$. However, $a(1 - 3^n l) = 3^n bk$ whence $3^n(al + bk) = a$. Multiplying both sides by 2^n, we derive that $0 = 2^n a = x$, as required. Furthermore, $3^nR \cong R/2^nR$ as well as $2^nR \cong R/3^nR$, so that $R \cong R_1 \times R_2$, where we put $R_1 = R/2^nR$ and $R_2 = R/3^nR$. Certainly, using the same trick, one can also decompose R as $R \cong (R/8R) \times (R/3R)$ because $(8, 3) = 1$. Next, since $R \to R/2^nR = R_1$ and $R \to R/3^nR = R_2$ are epimorphisms, it follows from Lemma 2.1 that both R_1 and R_2 are invo-clean. Hence, in view of Lemma 2.2, $6 \in Nil(R_1)$ and $6 \in Nil(R_2)$. But it is obviously true that $2 \in J(R_1)$ whence $3 \in U(R_1)$ which assures that $2 \in Nil(R_1)$ and even employing the second part of Lemma 2.2 we will have $2^3 = 8 = 0$ in R_1. In accordance with Proposition 2.9, the ring R_1 has to be nil-clean.

Regarding the second direct factor, $3 \in J(R_2)$ ensures that $2 \in U(R_2)$ and thus owing to Proposition 2.7 we obtain $3 \in Nil(R_2) = J(R_2) = \{0\}$ which amounts to $3 = 0$ in R_2. Next, given arbitrary $a \in R_2$, we write $2a = v + e$ where $v \in Inv(R_2)$ and $e \in Id(R_2)$ whence $a = \frac{v+1}{2} + \frac{e+2}{2}$. It is readily verified that both $\frac{v+1}{2}$ and $\frac{e+2}{2}$ are idempotents, as asserted. But R_2 being reduced is necessarily abelian whence commutative. On the other side, we can write $a - 1 = v + e$ with v, e as above, which means that $a = v + (1 + e)$. Since $(1 + e)^2 = 1 + 3e = 1$, we are done. That is why, with a modification of the Chinese Reminder Theorem at hand, we deduce that $R_2 \cong R_2/J(R_2)$ can be embedded in the direct product of invo-clean domains of characteristic 3 which, in conjunction with Proposition 2.8, are isomorphic to $\mathbb{Z}_3$, as claimed.

The sufficiency follows immediately from Proposition 2.13. □

As a nontrivial immediate consequence, which seems not to have a direct proof, is the following one:

Corollary 2.16. *If R is an invo-clean ring, then $J(R)$ is nil with index of nilpotence not exceeding* 3.

Proof. According to Theorem 2.15, one can decompose the invo-clean ring R as $R \cong R_1 \times R_2$, where R_1 is a nil-clean ring and R_2 is a ring with zero Jacobson radical. Since $J(R) \cong J(R_1)$, we next just apply [7] to get the desired claim.

Concerning the second part that the index of nilpotence of R is at most 3, writing $j = v + e$ for any $j \in J(R)$ with $v \in Inv(R)$ and $e \in Id(R)$, we have $j - v = e \in U(R) \cap Id(R) = \{1\}$. Hence $j = v + 1$ and so $j^2 = 2j$ which yields $j^3 = 4j$ and $4j^2 = 8j$. Since $2j \in J(R)$, repeating the same procedure for this element, we deduce that $(2j)^2 = 2(2j)$, that is, $4j^2 = 4j$. Thus $8j = 4j$, i.e., $4j = 0 = j^3$, as required. □

Another consequence gives a comprehensive description of invo-clean rings having the strongly property in the following manner:

Corollary 2.17. *A ring R is strongly invo-clean if and only if $R \cong R_1 \times R_2$, where R_1 is a strongly invo-clean ring of characteristic less than or equal to* 8 *which is strongly nil-clean, and R_2 is either* $\{0\}$ *or a commutative semiprimitive* (*and hence reduced*) *invo-clean ring of characteristic* 3 *which can be embedded as an isomorphic copy in the direct product of copies of $\mathbb{Z}_3$.*

Proof. To treat the necessity, the first part concerning the full classification of the subring R_1 as well as some facts for the subring R_2 follow directly from Theorem 2.15. As for the more concrete classification of the subring R_2 having characteristic 3, one observes that since any involution v and any idempotent e satisfy $v^3 = v$ and $e^3 = e$ and they also commute, every element $y = v + e$ in R_2 satisfies the equation $y^3 = y$. Therefore, applying [8], the ring R_2 must be commutative. (Note that it is trivially seen that R_2 is also a von Neumann regular ring and this also yields that $J(R_2) = \{0\}$.) Further on, the description of R_2 follows repeating the same trick as that from Theorem 2.15.

The sufficiency follows directly from Proposition 2.13. □

Remark 2.18. In regard to Theorem 2.15 and its Corollary 2.17, does it follow that (strongly) nil-clean rings of characteristic ≤ 8 and index of nilpotence 2 are (strongly) invo-clean? It is also worthwhile noticing that it was somewhat surprising the fact that R_2 is a commutative ring.

It was established in [10] that a ring R is uniquely clean if, and only if, R is abelian, $R/J(R)$ is boolean and idempotents of R lift modulo $J(R)$. Thus we are now ready to proceed by proving with the following.

Theorem 2.19. *Any uniquely invo-clean ring is strongly nil-clean, and hence $R/J(R)$ is boolean and $J(R)$ is nil.*

Proof. It follows accomplishing Corollary 2.16, the fact from [10] quoted above and [5, Theorem B]. □

Enlarging now the main concept of UU rings from [5] as those rings R such that $U(R) = 1 + Nil(R)$, we define R to be a ring with *unipotent involutions* or briefly *a UI ring*, provided $Inv(R) \subseteq 1 + Nil(R)$ (see [4, Problem 6]). Surprisingly, this condition does not give nothing new. Specifically, the following holds:

Proposition 2.20. *A ring R is a UI ring if and only if $2 \in Nil(R)$.*

Proof. Since $-1 \in Inv(R)$, it follows that $-1 \in 1 + Nil(R)$ and hence $2 \in Nil(R)$. Conversely, for any $u \in Inv(R)$ we have that $(1-u)^2 = 2(1-u)$ and thus by induction $(1-u)^{n+1} = 2^n(u-1)$ for any $n \in \mathbb{N}$, so that $2^k = 0$ for some $k \geq 1$ implies that $(1-u)^{k+1} = 0$ whence $u - 1 \in Nil(R)$ and $u \in 1 + Nil(R)$, as required. □

Referring to Corollary 2.6, or to Proposition 2.20 accomplished with [7], or to [11], a valuable example of UI rings are all nil-clean rings. As aforementioned, so are also all UU rings. The converse holds only if each unit is an involution.

In [5] it was proved that clean UU rings are strongly nil-clean. It is thereby reasonably adequate to conjecture that clean UI rings are nil-clean, but in view of Proposition 2.20 together with some standard facts this will be completely wrong. Nevertheless, we can currently offer the following:

Proposition 2.21. *Every invo-clean UI ring is nil-clean.*

Proof. An appeal to Theorem 2.15 allows us to write that $R \cong R_1 \times R_2$ where R_1 is nil-clean and R_2 is either $\{0\}$ or invo-clean with $2 \in Inv(R)$. In the first case for R_2 we are finished. In the second one, since it is trivial to verify that epimorphic images of UI rings are again UI rings, it must be that R_2 is UI and thus as observed above $2 \in Nil(R)$ which is impossible. Finally, $R_2 = \{0\}$ and therefore $R \cong R_1$ is nil-clean, as wanted. □

Remark 2.22. A direct proof can be deduced by combining Propositions 2.9 and 2.20.

In that way, owing to [2] and Proposition 2.20, it follows at once that weakly nil-clean UI rings are themselves nil-clean.

3. Concluding discussion

In the context considered above, we close the article with the following two problems.

Combining the notion of invo-clean rings with that of weakly nil-clean rings, one can state:

Problem 1. A ring R is called *weakly invo-clean* if, for each $a \in R$, there exist $v \in Inv(R)$ and $e \in Id(R)$ such that $a = v + e$ or $a = v - e$.

Describe the structure of these rings. Are they clean? However, they are obviously weakly clean in the sense of [1], considered not only in the commutative aspect.

Now, in order to expand the concept of UI rings in the light of WUU rings from [4] and the listed there Problem 7, one may also ask:

Problem 2. Classify all WUI rings R satisfying either $Inv(R) \subseteq \pm 1 + Nil(R)$.

Problem 3. Characterize *unitly invo-clean* rings R, that are rings R with $U(R) = Inv(R) + Id(R)$.

Acknowledgements. The author is deeply appreciated to the specialist referee and the handling editor for their given expert comments and suggestions.

References

[1] M.-S. Ahn and D. D. Anderson, *Weakly clean rings and almost clean rings*, Rocky Mount. J. Math. **36** (2006), no. 3, 783–798.

[2] S. Breaz, P. Danchev, and Y. Zhou, *Rings in which every element is either a sum or a difference of a nilpotent and an idempotent*, J. Algebra Appl. **15** (2016), no. 8, 1650148, 11 pp.

[3] A. Cîmpean and P. Danchev, *Weakly nil-clean index and uniquely weakly nil-clean rings*, Internat. Electr. J. Algebra **21** (2017).

[4] P. V. Danchev, *Weakly UU rings*, Tsukuba J. Math. **40** (2016), 101–118.

[5] P. V. Danchev and T. Y. Lam, *Rings with unipotent units*, Publ. Math. (Debrecen) **88** (2016), 449–466.

[6] P. V. Danchev and W. Wm. McGovern, *Commutative weakly nil clean unital rings*, J. Algebra **425** (2015), 410–422.

[7] A. J. Diesl, *Nil clean rings*, J. Algebra **383** (2013), 197–211.

[8] N. Jacobson, *Structure theory for algebraic algebras of bounded degree*, Ann. of Math. **46** (1945), 695–707.
[9] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.
[10] W. K. Nicholson and Y. Zhou, *Rings in which elements are uniquely the sum of an idempotent and a unit*, Glasg. Math. J. **46** (2004), no. 2, 227–236.
[11] J. Šter, *Nil-clean involutions*, arXiv preprint, 2015.

Peter V. Danchev
Department of Mathematics
Plovdiv University
Plovdiv 4000, Bulgaria
E-mail address: pvdanchev@yahoo.com