ABSTRACTS

OF

P A P E R S

ON

Non-Commutative Rings and Algebras

[1] P. V. Danchev, *A generalization of π-regular rings*, Turk. J. Math. (2) **43** (2019), 702-711.

ABSTRACT: We introduce the class of so-called regularly nil clean rings and systematically study their fundamental characteristic properties accomplished with relationships among certain classical sorts of rings such as exchange rings, Utumi rings etc. These rings of ours naturally generalize the long-known classes of π -regular and strongly π -regular rings. We show that the regular nil cleanness possesses a symmetrization which extends the corresponding one for strong π -regularity that was visualized by Dischinger [10]. Likewise, our achieved results substantially improve on establishments presented in two recent papers by Danchev and Šter [8] and Danchev [6].

[2] P. V. Danchev, *On a property of nilpotent matrices over an algebraically closed field*, Chebyshevskii Sbornik (3) **20** (2019), 276-279.

ABSTRACT: Suppose F is an algebraically closed field. We prove that the ring $\prod_{n=1}^{\infty} M_n(F)$ has a special property which is, somewhat, in sharp parallel with (and slightly better than) a property established by Ster (LAA, 2018) for the rings ˇ $\prod_{n=1}^{\infty} M_n(Z_2)$ and $\prod_{n=1}^{\infty} M_n(Z_4)$, where Z_2 is the finite simple field of two elements and Z_4 is the finite indecomposable ring of four elements.

[3] P. V. Danchev, *Representing matrices over fields as square-zero matrices and diagonal matrices*, Chebyshevskii Sbornik (3) **21** (2020), 84-88.

ABSTRACT: We prove that any square matrix over an arbitrary infinite field is a sum of a square-zero matrix and a diagonalizable matrix. This result somewhat contrasts recent theorem due to Breaz, published in Linear Algebra & Appl. (2018).

[4] P. V. Danchev, *Certain properties of square matrices over fields with applications to rings*, Rev. Colomb. Mat. (2) **54** (2020), 109-116.

ABSTRACT: We prove that any square nilpotent matrix over a field is a difference of two idempotent matrices as well as that any square matrix over an algebraically closed field is a sum of a nilpotent square-zero matrix and a diagonalizable matrix. We further apply these two assertions to a variation of π-regular rings. These results somewhat improve on establishments due to Breaz from Linear Algebra & Appl. (2018) and Abyzov from Siberian Math. J. (2019) as well as they also refine two recent achievements due to the present author, published in Vest. St. Petersburg Univ. - Ser. Math., Mech. & Astr. (2019) and Chebyshevskii Sb. (2019), respectively.

[5] P. V. Danchev and D. D. Anderson, *A note on a theorem of Jacobson related to periodic rings*, Proc. Amer. Math. Soc. (12) **148** (2020), 5087-5089.

ABSTRACT: We show that if R is a ring such that for each $x \in R$ there exist two natural numbers $n(x)$ and $m(x)$ of opposite parity with $x^{n(x)} = x^{m(x)}$, then R is commutative. This extends the classical famous theorem of Jacobson [Ann. of Math. 46 (1945), p. 695–707] for commutativity of potent rings.

[6] P. V. Danchev and J. Cui, *Some new characterizations of periodic rings*, J. Algebra & Appl. (12) **19** (2020).

ABSTRACT: A ring R is called periodic if, for every a in R, there exist two distinct positive integers m and n such that $a^m = a^n$. The paper is devoted to a comprehensive study of the periodicity of arbitrary unital rings. Some new characterizations of periodic rings and their relationship with strongly π-regular rings are provided as well as, furthermore, an application of the obtained main results to a $*$-version of a periodic ring is being considered. Our theorems somewhat considerably improved on classical results in this direction.

[7] P. V. Danchev, *On some decompositions of matrices over algebraically closed and finite fields*, J. Siberian Federal University - Mathematics & Physics (5) **14** (2021), 547-553.

ABSTRACT: Decomposition of every square matrix over an algebraically closed field or over a finite field into a sum of a potent matrix and a nilpotent matrix of order 2 is considered. This can be related to our recent paper, published in Linear & Multilinear Algebra (2022). The question of when each square matrix over an infinite field can be decomposed into a periodic matrix and a nilpotent matrix of order 2 is also completely considered.

[8] P. V. Danchev and A. Cimpean, *n-Torsion clean and almost n-torsion clean matrix rings*, Russian Math. (Iz. VUZ) (1) **65** (2021), 47-56.

ABSTRACT: We (completely) determine those natural numbers n for which the full matrix ring $M_n(F_2)$ and the triangular matrix ring $T_n(F_2)$ over the two elements field F_2 are either n-torsion clean or are almost n-torsion clean, respectively. These results somewhat address and settle a question, recently posed by Danchev–Matczuk in Contemp. Math. (2019) as well as they supply in a more precise aspect the nil-cleanness property of the full matrix $n \times n$ ring $M_n(F_2)$ for all naturals $n \geq 1$, established in Linear Algebra & Appl. (2013) by Breaz–Cˇalugˇareanu– Danchev–Micu and again in Linear Algebra & Appl. (2018) by Šter as well as in Indag. Math. (2019) by Shitov.

[9] P. V. Danchev and J. P. Bell, *Affine representability and decision procedures for commutativity theorems for rings and algebras*, Israel J. Math. (1) **249** (2022), 121-166.

ABSTRACT: We consider applications of a finitary version of the Affine Representability theorem, which follows from recent work of Belov-Kanel, Rowen, and Vishne. Using this result we are able to show that when given a finite set of polynomial identities, there is an algorithm that terminates after a finite number of steps which decides whether these identities force a ring to be commutative. We then revisit old commutativity theorems of Jacobson and Herstein in light of this algorithm and obtain general results in this vein. In addition, we completely characterize the homogeneous multilinear identities that imply the commutativity of a ring.

[10] P. V. Danchev and T.-K. Lee, *On n-generalized commutators and Lie ideals of rings*, J. Algebra & Appl. (12) **21** (2022).

ABSTRACT: Let R be an associative ring. Given a positive integer $n \geq 2$, for $a_1, \ldots, a_n \in R$ we define $[a_1,\ldots,a_n]_n := a_1a_2 \cdots a_n - a_na_{n-1} \cdots a_1$, the n-generalized commutator of $a_1,\ldots,a_n$. By an n-generalized Lie ideal of R (at the $(r + 1)$th position with $r \geq 0$) we mean an additive subgroup A of R satisfying $[x_1, ..., x_r, a, y_1, ..., y_s]_n \in A$ for all $x_i, y_j \in R$ and all $a \in A$, where $r + s = n - 1$. In the paper, we study n-generalized commutators of rings and prove that if R is a noncommutative prime ring and $n \geq 3$, then every nonzero n-generalized Lie ideal of R contains a nonzero ideal. Therefore, if R is a noncommutative simple ring, then $R = [R, . . . , R]_n$. This extends a classical result due to Herstein [Generalized commutators in rings, Portugal. Math. 13 (1954) 137–139]. Some generalizations and related questions on n-generalized commutators and their relationship with noncommutative polynomials are also discussed.

[11] P. V. Danchev, E. Garcia and M. G. Lozano, *Decompositions of matrices into potent and square-zero matrices*, Internat. J. Algebra & Computat. (2) **32** (2022), 251-263.

ABSTRACT: In order to find a suitable expression of an arbitrary square matrix over an arbitrary finite commutative ring, we prove that every such matrix is always representable as a sum of a potent matrix and a nilpotent matrix of order at most two when the Jacobson radical of the ring has zero-square. This somewhat extends results of ours in Linear Multilinear Algebra (2022) established for matrices considered on arbitrary fields. Our main theorem also improves on recent results due to Abyzov et al. in Mat. Zametki (2017), Ster in ˇ Linear Algebra Appl. (2018) and Shitov in Indag. Math. (2019).

[12] P. V. Danchev, E. Garcia and M. G. Lozano, *Decompositions of matrices into diagonalizable and square-zero matrices*, Linear & Multilinear Algebra (19) **70** (2022), 4056-4070.

ABSTRACT: In order to find a suitable expression of an arbitrary square matrix over an arbitrary field, we prove that every square matrix over an infinite field is always representable as a sum of a diagonalizable matrix and a nilpotent matrix of order less than or equal to two. In addition, each 2×2 matrix over any field admits such a representation. We, moreover, show that, for all natural numbers $n \geq 3$, every $n \times n$ matrix over a finite field having no less than $n + 1$ elements also admits such a decomposition. The latter completes a recent example due to Breaz [Matrices over finite fields as sums of periodic and nilpotent elements. Linear Algebra Appl. 2018;555:92–97]. As a consequence of these decompositions, we show that every nilpotent matrix over a field can be expressed as the sum of a potent matrix and a square-zero matrix. This somewhat improves on recent results due to Abyzov et al. [On some matrix analogues of the little Fermat theorem. Mat Zametki. 2017;101(2):163–168] and Shitov [The ring $M_{8k+4}(Z_2)$ is nil-clean of index four. Indag Math (N.S.). 2019;30:1077–1078].

[13] P. V. Danchev, *Weakly invo-clean rings having weak involution*, Vestnik Udmurtskogo Universiteta – Matematika, Mekhanika, Komp'yuternye Nauki (1) **32** (2022), 18-25.

ABSTRACT: We completely describe up to an isomorphism the structure of weakly invo-clean rings possessing weak involution. The obtained results expand two own establishments, namely those from Afrika Mat. (2017) concerning weakly invo-clean rings as well as those from Far East J. Math. Sci. (2021) concerning invo-clean rings with weak involution.

[14] P. V. Danchev and J. Cui, *On strongly π-regular rings with involution*, Commun. Math. (1) **31** (2023), 73-80.

ABSTRACT: Recall that a ring R is called strongly π-regular if, for every $a \in R$, there is a positive integer n such that $a^n \in a^{n+1}R \cap Ra^{n+1}$. In this paper we give a further study of the notion of a strongly π-$*$-regular ring, which is the $*$-version of strongly π-regular rings and which was originally introduced by Cui-Wang in J. Korean Math. Soc. (2015). We also establish various properties of these rings and give several new characterizations in terms of (strong) π-regularity and involution. Our results also considerably extend recent ones in the subject due to Cui-Yin in Algebra Colloq. (2018) proved for π-∗-regular rings and due to Cui-Danchev in J. Algebra Appl. (2020) proved for ∗-periodic rings.

[15] P. V. Danchev*, A symmetric generalization of π-regular rings*, Ric. Mat. (1) **73** (2024).

ABSTRACT: We introduce and investigate the so-called D-regularly nil clean rings by showing that these rings are, in fact, a non-trivial generalization of the classical π-regular rings (in particular, of the von Neumann regular rings and of the strongly π-regular rings). Some other close relationships with certain well-known classes of rings such as exchange rings, clean rings, nil-clean rings, etc., are also demonstrated. These results somewhat supply a recent publication of the author in Turk J Math (2019) as well as they some8 what expand the important role of the two examples of nil-clean rings obtained by Šter in Linear Algebra Appl (2018). Likewise, the obtained symmetrization supports that similar property for exchange rings established by Khurana et al. in Algebras Represent Theory (2015).