Journal of Algebra and Its Applications
Vol. 19, No. 12 (2020) 2050235 (15 pages)

DOI: 10.1142/S0219498820502357

Some new characterizations of periodic rings

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Received 12 May 2019
Accepted 16 October 2019
Published 29 November 2019

Communicated by D. Herbera

A ring R is called *periodic* if, for every a in R, there exist two distinct positive integers m and n such that $a^m = a^n$. The paper is devoted to a comprehensive study of the periodicity of arbitrary unital rings. Some new characterizations of periodic rings and their relationship with strongly π-regular rings are provided as well as, furthermore, an application of the obtained main results to a $*$-*version* of a periodic ring is being considered. Our theorems somewhat considerably improved on classical results in this direction.

Keywords: Periodic rings; strongly π-regular rings; π-UU rings; strongly nil clean rings; $*$-periodic rings.

Mathematics Subject Classification 2020: 16U99, 16E50, 16W10

1. Introduction

All rings into consideration in the present paper are associative with identity. Let R be a ring. The letters $U(R)$, $J(R)$ and $\mathrm{Nil}(R)$ stand for the set of all units, the Jacobson radical and the set of all nilpotents of R, respectively. As usual, the symbol $R[t]$ denotes the polynomial ring over R, $\mathbb{Z}$ denotes the ring of integers, and $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}$ modulo the ideal generated by n.

For a ring R, the inclusion $1 + \mathrm{Nil}(R) \subseteq U(R)$ always holds. So, a unit u of a ring R is called *unipotent* if $u \in 1+\mathrm{Nil}(R)$. Due to Călugăreanu [9], and to Danchev

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and Lam [19], R is called a *UU-ring* if every unit of R is unipotent, or equivalently $U(R) = 1 + \mathrm{Nil}(R)$. These UU-rings are closely related to strongly nil clean rings (herein, a ring R is *strongly nil clean* [20] if, for each $a \in R$, there exist $e^2 = e \in R$ and $b \in \mathrm{Nil}(R)$ such that $a = e + b$ and $be = eb$), and they have been extensively investigated recently (see [19, 26, 29], etc.). In [17], Danchev introduced the notion of a π-UU ring. A ring R is called *π-UU* if, for every $u \in U(R)$, there exists some positive integer n such that $u^n = 1 + b$, where $b \in \mathrm{Nil}(R)$. Recall that a ring R is *potent* if, for every $a \in R$, $a^n = a$ for some integer $n \geq 2$; and R is called *periodic* if, for every $a \in R$, there exist distinct positive integers m and n such that $a^m = a^n$. Periodic rings were introduced by Chacron in [10] (see also [11]). Clearly, potent rings are periodic and the latter are π-UU (indeed, if R is a periodic ring and u is a unit in R, then there is a positive integer n such that $u^n = 1$). Recall that a ring is *strongly π-regular* if, for every $a \in R$, there exists a positive integer m such that $a^m \in a^{m+1}R \cap Ra^{m+1}$. In fact, Dischinger in his PhD thesis [22] proved that a ring R is strongly π-regular if, and only if, for each $a \in R$, there exists an integer $m \geq 1$ such that $a^m \in a^{m+1}R$ (see also [21] and [1]). As it is well known, periodic rings are strongly π-regular.

In this article, some examples and basic properties of π-UU rings are investigated. Properties such as being π-UU, strongly π-regularity and strongly nil cleanness are subsequently applied to characterize periodic rings as some equivalent statements are obtained. Further, by extending periodic rings to their $*$-versions, we introduce the notion of *$*$-periodic rings*. Various characterizations of $*$-periodic rings are provided. In particular, it is shown that a ring R is $*$-periodic if, and only if, R is periodic and idempotents of R are projections if, and only if, R is a strongly π-regular π-UU ring and idempotents of R are projections if, and only if, R is an abelian π-UU ring, $R/J(R)$ is $*$-regular and $J(R)$ is nil.

2. π-UU Rings

In this section, some examples and properties of π-UU rings are provided.

Example 2.1. (1) All periodic rings, UU rings and rings with finite units are π-UU rings.

(2) A division ring R is a π-UU ring if and only if R is a potent field.

Proof. As point (1) is self-evident, it suffices to show only (2). To do that, assume that R is a division π-UU ring. Then, for any nonzero $a \in R$, we have $a^n = 1$ for some integer n as $\mathrm{Nil}(R) = 0$. By the well-known Jacobson's Theorem [30, Theorem 12.10], the ring R is commutative. So, R has to be a potent field. The converse is clear. $\square$

It is worthwhile noticing that there exists a noncommutative π-UU ring which is neither periodic nor UU. Motivated by [19, Example 2.5], we let $p \geq 3$ be a prime and let $\mathbb{Z}_p\langle x, y\rangle$ be the free algebra over the simple p element field $\mathbb{Z}_p$, generated by noncommutative variables x and y, and set $R = \mathbb{Z}_p\langle x, y\rangle/(x^2)$. Then, one may calculate that $\mathrm{Nil}(R) = \mathbb{Z}_p x + xRx$ and $U(R) = \{r + \mathbb{Z}_p x + xRx \mid r \in \mathbb{Z}_p\backslash\{0\}\}$. We claim now that R is π-UU. Indeed, for any nonzero $r \in \mathbb{Z}_p$, $r^k = 1 \in \mathbb{Z}_p$ for some $k \geq 1$; so $(r + \mathbb{Z}_p x + xRx)^k \subseteq 1 + \mathrm{Nil}(R)$. Clearly, by an inspection, we deduce that R is neither a periodic ring nor a UU-ring.

Proposition 2.2. *Let R be a π-UU ring. Then the following statements hold*:

(1) *If S is a factor ring of R such that units of S lift to units of R, then S is π-UU.*
(2) *Any* (*unital*) *subring of R is π-UU.*
(3) *If $u_i \in U(R)$ for $i = 1, \ldots, k$, then $u_i^n \in 1 + \mathrm{Nil}(R)$ for some integer $n \geq 1$.*

Proof. (1) Suppose that $f : R \to S$ is an epimorphism of rings. Let $v \in U(S)$. Then there exist $u \in U(R)$ and an integer n such that $v = f(u)$ and $u^n = 1 + b \in 1 + \mathrm{Nil}(R)$. So one has $v^n = f(u^n) = 1 + f(b) \in 1 + \mathrm{Nil}(S)$, as desired.

(2) The proof is similar to that of [19, Theorem 2.6(3)].

(3) By assumption, for each i, we may let $u_i^{n_i} \in 1 + \mathrm{Nil}(R)$, where $n_i \geq 1$. Let n be a common multiple of all n_i. It is easy to see that $u_i^n \in 1 + \mathrm{Nil}(R)$. $\square$

Lemma 2.3. *Let I be a nil ideal of a ring R. Then R is π-UU if and only if so is R/I.*

Proof. Assume that R/I is a π-UU ring. Let $u \in U(R)$. Then $\overline{u} := u + I \in U(R/I)$. So there is an integer k satisfying $\overline{u}^k - \overline{1} = \overline{u^k - 1} \in \mathrm{Nil}(R/I)$. As I is nil, we have $u^k - 1 \in \mathrm{Nil}(R)$, which implies R is π-UU. The converse follows with the aid of Proposition 2.2(1). $\square$

The condition I is nil in Lemma 2.3 is necessary. For example, let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at the prime ideal (p). Then $\mathbb{Z}_{(p)}/J(\mathbb{Z}_{(p)}) \cong \mathbb{Z}_p$ is obviously a π-UU ring, but $\mathbb{Z}_{(p)}$ is manifestly not π-UU.

Lemma 2.4 ([17, Lemma 2.3]). *If R is a π-UU ring, then so is eRe for any $e^2 = e \in R$.*

Proposition 2.5. *The finite direct product $\prod_{i=1}^m R_i$ is π-UU if and only if each component R_i is π-UU.*

Proof. Suppose that each R_i is a π-UU ring. Let $\alpha = (u_1, u_2, \ldots, u_m) \in U(\prod_{i=1}^m R_i)$. Then $u_i \in U(R_i)$ for all i. By hypothesis, there exists an integer n_i such that $u_i^{n_i} \in 1 + \mathrm{Nil}(R_i)$. In view of the proof of Proposition 2.2(3), we deduce that $u_i^n \in 1 + \mathrm{Nil}(R_i)$ for some integer n and $i = 1, \ldots, m$. This implies that $\alpha^n \in (1, 1, \ldots, 1) + (\mathrm{Nil}(R_1), \mathrm{Nil}(R_2), \ldots, \mathrm{Nil}(R_m)) = 1 + \mathrm{Nil}(\prod_{i=1}^m R_i)$. The converse follows by Proposition 2.2(1). $\square$

Remark 2.6. There exists an infinite product of π-UU rings that is not π-UU. In fact, let p_i be primes with $p_1 < p_2 < p_3 < \cdots$. Clearly, each $R_i := \mathbb{Z}_{p_i}$ is π-UU.

Let $u_i \in U(R_i)\backslash\{1\}$ with $u_i^{p_i-1} = 1$. As $\mathrm{Nil}(\prod_{i=1}^{\infty} R_i) = \{0\}$ and any power of $(u_1, u_2, \ldots, u_i, \ldots) \in U(\prod_{i=1}^{\infty} R_i)$ does not equal to the identity of $\prod_{i=1}^{\infty} R_i$, the direct product $\prod_{i=1}^{\infty} R_i$ is not π-UU.

For a ring R, let $T_n(R)$ be the upper triangular matrix ring over R, and $R_n = \{(a_{ij}) \in T_n(R) \mid a_{11} = a_{22} = \cdots = a_{nn}\}$.

Theorem 2.7. *Let R be a ring. The following are equivalent*:

(1) *R is a π-UU ring.*
(2) *$T_n(R)$ is π-UU for any positive integer n.*
(3) *R_n is π-UU for any positive integer n.*
(4) *$R[t]/(t^n)$ is π-UU for any integer $n \geq 2$.*

Proof. Note that $R[x]/(x^n)$ can be viewed as a subring of R_n. By Proposition 2.2(2), (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows, and (4) $\Rightarrow$ (1) follows by making use of Proposition 2.2(1).

(1) $\Rightarrow$ (2). Let $I(R) \subseteq T_n(R)$ be the set of all upper triangular matrices whose diagonals are zeros. Then $I(R)$ is known to be a nil ideal of $T_n(R)$. So, the n-times direct product, $T_n(R)/I(R) \cong R \times R \times \cdots \times R$ is π-UU by Proposition 2.5 since R is a π-UU ring. In view of Lemma 2.3, $T_n(R)$ is π-UU. $\square$

Proposition 2.8. *Let R, S be rings and N be an (R, S)-bimodule. Then the formal triangular matrix ring $\left(\begin{smallmatrix} R & N \\ 0 & S \end{smallmatrix}\right)$ is a π-UU ring if and only if R and S are π-UU rings.*

Proof. Assume that $K := \left(\begin{smallmatrix} R & N \\ 0 & S \end{smallmatrix}\right)$ is a π-UU ring. Then by Lemma 2.4, R and S are π-UU rings. For the converse, let $I = \left(\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right) \subseteq K$. Then I is a nil ideal of K, and $K/I \cong R \times S$ is π-UU by Proposition 2.5. So the result follows by Lemma 2.3. $\square$

It is clear that the matrix ring $M_n(R)$ over any finite ring R is π-UU; e.g., $M_n(\mathbb{Z}_2)$ is π-UU. However, there exists an infinite π-UU ring over which the matrix ring is no longer π-UU.

Example 2.9. The matrix ring $M_n(\mathbb{Z})$ is not π-UU for any integer $n \geq 2$.

Proof. Clearly, $\mathbb{Z}$ is a π-UU ring as $U(\mathbb{Z}) = \{-1, 1\}$. By virtue of Lemma 2.4, it suffices to show that $M_2(\mathbb{Z})$ is not π-UU. Assume on the contrary, let $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in U(M_2(\mathbb{Z}))$ and $A^k - I_2 \in \mathrm{Nil}(M_2(\mathbb{Z}))$ for some integer k. Note that the minimal polynomial of A is x^2-x-1. Since $A^k - I_2 \in \mathrm{Nil}(M_2(\mathbb{Z}))$, we have that $(A^k - I_2)^2 = 0$ and hence, $(x^k-1)^2$ is a multiple of x^2-x-1. But all the complex roots of $(x^k-1)^2$ have module 1, whereas the complex roots of $x^2 - x - 1$ do not have module 1, a contradiction. $\square$

In view of Example 2.9, one may observe that for an idempotent $e \in R$, R is not necessarily a π-UU ring even if eRe and $(1-e)R(1-e)$ are both π-UU rings.

Proposition 2.10. *The power series ring $R[[t]]$ is not π-*UU *for any ring R.*

Proof. In view of [30, Ex. 5.6], $J(R[[t]]) = J(R) + tR[[t]]$. So $1 + t \in U(R[[t]])$. Clearly, for any positive integer n, $(1+t)^n - 1$ is not nilpotent. Hence, $R[[t]]$ is not a π-UU ring. □

Recall that a ring R is *reduced* if it contains no nonzero nilpotents. The following assertion considerably extends the corresponding result from [9] established for UU-rings.

Proposition 2.11. *If R is a commutative ring, then R is π-*UU *if and only if so is $R[t]$.*

Proof. Assume that R is a commutative π-UU ring. In view of [30, Theorem 5.1], $J(R[t]) = \mathrm{Nil}(R[t]) = \mathrm{Nil}(R)[t]$. So, $J(R[t])$ is nil and $R[t]/J(R[t]) = R[t]/\mathrm{Nil}(R)[t] \cong (R/\mathrm{Nil}(R))[t]$. Note that $R/\mathrm{Nil}(R)$ is reduced. Then $(R/\mathrm{Nil}(R))[t]$ is reduced. By [2, Exercise 1.2], $U(R/\mathrm{Nil}(R)) = U((R/\mathrm{Nil}(R))[t])$ (see also [25, Corollary 1.7]). It follows that $(R/\mathrm{Nil}(R))[t]$ is a π-UU ring since $R/\mathrm{Nil}(R)$ is π-UU. Thus, $R[t]/J(R[t])$ is π-UU, and therefore, $R[t]$ is a π-UU ring by Lemma 2.3. The other direction is clear. □

We ending this section by asking if R is a π-UU ring, does it follow that $J(R)$ is nil (compare with [17] and [19], where some partial cases are being considered)? If yes, some reduction to the periodicity of the factor-ring $R/J(R)$ will be possible, which fact will considerably help us to simplify the proofs by considering semiprimitive (i.e. Jacobson semi-simple) periodical rings.

3. Periodic Rings

Recall that an element a of a ring R is *strongly nil clean* [20] if there exist $e^2 = e$ and $b \in \mathrm{Nil}(R)$ such that $a = e + b$ and $ae = ea$; in this case, we say that $a = e + b$ is a strongly nil clean expression of a; R is *strongly nil clean* if all of its elements are strongly nil clean (equivalently, $a - a^2 \in \mathrm{Nil}(R)$ for each $a \in R$ [35]). In [13], a ring R is called *strongly 2-nil clean* if, for each $a \in R$, $a - a^3 \in \mathrm{Nil}(R)$ (equivalently, a^2 is strongly nil clean in R). Recall that a unit u of a ring R is *n-UU* if $u^n \in 1 + \mathrm{Nil}(R)$ (see [17]).

For an element $a \in R$, write $\mathrm{comm}(a) = \{x \in R \mid xa = ax\}$.

Lemma 3.1. *Let R be a ring, $e^2 = e \in R$, $a \in U(eRe)$ and an integer $n \geq 1$. The following are equivalent*:

(1) *a is n-*UU *in eRe.*
(2) *a^n is strongly nil clean in R.*

Proof. (1) ⇒ (2). By assumption, $b := a^n - e \in \mathrm{Nil}(eRe) \subseteq \mathrm{Nil}(R)$. So, $a^n = e + b$ is a strongly nil clean expression of a^n in R as $ea = ae$.

(2) $\Rightarrow$ (1). Let $a^n = f + b$ with $f^2 = f \in R$, $b \in \mathrm{Nil}(R)$ and $a^n f = f a^n$. In view of [29, Proposition 2.4], $\mathrm{comm}(a^n) \subseteq \mathrm{comm}(f)$. As $e \in \mathrm{comm}(a^n)$, one has $ef = fe$ is an idempotent of eRe, and whence $be = eb \in \mathrm{Nil}(eRe)$. Thus, $a^n = efe + ebe$ is a strongly nil clean expression of a^n in eRe. So, we have $efe = a^n - ebe \in U(eRe)$ since $a^n \in U(eRe)$, which implies that $efe = e$. This proves that a is n-UU in eRe. □

Proposition 3.2. *Let R be a ring, $a \in R$ and an integer $n \geq 1$. The following are equivalent*:

(1) *a^n is strongly nil clean in R.*
(2) *There exists $e^2 = e \in \mathrm{comm}(a)$ such that $a = x + y$, where x is an n-UU element in eRe and $y \in \mathrm{Nil}((1-e)R(1-e))$.*
(3) *There exists $e^2 = e \in \mathrm{comm}(a)$ such that $a = x + y$, where $x \in U(eRe)$ with x^n strongly nil clean in R, and $y \in \mathrm{Nil}((1-e)R(1-e))$.*

Proof. (1) $\Rightarrow$ (2). Let $a^n = e + b$ be a strongly nil clean expression of a^n, with $e^2 = e \in R$, $b \in \mathrm{Nil}(R)$ and $eb = be$. Then, we have $ea = ae$ by [29, Proposition 2.4]. It follows that $(ae)^n = a^n e = e + ebe \in e + \mathrm{Nil}(eRe)$ and $(a(1-e))^n = b(1-e)$ is nilpotent in $(1-e)R(1-e)$. Write $x = ae$ and $y = a(1-e)$. Thus, $a = x + y$, where $x \in eRe$ is an n-UU element and $y \in \mathrm{Nil}((1-e)R(1-e))$.

(2) $\Rightarrow$ (1). By hypothesis, let $x^n = e + q$ with $q \in \mathrm{Nil}(eRe)$. Noting that $xy = yx = 0$, we have $a^n = x^n + y^n = e + (q + y^n)$. Clearly, $b := q + y^n \in \mathrm{Nil}(R)$. So, $a^n = e + b$ is a strongly nil clean expression of a^n in R.

(2) $\Leftrightarrow$ (3) follows by Lemma 3.1. □

An element a of a ring R is called *strongly regular* if $a = eu = eu$ for some $e^2 = e \in R$ and $u \in U(R)$; R is *strongly regular* if every element of R is strongly regular, or equivalently R is abelian regular (herein, a ring is *abelian* if every idempotent of the ring is central).

Lemma 3.3 ([33, Theorem 2(1)]). *A ring R is strongly π-regular if and only if for each $a \in R$, $a = ev + b = ve + b$, where $e^2 = e$, $v \in U(R)$ and $b \in \mathrm{Nil}(R)$ with $ab = ba$.*

Next, we give some new characterizations of periodic rings. These will be applied at the end of this section to exhibit some concrete examples of such rings.

Theorem 3.4. *Let R be a ring. The following are equivalent*:

(1) *R is a periodic ring.*
(2) *For each $a \in R$, $a = f + b$, where $f^n = f$ for some integer $n \geq 2$, $af = fa$ and $b \in \mathrm{Nil}(R)$.*
(3) *For each $a \in R$, $a = ev + b = ve + b$, where $e^2 = e \in R$, $v^{n-1} = 1$ for some integer $n \geq 2$ and $b \in \mathrm{Nil}(R)$ with $ab = ba$.*
(4) *For each $a \in R$, $a - a^n \in \mathrm{Nil}(R)$ for some integer $n \geq 2$.*

(5) *For each $a \in R$, there exists an integer $m \geq 1$ such that a^m is strongly nil clean in R.*
(6) *For each $a \in R$, there exists $e^2 = e \in \mathrm{comm}(a)$ such that eRe is a π-UU ring, and $a = x + y$ with $x \in U(eRe)$ and $y \in \mathrm{Nil}((1-e)R(1-e))$.*
(7) *R is a strongly π-regular π-UU ring.*

Proof. (1) $\Rightarrow$ (2) follows from [4, Lemma 1(c)].

(2) $\Rightarrow$ (3). Assume that (2) holds. By [27, Lemma 2.1], let $e = f^{n-1}$ and $v = 1 + f - f^{n-1}$. Then $e^2 = e$ and $(1-e)f = f(1-e) = 0$ as $f = f^n$. So, $v^{n-1} = ((1-e)+f)^{n-1} = 1 - e + f^{n-1} = 1$, and $f = ve = ev$. Thus $a = f + b = ev + b = ve + b$, where $e^2 = e \in R$, $v^{n-1} = 1$ and $b \in \mathrm{Nil}(R)$ with $ab = ba$.

(3) $\Rightarrow$ (4). Given $a \in R$ in (3). Since $(ev)^n = ev^n = evv^{n-1} = ev$ and $(ev)b = b(ev)$, we obtain $a - a^n = (ev+b) - (ev+b)^n = (ev - (ev)^n) - b(1 + \sum_{i=1}^n C_n^i (ev)^{n-i} b^{i-1}) = -b(1 + \sum_{i=1}^n C_n^i (ev)^{n-i} b^{i-1}) \in \mathrm{Nil}(R)$ as $b \in \mathrm{Nil}(R)$.

(4) $\Rightarrow$ (5). By hypothesis, we may let $(a - a^{m+1})^k = 0$, where $m,\ k \geq 1$. Then we have $0 = a^k(1-a^m)^k$, which implies that $0 = a^{km}(1-a^m)^k = (a^m(1-a^m))^k$. So, $a^m - a^{2m} \in \mathrm{Nil}(R)$. In view of [35, Lemma 3.5], a^m is strongly nil clean in R.

(5) $\Rightarrow$ (6). In view of Proposition 3.2, it suffices to prove that eRe is a π-UU ring. Let $u \in U(R)$. Then there is a positive integer n such that $u^n = e + b$ where $e^2 = e,\ b \in \mathrm{Nil}(R)$ and $eb = be$. So, $e = u^n - b \in U(R)$, which implies $e = 1$, whence $u^n \in 1 + \mathrm{Nil}(R)$. This proves that R is a π-UU ring. By making use of Lemma 2.4, eRe is a π-UU ring.

(6) $\Rightarrow$ (7). By (6), let $x^m = e + b \in e + \mathrm{Nil}(eRe)$ for some integer m. Note that $xy = yx = 0$. So, one has $a^m = x^m + y^m = e + (b + y^m)$, where $e^2 = e$, $b + y^m \in \mathrm{Nil}(R)$ and $ae = ea$. We conclude that R is a π-UU ring. Indeed, if $a \in U(R)$ then $e = 1$, and therefore, $a^m \in 1 + \mathrm{Nil}(R)$. Further, $x = xe = (x + (1-e))e = e(x + (1-e))$ is strongly regular in R since $x + (1-e) \in U(R)$. So, $a = x + y$ is a sum of a strongly regular element and a nilpotent that commute. By virtue of Lemma 3.3, R is strongly π-regular.

(7) $\Rightarrow$ (1). Let $a \in R$. Since R is strongly π-regular, by [32, Proposition 1], there exists an integer $n \geq 1$ such that $a^n = eu = ue$, where $e^2 = e$ and $u \in U(R)$. By assumption, let $u^k = 1 + b \in 1 + \mathrm{Nil}(R)$ for some $k \geq 1$. It follows that $a^{nk} = eu^k = e + eb$ and $a^{2nk} = eu^{2k} = e + (2e + eb)b$ as $eb = be$. Since $b \in \mathrm{Nil}(R)$, $a^{nk} - a^{2nk} = -(e + eb)b \in \mathrm{Nil}(R)$. So, one has $a^l = a^{l+1}f(a)$ for some integer $l \geq 1$ and $f(t) \in \mathbb{Z}[t]$. By virtue of [5, Theorem 1], R is a periodic ring. $\square$

We note that the equivalence "(1) $\Leftrightarrow$ (4)" in Theorem 3.4 can also be deduced by the main theorem in [11] (see also [12, Theorem 1.1]). Moreover, in [28, Question 3.17], the authors asked what can be said about rings such that $a^n - a$ is nilpotent for all a and a fixed integer n. By Theorem 3.4(4), these rings are obviously periodic. However, the converse need not be true and so, at this stage, the posed question is not completely settled.

Remark 3.5. Note that it is easy to see that the direct product of two strongly π-regular rings is a strongly π-regular ring, and the direct product of two π-UU rings is a π-UU ring in accordance with Proposition 2.5. Hence, by Theorem 3.4, the direct product of two periodic rings R and S is a periodic ring. This is also easy to be seen by [11, Proposition 1].

Nevertheless, the infinite direct product of periodic rings need not be again a periodic ring (indeed, this can be infered from Remark 2.6).

The following result somewhat addresses the main question of whether or not periodicity is retained by the full matrix ring if the former ring is periodic (the same question appeared to be actual for strongly π-regular rings as well – compare with [7] and [24]).

Corollary 3.6. *Let R be a ring such that $a^m = a$, for all $a \in R$ and a fixed integer $m > 1$. Then, for any positive integer n, the matrix ring $M_n(R)$ is periodic.*

Proof. Since R is commutative strongly π-regular, it follows from [7] that $M_n(R)$ is strongly π-regular.

We shall prove that $M_n(R)$ is a π-UU ring. By the arguments on [30, p. 197], R is a subdirect product of its left primitive homomorphic images R_i $(i \in I)$ and every R_i is a field such that $a_i^m = a_i$, for all $a_i \in R_i$. Therefore, $|R_i| \leq m$, for all i. Now, to prove that $M_n(R)$ is a π-UU ring, by Proposition 2.2, it is enough to prove that the product $M_n(\prod_{i\in I} R_i) \cong \prod_{i\in I} M_n(R_i)$ is a π-UU ring. Let $t = m^{n^2}!$ and let $U = (U_i)_{i\in I} \in \prod_{i\in I} M_n(R_i)$ be a unit. Note that $U_i^t = I_n$, for all $i \in I$, because U_i is a unit in $M_n(R_i)$ and $|M_n(R_i)| \leq m^{n^2}$. Thus, we have that $U^t = 1$. Hence, $M_n(R)$ is a π-UU ring. By Theorem 3.4, one concludes that $M_n(R)$ is a periodic ring, as asserted. $\square$

On the other side, imitating the proof of the above characterization theorem, one directly deduces the following:

Corollary 3.7. *Let I be a nil-ideal of a ring R. Then R is periodic if and only if R/I is periodic. In particular, R is periodic if and only if $J(R)$ is nil and $R/J(R)$ is periodic.*

The next technicality is useful, and it will be applied in the sequel.

Lemma 3.8 ([3, Lemma 5]). *If R is an abelian strongly π-regular ring, then* $\mathrm{Nil}(R) = J(R)$.

In the abelian case, when all idempotents are central, one can say even something more as follows:

Corollary 3.9. *Let R be an abelian ring. The following are equivalent*:

(1) *R is a periodic ring.*
(2) *$R/J(R)$ is potent and $J(R)$ is nil.*

Proof. (1) $\Rightarrow$ (2). Clearly, R is an abelian strongly π-regular ring. So, $J(R) = \mathrm{Nil}(R)$ by Lemma 3.8. Combining this with Corollary 3.7, we have $R/J(R)$ is a reduced periodic ring. Let $x \in R/J(R)$. In view of Theorem 3.4, $x - x^n \in \mathrm{Nil}(R/J(R)) = 0$ for some integer $n \geq 2$. Therefore, $R/J(R)$ is potent.

(2) $\Rightarrow$ (1). For any $a \in R$, write $\overline{a} = a + J(R) \in R/J(R)$. Then there is an integer $n \geq 2$ such that $\overline{a} = \overline{a}^n$. So, $a - a^n \in J(R) \subseteq \mathrm{Nil}(R)$. Thus, R is periodic employing Theorem 3.4. □

In general, (1) cannot imply (2) in Corollary 3.9 if R is not abelian. For instance, to see that, let $R = M_2(\mathbb{Z}_2)$. Then R is a periodic ring with $J(R) = 0$. However, $R/J(R) \cong R$ is obviously not potent.

As a consequence of [19, Theorem B] or [29, Theorem 2.7], which both say that a ring R is strongly nil clean if, and only if, $R/J(R)$ is boolean and $J(R)$ is nil, accomplishing this with Theorem 3.4, we have the following result immediately.

Corollary 3.10. *A ring R is strongly nil clean if and only if R is periodic and $R/J(R)$ is boolean.*

According to [18], a ring R is called *strong regularly nil clean* if, for every $a \in R$, there exists $e^2 = e \in Ra$ such that $ae = ea$ and $a - ae$ is nilpotent. It was shown in [18, Proposition 2.2] that these are actually strongly π-regular rings. So, the next statement is immediate as periodic rings are always strongly π-regular. We, however, will give a more transparent proof.

Proposition 3.11. *Every periodic ring is strong regularly nil clean.*

Proof. Suppose that R is a periodic ring. Let $a \in R$. Utilizing Theorem 3.4(4), $a - a^n$ is nilpotent for some integer $n \geq 2$. Assume that $(a - a^n)^m = 0$. So one has $(a^{n-1} - (a^{n-1})^2)^m = (a^{n-1})^m(1 - a^{n-1})^m = 0$. Then by [35, Lemma 3.5], we can find an idempotent $e \in \mathbb{Z}[a]$ such that $a^{n-1} = e + w$, where $w \in \mathrm{Nil}(R)$. Hence $(a(1-e))^{n-1} = a^{n-1}(1-e) = w(1-e) \in \mathrm{Nil}(R)$, whence $a - ae$ is nilpotent. □

A ring R is *nil clean* [20] if every element of R is a sum of an idempotent and a nilpotent; if these two elements commute, the ring is called *strongly nil clean*. As strongly nil clean rings are strongly π-regular, Diesl asked whether there is a nil clean ring that is not strongly π-regular [20]. Note that Šter gave an affirmative answer (see [34]) to that question, but a detailed analysis shows that the constructed ring is manifestly *not* π-UU. So, it arises here the following natural and rather difficult problem:

Question. *Is a nil clean π-*UU *ring also strongly π-regular* (*and hence periodic*)?

We just mention that it follows from [19, Theorem 4.3] that nil clean UU rings are strongly nil clean (and hence strongly π-regular and thus periodic). Likewise, if B is a boolean ring, then it was shown in [7] and [8] that, for any $n \geq 1$, the matrix ring $M_n(B)$ is simultaneously strongly π-regular and nil-clean, respectively.

Besides, it is not too hard to verify that $M_n(B)$ is also a π-UU ring (see, for more details, Corollary 3.6 alluded to above). Therefore, point (7) in Theorem 3.4 applies to get that $M_n(B)$ is periodic.

4. $*$-Periodic Rings

The results established in the previous section will be now applied to a $*$-version of periodicity. A ring R is called a *$*$-ring* (or, more precisely, a *ring with involution*) if there exists a map $* : R \to R$ such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x,\ y \in R$. Recall that an element p of a $*$-ring R is said to be a *projection* if $p^2 = p = p^*$ (see [6]). This section focuses on the study of $*$-periodic rings which can be viewed as $*$-versions of periodic rings.

Lemma 4.1 ([31, Lemma 2.1]). *Let R be a $*$-ring. If every idempotent of R is a projection, then R is abelian.*

Lemma 4.2. *Let R be a $*$-ring, $f \in R$ and an integer $n \geq 2$. Then $f^n = f$ and f^{n-1} is a projection if and only if $f = pv = vp$, where $p^2 = p^* = p \in R$ and $v^{n-1} = 1$.*

Proof. Assume that $f = pv = vp$, where $p^2 = p^* = p \in R$ and $v^{n-1} = 1$. Then $p = v^{n-1}p = (vp)^{n-1} = f^{n-1}$. So, f^{n-1} is a projection, and $f^n = v^n p = vp = f$.

Conversely, let $v = 1 + f - f^{n-1}$ and $p = f^{n-1}$. Then $p^2 = p = p^*$. As shown in the proof [27, Lemma 2.1], $f = vp = pv$ and $v^{n-1} = 1$. $\square$

Notice that Lemma 4.2 leads us to the following statements.

Lemma 4.3. *Let R be a $*$-ring and $a \in R$. The following are equivalent:*

(1) *$a = f + b$, where $f^n = f$ and f^{n-1} is a projection for some integer $n \geq 2$, $b \in \mathrm{Nil}(R)$ and $af = fa$.*
(2) *$a = pv + b = vp + b$, where $p^2 = p^* = p \in R$, $v^{n-1} = 1$ for some integer $n \geq 2$ and $b \in \mathrm{Nil}(R)$ with $ab = ba$.*

Definition 4.4. A $*$-ring R is called $*$-periodic if every element of R satisfies the conditions in Lemma 4.3.

Let R be a $*$-ring. Then $(J(R))^* \subseteq J(R)$. In particular, $R/J(R)$ is still a $*$-ring.

Theorem 4.5. *Let R be a $*$-ring. The following are equivalent:*

(1) *R is a $*$-periodic ring.*
(2) *R is a periodic ring and every idempotent of R is a projection.*
(3) *For each $a \in R$, there exists an integer $n \geq 1$ such that $a^n = p + b$, where $p^2 = p^* = p$, $b \in \mathrm{Nil}(R)$ and $pb = bp$.*
(4) *R is a periodic ring and $Re = Re^*$ for every idempotent $e \in R$.*
(5) *R is abelian, and for each $a \in R$ there exists an integer $n \geq 2$ such that $a - a^n \in \mathrm{Nil}(R)$ and $a^{n-1} - (a^{n-1})^* \in \mathrm{Nil}(R)$.*

Proof. (1) $\Rightarrow$ (2). Clearly, R is periodic. Let $e^2 = e \in R$. Then $e = f + b$, where $f^n = f$, f^{n-1} is a projection for some integer $n \geq 2$, $b \in \text{Nil}(R)$ and $ef = fe$. It follows that $e(1 - f^{n-1}) = b(1 - f^{n-1}) \in \text{Nil}(R)$. So, $e = ef^{n-1}$ as $e(1 - f^{n-1})$ is an idempotent. Moreover, $0 = (1 - e)e = (1 - e)f + (1 - e)b$, and then $(1 - e)f^{n-1} = -(1 - e)bf^{n-2} \in \text{Nil}(R)$. Since $(1 - e)f^{n-1}$ is an idempotent, it must be that $f^{n-1} = ef^{n-1}$. Thus, $e = f^{n-1}$ is a projection.

(2) $\Rightarrow$ (3). Let $a \in R$. Owing to Theorem 3.4, a^n is strongly nil clean for some integer $n \geq 1$. As idempotents coincide with projections, the result follows.

(3) $\Rightarrow$ (4). According to Theorem 3.4, R is a periodic ring. For any $e^2 = e \in R$, $e = p + b$ for some $p^2 = p^* = p \in R$, $b \in \text{Nil}(R)$ and $pe = ep$. Then $(e - p)^3 = e - p = b \in \text{Nil}(R)$, which yields that $e = p$. Thus, $Re = Rp = Re^*$.

(4) $\Rightarrow$ (1). In view of Lemma 4.3, it suffices to show that every idempotent of R is a projection. Let $e^2 = e \in R$. From $Re = Re^*$, we obtain $e = ee^*$ as e^* is also an idempotent. So, $e = (ee^*)^* = e^*$ is a projection, as desired.

(2) $\Rightarrow$ (5). By Lemma 4.1, R is abelian. So, $J(R) = \text{Nil}(R)$ by applying Lemma 3.8. Let $a \in R$. Then there exists $b \in \text{Nil}(R)$ such that $a = f + b$, where $af = fa$, $f^n = f$ and f^{n-1} is a projection for some integer $n \geq 2$. So, $a - a^n = (f + b) - (f + b)^n = (f + b) - (f^n + \sum_{i=1}^{n} C_n^i f^{n-i} b^i) = b(1 - \sum_{i=1}^{n} C_n^i f^{n-i} b^{i-1}) \in \text{Nil}(R)$. Notice that $a^{n-1} = (f + b)^{n-1} = f^{n-1} + b \sum_{i=1}^{n-1} C_{n-1}^i f^{n-1-i} b^{i-1}$. As both b and b^* are contained in $\text{Nil}(R) = J(R)$ and $f^{n-1} = (f^{n-1})^*$, it follows that $a^{n-1} - (a^{n-1})^* = b \sum_{i=1}^{n-1} C_{n-1}^i f^{n-1-i} b^{i-1} - b^* (\sum_{i=1}^{n-1} C_{n-1}^i f^{n-1-i} b^{i-1})^* \in \text{Nil}(R)$.

(5) $\Rightarrow$ (2). Assume (5) holds. Then, by Theorem 3.4, R is periodic. We only need to show that every idempotent of R is a projection. For any $e^2 = e \in R$, $e^* = (e^*)^2$. So, the hypothesis implies that $e - e^* \in \text{Nil}(R)$. Since R is abelian, $ee^* = e^*e$ and $(e - e^*)^3 = e - e^*$. It follows that $e - e^* = 0$, whence $e = e^*$, as required. □

For a $*$-ring R, the matrix ring $M_n(R)$ has a natural involution inherited from R: if $A = (a_{ij}) \in M_n(R)$, A^* equals (a_{ji}^*) (i.e. $A^* = (a_{ij}^*)^T = (a_{ji}^*)$). So we consider $M_n(R)$ as a $*$-ring with respect to this natural involution.

Corollary 4.6. *Let R be a $*$-ring. Then $M_n(R)$ is not $*$-periodic for any $n \geq 2$.*

For a $*$-ring R and $p^2 = p^* = p \in R$. Let $S = pRp$. Then the restriction of $*$ on S will be an involution of S, which is also denoted by $*$.

Corollary 4.7. *If R is a $*$-periodic ring, then so is eRe for any $e^2 = e \in R$.*

Proof. Let $S = eRe$. By Theorem 4.5, every idempotent of R is a projection. Thus, S is a $*$-ring. Note that S is a subring of R. So S is periodic, and therefore, S is a $*$-periodic ring since every idempotent of S is a projection. □

A $*$-ring R is called *π-$*$-regular* [16] if, for each $a \in R$, there exists $p^2 = p^* = p \in R$ such that $a^n R = pR$ for some integer $n \geq 1$; and further, R is called *$*$-regular* [6] if $aR = pR$.

Proposition 4.8. *Let R be a $*$-ring. The following are equivalent*:

(1) *R is a $*$-periodic ring.*
(2) *R is a strongly π-regular π-UU ring and idempotents of R are projections.*
(3) *R is an abelian π-$*$-regular π-UU ring.*

Proof. (1) $\Rightarrow$ (2). Using Theorem 4.5, R is periodic and every idempotent of R is a projection. The rest of this implication follows from Theorem 3.4.

(2) $\Rightarrow$ (3). In view of Lemma 4.1, R is abelian. Let $a \in R$. Since R is strongly π-regular, by [32, Proposition 1] $a^m = eu = ue$ for some integer $m \geq 1$, $e^2 = e$ and $u \in U(R)$. So, $a^m R = eR$, and therefore, R is π-$*$-regular since every idempotent of R is a projection.

(3) $\Rightarrow$ (1). For any $a \in R$, there exist a projection $p \in R$ and an integer $n \geq 1$ such that $a^n R = pR$. Then $a^n = pa^n$ and $p = a^n r$ with $r \in R$. As R is abelian, $a^n = pa^n = a^n p = a^{2n} r \in a^{n+1}R$. So, R is strongly π-regular. By virtue of Theorem 3.4, R is periodic. Further, we show that all idempotents of R are projections. Let $e^2 = e \in R$. Since R is π-$*$-regular, $eR = qR$ for $q^2 = q^* = q \in R$. So, $e = qe = eq = q$. Therefore, R is $*$-periodic. □

We say that a $*$-ring R is *$*$-potent* if, for each $a \in R$, there exists an integer $n \geq 2$ such that $a^n = a$ and a^{n-1} is a projection. By Theorem 4.5, every $*$-potent ring is necessarily $*$-periodic.

Theorem 4.9. *Let R be a $*$-ring. The following are equivalent*:

(1) *R is a $*$-periodic ring.*
(2) *R is abelian, $R/J(R)$ is $*$-potent and $J(R)$ is nil.*
(3) *R is abelian, $R/J(R)$ is $*$-periodic and $J(R)$ is nil.*
(4) *R is an abelian π-UU ring, $R/J(R)$ is $*$-regular and $J(R)$ is nil.*

Proof. (1) $\Rightarrow$ (2). Clearly, R is an abelian periodic ring. By Lemma 3.8, $J(R) = \mathrm{Nil}(R)$ is nil. Let $a \in R$. Applying Theorem 4.5, there exists an integer $n \geq 2$ such that $a - a^n \in J(R)$ and $a^{n-1} - (a^{n-1})^* \in J(R)$, which implies that $\overline{a} = \overline{a^n} = \overline{a}^n$ and $\overline{a^{n-1}} = \overline{(a^{n-1})^*} = \overline{a^{n-1}}^*$ is a projection of $R/J(R)$. Thus, $R/J(R)$ is $*$-potent.

(2) $\Rightarrow$ (3) is an obviously weakened implication.

(3) $\Rightarrow$ (4). Since $R/J(R)$ is periodic and $J(R)$ is nil, by Corollary 3.7 R is a periodic ring. Thus, R is π-UU by Theorem 3.4. To show that $R/J(R)$ is $*$-regular, we only need to verify that a $*$-periodic ring R with $J(R) = 0$ is $*$-regular. Under this assumption, R is abelian strongly π-regular, and so $\mathrm{Nil}(R) = J(R) = 0$. By Lemma 3.3, R is strongly regular. Since all idempotents are projections, R is $*$-regular, as required.

(4) $\Rightarrow$ (1). With [3, Theorem 4] at hand, R is strongly π-regular. Since $J(R)$ is nil, every idempotent of $R/J(R)$ can be lifted to an idempotent of R. It follows from R is abelian that so is $R/J(R)$. Then, by Proposition 4.8, it suffices to show that every idempotent of R is a projection. Let $e^2 = e \in R$. It is clear that $\overline{e}$ is

an idempotent of $R/J(R)$. Note that $R/J(R)$ is an abelian $*$-regular ring. By [15, Theorem 2.10], idempotents coincide with projections in $R/J(R)$. So, $\overline{e} = \overline{e}^* = \overline{e}^2$, whence $e - e^* \in J(R)$. As $ee^* = e^*e$, $(e - e^*)^3 = (e - e^*) \in J(R)$, which implies that $e = e^*$ is a projection. Therefore, R is $*$-periodic, as stated. □

Based on the established above results, one may expect that all $*$-periodic rings are commutative, but this is not true in general which can be substantiated via the construction in point (2) below.

Example 4.10. (1) Let $R = \mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is a prime. An involution $*$ of R is given by $(a, b) \mapsto (b, a)$. Then R is a commutative periodic ring, but it is not $*$-periodic since idempotents do not coincide with projections.

(2) Let $R = \mathbb{Z}_4$ with involution $* = 1_R$, and let G be quaternion group of order eight. Then the group ring RG is periodic as it is finite. Now, the map $* : RG \to RG$ given by $(\sum_g a_g g)^* = \sum_g a_g g^{-1}$ is an involution of RG, and it is denoted by $*$ again. In view of [14, Lemma 11], the idempotents in RG are same as these in R. So, every idempotent of RG is a projection. By Theorem 4.5, RG is a $*$-periodic ring. However, the ring RG is demonstrably not commutative.

We close with the simple but useful observation that if R is a $*$-periodic ring and $J(R)$ is central, then R is commutative. Indeed, by Proposition 4.8, R is an abelian strongly π-regular ring. In view of Lemma 3.8, one derives that $\mathrm{Nil}(R) = J(R)$ is central. Thus R is commutative by the chief result in [23] (see [5, Theorem 2] too).

Acknowledgments

The authors would like to express their special deep thanks to the expert referee for the very competent referee's report containing numerous valuable comments and insightful suggestions. They are also very grateful to the handling editor, Professor Dolors Herbera, for her professional editorial management.

The first named author was supported by the National Natural Science Foundation of China (No. 11401009) and the Key Project of Anhui Education Committee (No. gxyqZD2019009). The second named author was partially supported by the Bulgarian National Science Fund under Grant KP-06 No 32/1 of December 07, 2019.

References

[1] G. Azumaya, Strongly π-regular rings, *J. Fac. Sci. Hokkaido Univ.* **13** (1954) 34–39.

[2] M. F. Atiyah, I. G. Macdonald, *Introduction to Commutative Algebra* Addison-Wesley Series in Mathematics (Westview Press, 1994), 139 pp.

[3] A. Badawi, On abelian π-regular rings, *Comm. Algebra* **25**(4) (1997) 1009–1021.

[4] H. E. Bell, A commutativity study for periodic rings, *Pacific J. Math.* **70**(1) (1977) 29–36.

[5] H. E. Bell, On commutativity of periodic rings and near-rings, *Acta Math. Acad. Sci. Hung.* **36** (1980) 293–302.
[6] S. K. Berberian, *Baer ∗-Rings*, Die Grundlehren der Mathematischen Wissenschaften, Vol. 195 (Springer-Verlag, New York-Berlin, 1972).
[7] G. Borooah, A. J. Diesl and T. Dorsey, Strongly clean matrix rings over commutative local rings, *J. Pure Appl. Algebra* **212** (2008) 281–296.
[8] S. Breaz, G. Călugăreanu, P. Danchev and T. Micu, Nil-clean matrix rings, *Linear Algebra Appl.* **439**(10) (2013) 3115–3119.
[9] G. Călugăreanu, UU rings, *Carpathian J. Math.* **31**(2) (2015) 157–163.
[10] M. Chacron, Certains anneaux périodiques, *Bull. Soc. Math. Belg.* **20** (1968) 66–78.
[11] M. Chacron, On a theorem of Herstein, *Canad. J. Math.* **21** (1969) 1348–1353.
[12] H. Chen and M. Sheibani, Periodicity and J-clean-like rings, *Math. Rep.* **18**(4) (2016) 515–533.
[13] H. Chen and M. Sheibani, Strongly 2-nil-clean rings, *J. Algebra Appl.* **16**(9) (2017) 1750178.
[14] J. Chen, W. K. Nicholson and Y. Zhou, Group rings in which every element is uniquely the sum of a unit and an idempotent, *J. Algebra* **306** (2006) 453–460.
[15] J. Cui and X. B. Yin, Some characterizations of ∗-regular rings, *Comm. Algebra* **45**(2) (2017) 841–848.
[16] J. Cui and X. B. Yin, On π-regular rings with involution, *Algebra Colloq.* **25**(3) (2018) 509–518.
[17] P. V. Danchev, On exchange π-UU unital rings, *Toyama Math. J.* **39** (2017) 1–7.
[18] P. V. Danchev, A generalization of π-regular rings, *Turkish J. Math.* **43** (2019) 702–711.
[19] P. V. Danchev and T. Y. Lam, Rings with unipotent units, *Publ. Math. Debrecen* **88** (2016) 449–466.
[20] A. J. Diesl, Nil clean rings, *J. Algebra* **383** (2013) 197–211.
[21] F. Dischinger, Sur les anneaux fortement π-réguliers, *C. R. Acad. Sci. Paris Sér. A-B* **283**(8) (1976) A571–A573.
[22] F. Dischinger, Stark π-reguläre Ringe, Dissertation, Ludwig-Maximilians Universität München (1977).
[23] I. N. Herstein, A generalization of a theorem of Jacobson III, *Amer. J. Math.* **75** (1953) 105–111.
[24] Y. Hirano, Some studies on strongly π-regular rings, *Math. J. Okayama Univ.* **20** (1978) 141–149.
[25] P. Kanwar, A. Leroy and J. Matczuk, Clean elements in polynomial rings, *AMS Contemp. Math.* **634** (2015) 197–204.
[26] A. Karimi-Mansoub, T. Koşan and Y. Zhou, Rings in which every unit is a sum of a nilpotent and an idempoten, in *Advances in Rings and Modules*, AMS Contempary Mathematics Vol. 715 (Springer, 2018), pp. 189–203.
[27] T. Koşan, T. Yildirim and Y. Zhou, Rings whose elements are the sum of a tripotent and an element from the Jacobson radical, to appear in *Canad. Math. Bull.* (2020).
[28] T. Koşan, T. Yildirim and Y. Zhou, Rings with $x^n - x$ nilpotent, to appear in *J. Algebra Appl.* (2020).
[29] T. Koşan, Z. Wang and Y. Zhou, Nil-clean and strongly nil-clean rings, *J. Pure Appl. Algebra* **220** (2016) 633–646.
[30] T. Y. Lam, *A First Course in Noncommutative Rings* (Springer-Verlag, New York, 2001).
[31] C. N. Li and Y. Zhou, On strongly ∗-clean rings, *J. Algebra Appl.* **10**(6) (2011) 1363–1370.

[32] W. K. Nicholson, Strongly clean rings and Fitting's lemma, *Comm. Algebra* **27** (1999) 3583–3592.

[33] M. Ôhori, On strongly π-regular rings and periodic rings, *Math. J. Okayama Univ.* **27** (1985) 49–52.

[34] J. Šter, On expressing matrices over $\mathbb{Z}_2$ as the sum of an idempotent and a nilpotent, *Linear Algebra Appl.* **544** (2018) 339–349.

[35] Z. L. Ying, T. Koşan and Y. Zhou, Rings in which every element is a sum of two tripotents, *Canad. Math. Bull.* **59**(3) (2016) 661–672.