PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, Number 12, December 2020, Pages 5087–5089
https://doi.org/10.1090/proc/15246
Article electronically published on September 4, 2020

A NOTE ON A THEOREM OF JACOBSON RELATED TO PERIODIC RINGS

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(Communicated by Jerzy Weyman)

ABSTRACT. We show that if R is a ring such that for each $x \in R$ there exist two natural numbers $n(x)$ and $m(x)$ of opposite parity with $x^{n(x)} = x^{m(x)}$, then R is commutative. This extends the classical famous theorem of Jacobson [Ann. of Math. 46 (1945), p. 695–707] for commutativity of potent rings.

1. Introduction and fundamentals

Throughout, all rings considered in this paper are assumed associative and to have an identity element. A celebrated result of Jacobson states that if a ring R satisfies the property that for each $x \in R$ there exists a natural number $n(x) > 1$ with $x^{n(x)} = x$ (such rings are called *potent*), then R is commutative [1]. In this case, $x^{n(x)+1} = x^2$ for every $x \in R$. This raises the question: *If a ring R satisfies the property that for each $x \in R$, there exists a natural number $n(x) \neq 2$ with $x^{n(x)} = x^2$, must R be potent or even commutative*? Both questions have an easy answer “no”. Let us say a ring R *satisfies* (n, m), where $n > m$ are fixed natural numbers, if $x^n = x^m$ for all $x \in R$. Consider the four rings (1) $\mathbb{Z}_2$, (2) $\mathbb{Z}_2[X]/(X^2)$, (3) $\mathbb{Z}_4$, and (4) $\mathbb{T}_2(\mathbb{Z}_2)$, the ring of 2×2 upper triangular matrices over $\mathbb{Z}_2$. In fact, all four rings satisfy $(4, 2)$, but the last three ones are not potent, as, moreover, the last one is not even commutative. However, the full 2×2 matrix ring $\mathbb{M}_2(\mathbb{Z}_2)$ over $\mathbb{Z}_2$ as well as the upper triangular matrix ring $\mathbb{T}_2(\mathbb{Z}_4)$ over $\mathbb{Z}_4$ do not satisfy $(4, 2)$ because, for instance, there is an invertible matrix of order 3 and of order 4, respectively. Nevertheless, we prove in what follows that if an arbitrary ring R satisfies the property that $x^{n(x)} = x^2$, with $n(x)$ an odd integer, for all $x \in R$, then R is potent and hence commutative. Note that such a ring is necessarily of characteristic 2. Our proof also shows that if R satisfies $(n, 2)$ for n odd, then R satisfies $(n - 1, 1)$ and hence it is potent. Of course, in this case R again has characteristic 2. Example (3) shows that the rings satisfying $(4, 2)$ need not have characteristic 2 as opposed to examples (1), (2), and (4). So the requirement of having characteristic 2 is not enough to deduce the potent property and thus the restriction on $n(x)$ to be odd cannot be eliminated. Moreover, we also prove that if R is an arbitrary ring satisfying the property that $x^{n(x)} = x^3$ with $n(x)$ an even integer for all $x \in R$, then R is potent and thus commutative as well. These two

Received by the editors February 24, 2020.
2010 *Mathematics Subject Classification.* Primary 16D60, 16S34, 16U60.
Key words and phrases. Jacobson theorem, Jacobson radical, periodic rings, commutativity.
The work of the second named author was supported in part by the Bulgarian National Science Fund under Grant KP-06 N 32/1 of Dec. 07, 2019.

assertions alluded to above are, actually, simple consequences of a more general statement (see, for instance, Theorem 2.1 stated below).

On the other hand, a ring R is called *periodic* if for every $r \in R$ there exist two distinct naturals m, n depending on r such that the equality $r^m = r^n$ holds. It is not too hard to check that these rings are, in general, *not* commutative by looking at the equation $r^2 = r^4$, which as mentioned above is always satisfied by the upper 2×2 triangular matrix ring over $\mathbb{Z}_2$. Our purpose in this short paper is to extend the previously mentioned Jacobson's theorem in the viewpoint of periodic rings. Specifically, we determine when certain periodic rings are, in fact, potent.

2. The main result

We are now ready to establish the following curious statement, which improves the aforementioned fundamental theorem due to Jacobson.

Theorem 2.1. *Let R be a ring. For a fixed natural number m let $P(m)$ (resp., $P'(m)$) be the statement:*

$P(m)$: For each $x \in R$, there exists a natural number $n(x) > m$ of opposite parity of m with $x^{n(x)} = x^m$.

$P'(m)$: For each $x \in R$, there exists a natural number $n(x)$ of opposite parity of m with $x^{n(x)} = x^m$.

Then the following conditions are equivalent:

(1) *Given $x \in R$, there exist natural numbers $n(x)$ and $m(x)$ of opposite parity with $x^{n(x)} = x^{m(x)}$.*

(2) *Given $x \in R$, there exist natural numbers $n(x) > m(x)$, $n(x)$ even, $m(x)$ odd with $x^{n(x)} = x^{m(x)}$.*

(3) *Given $x \in R$, there exist natural numbers $n(x) > m(x)$, $n(x)$ odd, $m(x)$ even with $x^{n(x)} = x^{m(x)}$.*

(4) *R satisfies $P(1)$, that is, R is potent and, in the presence of (1), $x^{l(x)+1} = x$, where $l(x) = |n(x) - m(x)|$ is odd.*

(5) *R satisfies $P(2)$ and, in the presence of (1), $x^{l(x)+2} = x^2$, where $l(x) = |n(x) - m(x)|$ is odd.*

(6) *R satisfies $P(m)$ for some m.*

(7) *R satisfies $P'(m)$ for some m.*

(8) *R satisfies $P(m)$ for all m.*

(9) *R satisfies $P'(m)$ for all m.*

Proof. We will concentrate on the most difficult implication, (1) $\Rightarrow$ (4): to that goal, notice that $\mathrm{char}(R) = 2$ since $(-1)^{n(-1)} = (-1)^{m(-1)}$. Next note that $J(R) = 0$. In fact, let $x \in J(R)$. Suppose that $(1+x)^{n(x+1)} = (1+x)^{m(x+1)}$, where $n(x+1)$ is odd and $m(x+1)$ is even. So

$$1 + x + \binom{n(x+1)}{2}x^2 + \cdots + x^{n(x+1)} = 1 + \binom{m(x+1)}{2}x^2 + \cdots + x^{m(x+1)}.$$

Thus $x(1 + (\binom{n(x+1)}{2} - \binom{m(x+1)}{2})x + \cdots + x^{n(x+1)-1} - x^{m(x+1)-1}) = 0$, and hence $x = 0$ since the expression in the outer parentheses is invertible because x lies in $J(R)$. Consequently, R is a subdirect product of primitive rings each of which has characteristic 2 satisfying the condition that $x^{n(x)} = x^{m(x)}$, where $n(x)$ and $m(x)$ have opposite parity. Note that a division ring satisfying this condition is always potent and hence is a field. If we assume the contrary that the given primitive ring is not commutative, then there is a matrix ring $\mathbb{M}_n(F)$, where F is a field and $n > 1$ is an integer, such that the matrix ring also has characteristic 2 and satisfies the same equation $x^{n(x)} = x^{m(x)}$. Setting $0 \neq A \in \mathbb{M}_n(F)$ with $A^2 = 0$, one easily obtains for $n(A)$ odd and $m(A)$ even that

$$I + A = (I + A)^{n(A)} = (I + A)^{m(A)} = I,$$

whence $A = 0$, which is the desired contradiction. So R must be commutative too. To substantiate our claim that R is potent, let M be a maximal ideal of R. Take $x \in R$ so that $x^{n(x)} = x^{m(x)}$, where $n(x)$ and $m(x)$ have opposite parity. Letting $l(x) = |n(x) - m(x)|$, we observe that $l(x) \geq 1$ is odd. Now $\overline{x}^{n(x)} = \overline{x}^{m(x)}$ in $\overline{R} = R/M$. Suppose $\overline{x} \neq \overline{0}$. Then $\overline{x}^{l(x)} = \overline{1}$, so $\overline{x}^{l(x)+1} = \overline{x}$ where $l(x) + 1 \geq 2$ is even. Likewise, if $\overline{x} = \overline{0}$, it is obvious that $\overline{x}^{l(x)+1} = \overline{x}$. Therefore, $x^{l(x)+1} - x \in M$. Hence $x^{l(x)+1} - x \in \bigcap_{M \in Max(R)} M = J(R) = 0$. Consequently, $x^{l(x)+1} = x$ where $l(x) + 1$ is even, assuring that R is potent, as claimed.

As for validity of the remaining implications, one clearly sees that (9) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (1) and (9) $\Rightarrow$ (8) $\Rightarrow$ (6). Also, (4) $\Rightarrow$ (9) as for if $x^{n(x)} = x$ with $n(x)$ even, $x^{n(x)+n-1} = x^n$, where $n(x) + n - 1$ has the opposite parity of n. By what we have already shown above, that (1) implies (4), one deduces that points (1), (4), (6), (7), (8), and (9) are all equivalent. Evidently, (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (1), so as we just have that (1) $\Rightarrow$ (4), the other points (1), (2), (3), (4), and (5) are also equivalent, which finishes the proof. $\square$

As an immediate consequence, we yield the following.

Corollary 2.2. *Let R be a ring satisfying (n, m) where n and m are naturals having opposite parity. Then R satisfies $(n - m + 1, 1)$.*

Remark. This relationship is rather optimal since surprisingly there exists a commutative ring of characteristic 2 which satisfies $(4, 2)$ but, however, does *not* satisfy neither of $(3, 2)$ and $(3, 1)$. In fact, a direct inspection shows that such a ring is the group ring $R = B[G]$, where B is a Boolean ring possessing more than two elements (and thus it has non-trivial idempotents), and G is a torsion abelian group with elements of order at most 2.

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