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Представление матриц над полями в виде матриц с нулевым квадратом и диагональных матриц

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Аннотация

Мы доказываем, что любая квадратная матрица над произвольным бесконечным полем является суммой матрицы с нулевым квадратом и диагонализуемой матрицы. Этот результат несколько контрастирует с недавней теоремой Бреза, опубликованной в Linear Algebra & Appl. (2018).

Ключевые слова: матрицы, рациональная форма, диагональная форма, нильпотенты.

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Representing Matrices over Fields as Square-Zero Matrices and Diagonal Matrices

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Abstract

We prove that any square matrix over an arbitrary infinite field is a sum of a square-zero matrix and a diagonalizable matrix. This result somewhat contrasts recent theorem due to Breaz, published in Linear Algebra & Appl. (2018).

Keywords: matrices, rational form, diagonal form, nilpotents.

Bibliography: 13 titles.

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1. Introduction and Basic Facts

The presentation of an arbitrary matrix over a ring as a sum/difference of some special elements like units, nilpotents, idempotents, potents, etc., always plays a central role in the matrix ring theory. A brief collection of principally known historical facts in this branch is as follows: In [4] it was shown that any square matrix over the finite two elements field $\mathbb{Z}_2$ is a sum of a nilpotent matrix and an idempotent matrix; thereby the full matrix $n \times n$ ring $\mathbb{M}_n(\mathbb{Z}_2)$ is called *nil-clean*. This important fact was strengthened in [13] by showing that, for any $n \in \mathbb{N}$ and for every $n \times n$ matrix A over $\mathbb{Z}_2$, there exists an idempotent matrix E such that $(A - E)^4 = 0$, while over the finite indecomposable ring $\mathbb{Z}_4$ consisting of four elements this relation is precisely $(A - E)^8 = 0$ (see [2] and [12] for some further generalizations and specifications, too). In [13] is showed also that the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_2)$ is both nil-clean and von Neumann regular but *not* strongly π-regular, whereas the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{Z}_4)$ is both nil-clean and regularly nil clean in the sense of [5] but *not* π-regular (see [7], as well). Likewise, in [6] was established that the ring $\prod_{n=1}^{\infty} \mathbb{M}_n(K)$ over an algebraically closed field K is regularly nil clean even in a more thin setting by viewing that the required nilpotent is of exponent 2.

Moreover, a rather actual question is definitely the following one: *Is every matrix over each field presentable as the direct sum of a nilpotent and a potent*? In that aspect, it was proved in [3] that every $n \times n$ matrix M over a field of odd cardinality q has a decomposition of the form $M = P + N$, where $P^q = P$ is q-potent and N is nilpotent with $N^3 = 0$ but $N^2 \neq 0$ in general (compare also with the results obtained in [1]).

We, however, conjecture that this is not always true; it is rather a sum of a non-singular matrix and a nilpotent matrix – see, e. g., [10]. But over the four element field $\mathbb{F}_4$, which case is in sharp contrast to the aforementioned result from [3], this surely implies that it is a sum of a potent and a nilpotent, not knowing what are the exact degrees neither of the potent nor the nilpotent, however. In this way, a rather eluding question is *of whether or not* $\mathbb{M}_n(\mathbb{Z}_4)$ *is the sum of a nilpotent of order at most* 4 *and a potent*?

So, we come to the following basic and intriguing problem, which complete resolution seems to be extremely difficult:

Conjecture. Every square matrix A over a field F with at least four elements can be represented as $A = D + Q$ with $Q^2 = 0$ and D being diagonalizable over F.

It is worthwhile noticing that, for fields of three elements (i.e., over $\mathbb{F}_3 = \mathbb{Z}_3$), the conjecture fails as illustrated in [3, Example 6]. Nevertheless, concerning the fields with $|F| = 3$, we are believing that the same conjecture holds, but only for matrices A such that the exceptional 3×3 matrix from [3] does not appear as a rational normal form - block of A, $A + I$ and $A - I$, where I stands for the standard matrix identity, and also it may be the case that the matrices with such a block have to require index three nilpotents instead of these in the stated above conjecture.

The aim of this very short article is to settle this conjecture in the case of infinite fields (in particular, for algebraically closed fields). This will be successfully done in the sequel by stating and proving Theorem 1 presented below. The case of finite fields seems to be rather more difficult than we anticipate and so it remains still left-open (see Problem 1 stated at the end of the paper).

2. The Result and a Problem

We start here by completely solving the quoted above Conjecture for matrices over infinite fields (compare also with the corresponding result from [8], where a different approach is used).

Theorem 1. *Each square matrix over an infinite field is the sum of a nilpotent square-zero matrix and of a diagonalizable matrix.*

Proof. We shall separate our arguments related to the arbitrary square matrix A over a field of infinite cardinality into four points as presented below:

Point (1). Without loss of generality, we may assume that A has the rational normal form.

Point (2). Without loss of generality, we will assume that A has exactly one block, that is, A is a non-derogatory matrix.

Point (3). If A has characteristic polynomial $f(t) = t^n - s_{n-1}t^{n-1} + s_{n-2}t^{n-2} + ... + s_0$, we write the coefficient s_{n-1} as a sum of n pairwise different numbers $a_1 + ... + a_n$. The presentation below shows two matrices of which the first is diagonalizable, and the second one is square-zero. An appropriate choice of $x_1, ..., x_{n-1}$ can make their sum equal precisely to $f(t)$ as being its characteristic polynomial (this can be seen by determining the values of all x_i one-by-one from the bottom to the top; $1 \leq i \leq n-1$).

$$S = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & 0 \\ 1 & a_2 & 0 & \dots & 0 & 0 \\ 0 & 1 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_{n-2} & 0 & 0 \\ 0 & 0 & \dots & 1 & a_{n-1} & 0 \\ 0 & 0 & 0 & \dots & 1 & a_n \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & x_1 \\ 0 & 0 & 0 & \dots & 0 & x_2 \\ 0 & 0 & 0 & \dots & 0 & x_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & x_{n-1} \\ 0 & 0 & 0 & \dots & 0 & x_n \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Point (4). Note that the sum S of the matrices is a non-derogatory matrix – in fact, the point being that the 0-th, 1-st, ... , $(n-1)$-st powers of S are linearly independent. To see this, just look at what happens below the main diagonal in these powers, which inspection we leave to the interested reader.

Point (5). By what we have established above in points 2, 3 and 4, this S has the same rational normal form as A, and hence there exists a matrix C over the same field F such that the equality $A = C^{-1}SC$ holds.

This completes the proof after all. □

We end our work with the following challenging question of some interest and importance.

Problem 1. *Extend, if possibly, the considered above property from Theorem* 1 for any finite field.

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