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Numerical Determination of the Right Boundary Condition for Regime–Switching Models of European Options from Point Observations

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Abstract. This paper is devoted to develop a robust numerical algorithm to solve inverse problems of determining the right boundary conditions according to measurements inside a truncated domain for regime-switching models of European options. Difference schemes on Tavella-Randall grids are derived. We propose and discuss results of computational experiments for a several European options.

INTRODUCTION

Financial practitioners are untimely aware of the short comings of the now ubiquitous Black-Scholes framework [1,14]. Many extensions to the Black-Scholes model have been introduced to provide more realistic description for asset price dynamics. In particular, the Black-Scholes (BS) model has been extended to account for empirical behaviour of implied volatility smile. Among the most popular extensions are *regime-switching* processes. They have been applied to problems in electricity markets long term insurance guarantees, forestry valuation, and gas storage, see e.g. [12]. This gives rise to weakly coupled systems of *degenerate* parabolic equations on real semi-axes.

In case we use implicit numerical scheme we must truncate the domain to $[0, S_{max}]$ and then impose boundary conditions at $S = S_{max}$. To avoid generating large errors in the solution due to the approximation of the boundary conditions, the truncated domain must be large enough, which results in a large cost. There are a number of papers for the scalar case devoted to this problem, see e. g. [2,3,4,5,11].

Calibrating local regime-switching models is a challenging problem. For accurate solving the inverse problem of determining local volatility function the semiinfinite domain firstly is truncated posing correct right boundary condition [6].

The remainder of the paper proceeds as follows. Section 2 introduces the direct and inverse problems of the regime-switching model. The discretization of the European option problem is presented in Section 3. Also, fast algorithm for solution of the difference system of equations is proposed and analyzed. Section 4 provides the algorithms for solution of the discrete inverse problem. Numerical tests appear in Section 5 followed by a concluding section.

Direct and inverse problems

Let $V_k(S,t)$ be the value of an European option in regime k with striking price K with expiry date T. The European call option price $V_k(S,t)$ in each regime k satisfies the following backward problem

$$\frac{\partial V_k}{\partial t} + \frac{1}{2}\sigma_k^2 S^2 \frac{\partial^2 V_k}{\partial S^2} + r_k S \frac{\partial V_k}{\partial S} + \sum_{k,l=1;k\neq l}^{M} q_{kl}(V_l - V_k) = 0, \tag{1}$$

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$$V_k(S,T) = \max(S-K,0), \ \ k = 1,\dots,M. \tag{2}$$

For computational purposes the system (1) will be posed on the localized domain

$$(S,\tau) \in (0,S_{max}) \times [0,T]. \tag{3}$$

No boundary condition is required at $S = 0$ because setting $S = 0$ in (1) we obtain for $k = 1,\dots,M$

$$\frac{\partial V_k(0,t)}{\partial t} = q_{kk}V_k(0,t) - \sum_{k\neq l} q_{kl}V_l(0,t). \tag{4}$$

One very simple way to handle the asymptotic behaviour at infinity is to specify a Dirichlet condition that corresponds to the terminal function $V_0(T)$ at $S = S_{max}$, e. g. see for a regime–switching system [13,14]. For a call option this specification follows from the argument if $S \gg K$, it is unlikely that the option would expire worthless, so that holding this option is roughly equivalent to owning the underlying asset, reduced with the strike price K.

A better boundary condition for a call option is

$$V_k(S_{max},t) = S_{max} - Ke^{-rt}, \ k = 1,\dots,M. \tag{5}$$

In the case of a digital call option pricing, if $S \geq K$ and nothing otherwise, a reasonable boundary condition is

$$V_k(S_{max},t) = Ke^{-rt}, \ k = 1,\dots,M. \tag{6}$$

The problem (1)-(4) and (5) (or (6)) with unknown function $\{V_k\}_{k=1}^{M}$ is called *direct problem*.

All these methods *require some knowledge about the behaviour of the solution for large S*: either a known value for the solution or an assumption for linearity. Here, for a fixed in advance S_{max} with given *observations* at the points $S = S_1^\star, \ S = S_2^\star, \ \ 0 < S_k^\star < S_{max}$

$$V_k(S_k^\star,t) = \varphi_k(t), \ \ 0 < t < T, \ \ k = 1,2 \tag{7}$$

we wish to obtain approximately $V_k(S_{max},t), \ \ 0 < t < T, \ \ k = 1,2$. Thus, finding the optimal right boundary conditions we solve the whole *inverse* problem.

Note that posing such a formulation we face two variations of the problem: we have either $S_1^\star = S_2^\star$ or $S_1^\star \neq S_2^\star$. In this paper both cases will be considered.

Solution of the direct problem

The results in this paper hold for arbitrary spatial grid, but we shall particularly consider grids that are obtained by Tavella–Randall like grid [7,10]. Let $\psi : [a,b] \to [L,S]$ be any given continuous function that is strictly increasing and satisfies $\psi(a) = L, \ \psi(b) = S$. Let integer $I \geq 3$ and

$$\xi = a + i\triangle\xi\,(0 \leq i \leq I+1) \ \text{ with } \ \triangle\xi = (b-a)/(I+1).$$

Then a grid $L = S_0 < S_1 \cdots < S_I < S_{I+1} = S_{max}$ is defined by the transformation $S = \psi(\xi_i)$ $(0 \leq i \leq I+1)$ if the function ψ is sufficiently smooth, then also the grid is smooth in the sense that there exist real constants $C_0,C_1,C_2 > 0$ (independent of i and m) such that the mesh width is $h_i = S_i - S_{i-1}$ satisfy

$$C_0\triangle\xi \leq h_i \leq C_1\triangle\xi \ \text{ and } \ |h_{i+1} - h_i| \leq C_2(\triangle\xi)^2. \tag{8}$$

We next formulate the FDS that we consider for the semidiscretization of (1). Let $f : [L,S_{max}] \to R$ by any given function and $1 \leq i \leq I$. Write $H_i = h_i + h_{i+1}$. For approximating of the first derivative $f'(S_i)$ we deal with two central FDS:

$$f'(S_i) \approx \frac{h_{i+1}}{h_i H_i} f(S_{i-1}) + \frac{h_i - h_{i-1}}{h_i h_{i+1}} f(S_i) + \frac{h_i}{h_i H_i} f(S_{i+1}), \tag{9}$$

$$f'(S_i) \approx \frac{f(S_{i+1}) - f(S_{i-1})}{H_i}. \tag{10}$$

For approximating the second derivative $f''(S_i)$ we consider the central FDS

$$f''(S_i) \approx \frac{2}{h_i H_i} f(S_{i-1}) - \frac{2}{h_i h_{i+1}} f(S_i) + \frac{2}{h_{i+1} H_i} f(S_{i+1}) \tag{11}$$

We will make the simplifying assumptions that $\sigma_k = \text{const} > 0$, and that r_k, q_k are constants, with $r_k > 0$ for $k = 1, 2$. These assumptions, together with the notation $v_{k,i}^j \cong V_k(S_i, \tau^j)$ are used for simplifying the exhibition which follows.

By applying (9), (10) and (11) to Eq. (1), we have the implicit scheme

$$\frac{v_{k,0}^{j+1} - v_{k,0}^j}{\triangle\tau} = -(r_k + q_k) v_{k,0}^{j+1} + q_k v_{3-k,0}^{j+1},$$

$$\frac{v_{k,i}^{j+1} - v_{k,i}^j}{\triangle\tau} = \sigma_k^2 S_i^2 \left(\frac{v_{k,i-1}^{j+1}}{h_i H_i} - \frac{v_{k,i}^{j+1}}{h_i h_{i+1}} + \frac{v_{k,i+1}^{j+1}}{h_{i+1} H_i} \right) \tag{12}$$

$$+ r_k S_i \left\{ \begin{array}{ll} \frac{v_{k,i+1}^{j+1} - v_{k,i-1}^{j+1}}{H_i}, & \text{for (10)}, \\ -\frac{h_{i+1}}{h_i H_i} v_{k,i-1}^{j+1} + \frac{h_{i+1} - h_i}{h_i h_{i+1}} v_{k,i}^{j+1} + \frac{h_i}{h_{i+1} H_i} v_{k,i+1}^{j+1} & \text{for (9)} \end{array} \right\}$$

$$-(r_k + q_k) v_{k,i}^{j+1} + q_k v_{3-k,i}^{j+1}, \ (1 \le i < I),$$

$$v_{k,I+1}^{j+1} = \left(S_{I+1} - K e^{-r(j+1)\triangle\tau} \right)^+, \ (0 \le j < J). \tag{13}$$

Then we define $v_k^j = \left[v_{k,1}^j, \ldots, v_{k,I-1}^j \right]^{\mathrm{T}}$

and two $(I+1) \times (I+1)$ matrices $M_k,\ k = 1, 2$ given by

$$M_k = \begin{bmatrix} C_{k,0} & B_{k,0} & & & \\ A_{k,1} & C_{k,1} & B_{k,1} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{k,I-1} & C_{k,I-1} & B_{k,I-1} \\ & & & A_{k,I} & C_{k,I} \end{bmatrix}.$$

If $\mathbf{v}^j = \left[v_1^j, v_2^j \right]^{\mathrm{T}}$, then the implicit scheme system could be rewritten in matrix form given below

$$\mathbf{M}\mathbf{v}^{j+1} = \mathbf{v}^j, \ \text{where } \mathbf{M} = \begin{bmatrix} M_1 & -\triangle\tau q_1 I \\ -\triangle\tau q_2 I & M_2 \end{bmatrix},$$

$$C_{k,0} = 1 + \triangle\tau(r_k + q_k), \qquad B_{k,0} = 0, \ A_{k,i} = -\frac{\triangle\tau\sigma_k^2 S_i^2}{h_i H_i} + \frac{\triangle\tau r_k S_i}{H_i},$$

$$C_{k,i} = 1 + \frac{\triangle\tau\sigma_k^2 S_i^2}{h_i h_{i+1}} + \triangle\tau(r_k + q_k),$$

$$B_{k,i} = -\frac{\triangle\tau\sigma_k^2 S_i^2}{h_{i+1} H_i} - \frac{\triangle\tau r_k S_i}{H_i}, \ (1 \le i < I), \qquad A_{k,I} = 0, \qquad C_{k,I} = 1.$$

Solution of the inverse problem

We now present an algorithm for solving the inverse problem (1)-(4) and (5) (or (6)), adopt the method developed in [8].

Algorithm

We assume that the points of observation $S^\star$ as the grid mode with number $i^\star : S^\star = S_{i^\star}$, so that

$$v_{k,i^\star}^{j+1} = \varphi_k(\tau^{j+1}),\ (1 \le j < J).$$

To find an approximate solution $v_{k,i}^{j+1}$ of problem (12)-(13) at the new time level, we use the decomposition

$$V_k(S_i,\tau^{j+1}) := v_{k,i}^{j+1} = z_{k,i}^{j+1} + v_{1,I}^{j+1}.u_{k,i}^{j+1} + v_{2,I}^{j+1}.w_{k,i}^{j+1}, \quad (1 \le i \le I),\ (k = 1,2). \tag{14}$$

Step 1

Solve for the grid function $z_{k,i}^{j+1}$ the systems

$$\frac{z_{k,0}^{j+1} - v_{k,0}^{j}}{\triangle\tau} = -(r_k + q_k) z_{k,0}^{j+1} + q_k z_{3-k,0}^{j+1},$$

$$\frac{z_{k,i}^{j+1} - v_{k,i}^{j}}{\triangle\tau} = \sigma_k^2 S_i^2 \left(\frac{z_{k,i-1}^{j+1}}{h_i H_i} - \frac{z_{k,i}^{j+1}}{h_i h_{i+1}} + \frac{z_{k,i+1}^{j+1}}{h_{i+1} H_i} \right)$$

$$+ r_k S_i \left\{ \begin{array}{ll} \frac{z_{k,i+1}^{j+1} - z_{k,i-1}^{j+1}}{H_i}, & \text{for } (10), \\ -\frac{h_{i+1}}{h_i H_i} z_{k,i-1}^{j+1} + \frac{h_{i+1}-h_i}{h_i h_{i+1}} z_{k,i}^{j+1} + \frac{h_i}{h_{i+1} H_i} z_{k,i+1}^{j+1} & \text{for } (9) \end{array} \right\}$$

$$-(r_k + q_k) z_{k,i}^{j+1} + q_k z_{3-k,i}^{j+1},\ (1 \le i < I),\ z_{k,I}^{j+1} = 0.$$

Step 2.1

Solve for the auxiliary unknowns $u_{k,i}^{j+1}$ the systems

$$\frac{u_{k,0}^{j+1}}{\triangle\tau} = -(r_k + q_k) u_{k,0}^{j+1} + q_k u_{3-k,0}^{j+1},$$

$$\frac{u_{k,i}^{j+1}}{\triangle\tau} = \sigma_k^2 S_i^2 \left(\frac{u_{k,i-1}^{j+1}}{h_i H_i} - \frac{u_{k,i}^{j+1}}{h_i h_{i+1}} + \frac{u_{k,i+1}^{j+1}}{h_{i+1} H_i} \right)$$

$$+ r_k S_i \left\{ \begin{array}{ll} \frac{u_{k,i+1}^{j+1} - u_{k,i-1}^{j+1}}{H_i}, & \text{for } (10), \\ -\frac{h_{i+1}}{h_i H_i} u_{k,i-1}^{j+1} + \frac{h_{i+1}-h_i}{h_i h_{i+1}} u_{k,i}^{j+1} + \frac{h_i}{h_{i+1} H_i} u_{k,i+1}^{j+1} & \text{for } (9) \end{array} \right\}$$

$$-(r_k + q_k) u_{k,i}^{j+1} + q_k u_{3-k,i}^{j+1},\ (1 \le i < I),\ u_{k,I}^{j+1} = \left\{ \begin{array}{ll} 1 & \text{for} \quad k = 1 \\ 0 & \text{for} \quad k = 2 \end{array} \right. .$$

Step 2.2

Solve for the auxiliary unknowns $w_{k,i}^{j+1}$ the systems

$$\frac{w_{k,0}^{j+1}}{\triangle\tau} = -(r_k + q_k) w_{k,0}^{j+1} + q_k w_{3-k,0}^{j+1},$$

$$\frac{w_{k,i}^{j+1}}{\triangle\tau} = \sigma_k^2 S_i^2 \left(\frac{w_{k,i-1}^{j+1}}{h_i H_i} - \frac{w_{k,i}^{j+1}}{h_i h_{i+1}} + \frac{w_{k,i+1}^{j+1}}{h_{i+1} H_i} \right)$$

$$+r_k S_i \left\{ \begin{array}{ll} \frac{w_{k,i+1}^{j+1}-w_{k,i-1}^{j+1}}{H_i}, & \text{for } (10), \\ -\frac{h_{i+1}}{h_i H_i} w_{k,i-1}^{j+1} + \frac{h_{i+1}-h_i}{h_i h_{i+1}} w_{k,i}^{j+1} + \frac{h_i}{h_{i+1} H_i} w_{k,i+1}^{j+1} & \text{for } (9) \end{array} \right\}$$

$$-(r_k+q_k)w_{k,i}^{j+1} + q_k w_{3-k,i}^{j+1}, \ (1 \le i < I), \ w_{k,I}^{j+1} = \left\{ \begin{array}{lll} 0 & \text{for} & k=1 \\ 1 & \text{for} & k=2 \end{array} \right. .$$

Step 3

To compute $v_{k,I}^{j+1}$, we substitute $v_{k,i^\star}^{j+1} = z_{k,i^\star}^{j+1} + v_{1,I}^{j+1}.u_{k,i^\star}^{j+1} + v_{2,I}^{j+1}.w_{k,i^\star}^{j+1}$ into (14) to obtain

$$\left(v_{1,I}^{j+1}, \quad v_{2,I}^{j+1} \right) = \left[\left(\varphi_1(\tau^{j+1}), \quad \varphi_2(\tau^{j+1}) \right) - \left(z_{1,i^\star}^{j+1}, \quad z_{2,i^\star}^{j+1} \right) \right] \left(\begin{array}{cc} u_{1,i^\star}^{j+1} & u_{2,i^\star}^{j+1} \\ w_{1,i^\star}^{j+1} & w_{2,i^\star}^{j+1} \end{array} \right)^{-1} .$$

COMPUTATIONAL RESULTS

In the following, we present results in order to illustrate the accuracy and efficiency of the proposed computational algorithm of the solution of the boundary condition problem for regime-switching models of European options.

Direct Problem

First we solve the direct problem defined as the system of weakly coupled PDEs (1), the left boundary conditions (4), the initial conditions (2) and right boundary conditions which values match the discounted payoffs on each temporal layer.

In our paper we take $S_{max} = 50$, $K = 9$, $r = (10\%,\ 5\%)$, $\sigma = (80\%, 30\%)$, $q = (6,\ 9)$ and $T = 1$ year to option expiration. Despite the unconditional stability of the fully implicit difference scheme we make use of, for better results we take relatively small temporal step as $\triangle\tau = 1/1280$.

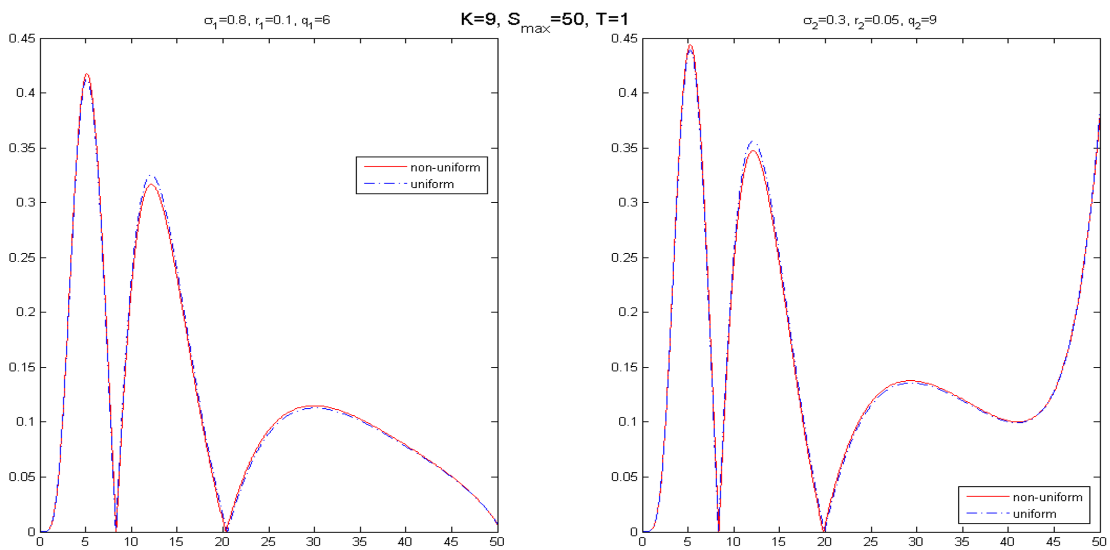

FIGURE 1: Absolute error in the direct problem solution to a cash-or-nothing call option using different grids

In Fig. 1 we compare the absolute error in the numerical solution derived by an uniform grid with spatial step $h = 0.03125$ and a non-uniform grid defined by the transformation of type (7), (8)

$$\psi(\xi) = K + \alpha \sinh(\xi), \qquad \xi \in [a,b],$$

where $a = \sinh^{-1}\left(\frac{-K}{\alpha}\right)$, $b = \sinh^{-1}\left(\frac{S_{max}-K}{\alpha}\right)$ and $\alpha = 0.5$ is a prescribed uniformity parameter (Fig. 2) [9, 5]. For the purpose of the experiment, a cash-or-nothing call option with payoff

$$g(S) = \begin{cases} K, & S > K \\ 0, & S \leq K \end{cases}$$

was taken into consideration. In contrast to [3], the uniform and non-uniform grids for the forward problem produce similar results.

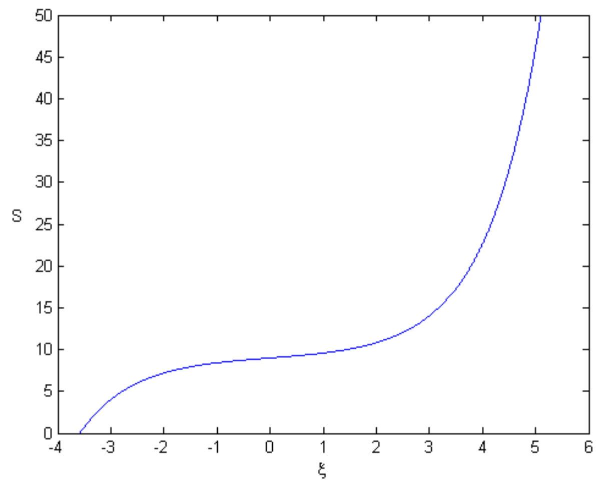

FIGURE 2: Graph of the coordinate transformation with $K = 9$ and $\alpha = 0.5$

Inverse Problem

After solving the direct problem, we define the observation at the point $S = S_k^\star, \quad 0 < S_k^\star < S_{max}$ for $k = 1,2$ and formulate the boundary inverse problem as follows: the system of PDEs (2), the conditions (2) and (4) and the quasi–real condition e. g. (5), (6) and so on.

As we are inverting the square matrix $\begin{pmatrix} u_{1,i^\star}^{j+1} & u_{2,i^\star}^{j+1} \\ w_{1,i^\star}^{j+1} & w_{2,i^\star}^{j+1} \end{pmatrix}$, we are interested in its determinant

$$D(i^\star, C) := \begin{vmatrix} u_{1,i^\star}^{j+1} & u_{2,i^\star}^{j+1} \\ w_{1,i^\star}^{j+1} & w_{2,i^\star}^{j+1} \end{vmatrix} = u_{1,i^\star} w_{2,i^\star} - u_{2,i^\star} w_{1,i^\star}.$$

In Table 1 we present the values of the auxiliary function D at the points $i^\star$ at different values of the discrete Courant number $C := \frac{\triangle\tau}{2h^2}$ for the Black–Scholes parabolic equations (1).

It is obvious that the values $D(i^\star, C)$ are the same for all types of Europeans options, regardless of their payoff functions $g(S)$.

TABLE 1: Values of $W_{i^\star}$ for various $i^\star$ and C

i^* \ C	0.4	0.8	1.6	3.2	6.4	12.8	25.6	51.2	102.4
2	4,9e-324	9,233e-248	2,154e-189	4,862e-146	1,187e-113	1,2909e-89	-3,377e-73	0	5,2972e-49
201	2,509e-135	-7,884e-102	2,2383e-77	5,0403e-61	2,6466e-48	1,1233e-39	7,2074e-32	1,5489e-25	1,8187e-21
401	8,2166e-97	-1,3719e-74	8,0942e-58	1,6133e-46	2,0275e-37	6,1044e-28	1,7174e-21	2,879e-17	1,4255e-14
601	1,4672e-74	-1,9093e-58	2,1692e-46	1,1893e-36	9,1043e-27	5,5562e-20	2,0325e-15	1,9814e-12	1,5981e-10
801	1,1398e-58	-4,0779e-47	3,7376e-36	4,0806e-26	3,9512e-19	2,4676e-14	4,1418e-11	5,3641e-09	1,1937e-07
1001	2,7342e-46	1,8407e-34	9,436e-25	6,0794e-18	3,3198e-13	5,9282e-10	9,1135e-08	2,466e-06	2,0213e-05
1201	8,1508e-30	2,2396e-21	1,9685e-15	2,8981e-11	2,3004e-08	2,2502e-06	4,9059e-05	0,000369	0,001339
1401	3,1411e-14	2,5988e-10	1,4927e-07	1,2843e-05	0,000285	0,00239	0,00999	0,02551	0,046376

As a special feature of the inverse problems is the fact that the bigger the Courant number, the more stable the inverse problem computations become. What is more, the same phenomenon is observed when the point of measurement $i^\star$ is increased. In Table 1, in the cases below the jagged line the algorithm gives satisfactory results with acceptable error, while in the cases above the line the algorithm diverges. Of course, the bigger Courant number, the bigger the error in the implied right boundary condition.

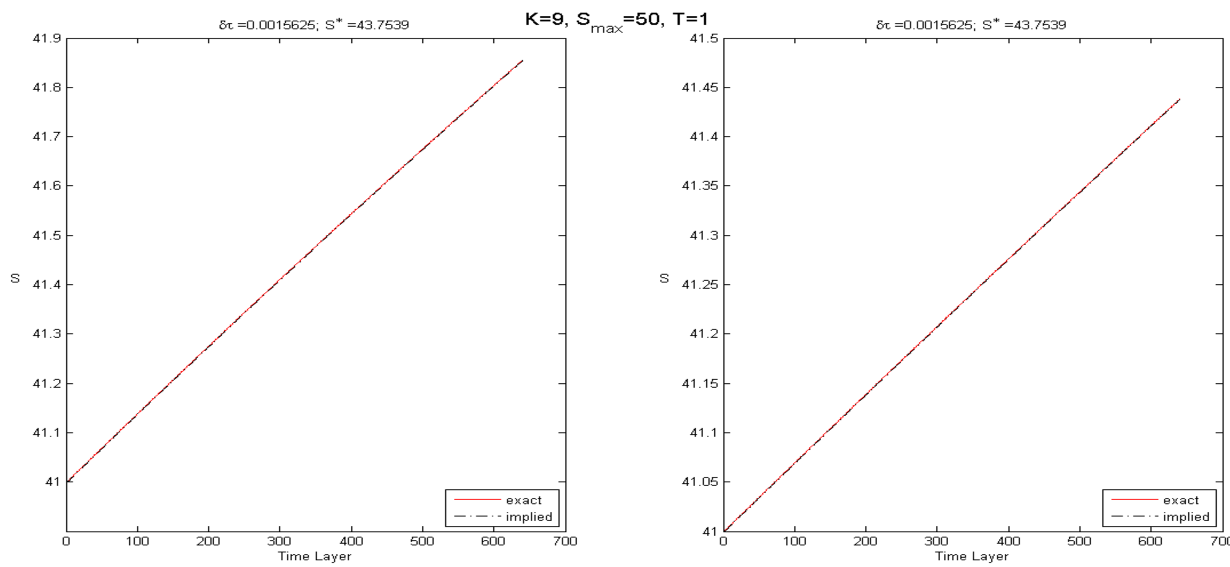

FIGURE 3: The values of the exact solution and the implied solution at the right boundary, for $S^\star = 43.75$, $C = 0.8$.

The numerical experiments approve these conclusions. In Fig. 3 and Fig. 4 we see that the exact solution and the solution to the inverse problem for the European call option practically coincide for $C = 0.8$, and they gradually go farther when C increases. On the other hand, we have gained convergence for smaller values of the measurement $S^\star$.

CONCLUSION

The problem of finding the right boundary condition for regime-switching models of European options is considered. We implement the decomposition technique proposed for the heat equation in [8]. Numerical experiments confirm the

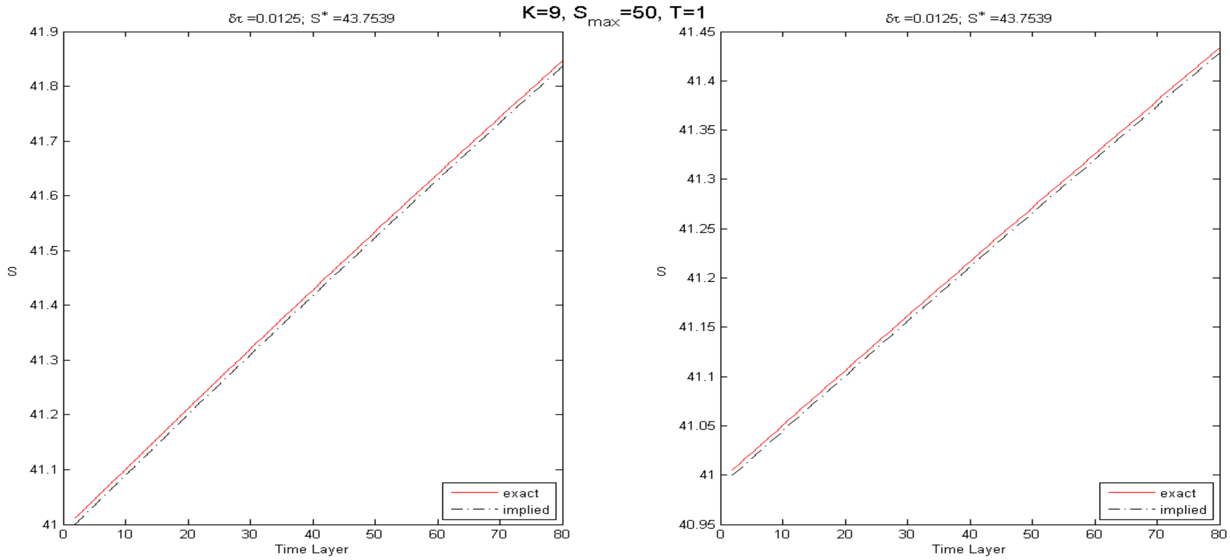

FIGURE 4: The values of the exact solution and the implied solution at the right boundary, for $S^{\star} = 43.75$, $C = 6.4$.

efficiency of the algorithm.

An open question remains finding sufficient conditions for the stability of the algorithm. Further investigations concern two-asset Black–Scholes equations, time-dependent volatility models, etc.

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