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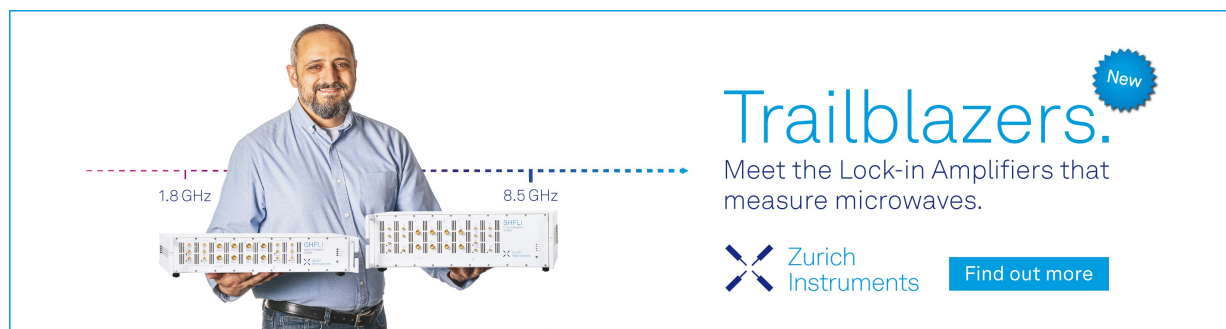
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Numerical Parameter Identification in a Three-Dimensional Honeybee – Mite Model

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Abstract. The recently observed honey bee colony losses have already raised concerns about the future of the managed honey bees and the bees in general. One of the most powerful approach to simulate and predict the dynamics of a complex system, which a honey bee colony undoubtedly is, is the mathematical modelling. We have adopted a compartment model to study the behaviour in a honey bee colony. A system of three ordinary differential equations with a control on the mites was introduced to model the populations of the hive bees, forager bees and mites. We solve a parameter identification inverse problem to reconstruct the values, which are directly unobservable but vital in honeybee management. We apply an adjoint equation optimization approach to solve the inverse problem. Numerical test examples are discussed and the paper is concluded with important implications about the honey bee management.

INTRODUCTION

In this section we first provide the reader with some basic knowledge concerning honeybee habitat. Next, we briefly describe the *Varroa mite* which is the most adverse parasite of the honeybee associated with a high percentage of colony losses over the winter.

Basic facts

The Varroa destructor mite has played a significant role in the collapse of honey bee colonies. Honeybee colonies are complex societies in which different members of the colony play specialized functions that serve the entire colony, thus making members of the colony highly dependent on each other. There are three basic castes of honey bees: **queen**, **drone** and **worker**. Prior to birth, a queen bee is fed royal jelly, which distinguish her from the rest of the bees. Each hive has only *one queen bee*. The drone bees are all male and their main purpose is to mate with the queen bee. After mating with the queen bee, these particular drones die. The rest are removed from the hive to die in the fall. Workers bees are divided in *hive* and *foragers* bees. The hive bees are the *younger worker bees* and work to maintain the hive, clean the cells and care for the brood. When the hive bees turn approximately *eight days old*, their responsibilities change as they begin to receive nectar, handle pollen, build comb, and clean the hive. As the hive bees get closer to foraging age, they begin to work outside the hive by ventilating, patrolling, duty guarding, and going on orientation flights to become aware with their surroundings once they begin foraging [15]. The hive bees overall have a *low mortality rate*, with the majority of hive bees living to forager age [8].

At about age of *18 days*, the *hive bees turn to foraging* [14]. Their responsibility consist of foraging nectar, pollen, and propolis, a sealant for the hive. However, this behavioral development process is dependent on *social feedback*. If there is a decline in the number of foraging bees, the hive bees will accelerate their behavioral development and begin foraging at earlier age to compensate for the lower forager numbers [8]. Hive bees switching to foraging earlier, known as *precocious foraging*, is associated with overall shorter lifespan because they are not mature enough and

thus less effective [8]. Precocious foragers could experience lifespan of less than seven days, approximately four to five, as foraging is more dangerous [8]. Colonies maintain a forager population of approximately 25 percent and will compensate to preserve that ratio [14]. The reverse change could occur as well, with foragers reverting to hive bees if there is deficit in the number of hive bees. The reverse process is known as a *social inhibition* and is accomplished through the process of old forager bees delivering ethyl oleate to young hive bees by way of trophallaxis, or mouth-to-mouth feeding [8].

The lifespan of the honey bee changes depending on the season and also depending on the caste which the bee belongs to. In the summer season, honey bees have the shortest lifespan, with the longest lifespan occurring during the winter months. During the winter, on average, a worker bee lives *around 140 days*. In the spring and fall this average drops to *30 to 60*, and in the summer it decreases to *15 to 38*.

However, the queen bee's lifespan is typically between *1 and 3 years* [15]. The differences in worker bee's lifespan, depending on the season, could be attributed to their activity level in each season [15]. In the winter, worker bees are less active and have slower metabolic rates, in comparison to the summer. Other exogenous factors, such as aforementioned precocious foraging or social inhibition, viruses and food availability, all affect the lifespan of the honeybee as well.

Varroa destructor mite

The **Varroa destructor** mites' natural host is the *Apis cerana*, the Asiatic honey bee. Its host began to shift in the late 1950's and 1960's to the *Apis mellifera*, the Western honey bee. Since then the *Varroa destructor* population has grown quickly [5, 6]. The *Varroa destructor* female utilizes a sealed honey bee brood cell to reproduce [12]. To achieve this, the female enters the brood cell prior to capping. Once the cell is capped, the Varroa destructor feeds on the developing bee and begins to lay eggs. First, one male egg is laid along with several female eggs on 30-hour intervals [12]. During this time, the mother mite prepares a place on the host for her offspring to feed, mature, and mate within the cell. The male mites take approximately 5 to 6 days to develop, and the female about 7 to 8 [12, 14, 15]. Once the host bee is ready to leave the cell, the adult female mite becomes attached to the bee [12]. Mature female mites attaches to the bee during the process known as phoretic phase when the mite pierces the intersegmental membrane of the bees to feed on the bee's haemolymph [12]. After two weeks, the mites would lay eggs in other brood cells, beginning the cycle again [12, 14, 15].

In the literature a number of models have been built, which account for different elements of the environment of the hive bees, such as demographic factors, overpopulation [1], dynamics of infections, seasonal effects and population of mites and viruses. The latter are recently developing and we adopt the model in [12]. Considering inverse problems, such methods concerning honeybee population with absence of mites in the hive are studied in [2], while unhealthy dynamics is examined in [3] and a fractional-order derivative model is considered in [4].

The primary aim of the present study is to propose an algorithm to solve the inverse problem of reconstructing parameters of the model including mites dynamics. The rest of the paper is organized as follows. In Section 2 the mathematical model is described and then the inverse problem is formulated. In the next section the adjoint optimization method is applied. Section 4 contains the numerical solution to the direct and inverse problems. Computational simulations are presented in Section 5 and then the paper is finalized with some conclusions.

THE THREE-DIMENSIONAL MODEL

In [8] it is investigated the population of both hive and forager bees. The authors of [12] have expanded the model to include a third equation in the system that accounts for the population of the *Varroa destructor* mite in the hive. The model accounts for the death of hive and forager bees due to *Varroa destructor* by adding an extra term to the equations in [8], as well as an extra equation for the mite population. The altered model is represented by the following equations on the time interval $(0, T)$:

$$\frac{dH}{dt} = L \frac{H+F}{\omega + H + F} - H \left(\alpha - \sigma \frac{F}{H+F} \right) - \rho MH \equiv f(H, F, M), \quad (1)$$

$$\frac{dF}{dt} = H \left(\alpha - \sigma \frac{F}{H+F} \right) - mF - \rho MF \equiv g(H, F, M), \quad (2)$$

$$\frac{dM}{dt} = rM \left(1 - \frac{M}{\alpha_2 H} \right) - \rho_2 M \equiv h(H, M). \quad (3)$$

In this model, H and F represent the number of hive bees and forager bees, respectively. The independent parameter L corresponds to the maximum egg laying rate of the queen, ω is a half-saturation constant that reflects the brood mortality, α is the maximum rate at which hive bees become foragers, σ represents the social inhibition, and m is the per capita death rate of foragers.

The additional third equation models the population of the *Varroa destructor* mite as logistic growth, with ρ_2 accounting for the per capita death rate of the mites, M . Similarly, the additional ρMH and ρMF terms in the first and second equations, respectively, represent the per capita deaths of the hive and forager bees due to the mite. The first term of (3) is a logistic growth rate of the mite population with r representing the growth rate of the mites. In this equation the term $\alpha_2 H$ is the carrying capacity of the bees in a hive.

We solve the system (1)-(3) with initial data

$$H(0) = H_0, F(0) = F_0, M(0) = M_0. \quad (4)$$

We will define the inverse problem for identification of the unknown parameters $\mathbf{p} = \{\rho, r, \alpha_2, \rho_2\}$. In absence of mites, i. e. $M(t) \equiv 0$, the system (1)-(3) is reduced to a system of two ODEs derived in [8]. For this model, the inverse problem of identification of the parameters ω, α, σ and m is solved in the paper [2]. Now, we assume these parameters to be known. In addition, if all constant coefficients of the system (1)-(3) are known, then the problem (1)-(4) is called *direct (forward)* problem.

Let us now assume now that the coefficients $\mathbf{p} = \{p^1 = \rho, p^2 = r, p^3 = \alpha_2, p^4 = \rho_2\}$ are unknown. We explore the inverse problem of reconstructing the parameters $\mathbf{p} \in S_{\text{adm}} = \{0 < p^i < p_{\text{max}}^i, i = 1, 2, 3, 4\}$ through the observed behavior

$$U(t^i) := \{H(t^i), F(t^i), M(t^i), i = 1, \dots, I_{\text{obs}}; t_0 = t^1 < \dots < t^{I_{\text{obs}}} = T\} \quad (5)$$

of the dynamics system (1)-(4).

ADJOINT OPTIMIZATION METHOD

We solve the point observation problem (1)-(5) via the minimization of an appropriate functional, see e. g. [7, 10]. We are going to minimize the least-square error functional

$$J(\mathbf{p}) = J(\rho, r, \alpha_2, \rho_2) = J_H(\mathbf{p}) + J_F(\mathbf{p}) + J_M(\mathbf{p}),$$

where

$$\begin{aligned} J_H(\mathbf{p}) &= \sum_{k=1}^{K_H} (H(t_k; \mathbf{p}) - X_k)^2, \\ J_F(\mathbf{p}) &= \sum_{k=1}^{K_F} (F(t_k; \mathbf{p}) - Y_k)^2, \\ J_M(\mathbf{p}) &= \sum_{k=1}^{K_M} (M(t_k; \mathbf{p}) - Z_k)^2. \end{aligned} \quad (6)$$

Theorem 1 The gradient $J'_{\mathbf{p}} \equiv \{J'_{\rho}, J'_{r}, J'_{\alpha_2}, J'_{\rho_2}\}$ of the functional $J(\mathbf{p})$ is given by

$$\begin{aligned}
J'_\rho(\mathbf{p}) &= \int_0^T \varphi_H H M + \varphi_F F M dt, \\
J'_r(\mathbf{p}) &= \int_0^T \varphi_M M \left(\frac{1}{\alpha_2} \frac{M}{H} - 1 \right) dt, \\
J'_{\alpha_2}(\mathbf{p}) &= -\frac{r}{\alpha_2} \int_0^T \varphi_M \frac{M^2}{H} dt, \\
J'_{\rho_2}(\mathbf{p}) &= \int_0^T \varphi_M M dt,
\end{aligned} \tag{7}$$

where the functions $\varphi_H = \varphi_H(t)$, $\varphi_F = \varphi_F(t)$, $\varphi_M = \varphi_M(t)$ are the unique solution to the adjoint final value problem

$$\begin{aligned}
\frac{d\varphi_H}{dt} &= a_{11}\varphi_H + a_{12}\varphi_F + a_{13}\varphi_M + 2 \sum_{k=1}^{K_H} (H(t; \mathbf{p}) - X(t)) \delta(t - t_k), \\
\frac{d\varphi_F}{dt} &= a_{21}\varphi_H + a_{22}\varphi_F + a_{23}\varphi_M + 2 \sum_{k=1}^{K_F} (F(t; \mathbf{p}) - Y(t)) \delta(t - t_k), \\
\frac{d\varphi_M}{dt} &= a_{31}\varphi_H + a_{32}\varphi_F + a_{33}\varphi_M + 2 \sum_{k=1}^{K_M} (M(t; \mathbf{p}) - Z(t)) \delta(t - t_k),
\end{aligned} \tag{8}$$

$$\varphi_H(T) = 0, \varphi_F(T) = 0, \varphi_M(T) = 0$$

and $X(t), Y(t), Z(t)$ are interpolants of the discrete functions taking values X_k at $t = t_k, k = 1, \dots, K_H$, $Y_k, k = 1, \dots, K_F$, $Z_k, k = 1, \dots, K_M$, respectively, and

$$\begin{aligned}
a_{11} &= -\frac{\partial f}{\partial H} = -L \frac{\omega}{(\omega + H + F)^2} + \alpha - \sigma \frac{F^2}{(H + F)^2} + \rho M, \\
a_{12} &= -\frac{\partial g}{\partial H} = -\alpha + \sigma \frac{F^2}{(H + F)^2}, \quad a_{13} = -\frac{\partial h}{\partial H} = -\frac{r}{\alpha_2} \cdot \frac{M^2}{H^2}, \\
a_{21} &= -\frac{\partial f}{\partial F} = -L \frac{\omega}{(\omega + H + F)^2} - \sigma \frac{H^2}{(H + F)^2}, \quad a_{22} = -\frac{\partial g}{\partial F} = \sigma \frac{H^2}{(H + F)^2} + \rho M + m, \\
a_{23} &= -\frac{\partial h}{\partial F} = 0, \\
a_{31} &= -\frac{\partial f}{\partial M} = \rho H, \quad a_{32} = -\frac{\partial g}{\partial M} = \rho F, \quad a_{33} = -\frac{\partial h}{\partial M} = -r + 2 \frac{r}{\alpha_2} \cdot \frac{M}{H} + \rho_2.
\end{aligned}$$

Proof We denote $\delta \mathbf{p} = (\delta \rho, \delta r, \delta \alpha_2, \delta \rho_2)$, $\delta \rho = \varepsilon h_1, \delta r = \varepsilon h_2, \delta \alpha_2 = \varepsilon h_3, \delta \rho_2 = \varepsilon h_4$ and $\delta H(t; \mathbf{p}) = H(t; \mathbf{p} + \delta \mathbf{p}) - H(t; \mathbf{p})$, $\delta F(t; \mathbf{p}) = F(t; \mathbf{p} + \delta \mathbf{p}) - F(t; \mathbf{p})$, $\delta M(t; \mathbf{p}) = M(t; \mathbf{p} + \delta \mathbf{p}) - M(t; \mathbf{p})$. Then, we rewrite the system (1)-(3) at $\mathbf{p} := \mathbf{p} + \delta \mathbf{p}$ for the triple $\{H(t; \mathbf{p} + \delta \mathbf{p}), F(t; \mathbf{p} + \delta \mathbf{p}), M(t; \mathbf{p} + \delta \mathbf{p})\}$ with initial values H_0, F_0, M_0 , respectively. Next, we perform the differences between the corresponding equations to obtain a system for the triple $\{\delta H(t; \mathbf{p}), \delta F(t; \mathbf{p}), \delta M(t; \mathbf{p})\}$ with zero initial conditions, which reads

$$\begin{aligned}
\frac{d}{dt} \delta H &= L \frac{\omega(\delta H + \delta F)}{(\omega + H + F)^2} + H \sigma \frac{H \delta F - F \delta H}{(H + F)^2} - \delta H \left(\alpha - \sigma \frac{F}{H + F} \right) - \rho M \delta H - \rho H \delta M - M H \delta \rho + \mathcal{O}(\varepsilon), \\
\frac{d}{dt} \delta F &= H \sigma \frac{F \delta H - H \delta F}{(H + F)^2} + \delta H \left(\alpha - \sigma \frac{F}{H + F} \right) - m \delta F - \rho M \delta F - \rho F \delta M - F M \delta \rho + \mathcal{O}(\varepsilon),
\end{aligned}$$

$$\frac{d}{dt}\delta M = rM \frac{M\alpha_2\delta H - H\alpha_2\delta M + HM\delta\alpha_2}{\alpha_2^2 H^2} + (r\delta M + M\delta r) \left(1 - \frac{M}{\alpha_2 H}\right) - \rho_2\delta M - M\delta\rho_2 + \mathcal{O}(\varepsilon).$$

On the second stage, we calculate the increment of the functional $J(\mathbf{p})$ to obtain:

$$\begin{aligned} J(\mathbf{p} + \delta\mathbf{p}) - J(\mathbf{p}) &= \sum_{k=1}^{K_H} (\delta H(t_k; \mathbf{p}) + H(t_k; \mathbf{p}) - X_k)^2 - \sum_{k=1}^{K_H} (H(t_k; \mathbf{p}) - X_k)^2 \\ &+ \sum_{k=1}^{K_F} (\delta F(t_k; \mathbf{p}) + F(t_k; \mathbf{p}) - Y_k)^2 - \sum_{k=1}^{K_F} (F(t_k; \mathbf{p}) - Y_k)^2 + \sum_{k=1}^{K_M} (\delta M(t_k; \mathbf{p}) + M(t_k; \mathbf{p}) - Z_k)^2 - \sum_{k=1}^{K_M} (M(t_k; \mathbf{p}) - Z_k)^2 \\ &= \sum_{k=1}^{K_H} \delta H(t_k; \mathbf{p}) (\delta H(t_k; \mathbf{p}) + 2(H(t_k; \mathbf{p}) - X_k)) + \sum_{k=1}^{K_F} \delta F(t_k; \mathbf{p}) (\delta F(t_k; \mathbf{p}) + 2(F(t_k; \mathbf{p}) - Y_k)) \\ &+ \sum_{k=1}^{K_M} \delta M(t_k; \mathbf{p}) (\delta M(t_k; \mathbf{p}) + 2(M(t_k; \mathbf{p}) - Z_k)) = 2 \sum_{k=1}^{K_H} \delta H(t_k; \mathbf{p}) (H(t_k; \mathbf{p}) - X_k) + \sum_{k=1}^{K_H} (\delta H(t_k; \mathbf{p}))^2 \\ &+ 2 \sum_{k=1}^{K_F} \delta F(t_k; \mathbf{p}) (F(t_k; \mathbf{p}) - Y_k) + \sum_{k=1}^{K_F} (\delta F(t_k; \mathbf{p}))^2 + 2 \sum_{k=1}^{K_M} \delta M(t_k; \mathbf{p}) (M(t_k; \mathbf{p}) - Z_k) + \sum_{k=1}^{K_M} (\delta M(t_k; \mathbf{p}))^2 \\ &= 2 \sum_{k=1}^{K_H} \int_0^T \delta H(t_k; \mathbf{p}) (H(t_k; \mathbf{p}) - X_k) \delta(t - t_k) dt + 2 \sum_{k=1}^{K_F} \int_0^T \delta F(t_k; \mathbf{p}) (F(t_k; \mathbf{p}) - Y_k) \delta(t - t_k) dt \\ &+ 2 \sum_{k=1}^{K_M} \int_0^T \delta M(t_k; \mathbf{p}) (M(t_k; \mathbf{p}) - Z_k) \delta(t - t_k) dt + \mathcal{O}(\varepsilon). \quad (9) \end{aligned}$$

Following the technology of paper [10], we multiply the equations for $\{\delta H(t; \mathbf{p}), \delta F(t; \mathbf{p}), \delta M(t; \mathbf{p})\}$ by smooth functions $\varphi_H(t)$ such that $\varphi_H(T) = 0$, $\varphi_F(t)$ such that $\varphi_F(T) = 0$, $\varphi_M(t)$, such that $\varphi_M(T) = 0$. We integrate both sides of the results from 0 to T and add them together.

$$\begin{aligned} \int_0^T \varphi_H \frac{d}{dt} \delta H + \varphi_F \frac{d}{dt} \delta F + \varphi_M \frac{d}{dt} \delta M dt &= \int_0^T \varphi_H \left[\delta H \left(L \frac{\omega}{(\omega + H + F)^2} - \sigma \frac{HF}{(H + F)^2} - \alpha + \sigma \frac{F}{H + F} - \rho M \right) \right. \\ &+ \delta F \left(L \frac{\omega}{(\omega + H + F)^2} + \sigma \frac{H^2}{(H + F)^2} \right) - \delta M (\rho H) - \delta \rho (HM) \Big] dt \\ &+ \int_0^T \varphi_F \left[\delta H \left(\sigma \frac{HF}{(H + F)^2} + \alpha - \sigma \frac{F}{H + F} \right) - \delta F \left(\sigma \frac{H^2}{(H + F)^2} - m - \rho M \right) \right. \\ &- \delta M (\rho F) - \delta \rho (FM) \Big] dt + \int_0^T \varphi_M \left[\delta H \left(\frac{r}{\alpha_2} \cdot \frac{M^2}{H^2} \right) + \delta M \left(r - 2 \frac{r}{\alpha_2} \cdot \frac{M}{H} - \rho_2 \right) \right. \\ &\left. + \delta r \left(M \left(1 - \frac{M}{\alpha_2 H} \right) \right) + \delta \alpha_2 \left(\frac{r}{\alpha_2^2} \cdot \frac{M^2}{H} \right) - \delta \rho_2 (M) \right] dt + \mathcal{O}(\varepsilon). \end{aligned}$$

On the other hand, from the equalities $\varphi_H(T) = 0$, $\delta H(0) = 0$, $\varphi_F(T) = 0$, $\delta F(0) = 0$, $\varphi_M(T) = 0$, $\delta M(0) = 0$ the integration by parts yields

$$\int_0^T \varphi_H \frac{d}{dt} \delta H + \varphi_F \frac{d}{dt} \delta F + \varphi_M \frac{d}{dt} \delta M dt = - \int_0^T \delta H \frac{d\varphi_H}{dt} dt - \int_0^T \delta F \frac{d\varphi_F}{dt} dt - \int_0^T \delta M \frac{d\varphi_M}{dt} dt. \quad (10)$$

Also, using the expressions for $\frac{d\varphi_H}{dt}$, $\frac{d\varphi_F}{dt}$, $\frac{d\varphi_M}{dt}$ from (8) in (10) and employing (9), after some tedious algebra we find

$$J(\mathbf{p} + \delta \mathbf{p}) - J(\mathbf{p}) = J(\rho + \varepsilon h_1, r + \varepsilon h_2, \alpha_2 + \varepsilon h_3, \rho_2 + \varepsilon h_4) - J(\rho, r, \alpha_2, \rho_2) = \\ \delta \rho \int_0^T \varphi_H H M + \varphi_F F M dt - \delta r \int_0^T \varphi_M M \left(1 - \frac{M}{\alpha_2 H}\right) dt - \delta \alpha_2 \int_0^T \varphi_M \left(\frac{r}{\alpha_2^2} \cdot \frac{M^2}{H}\right) dt + \delta \rho_2 \int_0^T \varphi_M M dt + o(\varepsilon).$$

Now, taking $h_2 = h_3 = h_4 = 0$, dividing the both sides of the last expression by εh_1 and passing to the limit $\varepsilon \rightarrow 0$, we obtain the formula for $J'_\rho(p)$ in (7). In the same manner one can check the validity of the other formulae in (7). \square

NUMERICAL SOLUTIONS

In this section, we will briefly present the numerical solutions to the direct and inverse problems, respectively. While the former will be given in brief, the latter will be described by a computational algorithm, which core is constituted by the adjoint optimization method, proposed in the section before.

Solution to the direct problem

Now we will briefly mention how to solve the direct problem (1)-(4). Such models generally do not feature a closed-form solution, so we have to solve it numerically. For this purpose, we introduce the piecewise-equidistant temporal mesh (11)

$$\overline{\omega}_\tau = \{t_0, t^i = t^{i-1} + \tau_i J_i, t^{I_{\text{obs}}} = T\} \text{ for } i = 1, \dots, I_{\text{obs}} - 1, \quad (11)$$

and the following subinterval splitting:

$$t_j^i = t^{i-1} + j \tau_i, \quad j = 1, \dots, J_i,$$

where $\forall i = 1, \dots, I_{\text{obs}} - 1$, t^i are the time instances at which observations are taken, $t_j^i, j = 1, \dots, J_i$ and τ_i are the time nodes and the time step corresponding to $(t^{i-1}, t^i]$.

There are a number of numerical methods to solve the initial value problem (1)-(4). Some of them are applied in [2, 3]. A regular Runge–Kutta method also does the job.

Solution to the inverse problem

After solving the direct problem, we use the solution to take measurements and to test the inverse problem solution approach. Here we will provide the numerical algorithm to solve the inverse problem (1)-(5):

1. Choose an initial approximation $\mathbf{p}_0 \in \mathbb{S}_{\text{adm}}$ and set $l := 0$.
2. Solve the direct problem (1)-(4) using the current value \mathbf{p}_l .
3. Find the observations (5) and interpolate them to obtain $X(t), Y(t), Z(t)$.
4. Solve the adjoint problem (8).
5. Calculate the gradient (7).
6. Using a gradient-based method, compute the new parameter values \mathbf{p}_{l+1} by the iterative formula

$$\mathbf{p}_{l+1} = \mathbf{p}_l - \mathbf{r} J'(\mathbf{p}_l). \quad (12)$$

7. If $\|\Delta \mathbf{p}_l\| < \varepsilon_{\mathbf{p}}$, then return $\check{\mathbf{p}} := \mathbf{p}_{l+1}$ and stop the procedure; else set $l := l + 1$ and go to Step 2.

In the algorithm the optimum found is denoted by $\check{\mathbf{p}}$; $\mathbf{r} > 0$ (12) is a descent vector parameter and it is derived empirically, please see the next section for an example. The tolerance $\varepsilon_{\mathbf{p}}$ is chosen according to the purpose of the algorithm, and $\Delta \mathbf{p}_l := \mathbf{p}_{l+1} - \mathbf{p}_l$.

COMPUTATIONAL EXPERIMENTS

In this section we supply numerical results to demonstrate the features of the algorithm. We first solve the direct problem.

Let us use the data provided in [8] and set the eggs laid by the queen $L = 1500$ and so is the maximal eclosion rate. The constant $\omega = 12000$, the maximal recruitment rate $\alpha = 0.25$ and the social inhibition constant $\sigma = 0.75$. We assume moderate forager mortality rate $m = 0.154$. As we mentioned earlier, these values are known for all experiments.

Further, let the mite-induced death rate $\rho = 1e-7$, the mite growth rate $r = 0.0165$, the carrying capacity coefficient $\alpha_2 = 0.5$ and death rate of the mites $\rho_2 = 0.1$. An average-sized colony is assumed with $H_0 = F_0 = 15000$ and $M_0 = 10000$. The considered time window is $T = 100$ days. The results of the simulation are given on Fig. 1, left.

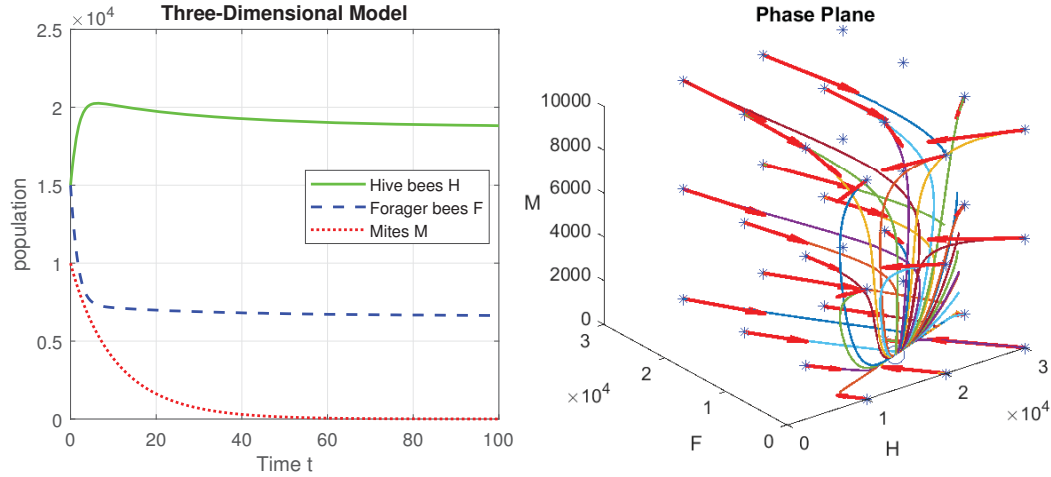


FIGURE 1: Honey bee population dynamics (left) and phase plane portrait (right)

The graph demonstrates that the colony approaches its equilibrium state, which is disease-free. Better understanding could be provided by the phase plane portrait on Fig. 1, right. There exist a certain threshold of the initial population of mites M_0 , beneath which the colony thrives in the context of the current parameter values.

We are ready to proceed to the inverse problem. Following the direct problem setting, we aim to recover the unknown parameters $\mathbf{p} = \{\rho, r, \alpha_2, \rho_2\}$ in case we are provided with observations of type (5). We require $I_{\text{obs}} = 21$ equidistant observations, which means we have to take measurements at every 5 days. We test the algorithm with initial values $\mathbf{p}_0 = (1e-4, 0.01, 1, 0.01)^\top$. The results are presented in Table 1.

TABLE 1: Test with $\varepsilon_p = 5e-6$

Param	p_0^i	p^i	\check{p}^i	$ p^i - \check{p}^i $	$\frac{ p^i - \check{p}^i }{p^i}$	r^i
ρ	1e-4	1e-7	6.2704e-8	3.7296e-8	0.3730	1.4e-20
r	0.01	0.0165	0.0167	2.2676e-4	0.0137	1.2e-16
α_2	1.00	0.5	0.4554	0.0446	0.0892	1.5e-12
ρ_2	0.01	0.1	0.0996	3.8618e-4	0.0039	1.3e-14

The implied parameters are relatively accurately reconstructed. The relative errors in the identification of r and ρ_2 are around 1%. The error associated with α_2 is below 9%. The relative error of the recovery of ρ is bigger, but this is due to the magnitude of the parameter.

The values of the residuals (6) are $J_H(\check{\mathbf{p}}) = 1.6454e4$, $J_F(\check{\mathbf{p}}) = 2.2868e3$, $J_M(\check{\mathbf{p}}) = 1.5619e3$, and the root mean square errors are $\text{RMSE}_H(\check{\mathbf{p}}) = 27.9913$, $\text{RMSE}_F(\check{\mathbf{p}}) = 10.4353$, $\text{RMSE}_M(\check{\mathbf{p}}) = 8.6242$, which are relatively small. All these imply a stable and robust algorithm for parameter identification.

CONCLUSION

We have adopted a compartment model for honey bee population dynamics which is both simple and useful. The model could be used from the academicians and professional beekeepers for simulation to predict colony losses. The equilibria analysis describes how the colony approaches the disease-free, endemic or extinction equilibrium for a given set of parameters. In all cases, the ratio between the hive and forager bees is kept stable, and the model accounts for this feature.

The novelty of the paper consists of the suggestion of a computational algorithm for identification of the important parameters concerning the mites impact on the colony life. To begin with, these parameters are unable to be measured in practice. On the other hand, their values are vital because they could indicate for a possible problem before being too late and give a hint for adequate precocious measures.

The inverse problem is solved via the adjoint equation optimization approach. After deriving the gradient of the cost error functional, it is incorporated in the supplied numerical algorithm. Its properties are demonstrated through computational experiments.

There are many possible ways to further develop the results. One is to use more sophisticated models, see e. g. [3], or to activate the hereditary properties of the dynamics system [4]. We hope that the algorithm and its further extensions are of help to the professionals to manage their apiaries in a right way, to prevent colony collapses if needed, and to deal with various diseases and epidemics.

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