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More on exposed points and extremal points of convex sets in $\mathbb{R}^n$ and Hilbert space

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Abstract. Let $\mathbb{V}$ be a separable real Hilbert space, $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, and let B be convex and closed in $\mathbb{V}$. Let $\mathcal{P}$ be a collection of linear k-subspaces of $\mathbb{V}$. A point $w \in B$ is called exposed by $\mathcal{P}$ if there is a $P \in \mathcal{P}$ so that $(w+P) \cap B = \{w\}$. We show that, under some natural conditions, B can be reconstituted as the convex hull of the closure of all its exposed by $\mathcal{P}$ points whenever $\mathcal{P}$ is dense and G_δ. In addition, we discuss the question when the set of exposed by some $\mathcal{P}$ points forms a G_δ-set.

Keywords: convex set; extremal point; exposed point; Hilbert space; Grassmann manifold

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1. Introduction

Throughout this paper $\mathbb{V}$ stands for a separable real Hilbert space. Thus $\mathbb{V}$ is isomorphic to either $\mathbb{R}^n$ or l^2. Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, B be convex and closed in $\mathbb{V}$ and let $\mathcal{G}_k(\mathbb{V})$ consist of all k-dimensional linear subspaces of $\mathbb{V}$ with the natural topology; see Definition 1. Let $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ and $w \in B$. We say that w is *exposed by* $\mathcal{P}$ if $(w+P) \cap B = \{w\}$ for some $P \in \mathcal{P}$. This definition generalizes each of the both concepts—an exposed point and a 0-exposed point—as defined in [6] and [1] respectively, that is, a point of $B \subset \mathbb{R}^n$ that is exposed by $\mathcal{G}_{n-1}(\mathbb{R}^n)$. By $\mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})$ we denote the set of all exposed by $\mathcal{P}$ points in B. Next, if $C \subset \mathbb{V}$ then we say that C is a $\mathcal{P}$*-imitation* of B if $B + P = C + P$ for every $P \in \mathcal{P}$. Further, $\mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$ stands for the set of *extremal* points of B with respect to $\mathcal{P}$ and is defined as $\mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P}) = \bigcap\{C \subset B \colon C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}$. The following exposed point theorem is proved in [5, Theorem 10].

Theorem 1. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex, and let $\mathcal{P}$ be a G_δ-subset of $\mathcal{G}_k(\mathbb{V})$ such that $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$. Then $\mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})$ is dense in $\mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$.*

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One of the goals of the current paper is to make use of the exposed point theorem and to prove the following theorem of Krein–Milman type; for example, see [15, Theorem 9.4.6]. It allows us, under some natural conditions, to reconstitute a closed convex set B in $\mathbb{V}$ as the convex hull of the closure of the set of all exposed by $\mathcal{P}$—a dense G_δ-subset of $\mathcal{G}_k(\mathbb{V})$—points in B. In this connection, let us mention the theorem of V. L. Klee, see [12, Theorem 2.3], which is about a reconstruction of a locally compact closed convex set B in a normed linear space, and B contains no line. Further, it is worth pointing out the theorem of V. Kanellopoulos, see [11, Theorem 1.1], that is of a similar type and is also an extension of Asplund's theorem, see [1], and Straszewicz theorem, see [16]. Recall that a *k-hyperplane* is a plane with codimension k and a *halfspace* of a plane L in $\mathbb{V}$ is any subset of L that consists of a hyperplane of L along with one of its sides. For the concept of a *derived face* the reader can refer to Definition 2. We have the following reconstitution theorem.

Theorem 2. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex that contains no k-hyperplane and let $\mathcal{P}$ be a dense G_δ-subset of $\mathcal{G}_k(\mathbb{V})$. If there is no derived face of B that is a halfspace of a k-hyperplane then*

$$\overline{\langle \mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P}) \rangle} = \left\langle \overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})} \right\rangle = B.$$

Let us point out that the requirement for $\mathcal{P}$ to be G_δ in both Theorem 1 and Theorem 2 cannot be omitted as Example 1 shows. Now, we need to make a couple of definitions. If $H \subset \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$ and $k \in \mathbb{N}$ with $k \le \dim H$ then we define $\mathcal{G}_k(H)$ as $\mathcal{G}_k(H) = \{L \in \mathcal{G}_k(\mathbb{R}^n) \colon L \subset H\}$. A compact and convex set B in $\mathbb{R}^n$ is called a *convex body* if $\dim B = n$. Next, let us discuss the following question: given $B \subset \mathbb{R}^n$ closed and convex and $1 \le k < n$ when can we find a nonempty subset $\mathcal{P}$ in $\mathcal{G}_k(\mathbb{R}^n)$ so that $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})$ is a G_δ-set? Here, we should mention the example of V. L. Klee, see [12, Example (6.10)], that is, a convex body B in $\mathbb{R}^3$ such that $\mathcal{X}_{\mathrm{p}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$ is not G_δ. More refined example is constructed by H. H. Corson in [7]—a convex body $B \subset \mathbb{R}^3$ such that $\mathcal{X}_{\mathrm{p}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$ is of the first category and hence does not contain a dense G_δ-subset of $\mathcal{X}_{\mathrm{t}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$. Further, S. Barov and J. J. Dijakstra in [5, Example 2] show that there is a convex body B in $\mathbb{R}^3$ for which the set of points exposed by $\mathcal{G}_1(\mathbb{R}^3) \setminus \mathcal{G}_1(H)$, for some linear two-dimensional plane H in $\mathbb{R}^3$, is not a G_δ-set. Moreover, [5, Example 3] is an expansion of Corson's example, namely, there is a convex body B in $\mathbb{R}^n$ such that $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{G}_k(\mathbb{R}^n))$ does not contain a dense G_δ-subset of the complete space $\mathcal{X}_{\mathrm{t}}^k(B, \mathcal{G}_k(\mathbb{R}^n))$ whenever $2 \le k < n$. In view of all those examples the following Straszewicz-type theorem is on the "positive" side of the discussion and is a slight improvement over [5, Theorem 3].

Theorem 3. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let B be closed and convex in $\mathbb{R}^n$. Let $\mathcal{P} \subset \mathcal{G}_1(\mathbb{R}^n)$ such that $\mathcal{G}_1(H) \setminus \mathcal{P}$ is countable for every $H \in \mathcal{G}_2(\mathbb{R}^n)$. Then $\mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P})$ is a dense G_δ-set in $\mathcal{X}^1_{\mathrm{t}}(B, \mathcal{P})$.*

Our paper is arranged as follows. In the introduction section we present and discuss our main results. In Section 2 we introduce the main concepts and give some basic properties and in Section 3 we prove our main theorems.

2. Definitions and preliminaries

The inner product in $\mathbb{V}$ is denoted by $x \cdot y$ and $\mathbf{0}$ always stands for the zero vector. The norm on $\mathbb{V}$ is given by $\|u\| = \sqrt{u \cdot u}$ and the metric d is given by $d(u, v) = \|v - u\|$. Let A be a subset of $\mathbb{V}$. We have that $\operatorname{aff} A$ denotes the affine hull of A, $\bar{A}$ the closure, and $\operatorname{int} A$ the interior of A in $\mathbb{V}$. Next, $\langle A \rangle$ stands for the convex hull of A, ∂A means the relative boundary of A, that is, the boundary with respect to the affine hull of A and we define $A^\circ = A \setminus \partial A$. Note that if A is convex and nonempty in a finite-dimensional space then $A^\circ \neq \emptyset$ and $\bar{A}^\circ \subset A$. We also define the linear space

$$A^\perp = \{v \in \mathbb{V}\colon\ v \cdot x = v \cdot y \ \text{ for all } \ x, y \in A\}.$$

In addition, if A is a closed linear subspace of $\mathbb{V}$, then $(A^\perp)^\perp = A$ and $A^\perp$ is called the *orthocomplement* of A. Also, we define $\operatorname{codim} A = \dim A^\perp \in \{0, 1, 2, \ldots, \infty\}$. Notice that $\operatorname{codim} A = \operatorname{codim} \operatorname{aff} A$. A *plane* in $\mathbb{V}$ is a closed affine subspace of $\mathbb{V}$; a *k-plane* in $\mathbb{V}$ is a k-dimensional affine subspace of $\mathbb{V}$. Now, let L be a plane in $\mathbb{V}$. A plane $H \subset L$ is called a *k-hyperplane* in L if $\dim(H^\perp \cap L) = k$. In other words, a k-hyperplane is a plane with codimension k in the ambient space. A *hyperplane* H of L is a plane of L of codimension 1. The two components of $L \setminus H$ are called the *sides* of the hyperplane H and the union of H with one of its sides is called a *halfspace* of L. A halfspace of a line is called a *halfline* or a *ray*. We say that H *supports* a subset A of L at x if $x \in H \cap A$ and A is contained in a halfspace that is associated with H.

Definition 1. Let $\mathbb{B} = \{v \in \mathbb{V}\colon \|v\| \leq 1\}$ be the unit ball in $\mathbb{V}$ and let $\mathcal{G}_m(\mathbb{V})$ stand for the collection of all m-dimensional linear subspaces of $\mathbb{V}$. As in [5], we topologize $\mathcal{G}_m(\mathbb{V})$ by defining a metric ϱ on $\mathcal{G}_m(\mathbb{V})$:

$$\varrho(L_1, L_2) = d_{\mathrm{H}}(L_1 \cap \mathbb{B}, L_2 \cap \mathbb{B}),$$

where d_{H} is the Hausdorff distance, associated with d, between two nonempty compact subsets of $\mathbb{B}$; see also [14, 1.11, page 95]. With the generated topology $\mathcal{G}_m(\mathbb{V})$ is complete; when $\mathbb{V}$ is finite-dimensional then $\mathcal{G}_m(\mathbb{V})$ is even compact and is called *Grassmann manifold*.

Definition 2. Let B be a closed and convex set in $\mathbb{V}$. A nonempty subset F of B is called a *face* of B if there is a hyperplane H of $\operatorname{aff} B$ that supports B with the property $F = B \cap H$. Note that F is also closed and convex and that $\operatorname{codim} F > \operatorname{codim} B$. If F is a face of B we write $F \prec B$. We say that a subset F of B is a *derived face* of B if $F = B$ or there exists a sequence $F = F_1 \prec F_2 \prec \cdots \prec F_m = B$ for some m. Furthermore, if $B \subset \mathbb{R}^n$ and $F \prec B$ then we say that F is a facet of B if $\dim F = \dim B - 1$. Observe that, in this case, F has a nonempty interior in ∂B. Besides, these interiors are disjoint for different facets of B. Therefore, by separability, a closed convex set in $\mathbb{R}^n$ can have only countably many facets.

Definition 3. Let $\mathcal{P}$ be a collection of linear subspaces of a vector space $\mathbb{V}$. We say that an affine subspace H of $\mathbb{V}$ is *consistent with* $\mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $z + P \subset H$ for some $z \in H$. Let B be a convex and closed subset of $\mathbb{V}$. A nonempty subset F of B is called a $\mathcal{P}$*-face* of B if $F = B \cap H$ for some hyperplane H of $\mathbb{V}$ that supports B and that is consistent with $\mathcal{P}$. A *derived* $\mathcal{P}$*-face* is a derived face of a $\mathcal{P}$-face. If $k \in \mathbb{N}$ and $k < \dim \mathbb{V}$ then we define the set $\mathcal{E}^k(B, \mathcal{P})$ as the closure of

$$\bigcup \{F \colon\ F \text{ is a derived } \mathcal{P}\text{-face of } B \text{ with } \operatorname{codim} F > k\}.$$

We finish this section with one more definition. A continuous map $f \colon X \to Y$ is called *proper* if the pre-image of every compactum in Y is compact. Recall that in metric spaces a continuous map is proper if and only if it is closed and every fibre is compact; see [8, Theorem 3.7.18].

3. Proofs of the main results

We are going to establish our main theorems. As the following theorem shows if $B^\circ = \emptyset$ or $\operatorname{codim} B \geq k$ then we have a stronger result than Theorem 2.

Theorem 4. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex, and let $\mathcal{P}$ be somewhere dense in $\mathcal{G}_k(\mathbb{V})$.*

(a) *If $B^\circ = \emptyset$ and $\mathcal{P}$ is G_δ, or*

(b) *if $\operatorname{codim} B \geq k$*

then $B = \mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})$.

Proof: The theorem follows directly from [5, Theorem 12] and [5, Remark 2]. □

Let $\mathcal{D}_k(B)$ be the union of all derived faces of B that are halfspaces of k-hyperplanes. Theorem 2 follows immediately from the following more general result having in mind that $\mathcal{D}_k(B) = \emptyset$ by assumption of Theorem 2, and that $\langle\, \overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})}\,\rangle \subset \overline{\langle \mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P}) \rangle}$ holds generally.

Theorem 5. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex that contains no k-hyperplane and let $\mathcal{P}$ be a dense G_δ-subset of $\mathcal{G}_k(\mathbb{V})$. Then*

$$\big\langle\, \overline{\mathcal{X}^k_{\mathrm{p}}(B,\mathcal{P}) \cup \mathcal{D}_k(B)}\,\big\rangle = \big\langle\, \overline{\mathcal{X}^k_{\mathrm{p}}(B,\mathcal{P})} \cup \mathcal{D}_k(B)\big\rangle = B.$$

Proof: If $\operatorname{codim} B \geq k$ then the theorem follows from Theorem 4. So, without loss of generality, we can assume that $\operatorname{codim} B < k$. Next, we will show the following key claim.

Claim 1. *We have $B = \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$.*

Proof: Indeed, striving for a contradiction assume that $B \not\subset \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$. Consider the collection

$$\mathcal{F} = \{F\colon\ F \text{ is a derived face of } B \text{ such that}\\ F \not\subset \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle\}.$$

Since B is a derived face of itself we have that $B \in \mathcal{F}$. By the definition of $\mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V}))$, we have that if $F \in \mathcal{F}$ then $\operatorname{codim} F \leq k$. Thus we can choose an $F \in \mathcal{F}$ with a maximal codimension. By [4, Lemma 17], we get that $F^\circ \neq \emptyset$. Set $L = \operatorname{aff} F$ and observe that $\operatorname{codim} L \leq k$. Next, since B contains no k-hyperplane we have that $F \neq L$. Therefore, we can pick a point $x \in \partial F$. By Hahn–Banach theorem, we consider a supporting hyperplane H_1 at x to F in L. Suppose that $H_1 \subset F$. Then we must have that $\operatorname{codim} H_1 = k+1$ and $\operatorname{codim} L = k$. By the structure of closed convex sets, see [10, §2.5], we have that if $y \in L$ then either $(y-x+H_1) \subset F$ or $(y-x+H_1) \cap F = \emptyset$. Next, let $\hat{l} \subset L$ be a line through x with $\hat{l} \perp H_1$. Observe that, $S = \hat{l} \cap F$ is either a nondegenerate line segment or a ray such that in both cases x is an end point. Clearly, $F = \bigcup\{z - x + H_1\colon z \in S\}$. Further, if S is a ray then we get that F is a halfspace of the k-hyperplane L. Hence $F \subset \mathcal{D}_k(B)$, a contradiction. If S is a line segment then there is a $w \in L$ such that $S = \langle\{x,w\}\rangle$. In this case $\partial F = H_1 \cup (w - x + H_1)$. Consequently, $\partial F \subset \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V}))$ since $\operatorname{codim} H_1 = \operatorname{codim}(w - x + H_1) = k+1$. Hence $F = \langle \partial F\rangle \subset \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V}))\rangle$, a contradiction again. Therefore, $H_1 \not\subset F$ and we can pick an $y \in H_1 \backslash F$. Further, since F is closed and convex, we can find the (unique) F-supporting hyperplane H_2 through y in L so that $d(H_2,F) = d(y,F) > 0$; see [13, page 347]. Notice that $H_1 \neq H_2$ and $y \in H_1 \cap H_2$. Furthermore, by [3, Lemma 8], there is a line $l \in \mathcal{G}_1$ with $y + l \subset L$ and $\psi_l \restriction F \to \mathbb{V}$ is proper, where $\psi_l\colon \mathbb{V} \to l^\perp$ denotes the orthogonal projection along l onto $l^\perp$. Now, let $z \in F$. If $z \in \partial F$ then, by Hahn–Banach theorem, there is a face F' of F that contains z. Clearly, F' is a derived face of B with $\operatorname{codim} F' > \operatorname{codim} F$. By the choice of F we get that $F' \subset \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$. Hence $z \in \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$. That argument also implies that $\partial F \subset \langle \mathcal{E}^k(B,\mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$. Now, suppose that $z \in F^\circ$. Since $\psi_l \restriction F \to \mathbb{V}$ is proper, we get that $K = (z+l) \cap F$ is a line segment. So $K \subset \langle \mathcal{E}^k(B, \mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$ since the end points of K are in ∂F. Hence $F \subset \langle \mathcal{E}^k(B, \mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$. We arrive at a contradiction. Consequently, we obtain that $B \subset \langle \mathcal{E}^k(B, \mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle$. Thus the claim holds. □

Further, since $\operatorname{codim} B < k$, by [5, Theorem 4] and [5, Lemma 9], we have that $\mathcal{E}^k(B, \mathcal{P}) = \mathcal{E}^k(B, \mathcal{G}_k(\mathbb{V})) = \mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P}) = \mathcal{X}_{\mathrm{t}}^k(B, \mathcal{G}_k(\mathbb{V}))$. Now, we can apply the exposed point theorem, see [5, Theorem 10], to get that $\overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})} = \mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P})$. Consequently, $B = \langle \mathcal{E}^k(B, \mathcal{G}_k(\mathbb{V})) \cup \mathcal{D}_k(B)\rangle = \langle \overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})} \cup \mathcal{D}_k(B)\rangle$. Since $\left\langle \overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})} \cup \mathcal{D}_k(B)\right\rangle \subset \left\langle \overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P}) \cup \mathcal{D}_k(B)} \right\rangle$, the theorem follows. □

Example 1. A convex body in $\mathbb{R}^n$ is *smooth* if there is a unique supporting hyperplane at each point of its boundary; see [9]. In [2, Section 5], for every $n \geq 2$ smooth symmetric convex bodies $B(n)$ in $\mathbb{R}^n$ and dense sets $\mathcal{P}(n)$ in $\mathcal{G}_{n-1}(\mathbb{R}^n)$ are constructed such that the union of all facets of $B(n)$ is dense in the boundary of $B(n)$ and $\mathcal{X}_{\mathrm{p}}^{n-1}(B(n), \mathcal{P}(n)) = \emptyset$ for $n \geq 2$. This example is closely related to Theorem 2 and Theorem 5 and shows that the G_δ-condition in both theorems cannot be omitted.

We have the following corollary that is closely related to the finite-dimensional version of Krein–Milman theorem in [15, Theorem 9.4.6], along with [16] as well as to [12, Theorem 2.3].

Corollary 6. *Let $n \in \mathbb{N}$ with $n \geq 2$, let $B \subsetneqq \mathbb{R}^n$ be closed and convex, and let $\mathcal{P}$ be a dense G_δ-subset of $\mathcal{G}_{n-1}(\mathbb{R}^n)$. If every face of B is compact then*

$$B = \left\langle \overline{\mathcal{X}_{\mathrm{p}}^{n-1}(B, \mathcal{P})} \right\rangle.$$

Example 2. Let $C = \{(x,y)\colon\ x \in \mathbb{R} \text{ and } y = x^2\}$ and $B = \langle C\rangle$. Then B is a closed and convex set in $\mathbb{R}^2$. Notice that at every point x of the boundary there is a unique supporting line to B that, in fact, exposes x. Thus $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{G}_1(\mathbb{R}^2)) = C$. Although B itself contains a ray, Corollary 6 is applicable since every face of B is compact.

Further, we are going to prove Theorem 3. Before that we need a lemma.

Lemma 7. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let B be closed and convex in $\mathbb{R}^n$. Let $\mathcal{P} \subset \mathcal{G}_1(\mathbb{R}^n)$ such that $\mathcal{G}_1(L) \cap \mathcal{P}$ is a dense G_δ-subset of $\mathcal{G}_1(L)$ for every $L \in \mathcal{G}_2(\mathbb{R}^n)$. Then $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{P})$ is dense in $\mathcal{X}_{\mathrm{t}}^1(B, \mathcal{P})$.*

Proof: Let $\varepsilon > 0$. First of all, observe that $\mathcal{P}$ must be dense in $\mathcal{G}_1(\mathbb{R}^n)$. If $n = 2$ then we are done by [5, Theorem 10]. So assume that $n \geq 3$ and, in view of Theorem 4, we may assume that $\dim B = n$. By [5, Theorem 4] and [5,

Lemma 9], we have that $\mathcal{E}^1(B,\mathcal{G}_1(\mathbb{R}^n)) = \mathcal{X}^1_{\mathrm{t}}(B,\mathcal{P})$. Let $F = H\cap B$ be a face of B, where H is a supporting hyperplane to B.
Case 1. Let $\dim F < n-1$. Then there is a hyperplane $\widehat{H}$ in H such that $F\subset \widehat{H}$. Let $x\in F$. Let L be a 2-plane in H with $x\in L$ and $L\setminus \widehat{H}\neq\emptyset$. Thus $\dim L\cap \widehat{H} = 1$. By [5, Remark 2] we can find an $l\in\mathcal{P}$ such that $(x+l)\cap\widehat{H}=\{x\}$ and $x+l\subset L$. This implies that $(x+l)\cap B=\{x\}$, i.e. $x\in\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P})$.
Case 2. Let $\dim F = n-1$. In this case F is a facet of B. Take an $x\in\partial F$. Let $y\in F^\circ$ and $z\in B^\circ$. Consider the 2-plane $L=\operatorname{aff}\{x,y,z\}$. Put $B_L = L\cap B$ and $\widehat{\mathcal{P}} = \mathcal{G}_1(L-x)\cap\mathcal{P}$. Now, we have that $\widehat{\mathcal{P}}$ is a dense G_δ-subset of $\mathcal{G}_1(L-x)$. Further, observe that $\widehat{F} = H\cap B_L$ is a facet of B_L and $x\in\partial\widehat{F}$. Hence $x\in\mathcal{E}^1(B_L,\mathcal{G}_1(L-x))$. Besides, by [5, Theorem 4] and [5, Lemma 9], we get that $x\in\mathcal{X}^1_{\mathrm{t}}(B_L,\widehat{\mathcal{P}})$. Thus we can apply [5, Theorem 10] for B_L in L to find an $l\in\widehat{\mathcal{P}}$ and $\hat{x}\in B_L$ so that $\|x-\hat{x}\|<\varepsilon$ and $(\hat{x}+l)\cap B_L=\{\hat{x}\}$. Now, clearly, we have $(\hat{x}+l)\cap B = \{\hat{x}\}$. Consequently, $\partial F\subset\overline{\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P})}$.

From both cases we obtain that $\overline{\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P})} = \mathcal{E}^1(B,\mathcal{G}_1(\mathbb{R}^n))$ and, therefore, $\overline{\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P})} = \mathcal{X}^1_{\mathrm{t}}(B,\mathcal{P})$. That completes the proof. □

Now, let us prove Theorem 3.

Proof of Theorem 3: If $\dim B<n$ then, by [5, Remark 2], $\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P}) = B$ and the theorem is proved. Besides, if $n=2$ then, by [5, Theorem 3], we are done as well. So we may assume that $\dim B = n$ with $n\geq 3$. By Lemma 7 we have that $\overline{\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P})} = \mathcal{X}^1_{\mathrm{t}}(B,\mathcal{P})$. Now, we are going to show that $\mathcal{X}^1_{\mathrm{p}}(B,\mathcal{P})$ is a G_δ-set. Let $F_m\prec F_{m-1}\cdots\prec F_1 = B$ be a sequence of derived faces. We call a sequence $F_m\prec F_{m-1}\cdots\prec F_1=B$ of derived faces *regular* if $\dim F_k-\dim F_{k+1}=1$ for every $1\leq k<m$. Also, we call a derived face F of B *regular* if for F exists a regular sequence. As it is noticed in Definition 2 the set B has countably many facets. Consequently, we can easily get that B has countably many regular derived faces and one of them is B itself. Next, let $x\in B$. Inductively, we construct a sequence $x\in F_m\prec F_{m-1}\prec\cdots\prec F_1 = B$ of derived faces such that the following two conditions hold:

(i) either $x\in F_m^\circ$ or $\operatorname{codim} F_m > m-1$ (or both) holds, and
(ii) if $m>2$ then $F_{m-1}\prec\cdots\prec F_1 = B$ is a regular sequence.

Set $F_1 = B$ and assume that we have constructed a regular sequence $x\in F_k\prec F_{k-1}\prec\cdots\prec F_1 = B$ for some $1\leq k$. Clearly, $\operatorname{codim} F_k = k-1$. If $x\in F_k^\circ$ we are done. Otherwise, we will have that $x\in\partial F_k$. So we are in a position to add one more element to the sequence under construction. We apply the Hahn–Banach theorem to find a supporting hyperplane $\hat{L}$ at x to F_k in $L=\operatorname{aff} F_k$. Set $F_{k+1} = \hat{L}\cap F_k$. Observe that, if $\operatorname{codim} F_{k+1} > k$ we are done. Otherwise, we would have that $\operatorname{codim} F_{k+1} = k$ and, therefore, $x\in F_{k+1}\prec F_k\cdots\prec F_1 = B$

would be a regular sequence. Obviously, after finitely many steps, we will have both conditions (i) and (ii) satisfied and we will get our sequence constructed.

Claim 2. *If* $\dim F_{m-1} \geq 3$ *and* $\dim F_{m-1} - \dim F_m \geq 2$ *then every* $y \in F_m$ *is an exposed by* $\mathcal{P}$ *point of* B*.*

Proof: Consider a coordinate system such that $y = \mathbf{0}$. Let H be a supporting hyperplane at $\mathbf{0}$ to F_{m-1} in $\operatorname{aff} F_{m-1}$ such that $F_m = H \cap F_{m-1}$. Then the codimension of F_m in H is at least 1. Therefore, we have room enough to find $P \in \mathcal{P} \cap \mathcal{G}_1(H)$ such that $P \cap F_m = \{\mathbf{0}\}$. Hence $P \cap B = \{\mathbf{0}\}$. The claim is proved. $\square$

The next claim is, in fact, [5, Claim 3] when $\mathcal{G}_1(\mathbb{R}^n)$ is replaced by $\mathcal{P}$. With this substitution its proof is virtually the same as the proof of [5, Claim 3] and, therefore, we omit it.

Claim 3. *Let* F *be a derived face of* B*. If there is a* $y \in \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P}) \cap F^\circ$ *then* $F \subset \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P})$*.*

Further, we go to the following important claim.

Claim 4. *The set*

$$T = \{x \in B \setminus \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P})\colon\ \dim F_{m-1} = 2 \ \textit{and} \ F_m = \{x\}\}$$

is countable.

Proof: Let $x \in T$ and let us consider the respective sequence $x \in F_m \prec F_{m-1} \prec F_{m-2} \cdots \prec F_1 = B$ of derived faces for x. Since $\dim B = n \geq 3$ we have that $m \geq 3$. Then $F_{m-1} \prec F_{m-2} \cdots \prec F_1 = B$ is a regular sequence of derived faces. Thus F_{m-1} is a regular derived face with $\dim F_{m-1} = 2$. In addition, since $F_m = \{x\}$ we get that $x \in \partial F_{m-1}$ and x is exposed by $\widehat{\mathcal{P}} = \mathcal{G}_1(\operatorname{aff} F_{m-1} - x) \setminus \mathcal{P}$. Further, since $\widehat{\mathcal{P}}$ is countable, we have that the set $\{y \in F_{m-1}\colon\ y \text{ is exposed by } \widehat{\mathcal{P}}\}$ is also countable. Now, having in mind that the set of all regular derived faces of B is countable, we get that T must be countable as well. That completes the proof. $\square$

Let $x \in B \setminus \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P})$. Suppose that the sequence $x \in F_m \prec F_{m-1} \cdots \prec F_1 = B$ is not regular. Then we have $\dim F_{m-1} - \dim F_m \geq 2$. Next, we have that $m > 2$. Indeed, if $m = 2$ then $\dim B - \dim F_2 > 1$ and, by Claim 2, we would have had $x \in \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P})$. Further, if $\dim F_{m-1} \geq 3$ then, by Claim 2, we would again get that $x \in \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{P})$. Consequently, we have that $\dim F_{m-1} = 2$, $F_m = \{x\}$ and $x \in \partial F_{m-1}$. So we are under the hypotheses of Claim 4. Hence, in this case, $x \in T$ with T countable. Now, let us assume that the sequence $x \in F_m \prec F_{m-1} \cdots \prec F_1 = B$ is regular. Then, notice that, $\operatorname{codim} F_m = m - 1$.

Therefore, we get that $x \in F_m^\circ$. Now, we apply the same argument as in the proof of [5, Theorem 3]. Namely, consider the countable set

$$\mathcal{L} = \{F^\circ\colon\ F \text{ is a regular derived face of } B \text{ with } F^\circ \cap \mathcal{X}_{\mathrm{p}}^1(B,\mathcal{P}) = \emptyset\}.$$

Since $x \in F_m^\circ \setminus \mathcal{X}_{\mathrm{p}}^1(B,\mathcal{P})$, by Claim 3, we have that $F_m^\circ \in \mathcal{L}$. Next, every $F^\circ \in \mathcal{L}$ is an open subset of a closed set in $\mathbb{R}^n$, hence σ-compact. Since $\mathcal{L}$ is countable, $\bigcup\mathcal{L}$ is also σ-compact with $\bigcup\mathcal{L} \subset B \setminus \mathcal{X}_{\mathrm{p}}^1(B,\mathcal{P})$. Consequently, we get that $\big(\bigcup\mathcal{L}\big) \cup T = B \setminus \mathcal{X}_{\mathrm{p}}^1(B,\mathcal{P})$ with $\big(\bigcup\mathcal{L}\big) \cup T$ being a σ-compact subset of B. Hence $\mathcal{X}_{\mathrm{p}}^1(B,\mathcal{P})$ is G_δ-subset in B and, of course, in $\mathcal{X}_{\mathrm{t}}^1(B,\mathcal{P})$ as well. That completes the proof of the theorem. □

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