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On a Question of M. Paoli and E. Ripoli.

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Sunto. – *H.-J. Schmidt in* [S] *introduce una classe di spazi — che in questo lavoro chiamiamo HS-spazi — ed afferma* [*Teorema* 11 (3*d*)] *che ogni HS-spazio di Hausdorff è regolare. M. Paoli e E. Ripoli, in* [PR1] *hanno però osservato che la dimostrazione di tale teorema non è corretta, senza però offrire controesempi all'affermazione. In questo articolo si fornisce una parziale risposta al problema, presentando una larga classe di spazi, che contiene tutti gli spazi di Hausdorff di carattere numerabile, in cui il teorema di Schmidt è vero.*

1. – Introduction and preliminary results and definitions.

The following definition is motivated by the results of H.-J. Schmidt in [S].

Definition 1.1. – A topological space X is called a *HS-space* if, for every subspace A of X, the map $i_A: 2^{A,T} \to 2^{X,T}$, defined by the formula $i_A(B) = cl_X B$, for every $B \in 2^A$, is a continuous map.

Here and below, for every topological space $(X, \mathcal{O})$, 2^X stands for the set of all non-empty closed subsets of X and $cl_X B$—for the closure of the subset B of X in the space X. The set 2^X is endowed with the Tychonoff topology $\mathcal{O}_T$, which is known also as upper semi-finite topology [M], generated by the base $\mathcal{B} = \{\langle U \rangle : U \in \mathcal{O}\}$, where $\langle U \rangle = \{F \in 2^X : F \subset U\}$. The topological space $(2^X, \mathcal{O}_T)$ is denoted briefly by $2^{X,T}$. The class of all *HS*-spaces (resp., all T_i-spaces, for $i = 1, 2, 3, 3.5, 4$ will be denoted by $\mathcal{HS}$ (resp., by $\mathcal{T}_i$, $i = 1, 2, 3, 3.5, 4$) and the class of all normal spaces—by $\mathcal{N}$ (1). In [S] H.-J. Schmidt proved the following theorem.

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(1) In this paper we assume that T_i-spaces $(1 = 3, 3.5, 4)$ are Hausdorff, while the regular and normal spaces are not assumed to be, in general, T_1-spaces.

THEOREM 1.2. ([S, Theorem 11(3d)]). – $\mathcal{HS} \cap \mathcal{T}_2 \subseteq \mathcal{T}_3$.

M. Paoli and E. Ripoli noted in [PR1] that the proof of this theorem is incorrect, but the question of the correctness of the statement remains open. In the present paper we give a partial solution of this question. More precisely: *a*) we give an internal (i.e. in terms of the space only) characterization of *HS*-spaces (see Theorem 2.10); *b*) we introduce a large class of spaces, denoted by $\mathcal{K}^*$ (see Definition 2.17), containing all Hausdorff spaces with countable character (see 2.24 and 2.22), where 1.2 holds (see Theorem 2.19, where a stronger result is proved) and we show that 1.2 is true iff (= if and only if) the statement «$\mathcal{T}_2 \subseteq \mathcal{K}^*$» is true (see Theorem 2.20); using Theorem 2.19, we demonstrate that the limit of an inverse sequence of *HS*-spaces needs not be a *HS*-space (see Example 2.26); *c*) we show that the class $\mathcal{HS}$ is invariant under closed mappings (see Theorem 2.11); *d*) we prove that 1.2 is true iff the statement «$\mathcal{HS} \cap \mathcal{T}_2 = \mathcal{T}_4$» does (see Theorem 2.14). Moreover, some new classes of spaces, closely related to the problem discussed here, are introduced and briefly investigated. These are the class of *F*-normal spaces (see Definition 2.4 and Theorems 2.12, 2.28) and the classes of *K*-, *K'*- and *K''*-spaces (see Definition 2.23, Theorems 2.30, 2.31 and Examples 2.32, 2.33).

Let us cite also the following corollary of Proposition 5 in [Se]:

PROPOSITION 1.3. ([Se]). – $\mathcal{N} \subseteq \mathcal{HS}$.

NOTATION 1.4. – If $\mathcal{P}$ is a class of topological spaces, then we denote by $\neg\mathcal{P}$ the class defined in the following way: $X \in \neg\mathcal{P}$ iff $X \notin \mathcal{P}$.

For all notions and notations undefined here see [E].

Let us finally note, that many of the results of this paper were announced (without proofs) in [BDN].

2. – The results.

NOTATION 2.1. – For any set X, we denote by $E(X)$ the set of all nonempty subsets of X.

CONVENTION 2.2. – Let $(X, \mathcal{O})$ be a topological space, $H \in 2^X$, $U \in \mathcal{O}$ and $H \subseteq U$. Then we will say that (H, U) is a *pair in X*.

DEFINITION 2.3. – A pair (H, U) in $(X, \mathcal{O})$ is said to be *F-embedded in X* if there exists a $V \in \mathcal{O}$ such that

i) $H \subseteq V$, and

ii) $(\Phi \in 2^U, \Phi \subseteq V)$ implies $(\Phi \in 2^X)$.

DEFINITION 2.4. – A topological space X is said to be *F-normal* if every pair (H, U) in X is F-embedded in X. The class of all F-normal spaces will be denoted by $\mathcal{FN}$.

DEFINITION 2.5. – A topological space X is said to be *LF-normal* if for every pair (H, U) in X and for every subspace Y of X such that $H \subseteq Y$, the pair $(H, U \cap Y)$ in Y is F-embedded in Y. The class of all LF-normal spaces is denoted by $\mathcal{LFN}$.

REMARK 2.6. – Obviously, $\mathcal{N} \subseteq \mathcal{LFN} \subseteq \mathcal{FN}$. The inclusions $\mathcal{N} \subset \mathcal{LFN}$ and $\mathcal{T}_4 = \mathcal{N} \cap \mathcal{T}_1 \subset \mathcal{LFN} \cap \mathcal{T}_1$ are strong, since for any infinite set X with the cofinite topology we have that $X \in (\mathcal{LFN} \cap \mathcal{T}_1) \setminus \mathcal{T}_2$ and, hence, $X \in (\mathcal{LFN} \cap \mathcal{T}_1) \setminus \mathcal{T}_4$ and $X \in \mathcal{LFN} \setminus \mathcal{N}$.

Now, we are going to show that the classes $\mathcal{HS}$ and $\mathcal{LFN}$ coincide (which, in particular, will imply 1.3).

CONVENTION 2.7. – Let X be a topological space and $\Phi \in E(X)$. The statement «for every subspace A of X such that $\Phi \in 2^A$, the mapping $i_A \colon 2^{A,T} \to 2^{X,T}$ is continuous at the point Φ of 2^A» will be shortened as «i is continuous at Φ».

REMARK 2.8. – $X \in \mathcal{HS}$ iff i is continuous at any $\Phi \in E(X)$.

LEMMA 2.9. – *Let $(X, \mathcal{O})$ be a topological space. Then $X \in \mathcal{HS}$ iff i is continuous at any $F \in 2^X$.*

PROOF. – ($\Rightarrow$) This is trivial.

($\Leftarrow$) Let $\Phi \in E(X)$. We will show that i is continuous at Φ, which will imply by 2.8, that $X \in \mathcal{HS}$. Let A be a subspace of X such that $\Phi \in 2^A$. We have to show that the map $i_A \colon 2^{A,T} \to 2^{X,T}$ is continuous at Φ. Let $F = cl_X \Phi$, $U \in \mathcal{O}$ and $F \subseteq U$. We put $C = A \cup F$. Then $F \subseteq C$ and $F \in 2^C$. Since the map $i_C \colon 2^{C,T} \to 2^{X,T}$ is continuous at F, there exists an open in C set V_1 such that $F \subseteq V_1$ and

$$(1) \qquad i_C(\langle V_1 \rangle) \subseteq \langle U \rangle.$$

Let $V = V_1 \cap A$. Then V is open in A, $\Phi \subseteq V$ and $i_A(\langle V \rangle) \subseteq \langle U \rangle$. Indeed, let $B \in 2^A$, $B \subseteq V$ and $B_1 = cl_C B$. Then $B_1 \subseteq B \cup F \subseteq V \cup V_1 = V_1$ and hence, by (1), $cl_X B = cl_X B_1 \subseteq U$. ■

THEOREM 2.10. – $\mathcal{HS} = \mathcal{LFN}$.

PROOF. – *A*) Let $(X, \mathcal{O}) \in \mathcal{HS}$. We will show that $X \in \mathcal{LFN}$. Let (H, U) be a pair in X and Y be a subspace of X such that $H \subset Y$. We have to show that the pair $(H, U \cap Y)$ in Y is F-embedded in Y.

Put $A = U \cap Y$. Then $H \in 2^A$ and $i_A(H) = H \in \langle U \rangle$. Since the map $i_A \colon 2^{A,T} \to 2^{X,T}$ is continuous, there exists an open set V in A such that $H \subseteq V$ and $i_A(\langle V \rangle) \subseteq \langle U \rangle$. Obviously, V is open also in Y. Let $\Phi \in 2^{U \cap Y} = 2^A$ and $\Phi \subseteq V$. Then $\Phi \in \langle V \rangle$ and hence $cl_X \Phi \subseteq U$. We obtain that $\Phi = cl_A \Phi = Y \cap U \cap cl_X \Phi = Y \cap cl_X \Phi = cl_Y \Phi$, i.e. $\Phi \in 2^Y$.

B) Let $(X, \mathcal{O}) \in \mathcal{LFN}$. We will show that $H \in \mathcal{HS}$. By 2.9, it is enough to prove that i is continuous at each $F \in 2^X$.

Suppose there exists a $F_0 \in 2^X$ such that i is not continuous at F_0. Then there exists a subspace B of X such that $F_0 \subseteq B$ and the map $i_B \colon 2^{B,T} \to 2^{X,T}$ is not continuous at the point F_0 of 2^B. Hence, there exists a $U_0 \in \mathcal{O}$ such that: *a*) $F_0 \subseteq U_0$ and *b*) for every open in B set V, containing F_0, there exists a $\Phi_V \in 2^B$ such that $\Phi_V \subset V$ and $(cl_X \Phi_V) \setminus U_0 \neq \emptyset$. Let us put $C = B \cap U_0$. Then the map $i_C \colon 2^{C,T} \to 2^{X,T}$ is not continuous at the point F_0 of 2^C (since every set, open in C, is open in B too). Let $Y = C \cup (X \setminus U_0)$. Then $F_0 \subset Y$. Since $X \in \mathcal{LFN}$, it follows that the pair $(F_0, Y \cap U_0)$ in Y is F-embedded in Y. But $Y \cap U_0 = (C \cup (X \setminus U_0)) \cap U_0 = C \cap U_0 = B \cap U_0 = C$ and hence C is open in Y and the pair (F_0, C) in Y is F-embedded in Y. So, there exists an open in Y set V_0 such that: (i) $F_0 \subseteq V_0$, and (ii) $(\Phi \subset V_0, \Phi \in 2^C) \Rightarrow (\Phi \in 2^Y)$. Then $V_1 = V_0 \cap C$ is open in Y and in C. Hence V_1 is open in B and $F_0 \subseteq V_1$. So, there exists a $\Phi_1 \in 2^B$ such that $\Phi_1 \subseteq V_1$ and

$$(cl_X \Phi_1) \setminus U_0 \neq \emptyset . \tag{2}$$

Since $\Phi_1 \subseteq C \subseteq B$, we obtain that $\Phi_1 \in 2^C$. Now, $\Phi_1 \subseteq V_1 \subseteq V_0$ and (ii) imply that $\Phi_1 \in 2^Y$. Hence $M = (cl_X \Phi_1) \cap (X \setminus U_0) \subset (cl_X \Phi_1) \cap Y = cl_Y \Phi_1 = \Phi_1 \subset C \subset U_0$, i.e. $M \subset (X \setminus U_0) \cap U_0 = \emptyset$, while (2) shows that $M \neq \emptyset$. ■

THEOREM 2.11. – *Let $f \colon X \to Z$ be a closed map, $f(X) = Z$ and $X \in \mathcal{HS}$. Then $Z \in \mathcal{HS}$.*

PROOF. – By 2.10, it is enough to show that $Z \in \mathcal{LFN}$. Let (Φ, U) be a pair in Z and let Y be a subspace of Z such that $\Phi \subseteq Y$. We have to prove that the pair $(\Phi, U \cap Y)$ in Y is F-embedded in Y.

Since $X \in \mathcal{HS}$ and hence, by 2.10, $X \in \mathcal{LFN}$, the pair $(f^{-1}\Phi, f^{-1}(U \cap Y)) = (f^{-1}\Phi, f^{-1}U \cap f^{-1}Y)$ is F-embedded in $f^{-1}Y$. Then there exists an open in $f^{-1}Y$ set V such that: (i) $f^{-1}\Phi \subseteq V$, and (ii) $B \subseteq V$ and $B \in 2^{f^{-1}(U \cap Y)}$ imply that $B \in 2^{f^{-1}Y}$. Obviously, the same holds for the set $V' = V \cap f^{-1}U$. The map $\varphi = f_Y \colon f^{-1}Y \to Y$ is closed since f is closed (see [E, 2.1.4]). Hence, there exists an open in Y set W such that $\Phi \subseteq W$ and $\varphi^{-1}(W) \subseteq V'$ (see [E, 1.4.12]). Then $f^{-1}\Phi \subseteq f^{-1}(W) = \varphi^{-1}(W) \subseteq V' \subseteq f^{-1}(U \cap Y)$ and, consequently, $\Phi \subseteq W \subseteq U \cap Y$. Let now $B' \subseteq W$ and $B' \in 2^{U \cap Y}$. Then $B = \varphi^{-1}(B') = f^{-1}(B') \subseteq V'$ and $B \in 2^{f^{-1}(U \cap Y)}$. Hence $B \in 2^{f^{-1}Y}$. Since φ is a quotient map, this shows that $B' \in 2^{Y}$. Therefore, $Z \in \mathcal{HS}$. ■

THEOREM 2.12. – *Let $f \colon X \to Z$ be a closed map, $f(X) = Z$ and $X \in \mathcal{FN}$. Then $Z \in \mathcal{FN}$.*

PROOF. – Put $Y = Z$ in the proof of 2.11. ■

The next definition will be used in the proof of Theorem 2.14 and further in the text.

DEFINITION 2.13. – A pair (H, U) in a topological space $(X, \mathcal{O})$ is called *nonseparable pair* if $(cl_X V) \setminus U \neq \emptyset$ for every $V \in \mathcal{O}$ such that $H \subseteq V$.

THEOREM 2.14. – *The following assertions are equivalent:*

a) $\mathcal{HS} \cap \mathcal{T}_2 \subseteq \mathcal{T}_3$;

b) $\mathcal{HS} \cap \mathcal{T}_2 \subseteq \mathcal{T}_{3.5}$;

c) $\mathcal{HS} \cap \mathcal{T}_2 \subseteq \mathcal{T}_4$;

d) $\mathcal{HS} \cap \mathcal{T}_2 = \mathcal{T}_4$.

PROOF. – Obviously, *d)*⇒*c)*⇒*b)*⇒*a)* and *c)*⇒*d)* (see 1.3 or 2.10 and 2.6). Hence in order to prove the theorem, we have to show that *a)*⇒*c)*.

Let $X \in \mathcal{HS} \cap \mathcal{T}_2$. Then, by *a)*, $X \in \mathcal{T}_3$. Suppose $X \notin \mathcal{T}_4$. This means there exists a nonseparable pair (F, U) in X. Let $Y = X/F$, i.e. the quotient space Y is obtained by identifying the points of the closed subset F of X, and let $\varphi \colon X \to X/F = Y$ be the natural map. Then φ is a closed map and, hence, by 2.11, $Y \in \mathcal{HS}$. Obviously, we have that $Y \in \mathcal{T}_2$ (since $X \in \mathcal{T}_3$). Now, the condition *a)* implies that $Y \in \mathcal{T}_3$. But the pair $(\varphi(F), \varphi(U))$ is, obviously, a nonseparable pair in Y, which shows that $Y \notin \mathcal{T}_3$—a contradiction. Hence $X \in \mathcal{T}_4$ ■

REMARK 2.15. – The inclusion $\mathcal{HS} \cap \mathcal{T}_1 \subsetneq \mathcal{T}_2$ doesn't hold: any infinite space X with the cofinite topology testifies to this (see 2.6).

DEFINITION 2.16. – Let $(X, \mathcal{O})$ be a topological space and (H, U) be a pair in X. The pair (H, U) is said to be *N-embedded in X* if for every $V \in \mathcal{O}$ with $H \subset V \subset U$, there exists a subset B_V of V such that $\emptyset \neq (cl_X B_V) \setminus V \subseteq X \setminus U$.

DEFINITION 2.17. – A space X is said to be a *K^*-space* if either $X \in \mathcal{K}$ or there exist a nonseparable pair (H, U) in X and a subspace Y of X such that $H \subseteq Y$ and the pair $(H, U \cap Y)$ in Y is N-embedded in Y. The class of all K^*-spaces is denoted by $\mathcal{K}^*$.

PROPOSITION 2.18. – *A pair (H, U) in $(X, \mathcal{O})$ is N-embedded in X iff (H, U) is not F-embedded in X.*

PROOF. – ($\Rightarrow$) Let (H, U) be N-embedded in X and suppose that (H, U) is also F-embedded in X. Then there exists a $V \in \mathcal{O}$ such that $H \subseteq V$ and $(\Phi \in 2^U, \Phi \subset V) \Rightarrow (\Phi \in 2^X)$. Since (H, U) is N-embedded in X, there exists a subset B of $V \cap U$ such that $\emptyset \neq (cl_X B) \setminus (V \cap U) \subset X \setminus U$. Then

$$(cl_U B) \setminus (V \cap U) = (cl_X B) \cap U \cap (X \setminus (V \cap U)) =$$

$$((cl_X B) \setminus (V \cap U)) \cap U \subset (X \setminus U) \cap U = \emptyset.$$

Hence, $\Phi = cl_U B \subset V \cap U \subset V$ and $\Phi \in 2^U$. This implies that $\Phi \in 2^X$. Thus $(cl_X B) \setminus (V \cap U) = \Phi \setminus (V \cap U) = \emptyset$, which is a contradiction. Hence, (H, U) is not F-embedded in X.

($\Leftarrow$) Let (H, U) be not F-embedded in X. We will show that (H, U) is N-embedded in X. Indeed, let $V \in \mathcal{O}$ and $H \subset V \subset U$. Then there exists a $\Phi \in 2^U$ such that $\Phi \subset V$ and $\Phi \notin 2^X$. Further, $(cl_X \Phi) \cap V = cl_V \Phi = (cl_U \Phi) \cap V = \Phi \cap V = \Phi$ and $(cl_X \Phi) \setminus \Phi \neq \emptyset$. Thus

$$(cl_X \Phi) \setminus V = (cl_X \Phi) \setminus ((cl_X \Phi) \cap V) = (cl_X \Phi) \setminus \Phi \neq \emptyset$$

and

$$(cl_X \Phi) \setminus V = (cl_X \Phi) \setminus \Phi = (cl_X \Phi) \setminus cl_U \Phi = (cl_X \Phi) \setminus ((cl_X \Phi) \cap U) =$$

$$(cl_X \Phi) \setminus U \subset X \setminus U.$$

Hence, the pair (H, U) is N-embedded in X. ∎

THEOREM 2.19. – *a)* $\mathcal{HS} \cap \mathcal{K}^* = \mathcal{N}$ *and hence* $\mathcal{HS} \cap \mathcal{T}_2 \cap \mathcal{K}^* = \mathcal{T}_4$;

b) $\mathcal{HS} = \mathcal{N} \cup (\neg \mathcal{K}^*)$ *and, equivalently,* $\mathcal{K}^* = \mathcal{N} \cup (\neg \mathcal{HS})$;

c) If $\mathcal{P}$ *is a class of spaces such that* $\mathcal{HS} \cap \mathcal{P} \subseteq \mathcal{N}$, *then* $\mathcal{P} \subseteq \mathcal{K}^*$.

PROOF. – *a)* Obviously, $\mathcal{N} \subset \mathcal{HS} \cap \mathcal{K}^*$. Hence, we have to show that $\mathcal{HS} \cap \mathcal{K}^* \subseteq \mathcal{N}$.

Let $X \in \mathcal{HS} \cap \mathcal{K}^*$. Suppose that $X \notin \mathcal{N}$. Since $X \in \mathcal{K}^*$, there exist a nonseparable pair (H, U) in X and a subspace Y of X such that $H \subseteq Y$ and the pair $(H, U \cap Y)$ in Y is N-embedded in Y. Since $X \in \mathcal{HS}$ and $\mathcal{HS} = \mathcal{LFN}$ (see 2.10), the pair $(H, U \cap Y)$ in Y is F-embedded in Y. But this contradicts 2.18. So, $X \in \mathcal{N}$.

b) In *a)* we have shown, in fact, that $(\mathcal{K}^* \setminus \mathcal{N}) \cap \mathcal{HS} = \emptyset$. This, obviously, implies that $\mathcal{HS} \subseteq \mathcal{N} \cup (\neg \mathcal{K}^*)$. Let us prove now that $\mathcal{N} \cup (\neg \mathcal{K}^*) \subseteq \mathcal{HS}$. Since $\mathcal{N} \subset \mathcal{HS}$ (see 2.6 and 2.10 or 1.3), we have only to show that $\neg \mathcal{K}^* \subseteq \mathcal{HS}$.

Let $X \in \neg \mathcal{K}^*$ and suppose that $X \notin \mathcal{HS}$, i.e. that $X \notin \mathcal{LFN}$ (by 2.10). Then there exist a pair (H, U) in X and a subspace Y of X such that $H \subseteq Y$ and the pair $(H, U \cap Y)$ in Y is not F-embedded in Y. The pair (H, U) in X is a nonseparable pair (otherwise the pair $(H, U \cap Y)$ in Y should be F-embedded in Y) and, by 2.18, the pair $(H, U \cap Y)$ in Y is N-embedded in Y. Hence, $X \in \mathcal{K}^*$—a contradiction.

c) Let $\mathcal{P}$ be a class of spaces such that $\mathcal{HS} \cap \mathcal{P} \subseteq \mathcal{N}$. Then, by *b)*, $\mathcal{P} \cap (\neg \mathcal{K}^*) = \mathcal{P} \cap (\mathcal{HS} \setminus \mathcal{N}) \subseteq \mathcal{N} \setminus \mathcal{N} = \emptyset$, i.e. $\mathcal{P} \subseteq \mathcal{K}^*$. ■

THEOREM 2.20. – *The following assertions are equivalent:*

a) $\mathcal{HS} \cap \mathcal{T}_2 \subseteq \mathcal{T}_3$;

b) $\mathcal{T}_2 \subseteq \mathcal{K}^*$.

PROOF. – *a)*$\Rightarrow$*b)*. By 2.14, the assertion *a)* implies that $\mathcal{HS} \cap \mathcal{T}_2 = \mathcal{T}_4$. Hence, by 2.19*c*), we obtain that $\mathcal{T}_2 \subseteq \mathcal{K}^*$.

b)$\Rightarrow$*a)* If $\mathcal{T}_2 \subseteq \mathcal{K}^*$, then, by 2.19*a*), $\mathcal{HS} \cap \mathcal{T}_2 = \mathcal{HS} \cap \mathcal{T}_2 \cap \mathcal{K}^* = \mathcal{T}_4 \subset \mathcal{T}_3$. ■

DEFINITION 2.21. ([DIT], [O]). – A topological space X is called a *gF-space* if for every subset A of X and for every $x \in (cl_X A) \setminus A$, there exists a subset B of A such that $\{x\} = (cl_X B) \setminus B$. The class of all *gF*-spaces will be denoted by $\mathcal{GF}$.

REMARK 2.22. ([DIT], [O]). – Every Frechet-Urysohn T_2-space (and hence every T_2-space with countable character) is a *gF*-space.

DEFINITIONS 2.23. – A topological space $(X, \mathcal{O})$ is called a

i) *K-space* if for every $U \in \mathcal{O}$ and for every $x \in (cl_X U) \setminus U$ there exists a subset B of U such that $\{x\} = (cl_X B) \setminus B$;

ii) *K'-space* if every nonseparable pair (H, U) in X is N-embedded in X;

iii) *K''-space* if either $X \in \mathcal{N}$ or there exists a pair (H, U) in X which is N-embedded in X.

The class of all K-spaces (resp., K'-spaces; K''-spaces) will be denoted by $\mathcal{K}$ (resp., $\mathcal{K}'$; $\mathcal{K}''$).

REMARK 2.24. – $\mathcal{GF} \subseteq \mathcal{K} \subseteq \mathcal{K}' \subseteq \mathcal{K}'' \subseteq \mathcal{K}^*$.

REMARK 2.25. – The theorem from [PR2, n. 1] asserts that if X is a Hausdorff countably compact space with countable character, then $X \in \mathcal{HS}$. We will show that this assertion is not true. (The fact that the proof of Theorem 1 from n.1 of [PR2] is incorrect was mentioned in MR # 88a:54020.) Indeed, J. Vaughan constructed in [V] a Hausdorff countably compact space X_0 with countable character which is not normal. By Remarks 2.22 and 2.24 we get that $X_0 \in \mathcal{K}^*$. So, if X_0 were a HS-space, then, by Theorem 2.19*a*), X_0 should be a normal space—a contradiction. Thus, X_0 is not a HS-space.

REMARK 2.26. – Theorem 2.19*a*) together with Remarks 2.22 and 2.24 show that any space $X \in \mathcal{T}_2 \setminus \mathcal{T}_4$ with countable character is not a HS-space (i.e. X contains a subspace A for which the map $i_A: 2^{A,T} \to 2^{X,T}$ is not continuous). Since the square of the Sorgenfrey line L is such a space, we obtain that $L^2 = X \notin \mathcal{HS}$, which is the content of Theorem (f) in n. 3 of [PR1].

EXAMPLE 2.27. – A limit of an inverse sequence of T_4-spaces, which is not a HS-space.

PROOF. – Let $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{P}$ be the subspaces of the real line $\mathbb{R}$ (endowed with its natural topology) consisting of all natural, all rational and all irrational numbers respectively and let $X = \mathbb{R}_{\mathbb{Q}}$ be the Michael line (see [E, 5.1.32]). Then, as shown by E. Michael, the space $Y = X \times \mathbb{P}$ is not normal (see [E, 5.1.32]). Since $\mathbb{P}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and Y has a countable character, the standard representation of the infinite Cartesian product $X \times \mathbb{N}^{\mathbb{N}}$ as the limit of an inverse sequence (see [E, 2.5.3]) and our Theorem 2.19*a*) (together with 2.22 and 2.24) show that the space Y is the desired example. ■

THEOREM 2.28. – *a)* $\mathcal{FN} \cap \mathcal{K}'' = \mathcal{N}$;

b) $\mathcal{FN} = \mathcal{N} \cup (\neg \mathcal{K}'')$;

c) If $\mathcal{P}$ *is a class of spaces such that* $\mathcal{FN} \cap \mathcal{P} \subseteq \mathcal{N}$, *then* $\mathcal{P} \subseteq \mathcal{K}''$.

PROOF. – The proof of this theorem can be obtained from the proof of Theorem 2.19 putting there $Y = X$. ■

Now, we are going to present examples testifying that the first three inclusions in 2.24 are strong; as a first step in this direction we prove an auxiliary theorem (which generalizes the construction of one of the examples), using the following definition.

DEFINITION 2.29 ([A]). – A topological space X is called a *funnel-shaped space* if for every point x of X there exists a well ordered by inclusion local base $\mathcal{B}(x)$ at x.

THEOREM 2.30. – *Let* $(X, \mathcal{O})$ *be a topological space which can be mapped by a continuous one-to-one map* f *onto a Hausdorff funnel-shaped space* $(Y, \mathcal{O}')$ *such that* $\chi(y, Y) = \tau$ *for every* $y \in Y$, *where* τ *is an infinite regular cardinal number. Then the space* X *can be homeomorphically embedded as a closed nowhere dense subset of a Hausdorff K-space Z.*

PROOF. – Obviously, there is no loss of generality in assuming that the set Y coincides with the set X and that $f(x) = x$ for every $x \in X$. Then $\mathcal{O}' \subseteq \mathcal{O}$. Denoting by λ the initial ordinal number of cardinality τ, let W be the set of all ordinal numbers less than or equal to λ.

Since the space $(Y, \mathcal{O}')$ is funnel-shaped and the character of Y at any point $y \in Y$ is equal to τ, we can fix, for every $y \in Y$, a well-ordered by inclusion local base $\mathcal{B}(y) = \{V_{\alpha,y} : \alpha < \lambda\}$ for Y at y.

Let Z be the Cartesian product of the sets Y and W. We will define a topology $\mathcal{O}''$ on the set Z in the following way: *a)* if $(y, \alpha) \in Z$ and $\alpha \neq \lambda$ then (y, α) is an isolated point of Z; *b)* for every point $z = (y, \lambda)$ of Z, the local base $\mathcal{B}'(z)$ for Z at z is the family $\{(V_{\alpha,y} \times (\alpha, \lambda)) \cup (U \times \{\lambda\}) : \alpha < \lambda,\ y \in U \in \mathcal{O}$ and $U \subseteq V_{\alpha,y}\}$, where $(\alpha, \lambda) = \{\beta \in W : \alpha < \beta < \lambda\}$. Then, obviously, $(Z, \mathcal{O}'')$ is a Hausdorff space and the map $i : (X, \mathcal{O}) \to (Z, \mathcal{O}'')$, $x \to (x, \lambda)$, is a homeomorphic embedding and $i(X)$ is a closed nowhere dense subset of $(Z, \mathcal{O}'')$. We will show that $(Z, \mathcal{O}'')$ is a K-space.

Let $O \in \mathcal{O}''$ and $z \in (cl_Z O) \setminus O$. Then, obviously, $z \in Y \times \{\lambda\}$, i.e. $z = (y, \lambda)$ for some $y \in Y$. Let us put $O_1 = O \setminus (Y \times \{\lambda\})$. We will show that $z \in cl_Z O_1$. Indeed, supposing that $z \notin cl_Z O_1$, we will obtain a neighbourhood V of z in Z such that $V \cap O_1 = \emptyset$ and $V \cap O \cap (Y \times \{\lambda\}) \neq \emptyset$.

Let $V'=V\cap O$. Then $V'\in\mathcal{O}''$, $V'\subset O$, $V'\neq\emptyset$ and $V'\cap O_1=\emptyset$. Hence $V'\subset Y\times\{\lambda\}=i(X)$ and this is a contradiction since $\mathrm{Int}_Z(i(X))=\emptyset$. So, $z\in cl_Z O_1$.

Now, for every $\alpha<\lambda$, we put $V_\alpha=V_{\alpha,y}\times(\alpha,\lambda]$ and choose a point $b_\alpha=(y_\alpha,\xi_\alpha)\in O_1$ in the following way:

1) if $\alpha=1$, then we choose some point from $V_1\cap O_1$ and we denote it by b_1;

2) let $\beta<\lambda$ and assume that b_α has already been defined for every $\alpha<\beta$ in such a way that $b_\alpha\in V_{\varphi(\alpha)}\cap O_1$, where $\varphi:[1,\beta)\to W$ is some increasing function and $b_{\alpha'}\neq b_{\alpha''}$ for $\alpha'\neq\alpha''$. We shall define the point b_β. Let us prove first that the set $F_\beta=\{b_\alpha:\ \alpha<\beta\}$ is closed in Z. Indeed, since τ is a regular cardinal number, there exists a $\gamma_\beta\in W\setminus\{\lambda\}$ such that $\gamma_\beta>\xi_\alpha$ for every $\alpha<\beta$. Then, for every $z'=(y',\lambda)\in Y\times\{\lambda\}$, we have that $O_{z'}=V_{\gamma_\beta,y'}\times(\gamma_\beta,\lambda]$ is a neighbourhood of z' in Z and $O_{z'}\cap F_\beta=\emptyset$. Hence, F_β is a closed subset of Z and $F_\beta\subset O_1$. Since $b_\alpha=(y_\alpha,\xi_\alpha)\in V_{\varphi(\alpha)}$, we have that $\xi_\alpha>\varphi(\alpha)$ and, consequently, $\gamma_\beta>\varphi(\alpha)$ for every $\alpha<\beta$. Putting $\varphi(\beta)=\gamma_\beta$ and choosing a point b_β from the set $V_{\varphi(\beta)}\cap O_1$, we complete the construction of the points $\{b_\alpha:\ \alpha<\lambda\}$.

If we put now $B=\{b_\alpha:\ \alpha<\lambda\}$ then, obviously, $z\in cl_Z B$ and $B\subset O_1\subset O$. We shall show that $B\cup\{z\}=cl_Z B$. Indeed, let $y''\in Y\setminus\{y\}$. Then there exists a $\gamma\in W\setminus\{\lambda\}$ such that $V_{\gamma,y}\cap V_{\gamma,y''}=\emptyset$. By the construction of the points b_α, we have that $b_\alpha\in V_\gamma\cap O_1$ for every $\alpha>\gamma$. Let $\gamma'=\sup\{\xi_\alpha:\ \alpha\leqslant\gamma\}$. Then $\gamma'<\lambda$, $\gamma'\geqslant\gamma$ and $V_{\gamma',y''}\times(\gamma',\lambda]$ is a neighbourhood of (y'',λ) in Z which has no common points with B. Hence, $cl_Z B=B\cup\{z\}$. ■

THEOREM 2.31. – *All of the inclusions $\mathcal{GF}\subset\mathcal{K}\subset\mathcal{K}'\subset\mathcal{K}''$ are strong even in the class of Hausdorff spaces.*

PROOF. – *A*). Construction of a space $Z\in(\mathcal{K}\setminus\mathcal{GF})\cap\mathcal{T}_2$.

Let $X_0=(0,1/2]\cup\{1\}\cup\{1+1/n:\ n\in\mathbb{N}\}$ with the topology of a subspace of $\mathbb{R}$ (see 2.27 for $\mathbb{N}$ and $\mathbb{R}$) and let E be the equivalence relation on X_0 defined by letting $x\,E\,y$ iff either $x=y$ or $|x-y|=1$. Let $X=X_0/E$ and $q:\ X_0\to X$ be the natural quotient mapping. Then q is not hereditarily quotient (= pseudoopen) mapping (see [E, 2.4.17 and 2.4.F]). Since $\chi(X_0)=\aleph_0$ we have, by 2.22, that $X_0\in\mathcal{GF}$. These facts, together with [DIT, Theorem 3.48], show that $X\notin\mathcal{GF}$. Let Y be the set $[0,1/2]$ endowed with the topology of a subspace of $\mathbb{R}$. Then the map $f:\ X\to Y$, defined by the formula $f(q(t))=t$, for any $t\in(0,1/2]$, and $f(q(1))=0$, is obviously a continuous bijection.

Since $Y \in \mathcal{T}_2$ and $\chi(y,Y)=\aleph_0$ for every $y \in Y$, we obtain, using Theorem 2.30, that there exist a Hausdorff K-space Z and a homeomorphic embedding $\varphi: X \to Z$. Then $\varphi(X)$ is a subspace of Z which is not a gF-space. This implies, by [DIT, 3.46e)], that $Z \notin \mathcal{GF}$. So, $Z \in (\mathcal{K} \setminus \mathcal{GF}) \cap \mathcal{T}_2$.

B) Construction of a space $X \in (\mathcal{K}' \setminus \mathcal{K}) \cap \mathcal{T}_2$.

Let $X = \beta\mathbb{N}$—the Stone-Ĉech compactification of $\mathbb{N}$. Then X is a normal space and hence, by Definition 2.23ii), $X \in \mathcal{K}'$. We shall prove that $X \notin \mathcal{K}$. Indeed, let $U = \mathbb{N} \subset \beta\mathbb{N}$ and $x \in (cl_X U) \setminus U = X \setminus U$. Suppose that $X \in \mathcal{K}$. Then there exists a subset B of $\mathbb{N} = U$ such that $cl_X B = B \cup \{x\}$. Since, obviously, B is an infinite set, we have that $cl_X B$ is homeomorphic with $X = \beta\mathbb{N}$. Hence $|cl_X B| > \aleph_0$ and $|\{x\}| = |(cl_X B) \setminus B| > \aleph_0$—a contradiction. So, $X \in (\mathcal{K}' \setminus \mathcal{K}) \cap \mathcal{T}_2$.

C) Construction of a space $X \in (\mathcal{K}'' \setminus \mathcal{K}') \cap \mathcal{T}_2$.

Let W be the space of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1 with the usual order topology, Z be the subset of W consisting of all isolated points in W and $\mathbb{N}$ be the space of all natural numbers with the discrete topology. Let $X = (W \times \mathbb{N}) \cup \{p\}$, where $p \notin W \times \mathbb{N}$. Endow the set X with the following topology $\mathcal{O}$: all sets which are open in the space $W \times \mathbb{N}$ belong to $\mathcal{O}$; the local base $\mathcal{B}(p)$ for X at the point p consists of all subsets of X of the form $U_{A,i} = \{p\} \cup \bigcup\{A \times \{j\}: j \geq i\}$, where $i \in \mathbb{N}$ and A is a subset of Z such that $|Z \setminus A| < \aleph_1$. The space $(X, \mathcal{O})$ is, obviously, a Hausdorff space, but $(X, \mathcal{O}) \notin \mathcal{T}_3$. Indeed, let $H = \{(\omega_1, i) \in W \times \mathbb{N}: i \in \mathbb{N}\}$. Then $H = cl_X H$ and hence $O = X \setminus H \in \mathcal{O}$. The pair (p, O) is a nonseparable pair in X since for every $U_{A,i} \in \mathcal{B}(p)$ we have that $(\omega_1, j) \in H \cap cl_X U_{A,i}$ for every $j \geq i$, i.e. $(cl_X U_{A,i}) \setminus O \neq \emptyset$. So, $X \notin \mathcal{T}_3$.

In order to prove that $X \notin \mathcal{K}'$, it suffices to show that the nonseparable pair (p, O) is not N-embedded in X.

Let $U = U_{A,i} \in \mathcal{B}(p)$. Then $p \in U \subset O$. We will show that, for every subset B of U, either $(cl_X B) \setminus U = \emptyset$ or $(cl_X B) \setminus U$ is not a subset of $X \setminus O = H$. Indeed, let $B \subset U$ and $(cl_X B) \setminus U \neq \emptyset$. Then there exists a $j \geq i$ such that $|B \cap (W \times \{j\}| \geq \aleph_0$. Let C be an infinite countable subset of $B \cap (W \times \{j\})$. Then there exists an $\alpha \in (cl_{W \times \{j\}} C) \setminus (U \cup H) = (cl_X C) \setminus (U \cup H)$. Hence $(cl_X B) \setminus U$ is not a subset of $X \setminus O$. So, the pair (p, O) is not N-embedded in X and hence $X \notin \mathcal{K}'$.

Let $U = U_{A,i} \in \mathcal{B}(p)$. We will show that the pair (p, U) is not F-embedded in X. Indeed, it is enough to prove that for every $U' = U_{A',i'} \in \mathcal{B}(p)$ with $U' \subset U$, there exists a $\Phi_{U'} \in U'$ such that $\Phi_{U'} \in$

$2^U \setminus 2^X$. But, obviously, if, for every $U' \in \mathcal{B}(p)$ such that $U' \subset U$, we put $\Phi_{U'} = U'$, then we will get that $\Phi_{U'} \in 2^U \setminus 2^X$. Hence, the pair (p, U) is not F-embedded in X, which implies that $X \notin \mathcal{FN}$. Now, we get, by Theorem 2.28*b*), that $X \in \mathcal{K}''$. So, $X \in (\mathcal{K}'' \setminus \mathcal{K}') \cap \mathcal{T}_2$. ∎

The Hausdorff space $X \in \mathcal{K}'' \setminus \mathcal{K}'$ which was constructed in the proof of Theorem 2.31 seems to be a natural and simple example with such properties, but it has cardinality $\aleph_1$. Now, we will describe a countable T_2-space $Y \in \mathcal{K}'' \setminus \mathcal{K}'$.

EXAMPLE 2.32. – A countable, sequential, Hausdorff space Y which is a K''-space and is not a K'-space.

PROOF. – Denote by $AF(x)$ the Arhangel'skiĭ-Franklin space S_ω with basic point x (see [AF]). Since we use it, we shall describe its construction for the convenience of the reader.

The set $AF(x)$ is of the form $AF(x) = \bigoplus\{AF_i(x) : i \in \mathrm{N} \cup \{0\}\}$, where the set $AF_i(x)$ is called *the* i^{th} *level of the set* $AF(x)$, for every $i \in \mathrm{N}$. The levels $AF_i(x)$ will be constructed by induction. Put $AF_0(x) = \{x\}$. Assuming that all levels $AF_i(x)$, for $i = 0, 1, 2, \dots, k$, have already been defined, we will construct the set $AF_{k+1}(x)$. With every point $y \in AF_k(x)$, we associate an infinite countable set M_y (called *a sequence corresponding to* y) in such a way that $M_y \cap \bigcup\{AF_i(x) : i = 0, 1, \dots, k\} = \emptyset$ and $M_y \cap M_z = \emptyset$ for $y, z \in AF_k(x)$, $y \neq z$. Then we put $AF_{k+1}(x) = \bigcup\{M_y : y \in AF_k(x)\}$.

For every point $z \in AF(x)$ we denote by Φ_z the Frechet filter on the sequence M_z corresponding to z.

The topology $\mathcal{O}_x$ on the set $AF(x)$ is defined as follows. Let $y \in AF(x)$. Then there exists a unique $i_y \in \mathrm{N} \cup \{0\}$ such that $y \in AF_{i_y}(x)$. The local base $\mathcal{B}_y$ for $AF(x)$ at the point y consists of all subsets U of $AF(x)$ which satisfy the following two conditions:

1) $\{y\} = U \cap \bigcup\{AF_j(x) : j \leq i_y\}$;

2) $U \cap AF_{k+1}(x) = \bigcup\{A_z : z \in U \cap AF_k(x)\}$, for every $k \geq i_y$ (where, for every $z \in U \cap AF_k(x)$, A_z is some element of Φ_z).

It is easy to check that in such a way we define a topology $\mathcal{O}_x$ on the set $AF(x)$ and that the space $(AF(x), \mathcal{O}_x)$ is a countable, Hausdorff, sequential, zero-dimensional space.

Let now $\{x_i : i \in \mathrm{N}\}$ be a sequence such that $x_i \neq x_j$ for $i \neq j$, $i, j \in \mathrm{N}$. We put $Y' = \bigoplus\{AF(x_i) : i \in \mathrm{N}\}$, $Y = Y' \cup \{c\}$, where $c \notin Y'$, and $T_{i,k} = \bigcup\{AF_s(x_i) : s \geq k\}$, for every $i, k \in \mathrm{N}$. Obviously, the sets $T_{i,k}$ are open subsets of $(AF(x_i), \mathcal{O}_{x_i})$, for every $i, k \in \mathrm{N}$. Let's introduce a topology $\mathcal{O}$ on the set Y. The local base $\mathcal{B}_c$ for $(Y,\mathcal{O})$ at the point c consists of all subsets $U_{i,j}$, $i,j \in \mathbb{N}$, of Y which have the form: $U_{i,j} = \{c\} \cup \bigcup\{T_{m,j}\colon m \geq i\}$. Further, for every point $y \in Y'$ there exists a unique $i \in \mathbb{N}$ such that $y \in AF(x_i)$. Then the local base for $(Y,\mathcal{O})$ at the point y coincides with the local base $\mathcal{B}_y$ for $(AF(x_i), \mathcal{O}_{x_i})$ at y. It is easy to see that in such a way we define a topology $\mathcal{O}$ on Y and that the space $(Y,\mathcal{O})$ is a countable, Hausdorff, sequential space. We are going to show that $Y \in \mathcal{K}'' \setminus \delta\mathcal{K}'$.

Let us first prove that $Y \in \mathcal{K}''$.

Put $H = \{x_i\colon i \in \mathbb{N}\}$. Then the pair (H, Y') in Y is not F-embedded in Y. Indeed, let $V = \bigcup\{U_i\colon i \in \mathbb{N}\}$, where $U_i \in \mathcal{B}_{x_i}$ for every $i \in \mathbb{N}$. Putting $\Phi = V$, we obtain that $\Phi \in 2^{Y'}$, $\Phi \subseteq V$ and $\Phi \notin 2^Y$. Since the open sets like V form a local base for $(Y,\mathcal{O})$ at the set H, we get that the pair (H, Y') is not F-embedded in Y. Hence, $Y \notin \mathcal{FN}$. This implies, by 2.28b), that $Y \in \mathcal{K}''$.

Next, let us show that $Y \notin \mathcal{K}'$.

Put $O = Y \setminus H$. Then the pair (c, O) in Y is nonseparable one since any pair of neighbourhoods of c and H has nonvoid intersection. Further, the pair (c, O) in Y is not N-embedded in Y. Indeed, put $V = U_{1,2} \in \mathcal{B}_c$. Then $c \in V \subset O$. We will show that there is no subset B_V of V such that $\emptyset \neq (cl_Y B_V) \setminus V \subseteq H$. For proving this, consider a subset B of V such that $(cl_Y B) \setminus V \neq \emptyset$. We have that $(cl_Y B) \setminus V \subset Y \setminus V = H \cup \bigcup\{AF_1(x_i)\colon i \in \mathbb{N}\}$. Suppose that $(cl_Y B) \setminus V \subseteq H$ and let $x_i \in (cl_Y B) \setminus V$. Then, for every $y \in AF_1(x_i)$, there exists a $U_y \in \mathcal{B}_y$ such that $U_y \cap B = \emptyset$. Let $W = \{x_i\} \cup \bigcup\{U_y\colon y \in AF_1(x_i)\}$. Then W is a neighbourhood of x_i in Y and, hence, $W \cap B \neq \emptyset$. But $W \cap B \subseteq \{x_i\}$ and $x_i \notin B$ since $x_i \notin V$. Hence, $W \cap B = \emptyset$. This is a contradiction, showing that $(cl_Y B) \setminus V \subseteq H$. So, the nonseparable pair (c, O) in Y is not N-embedded in Y. This implies that $Y \notin \mathcal{K}'$. ■

EXAMPLE 2.33. – A Hausdorff non-normal space $(Z,\mathcal{O}) \in \mathcal{K}''$ such that, for each pair (H, U) in Z, every local base $\mathcal{B}_H$ for Z at H has non-clopen in U elements (in contrast with the Hausdorff spaces X and Y constructed in the part C) of the proof of 2.31 and in 2.32, respectively).

PROOF. – Let $\mathcal{O}'$ be the natural Euclidean topology on the real line $\mathbb{R}$ and $\mathcal{O}''$ be the cocountable topology on $\mathbb{R}$. Let Z coincides with $\mathbb{R}$ as a set and $\mathcal{O}$ be the suprema of $\mathcal{O}'$ and $\mathcal{O}''$. We will show that $(Z,\mathcal{O})$ is the desired example.

It is easy to see that: 1) a set O is open in $(Z, \mathcal{O})$ iff $O = U \setminus A$, where $U \in \mathcal{O}'$ and $|A| \leq \aleph_0$, and 2) if $O = U \setminus A$, where $U \in \mathcal{O}'$ and $|A| \leq \aleph_0$, then $cl_{(Z,\mathcal{O})} O = cl_{(\mathbb{R},\mathcal{O}')} U$ (see [SS, Example 63]). Using these two facts and the local connectedness of $(\mathbb{R}, \mathcal{O}')$, one easily realizes that $(Z, \mathcal{O})$ has the desired local base property described above. Since $(Z, \mathcal{O})$ is obviously a Hausdorff space and $(Z, \mathcal{O}) \notin \mathcal{T}_3$ (see [SS]), we have only to prove that $Z \in \mathcal{K}''$. We will show that the pair $(\sqrt{2}, \mathbb{P})$, where $\mathbb{P}$ is the set of irrationals (see 2.27 for the notations), is not F-embedded in $(Z, \mathcal{O})$, which, by 2.18, will imply that $(\sqrt{2}, \mathbb{P})$ is N-embedded in $(Z, \mathcal{O})$, i.e. that $Z \in \mathcal{K}''$. For doing this, it is enough to prove that for every $n \in \mathbb{N}$ and for every countable set $A \subset Z$ such that $\mathbb{Q} \cap V_n \subset A \subset V_n$, where $V_n = (\sqrt{2} - 1/n, \sqrt{2} + 1/n)$, there exists a subset Φ of $V_n \setminus A$ which is closed in $(\mathbb{P}, \mathcal{O} | \mathbb{P})$, but which is not closed in $(Z, \mathcal{O})$. So, let $n \in \mathbb{N}$ and $V_n \cap \mathbb{Q} \subset A$. If $A \cap \mathbb{P}$ is not dense in $(V_n, \mathcal{O}' | V_n)$, then we can find a closed interval $[r_1, r_2]$, where $r_1, r_2 \in \mathbb{Q}$, such that $\mathbb{P} \cap [r_1, r_2] \subset V_n \setminus A$, and put $\Phi = [r_1, r_2] \cap \mathbb{P}$. This Φ will do the job (see 2) above).

Let now $A \cap \mathbb{P}$ be dense in $(V_n, \mathcal{O}' | V_n)$. Then $|A \cap \mathbb{P}| = \aleph_0$, so we can let: $A \cap \mathbb{P} = \{a_i : i \in \mathbb{N}\}$. To obtain the subset Φ of $V_n \setminus A$ under question, we exploit a construction similar to that of the Cantor set. First of all, let $l_0, r_0 \in \mathbb{Q}$ be such that $\sqrt{2} - 1/2n < l_0 < \sqrt{2} < r_0 < \sqrt{2} + 1/2n$. Then we start with a_1 and find $l_1, r_1 \in \mathbb{Q}$ such that $l_0 < l_1 < a_1 < r_1 < r_0$ and $l_1 - l_0 < 1/3n$, $r_0 - r_1 < 1/3n$. We put $F_1 = [l_0, l_1] \cup [r_1, r_0]$. Let $i_2 = \min\{i \in \mathbb{N} : a_i \notin (l_1, r_1)\}$, $i_{2,1} = \min\{i \in \mathbb{N} : a_i \in (l_0, l_1)\}$ and $i_{2,2} = \min\{i \in \mathbb{N} : a_i \in (r_1, r_0)\}$. Then $i_2 \in \{i_{2,1}, i_{2,2}\}$. Obviously, there exist $l_{2,1}, r_{2,1}, l_{2,2}, r_{2,2} \in \mathbb{Q}$ such that $l_0 < l_{2,1} < a_{i_{2,1}} < r_{2,1} < l_1$, $r_1 < l_{2,2} < a_{i_{2,2}} < r_{2,2} < r_0$ and $l_{2,1} - l_0 < 1/3^2 n$, $l_1 - r_{2,1} < 1/3^2 n$, $l_{2,2} - r_1 < 1/3^2 n$, $r_0 - r_{2,2} < 1/3^2 n$. We put $F_2 = [l_0, l_{2,1}] \cup [r_{2,1}, l_1] \cup [r_1, l_{2,2}] \cup [r_{2,2}, r_0]$. Further, we define F_j, for every $j \in \mathbb{N}$, in the similar way. Put $\Phi' = \bigcap\{F_j : j \in \mathbb{N}\}$. Then Φ' is homeomorphic to the Cantor set and, hence, $|\Phi'| = 2^{\aleph_0}$. Thus $\Phi = \Phi' \cap \mathbb{P}$ is a non-void closed subset of $(\mathbb{P}, \mathcal{O} | \mathbb{P})$ and $\Phi \subset V_n \setminus A$. It is well known that every point of the Cantor set is a complete accumulation point of it. Hence, $l_0 \in cl_{(Z,\mathcal{O})} \Phi$. Thus Φ is not closed in $(Z, \mathcal{O})$. So, we proved that $(Z, \mathcal{O}) \in \mathcal{K}''$. ■

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