Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 46, N° 3, 1993

MATHEMATIQUE
Topologie

ON A THEOREM OF H.-J. SCHMIDT*

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(Submitted by Academician L. Iliev on November 18, 1992)

For any topological space (X, Θ) we denote by 2^X the set of all non-empty closed subset of X. The topology Θ_T on the set 2^X generated by the base $B=\{\langle U\rangle : U\in\Theta\}$, where $\langle U\rangle=\{F\in 2^X : F\subset U\}$, is called *Tychonoff topology.* The topological space $(2^X, \Theta_T)$ will be denoted briefly by 2^{XT}.

We shall say that a topological space X is a *HS-space* if, for every subspace A of X, the map $i_A : 2^{AT}\to 2^{XT}$ defined by the formula $i_A(B)=\mathrm{cl}_X B$, for every $B\in 2^A$, is a continuous map. (By $\mathrm{cl}_X B$ we denote, as usual, the closure of the subset B of X in the space X.) The class of all HS-spaces (resp., all T_i-spaces, for $i=1, 2, 3, 3.5, 4$) will be denoted by $\mathscr{HS}$ (resp., by T_i, $i=1, 2, 3, 3.5, 4$).[1]

In [6], Theorem 11 H. -J. Schmidt proved that $\mathscr{HS}\cap T_2\subset T_3$. Paoli and Ripoli noted in [5] that the proof of this theorem is incorrect, but the question of the correctness of the statement is open. In this paper we give a partial solution of this problem. More precisely: a) we give an internal (i. e. in terms of the space only) characterization of HS-spaces (see Theorem 3); b) we introduce a large class of spaces, called $\mathscr{K}^*$ (see Definition 8), where Schmidt's statement is true (see Theorem 9, where a stronger result is proved); as a corrollary we obtain, using an example constructed by Vaughan in [7], that the Theorem from n. 1 of [5] is not true (see Remark 15); c) we show that the class $\mathscr{HS}$ is invariant under closed mappings (see Theorem 4); d) we prove that Schmidt's statement is true iff the statement '$\mathscr{HS}\cap T_2 = T_4$' is true (see Theorem 6). Moreover, some new classes of spaces are introduced (see Def. 1c, Def. 13) and some new problems are formulated. For all notions and notations undefined here see [3].

1. **Definitions.** a) Let (X, Θ) be a topological space, H be a closed subset of X, $U\in\Theta$ and $H\subset U$. Then we shall say that (H, U) *is a pair in* X.

b) We shall say that a pair (H, U) in (X, Θ) is *F-embedded in* X if there exists a $V\in\Theta$ such that

i) $H\subset V$, and

ii) $\Phi\in 2^U$ and $\Phi\subset V$ imply that $\Phi\in 2^X$.

c) A space X is said to be *F-normal* if every pair (H, U) in X is F-embedded in X.

d) A space X is said to be *LF-normal* if for every pair (H, U) in X and for every subspace Y of X such that $H\subset Y$, the pair $(H, U\cap Y)$ in Y is F-embedded in Y.

The class of all F-normal (resp. LF-normal, normal (and not necessarily T_1)) spaces will be denoted by $\mathscr{FN}$ (resp. $\mathscr{LFN}$, $\mathscr{N}$).

2. **Remarks.** a) $\mathscr{N}\subset\mathscr{LFN}\subset\mathscr{FN}$;

[1] In this paper we assume that T_i-spaces ($i=3, 3.5, 4$) are Hausdorff

* This work was partially supported by the Ministry of Science and Education under grant MM-28/91

b) For any infinite space X with the cofinite topology we have that $X\in(\mathcal{LFN}\cap T_1$ and hence $X\in\mathcal{LFN}\setminus\mathcal{N}$.

3. Theorem. $\mathcal{HS}=\mathcal{LFN}$.

4. Theorem. Let $f\colon X\xrightarrow{onto}Y$ be a closed map and $X\in\mathcal{HS}$ (resp., $X\in\mathcal{FN}$). Then $Y\in\mathcal{HS}$ (resp., $Y\in\mathcal{FN}$).

5. Open question. Is it true that $\mathcal{FN}\cap T_2=T_4$?

6. Theorem. The following assertions are equivalent:

a) $\mathcal{HS}\cap T_2\subset T_3$;

b) $\mathcal{HS}\cap T_2\subset T_{3.5}$;

c) $\mathcal{HS}\cap T_2\subset T_4$;

d) $\mathcal{HS}\cap T_2=T_4$.

7. Definitions. Let (X,Θ) be a topological space and (H,U) be a pair in X. We shall say that:

a) (H,U) is a *nonseparated pair* if $\mathrm{cl}_X V\setminus U\neq\emptyset$ for every $V\in\Theta$ such that V;

b) (H,U) is *N-embedded in* X if for every $V\in\Theta$ such that $H\subset V\subset U$, there is a subset B of V for which $\emptyset\neq\mathrm{cl}_X B\setminus V\subset X\setminus U$ holds.

8. Definition. A space X is said to be a *K^*-space* if either $X\in N$ or there exists a nonseparated pair (H,U) in X and a subspace Y of X such that $H\subset Y$ and the pair $(H, U\cap Y)$ is N-embedded in Y.

The class of all K^*-spaces will be denoted by $\mathcal{K}^*$.

9. Theorem. a) $\mathcal{HS}\cap\mathcal{K}^*=\mathcal{N}$ and hence $\mathcal{HS}\cap T_2\cap\mathcal{K}^*=T_4$;

b) If $\mathcal{P}$ is a class of spaces such that $\mathcal{HS}\cap\mathcal{P}=\mathcal{N}$, then $\mathcal{P}\subseteq\mathcal{K}^*$, i.e. $\mathcal{HS}=\mathcal{N}\cup(\neg\mathcal{K}^*)$, where the class $\neg\mathcal{K}^*$ is defined in the following way: $X\in\neg\mathcal{K}^*$ iff $X\notin\mathcal{K}^*$.

10. Corollary. The following assertions are equivalent:

a) $\mathcal{HS}\cap T_2\subset T_3$;

b) $T_2\subset\mathcal{K}^*$.

11. Definition ([2, 4]). A topological space X is called a *gF*-space if for every set A of X and for every $x\in\mathrm{cl}_X A\setminus A$, there exists a subset B of A such that $\{x\}=\mathrm{cl}_X B\setminus B$.

The class of all *gF*-spaces will be denoted by $g\mathcal{F}$.

12. Remark. Every Frechet-Urysohn T_2-space (and hence every T_2-space with countable character) is a *gF*-space.

13. Definitions. A topological space (X,Θ) is called a

a) *K-space* if for every $U\in\Theta$ and for every $x\in\mathrm{cl}_X U\setminus U$ there exists a subset B of U such that $\{x\}=\mathrm{cl}_X B\setminus B$;

b) *K'-space* if every nonseparated pair (H,U) in X is N-embedded in X;

c) *K''-space* if either $X\in\mathcal{N}$ or there exists a pair (H,U) in X which is N-embedded in X.

The class of all K-spaces (resp., K'-spaces, K''-spaces) will be denoted by $\mathcal{K}$ (resp., $\mathcal{K}'$, $\mathcal{K}''$).

14. Remark. $g\mathcal{F}\subseteq\mathcal{K}\subseteq\mathcal{K}'\subseteq\mathcal{K}''\subseteq\mathcal{K}^*$.

15. Remark. J. Vaughan constructed in [7] a Hausdorff countably compact space with countable character which is not normal. By Remarks 12 and 14 we get that $X\in\mathcal{K}^*$. Hence, using our Theorem 9, we obtain that for this space X the Theorem n. 1 of [5] fails.

16. Theorem. a) $\mathcal{FN}\cap\mathcal{K}''=\mathcal{N}$;

b) If $\mathcal{P}$ is a class of spaces such that $\mathcal{FN}\cap\mathcal{P}=\mathcal{N}$ then $\mathcal{P}\subseteq\mathcal{K}''$, i.e. $\mathcal{FN}=\mathcal{N}\cup(\neg\mathcal{K}'')$, where the class $\neg\mathcal{K}''$ is defined in the following way: $X\in\neg\mathcal{K}''$ iff $X\notin\mathcal{K}''$.

17. Definition ([1]). A topological space X is called a *funnel-shaped space* if for every point x of X there exists a well-ordered by inclusion local base $B(x)$ for X

18. Theorem. Let $f\colon X\to Y$ be a continuous one-to-one map from the topological

... X onto the Hausdorff space Y and let τ be an infinite regular cardinal number. ... Y is a funnel-shaped space and $\chi(y, Y)=\tau$, for every $y \in Y$, then there exist a K-... Z and a homeomorphic embedding $\varphi: X \to Z$ such that $\varphi(X)$ is a closed nowhere ... subset of Z.

19. **Theorem.** All of the inclusions $g\mathscr{F} \subset \mathscr{K} \subset \mathscr{K}' \subset \mathscr{K}''$ are strong even in the ... of Hausdorff spaces.

20. **Open question.** Is it true that $\mathscr{K}''=\mathscr{K}^*$ or, equivalently, is it true that ...$=\mathscr{G}\mathscr{F}\mathscr{N}$?

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