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A NOTE ON THE SPACES WHICH ARE QUOTIENT COMPACT-COVERING s-IMAGES OF METRIC SPACES[1]

S. Barov

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Throughout this paper all spaces are assumed to be Hausdorff and all maps are continuous surjections. If A is a set then $|A|$ stands for the cardinality of A. For a topological space X we define the following conditions:

(i) X has a point-countable cover $\mathcal{F}$ such that for each $x \in K \bigcap U$, with K compact and U open in X, there is a finite $\mathcal{V} \subset \mathcal{F}$ and a finite closed cover $\mathcal{W}$ of a neighborhood of x in K for which $\mathcal{W}$ refines $\mathcal{V}$ and $\bigcup \mathcal{V} \subset U$.

(ii) X has a point-countable cover $\mathcal{F}$ such that for each compact K and an open cover $\mathcal{U}$ of K there is a finite $\mathcal{V} \subset \mathcal{F}$ that refines $\mathcal{U}$ and is refined by a finite closed cover of K.

(j) X has a family $\mathcal{F} = \bigcup_n \mathcal{F}_n, \mathcal{F}_n$ is a point-finite cover for X for all n, such that the following holds:

(j1) for each compact $K \subset X$ and $n \in \mathbf{N}\{K \bigcap F : F \in \mathcal{F}_n\}$ admits a finite closed refinement covering K;

(j2) if $x \in F_n$ and $F_n \in \mathcal{F}_n$ then (F_n) is an outer base (for the definition see [3]) at x in X.

Now, if X is a space and $\mathcal{V}$ be a finite cover of it, then let us call $\mathcal{V}$ to be a minimal cover with respect to a finite closed refinement if $\mathcal{V}$ admits a finite closed refinement covering X and $\mathcal{W}$ does not for each $\mathcal{W} \subset \mathcal{V}, \mathcal{W} \neq \mathcal{V}$.

The main purpose of this note is to establish the following theorems.

Theorem 1.1 ([1]). The following properties of a space X are equivalent:

(a) X is a quotient compact-covering $s-$image of a metric space;

(b) X is a $k-$space satisfying (i);

(c) X is a $k-$space satisfying (ii).

Theorem 1.2. For a topological space X the following are equivalent:

(a) X is an image under a quotient compact-covering map from a metric space with compact fibers;

(b) X is a $k-$space satisfying (j).

To prove Theorem 1.1 and Theorem 1.2 we apply the lemma below. This lemma may hold some independent interest.

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Lemma 1.3. Suppose that a space X, with $d(X) \le \tau$, has a τ-point[1] cover $\mathcal{F}$. Then $|\mathcal{P}| \le \tau$, where $\mathcal{P}$ is the set of all finite subcollections of $\mathcal{F}$ that cover X and are minimal with respect to a finite closed refinement.

In Section 2 we prove our results, and in Section 3 a lemma of inductively perfect maps is recorded. The last section is devoted to examples.

2. Proofs of the results. Lemma 2.1. Let $\mathcal{V}$ be a minimal finite closed cover of a topological space X. Then Int$V \ne \phi$ for each $V \in \mathcal{V}$.

Proof. Let $V \in \mathcal{V}$ be arbitrary. Set $W = V \setminus \bigcup\{\mathcal{V} - \{V\}\}$. Then W is open and $W \ne \phi$ since $\mathcal{V}$ is minimal. That completes the proof.

Proof of Lemma 1.3. Let E be a dense subspace of X, with $|E| = d(X)$, and $\mathcal{V}$ be an arbitrary finite cover of X that is minimal with respect to a finite closed refinement. By Lemma 2.1 if $V \in \mathcal{V}$ then Int $V \ne \phi$ and hence $V \bigcap E \ne \phi$. Since $\mathcal{F}$ is a τ-point cover it follows that $|\mathcal{P}| \le \tau$, where $\mathcal{P}$ is the set of all finite subcollections of $\mathcal{F}$ that covers X and are minimal with respect to a finite closed refinement. That completes the proof.

Proof of Theorem 1.1. The equivalency [(a) ⇔ (c)] is proved in [1]. For reader's convenience we outline the proof.
(a)⇒ (c). Let $f : (M, \mathcal{B}) \mapsto X$ be a quotient compact-covering s-image from a metric space with a point-countable base $\mathcal{B}$. If $K \subset X$ is compact then there exists a compact C in M such that $f(C) = K$. Let $(\mathcal{U}_n)$ be all finite subcollections of $\mathcal{B}$ that are minimal covers for C, arranged in a sequence. Let

$$\mathcal{V}_n = \{f(U) : U \in \mathcal{U}_n\}, n \in \mathbf{N}.$$

Now, setting $\mathcal{F} = \{f(B) : B \in \mathcal{B}\}$ we can pick some $\mathcal{V}_n$ for $\mathcal{V}$ in the condition (ii).
(c) ⇒ (a). Giving $\mathcal{F}$ the discrete topology, the countable product $\mathcal{F}^\omega$ is metrizable. Let $M(\mathcal{F})$ be the set of all $(V_n) \in \mathcal{F}^\omega$ such that, for some $x \in \bigcap_n V_n$, every neighbourhood of x contains some V_k. In fact, $\{x\} = \bigcap_n V_n$, since X is Hausdorff. In this way we define a map $\varphi(\mathcal{F}, X) : M(\mathcal{F}) \mapsto X$, which is continuous and onto (see [4] or [3]). It is easy to verify that $\varphi(\mathcal{F}, X)$ is an s-map. By ([2], Footnote 16) it suffices to show that $\varphi(\mathcal{F}, X)$ is compact-covering. Indeed, let K be a compact in X and $\mathcal{S}$ be the set of all finite subcollections of $\mathcal{F}$ that cover K and are minimal with respect to a finite closed refinement. Applying Lemma 1.3 we arrange $\mathcal{S}$ in a sequence $(\mathcal{V}_n)$. Denote by $(\mathcal{V}_n')$ a sequence of closed finite covers of K for which $\mathcal{V}_n'$ refines $\mathcal{V}_n$ for all n. Set $C = \{(V_n) \in \prod_n \mathcal{V}_n : \exists (V_n')$ with the finite intersection property and $V_n' \subset V_n, V_n' \in \mathcal{V}_n'$ for all $n\}$.

Now, let us see that if $(V_n) \in C$ then $(V_n) \in M(\mathcal{F})$. Obviously, $\bigcap_n V_n' \ne \phi$, where (V_n') is as in the definition of C. Suppose that there are $x, y \in \bigcap_n V_n$. We find an open cover $\mathcal{U}$ of K such that if $\mathcal{V}$ is a cover of K refining $\mathcal{U}$ then $x \notin \bigcup\{W : y \in W, W \in \mathcal{V}\}$. Using (ii) we can find $\mathcal{V}_n \in \mathcal{S}$ for some n, such that $x \notin \bigcup\{W : y \in W, W \in \mathcal{V}_n\}$. Thus, we obtain that if $(V_n) \in C$ then $(V_n) \in M(\mathcal{F})$. Let us chech that C is closed in the compact $\prod_n \mathcal{V}_n$. Get an arbitrary $(P_n) \in \prod_n \mathcal{V}_n \setminus C$. Set $F_k = \{(V_n') \subset \prod_n \mathcal{V}_n' : V_i' \subset P_i, i = \overline{1,k}$ & $\bigcap_{i=1}^k V_i' \ne \phi\}$. Clearly, $F_{i+1} \subset F_i$ and F_i is closed in $\prod_n \mathcal{V}_n'$ for all i. If $F_i \ne \phi$ for all i, then $\bigcap_{i=1}^\infty F_i \ne \phi$, which leads to $(P_n) \in C$ - contradiction. Thus, $F_{k_0} = \phi$ for some k_0. Then $U = \prod_{i=1}^{k_0} P_i \times \prod_{i=k_0+1}^\infty \mathcal{V}_i$ is a neighbourhood of (P_n) such that $U \bigcap C = \phi$. Hence, C is closed in $\prod_n \mathcal{V}_n$. It is not difficult to verify that $\varphi(\mathcal{F}, X)(C) = K$ and we leave that to the reader. Hence, $\varphi(\mathcal{F}, X)$ is compact-covering and that completes the proof of the implication.

[1] $\mathcal{U}$ is a τ-point ($\tau \ge \aleph_0$) cover of X if for each $x \in X$ the cardinality of $\{U : U \in \mathcal{U}, x \in U\}$ is at most τ.

an arbitrary compact and $\mathcal{U}$ is an open cover of it then for every
e $U \in \mathcal{U}$ and we pick a finite $\mathcal{V}_x$ as $\mathcal{V}$ in (i). From $\{\bigcup_{x \in K} \mathcal{V}_x\}$ we
vers K and admits a finite closed refinement.
U, with K compact and U open, then construct an open cover $\mathcal{U}$
and U is the only element of $\mathcal{U}$ that contains x. Pick a finite $\mathcal{V}$ as
$\{V : V \in \mathcal{V}, x \in V\}$ satisfies all our requirements. That completes

proof of Theorem 1.2. (a)$\Rightarrow$ (b). Assume that $f : (M, \mathcal{B}) \to X$
t-covering map from a metric space onto X all whose fibers $f^{-1}(x)$
a metric on M, we find a sequence $(\mathcal{B}_n)$ of locally finite open covers
$\in \mathcal{B}_n$ then diam $B < \frac{1}{n}$ for all n. Then let

$$\mathcal{F}_n = \{f(B) : B \in \mathcal{B}_n\} \text{ and } \mathcal{F} = \bigcup_n \mathcal{F}_n.$$

$\mathcal{F}$ satisfies (j).
ng each $\mathcal{F}_n$ with the discrete topology we let

$$M = \{(S_n) \in \prod_n \mathcal{F}_n : \bigcap S_n \neq \emptyset\}.$$

$M \mapsto X$ by $\varphi((S_n)) = \bigcap_n S_n$. This map is correctly defined since (j2)
K is compact in X then from $\{S : S \bigcap K \neq \emptyset, S \in \mathcal{F}_n\}$ we can get a
that admits a finite closed refinement. Further, we follow the way of
orem 1.1.

na on inductively perfect maps. In this section we would like to
emma that could be applied in various ways. Probably this lemma has
ed directly or indirectly in a lot of proofs so far, but nevertheless its
venient for a later use.
.1. For the map $f : X \mapsto Y$ the following are equivalent:
ductively perfect;
a closed locally finite cover $\{Y_\alpha\}_{\alpha \in A}$ such that $f|f^{-1}(Y_\alpha) : f^{-1}(Y_\alpha) \mapsto Y_\alpha$
perfect for all $\alpha \in A$.
nce $f|f^{-1}(Y_\alpha) : f^{-1}(Y_\alpha) \mapsto Y_\alpha$ is inductively perfect for all $\alpha \in A$ there
such that $f_\alpha = f|X_\alpha : X_\alpha \mapsto Y_\alpha$ is perfect. Let $X' = \bigcup_{\alpha \in A} X_\alpha$. We will
$f|X' : X' \mapsto Y$ is a perfect map. Indeed, let $y \in Y$ be arbitrary. Clearly,
pact as a union of finitely many compacts. If V is a neighborhood of
and $Y_{\alpha_1}, Y_{\alpha_2}, \ldots, Y_{\alpha_k}$ are all sets of $\{Y_\alpha\}_{\alpha \in A}$ that contain y then for each
et a neighbourhood U_{α_i} of y in Y_{α_i} such that $f_{\alpha_i}^{-1}(U_{\alpha_i}) \subset V \bigcap X_{\alpha_i}$. Now,
bourhood U of y in Y such that $U \bigcap Y_{\alpha_i} \subset U_{\alpha_i}$ and $U \subset \bigcup_{i=1}^k Y_{\alpha_i}$. Then
y verify that $g^{-1}(U) \subset V$. Hence, g is closed and that completes the proof.
k 3.2. (a) The requirement of $\{Y_\alpha\}_{\alpha \in A}$ to be closed is essential (see Ex-

o, the following lemma holds:
a 3.1′. A map $\varphi : X \mapsto 2^Y$, with each $\varphi(x)$ compact, is upper semi-
(u.s.c.) iff X has a closed locally finite cover $\{X_\alpha\}_{\alpha \in A}$ such that $\varphi|X_\alpha$:
s u.s.c. for all $\alpha \in A$.

Corollary 3.3. Suppose that $f : X \mapsto Y$ is a quotient map from a locally compact space onto a first-countable paracompact with each fiber $f^{-1}(y)$ Lindelöff. Then f is inductively perfect.

Proof. Let $y \in Y$ be an arbitrary point. For each $x \in f^{-1}(y)$ we get a neighbourhood U_x of x such that $\overline{U}_x$ is compact. Since $f^{-1}(y)$ is Lindelöff we can get from $\{U_x : x \in f^{-1}(y)\}$ a countable subcover $\mathcal{F}^*$. Further, $\mathcal{F}$ determines (see [2] for the definition) X since $\mathcal{F}^* \bigcup (X \setminus f^{-1}(y)) = \mathcal{F}$ is an open cover of X. By ([2], Lemma 1.7) and ([2], Lemma 2.6) there is a finite $\mathcal{V} \subset \mathcal{F}^*$ such that

$$y \in \text{Int } W, \text{ where } W = \bigcup\{f(V) : V \in \mathcal{V}\}.$$

It means that for $y \in Y$ there is a compact K_y for which the following holds: $f|X_y : X_y \mapsto K_y$ is onto for some compact $X_y \subset X$;

(1) $f|X_y : X_y \mapsto K_y$ is onto for some compact $X_y \subset X$;

(2) $y \in$ Int K_y.

Now, our corollary follows from Lemma 3.1 taking into consideration that Y is paracompact.

4. Examples. Example 4.1. Denote $\mathbf{I} = [0,1]$ with the standard topology. Pick a point-countable base $\mathcal{B}$ of $\mathbf{I}$. For $B \in \mathcal{B}$ let

$$V_B = \{x : x \text{ is a rational point of } B\},$$

$$W_B = \{B \backslash V_B : B \in \mathcal{B}\},$$

$$\mathcal{F} = \{V_B : B \in \mathcal{B}\} \bigcup \{W_B : B \in \mathcal{B}\}.$$

By Theorem 1.1 $\varphi(\mathcal{F}, X) : M(\mathcal{F}) \mapsto \mathbf{I}$ is not compact-covering. Moreover, if U is open in $\mathbf{I}$ we can find a sequence $K \subset U$ such that $\varphi(\mathcal{F}, X)(C) \neq K$ for every compact $C \in M(\mathcal{F})$.

Example 4.2. Let Y be the set $\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$ with the standard topology and $f : X \mapsto Y$ be one-to-one map from the discrete space X. Let $Y_1 = \{0\}$ and $Y_2 = Y \backslash Y_1$. Obviously $f|f^{-1}(Y_1) : f^{-1}(Y_1) \mapsto Y_1$ and $f|f^{-1}(Y_2) : f^{-1}(Y_2) \mapsto Y_2$ are perfect maps whereas $f : X \mapsto Y$ is not (see Remark 3.2(a)).

REFERENCES

[1] BAROV S. Covers of topological spaces and compact-covering maps, in preparation. [2] GRUENHAGE G., E. MICHAEL, Y. TANAKA. Pacific J. Math., **113**, 1984, No 2, 303-332. [3] MICHAEL E., K. NAGAMI. Proc. of Amer. Math. Soc., **37**, 1973, No 1, 260-266. [4] PONOMAREV V. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **8**, 1960, 127-134 (Russian).

Institute of Mathematics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: stoyu@math.bas.bg