http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 58, 2021

Pages 71–83

http://topology.nipissingu.ca/tp/

Smooth Convex Bodies in $\mathbb{R}^n$ with Dense Union of Facets

by

Stoyu T. Barov

Electronically published on June 25, 2020

Topology Proceedings

Web: http://topology.auburn.edu/tp/

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: (Online) 2331-1290, (Print) 0146-4124

http://topology.auburn.edu/tp/
http://topology.nipissingu.ca/tp/

TOPOLOGY PROCEEDINGS
Volume 58 (2021)
Pages 71-83

E-Published on June 25, 2020

SMOOTH CONVEX BODIES IN $\mathbb{R}^n$ WITH DENSE UNION OF FACETS

STOYU T. BAROV

Abstract. Let B be closed and convex in $\mathbb{R}^n$; B is called a *convex body* if B is compact and has a nonempty interior with respect to $\mathbb{R}^n$. In addition, B is *smooth* if B has a unique supporting hyperplane at every boundary point. Let $k, n \in \mathbb{N}$ with $k < n$ and let $\mathbb{L}_k^n$ denote the *Grassmann manifold* consisting of all k-dimensional linear subspaces in $\mathbb{R}^n$. An intersection F of B and a supporting hyperplane is called a *facet* if $\dim F = n-1$. A point x of B is called *exposed* by $\mathcal{P} \subset \mathbb{L}_k^n$ if there is a $P \in \mathcal{P}$ such that $(x+P) \cap B = \{x\}$. In this paper, for every $n \geq 2$, we have constructed symmetric smooth convex bodies $B(n)$ in $\mathbb{R}^n$ whose union of all facets is dense in the boundary of $B(n)$ and so that the set of its facets defines a dense set $\mathcal{P}$ in $\mathbb{L}_k^n$ such that the set of all points in $B(n)$ exposed by $\mathcal{P}$ is empty.

1. Introduction

Let B be convex and closed in $\mathbb{R}^n$ and let $\mathbb{L}_k^n$ denote the Grassmann manifold consisting of all k-dimensional linear subspaces of $\mathbb{R}^n$; see Definition 2.3. Let $\mathcal{P} \subset \mathbb{L}_k^n$. A point $x \in B$ is *exposed by* $\mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $(x+P) \cap B = \{x\}$. In [6], the concept of an exposed point is defined, that is, a point exposed by $\mathbb{L}_{n-1}^n$. In principle, our definition generalizes that concept. By $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})$ we denote the set of all points in B exposed by $\mathcal{P}$. A set $C \subset \mathbb{R}^n$ is called a $\mathcal{P}$-imitation of B if $C + P = B + P$ for every $P \in \mathcal{P}$. Let $\mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P}) = \bigcap\{C \subset B \colon C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}$. In general, under some conditions, if $\mathcal{P} \subset \overline{\operatorname{int} \mathcal{P}}$ is not empty, then $\mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P})$ contains

2020 *Mathematics Subject Classification.* 52A20, 52A07.

Key words and phrases. convex body, Euclidean space, exposed point, Grassmann manifold.

a manifold of dimension at least $n-k-1$ (see [3, Theorem 1] and [3, Theorem 16] for more details). Let $\mathcal{W}(B)$ denote the set of all elements H of $\mathbb{L}^n_{n-1}$ such that there exists an x in B so that $x+H$ is a supporting hyperplane to B and $(x+H)\cap B$ is a facet of B, i.e., $\dim(x+H)\cap B = n-1$. The starting point of this paper is the following exposed point theorem, as stated in [5, Theorem 10] when the underlying space is $\mathbb{R}^n$.

Theorem 1.1. *Let $k, n \in \mathbb{N}$ with $k < n$, let $B \subset \mathbb{R}^n$ be closed and convex, and let $\mathcal{P}$ be a G_δ-subset of $\mathbb{L}^n_k$ such that $\mathcal{P} \subset \operatorname{int}\overline{\mathcal{P}}$. Then $\mathcal{X}^k_{\mathrm{p}}(B,\mathcal{P})$ is dense in $\mathcal{X}^k_{\mathrm{t}}(B,\mathcal{P})$.*

In addition, [5, Example 1] shows that G_δ-property for $\mathcal{P}$ can not be omitted. In view of that example, it is natural to ask whether there is a convex body B in $\mathbb{R}^n$ and a dense $\mathcal{P}$ in $\mathbb{L}^n_{n-1}$ such that $\mathcal{X}^{n-1}_{\mathrm{p}}(B,\mathcal{P}) = \emptyset$. The main purpose of this note is to give a positive answer to that question by constructing the convex bodies $B(n)$ for every natural $n \geq 2$. Besides, the following theorem is of independent interest because the constructed sets $B(n)$ have additional interesting properties and the construction itself is very elegant.

Theorem 1.2. *For every $n \in \mathbb{N}$ with $n \geq 2$ there exist smooth convex bodies $B(n)$ in $\mathbb{R}^n$ such that*

(1) *$\mathcal{W}(B(n))$ is dense in $\mathbb{L}^n_{n-1}$,*
(2) *$\bigcup\{F\colon F$ is a facet in $B\}$ is dense in ∂B,*
(3) *$B(n) = -B(n)$,*
(4) *$\mathcal{X}^{n-1}_{\mathrm{p}}(B(n),\mathcal{W}(B(n))) = \emptyset$.*

Clearly, $\mathcal{W}(B(n))$ is not G_δ in $\mathbb{L}^n_{n-1}$. Just to recall and make a contrast for B compact in $\mathbb{R}^n$, we have that $\overline{\langle \mathcal{X}^{n-1}_{\mathrm{p}}(B,\mathbb{L}^n_{n-1})\rangle} = B$ by the Krein–Milman theorem. Moreover, as it is shown in [1], for a compact B in $\mathbb{R}^n$, if $\mathcal{P} \subset \mathbb{L}^n_{n-1}$ is dense and G_δ, then we still have $\overline{\langle \mathcal{X}^{n-1}_{\mathrm{p}}(B,\mathcal{P})\rangle} = B$.

Our paper is arranged as follows. In §2 we define the main concepts and give some basic properties. In §3 we discuss smooth convex sets and establish some properties of those sets that are of independent interest. In §4 and §5 we establish and develop our construction of the sets $B(n)$, $n \geq 2$, and, consequently, we prove our main theorem.

2. Definitions and Preliminaries

Throughout this paper, the underlying space is the Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$. In $\mathbb{R}^n$, we use the standard inner product $x \cdot y = \sum_{i=1}^n u_i v_i$ for $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ elements in $\mathbb{R}^n$. The zero vector is denoted by $\mathbf{0}$ and $S^{n-1} = \{u \in \mathbb{R}^n \colon \|u\| = 1\}$ stands for the standard unit sphere. The norm on $\mathbb{R}^n$ is given by $\|u\| = \sqrt{u \cdot u}$ and the metric

d is given by $d(u,v) = \|v-u\|$. Let A be a subset of $\mathbb{R}^n$. We have that $\operatorname{aff} A$ denotes the affine hull, $\langle A \rangle$ the convex hull, $\overline{A}$ the closure, and $\operatorname{int} A$ the interior of A in $\mathbb{R}^n$. Also, ∂A means the relative boundary of A, that is, the boundary with respect to $\operatorname{aff} A$, and we define $A^\circ = A \setminus \partial A$. Note that if A is convex and nonempty, then $A^\circ \neq \emptyset$ and $(\overline{A})^\circ \subset A$. Further, if A is compact and convex with $\operatorname{int} A \neq \emptyset$, then we call A a *convex body*.

Definition 2.1. Let B be a closed and convex set in $\mathbb{R}^n$. A nonempty subset F of B is called a *face* of B if there is a hyperplane H of $\operatorname{aff} B$ that supports B and $F = H \cap B$. If $\dim F = n-1$, then the face F is called a *facet*.

Definition 2.2 ([8])**.** A closed convex subset of $\mathbb{R}^n$ is called *smooth* if there is a unique supporting hyperplane at each point of its boundary.

Definition 2.3. Consider the closed unit ball $\mathbb{B} = \{u \in \mathbb{R}^n \colon \|u\| \leq 1\}$. Let $\mathcal{K}(\mathbb{B})$ stand for the hyperspace of all nonempty compact subsets of $\mathbb{B}$. Recall that the Hausdorff metric d_{H} on $\mathcal{K}(\mathbb{B})$ associated with d is defined as

$$d_{\mathrm{H}}(A,B) = \sup\{d(x,A), d(y,B) : x \in B \text{ and } y \in A\}.$$

According to [10, Theorem 1.11.3], $\mathcal{K}(\mathbb{B})$ is compact. We let $\mathbb{L}^n_m$ stand for the collection of all m-dimensional linear subspaces of $\mathbb{R}^n$. We topologize $\mathbb{L}^n_m$ by defining a metric ρ on $\mathbb{L}^n_m$:

$$\rho(L_1, L_2) = d_{\mathrm{H}}(L_1 \cap \mathbb{B}, L_2 \cap \mathbb{B}).$$

One can easily see that $\mathbb{L}^n_m$ corresponds to a closed subset of $\mathcal{K}(\mathbb{B})$ and, therefore, it is compact too. Also, $\mathbb{L}^n_m$ is known as a *Grassmann manifold*.

Definition 2.4. Let B be a convex body in $\mathbb{R}^n$. We define the set $\mathcal{W}(B) \subset \mathbb{L}^n_{n-1}$ as

$$\mathcal{W}(B) = \{H \in \mathbb{L}^n_{n-1} \colon \exists \text{ a facet } F \text{ of } B \text{ such that } \operatorname{aff} F \parallel H\}.$$

Definition 2.5. Let X and Y be topological spaces and let 2^Y stand for the collection of all nonempty subsets of Y. A set-valued $\varphi \colon X \to 2^Y$ is called *upper semi-continuous* (*USC*) if $\varphi^{-1}(U) = \{x \in X : \varphi(x) \subset U\}$ is open in X for every open U in Y.

Definition 2.6. A subset A of S^{n-1} is called *convex* if $w \in A$ whenever $w = \alpha u + \beta v \in S^{n-1}$ with $\alpha, \beta \geq 0$ and $u, v \in A$.

Definition 2.7. Let B be a convex and closed subset of $\mathbb{R}^n$, and we define a set-valued function $\Phi : \mathbb{R}^n \setminus \operatorname{int} B \to 2^{S^{n-1}}$ as

$$\Phi(x) = \{a \in S^{n-1} : a \cdot (y - x) \leq 0 \text{ for every } y \in B\}.$$

In other words, $\Phi(x)$ consists of all unit vectors a such that $x + H_a$ is supporting to B and a points towards a side of $x + H_a$ that does not contain points of B, where $H_a = \{x \in \mathbb{R}^n \colon a \cdot x = 0\}$. Observe that by the Hahn–Banach theorem $\Phi(x) \neq \emptyset$ for every x.

We need the following lemma, which is actually [2, Lemma 5].

Lemma 2.8. *Let B be a closed and convex subset in $\mathbb{R}^n$. Then each $\Phi(x)$ is nonempty, closed, and convex in S^{n-1}, and Φ is a USC set-valued map. If B is a convex body, then no $\Phi(x)$ contains antipodal vectors.*

Remark 2.9. Let us point out that in the Hilbert space ℓ^2 the function $\Phi(x)$ may not be a USC set-valued map, as [4, Example 1] shows.

Definition 2.10. Let B be a closed and convex subset of $\mathbb{R}^n$ with $\dim B \geq 1$. In addition, let B be smooth in $\operatorname{aff} B$. Then we can define a map $\Phi_B : \partial B \to S^{n-1}$ as follows: If $x \in \partial B$, then $\Phi_B(x) \in S^{n-1} \cap (\operatorname{aff} B - x)$ such that $\Phi_B(x) \cdot (y - x) \leq 0$ for every $y \in B$.

Remark 2.11. Observe that, according to Definition 2.7 and Lemma 2.8, Φ_B must be continuous.

Remark 2.12. We list a few facts concerning closed convex sets and hyperplanes; see [9, §2.2]. Let B be a closed and convex set in $\mathbb{R}^n$. If the interior of B is nonempty, then a hyperplane H cuts B if and only if H meets the interior of B. Every point in ∂B is contained in a hyperplane H of $\operatorname{aff} B$ that does not cut B. In other words, ∂B equals the union of the faces of B.

3. Enlarging and Shrinking of Smooth Convex Sets

In this section we establish some properties of smooth convex bodies that are of independent interest. However, some of the results are used in the sequel.

Lemma 3.1. *Let $n \in \mathbb{N}$ and let B be a smooth convex body in $\mathbb{R}^n$. If the union of all facets of B is dense in ∂B, then $\mathcal{W}(B)$ is dense in $\mathbb{L}^n_{n-1}$.*

Proof. If $n = 1$, then the lemma is trivial. Thus, we may assume that $n \geq 2$. Let $P \in \mathbb{L}^n_{n-1}$ and let a be the unit vector orthogonal to P. Pick an $x \in \partial B$ such that $\Phi_B(x) = a$. Since the union of facets of B is dense, we can find a convergent to x sequence $(x_i)_i$ with $x_i \in P_i$ for every $i \in \mathbb{N}$, where each P_i is a facet of B. Since Φ_B is continuous, we obtain that $\lim_{i\to\infty} \Phi_B(x_i) = a$. Further, since $\Phi_B(x_i)$ is perpendicular to $H_i = \operatorname{aff} P_i - x_i$ for every $i \in \mathbb{N}$, we get that $\lim_{i\to\infty} H_i = (\mathbb{R}a)^\perp = P$ with every $H_i \in \mathcal{W}(B)$. Hence, $\mathcal{W}(B)$ is dense in $\mathbb{L}^n_{n-1}$. □

Lemma 3.2. *Let $n \in \mathbb{N}$, let $B \subset \mathbb{R}^n$ be a smooth convex body, and let $x \notin B$. If $y \in B$ such that $d(x, B) = d(x, y)$, then $\frac{x-y}{\|x-y\|} = \Phi_B(y)$.*

Proof. If $n = 1$, then the lemma follows immediately. Let us assume that $n \geq 2$. Let H be a hyperplane at y such that H is orthogonal to $x - y$. We need to show that H is a supporting hyperplane to B. Suppose not. Then H cuts B and we can take $z \in B$ such that $z \notin H$ and z and x are from the same side with respect to H, i.e., $(x-y)(z-y) > 0$. Let $\hat{z}$ be the orthogonal projection of x onto the ray $r = y + [0, +\infty)(z - y)$. Now, $\hat{z}$ is not in the line segment $\langle\{y, z\}\rangle$ because, otherwise, we would have $\hat{z} \in B$ with $\|x - \hat{z}\| < \|x - y\|$—a contradiction. Hence, z is between $\hat{z}$ and y. Observe that, in this case, $\|x - z\| < \|x - y\|$ with $z \in B$, a contradiction with the choice of y. We are done. $\square$

Lemma 3.3. *Let $n \in \mathbb{N}$, let $\varepsilon > 0$, and let $B \subset \mathbb{R}^n$. If B is a smooth convex body with dense union of facets, then so is $\{x \in \mathbb{R}^n : d(x, B) \leq \varepsilon\}$.*

Proof. Observe that if $n = 1$ the lemma is trivial. So we may assume that $n \geq 2$. Set $B_\varepsilon = \{x \in \mathbb{R}^n : d(x, B) \leq \varepsilon\}$. It is trivial to see that B_ε is a convex body in $\mathbb{R}^n$ as well. First, we are going to show that B_ε is smooth. Let $x \in \partial B_\varepsilon$. Take $x' \in B$ such that $d(x, B) = d(x, x')$.

Claim 1. If H is a supporting hyperplane at x' to B then $x - x' + H$ is the only supporting hyperplane at x to B_ε.

Proof of Claim 1. By Lemma 3.2, we have that H is orthogonal to $x - x'$. Let $\hat{H}$ be an arbitrary supporting hyperplane at x to B_ε such that $\hat{H} \neq x - x' + H$. Then $d(x', \hat{H}) < d(x, x') = \varepsilon$. Thus, we can find a $y \in B_\varepsilon$ such that $\hat{H}$ cuts the line segment $\langle\{x', y\}\rangle$ and $\|x' - y\| = \varepsilon$. Consequently, $\hat{H}$ cuts B_ε—a contradiction. Since B_ε is a convex body, by the Hahn–Banach theorem, there is a supporting hyperplane L at x to B_ε. So we get that L must be parallel to H, and we are done.

Next, let $F \subset \partial B$ be a facet of B. So $\dim F = n - 1$ and $F = H \cap B$, where H is a supporting hyperplane to B. Clearly, if $x_1, x_2 \in F$, then $\Phi_B(x_1) = \Phi_B(x_2)$. Set

$$A_F = \{x + \varepsilon\Phi_B(x) : x \in F\}.$$

Claim 2. A_F is a facet of B_ε.

Proof of Claim 2. First of all, observe that $\dim A_F = n - 1$ because A_F is a translate of F. Let $x \in \operatorname{aff} A_F$. Find $x^* \in \operatorname{aff} F = H$ such that $x - x^*$ is orthogonal to H. Observe that $d(x, x^*) = \varepsilon$ and $d(x, z) > \varepsilon$ if $z \in (H \cup B) \setminus \{x^*\}$. Thus, $x \in A_F$ if and only if $x^* \in F$; i.e., $A_F = \operatorname{aff} A_F \cap B_\varepsilon$ and $\operatorname{aff} A_F$ is a supporting hyperplane to B_ε. In addition, since $\dim A_F = n - 1$, we have that A_F is a facet. The proof of Claim 2 is done.

Finally, we show that the union of facets of B_ε is dense in ∂B_ε. Indeed, let $x \in \partial B_\varepsilon$ be arbitrary. Let $\hat{x} \in B$ be such that $\|x - \hat{x}\| = d(x, B) =$

ε. By Lemma 3.2, we have that $\frac{x-\hat{x}}{\|x-\hat{x}\|} = \Phi_B(x)$. Take a convergent sequence $\{\hat{x}_i\}_{i=1}^{\infty}$ to $\hat{x}$ such that each $\hat{x}_i$ is in some facet of B. Set $x_i = \hat{x}_i + \varepsilon\Phi_B(\hat{x}_i)$. By Claim 2, x_i belongs to some facet of B_ε for every $i \in \mathbb{N}$. It suffices to show that $\lim_{i\to\infty} x_i = x$. By Remark 2.11, we have that $\lim_{i\to\infty} \varepsilon\Phi_B(\hat{x}_i) = \varepsilon \lim_{i\to\infty} \Phi_B(\hat{x}_i) = \varepsilon\frac{x-\hat{x}}{\|x-\hat{x}\|} = x - \hat{x}$. Thus,

$$\begin{aligned}\lim_{i\to\infty} x_i &= \lim_{i\to\infty} (\hat{x}_i + \varepsilon\Phi_B(\hat{x}_i)) \\ &= \lim_{i\to\infty} \hat{x}_i + \lim_{i\to\infty} \varepsilon\Phi_B(\hat{x}_i) = \hat{x} + x - \hat{x} = x.\end{aligned}$$

This completes the proof of the lemma. □

Theorem 3.4. *Let $n \in \mathbb{N}$ and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear and onto mapping. If B is a smooth convex body in $\mathbb{R}^n$ with dense union of facets, then so is $T(B)$.*

Proof. Let $T(B) = B_T$. It is well known that T, being a linear map from $\mathbb{R}^n$ onto $\mathbb{R}^n$, must be continuous and one-to-one. Likewise, T^{-1} is linear and continuous, and, therefore, both T and T^{-1} are linear homeomorphisms. By [10, Corollary 3.6.7], we have that $T(\partial B) = \partial B_T$. Moreover, observe that $T(\operatorname{int} B) = \operatorname{int} T(B)$. In addition, $T^{-1}(\partial B_T) = \partial B$ and $T^{-1}(\operatorname{int} B_T) = \operatorname{int} B$. Further, because T is linear and onto, we have that the image of a plane is a plane with the same dimension.

Claim 1. B_T is a smooth convex body.

Proof of Claim 1. It is trivial to see that B_T is a convex body. Let $y \in \partial B_T$ and suppose that there are two distinctive supporting hyperplanes H_1 and H_2 at y to B_T. Then $T^{-1}(H_1)$ and $T^{-1}(H_2)$ are supporting hyperplanes at $x = T^{-1}(y) \in \partial B$ to B. Thus, since B is a smooth convex body, we get that $T^{-1}(H_1) = T^{-1}(H_2)$, a contradiction, with T being bijective and having $H_1 \neq H_2$. So there is only one supporting hyperplane at y to B_T. Hence, B_T is a smooth convex body. The proof is complete.

Now, we will show that the union of all facets of B_T is dense in ∂B_T. Let F be a facet of B; i.e., $\dim F = n-1$ and $F = H \cap \partial B$, where H is a supporting hyperplane to B. Let $F_T = T(F)$.

Claim 2. F_T is a facet of B_T.

Proof of Claim 2. Observe that $H_T = T(H)$ is a supporting hyperplane to B_T. We have that $F_T \subset H_T \cap \partial B_T$. On the other hand, if $y \in H_T \cap B_T$, then there is an $x \in F$ such that $T(x) = y$. Hence, $y \in F_T$ and, therefore, $F_T = H_T \cap \partial B_T$. Since F is a facet, we can take a U open in H with $U \subset \partial B$. Since $T{\restriction}H$ is a homeomorphism, we have that $\dim T(U) = n-1$. In addition, $T(U) \subset \partial B_T$. Thus, $T(U) \subset H_T \cap \partial B_T = F_T$. Consequently,

$\dim F_T = n-1$ and, therefore, F_T is a facet. The proof of the claim is complete.

Now, consider $\mathcal{F} = \{F\colon F \text{ is a facet of } B\}$. Since T is onto and $\bigcup\mathcal{F}$ is dense in ∂B, we get that $T(\bigcup\mathcal{F}) = \bigcup_{F\in\mathcal{F}} T(F)$ is also dense in ∂B_T. Further, by Claim 2, each $T(F)$, $F\in\mathcal{F}$, is a facet of B_T. Consequently, the union of all facets of B_T is dense in ∂B_T. Thus, B_T is as required. That completes the proof of the theorem. □

Let B be a smooth convex body in $\mathbb{R}^n$ with dense union of facets and let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be any finite composition of translations and linear bijections. Then, in view of Theorem 3.4, $T(B)$ must also be a smooth convex body in $\mathbb{R}^n$ with dense union of facets. In particular, if $w \in \mathbb{R}^n$, $r > 0$ and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is defined as $T(x) = w + (x-w)r$, a radial map with a pole w, then $T(B)$ must be a smooth convex body in $\mathbb{R}^n$ with dense (in $\partial T(B)$) union of facets. Thus, we prove the following lemma.

Lemma 3.5. *Let $n\in\mathbb{N}$, let B be a smooth convex body in $\mathbb{R}^n$ with dense union of facets, and let $w\in\mathbb{R}^n$. Then for every positive real number r, the set $B(w,r) = \{w + (x-w)r\colon x\in B\}$ is a smooth convex body with dense union of facets.*

4. Constructions in $\mathbb{R}^2$

In this section, we prove Theorem 1.2 for $n=2$ by constructing $B(2)$ as required. First, we define a continuous and decreasing function $f(x)\colon [0,1]\to[1,0]$, inductively, as follows. Set $H_1 = (1/3, 2/3)$ and $C_1 = [0,1]\setminus H_1 = [0,1/3]\cup[2/3,1]$ and define $f(x) = 1/2$ for $x\in H_1$. Suppose that we have constructed H_k and C_k and have defined $f(x)$ on $\bigcup_{m=1}^k H_m$. Assume that C_k is a union of 2^k disjoint closed subintervals of $[0,1]$, each of length 3^{-k}, i.e.,

$$C_k = \bigcup_{m=1}^{2^k} [a_m, b_m]$$

with $b_m - a_m = 3^{-k}$, $1\le m\le 2^k$, and $a_1 < b_1 < \cdots < a_{2^k} < b_{2^k}$. For each $[a_m, b_m]$, we consider its "middle third" (a_{m_1}, b_{m_1}). Define

$$f\restriction(a_{m_1}, b_{m_1}) = 1 + \frac{1-2m}{2^{k+1}}, 1\le m\le 2^k,$$

$$H_{k+1} = \bigcup\{(a_{m_1}, b_{m_1})\colon 1\le m\le 2^k\},$$

$$C_{k+1} = [0,1]\setminus \bigcup_{m=1}^{k+1} H_m = \bigcup_{m=1}^{2^k}[a_m, a_{m_1}] \cup \bigcup_{m=1}^{2^k}[b_{m_1}, b_m].$$

Clearly, C_{k+1} is a union of 2^{k+1} disjoint closed subintervals of $[0,1]$, each of length 3^{-k-1}. In addition, $C_{k+1} \subset C_k$. It is well known that $C = \bigcap_{m=1}^{\infty} C_m$ is the Cantor middle-third set. We have defined $f\restriction[0,1]\setminus C$, and observe that it is uniformly continuous. That allows us to continuously extend $f\restriction[0,1]\setminus C$ over $[0,1]$ (see, for example, [7, Theorem 4.3.17]) since the image set $[0,1]$ is complete. By the construction, one can easily see that f on $[0,1]$ must be decreasing. Moreover, since the Lebesque measure of C is zero, we can estimate that $\int_0^1 f(x)dx = 1/2$. Set $\hat{F}(x) = \int_0^x f(t)dt$ for $x \in [0,1]$. Observe that $\hat{F}(x)$ is differentiable and concave down since $\hat{F}'(x) = f(x)$ is decreasing. Moreover, there is a dense set of intervals on which f is constant and, therefore, the nondegenerate linear segments on the graph of $\hat{F}(x)$ form a dense subset. Next, we will define a function $F^* : [-3/2, 3/2] \to [0, 3/2]$ as follows:

(1) $F^*(x) = 1 + \hat{F}(x+1)$ for $x \in [-1,0]$; i.e., the graph of F^* on $[-1,0]$ is the graph of $\hat{F}$ translated with the vector $a = (-1,1)$;
(2) the graph of F^* on the interval $[-3/2,-1])$ is symmetric to the graph of F^* on the interval $[-1,0])$ with respect to the line $\mathbb{R}(-1,1)$;
(3) finally, we make $F^*(x)$ an even function by setting $F^*(x) = F^*(-x)$ for $-3/2 \le x \le 3/2$.

Notice that at the points $x = -1$ and $x = 0$, the function F^* is differentiable; i.e., $(F^*)'(-1) = 1$ and $(F^*)'(0) = 0$. Furthermore, observe that

(1) the union of all nondegenerate line segments on the graph of F^* is dense in the graph of F^*;
(2) $\lim_{x\to -3/2^+}(F^*)'(x) = +\infty$ and $\lim_{x\to 3/2^-}(F^*)'(x) = -\infty$;
(3) F^* is concave down.

Now, for sake of convenience, we define a function $\Theta : [-1,1] \to [0,1]$ by shrinking $F^*(x)$ slightly as follows. The graph $G(\Theta)$ is defined in the following way:

$$G(\Theta) = \{(2x/3, 2F(x)/3) \colon x \in [-3/2, 3/2]\}.$$

Remark 4.1. Define the map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ as $T(x) = 2x/3$ for any $x \in \mathbb{R}^n$. Thus, the graph of the function $F^*(x)$ goes to the graph of the function $\Theta(x)$. By Lemma 3.5, we have that $\Theta(x)$ preserves all the above mentioned properties of $F^*(x)$. In particular, we have that $\lim_{x\to -1^+}\Theta'(x) = +\infty$ and $\lim_{x\to 1^-}\Theta'(x) = -\infty$.

Now we can define the required smooth convex body $B(2)$. The boundary of $B(2)$ consists of all points of the form $(x, \Theta(x))$ and $(x, -\Theta(x))$ for $x \in [-1,1]$. Notice that at both points $(-1,0)$ and $(1,0)$ there is a unique supporting line, namely the vertical line. Clearly, $B(2)$ is compact, convex, and smooth with a dense set of linear segments on its boundary. By Lemma 3.1, we get that $\mathcal{W}(B(2))$ is dense in $\mathbb{L}_1^2$. Furthermore, by the construction, $B(2)$ is symmetric with respect to the origin and both axes and, at each boundary point, there is only one supporting line to $B(2)$. Finally, let us show that the last condition of Theorem 1.2 holds. Let $H \in \mathcal{W}(B(2))$. Then there is an $x \in \partial B(2)$ such that $F = (x+H) \cap B(2)$ is a facet. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\phi(x) = -x$. Now, we have that $\phi(B(2)) = B(2)$. In addition, clearly, ϕ is linear and injective and, by Claim 2 in the proof of Theorem 3.4, we get that $-F$ is also a facet. Since $F \neq -F$, we obtain that there is no exposed point by $\{H\}$. Hence, $\mathcal{X}_{\mathrm{p}}^1(B(2), \mathcal{W}(B(2))) = \emptyset$. That completes the construction of $B(2)$ in $\mathbb{R}^2$.

5. Constructions in $\mathbb{R}^n$

We use an induction to construct $B(n) \subset \mathbb{R}^n$, $n \geq 3$, satisfying Theorem 1.2. To do this, we are going to make significant use of the function $\Theta : [0,1] \to [0,1]$, defined in the previous section. Suppose that $B(n-1)$, $n \geq 3$, has been already constructed. We need to construct $B(n)$ in $\mathbb{R}^n$. Set $e = (0,0,\dots,1) \in \mathbb{R}^n$. For $t \in \mathbb{R}$ set

$$H_t = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \colon x_n = t\}.$$

Identifying H_0 with $\mathbb{R}^{n-1}$, we let $B(n-1)$ be a subset of H_0 such that $B(n-1) = -B(n-1)$. For $t \in [-1,1]$, define

$$A_t = \{x \in H_0 \colon d(x, B(n-1)) \leq \Theta(t)\},$$

$$B(n) = \bigcup_{t \in [-1,1]} (A_t + te).$$

From the definition, we immediately have that

$$H_t \cap B(n) = (A_t + te) \text{ for } t \in [-1,1] \text{ and}$$

$$H_t \cap B(n) = \emptyset \text{ if } t \notin [-1,1].$$

Claim 5.1. *$B(n)$ is a symmetric convex body in $\mathbb{R}^n$.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $B(n)$ that converges to x. Let $(x_i)_n = t_i$ for every $i \in \mathbb{N}$. So $\lim_{i\to\infty} t_i = x_n = t$. Clearly, $x_i \in A_{t_i} + t_i e$, $i \in \mathbb{N}$ and $t \in [-1,1]$; therefore, $x_i = \hat{x}_i + t_i e$ with $\hat{x}_i \in A_{t_i}$. Consequently, $d(\hat{x}_i, B(n-1)) \leq \Theta(t_i)$ for every $i \in \mathbb{N}$. Also, $\lim_{i\to\infty} \Theta(t_i) = \Theta(t)$ because Θ is continuous. Further,

$$\begin{aligned} d(x - te, B(n-1)) &= d(\lim_{i\to\infty} x_i - e \lim_{i\to\infty} t_i, B(n-1)) \\ &= \lim_{i\to\infty} d(x_i - t_i e, B(n-1)) \leq \lim_{i\to\infty} \Theta(t_i) = \Theta(t). \end{aligned}$$

Hence, $x - te \in A_t$ and, therefore, $x \in A_t + te \subset B(n)$ for $t \in [-1,1]$. Thus, $B(n)$ is closed.

Let us show that $B(n)$ is symmetric, i.e., $B(n) = -B(n)$. Take $t \in [-1,1]$ and $x \in A_t + te$. So $x = \hat{x} + te$ for some $\hat{x} \in A_t$. Observe that $A_t = A_{-t}$ and, also, A_t is symmetric since $B(n-1)$ is symmetric. Then $-x = -\hat{x} - te = (-\hat{x}) + (-t)e \in A_{-t} - te$. Since $A_{-t} - te \subset B(n)$, we are done.

Now, let $w \in A_{t_1} + t_1 e$ and $v \in A_{t_2} + t_2 e$ for some $t_1, t_2 \in [-1,1]$. Let $p = mw + (1-m)v$ for some $m \in [-1,1]$. It suffices to show that $p \in A_t + te$, where $t = mt_1 + (1-m)t_2$. Indeed, $w = w^* + a(w) + t_1 e$ and $v = v^* + a(v) + t_2 e$ with $w^*, v^* \in B(n-1)$, $\|a(w)\| \leq \Theta(t_1)$, and $\|a(v)\| \leq \Theta(t_2)$. Thus,

$$p = (mw^* + (1-m)v^*) + ma(w) + (1-m)a(v) + te.$$

Since $\Theta(x)$ is concave down, we get

$$\|ma(w) + (1-m)a(v)\| \leq m\|a(w)\| + (1-m)\|a(v)\|$$

$$\leq m\Theta(t_1) + (1-m)\Theta(t_2) \leq \Theta(mt_1 + (1-m)t_2) = \Theta(t).$$

Furthermore, since $B(n-1)$ is convex, we have that $mw^* + (1-m)v^* \in B(n-1)$. Therefore, we conclude that $(mw^* + (1-m)v^*) + ma(w) + (1-m)a(v) \in A_t$. Hence, $p \in A_t + te \subset B(n)$. Thus, $B(n)$ is convex. The fact that $\dim B(n) = n$ is obvious. The claim is proved. □

Let $w \in \partial B(n) \setminus ((A_1 + e)^\circ \cup (A_{-1} - e)^\circ)$; i.e., $w \in \partial B(n) \setminus ((B(n-1) + e)^\circ \cup (B(n-1) - e)^\circ)$ since $A_1 = A_{-1} = B(n-1)$ $(\Theta(1) = \Theta(-1) = 0)$. Then there is a $t(w) \in [-1,1]$ such that $w \in \partial(A_{t(w)} + t(w)e)$. Find $w_0 \in \partial(B(n-1) + t(w)e)$ such that $\|w - w_0\| = d(w, B(n-1) + t(w)e)$. Notice that, in fact, $\|w - w_0\| = \Theta(t(w))$. Next, there is a $\hat{w} \in \partial B(n-1)$ such that $w_0 = \hat{w} + t(w)e$. Since $B(n-1)$ is a smooth convex body with $\dim B(n-1) \geq 2$, we can set $a = \Phi_{B(n-1)}(\hat{w})$. Observe, by Lemma 3.2, we have that if $w \neq w_0$, then $\frac{w-w_0}{\|w-w_0\|} = a = \frac{w-w_0}{\Theta(t(w))}$. Consequently, $\Theta(t(w))a = w - w_0$ since $w = w_0$ if and only if $\Theta(t(w)) = 0$. Now, we define

$$M_w = \{\hat{w} + te + \Theta(t)a \colon t \in [-1,1]\}.$$

For $t = t(w)$, we get $\hat{w} + t(w)e + \Theta(t(w))a = \hat{w} + w - w_0 + t(w)e = w$. Thus, $w \in M_w$. By the definition, M_w is the graph of the function $\Theta(x)$ in the 2-dimensional orthogonal coordinate system with origin $\hat{w}$ and perpendicular axes $\hat{w} + \mathbb{R}e$ and $\hat{w} + \mathbb{R}a$ $(a \perp e)$.

Claim 5.2. *If $v \in \partial(A_{t(w)} + t(w)e)$, then the coordinates of v on M_v and w on M_w with respect to their coordinate systems are the same.*

Proof. Observe that the first coordinate of v on M_v is the same as the first coordinate of w on M_w, namely $t(w)$. Then, being on the graph of the same function, their second coordinates are the same, namely $\Theta(t(w))$. $\square$

Claim 5.3. $\operatorname{aff} M_w \cap \operatorname{int} B(n) \neq \emptyset$.

Proof. Observe that $\hat{w} \in \operatorname{aff} M_w \cap \operatorname{int} B(n)$. $\square$

From the construction of M_w, we have the following interesting claim.

Claim 5.4. *If $v \in M_w$, then $M_v = M_w$.*

Claim 5.5. *If $v \in M_w$, then there is only one supporting hyperplane at v to $B(n)$.*

Proof. We find $t \in [-1, 1]$ such that $v \in A_t + te$. Let H be a supporting hyperplane at v to $A_t + te$ in $\operatorname{aff}(A_t + te)$. We have two cases.

Case 1: $t \in \{-1, 1\}$; i.e., v is an end point of M_w. Let $v = \hat{w} + e$. Clearly, $v \in A_1 + e$. Observe that $\operatorname{aff}(A_1 + e)$ is a supporting hyperplane at v to $B(n)$. Suppose $L \neq \operatorname{aff}(A_1 + e)$ is also a supporting hyperplane at v to $B(n)$. First, let us see that $H \subset L$. Indeed, since $\dim(A_1 + e) = n-1$ and $L \neq \operatorname{aff}(A_1+e)$, we have that $L \cap \operatorname{aff}(A_1+e)$ is a supporting hyperplane at v to A_1+e in $\operatorname{aff}(A_1+e)$. Since A_1+e is a smooth body in $\operatorname{aff}(A_1+e)$, H is the only one with this property. Thus, $H = L \cap \operatorname{aff}(A_1 + e)$. Now, consider $\ell = v + \mathbb{R}a$. By definition, $\ell \subset \operatorname{aff}(A_1+e)$, $\ell \perp H$, and $\ell \subset \operatorname{aff} M_w$. Moreover, by Remark 4.1, we have that $\lim_{x\to 1^-} \Theta'(x) = -\infty$. Thus, we get that ℓ is a one-sided tangent line to M_w in $\operatorname{aff} M_w$. That implies that $L \cap \operatorname{aff} M_w$ must contain ℓ. Consequently, $(H \cup \ell) \subset L$ and, therefore, $\operatorname{aff}(A_1 + e) = \operatorname{aff}(H \cup \ell) \subset L$. Hence, $L = \operatorname{aff}(A_1 + e)$—a contradiction. The case when $v = \hat{w} - e$ can be proved analogously.

Case 2: $t \in (-1, 1)$. Let L be a supporting hyperplane at v to $B(n)$. Then $L \neq \operatorname{aff}(A_t + te)$ since $\operatorname{int} B(n) \cap \operatorname{aff}(A_t + te) \neq \emptyset$. As in Case 1, we can show that $H \subset L$. Further, by Claim 5.3, $\operatorname{aff} M_w \not\subset L$. Clearly, $\dim \operatorname{aff} M_w = 2$ and $\dim L = n - 1$ and, therefore, $\dim(L \cap \operatorname{aff} M_w) = 1$. Let $\ell = \operatorname{aff} M_w \cap L$. Then ℓ is a tangent line to M_w in $\operatorname{aff} M_w$. Moreover, since ℓ is not perpendicular to $\mathbb{R}e$, we have that $\ell \not\subset A_t + te$. Hence, $\ell \cap H = \{v\}$ with $\operatorname{aff}(\ell \cup H) \subset L$. Next, $\dim(\ell \cup H) = n - 2 + 1 = n - 1$. Hence, $\operatorname{aff}(\ell \cup H) = L$ and L is uniquely determined by ℓ and H. The claim is proved. $\square$

Lemma 5.6. *$B(n)$ is a smooth convex body.*

Proof. Let $v \in \partial B(n)$. Observe that $\operatorname{aff}(A_1 + e)$ and $\operatorname{aff}(A_{-1} - e)$ are supporting hyperplanes to $B(n)$. Thus, both $(A_1 + e)$ and $(A_{-1} - e)$ are facets of $B(n)$. So if $v \in (A_1 + e)^\circ$ or $v \in (A_{-1} - e)^\circ$, then there are unique supporting hyperplanes at v to $B(n)$. Thus, we may assume that $v \in \partial B(n) \setminus ((A_1+e)^\circ \cup (A_{-1}-e)^\circ)$. Let us consider M_v. Then $v \in M_v$. Now, by Claim 5.5, we get that there is only one supporting hyperplane at v to $B(n)$. Consequently, the lemma holds. □

Lemma 5.7. *The union of all facets of $B(n)$ is dense in $\partial B(n)$.*

Proof. It is easy to notice that both A_1+e and $A_{-1}-e$ are facets of $B(n)$. So let $v \in \partial B(n)\setminus((A_1+e)\cup(A_{-1}-e))$ and let $\varepsilon > 0$. Considering M_v, we have that $v \in M_v$. Since nondegenerate linear segments on M_v are dense, we find a $v_1 \in M_v$ such that $\|v-v_1\| < \varepsilon/2$ and v_1 is an interior point of a linear segment of M_v. By Claim 5.4, we have $M_v = M_{v_1}$. Further, there is a $t \in (-1,1)$ such that $v_1 \in A_t + te = B_t$. By Lemma 3.3, the union of all facets of B_t in aff B_t is dense in ∂B_t. Thus, we can find a $v_2 \in B_t$ such that $\|v_1 - v_2\| < \varepsilon/2$ and v_2 is an interior point of some facet F of B_t in aff B_t. By Claim 5.2, since $M_v = M_{v_1}$, v_2 is also an interior point of a linear segment L of M_{v_2}. Obviously, aff L is the tangent line at v_2 to M_{v_2} in the 2-dimensional plane aff M_{v_2}. Since aff L is not perpendicular to $\mathbb{R}e$, we have that $\operatorname{aff} L \cap \operatorname{aff} B_t = \{v_2\}$. Thus, $\operatorname{aff} L \cap \operatorname{aff} F = \{v_2\}$ because $F \subset B_t$. By Lemma 5.6, we can consider the unique supporting hyperplane H at v_2 to $B(n)$. Next, since $v_2 \in L^\circ$ and $v_2 \in F^\circ$, we obtain that

$$v_2 \in L \cup F \subset \langle L \cup F\rangle \subset \operatorname{aff}(L \cup F) \subset H.$$

Moreover, $\dim\langle L \cup F\rangle = 1 + n - 2 = n - 1$ with $v_2 \in \langle L \cup F\rangle^\circ$. Now, we get that $\hat{F} = H \cap B(n)$ is a facet of $B(n)$ with $v_2 \in \hat{F}^\circ$ because $\langle L \cup F\rangle \subset B(n) \cap H$. In addition, $\|v - v_2\| \leq \|v - v_1\| + \|v_1 - v_2\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. We are done. □

Now, we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. By Claim 5.1, $B(n)$ is a symmetric convex body in $\mathbb{R}^n$. Next, $B(n)$ is smooth by Lemma 5.6 and the union of its facets is dense in $\partial B(n)$ by Lemma 5.7. Applying Lemma 3.1, we get that $\mathcal{W}(B(n))$ is dense in $\mathbb{L}^n_{n-1}$. Further, we show that the last condition of Theorem 1.2 holds. Let $H \in \mathcal{W}(B(n))$. We find an $x \in \partial B(n)$ such that $F = (x+H)\cap B(n)$ is a facet. Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $\phi(x) = -x$. Clearly, ϕ is linear and injective and satisfies $\phi(B(n)) = B(n)$. Now, by Claim 2 of the proof of Lemma 3.3, we get that $-F$ is also a facet. Moreover, $F \neq -F$ and, therefore, there is not an exposed point by $\{H\}$. Hence, $\mathcal{X}^{n-1}_{\mathrm{p}}(B(n), \mathcal{W}(B(n))) = \emptyset$. Thus, for every $n \in \mathbb{N}$, $n \geq 2$, we have constructed the required smooth convex body $B(n)$. That completes the proof. □

Acknowledgment. The author would like to thank the referee for careful reading of the paper and for his/her valuable suggestions and comments.

References

[1] Stoyu Barov, *More on exposed points and extremal points of convex sets in* $\mathbb{R}^n$ *and Hilbert space*. Preprint.

[2] Stoyu Barov and Jan J. Dijkstra, *On closed sets with convex projections under narrow sets of directions*, Trans. Amer. Math. Soc. **360** (2008), no. 12, 6525–6543.

[3] Stoyu Barov and Jan J. Dijkstra, *On closed sets with convex projections under somewhere dense sets of directions*, Proc. Amer. Math. Soc. **137** (2009), no. 7, 2425–2435.

[4] Stoyu Barov and Jan J. Dijkstra, *On closed sets in Hilbert space with convex projections under somewhere dense sets of directions*, J. Topol. Anal. **2** (2010), no. 1, 123–143.

[5] Stoyu Barov and Jan J. Dijkstra, *On exposed points and extremal points of convex sets in* $\mathbb{R}^n$ *and Hilbert space*, Fund. Math. **232** (2016), no. 2, 117–129.

[6] Gustave Choquet, Harry Corson, and Victor Klee, *Exposed points of convex sets*, Pacific J. Math. **17** (1966), 33–43.

[7] Ryszard Engelking, *General Topology*. Translated from the Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[8] Richard J. Gardner, *Geometric Tomography*. 2nd ed. Encyclopedia of Mathematics and its Applications, 58. New York: Cambridge University Press, 2006.

[9] Branko Grünbaum, *Convex Polytopes*. Pure and Applied Mathematics, Vol. 16. New York: Interscience Publishers John Wiley & Sons, Inc., 1967.

[10] Jan van Mill, *The Infinite-Dimensional Topology of Function Spaces*. North-Holland Mathematical Library, 64. Amsterdam: North-Holland Publishing Co., 2001.

Institute of Mathematics and Informatics; Bulgarian Academy of Sciences; 8 Acad. G. Bonchev Str.; 1113 Sofia, Bulgaria

Email address: `stoyu@yahoo.com`