TOPOLOGY PROCEEDINGS
Volume 30, No. 1, 2006
Pages 59-68

COVERS OF TOPOLOGICAL SPACES AND COMPACT-COVERING MAPS

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Abstract. In this paper, we characterize spaces which are quotient compact-covering s-images of metric spaces. We show that X is a quotient compact-covering s-image of a metric space if and only if X is a quotient countable-compact-covering s-image of a metric space.

1. Introduction

Throughout this paper, we denote by $(M, \mathcal{B})$ a space M with a point-countable base $\mathcal{B}$. All maps are assumed to be continuous surjections and all spaces are Hausdorff.

The starting point of this note is the question posed by E. Michael and K. Nagami [11]: Is every quotient s-image of a metric space also a compact-covering quotient s-image of a metric space?

Huaipeng Chen [3] answered that question in the negative. G. Gruenhage, E. Michael, and Y. Tanaka [6, Theorem 6.1] showed that X is a quotient s-image of a metric space if and only if X is a sequence-covering quotient s-image of a metric space. In light of the above results, it is natural to ask whether every quotient countable-compact-covering s-image of a metric space is also a quotient compact-covering s-image of a metric space. The main purpose of this paper is to give a positive answer to this question.

2000 *Mathematics Subject Classification.* 54C10, 54E40, 54C05.

Key words and phrases. compact-covering, point-countable, s-map.

The author was supported in part by a research grant given by Ball State University.

Now, we state our main theorem. For the definitions of all concepts, see section 2.

Theorem 1.1. *The following are equivalent for a topological space X.*

(a) *X is a quotient countable-compact-covering s-image of a metric space.*
(b) *X is a k-space satisfying (i).*
(c) *X is a k-space satisfying (ii).*
(d) *X is a quotient compact-covering s-image of a space having a point-countable base.*
(e) *X is a quotient (ω_0+1)-compact-covering s-image of a metric space.*

To show Theorem 1.1, we use the idea of the construction originated in the proof of [6, Theorem 6.1]. However, to get our theorem, we extend significantly the argument. Also, let us point out that the equivalency of (c) and (d) in the above theorem can be derived directly from [8, Theorem 1] in combination with Lemma 3.2.

We establish our terminology in section 2. In section 3, we prove some lemmas. In section 4, we establish Theorem 1.1.

2. Notations and definitions

The cardinality of an arbitrary set A is denoted by $|A|$; if X is a space and $\mathcal{V}$ is a family of subsets of X then we say that $\mathcal{V}$ covers $A \subset X$ if $A \subset \bigcup \mathcal{V}$. Now, suppose X is a topological space and $\mathcal{F}$ is a network for X such that for each $x \in X$ there exists a countable $\mathcal{V} \subset \mathcal{F}$, which is a network at x. Giving $\mathcal{F}$ the discrete topology, the countable product $\mathcal{F}^{\aleph_0}$ is metrizable. Let $M(\mathcal{F}) \subset \mathcal{F}^{\aleph_0}$ be the set of all $(V_n) \in \mathcal{F}^{\aleph_0}$ such that, for some $x \in \bigcap_n V_n$, every neighborhood of x contains some V_k. Actually, $\{x\} = \bigcap_n V_n$, since X is Hausdorff. In this way, we define a map $\varphi_{X,\mathcal{F}} : M(\mathcal{F}) \to X$ which is continuous and onto. This construction is well known. (For instance, see [12] or [11].) As far as other topological concepts are concerned, we follow [5].

Further, let us define the following conditions.

(*) X has a point-countable family $\mathcal{F}$ such that if $x \in K \cap U$, with K compact in X and U open in X, then there are finite collections $\mathcal{V}_x \subset \mathcal{F}$ and $\mathcal{V}'_x$ such that $\bigcup \mathcal{V}_x \subset U$ and

$\mathcal{V}'_x$ is a finite closed refinement of $\mathcal{V}_x$ that is a cover for a neighborhood of x in K.

(i) X has a point-countable family $\mathcal{F}$ such that, if K is a countable and compact subspace of X and $\mathcal{P}$ is an open (in X) cover of K then there exists a finite $\mathcal{V} \subset \mathcal{F}$ refining $\mathcal{P}$ and which is refined by a closed finite cover $\mathcal{V}'$ of K.

(ii) X has a point-countable family $\mathcal{F}$ such that, if $K \subset X$ is compact and $\mathcal{P}$ is an open (in X) cover of K, then there exists a finite $\mathcal{V} \subset \mathcal{F}$ refining $\mathcal{P}$ and which is refined by a closed finite cover $\mathcal{V}'$ of K.

To this end, one can see that actually (*) and (ii) are equivalent. Also, let us point out that by Lemma 3.2, $\mathcal{F}$ in (*) (and, also, $\mathcal{F}$ in (ii)) is a point-countable strong k-network in the sense of [8].

Definition 2.1. (see [4] or [9]) Let X be a topological space and let α be an ordinal. Then *α-th derivative of* X, denoted by $D^{(\alpha)}X$, is defined inductively as follows:

$$D^{(0)}X = X,$$
$$D^{(\alpha+1)}X = D^{(\alpha)}X \backslash \{x : x \text{ is an isolated point in } D^{(\alpha)}X\},$$
$$D^{(\alpha)}X = \bigcap\nolimits_{\beta<\alpha} D^{(\beta)}X \text{ for limit ordinals } \alpha.$$

The smallest α for which $D^{(\alpha)}X = D^{(\alpha+1)}X$ is called the *Cantor-Bendixson height* of X and denoted by $CB(X)$.

Definition 2.2. ([4]) For $\alpha < \omega_1$, a map $f : X \to Y$ is called *α-compact-covering* if every countable and compact K in Y, with $D^{(\alpha)}K = \emptyset$, is the image of some compact C in X.

Definition 2.3. ([10]) A map $f : X \to Y$ is compact-covering (countable-compact-covering, respectively) if every compact (countable and compact, respectively) $K \subset Y$ is the image of some compact $C \subset X$.

Definition 2.4. ([11]) A map $f : X \to Y$ is called an *s-map* if each fiber $f^{-1}(y)$ is separable.

3. Some lemmas

In this section, we establish two lemmas to prepare the groundwork for the proof of our main results. We begin with a lemma that is fundamental for proving our main theorem.

Lemma 3.1. *Let X be a metrizable space and $\mathcal{U}$ be a finite non-empty collection of subsets of X. Then the following are equivalent.*

(a) *$\mathcal{U}$ admits a finite closed refinement that covers X.*
(b) *If E is a countable and compact subspace of X with $CB(E) \leq |\mathcal{U}|$, then $\{U \bigcap E : U \in \mathcal{U}\}$ admits a finite closed refinement that is a cover of E.*

Proof: $(a) \Rightarrow (b)$ is trivial.

$(b) \Rightarrow (a)$. We will prove this by induction on n, where $n = |\mathcal{U}|$.

Step 1. Assume that $|\mathcal{U}| = 1$, i.e., $\mathcal{U} = \{U\}$. If for every countable and compact $E \subset X$, with $CB(E) = 1$, we have that $U \cap E$ covers E, then obviously $U = X$. Hence, the lemma holds when $|\mathcal{U}| = 1$.

Step 2. Let $\mathcal{U} = \{U_0, ..., U_n\}$ and let us assume that for every metrizable space Y and for every finite collection $\mathcal{U}^*$ of subsets of Y, with $1 \leq |\mathcal{U}^*| \leq n$, we have that the lemma holds. Clearly, $\mathcal{U}$ must be a cover of X. Let us suppose that for every $0 \leq i \leq n$ there is a finite closed refinement $\mathcal{W}_i$ of $\mathcal{U}$ that covers U_i. Then we consider $\mathcal{W} = \bigcup_{k=0}^{n} \mathcal{W}_k$ and $V_{U_i} = \bigcup\{W \in \mathcal{W} \colon W \subset U_i\}$. Obviously, $\{V_{U_i} \colon 0 \leq i \leq n\}$ would be a finite closed refinement of $\mathcal{U}$ that covers X. This argument shows that it suffices to prove that there is a finite closed refinement $\mathcal{W}_0$ of $\mathcal{U}$ that is a cover of U_0. Let us find $\mathcal{W}_0$ with the required properties.

Set

$$\mathcal{U}_i = \mathcal{U} \setminus \{U_i\},$$
$$P^* = \overline{U_0} \setminus U_0,$$
$$C = \overline{P^*} \cap (\bigcap_{i=0}^{n} U_i),$$
$$P = \overline{P^*} \setminus C.$$

Now, we prove that

$$P^* \subset P \text{ and } \overline{P} = \overline{P^*}.$$

Indeed, since $P^* \cap U_0 = \emptyset$, we have that $P^* \cap C = \emptyset$ and therefore, $P^* \subset P$. In addition, since $\overline{P} \subset \overline{P^*}$, we get that $\overline{P} = \overline{P^*}$.

To this end, we can assume that $P^* \neq \emptyset$, since otherwise we are done.

Claim 1. For every $x \in P$ there is a finite closed refinement of $\mathcal{U}$ that covers a neighborhood of x in X.

Proof of Claim 1: Let $x \in P$. Then $x \notin \bigcap_{i=0}^{n} U_i$. So, there is $0 \le j \le n$ such that $x \notin U_j$. We show that there is a finite closed refinement $\mathcal{V}$ of $\mathcal{U}_j$ that covers a neighborhood of x in X. Indeed, suppose that every finite closed refinement of $\mathcal{U}_j$ does not cover a neighborhood of x. We take a decreasing base (O_i) at x. By the inductive hypotheses for every $i \in \mathbb{N}$, we pick a countable and compact space S_i such that

$$S_i \subset O_i,\ CB(S_i) \le n, \text{ and}$$

$$\mathcal{U}_j \text{ does not have a finite closed refinement that covers } S_i.$$

Next, we consider $E = \bigcup_i S_i \bigcup \{x\}$. Clearly, E is a countable and compact space with $CB(E) \le n + 1$. Consequently, there is a finite closed refinement $\{V_U : U \in \mathcal{U}\}$ of $\mathcal{U}$ that covers E. We may also assume that $V_U \subset U$ for all $U \in \mathcal{U}$. Obviously, $x \notin V_{U_j}$, and we can find $l \in \mathbb{N}$ such that

$$O_l \cap V_{U_j} = \emptyset.$$

Now, S_l must have a finite closed cover that refines $\mathcal{U}_j$. That, however, contradicts the choice of S_l. Hence, the claim results.

By Claim 1, for every $x \in P$, we can pick a neighborhood V_x such that V_x is covered by a finite closed refinement of $\mathcal{U}_j$ for some $0 \le j \le n$. Let

$$V = \bigcup_{x \in P} V_x,$$

$$A = \overline{P} \setminus V \text{ and } B = \overline{P} \cap V.$$

Observe that $P \subset B$ and A is closed in X. Furthermore,

$$A = \overline{P} \setminus V \subset \overline{P} \setminus P = \overline{P^*} \setminus P = C.$$

Hence, $A \subset C$ and therefore,

$$A \subset \bigcap_{i=0}^{n} U_i.$$

Claim 2. There is a locally finite family $\mathcal{W}$ in $X \setminus A$ consisting of open sets such that the following hold.

(a) $\{\overline{W} \colon W \in \mathcal{W}\}$ refines $\{V_x \colon x \in P\}$.
(b) $\mathcal{W}$ covers B.

Proof of Claim 2: For every $x \in B$, we choose a neighborhood O_x of x such that $\overline{O_x} \subset V_y$ for some $y \in P$. Consider

$$\mathcal{V} = \{O_x \colon x \in B\} \cup \{X \setminus \overline{P}\}.$$

Observe that $\mathcal{V}$ is an open cover of $X \setminus A$. Thus, we can find a locally finite (in $X \setminus A$) cover $\mathcal{W}^*$ of $X \setminus A$ such that $\mathcal{W}^*$ refines $\mathcal{V}$. Set

$$\mathcal{W} = \{W \in \mathcal{W}^* \colon W \cap B \neq \emptyset\}.$$

Now, notice that $\mathcal{W}$ is a locally finite family in $X \setminus A$ that refines $\{O_x \colon x \in B\}$. Therefore, $\{\overline{W} \colon W \in \mathcal{W}\}$ refines $\{V_x \colon x \in P\}$. Moreover, we have that $\mathcal{W}$ covers B. That completes the proof of the claim.

By Claim 2, if $W \in \mathcal{W}$, then $\overline{W} \subset V_x$ for some $x \in P$. Therefore, for each $W \in \mathcal{W}$, we can pick a finite closed refinement $\mathcal{O}_W$ of $\mathcal{U}$ such that

$$\bigcup \mathcal{O}_W = \overline{W}.$$

Let

$$\mathcal{O}^* = \bigcup_{W \in \mathcal{W}} \mathcal{O}_W,$$

$$V^*_{U_i} = \bigcup \{O^* \in \mathcal{O}^* \colon O^* \subset U_i\} \text{ and } V_{U_i} = V^*_{U_i} \cup A \text{ for } 0 \leq i \leq n.$$

Observe that $V^*_{U_i}$ is closed in $X \setminus A$ for every $0 \leq i \leq n$. Furthermore, we have that every V_{U_i} is closed in X because A is closed in X. Moreover, since $A \subset \bigcap_{i=0}^n U_i$, we have that

$$V_{U_i} \subset U_i.$$

Claim 3. $P^* \subset P \subset \operatorname{int}(\bigcup_{i=0}^n V_{U_i})$.

Proof of Claim 3: By our construction, we have that $\bigcup_{i=0}^n V^*_{U_i} = \bigcup \mathcal{O}^* = \bigcup_{W \in \mathcal{W}} \overline{W}$, $B \subset \bigcup \mathcal{W}$, and $P \subset B$. Thus,

$$P^* \subset P \subset B \subset \bigcup \mathcal{W} = \operatorname{int}(\bigcup \mathcal{W}) \subset \bigcup \mathcal{O}^* = \bigcup_{i=0}^n V^*_{U_i} \subset \bigcup_{i=0}^n V_{U_i}.$$

That completes the proof of the claim.

Let

$$F_{U_0} = \overline{U_0} \setminus \operatorname{int}(\bigcup_{i=0}^n V_{U_i}).$$

Now, by Claim 3, we have that

$$F_{U_0} \subset \overline{U_0} \setminus P^* = \overline{U_0} \setminus (\overline{U_0} \setminus U_0) = U_0.$$

We let

$$\mathcal{W}_0 = \{V_{U_i} \colon 0 \leq i \leq n\} \cup \{F_{U_0}\}.$$

Observe that $\mathcal{W}_0$ is a finite closed refinement of $\mathcal{U}$ as well as a cover of U_0.

So, that completes Step 2 and the proof of the lemma. □

Now, for the lemma below we need the following concept. Let Y be a space and $X \subset Y$. If $\mathcal{V}$ is a finite cover of X, then let us call $\mathcal{V}$ *a minimal cover of* X *with respect to a finite closed refinement* if $\mathcal{V}$ admits a finite closed (in X) refinement covering X and $\mathcal{W}$ does not admit such a refinement whenever $\mathcal{W} \subset \mathcal{V}$ and $\mathcal{W} \neq \mathcal{V}$ (see [1]).

The proof of the next lemma follows the idea of the proof of [2, Lemma 3.2].

Lemma 3.2. *Suppose that a topological space* Y *has a point-countable cover* $\mathcal{F}$ *and* $X \subset Y$ *is a metrizable subspace of* Y*. Then* $|\mathcal{P}| \leq \aleph_0$*, where* $\mathcal{P}$ *is the set of all finite subcollections of* $\mathcal{F}$ *that are minimal covers of* X *with respect to a finite closed refinement.*

Proof: For each $n \in \mathbb{N}$, let

$$G_n = \{\mathcal{V} : \mathcal{V} \in \mathcal{P}, |\mathcal{V}| = n\}.$$

Obviously, it suffices to show that G_n is countable for all n. Suppose that G_n is uncountable for some n. Let us get a maximal $\mathcal{S} \subset \mathcal{F}$ such that $\mathcal{S} \subset \mathcal{V}$ for uncountably many $\mathcal{V} \in G_n$ and let $H = \{\mathcal{V} \in G_n : \mathcal{S} \subset \mathcal{V}\}$. Clearly, $0 \leq |\mathcal{S}| < n$. Then any finite closed (in X) refinement of $\mathcal{S}$ does not cover X. By Lemma 3.1, applied to X and $\mathcal{U} = \{S \cap X \colon S \in \mathcal{S}\}$, we can find a countable set $E \subset X$ for which $\{S \bigcap E : S \in \mathcal{S}\}$ does not admit a finite closed refinement covering E. (If $|\mathcal{S}| = 0$ then we let $E = \{x\}$ for some $x \in X$.) Clearly, if $\mathcal{V} \in G_n$, then $\{V \cap E \colon V \in \mathcal{V}\}$ admits a finite closed refinement that is a cover of E. Consequently, for every $\mathcal{V} \in G_n$, we can choose some $F_{\mathcal{V}} \in \mathcal{V}$ such that $F_{\mathcal{V}} \notin \mathcal{S}$ and $F_{\mathcal{V}} \cap E \neq \emptyset$. Set

$$\mathcal{F}^* = \{F_{\mathcal{V}} \colon \mathcal{V} \in H\}.$$

But E is countable, so E intersects only countably many elements of $\mathcal{F}$. Hence, the set $\mathcal{F}^*$ is countable. So, we can find $F \in \mathcal{F}^*$ that lies in uncountably many $\mathcal{V} \in H$. Let $\mathcal{S}_1 = \mathcal{S} \bigcup \{F\}$. Obviously,

$\mathcal{S} \subset \mathcal{S}_1, \mathcal{S} \neq \mathcal{S}_1$ (since $\mathcal{F}^* \cap \mathcal{S} = \emptyset$) and $\mathcal{S}_1 \subset \mathcal{V}$ for uncountably many $\mathcal{V} \in G_n$, a contradiction with the choice of $\mathcal{S}$. That completes the proof. □

Remark 3.3. (a) Lemma 3.2 is valid if we assume only that X is separable. The proof in this case is very simple and it follows directly from [1, Lemma 1.3 and Lemma 2.1].For our goals, this modification of Lemma 3.2 is sufficient since it is applied to a metrizable compact in a space Y that is a quotient s-image of a metric space. Let us recall that by [6] if Y is a quotient s-image of a metric space, then every compact in Y is metrizable and therefore is separable.

(b) Let us point out that [7, Lemma 7] is similar in spirit to our Lemma 3.2.

4. Proof of Theorem 1.1

Having proved the lemmas in section 3, we can go to the proof of Theorem 1.1.

Proof of Theorem 1.1: (a)⇒(b). Let $f : (M, \mathcal{B}) \to X$ be a quotient countable-compact-covering s-map from a metric space. Denote $\mathcal{F} = \{f(B) : B \in \mathcal{B}\}$ and pick a countable and compact subspace K of X. Let $\mathcal{P}$ be a finite open (in X) cover of K. Find a compact $C \subset M$ such that $f(C) = K$. Consider $\mathcal{P}' = \{f^{-1}(P) \colon P \in \mathcal{P}\}$. Clearly, $\mathcal{P}'$ is an open cover of C. So, we can find a finite $\mathcal{W} \subset \mathcal{B}$ that covers C and refines $\mathcal{P}'$. Let $\mathcal{W}'$ be a finite closed refinement of $\mathcal{W}$ such that $\bigcup \mathcal{W}' = C$. Set $\mathcal{V} = \{f(W) \colon W \in \mathcal{W}\}$ and $\mathcal{V}' = \{f(W') \colon W' \in \mathcal{W}'\}$. For the compact K and the cover $\mathcal{P}$, clearly $\mathcal{V}$ and $\mathcal{V}'$ are as required. The fact that X is a k-space follows directly from [6, Theorem 6.1]. We are done.

(b)⇒(c). Let K be compact in X. Let $x \in K$ and let U be open in X such that $x \in U$. To this end, observe that it suffices to find finite collections $\mathcal{V}_x$ and $\mathcal{V}'_x$ as in (*). Arrange the set $\mathcal{F}_x = \{V : V \in \mathcal{F}, x \in V \subset U\}$ in a sequence (V_n). Let us fix a base (W_n) at x in K such that

$$W_{n+1} \subset W_n \text{ and } W_n \subset U \text{ for all } n.$$

Suppose that there are no collections $\mathcal{V}_x$ and $\mathcal{V}'_x$ as in (*). For $i \in \mathbb{N}$ set

$$\mathcal{P}_i = \{V_j \colon 1 \leq j \leq i\}.$$

By Lemma 3.1, for every $i \in \mathbb{N}$ we find a countable and compact $K_i \subset W_i$ such that $\{K_i \cap P \colon P \in \mathcal{P}_i\}$ does not admit a finite closed (in K) refinement covering K_i. Let

$$K^* = \bigcup_n K_n \bigcup \{x\}.$$

Then for K^* and U, we can not find $\mathcal{V}$ and $\mathcal{V}'$ as in (i), a contradiction with the hypothesis. Hence, the implication holds.

(e)$\Rightarrow$(c). Using the proof of (a)$\Rightarrow$(b), we see that X satisfies condition (i$'$), a modification of (i), namely, (i$'$) is the same as (i) providing that the countable and compact space K satisfies $D^{(\omega_0+1)}K = \emptyset$ instead of being only countable and compact. Then we follow the proof of (b)$\Rightarrow$(c), choosing K_i's to satisfy $CB(K_i) \leq i$ for every $i \in \mathbb{N}$.

(c)$\Rightarrow$(d). Here we can directly apply [8, Theorem 1], along with Lemma 3.2. Also, one may use [7, Theorem 2]. Alternatively, we could apply [1, Theorem 1.1] which is established independently from [8, Theorem 1]. However, for the sake of completeness, we give the proof in detail. Indeed, we show that $\varphi_{X,\mathcal{F}} : M(\mathcal{F}) \to X$ is a quotient compact-covering s-map. It is an s-map because $\mathcal{F}$ is a point-countable family. Now, by [6, footnote 16], it suffices to show that $\varphi_{X,\mathcal{F}}$ is compact-covering. Let K be non-empty and compact in X. Consider the set $\mathcal{S}$ which contains all finite subcollections of $\mathcal{F}$ that cover K and that are minimal with respect to a finite closed refinement.

By Lemma 3.2 or by Remark 3.3, $\mathcal{S}$ is a countable set. Arrange all elements of $\mathcal{S}$ in a sequence $(\mathcal{V}_n)$. Denote by $(\mathcal{V}'_n)$ a sequence of closed finite covers of K such that $\mathcal{V}'_n$ refines $\mathcal{V}_n$ and $\bigcup \mathcal{V}'_n = K$ for all n. Set

$$\begin{aligned} C = \{(V_n) \in \textstyle\prod_n \mathcal{V}_n : \exists (V'_n) \in \prod_n \mathcal{V}'_n \text{ such that } (V'_n) \text{ has} \\ \text{the finite intersection property and } V'_n \subset V_n \text{ for all } n\}. \end{aligned}$$

Let us check that C is closed in the compact space $\prod_n \mathcal{V}_n$. Let $(W_n) \in \prod_n \mathcal{V}_n \backslash C$. Set

$$F_k = \{(V'_n) \in \textstyle\prod_n \mathcal{V}'_n : V'_i \subset W_i, i = \overline{1,k} \text{ and } \bigcap_{i=1}^k V'_i \neq \emptyset\}.$$

Clearly, $F_{i+1} \subset F_i$ and F_i is closed in $\prod_n \mathcal{V}'_n$ for all i. If $F_i \neq \emptyset$ for all i, then $\bigcap_{i=1}^\infty F_i \neq \emptyset$, which leads to $(W_n) \in C$, a contradiction.

Thus, $F_{k_0} = \emptyset$ for some k_0. Then $U = \prod_{i=1}^{k_0} W_i \times \prod_{i=k_0+1}^{\infty} \mathcal{V}_i$ is a neighborhood of (W_n) such that $U \bigcap C = \emptyset$. Hence, C is closed in $\prod_n \mathcal{V}_n$. It is easy to verify that $C \subset M(\mathcal{F})$ and $\varphi_{X,\mathcal{F}}(C) = K$.

(d)⇒(a) follows from [11, Theorem 1.4] and (d)⇒(e) is trivial. That completes the proof. □

References

[1] S. Barov, *A note on the spaces which are quotient compact-covering s-images of metric spaces,* C. R. Acad. Bulgare Sci. **52** (1999), no. 5-6, 11–14.

[2] Dennis Burke and Ernest Michael, *On a theorem of V. V. Filippov,* Israel J. Math. **11** (1972), 394–397.

[3] Huaipeng Chen, *Weak neighborhoods and Michael-Nagami's question,* Houston J. Math. **25** (1999), no. 2, 297–309.

[4] Myung-Hyun Cho and Winfried Just, *Countable-compact-covering maps and compact-covering maps,* Topology Appl. **58** (1994), no. 2, 127–143.

[5] Ryszard Engelking, *General Topology.* Translated from Polish by the author. 2nd ed. Sigma Series in Pure Mathematics, 6. Berlin: Heldermann Verlag, 1989.

[6] G. Gruenhage, E. Michael, and Y. Tanaka, *Spaces determined by point-countable covers,* Pacific J. Math. **113** (1984), no. 2, 303-332.

[7] Shou Lin and Pengfei Yan, *Notes on cfp-covers,* Comment. Math. Univ. Carolin. **44**, (2003), no. 2, 295–306.

[8] Chuan Liu and Mu Min Dai, *Compact-covering s-images of metric spaces* (Chinese), Acta Math. Sinica **39** (1996), no. 1, 41–44.

[9] Winfried Just and Howard Wicke, *Some conditions under which tri-quotient or compact-covering maps are inductively perfect,* Topology Appl. **55** (1994), no. 3, 289–305.

[10] E. Michael, *Inductively perfect maps and tri-quotient maps,* Proc. Amer. Math. Soc. **82** (1981), no 1, 115–119.

[11] E. Michael and K. Nagami, *Compact-covering images of metric spaces,* Proc. of Amer. Math. Soc. **37** (1973), no. 1, 260–266.

[12] V. Ponomarev, *Axioms of countability and continuous mappings* (Russian), Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 127–134.

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