REVIEW OF MY RESEARCH WORK

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In the current review I will briefly discuss my work as a researcher and outline the main results in the presented research papers. I have been working in three research areas: General Topology, Selection Theory and Geometric Tomography/Topology. I should say that my research in Geometric Topology overlaps with areas in Geometric Tomography, Infinite-Dimensional Topology, Measure Theory and Integration, Convex Geometry, Convex Analysis, Topology of Manifolds. Thus Geometric Tomography is closely related to convex geometry in topological vector spaces.

My research in Geometric Topology/Tomography deals with finding important characteristics or retrieving information about an object in a space from information concerning its projections onto planes, called "shadows". In part, it has to do with reconstructing an object from X-rays. The problem of retrieving information about an object based on its projections in lower dimensional spaces is one of importance in many areas of science. Consider, for example, magnetic resonance imaging (MRI), where a three-dimensional object is visualized based on reflected rays. The problem is only going to grow in importance as technology advances. The other area, Selection Theory (connected to Topology, Analysis, Convexity, Banach spaces), has to do with finding different types of selections for a given set-valued map and characterizing different concepts via selections. I present 5 research papers in the area of general topology and the rest 10 (2 in Selection theory and 8 in Convex geometry and tomography) have to do with convex geometry in TVS (topological vector spaces).

1. General Topology

[1] S. Barov, G. Dimov and St. Nedev, On a theorem of H.-J. Schmidt, C. R. Acad. Bulgare Sci., 46 (1993), 9-11.

For every topological space X let 2^X stand for the set of all non-empty closed subsets of X and $cl_X B$—for the closure of the subset B of X in the space X. The topological space $2^{X,T}$ is the set 2^X, endowed with the Tychonoff topology, generated by the base $\{< U >: U \text{ is open}\}$, where $< U >= \{F \in 2^X : F \subset U\}$. A topological space X is called a *HS-space* if, for every subspace A of X, the map $i_A : 2^{A,T} \to: 2^{X,T}$, defined by the formula $i_A(B) = cl_X B$, for every $B \in 2^A$, is a continuous map. In this paper we discuss the assertion of H. J. Schmidt, that is, whether every Hausdorff HS-space is a T_3-space. We give an internal characterization of HS-spaces and define a class of spaces, called $\mathcal{K}^*$, and show that if a Hausdorff space X belongs to that class then X is a normal space (T_4).

[2] S. Barov, G. Dimov and St. Nedev, On a question of M. Paoli and E. Ripoli, Bollettino U. M. I. (7), 10-A (1996), 127-141.

For every topological space X let 2^X stand for the set of all non-empty closed subsets of X and $cl_X B$—for the closure of the subset B of X in the space X. The topological space $2^{X,T}$ is the set 2^X, endowed with the Tychonoff topology, generated by the base $\{< U >: U \text{ is open}\}$, where $< U >= \{F \in 2^X : F \subset U\}$. A topological space X is called a *HS-space* if, for every subspace A of X, the map $i_A : 2^{A,T} \rightarrow: 2^{X,T}$, defined by the formula $i_A(B) = cl_X B$, for every $B \in 2^A$, is a continuous map. In this paper we discuss the assertion of H. J. Schmidt, that is, whether every Hausdorff HS-space is a T_3-space. In the current paper we give a partial solution to this question. More precisely: a) we give an internal characterization of HS-spaces and show that the class HS is invariant under closed mappings; b) we introduce a large class of spaces, called $\mathcal{K}^*$, containing all Haussdorff spaces with a countable character, where we answer in positive to the Schmidt's conjecture; c) we state an equivalent statement to the Schmidt's conjecture; d) we show that if X is Hausdorff and a $\mathcal{K}^*$-space then X is even a normal space (T_4).

[3] S. Barov, A note on spaces which are quotient compact-covering s-images of metric spaces, C. R. Acad. Bulgare Sci., 52, No. 5-6, (1999), 11-14.

For the main terminology the reader can see the definitions in paper [8]. We characterize spaces that are countable-compact-covering s-images of metric spaces via covers with certain properties. One of those characterizations is applied later on to prove the main theorem in [8].

[4] S. Barov, Some properties of star-countable covers, C. R. Acad. Bulgare Sci., 52, No. 7-8, (1999), 5-8.

A family $\mathcal{F}$ of subsets of a space X is called *star-countable* if for every $V \in \mathcal{F}$ the set $\{U \in \mathcal{F} : U \cap V \neq \emptyset\}$ is countable. Our interest in star-countable covers comes from two directions. An open problem of Michael and Nagami (see [8]) can be restated in terms of point-countable coverings. Here we give a positive answer to this question for a star-countable cover instead of a point-countable one. The second direction is related to various assumptions made on star-countable covers. We show some properties for a space X having certain star-countable covers that are not valid or it is not known whether they are valid for the respective point-countable covers.

[8] S. Barov, Covers of topological spaces and compact-covering maps, Topology Proceedings, Vol. 30, No. 1, 2006, 1-10.

A map $f : X \to Y$ is compact-covering (sequence-covering, countable-compact-covering, respectively) if every compact (convergent sequence, countable and compact, resp.) K in Y is the image of some compact C in X. A map $f : X \to Y$ is called an s-map if each fiber $f^{-1}(y)$ is separable. The starting point of this note is the question posed by E. Michael and K. Nagami: Is every quotient s-image of a metric space also a compact-covering quotient s-image of a metric space? G. Gruenhage, E. Michael, and Y. Tanaka showed that X is a quotient s-image of a metric space if and only if X is a sequence-covering quotient s-image of a metric space. Huaipeng Chen answered above question in negative. In light of the above results, it is natural to ask whether every quotient countable-compact-covering s-image of a metric space is also a quotient compact-covering s-image of a metric space. In the current paper we give a positive answer to this question.

2. Selection Theory

Let us set up some basic terminology. A topological space X is called *paracompact* (*countably paracompact*) if X is a Housdorff space and every open cover (resp. every countable open cover) of X has a locally finite (every point $x \in X$ has a neighborhood that intersects only finitely many elements of the cover) open refinement. If Y is a set then 2^Y stands for the set of all nonempty subsets of Y. Let X and Y be topological spaces and let $\varphi : X \to 2^Y$ be a set-valued function. If $A \subset Y$ then we put $\varphi^{-1}[A] = \{x \in X : \varphi(x) \cap A \neq \emptyset\}$ and denote $\operatorname{int} A$ to be the interior of A in Y. The *geometric interior* A° of A is the interior of A relative to the affine hull of A. The function φ is called *lower semi-continuous* (LSC for short) if for each open set O in Y the set $\varphi^{-1}[O]$ is open in X. A function $f : X \to Y$ is called a *selection* of φ if $f(x) \in \varphi(x)$ for every $x \in X$.

[6] S. Barov and J. J. Dijkstra, On boundary avoiding selections and some extension theorems, Pacific J. Math., Vol. 203, No. 1, 2002, 79-87.

We have two general results concerning boundary avoiding continuous selections into Banach spaces. In addition, with relatively simple means we improve upon some other results of Marc Frantz involving extensions of products and of disjoint families of functions. All spaces are assumed to be Tychonoff. We prove two characterization theorems via boundary avoiding selections.

Theorem 1. *The following statements are equivalent:*

(1) *X is a normal and countably paracompact space.*
(2) *For every separable Banach space $\mathbb{B}$, every convex subset C of $\mathbb{B}$, every LSC function $\varphi : X \to 2^C$ such that each $\varphi(x)$ is* compact *and convex in $\mathbb{B}$, and every $A \subset \varphi^{-1}[\operatorname{int} C]$ that is an F_σ-subset of X there exists a continuous selection f of φ with $A \subset f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$.*
(3) *For every separable Banach space $\mathbb{B}$, every convex subset C of $\mathbb{B}$, every LSC function $\varphi : X \to 2^C$ such that each $\varphi(x)$ is* closed *and convex in $\mathbb{B}$, and every $A \subset \varphi^{-1}[\operatorname{int} C]$ that is an F_σ-subset of X there exists a continuous selection f of φ with $A \subset f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$.*

Theorem 2. *The following statements are equivalent:*

(1) *X is a paracompact space.*
(2) *For every Banach space $\mathbb{B}$, every convex subset C of $\mathbb{B}$, every LSC function $\varphi : X \to 2^C$ such that each $\varphi(x)$ is closed and convex in $\mathbb{B}$, and every $A \subset \varphi^{-1}[\operatorname{int} C]$ that is an F_σ-subset of X there exists a continuous selection f of φ with $A \subset f^{-1}(\operatorname{int} C) \subset \varphi^{-1}[\operatorname{int} C]$.*

We also have a product extending theorem. Put $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$.

Theorem 3. *Let X be a normal space and let A be a closed subset of X. If $f : A \to \mathbb{R}^+$, $g : A \to \mathbb{R}^+$, and $h : X \to \mathbb{R}^+$ are continuous functions such that $f \cdot g = h|A$ then there are continuous extensions $\hat{f}, \hat{g} : X \to \mathbb{R}^+$ of f and g with $\hat{f} \cdot \hat{g} = h$. If in addition $g^{-1}(0) \subset f^{-1}(0)$ then it can be arranged that $\hat{g}^{-1}(0) \subset \hat{f}^{-1}(0)$.*

[11] S. Barov, On a characterization of normal and countably paracompact spaces via set-valued selections, Comment. Math. Univ. Carolin. 49, 1 (2008) 45-52.

Here, in this part, all spaces are assumed to be T_1-spaces. We denote by $(\mathbb{B}, \rho)$ the Banach space $\mathbb{B}$ with the metric ρ generated by the given norm on $\mathbb{B}$; $2^{\mathbb{B}}$ stands for the set of all nonempty subsets of $\mathbb{B}$. In order to state our main result let us set up some terminology. For a space $\mathbb{B}$ we denote $E(\mathbb{B}) = \{A \in 2^{\mathbb{B}} \colon A \text{ is convex and } \dim A < \infty\}$. The *geometric interior* A° of $A \subset \mathbb{B}$ is the interior of A relative to the affine hull of A. An open ball with a radius $\varepsilon > 0$ and a center x in a given metric space will be denoted by $B(x, \varepsilon)$. Now, we formulate our main theorem.

Theorem 4. *For a T_1-space X the following are equivalent.*

(i) *X is normal and countably paracompact.*
(ii) *For every separable Banach space $\mathbb{B}$ and for every LSC map $\phi : X \to E(\mathbb{B})$ such that $\dim \phi(x) = \dim \phi(y)$ for every $x, y \in X$ there exists a continuous selection f for ϕ such that $f(x) \in [\phi(x)]^\circ$.*
(iii) *For every separable Banach space $\mathbb{B}$ and for every LSC map $\phi : X \to E(\mathbb{B})$ such that $\dim \phi(x) = \dim \phi(y)$ for every $x, y \in X$ there exists a continuous selection f for ϕ such that for every $x \in X$ there exist a neighborhood V_x of x and an $\varepsilon_x > 0$ with $B(f(y), \varepsilon_x) \cap \phi(y) \subset [\phi(y)]^\circ$ for every $y \in V_x$.*

3. Convex Geometry in TVS and Geometric Tomography

Let B be a closed set either in $\mathbb{R}^n$ or in the Hilbert space l^2 and let the projections of B onto all planes with codimension k be convex. What can one say about B? What topological properties does B possess? In addition, we are interested in finding for every closed convex set B "minimal imitations" C of , i.e. closed subsets C of B that have the same projections onto planes with codimension k, for a fixed k, and are minimal with respect to dimension.

Let us set some terminology in order to state our main results. Throughout this review $\mathbb{V}$ stands for a separable real Hilbert space. Thus $\mathbb{V}$ is isomorphic to either an $\mathbb{R}^n$ or ℓ^2. Let B be convex and closed in $\mathbb{V}$ and let $\mathcal{G}_k(\mathbb{V})$ consist of all k-dimensional linear subspaces of $\mathbb{V}$ with the natural topology, that is, endowed with the Hausdorff metric. Sometimes, we use $\mathcal{G}_k$ instead of $\mathcal{G}_k(\mathbb{V})$ if it is clear whether $\mathbb{V} = \mathbb{R}^n$ or $\mathbb{V} = \ell^2$; see, for example, [13]. Recall that the Hausdorff metric d_{H} associated with the metric d on $\mathbb{V}$ between two nonempty closed and bounded sets A and B in $\mathbb{V}$ is defined as follows:

$$d_{\mathrm{H}}(A, B) = \sup\{d(x, A), d(y, B) : x \in B \text{ and } y \in A\}.$$

We topologize $\mathcal{G}_k(\mathbb{V})$ by defining a metric ρ on $\mathcal{G}_k(\mathbb{V})$:

$$\rho(L_1, L_2) = d_{\mathrm{H}}(L_1 \cap \mathbb{B}_{\mathbb{V}}, L_2 \cap \mathbb{B}_{\mathbb{V}}),$$

where $\mathbb{B}_{\mathbb{V}}$ stands for the unit ball in $\mathbb{V}$. When $\mathbb{V} = \mathbb{R}^n$ then $\mathcal{G}_k(\mathbb{V})$ is called a *Grassmann manifold*. With that metric $\mathcal{G}_k(\mathbb{V})$ is complete and when $\mathbb{V} = \mathbb{R}^n$ it is even compact. Let $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$. We say that $x \in B$ *is exposed by* $\mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $(x + P) \cap B = \{x\}$. We denote by $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P})$ the set of all points of B exposed by $\mathcal{P}$. This definition generalizes the concept of an exposed point, that is, a point of $B \subset \mathbb{R}^n$ that is exposed by $\mathcal{G}_{n-1}(\mathbb{R}^n)$. If $B, C \subset \mathbb{V}$ and $\mathcal{P} \subset \mathcal{G}_k$ then B and C are called *$\mathcal{P}$-imitations* of each other if $B + P = C + P$ for each $P \in \mathcal{P}$, that is, B and C have identical projections along each element of $\mathcal{P}$. If $\overline{B + P} = \overline{C + P}$ for each $P \in \mathcal{P}$ then B and C are called *weak $\mathcal{P}$-imitations* of each other. If $A \subset \mathbb{V}$ then we define $A^\perp$ in the following way:

$$A^\perp = \{v \in \mathbb{V} \colon v \cdot x = v \cdot y \text{ for all } x, y \in A\}.$$

If L is a closed linear subspace of $\mathbb{V}$ then $L^\perp$ is called the *orthocomplement* of L and we have $L^{\perp\perp} = L$. Also, we define $\operatorname{codim} A = \dim A^\perp \in \{0, 1, \ldots, \infty\}$.

An *extremal* point of B with respect to the projections under $\mathcal{P}$ is a point that all closed subsets of B that are $\mathcal{P}$-imitations of B have in common. If $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ then the set of extremal points of B with respect to $\mathcal{P}$ is denoted by $\mathcal{X}^k_{\mathrm{t}}(B, \mathcal{P})$.

A *plane* in $\mathbb{V}$ is a closed affine subspace of $\mathbb{V}$ and a plane L is called a *k-plane* if $\dim L = k$; a *k-subspace* is a k-dimensional linear subspace of $\mathbb{V}$. The *affine hull* $\operatorname{aff} A$ of A is defined as the intersection of all planes that contain A; $\overline{A}$ denotes the closure and $\operatorname{int} A$ the interior of A in $\mathbb{V}$. Observe that $A^\perp = (\operatorname{aff} A)^\perp$ and $\operatorname{codim} A = \operatorname{codim}(\operatorname{aff} A)$. The *geometric interior* A° of A is the interior of A relative to the affine hull of A. The *geometric boundary* of A is $\partial A = \overline{A} \setminus A^\circ$. Also, $\langle A \rangle$ is the convex hull of A. A *k-hyperplane* H is a plane with $\operatorname{codim} H = k$. If L is a plane then $\mathfrak{p}_L : \mathbb{V} \to L$ denotes the orthogonal projection onto L, defined by $\{\mathfrak{p}_L(x)\} = L \cap (x + L^\perp)$ for $x \in \mathbb{V}$. If L is a linear subspace of $\mathbb{V}$ then ψ_L is the orthogonal projection of $\mathbb{V}$ along L onto $L^\perp$. A projection $\psi_L(A)$ is called *proper* if $\psi_L(A) \neq L^\perp$. Recall that a set $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ is *somewhere dense* if $\operatorname{int} \overline{\mathcal{P}} \neq \emptyset$. Finally, the unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}.

[9] S. Barov and Jan J. Dijkstra, On closed sets with convex projections in Hilbert space, Fundamenta Mathematicae, Vol. 197, No. 1,

2007, 17-33.

Let k be a fixed natural number. Our main results are: a) we show that if C is a closed and nonconvex set in Hilbert space l^2 such that the closures of the projections onto all k-hyperplanes (planes with codimension k) are convex and proper, then C must contain a closed topological copy of l^2. In order to prove this result we introduce for convex closed sets B the set $\mathcal{E}^k(B)$ consisting of all points of B that are extremal with respect to projections onto k-hyperplanes; b) we prove that $\mathcal{E}^k(B)$ is precisely the intersection of all k-imitations C of B, i.e., closed sets C that have the same projections as B onto all k-hyperplanes; c) for every closed convex set B in l^2 with non-empty interior we construct "minimal" k-imitations C, by which we mean that $\dim(C \setminus \mathcal{E}^k(B)) \leq 0$; d) finally, we show that whenever a compact set has convex projections onto finite-dimensional planes, then it must be convex.

[10] S. Barov and Jan J. Dijkstra, On closed sets with convex projections under a narrow set of directions, Transactions of Amer. Math. Soc., Vol. 360, No. 12 (2008), 6525-6543.

Here, we have the following main result: Let $k, n \in \mathbb{N}$ with $k < n$ and let $\mathcal{G}_k(\mathbb{R}^n)$ denote the Grassmann manifold consisting of all k-dimensional linear subspaces in $\mathbb{R}^n$ endowed with the Hausdorff metric. We show that if the projections of a nonconvex closed set $C \subset \mathbb{R}^n$ are convex and proper for projection directions from some nonempty open set $\mathcal{P} \subset \mathcal{G}_k(\mathbb{R}^n)$, then C contains a closed copy of an $(n-k-1)$-manifold.

[12] S. Barov and Jan J. Dijkstra, On closed sets with convex projections under somewhere dense sets of directions, Proc. Amer. Math. Soc., 137 (2009), 2425-2435.

We prove the following main theorem.

Theorem 5. *Let $0 < k < n$, let C be a closed non-convex subset of $\mathbb{R}^n$, and let $\mathcal{P}$ be open in $\mathcal{G}_{n-k}(\mathbb{R}^n)$. Let $\psi_{P^*}(\langle C\rangle) \neq (P^*)^\perp$ for some $P^* \in \mathcal{P}$ and let $\overline{\psi_P(C)}$ be convex for every $P \in \mathcal{P}$. If $\langle C\rangle$ contains no k-plane then C contains a closed set that is homeomorphic to either*
(i) $\mathbb{R}^{k-1}$ *or*
(ii) $S^i \times \mathbb{R}^{k-i-1}$ *for some* $i \in \{1, 2, \ldots, k-1\}$.

[13] S. Barov and Jan J. Dijkstra, On closed sets in Hilbert space with convex projections under somewhere dense sets of directions, Journal of Topology and Analysis, Vol. 2, No. 1 (2010), 123-143.

In the current paper, we have that the ambient space is ℓ^2 and that's why, we write $\mathcal{G}_k$ instead of $\mathcal{G}_k(\mathbb{V}) = \mathcal{G}_k(\ell^2)$. Next, the theorem in [9] (part a)) is strengthened significantly by making the much weaker assumption that the set of projection directions is somewhere dense. Recall that $\mathcal{P}$ is *somewhere dense*, if $\operatorname{int}\overline{\mathcal{P}} \neq \emptyset$. To show the sharpness of the main theorem we construct "minimal imitations" of closed convex sets in ℓ^2. We have the following results.

Theorem 6. *Let $k \in \mathbb{N}$, let B be a closed convex subset of ℓ^2 that contains no k-hyperplane, and let $\mathcal{P}$ be a subset of $\mathcal{G}_k$ such that B is no $(\operatorname{int}\overline{\mathcal{P}})$-imitation of ℓ^2. If C is a closed weak $\mathcal{P}$-imitation of B with $C \neq B$ then $C \cap B$ contains a closed set that is homeomorphic to ℓ^2.*

In order to prove Theorem 6 we introduce the set $\mathcal{E}^k(B,\mathcal{P})$ consisting of $\mathcal{P}$-extremal points of B. We prove that every closed weak $\mathcal{P}$-imitation of B contains the set $\mathcal{E}^k(B,\mathcal{P})$. Theorem 6 is then proved by finding the copy of ℓ^2 in the set $\mathcal{E}^k(B,\mathcal{P})$. The following theorem shows that for a closed set to imitate a convex set B it needs to contain very little besides $\mathcal{E}^k(B,\mathcal{P})$. A topological space is *zero-dimensional* if it has a basis consisting of clopen sets.

Theorem 7 (Imitation Theorem)**.** *Let $k \in \mathbb{N}$, let B be a closed convex subset of ℓ^2 with $\operatorname{codim} B \neq k$, and let $\mathcal{P}$ be a subset of $\mathcal{G}_k$. Then there exists a closed $\mathcal{P}$-imitation C of B such that $C \subset B$ and $C \setminus \mathcal{E}^k(B,\mathcal{P})$ is zero-dimensional.*

If $A \subset \ell^2$ then the *geometric interior* A° of A is the interior of A relative to its closed affine hull. If A is a finite-dimensional convex set then A° is nonempty (even dense in A) and this fact plays a key role when we are in $\mathbb{R}^n$. In ℓ^2 there are many closed and convex sets B with empty geometric interior (for instance, every infinite-dimensional compactum has this property). We devote a section on that kind of sets. We establish two major theorems—the Imitation Theorem and the Exposed Point Theorem. The Imitation Theorem says that those sets can not be imitated by other closed sets:

Theorem 8. *Let $k \in \mathbb{N}$ and let B be a closed convex subset of ℓ^2 with $B^\circ = \emptyset$. Let $\mathcal{P}$ be somewhere dense in $\mathcal{G}_k$. If C is a closed weak $\mathcal{P}$-imitation of B then $C = B$.*

To state the Exposed Point Theorem let us recall the following definition. Let $B \subset \mathbb{V}$, let $w \in B$, and let $\mathcal{P}$ be a collection of linear subspaces of $\mathbb{V}$. Then we say that w is *exposed by $\mathcal{P}$* if $B \cap (w + P) = \{w\}$ for some $P \in \mathcal{P}$.

Theorem 9 (Exposed Point Theorem)**.** *Let $k \in \mathbb{N}$, let B be a closed and convex set in ℓ^2 with $B^\circ = \emptyset$, and let $\mathcal{P}$ be a nonempty open set in $\mathcal{G}_k$. Then every $w \in B$ is exposed by $\mathcal{P}$.*

The following theorem shows the importance of the set $\mathcal{E}^k(B,\mathcal{P})$, namely, that is the set of all extremal points of B with respect to $\mathcal{P}$ when $\operatorname{codim} B \neq k$.

Theorem 10. *Let $k \in \mathbb{N}$, let B be a closed convex set in ℓ^2 with $\operatorname{codim} B \neq k$, and let $\mathcal{P} \subset \mathcal{G}_k$ be such that $\mathcal{P} \subset \operatorname{int}\overline{\mathcal{P}}$. Then*

$$\begin{aligned}\mathcal{E}^k(B,\mathcal{P}) &= \bigcap\{C : C \text{ is a closed weak } \mathcal{P}\text{-imitation of } B\}\\ &= \bigcap\{C : C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}.\end{aligned}$$

[14] S. Barov, Jan J. Dijkstra and Maurits van der Meer, On Cantor sets with shadows of prescribed dimension, Topology and its Applications, 159 (2012), 2736-2742.

Recall that a *shadow* of $A \subset \mathbb{V}$ is an orthogonal projection of A onto a hyperplane. Our main results read as follows.

Theorem 11. *Let m, k be integers with $m \geq 2$ and $0 \leq k \leq m-1$. Then there exists a Cantor set in $\mathbb{R}^m$ such that all its shadows are k-dimensional.*

The next two theorems concern projections of Cantor sets in ℓ^2 with prescribed dimension. The first theorem shows that Theorem 11 can not be expanded over the Hilbert space ℓ^2 while the second one gives us some positive result when projections onto m-planes, $m \in \mathbb{N}$, are concerned.

Theorem 12. *Let $m \in \mathbb{N}$, and let K be compact in ℓ^2. Then the set $\mathcal{P} = \{P \in \mathcal{G}_m(\ell^2) \colon (w+P) \cap K = \{w\}$ for every $w \in K\}$ is dense and G_δ in $\mathcal{G}_m(\ell^2)$.*

Theorem 13. *For $m \in \mathbb{N}$ there exists a Cantor set $C \subset \ell^2$ such that its projections onto all $m-$planes are exactly $(m-1)-$dimensional.*

[15] S. Barov and Jan J. Dijkstra, On exposed points and extremal points of convex sets in and Hilbert space , Fundamenta Mathematicae, 232 (2016), 117-129.

Let $k \in \mathbb{N}$, $B \subset \mathbb{V}$ be closed and convex and $\mathcal{P} \subset \mathcal{G}_k$. Recall that a point x of B is called *exposed* by $\mathcal{P} \subset \mathcal{G}_k$ if there is a $P \in \mathcal{P}$ such that $(x+P) \cap B = \{x\}$. We denote by $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P})$ the set of all points of B exposed by $\mathcal{P}$. The set of *extremal* points of B with respect to $\mathcal{P}$ is denoted by $\mathcal{X}^k_{\mathrm{t}}(B, \mathcal{P})$ and is defined as the intersection of all closed subsets of B that are $\mathcal{P}$-imitations of B. Clearly, every exposed point is extremal as well. One of the main results in this paper is to establish the following connection between exposed and extremal points in the general setting.

Theorem 14 (Exposed Point Theorem)**.** *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex, and let $\mathcal{P}$ be a G_δ-subset of $\mathcal{G}_k(\mathbb{V})$ such that $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$. Then $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P})$ is dense in $\mathcal{X}^k_{\mathrm{t}}(B, \mathcal{P})$.*

Further, we have the following improvement over Theorem 9.

Theorem 15. *For closed and convex sets $B \subset \ell^2$ with empty geometric interior B° we have $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P}) = \mathcal{X}^k_{\mathrm{t}}(B, \mathcal{P}) = B$ for any $k \in \mathbb{N}$ and somewhere dense G_δ-set $\mathcal{P} \subset \mathcal{G}_k(\ell^2)$.*

We say that *generic* elements of a space have a certain property if the space has a dense G_δ-subset all elements of which have the property. In addition, we have the following theorem.

Theorem 16. *Let B be closed and convex in $\mathbb{R}^n$ and let $n \in \mathbb{N}$ with $n \geq 2$. Then $\mathcal{X}^1_{\mathrm{p}}(B, \mathcal{G}_1(\mathbb{R}^n))$ is a G_δ-set in $\mathcal{X}^1_{\mathrm{t}}(B, \mathcal{G}_1(\mathbb{R}^n))$. Consequently, in this case generic extremal points are exposed.*

Finally, in this paper, we show that if $k \neq 1$ then no dense subset of $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{G}_k(\mathbb{R}^n))$ is a G_δ-set in $\mathcal{X}^k_{\mathrm{t}}(B, \mathcal{G}_k(\mathbb{R}^n))$.

[16] S. Barov, Smooth convex bodies in with dense union of facets, Topology Proceedings, 58 (2021), 71-83.

Let B be closed and convex in $\mathbb{R}^n$; B is called a *convex body* if B is compact and has a nonempty interior with respect to $\mathbb{R}^n$. In addition, B is *smooth* if B has a unique supporting hyperplane at every boundary point. Let $k, n \in \mathbb{N}$ with $k < n$ and let $\mathcal{G}_k$ denote the *Grassmann manifold* consisting of all k-dimensional linear subspaces in $\mathbb{R}^n$. An intersection F of B and a supporting hyperplane is called a *facet* if $\dim F = n - 1$. A point x of B is called *exposed* by $\mathcal{P} \subset \mathcal{G}_k$ if there is a $P \in \mathcal{P}$ such that $(x + P) \cap B = \{x\}$. In this paper, for every $n \geq 2$ we have constructed symmetric smooth convex bodies $B(n)$ in $\mathbb{R}^n$ whose union of all facets is dense in the boundary of $B(n)$ and so that the set of its facets defines a dense set $\mathcal{P}$ in $\mathcal{G}_k$ such that the set of all points in $B(n)$ exposed by $\mathcal{P}$ is empty.

[17] S. Barov, More on exposed points and extremal points of convex sets in and Hilbert space, Comment. Math. Univ. Carolin., 64, 1 (2023), 63-72.

In this paper we have a theorem of Krein-Milman type. Recall that a *halfspace* of a plane L in $\mathbb{V}$ is any subset of L that consists of a hyperplane of L along with one of its sides. A nonempty subset F of B is called a *face* of B if there is a hyperplane H of $\operatorname{aff} B$ that supports B with the property $F = B \cap H$. If F is a face of B we write $F \prec B$. We say that a subset F of B is a *derived face* of B if $F = B$ or there exists a sequence $F = F_1 \prec F_2 \prec \cdots \prec F_m = B$ for some m. We have proved the following reconstitution theorem.

Theorem 17. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex that contains no k-hyperplane and let $\mathcal{P}$ be a dense G_δ-subset of $\mathcal{G}_k(\mathbb{V})$. If there is no derived face of B that is a halfspace of a k-hyperplane then*

$$\overline{\langle \mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P}) \rangle} = \left\langle \overline{\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P})} \right\rangle = B.$$

In addition to that theorem, we generalize Theorem 16.