

# ON CLOSED SETS IN HILBERT SPACE WITH CONVEX PROJECTIONS UNDER SOMEWHERE DENSE SETS OF DIRECTIONS

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Let  $k$  be a fixed natural number. In an earlier paper the authors show that if  $C$  is a closed and nonconvex set in the Hilbert space  $\ell^2$  such that the closures of the projections onto *all*  $k$ -hyperplanes (planes with codimension  $k$ ) are convex and proper, then  $C$  must contain a closed copy of  $\ell^2$ . Here this theorem is strengthened significantly by making the much weaker assumption that the set of projection directions is somewhere dense. To show the sharpness of the main theorem we construct “minimal imitations” of closed convex sets in  $\ell^2$ . In addition, we show that closed convex sets with an empty geometric interior cannot be imitated by other closed sets.

*Keywords:* Hilbert space; convex projection; hyperplane; imitation; Grassmann manifold.

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## 1. Introduction

Let  $C$  be a nonconvex closed set in the vector space  $\mathbb{R}^n$  for  $n \geq 3$ . Barov *et al.* [1] have shown that if *all* projections onto  $k$ -planes,  $1 \leq k \leq n-1$ , of  $C$  are convex and proper in a significant number of directions, then  $C$  contains a closed subset that is  $(k-1)$ -manifold without boundary. Subsequently, the authors have shown in [2] that if  $C$  is a closed and nonconvex set in the Hilbert space  $\ell^2$  such that the closures of

the projections onto *all*  $k$ -hyperplanes (planes with codimension  $k$ ) are convex and proper then  $C$  must contain a closed copy of  $\ell^2$ . Moreover, in [3, 4] the authors show that the above result in [1] remains valid if we make much weaker assumption that the collection of projection directions that produce convex projections is somewhere dense.

Having all this in mind, we naturally ask ourselves whether the results in Hilbert space, obtained in [2], are valid if we require convexity of the projections only for a somewhere dense set of directions. So the main purpose of this paper is to give a positive answer to that question and to generalize the Imitation Theorem [2, Theorem 2], that is, to find “minimal imitations” of closed and convex sets for arbitrary sets of projection directions.

In order to formulate our main theorem we need some definitions and notations. If  $k \in \mathbb{N}$ , then we let  $\mathcal{G}_k$  denote the infinite-dimensional “Grassmann manifold” consisting of all  $k$ -dimensional linear subspaces of  $\ell^2$ , see Definition 1. If  $A$  is a subset of a topological space, then  $\overline{A}$  is the closure of  $A$  and  $\text{int } A$  is the interior of  $A$ . If  $B, C \subset \ell^2$  and  $\mathcal{P} \subset \mathcal{G}_k$ , then  $B$  and  $C$  are called  $\mathcal{P}$ -imitations of each other if  $B + P = C + P$  for each  $P \in \mathcal{P}$ . If  $\overline{B + P} = \overline{C + P}$  for each  $P \in \mathcal{P}$ , then  $B$  and  $C$  are called *weak  $\mathcal{P}$ -imitations* of each other. If  $A \subset \ell^2$ , then

$$A^\perp = \{v \in \ell^2 : v \cdot x = v \cdot y \text{ for all } x, y \in A\},$$

where  $\cdot$  denotes the inner product. Also we define

$$\text{codim } A = \dim A^\perp \in \{0, 1, \dots, \infty\}.$$

A  $k$ -hyperplane in  $\ell^2$  is a closed affine subspace with  $\text{codim} = k$ .

**Theorem 1.** *Let  $k \in \mathbb{N}$ , let  $B$  be a closed convex subset of  $\ell^2$  that contains no  $k$ -hyperplane, and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_k$  such that  $B$  is not an  $(\text{int } \overline{\mathcal{P}})$ -imitation of  $\ell^2$ . If  $C$  is a closed weak  $\mathcal{P}$ -imitation of  $B$  with  $C \neq B$ , then  $C \cap B$  contains a closed set that is homeomorphic to  $\ell^2$ .*

In order to prove Theorem 1, we introduce the set  $\mathcal{E}^k(B, \mathcal{P})$  consisting of  $\mathcal{P}$ -extremal points of  $B$ ; see Definition 4. We proved in [4, Theorem 15] that every closed weak  $\mathcal{P}$ -imitation of  $B$  contains the set  $\mathcal{E}^k(B, \mathcal{P})$ ; see Theorem 22. Theorem 1 is then proved by finding the copy of  $\ell^2$  in the set  $\mathcal{E}^k(B, \mathcal{P})$ ; see Theorem 25. The following theorem shows that for a closed set to imitate a convex set  $B$ , it needs to contain very little besides  $\mathcal{E}^k(B, \mathcal{P})$ . A topological space is *zero-dimensional* if it has a basis consisting of clopen sets.

**Theorem 2.** (Imitation Theorem) *Let  $k \in \mathbb{N}$ , let  $B$  be a closed convex subset of  $\ell^2$  with  $\text{codim } B \neq k$ , and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_k$ . Then there exists a closed  $\mathcal{P}$ -imitation  $C$  of  $B$  such that  $C \subset B$  and  $C \setminus \mathcal{E}^k(B, \mathcal{P})$  is zero-dimensional.*

A consequence of the proof of this result is that  $\mathcal{E}^k(B, \mathcal{P})$  is precisely the intersection of all closed  $\mathcal{P}$ -imitations of  $B$ :

**Theorem 3.** *Let  $k \in \mathbb{N}$ , let  $B$  be a closed convex set in  $\ell^2$  with  $\text{codim } B \neq k$ , and let  $\mathcal{P} \subset \mathcal{G}_k$  be such that  $\mathcal{P} \subset \text{int } \overline{\mathcal{P}}$ . Then*

$$\begin{aligned} \mathcal{E}^k(B, \mathcal{P}) &= \bigcap \{C : C \text{ is a closed weak } \mathcal{P}\text{-imitation of } B\} \\ &= \bigcap \{C : C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}. \end{aligned}$$

In the process of proving our results we follow the general approach of [3]. There are, however, a number of significant differences, the most important of which is connected to the following definition. If  $A \subset \ell^2$ , then the *geometric interior*  $A^\circ$  of  $A$  is the interior of  $A$  relative to its closed affine hull. If  $A$  is a finite-dimensional convex set, then  $A^\circ$  is nonempty (even dense in  $A$ ) and this fact plays a key role in the proofs in [3]. In  $\ell^2$  there are many closed and convex sets  $B$  with empty geometric interior (for instance, every infinite-dimensional compactum has this property). The method of [3] breaks down for sets  $B$  with  $B^\circ = \emptyset$  and we are forced to deal with those sets separately. This is the subject of Sec. 3. We found that these sets cannot be imitated by other closed sets:

**Theorem 4.** *Let  $k \in \mathbb{N}$  and let  $B$  be a closed convex subset of  $\ell^2$  with  $B^\circ = \emptyset$ . Let  $\mathcal{P}$  be somewhere dense in  $\mathcal{G}_k$ . If  $C$  is a closed weak  $\mathcal{P}$ -imitation of  $B$ , then  $C = B$ .*

Another obstruction to us closely following the method in [3] is highlighted in Example 1. In addition, the proofs for  $\mathbb{R}^n$  in [1, 3] rely on the fact that closed sets are  $\sigma$ -compact. In particular, in the finite-dimensional analogue of Theorem 2, that is [3, Theorem 2], the zero-dimensional part of the minimal imitation  $C$  is a countable union of Cantor sets. That method clearly cannot work in  $\ell^2$  and the proof of Theorem 2 merges ideas from [3] with constructions from [2].

Note that Theorem 1 deals with the retrieval of information about a geometric object from data about its projections which places the result in the field of Geometric Tomography; see Gardner [12] for background information. Our line of investigation has come out of results by Borsuk [6], Cobb [7], and Dijkstra *et al.* [8] concerning projections of compacta in  $\mathbb{R}^n$ .

Our paper is arranged as follows. In Sec. 2 we define the main concepts and establish some basic properties. In Sec. 3 we deal with closed convex sets with empty geometric interiors and we establish Theorem 4. Section 4 is devoted to the proof of Theorem 1. Section 5 is centered around Theorem 2 and its consequences.

## 2. Definitions and Preliminaries

Throughout this paper  $\mathbb{V}$  stands for a separable real Hilbert space with an inner product  $x \cdot y$ . Thus  $\mathbb{V}$  is isomorphic to either an  $\mathbb{R}^n$  or  $\ell^2$ , the Hilbert space of square summable real sequences. The norm on  $\mathbb{V}$  is given by  $\|u\| = \sqrt{u \cdot u}$  and the metric

$d$  is given by  $d(u, v) = \|v - u\|$ . We let  $\mathbf{0}$  denote the zero vector of  $\mathbb{V}$  and  $\mathbb{S}$  stands for the unit sphere in  $\mathbb{V}$ . Let  $A$  be a subset of  $\mathbb{V}$ . We let  $[A]$  denote the linear hull and  $\langle A \rangle$  the convex hull of  $A$ . We define  $A^\perp$  in the following way:

$$A^\perp = \{v \in \mathbb{V} : v \cdot x = v \cdot y \text{ for all } x, y \in A\}.$$

If  $L$  is a closed linear subspace of  $\mathbb{V}$ , then  $L^\perp$  is called the *orthocomplement* of  $L$  and we have  $L^{\perp\perp} = L$ . Also, we define  $\text{codim } A = \dim A^\perp \in \{0, 1, \dots, \infty\}$ . A *plane* in  $\mathbb{V}$  is a closed affine subspace of  $\mathbb{V}$ ; a *k-subspace* is a  $k$ -dimensional linear subspace of  $\mathbb{V}$ . The *affine hull*  $\text{aff } A$  of  $A$  is defined as the intersection of all planes that contain  $A$ . Observe that  $A^\perp = (\text{aff } A)^\perp$  and  $\text{codim } A = \text{codim}(\text{aff } A)$ . The *geometric interior*  $A^\circ$  of  $A$  is the interior of  $A$  relative to the affine hull of  $A$ . The *geometric boundary* of  $A$  is  $\partial A = \overline{A} \setminus A^\circ$ . We set  $\text{diam } A = \sup\{\|x - y\| : x, y \in A\}$ . A closed and convex set  $A$  with  $\text{int } A \neq \emptyset$  is called a *convex body* in  $\mathbb{V}$ .

**Definition 1.** Let  $\mathcal{K}(\mathbb{V})$  stand for the collection of all non-empty compact subsets of  $\mathbb{V}$ . Recall that the Hausdorff metric  $d_H$  on  $\mathcal{K}(\mathbb{V})$  associated with  $d$  is defined as follows:

$$d_H(A, B) = \sup\{d(x, A), d(y, B) : x \in B \text{ and } y \in A\}.$$

We let  $\mathcal{G}_k(\mathbb{V})$  stand for the collection of all  $k$ -subspaces of  $\mathbb{V}$ . Consider the ball  $\mathbb{B} = \{v \in \mathbb{V} : \|v\| \leq 1\}$ . We topologize  $\mathcal{G}_k(\mathbb{V})$  by defining a metric  $\rho$  on  $\mathcal{G}_k(\mathbb{V})$ :

$$\rho(L_1, L_2) = d_H(L_1 \cap \mathbb{B}, L_2 \cap \mathbb{B}).$$

When  $\mathbb{V}$  is finite-dimensional,  $\mathcal{G}_k(\mathbb{V})$  is known as a *Grassmann manifold*. We use the notation  $\mathcal{G}_k = \mathcal{G}_k(\ell^2)$ . We also allow the degenerate cases  $\mathcal{G}_0(\mathbb{V}) = \{\{\mathbf{0}\}\}$  and  $\mathcal{G}_k(\mathbb{R}^k) = \{\mathbb{R}^k\}$ .

The next two lemmas, proved in [3] for  $\mathbb{R}^n$ , give us an alternative way to define the topology on  $\mathcal{G}_k(\mathbb{V})$ . The proofs for  $\ell^2$  are analogous.

**Lemma 5.** Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ ,  $\varepsilon > 0$ , and  $L \in \mathcal{G}_k(\mathbb{V})$ . Let  $v_1, \dots, v_k$  be a basis for  $L$ . Then there is a  $\delta > 0$  such that for every set  $F = \{v'_1, \dots, v'_k\} \subset \mathbb{V}$  with  $\|v'_i - v_i\| < \delta$  for every  $i$  we have  $\rho([F], L) < \varepsilon$ .

**Lemma 6.** Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ ,  $\varepsilon > 0$ , and  $L \in \mathcal{G}_k(\mathbb{V})$ . Let  $v_1, \dots, v_k$  be a basis for  $L$ . Then there is a  $\delta > 0$  such that for every  $P \in \mathcal{G}_k(\mathbb{V})$ , with  $\rho(L, P) < \delta$ , there is a basis  $\{v'_1, \dots, v'_k\}$  for  $P$  such that  $\|v'_i - v_i\| < \varepsilon$  for every  $1 \leq i \leq k$ .

Now, we prove the following useful lemma.

**Lemma 7.** Let  $m, n \in \mathbb{N} \cup \{0\}$  with  $m + n \leq \dim \mathbb{V}$ . Let  $N \in \mathcal{G}_n(\mathbb{V})$  and  $L \in \mathcal{G}_m(\mathbb{V})$  such that  $N \cap L = \{\mathbf{0}\}$ . Then for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that if  $\rho(N, N') < \delta$  and  $\rho(L, L') < \delta$ , then  $\rho(N + L, N' + L') < \varepsilon$ .

**Proof.** If  $m = 0$  or  $n = 0$ , then we simply take  $\delta = \varepsilon$ . Let  $m, n \in \mathbb{N}$ . Then there exist independent vectors  $v_1, v_2, \dots, v_{m+n}$  such that  $L = [\{v_1, \dots, v_m\}]$  and  $N = [\{v_{m+1}, \dots, v_{m+n}\}]$ . By Lemma 5 we can find an  $\varepsilon^*$  such that

$$\rho([F], L + N) < \varepsilon \quad \text{whenever } F = \{v'_1, v'_2, \dots, v'_{m+n}\} \text{ with } \|v_j - v'_j\| < \varepsilon^*.$$

Now, we can apply Lemma 6 and find a  $\delta > 0$  such that if  $L' \in \mathcal{G}_m$  with  $\rho(L, L') < \delta$  and  $N' \in \mathcal{G}_n$  with  $\rho(N, N') < \delta$ , then there are a basis  $\{v'_1, \dots, v'_m\}$  for  $L'$  and a basis  $\{v'_{m+1}, \dots, v'_{m+n}\}$  for  $N'$  such that  $\|v_j - v'_j\| < \varepsilon^*$  for  $1 \leq j \leq m + n$ . Now, one can easily observe that  $\delta$  is as required. That completes the proof.  $\square$

**Definition 2.** Let  $m, i \in \mathbb{N}$  with  $i \leq m \leq \dim \mathbb{V}$  and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_m(\mathbb{V})$ . If  $L \in \mathcal{G}_i(\mathbb{V})$ , then we define

$$\mathcal{P}_L = \{N \in \mathcal{G}_{m-i}(L^\perp) : N + L \in \mathcal{P}\}.$$

**Remark 1.** Let  $m, i \in \mathbb{N}$  with  $i \leq m \leq \dim \mathbb{V}$  and let  $\mathcal{P}$  be an open subset of  $\mathcal{G}_m(\mathbb{V})$ . Lemma 7 implies that if  $L \in \mathcal{G}_i(\mathbb{V})$ , then  $\mathcal{P}_L$  is open in  $\mathcal{G}_{m-i}(L^\perp)$ .

Now let  $L$  be a plane in  $\mathbb{V}$ . A plane  $H \subset L$  is called a  $k$ -hyperplane in  $L$  if  $\dim(H^\perp \cap L) = k$ . In other words, a  $k$ -hyperplane is a plane with codimension  $k$  in the ambient space. A *hyperplane*  $H$  of  $L$  is a plane of  $L$  of codimension 1. The two components of  $L \setminus H$  are called the *sides* of the hyperplane  $H$  and the union of  $H$  with one of its sides is called a *half space* of  $L$ . We say that  $H$  supports a subset  $A$  of  $L$  if  $A$  is contained in a half space that is associated with  $H$ .

**Definition 3.** Let  $B$  be a closed and convex set in  $\mathbb{V}$ . A nonempty subset  $F$  of  $B$  is called a *face* of  $B$  if there is a hyperplane  $H$  of  $\text{aff } B$  that supports  $B$  with the property  $F = B \cap H$ . Note that  $F$  is also closed and convex and that  $\text{codim } F > \text{codim } B$  whenever  $\text{codim } B$  is finite. If  $F$  is a face of  $B$ , we write  $F \prec B$ . We say that a subset  $F$  of  $B$  is a *derived face* of  $B$  if  $F = B$  or there exists a sequence  $F = F_1 \prec F_2 \prec \dots \prec F_m = B$  for some  $m \in \mathbb{N}$ .

**Definition 4.** Let  $\mathcal{P}$  be a collection of closed linear subspaces of  $\mathbb{V}$ . A hyperplane  $H$  in  $\mathbb{V}$  is said to be *consistent with*  $\mathcal{P}$  if  $H + P = H$  for some  $P \in \mathcal{P}$ . Let  $B$  be a convex and closed subset of  $\mathbb{V}$ . A nonempty subset  $F$  of  $B$  is called a  $\mathcal{P}$ -*face* of  $B$  if  $F = B \cap H$  for some hyperplane  $H$  of  $\mathbb{V}$  that supports  $B$  and that is consistent with  $\mathcal{P}$ . A *derived  $\mathcal{P}$ -face* is a derived face of a  $\mathcal{P}$ -face. If  $k \in \mathbb{N}$  and  $k < \dim \mathbb{V}$ , then we define the set  $\mathcal{E}^k(B, \mathcal{P})$  as the closure of

$$\bigcup \{F : F \text{ is a derived } \mathcal{P}\text{-face of } B \text{ with } \text{codim } F > k\}.$$

If  $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ , then the elements of  $\mathcal{E}^k(B, \mathcal{P})$  are called the  $\mathcal{P}$ -*extremal* points of  $B$ .

**Definition 5.** Let  $B, C \subset \mathbb{V}$  and let  $\mathcal{P}$  be a set of closed linear subspaces of  $\mathbb{V}$ .  $B$  and  $C$  are called  $\mathcal{P}$ -*imitations* of each other if  $B + P = C + P$  for each  $P \in \mathcal{P}$ .

If  $\overline{B+P} = \overline{C+P}$  for each  $P \in \mathcal{P}$ , then  $B$  and  $C$  are called *weak  $\mathcal{P}$ -imitations* of each other.

**Definition 6.** Let  $L$  be a plane in  $\mathbb{V}$ . Then  $\psi_L : \mathbb{V} \rightarrow L^\perp$  denotes the orthogonal projection along  $L$  onto  $L^\perp$  defined by the conditions  $\psi_L(x) - x \in L^{\perp\perp}$  and  $\psi_L(x) \in L^\perp$  for each  $x \in \mathbb{V}$ . Note that if  $\mathbf{0} \in L$  then  $\{\psi_L(x)\} = L^\perp \cap (x + L)$ .

**Remark 2.** Observe that  $B$  and  $C$  are (weak)  $\mathcal{P}$ -imitations of each other precisely if  $\psi_P(B) = \psi_P(C)$  ( $\overline{\psi_P(B)} = \overline{\psi_P(C)}$ ) for each  $P \in \mathcal{P}$ . If  $B$  and  $C$  are weak  $\mathcal{P}$ -imitations of each other, then  $B$  and  $C$  have precisely the same supporting hyperplanes that are consistent with  $\mathcal{P}$ ; see [4, Remark 1].

**Definition 7.** Let  $X$  and  $Y$  be topological spaces and let  $2^Y$  stand for the collection of nonempty subsets of  $Y$ . A set-valued  $\varphi : X \rightarrow 2^Y$  is called *USC (upper semi-continuous)* if  $\varphi^{-1}(U) = \{x \in X : \varphi(x) \subset U\}$  is open in  $X$  for every open  $U$  in  $Y$ . Equivalently,  $\{x \in X : \varphi(x) \cap U \neq \emptyset\}$  is closed in  $X$  for every open  $U$  in  $Y$ .

**Remark 3.** Let  $B$  be a convex body in  $\mathbb{V}$ . We define a set-valued function  $\Phi : \mathbb{V} \setminus \text{int } B \rightarrow 2^{\mathbb{S}}$  as follows:

$$\Phi(x) = \{a \in \mathbb{S} : a \cdot (y - x) \leq 0 \text{ for every } y \in B\}.$$

In other words,  $\Phi(x)$  consists of all unit vectors  $a$  such that  $x + H_a$  is supporting to  $B$  and  $a$  points towards the side of  $x + H_a$  that does not contain points of  $B$ , where  $H_a = \{x \in \mathbb{V} : x \cdot a = 0\}$ . Observe that by the Hahn–Banach Theorem,  $\Phi(x) \neq \emptyset$  for every  $x$ .

We have shown in [3] that in  $\mathbb{R}^n$ ,  $\Phi$  is USC. This fact is used in the proof of the key theorem [3, Theorem 17], which is the finite-dimensional version of Theorem 25 here. The following example shows that in  $\ell^2$ ,  $\Phi$  does not need to be a USC function which, as mentioned in the Introduction, influences the proof of Theorem 25.

**Example 1.** Consider the following convex compactum in  $\ell^2$ :

$$T = \{x \in \ell^2 : x_n \in [-2^{-n}, 2^{-n}] \text{ for all } n \in \mathbb{N}\}.$$

In the space  $\mathbb{R} \times \ell^2$  we define a closed and convex set  $B$  as follows.

$$B = \overline{\langle \{\mathbf{0}\} \times T \rangle \cup \langle \{1\} \times \mathbb{B} \rangle}.$$

Notice that  $\text{int } B \neq \emptyset$  because  $\text{int } \mathbb{B} \neq \emptyset$ . By [2, Example 1] we have that there is no supporting hyperplane at  $\mathbf{0}$  to  $T$  in  $\ell^2$ . Therefore,  $H = \{\mathbf{0}\} \times \ell^2$  is the only supporting hyperplane at  $\mathbf{0}$  to  $B$  in  $\mathbb{R} \times \ell^2$ . Clearly, we have  $\Phi(\mathbf{0}, \mathbf{0}) = \{a\}$ , where  $a = (-1, \mathbf{0})$ . Now, for  $n \in \mathbb{N}$  consider the vectors:

$$x_n = (0, (0, \dots, 0, 2^{-n}, 0, \dots)) \in B$$

and

$$a_n = (-\sqrt{2}/2, (0, \dots, 0, \sqrt{2}/2, 0, \dots)),$$

where the nonzero entry in the second factors is in the  $n$ th position. It is easily verified that  $a_n \in \Phi(x_n)$ . We clearly have that  $\lim_{n \rightarrow \infty} x_n = (0, \mathbf{0})$  but  $\lim_{n \rightarrow \infty} a_n \neq a$  thus  $\Phi$  is not USC for this  $B$ .

A continuous map  $f: X \rightarrow Y$  is called *proper* if the pre-image of every compactum in  $Y$  is compact. Recall that in metric spaces a continuous map is proper if and only if it is closed and every fibre is compact, see Engelking [10, Theorem 3.7.18]. In particular, if  $B \subset \mathbb{V}$  and a linear space  $L \subset \mathbb{V}$  are such that the restriction  $\psi_L|_B: B \rightarrow \mathbb{V}$  is proper, then  $\psi_L(B)$  and  $B + L = \psi_L^{-1}(\psi_L(B))$  are closed in  $\mathbb{V}$ .

The following four lemmas about proper maps are [3, Lemma 6], [4, Lemma 6], [4, Lemma 9], and [4, Lemma 10], respectively.

**Lemma 8.** *If  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are continuous, then  $g \circ f: X \rightarrow Z$  is proper if and only if both  $f$  and  $g|_{f(X)}: f(X) \rightarrow Z$  are proper.*

**Lemma 9.** *Let  $P$  be a finite-dimensional linear subspace of  $\mathbb{V}$  and let  $B$  be a closed and convex set in  $\mathbb{V}$ . Then  $\psi_P|_B$  is proper if and only if  $(z + P) \cap B$  is bounded for some  $z \in B$ .*

**Lemma 10.** *Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$  and let  $C$  be closed in  $\mathbb{V}$ . If  $P \in \mathcal{G}_k(\mathbb{V})$  and  $w \in \mathbb{V}$  are such that  $\psi_P|_{\overline{\langle C \rangle}}$  is proper and  $(w + P) \cap C = \emptyset$ , then there is a neighbourhood  $\mathcal{U}$  of  $P$  such that  $(w + P') \cap C = \emptyset$  for each  $P' \in \mathcal{U}$ .*

**Lemma 11.** *If  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$  and  $B$  is a closed convex set in  $\mathbb{V}$ , then  $\{P \in \mathcal{G}_k(\mathbb{V}) : \psi_P|_B \text{ is proper}\}$  is open in  $\mathcal{G}_k(\mathbb{V})$ .*

**Remark 4.** The following fact can be found in [13, Sec. 2.5] and [5, p. 93]. If  $B$  is a closed convex set in  $\ell^2$ , then there is a unique linear space  $\mathcal{L}_B \subset \ell^2$  such that  $\text{cs } B = B \cap (\mathcal{L}_B)^\perp$  is line-free and  $B = \mathcal{L}_B + \text{cs } B$ . Note that  $\text{cs } B = \psi_{\mathcal{L}_B}(B)$  and  $B = B + \mathcal{L}_B$ .

The following result is from [1, Lemma 4] and [2, Lemma 6].

**Lemma 12.** *If  $B$  is closed and convex in  $\mathbb{V}$ , then for every derived face  $F$  of  $B$  we have  $F = F + \mathcal{L}_B$ .*

**Remark 5.** We will need information about the topology of boundaries of convex bodies  $B$  in  $\ell^2$ . According to [5, Proposition III.6.1], the boundary of a convex body is either empty or homeomorphic to  $\ell^2$  or  $S^n \times \ell^2$  for some  $n$ -sphere  $S^n$ . Thus  $\partial B$  is either empty or it contains closed copies of  $\ell^2$ .

### 3. Sets with Empty Geometric Interiors

This section is devoted to the proof of Theorem 4. Several of the steps towards that goal are of independent interest. We mention the Exposed Point Theorem

(Theorem 14) and Theorem 19 which states that all the points in a closed convex set with empty geometric interior are  $\mathcal{P}$ -extremal whenever  $\mathcal{P}$  is somewhere dense in  $\mathcal{G}_k$ .

**Definition 8.** A real-valued function  $f$  defined on a topological space  $X$  is *lower semi-continuous (LSC)* if  $f^{-1}((r, \infty))$  is open for every  $r \in \mathbb{R}$ .  $f$  is called *upper semi-continuous (USC)* if  $-f$  is LSC. A real-valued function  $g$  defined on a convex set  $B$  is called *convex* if  $g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y)$  for  $t \in [0, 1]$  and  $x, y \in B$ . The function  $g$  is called *concave* if  $-g$  is convex.

**Lemma 13.** *Let  $B$  be a closed and convex set in  $\ell^2$  such that  $B^\circ = \emptyset$ . If  $P$  is a finite-dimensional subspace of  $\ell^2$ , then  $\psi_P(B)^\circ = \emptyset$ .*

**Proof.** We prove the lemma in three steps.

**Claim 1.** *The lemma is valid under the additional assumptions that  $\psi_P \upharpoonright B$  is proper and that  $P$  is one-dimensional.*

**Proof of Claim 1.** We have that  $P = \mathbb{R}u$  for some unit vector  $u$ . Set  $B_P = \psi_P(B)$  and  $V = P + \text{aff } B_P$ . Let the functions  $f, g: B_P \rightarrow \mathbb{R}$  be defined by  $f(x) = \min\{a \in \mathbb{R} : x + au \in B\}$  and  $g(x) = \max\{a \in \mathbb{R} : x + au \in B\}$ . Since  $\psi_P \upharpoonright B$  is proper, we have that every fibre of  $\psi_P \upharpoonright B$  is compact thus  $f$  and  $g$  are well defined. Note that  $f \leq g$ , that  $f$  is convex, and that  $g$  is concave.

We show that  $f$  is LSC. Let  $x \in B_P$  and  $r < f(x)$ . Define the closed subset  $C$  of  $B$  by  $C = \{y \in B : u \cdot y \leq r\}$ . Since  $\psi_P \upharpoonright B$  is proper, it is a closed mapping and hence  $\psi_P(C)$  is closed. Note that  $U = B_P \setminus \psi_P(C)$  is an open neighbourhood of  $x$  in  $B_P$  such that  $f(z) > r$  for each  $z \in U$ . We have shown that  $f$  is LSC and by symmetry that  $g$  is USC.

Striving for a contradiction, we assume that  $(B_P)^\circ \neq \emptyset$ . We have two subcases to consider.

**Case I.**  $f = g$ . Then  $f$  is continuous and both convex and concave on  $B_P$ . It is then easily verified that  $f$  extends to a continuous affine map  $\bar{f}: \text{aff } B_P \rightarrow \mathbb{R}$ . Then  $H = \{x + \bar{f}(x)u : x \in \text{aff } B_P\}$  is a hyperplane in  $V$  that contains  $B$ . Note that the open subset  $\{y \in H : \psi_P(y) \in (B_P)^\circ\}$  of  $H$  is contained in  $B$  thus  $B^\circ \neq \emptyset$  and Case I is complete.

**Case II.**  $f \neq g$ . Then there is a point  $z \in B_P$  such that  $f(z) < g(z)$ . Convexity of  $B$  implies that there are points in  $(B_P)^\circ$  with this property. By [9, Lemma 2.1] the functions  $f$  and  $g$  are continuous on  $(B_P)^\circ$ . Note that  $\{x + tu : x \in (B_P)^\circ, f(x) < t < g(x)\}$  is an open nonempty subset of  $V$  that is contained in  $B$ . We may conclude that  $B^\circ \neq \emptyset$ .  $\diamond$

**Claim 2.** *The lemma is valid under the additional assumption that  $\psi_P \upharpoonright B$  is proper.*



**Proof of Claim 2.** Let  $k > 1$  and let  $\{e_1, e_2, \dots, e_k\}$  be an orthogonal basis for  $P$ . Set  $\ell_i = \mathbb{R}e_i$  for  $1 \leq i \leq k$ . Recursively, we define the convex sets

$$B_0 = B \quad \text{and} \quad B_i = \psi_{\ell_i}(B_{i-1}) \quad \text{for } 1 \leq i \leq k.$$

Thus we have that  $\psi_P = \psi_{\ell_k} \circ \dots \circ \psi_{\ell_2} \circ \psi_{\ell_1}$  and by Lemma 8 we have that every  $\psi_{\ell_i}|_{B_{i-1}}$  is proper and hence every  $B_i$  is closed. If  $B^\circ = \emptyset$ , then by applying Claim 1  $k$  times we obtain  $(B_k)^\circ = \emptyset$  and the claim is proved.  $\diamond$

We are now ready to prove Lemma 13 in full generality. Assume that  $B^\circ = \emptyset$  and that  $\mathbf{0} \in B$ . Set  $B_P = \psi_P(B)$  and  $V = P + \text{aff } B_P$ . Define for  $n \in \mathbb{N}$  the convex closed set  $C_n = \{x \in B : \|x\| \leq n\}$ . Note that  $B \subset \text{aff } C_n$  and hence  $\text{aff } C_n = \text{aff } B$ . Thus we have that  $(C_n)^\circ = \emptyset$  for every  $n \in \mathbb{N}$ . Since  $C_n$  is bounded, we have that  $\psi_P|_{C_n}$  is proper by Lemma 9 and hence  $\psi_P(C_n)$  is closed. With Claim 2 we may conclude that  $\psi_P(C_n)^\circ = \emptyset$ . Thus  $\psi_P(C_n)$  is a nowhere dense subset of  $\text{aff } B_P$ . Consequently,  $B_P = \bigcup_{n=1}^{\infty} \psi_P(C_n)$  is a meagre set in  $\text{aff } B_P$  and hence  $(B_P)^\circ = \emptyset$  by the Baire Category Theorem.  $\square$

**Example 2.** Lemma 13 shows in particular that even when we look in just one direction a closed convex set  $B$  with  $B^\circ = \emptyset$  cannot imitate a set with nonempty geometric interior. However, the following example shows that such a  $B$  can *weakly* imitate even the whole space; cf. Theorem 4.

Let  $B = \{(x_i)_i \in \ell^2 : x_i \geq 0\}$ ,  $u = (1, \frac{1}{2}, \dots, \frac{1}{n}, \dots)$  and  $P = \mathbb{R}u \in \mathcal{G}_1$ . It is clear that  $B$  is closed and convex with  $B^\circ = \emptyset$ . We show that  $\overline{\psi_P(B)} = P^\perp$ . Let  $\varepsilon > 0$  and  $w \in P^\perp$ . We can find a  $v = (v_i)_i \in \ell^2$  such that  $\|v - w\| < \varepsilon$  and  $v_i = 0$  if  $i \geq N$  for some  $N \in \mathbb{N}$ . Observe that there is an  $\alpha > 0$  such that  $v + \alpha u \in B$  thus  $v' = \psi_P(v) \in \psi_P(B)$ . Note that  $\|v - v'\| = d(v, P^\perp) \leq \|v - w\| < \varepsilon$ . So we have  $\|v' - w\| < 2\varepsilon$  and  $B$  is a weak  $\{P\}$ -imitation of  $\ell^2$ .

Now we prove the Exposed Point Theorem for which we need the following definition. Let  $B \subset \mathbb{V}$ ,  $w \in B$ , and  $\mathcal{P}$  be a collection of linear subspaces of  $\mathbb{V}$ . Then we say that  $w$  is *exposed by*  $\mathcal{P}$  if  $B \cap (w + P) = \{w\}$  for some  $P \in \mathcal{P}$ .

**Theorem 14.** (Exposed Point Theorem) *Let  $k \in \mathbb{N}$ ,  $B$  be a closed and convex set in  $\ell^2$  with  $B^\circ = \emptyset$ , and let  $\mathcal{P}$  be a nonempty open set in  $\mathcal{G}_k$ . Then every  $w \in B$  is exposed by  $\mathcal{P}$ .*

**Proof.** We choose our coordinate system for  $\ell^2$  in such a way that  $w = \mathbf{0}$ . Define

$$\mathcal{A} = \{L : L \text{ is a linear subspace of some } P \in \mathcal{P} \text{ such that } L \cap B = \{\mathbf{0}\}\}.$$

Note that  $\{\mathbf{0}\} \in \mathcal{A}$  and hence we may define

$$m = \max\{\dim L : L \in \mathcal{A}\}.$$

It suffices to show that  $m = k$  so let us assume that  $m < k$ . Choose linear subspaces  $P_1 \in \mathcal{P}$  and  $L \subset P_1$  such that  $\dim L = m$  and  $L \cap B = \{\mathbf{0}\}$ . Define  $B_L = \psi_L(B)$  and  $P_2 = \psi_L(P_1)$ . Since  $L \cap B = \{\mathbf{0}\}$ , we have by Lemma 9 that  $\psi_L|_B$  is proper

thus  $B_L$  is closed. In addition, by Lemma 13, we get that  $(B_L)^\circ = \emptyset$ . Next, choose a 1-subspace  $\ell$  in  $P_2$  and set

$$T = \{u \in \mathbb{S} \cap L^\perp : \mathbb{R}u + L + \psi_\ell(P_2) \in \mathcal{P}\}.$$

Clearly,  $T \neq \emptyset$  and by Remark 1 we get that  $T$  is open in  $\mathbb{S} \cap L^\perp$ .

**Claim.** *There is a  $u^* \in T$  such that  $\mathbb{R}u^* \cap B_L = \{\mathbf{0}\}$ .*

**Proof of Claim.** Striving for a contradiction, assume that for every  $u \in T$  we have that  $\text{diam}(\mathbb{R}u \cap B_L) > 0$ . Define for each  $n \in \mathbb{N}$  the set

$$S_n = \left\{ u \in T : \frac{1}{n}u \in B_L \right\}.$$

Since  $B_L$  is closed, we clearly have that each  $S_n$  is closed in  $T$ . Moreover, it is easily verified that  $\bigcup_{n=1}^\infty (S_n \cup -S_n) = T$ . Thus, by the Baire Category Theorem there is an  $m \in \mathbb{N}$  such that  $S_m$  has a nonempty interior  $O$  in  $\mathbb{S} \cap L^\perp$  (if  $-S_m$  has interior points then so does  $S_m$ ). Note that

$$C = \left\{ x \in L^\perp : 0 < \|x\| < \frac{1}{m}, \frac{x}{\|x\|} \in O \right\}$$

is an open nonempty subset of  $L^\perp$ . Since  $B_L$  is convex and  $\mathbf{0} \in B_L$  we have that  $C \subset B_L$ . Thus we have  $(B_L)^\circ \neq \emptyset$  in contradiction with an earlier result.  $\diamond$

We define  $L' = L + \mathbb{R}u^*$  and  $P' = L' + \psi_\ell(P_2)$ . Now we have that  $L'$  is an  $(m+1)$ -subspace of  $P' \in \mathcal{P}$  such that

$$L' \cap B = \{\mathbf{0}\}.$$

That violates the maximality of  $m$  – hence  $L = P_1$  and that completes the proof.  $\square$

**Example 3.** Theorem 14 is no longer valid if  $\mathcal{P}$  is merely somewhere dense instead of open, even for compact  $B$ . Let  $B$  be the convex compactum  $\prod_{i=1}^\infty [-2^{-i}, 2^{-i}]$  in  $\ell^2$  and let

$$\mathcal{P} = \{\mathbb{R}u : u \in \mathbb{S}, \exists i \in \mathbb{N}, 0 = u_{i+1} = u_{i+2} = \dots\}.$$

Note that  $\mathcal{P}$  is dense in  $\mathcal{G}_1$  and that it is easily seen that  $\mathbf{0} \in B$  is not exposed by  $\mathcal{P}$ .

Let us recall two useful lemmas that we need in the sequel. The first lemma is [2, Lemma 3] and the second one is [2, Lemma 4].

**Lemma 15.** *Let  $B$  be a convex set in  $\ell^2$  with  $B^\circ = \emptyset$ . If  $A$  is a subset of  $B$  with finite codimension in  $\ell^2$ , then  $A^\circ = \emptyset$ .*

**Lemma 16.** *Let  $B$  be a convex closed set in  $\ell^2$  with  $B^\circ = \emptyset$ . Then the set  $\bigcup\{F : F \text{ is a face of } B\}$  is dense in  $B$ .*

**Lemma 17.** *Let  $k \in \mathbb{N}$ , let  $B$  be a closed and convex subset of  $\ell^2$ , let  $\mathcal{P} \subset \mathcal{G}_k$ , and let  $F$  be a derived  $\mathcal{P}$ -face of  $B$ . If  $B^\circ = \emptyset$  or  $F^\circ = \emptyset$ , then  $F \subset \mathcal{E}^k(B, \mathcal{P})$ .*

**Proof.** Assume that  $B^\circ = \emptyset$  or  $F^\circ = \emptyset$  and that  $F \not\subset \mathcal{E}^k(B, \mathcal{P})$ . Consider the collection

$$\mathcal{F} = \{F' : F' \text{ is a derived face of } F \text{ such that } F' \not\subset \mathcal{E}^k(B, \mathcal{P})\}.$$

Since  $F$  is a derived face of itself, we have  $F \in \mathcal{F}$ . By the definition of  $\mathcal{E}^k(B, \mathcal{P})$  we have that if  $F' \in \mathcal{F}$  then  $\text{codim } F' \leq k$ . So we can select an  $F'$  in  $\mathcal{F}$  with a maximal codimension. By Lemma 15 we have  $(F')^\circ = \emptyset$ . According to Lemma 16, there is a face  $G$  of  $F$  such that  $G \not\subset \mathcal{E}^k(B, \mathcal{P})$  and hence  $G \in \mathcal{F}$ . Since  $\text{codim } G > \text{codim } F'$ , we get a contradiction with the choice of  $F'$ . The proof is complete.  $\square$

**Lemma 18.** *Let  $k \in \mathbb{N}$  and let  $B$  be a closed and convex subset of  $\ell^2$  with  $B^\circ = \emptyset$ . Let  $\mathcal{P} \subset \mathcal{G}_k$  be such that  $\psi_P \upharpoonright B$  is proper for every  $P \in \mathcal{P}$ . Then  $\mathcal{E}^k(B, \mathcal{P})$  is a  $\mathcal{P}$ -imitation of  $B$ .*

**Proof.** Since  $\mathcal{E}^k(B, \mathcal{P}) \subset B$  it suffices to prove that  $\psi_P(B) \subset \psi_P(\mathcal{E}^k(B, \mathcal{P}))$  for every  $P \in \mathcal{P}$ . Let  $P \in \mathcal{P}$  and  $w \in B_P = \psi_P(B)$ . Observe that  $\psi_P(\mathcal{E}^k(B, \mathcal{P}))$  is closed because  $\mathcal{E}^k(B, \mathcal{P})$  is a closed subset of  $B$  and  $\psi_P \upharpoonright B$  is proper. By Lemma 13, we have that  $(B_P)^\circ = \emptyset$ . Let  $\varepsilon > 0$  and apply Lemma 16 to find a  $w_0 \in B_P$  with  $\|w - w_0\| < \varepsilon$  and a supporting hyperplane  $H$  at  $w_0$  to  $B_P$  in  $P^\perp$ . Then  $F = (H + P) \cap B$  is a  $\mathcal{P}$ -face of  $B$  such that  $d(w, \psi_P(F)) \leq d(w, w_0) < \varepsilon$ . By Lemma 17, we have that  $F \subset \mathcal{E}^k(B, \mathcal{P})$  and therefore  $d(w, \psi_P(\mathcal{E}^k(B, \mathcal{P}))) < \varepsilon$ . Thus  $w \in \psi_P(\mathcal{E}^k(B, \mathcal{P}))$  because  $\psi_P(\mathcal{E}^k(B, \mathcal{P}))$  is closed and  $\varepsilon$  is arbitrary. We have shown that  $\psi_P(B) \subset \psi_P(\mathcal{E}^k(B, \mathcal{P}))$ .  $\square$

Now we are in a position to prove the following key theorem.

**Theorem 19.** *Let  $k \in \mathbb{N}$  and  $B$  be a closed and convex subset of  $\ell^2$  with  $B^\circ = \emptyset$ . If  $\mathcal{P}$  is a somewhere dense subset of  $\mathcal{G}_k$ , then  $B = \mathcal{E}^k(B, \mathcal{P})$ .*

**Proof.** Since we trivially have  $\mathcal{E}^k(B, \mathcal{P}) \subset B$ , let  $w \in B$  be arbitrary. According to Theorem 14 there is a  $P \in \text{int } \overline{\mathcal{P}}$  such that  $(w + P) \cap B = \{w\}$ . Note that  $\psi_P \upharpoonright B$  is proper by Lemma 9. Let  $\varepsilon > 0$  and define  $C = \{x \in B : \|x - w\| \geq \varepsilon/2\}$ . We may apply Lemma 10 to  $w$ ,  $C$  and  $\psi_P \upharpoonright \overline{\langle C \rangle}$  to find a neighbourhood  $\mathcal{U}$  of  $P$  in  $\text{int } \overline{\mathcal{P}}$  such that  $(w + P') \cap C = \emptyset$  for each  $P' \in \mathcal{U}$ . Select a  $P' \in \mathcal{U} \cap \mathcal{P}$ . Since  $\text{diam}(B \setminus C) \leq \varepsilon$ , we have  $\text{diam}((w + P') \cap B) \leq \varepsilon$  and hence  $\psi_{P'} \upharpoonright B$  is proper by Lemma 9. By Lemma 18 we have that  $\psi_{P'}(B) = \psi_{P'}(\mathcal{E}^k(B, \{P'\})) \subset \psi_{P'}(\mathcal{E}^k(B, \mathcal{P}))$ . Thus there is a  $w_0 \in (w + P') \cap \mathcal{E}^k(B, \mathcal{P}) \subset (w + P') \cap B$ . Hence  $\|w - w_0\| \leq \varepsilon$ . Now, since  $\mathcal{E}^k(B, \mathcal{P})$  is closed and  $\varepsilon > 0$  is arbitrary, we get that  $w \in \mathcal{E}^k(B, \mathcal{P})$  and the proof is finished.  $\square$

**Lemma 20.** *Let  $k \in \mathbb{N}$  and let  $B$  be a closed and convex set in  $\ell^2$  with  $B^\circ = \emptyset$ . If  $\mathcal{P}$  is a nonempty open subset of  $\mathcal{G}_k$  and  $w \notin B$ , then there exists a  $P \in \mathcal{P}$  such that  $\psi_P \upharpoonright B$  is proper and  $(w + P) \cap B = \emptyset$ .*

**Proof.** By Theorem 14 and Lemmas 9 and 11 we may assume without loss of generality that  $\psi_L \upharpoonright B$  is proper for every  $L \in \mathcal{P}$ . Set  $B' = \overline{\langle \{w\} \cup B \rangle}$ . First, let us assume that  $B'$  is a  $\mathcal{P}$ -imitation of  $B$ . By Lemma 13 we have that  $(\psi_L(B'))^\circ = (\psi_L(B))^\circ = \emptyset$  for every  $L \in \mathcal{P}$ . Since projections are open we have  $(B')^\circ = \emptyset$ . Now we can apply the Exposed Point Theorem to  $B'$  to find an  $L \in \mathcal{P}$  such that  $(w + L) \cap B' = \{w\}$  – a contradiction with  $B \subset B' \setminus \{w\}$ . Thus there are a  $P \in \mathcal{P}$  and an  $x \in B'$  such that  $(x + P) \cap B = \emptyset$ . Since  $\psi_P \upharpoonright B$  is proper, the set  $B_P = \psi_P(B)$  is closed. Since  $\psi_P(x) \notin B_P$ , there exists a (unique) hyperplane  $H'$  in  $P^\perp$  through  $\psi_P(x)$  such that  $d(H', B_P) = d(\psi_P(x), B_P) > 0$ ; see [14, p. 347]. Thus, there is a parallel supporting hyperplane  $H$  to  $B_P$  in  $P^\perp$  that strictly separates  $B_P$  from  $\psi_P(x)$ . Since  $x \in \overline{\langle \{w\} \cup B \rangle}$ ,  $w$  and  $B$  must be on different sides of  $H + P$ . Therefore,  $(w + P) \cap B = \emptyset$  and the lemma is proved.  $\square$

We end this section with the proof of Theorem 4 for which we need the following results from [4, Lemma 11] and [4, Theorem 15].

**Lemma 21.** *Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ , let  $\mathcal{P}$  be an open subset of  $\mathcal{G}_k(\mathbb{V})$ , and let  $B$  be a closed and convex set in  $\mathbb{V}$  that contains no  $k$ -hyperplane. If  $P \in \mathcal{P}$  and  $w \in \mathbb{V}$  are such that  $(w + P) \cap B = \emptyset$ , then there is a nonempty open subset  $\mathcal{U}$  of  $\mathcal{P}$  such that  $\psi_L \upharpoonright B$  is proper and  $(w + L) \cap B = \emptyset$  for every  $L \in \mathcal{U}$ .*

**Theorem 22.** *Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ , let  $B$  be a convex and closed set in  $\mathbb{V}$ , and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_k(\mathbb{V})$  such that  $\mathcal{P} \subset \text{int } \overline{\mathcal{P}}$ . If  $C$  is a closed set that is a weak  $\mathcal{P}$ -imitation of  $B$ , then  $\mathcal{E}^k(B, \mathcal{P}) \subset C$ .*

**Proof of Theorem 4.** Let  $k \in \mathbb{N}$  and  $B$  be a closed convex subset of  $\ell^2$  with  $B^\circ = \emptyset$ . Let  $\mathcal{P}$  be somewhere dense in  $\mathcal{G}_k$  and let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$ . We need to prove that  $C = B$ . If we put  $\mathcal{P}^* = \mathcal{P} \cap \text{int } \overline{\mathcal{P}}$ , then we have that  $\mathcal{P}^* \subset \text{int } \overline{\mathcal{P}^*}$ . Theorem 22 now states that  $\mathcal{E}^k(B, \mathcal{P}^*) \subset C$ . According to Theorem 19 we have  $\mathcal{E}^k(B, \mathcal{P}^*) = B$  and hence  $B \subset C$ .

Now let  $w \in C \setminus B$ . By Lemma 20 we find a  $P \in \text{int } \overline{\mathcal{P}}$  such that  $(w + P) \cap B = \emptyset$ . By Lemma 15 we have that  $B$  contains no  $k$ -hyperplane. Now, Lemma 21 states that there is a nonempty open subset  $\mathcal{U}$  of  $\text{int } \overline{\mathcal{P}}$  such that  $\psi_{P'} \upharpoonright B$  is proper and  $(w + P') \cap B = \emptyset$  for every  $P' \in \mathcal{U}$ . Select an  $L \in \mathcal{U} \cap \mathcal{P}$ . Then  $\psi_L(w) \notin \psi_L(B) = \overline{\psi_L(B)}$ , contradicting the assumption that  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$ . The theorem is proved.  $\square$

#### 4. The Proof of Theorem 1

In this section we establish Theorem 1 and some related theorems. We begin with two lemmas.

**Lemma 23.** *Let  $B$  be a closed convex set in  $\mathbb{V}$ , let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ , and let  $\mathcal{P}$  be an open subset of  $\mathcal{G}_k(\mathbb{V})$ . If  $P \in \mathcal{P}$  and  $w$  is an interior point of  $B \cap (w + P)$  in  $w + P$ , then there is a neighbourhood  $U$  of  $w$  in  $\mathbb{V}$  such that every supporting hyperplane  $H$  to  $B$  that meets  $U$  is consistent with  $\mathcal{P}$ .*

**Proof.** We may assume that  $w = \mathbf{0}$ . Let  $\delta > 0$  be such that  $\{x \in P : \|x\| \leq \delta\} \subset B$ . Select a basis  $u_1, \dots, u_k$  for  $P$  consisting of unit vectors. According to Lemma 5, there is an  $\varepsilon > 0$  such that whenever we have for each  $i \in \{1, \dots, k\}$  a vector  $u'_i$  with  $\|u'_i - u_i\| < \varepsilon$ , then  $[\{u'_1, \dots, u'_k\}] \in \mathcal{P}$ . We define  $U = \{x \in \mathbb{V} : \|x\| < \delta\varepsilon\}$ . Let  $H$  be a supporting hyperplane at  $x \in U$  to  $B$  and consider an  $i \in \{1, \dots, k\}$ . If  $u_i \in H - x$ , we put  $u'_i = u_i$ . If  $u_i \notin H - x$ , the line  $\mathbb{R}u_i$  will intersect the hyperplane  $H$  at some point  $r_i u_i$ . Since  $H$  does not cut  $B$  it does not cut the line segment  $[-\delta, \delta]u_i$  and hence  $|r_i| \geq \delta$ . So  $r_i u_i - x$  is an element of the linear space  $H - x$  and hence  $u'_i = u_i - \frac{1}{r_i}x \in H - x$ . We have  $\|u_i - u'_i\| = \|x\|/|r_i| < \varepsilon$ . We now have  $P' = [\{u'_1, \dots, u'_k\}] \subset H - x$  and  $P' \in \mathcal{P}$ .  $\square$

The following lemma is [4, Lemma 12].

**Lemma 24.** *Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ , let  $B$  and  $C$  be convex closed subsets of  $\mathbb{V}$ , and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_k(\mathbb{V})$  such that  $\mathcal{P} \subset \text{int } \overline{\mathcal{P}}$  and  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$ . Then  $\mathcal{E}^k(C, \mathcal{P}) = \mathcal{E}^k(B, \mathcal{P})$ . If, in addition,  $B$  does not contain a  $k$ -hyperplane and  $B$  is not an  $(\text{int } \overline{\mathcal{P}})$ -imitation of  $\mathbb{V}$ , then also  $C$  does not contain a  $k$ -hyperplane and  $B$  and  $C$  have identical (derived)  $\mathcal{P}$ -faces.*

The following theorem is a key step in the proof of Theorem 1.

**Theorem 25.** *Let  $k \in \mathbb{N}$  and let  $B$  be a convex and closed subset of  $\ell^2$  such that  $\text{codim } B \leq k$  and  $B^\circ \neq \emptyset$ . Let  $\mathcal{P}$  be an open subset of  $\mathcal{G}_k$ . Then the following statements are equivalent:*

- (1)  *$B$  contains no  $k$ -hyperplane and  $B$  is not a  $\mathcal{P}$ -imitation of  $\ell^2$ .*
- (2)  *$B$  contains no  $k$ -hyperplane and  $B$  is not a  $\mathcal{P}$ -imitation of  $\text{aff } B$ .*
- (3) *There is a closed subset of  $\mathcal{E}^k(B, \mathcal{P})$  that is homeomorphic to  $\ell^2$ .*
- (4)  *$\mathcal{E}^k(B, \mathcal{P}) \neq \emptyset$ .*

**Proof.** The implication (3)  $\Rightarrow$  (4) requires no proof.

We show that (4)  $\Rightarrow$  (1). If  $B$  is a  $\mathcal{P}$ -imitation of  $\ell^2$ , then  $\mathcal{E}^k(B, \mathcal{P}) = \mathcal{E}^k(\ell^2, \mathcal{P}) = \emptyset$  by Lemma 24. Suppose now that  $B$  contains a  $k$ -hyperplane. According to Lemma 12, this means that every derived face  $F$  of  $B$  contains a  $k$ -hyperplane and hence  $\text{codim } F \leq k$ . Since every derived  $\mathcal{P}$ -face is a derived face, we again have  $\mathcal{E}^k(B, \mathcal{P}) = \emptyset$  which proves the point.

We turn to proving the implication (1)  $\Rightarrow$  (2). Assume that (1) is valid and that  $B$  is a  $\mathcal{P}$ -imitation of  $\text{aff } B$ . Apply Lemma 24 to obtain that  $\text{aff } B$  contains no  $k$ -hyperplane. Then  $\text{codim}(\text{aff } B) = \text{codim } B > k$  in violation of a premise of the theorem.

To prove (2)  $\Rightarrow$  (3) we assume property (2).

**Claim A.** *Without loss of generality we may assume that every  $\mathcal{P}$ -face of  $B$  is contained in  $\mathcal{E}^k(B, \mathcal{P})$ .*

**Proof of Claim A.** Consider the collection  $\mathcal{D}$  consisting of all derived  $\mathcal{P}$ -faces of  $B$  that are not contained in  $\mathcal{E}^k(B, \mathcal{P})$ . Note that every element of  $\mathcal{D}$  has codimension at most  $k$ . Now, assume that  $\mathcal{D} \neq \emptyset$  and select an  $F \in \mathcal{D}$  with maximal codimension. All the faces of  $F$  have a greater codimension than  $F$  and hence they are not members of  $\mathcal{D}$  which means that they are contained in  $\mathcal{E}^k(B, \mathcal{P})$ . If  $F^\circ = \emptyset$  then by Lemma 16 the union of its faces is dense in  $F$  and consequently,  $F \subset \mathcal{E}^k(B, \mathcal{P})$  because  $\mathcal{E}^k(B, \mathcal{P})$  is closed. Thus we get a contradiction with  $F \in \mathcal{D}$  and we may conclude that  $F^\circ \neq \emptyset$ . Then by the Hahn–Banach Theorem every point of  $\partial F$  is contained in some face of  $F$  and hence  $\partial F \subset \mathcal{E}^k(B, \mathcal{P})$ . Since  $\text{codim } F \leq k$ , we can select a  $k$ -hyperplane  $M$  of  $\mathbb{V}$  in  $\text{aff } F$  that meets  $F^\circ$ . Then the closed convex set  $G = F \cap M$  has codimension  $k$  and  $G^\circ \neq \emptyset$ . Note that  $M$ , being a  $k$ -hyperplane, is a copy of  $\ell^2$ . So  $G$  can be viewed as a convex body in  $\ell^2$ . Now if  $\partial G = \emptyset$  then  $G = M$  and  $B$  contains a  $k$ -hyperplane in contradiction to the premise (2). Thus  $\partial G \neq \emptyset$  and statement (3) is proved because  $\partial G \subset \partial F \subset \mathcal{E}^k(B, \mathcal{P})$  and  $\partial G$  contains a closed copy of  $\ell^2$  by Remark 5. We may now assume that  $\mathcal{D} = \emptyset$  which means that whenever  $H$  is a supporting hyperplane to  $B$  that is consistent with  $\mathcal{P}$ , then  $H \cap B \subset \mathcal{E}^k(B, \mathcal{P})$ .  $\diamond$

Let  $P' \in \mathcal{P}$  be such that  $\psi_{P'}(B) \neq \psi_{P'}(\text{aff } B)$ . Then there is a  $w \in \text{aff } B$  such that  $(w + P') \cap B = \emptyset$ . By Lemma 21 we can find a  $P \in \mathcal{P}$  such that  $(w + P) \cap B = \emptyset$  and  $\psi_P|_B$  is proper. Note that  $\psi_P(w) \in \text{aff}(\psi_P(B)) \setminus \psi_P(B)$ . Define  $\mathcal{A}$  as the collection of all linear subspaces  $L$  of  $P$  such that there are an open subset  $O$  of  $B$ , an  $\varepsilon > 0$ , and a  $y \in \partial\psi_L(B)$  with the following two properties:

- (i)  $(y + L) \cap B \subset O$  and
- (ii) for every supporting hyperplane  $H$  to  $B$  that meets  $O$  there exists a  $V \in \mathcal{G}_{k-\dim L}$  such that  $H = H + V$  and  $V + L' \in \mathcal{P}$  for every  $L'$  with  $\rho(L, L') < \varepsilon$ .

Next, let us show that  $P \in \mathcal{A}$ . Since  $\mathcal{P}$  is open, there is an  $\varepsilon > 0$  such that if  $L' \in \mathcal{G}_k$  with  $\rho(P, L') < \varepsilon$ , then  $L' \in \mathcal{P}$ . Set  $O = B$  and  $V = \{\mathbf{0}\}$  for any  $H$ . Since  $\text{aff}(\psi_P(B)) \neq \psi_P(B)$  we may choose a  $y \in \partial\psi_P(B)$ . It is now clear that the properties (i) and (ii) are satisfied and thus  $P \in \mathcal{A}$ .

We may define

$$l = \min\{\dim L : L \in \mathcal{A}\}.$$

Choose an  $L \in \mathcal{A}$  such that  $\dim L = l$ , and let  $O, \varepsilon, y$  be as in the definition of  $\mathcal{A}$  corresponding to  $L$ . We put  $B_L = \psi_L(B)$  and note that  $(B_L)^\circ \neq \emptyset$  because  $B^\circ \neq \emptyset$ . Since  $\psi_L|_B$  is proper by Lemma 8 and hence closed and since  $(y + L) \cap B \subset O$  the set  $U = B_L \setminus \psi_P(B \setminus O)$  is an open neighbourhood of  $y$  in  $B_L$  such that  $(U + L) \cap B \subset O$ .

**Claim B.**  *$B \cap (w + L)$  is a singleton for every  $w \in U \cap \partial B_L$ .*

**Proof of Claim B.** Let  $w \in U \cap \partial B_L$  be arbitrary and let  $C = (w + L) \cap B \subset O$ . Since  $\psi_L \upharpoonright B$  is proper, we have that  $B_L$  is closed and hence  $C \neq \emptyset$ . Striving for a contradiction, assume that the convex set  $C$  is not a singleton thus  $m = \dim C \geq 1$ . Since  $C$  is finite dimensional and convex we have  $C^\circ \neq \emptyset$  and we may assume that  $\mathbf{0} \in C^\circ$  thus  $w = \psi_L(\mathbf{0}) = \mathbf{0}$ . Set  $E = \text{aff } C \subset L$  and  $L^* = \psi_E(L) = L \cap E^\perp$ . Observe that  $\dim E = m$  and  $\dim L^* = l - m < l$ . We will show that  $L^* \in \mathcal{A}$  which gives the desired contradiction with the minimality of  $l$ . Put  $B_{L^*} = \psi_{L^*}(B)$  and note that  $C \subset E \cap B \subset L \cap B = C$  so  $C = E \cap B$ .

By Lemma 7, we find an  $\varepsilon^* > 0$  such that if  $\rho(E, \hat{E}) < \varepsilon^*$  for  $\hat{E} \in \mathcal{G}_m$  and  $\rho(L^*, \hat{L}) < \varepsilon^*$  for  $\hat{L} \in \mathcal{G}_{l-m}$ , then  $\rho(L^* + E, \hat{E} + \hat{L}) = \rho(L, \hat{E} + \hat{L}) < \varepsilon$ . By Lemma 23 we can find an open neighbourhood  $O^*$  of  $\mathbf{0}$  in  $O$  with the following property:

- (\*) If  $H$  is a supporting hyperplane to  $B$  that meets  $O^*$ , then there is an  $\hat{E} \in \mathcal{G}_m$  such that  $\rho(E, \hat{E}) < \varepsilon^*$  and  $H = H + \hat{E}$ .

We have now found  $O^*$  and  $\varepsilon^*$  for  $L^*$  and we put  $y^* = \mathbf{0}$ . Note that  $y^* \in B_{L^*}$  and  $\psi_E(y^*) = w \in \partial B_L$  thus  $y^* \in \partial B_{L^*}$ . We also have

$$(y^* + L^*) \cap B = L^* \cap B = E^\perp \cap L \cap B = E^\perp \cap E \cap B = \{\mathbf{0}\} \subset O^*$$

thus condition (i) is satisfied for  $L^*$ . For condition (ii), let  $H$  be a supporting hyperplane to  $B$  that meets  $O^*$ . Since  $L \in \mathcal{A}$  and  $O^* \subset O$ , there exists a  $V \in \mathcal{G}_{k-l}$  such that  $H = H + V$  and for every  $L'$  with  $\rho(L, L') < \varepsilon$  we have  $V + L' \in \mathcal{P}$ . By property (\*), we find an  $\hat{E} \in \mathcal{G}_m$  such that  $\rho(E, \hat{E}) < \varepsilon^*$  and  $H = H + \hat{E}$ . Thus we have  $H = H + V + \hat{E}$ . Moreover, if  $L' \in \mathcal{G}_{l-m}$  with  $\rho(L^*, L') < \varepsilon^*$  then, by the choice of  $\varepsilon^*$ , we have that  $\rho(L, \hat{E} + L') < \varepsilon$ . Consequently,

$$V + \hat{E} + L' \in \mathcal{P}.$$

Hence  $V + \hat{E}$  satisfies condition (ii) for  $L^*$  and we may conclude that  $L^* \in \mathcal{A}$ . Since  $\dim L^* < l$ , we have a contradiction with the minimality of  $l$ . The claim is proved.  $\diamond$

Now, let  $W = U \cap \partial B_L$  and  $W' = (V + L) \cap B = (\psi_L \upharpoonright B)^{-1}(W)$ . By Claim B and the fact that  $\psi_L \upharpoonright B$  is proper we have that  $\psi_L \upharpoonright W': W' \rightarrow W$  is a proper bijection and thus a homeomorphism. We prove that  $W' \subset \mathcal{E}^k(B, \mathcal{P})$ . Let  $w' \in W'$  and put  $w = \psi_L(w') \in W$ . Since  $(B_L)^\circ \neq \emptyset$  we can find a supporting hyperplane  $H$  at  $w$  to  $B_L$ . Then  $H + L$  is a supporting hyperplane at  $w'$  to  $B$ . By property (ii) we have that there exists  $V \in \mathcal{G}_{k-l}$  such that  $H + L = H + L + V$  and  $V + L \in \mathcal{P}$ . Therefore,  $H + L$  is consistent with  $\mathcal{P}$ . Hence  $w' \in (H + L) \cap B \subset \mathcal{E}^k(B, \mathcal{P})$  by Claim A. We have shown that  $W' \subset \mathcal{E}^k(B, \mathcal{P})$ . Furthermore, since  $\text{codim } B_L \leq \text{codim } B + \dim L \leq k + l$  and  $(B_L)^\circ \neq \emptyset$  we have that  $U$ , being a nonempty open subset of  $\partial B_L$ , contains a copy  $M$  of  $\ell^2$  that is closed in  $\ell^2$  by Remark 5. Thus also  $M' = (\psi_L \upharpoonright B)^{-1}(M)$  is a copy of  $\ell^2$  that is closed in  $\ell^2$ . Since  $M' \subset W' \subset \mathcal{E}^k(B, \mathcal{P})$ , the proof is complete.  $\square$

The following result was proved for  $\mathbb{R}^n$  in [3, Lemma 13]. The proof for  $\ell^2$  is virtually identical.

**Lemma 26.** *Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ , let  $B$  be a closed subset of  $\mathbb{V}$  such that  $\text{codim } B > k$ , and let  $\mathcal{P}$  be a nonempty open subset of  $\mathcal{G}_k(\mathbb{V})$ . Then the only closed weak  $\mathcal{P}$ -imitation of  $B$  is  $B$  itself.*

The following theorem is from [4, Theorem 13].

**Theorem 27.** *Let  $k \in \mathbb{N}$  with  $k < \dim \mathbb{V}$ , let  $B$  be a closed convex subset of  $\mathbb{V}$  that contains no  $k$ -hyperplane, and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_k(\mathbb{V})$  such that  $B$  is not an  $(\text{int } \overline{\mathcal{P}})$ -imitation of  $\mathbb{V}$ . If  $C$  is a closed weak  $\mathcal{P}$ -imitation of  $B$ , then there is an open subset  $\mathcal{U}$  of  $\text{int } \overline{\mathcal{P}}$  such that  $C$  is a  $\mathcal{U}$ -imitation of  $B$  and  $B$  is not a  $\mathcal{U}$ -imitation of  $\mathbb{V}$ .*

Now we are in a position to prove Theorem 1.

**Proof of Theorem 1.** Note that the premises of Theorem 1 are identical to those in Theorem 27. Consequently, for the purpose of proving Theorem 1 we may assume that  $\mathcal{P}$  is an open subset of  $\mathcal{G}_k$  such that  $C$  is a  $\mathcal{P}$ -imitation of  $B$  and  $B$  is not a  $\mathcal{P}$ -imitation of  $\ell^2$ . Note that  $\mathcal{P} \neq \emptyset$  because  $B$  is not a  $\mathcal{P}$ -imitation of  $\ell^2$ . Since  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$  with  $C \neq B$ , by Lemma 26 we have that  $\text{codim } B \leq k$  and by Theorem 4 we obtain that  $B^\circ \neq \emptyset$ . Now, we can apply Theorem 25 to  $B$  and  $\mathcal{P}$  to get that there is a non-empty closed subset  $A$  of  $\mathcal{E}^k(B, \mathcal{P})$  that is homeomorphic to  $\ell^2$ . It follows from Theorem 22 that  $A \subset \mathcal{E}^k(B, \mathcal{P}) \subset C \cap B$ .  $\square$

The following theorem about sets with convex projections follows easily from Theorem 1.

**Theorem 28.** *Let  $k \in \mathbb{N}$  and let  $C$  be a closed nonconvex subset of  $\ell^2$ , and let  $\mathcal{P}$  be a subset of  $\mathcal{G}_k$ . Let  $\psi_{P^*}(\overline{\langle C \rangle}) \neq (P^*)^\perp$  for some  $P^* \in \text{int } \overline{\mathcal{P}}$  and let  $\overline{\psi_P(C)}$  be convex for every  $P \in \mathcal{P}$ . If  $\langle C \rangle$  contains no  $k$ -hyperplane, then  $C$  contains a closed copy of  $\ell^2$ .*

**Proof.** Let  $B = \overline{\langle C \rangle}$ . Since  $C$  is nonconvex, we have that  $C \neq B$ . We have that  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$  because

$$\overline{\psi_P(C)} \subset \overline{\psi_P(B)} = \overline{\psi_P(\overline{\langle C \rangle})} \subset \overline{\langle \psi_P(C) \rangle} \subset \overline{\langle \overline{\psi_P(C)} \rangle} = \overline{\psi_P(C)}$$

for each  $P \in \mathcal{P}$ . Furthermore, we have that  $B$  is not an  $(\text{int } \overline{\mathcal{P}})$ -imitation of  $\ell^2$  because  $\psi_{P^*}(B) \neq (P^*)^\perp = \psi_{P^*}(\ell^2)$ . Moreover,  $B$  contains no  $k$ -hyperplane. Apply now Theorem 1.  $\square$

## 5. Imitations

Theorem 22 states that every weak  $\mathcal{P}$ -imitation of a convex set  $B$  contains the set of extremal points  $\mathcal{E}^k(B, \mathcal{P})$ . In this section we show that  $B$  has “minimal



imitations", that is, sets that contain little else besides  $\mathcal{E}^k(B, \mathcal{P})$ . This result was proved for closed sets  $B$  in  $\mathbb{R}^n$  in [3, Theorem 22].

**Lemma 29.** *Let  $k \in \mathbb{N}$ ,  $B$  be a closed convex subset of  $\ell^2$  with  $B^\circ \neq \emptyset$ , and  $\mathcal{P}$  be a subset of  $\mathcal{G}_k$ . Then there exists a closed subset  $\mathcal{K}^k(B, \mathcal{P})$  of  $B$  such that  $\mathcal{K}^k(B, \mathcal{P}) \cap \partial B \subset \mathcal{E}^k(B, \mathcal{P})$  and  $\psi_P(\mathcal{K}^k(B, \mathcal{P})) = \psi_P(B)$  for every  $P \in \mathcal{P}$  that satisfies  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$ .*

**Proof.** Despite being very similar to the proof of [3, Lemma 21], we include the entire construction because in certain places the fact that we are in  $\ell^2$  requires a bit of extra care in reasoning (this remark applies also to other results in this section). Choose a coordinate system for  $\ell^2$  such that  $\mathbf{0} \in B^\circ$ . We define  $S$  to be the closure of

$$\{tx : 0 \leq t \leq 1 \text{ and } x \in \mathcal{E}^k(B, \mathcal{P})\}$$

and we observe that  $S$  is a subset of  $B$ . Since  $\mathbf{0} \in B^\circ$ , we have  $S \cap \partial B \subset \mathcal{E}^k(B, \mathcal{P})$ . We now define the following subset of  $B^\circ$ :

$$T = \{\xi(x) : x \in B\},$$

where  $\xi(x) = \frac{\|x\|}{\|x\|+1}x$  for  $x \in \ell^2$ . As in the proof of [3, Lemma 21], one can easily show that  $T$  is closed. Set  $K = \mathcal{K}^k(B, \mathcal{P}) = S \cup T$  and we note that  $K \subset B$ . Moreover, we have that  $K \cap \partial B \subset \mathcal{E}^k(B, \mathcal{P})$  and that  $K$  is a closed set such that  $tx \in K$  whenever  $x \in K$  and  $t \in [0, 1]$ .

Let  $P \in \mathcal{P}$  such that  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$  and let  $x \in B$  be arbitrary. It suffices to show that  $\psi_P(x) \in \psi_P(K)$ . We define an  $s \in [1, \infty]$  by

$$s = \sup\{t : t \geq 1 \text{ and } t\psi_P(x) \in \psi_P(B)\}$$

and we consider the following two cases:

**Case I.**  $s = \infty$  or  $s\psi_P(x) \notin \psi_P(B)$ . Then  $s > 1$  and we may select sequences  $s_1, s_2, \dots$  in  $(1, s)$  and  $y_1, y_2, \dots$  in  $B$  such that  $\lim_{i \rightarrow \infty} s_i = s$  and  $y_i \in B$  with  $\psi_P(y_i) = s_i\psi_P(x)$  for each  $i \in \mathbb{N}$ . If the sequence  $y_1, y_2, \dots$  has a limit point  $y$ , then  $y \in B$  and  $\psi_P(y) = s\psi_P(x)$  if  $s < \infty$  or  $\psi_P(x) = \mathbf{0}$  if  $s = \infty$ . The first option is a violation of the premise of this case and when  $\psi_P(x) = \mathbf{0} = \psi_P(\mathbf{0})$  we are done because  $\mathbf{0} \in K$ . Thus we may assume that  $y_1, y_2, \dots$  has no limit point. Since the sequence lies in the finite-dimensional space  $P + \mathbb{R}x$ , we have that  $\lim_{i \rightarrow \infty} \|y_i\| = \infty$  and hence that  $\lim_{i \rightarrow \infty} s_i\|y_i\|/(\|y_i\| + 1) = s$ . Choose an  $i \in \mathbb{N}$  with  $r = s_i\|y_i\|/(\|y_i\| + 1) > 1$ . Note that  $\xi(y_i) \in T \subset K$  and hence  $\frac{1}{r}\xi(y_i) \in K$ . We have  $\psi_P(\frac{1}{r}\xi(y_i)) = \psi_P(\frac{1}{s_i}\xi(y_i)) = \psi_P(x)$  and Case I is complete.

**Case II.**  $s < \infty$  and  $s\psi_P(x) \in \psi_P(B)$ . In this case  $y = s\psi_P(x) \in \partial\psi_P(B)$ . Since  $B^\circ \neq \emptyset$ , we also have  $\psi_P(B)^\circ \neq \emptyset$  so we can find a hyperplane  $H$  in  $P^\perp$  that is supporting to  $\psi_P(B)$  at  $y$ . Then  $F = (H + P) \cap B$  is a  $\mathcal{P}$ -face of  $B$  that meets  $y + P$ . Define the collection

$$\mathcal{D} = \{F' : F' \text{ is a derived face of } F \text{ such that } F' \cap (y + P) \neq \emptyset\}.$$

Since  $F \in \mathcal{D}$ , we have that  $\mathcal{D} \neq \emptyset$ . If there is an  $F' \in \mathcal{D}$  with  $F' \subset \mathcal{E}^k(B, \mathcal{P})$ , then choose a point  $a \in F' \cap (y + P)$  and note that  $\frac{1}{s}a \in S \subset K$  and  $\psi_P(\frac{1}{s}a) = \psi_P(x)$ . That is the desired conclusion so we may assume that no element of  $\mathcal{D}$  is contained in  $\mathcal{E}^k(B, \mathcal{P})$ . By the definition of  $\mathcal{E}^k(B, \mathcal{P})$  this means that  $\text{codim } F' \leq k$  for each  $F' \in \mathcal{D}$  and hence there is an  $F_1 \in \mathcal{D}$  with maximal codimension in  $\ell^2$ . Lemma 17 guarantees that  $F_1^\circ \neq \emptyset$ . Note that the codimension of  $F_1$  in the hyperplane  $H + P$  is at most  $k - 1$ . Since  $\dim P = k$ , there exists a line  $\ell$  through  $\mathbf{0}$  such that  $y' + \ell \subset (y + P) \cap \text{aff } F_1$  for some  $y' \in (y + P) \cap F_1$ . Since  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$ , we have that  $\ell \cap \mathcal{L}_B = \{\mathbf{0}\}$  and hence  $\ell' = \psi_{\mathcal{L}_B}(y' + \ell)$  is also a line. We have that  $\ell'$  is not contained in  $\text{cs } B = \psi_{\mathcal{L}_B}(B)$  because  $\text{cs } B$  is line-free. Thus,  $y' + \ell$  is neither contained in  $B$  nor in its subset  $F_1$ . Since  $y' + \ell$  is a subset of  $\text{aff } F_1$  that meets  $F_1$  the line contains a point  $z$  of  $\partial F_1$ . Since  $F_1^\circ \neq \emptyset$ , the point  $z$  is contained in some face  $G$  of  $F_1$ . Note that  $z \in y + P$ , thus  $G \in \mathcal{D}$ . Since  $\text{codim } G > \text{codim } F_1$  we have a violation of the choice of  $F_1$ .  $\square$

Let us recall the following proposition (see [2, Theorem 19]).

**Proposition 30.** *For every  $\varepsilon > 0$  there is a zero-dimensional closed set  $\mathfrak{Z}_\varepsilon$  in  $\ell^2$  such that  $S \cap \mathfrak{Z}_\varepsilon \neq \emptyset$  whenever  $S$  is a convex subset of  $\ell^2$  with  $\text{diam } S \geq \varepsilon$ .*

Now we are ready to prove Theorem 2.

**Proof of Theorem 2.** Let  $k \in \mathbb{N}$ ,  $B$  be a closed convex subset of  $\ell^2$  with  $\text{codim } B \neq k$ , and  $\mathcal{P}$  be a subset of  $\mathcal{G}_k$ . We construct a closed  $\mathcal{P}$ -imitation  $C$  of  $B$  such that  $C \subset B$  and  $C \setminus \mathcal{E}^k(B, \mathcal{P})$  is zero-dimensional. If  $\mathcal{P} = \emptyset$ , then there is nothing to prove. Let  $P \in \mathcal{P}$ . If  $\text{codim } B > k$  then  $\text{codim}(P + \text{aff } B) \geq 1$  so  $P + \text{aff } B$  is contained in some hyperplane of  $\ell^2$ . This means that  $B$  is a  $\mathcal{P}$ -face of itself and that  $\mathcal{E}^k(B, \mathcal{P}) = B$  so we may choose  $C = B$ .

Now, let us assume that  $B^\circ = \emptyset$ . Set  $C = (\mathfrak{Z}_1 \cap B) \cup \mathcal{E}^k(B, \mathcal{P})$  and let us prove that

$$\psi_P(B) = \psi_P(C).$$

Indeed, if  $\psi_P \upharpoonright B$  is proper then, by Lemma 18, we have that  $\psi_P(B) = \psi_P(\mathcal{E}^k(B, \{P\})) = \psi_P(\mathcal{E}^k(B, \mathcal{P}))$ . If  $\psi_P \upharpoonright B$  is not proper then, by Lemma 11, all nonempty fibres are unbounded and must meet  $\mathfrak{Z}_1$ . Therefore we have

$$\psi_P(B) = \psi_P(\mathfrak{Z}_1 \cap B).$$

Now, we can assume that  $\mathcal{P} \neq \emptyset$ ,  $B^\circ \neq \emptyset$ , and  $\text{codim } B < k$ . As in [2, Theorem 2]  $C$  will have the form  $\mathcal{E}^k(B, \mathcal{P}) \cup Z_1 \cup Z_2$ , where  $Z_1$  and  $Z_2$  are zero-dimensional sets. Consider the open subset  $D = B \setminus \mathcal{E}^k(B, \mathcal{P})$  of  $B$  and its closed subset  $K = \mathcal{K}^k(B, \mathcal{P}) \setminus \mathcal{E}^k(B, \mathcal{P})$ , see Lemma 29. Precisely as in the proof of [2, Theorem 2] we can construct a zero-dimensional subset  $Z_1$  of  $D$  such that  $\mathcal{E}^k(B, \mathcal{P}) \cup Z_1$  is closed in  $\ell^2$  and every line in  $\text{aff } B$  that meets  $K$  also meets  $Z_1$ . The set  $Z_2$  is simply  $B \cap \mathfrak{Z}_1$ , thus for every line  $\ell$  in  $\mathcal{L}_B$  and point  $x \in B$  we have  $(x + \ell) \cap Z_2 \neq \emptyset$ .  $Z_1 \cup Z_2$  is zero-dimensional by [11, Theorem 1.3.1].

Let  $P \in \mathcal{P}$  and  $x \in B$  be arbitrary. It now suffices to show that  $\psi_P(x) \in \psi_P(C)$ . Assume first that  $P \cap \mathcal{L}_B \neq \{\mathbf{0}\}$  and hence that  $P \cap \mathcal{L}_B$  contains a line  $\ell$  through  $\mathbf{0}$ . Since  $x + \ell$  intersects  $Z_2$  we have  $\psi_P(x) \in \psi_P(Z_2) \subset \psi_P(C)$ . Now let  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$  and note that we may apply Lemma 29 to find that  $\psi_P(B) = \psi_P(\mathcal{K}^k(B, \mathcal{P}))$  thus  $(x + P) \cap \mathcal{K}^k(B, \mathcal{P}) \neq \emptyset$ . If  $(x + P) \cap \mathcal{E}^k(B, \mathcal{P}) \neq \emptyset$ , then we are done. So we may assume that  $(x + P) \cap K \neq \emptyset$ . Since  $\text{codim } B < k$ , we have  $\dim((x + P) \cap \text{aff } B) \geq 1$  and we can find a line  $\ell$  in  $(x + P) \cap \text{aff } B$  that meets  $K$ . Thus  $\ell$  must also meet  $Z_1$  and, therefore, we have  $\psi_P(x) \in \psi_P(Z_1) \subset \psi_P(C)$ . The proof is complete.  $\square$

**Proof of Theorem 3.** By Theorem 22 we have that

$$\begin{aligned} \mathcal{E}^k(B, \mathcal{P}) &\subset \bigcap \{C : C \text{ is a closed weak } \mathcal{P}\text{-imitation of } B\} \\ &\subset \bigcap \{C : C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}. \end{aligned}$$

On the other hand, Theorem 2 guarantees that there exist closed  $\mathcal{P}$ -imitations  $C$  of  $B$  with  $C = \mathcal{E}^k(B, \mathcal{P}) \cup Z_1 \cup Z_2$ , where  $Z_1$  and  $Z_2$  are zero-dimensional sets. As noted in [2], given any point  $x \in \ell^2$  it can be arranged that  $Z_1 \cup Z_2$  avoids that point. This observation proves the theorem.  $\square$

**Remark 6.** We now explain why the case  $\text{codim } B = k$  is excluded in Theorems 2 and 3. Let  $B$  be convex and closed in  $\ell^2$  such that  $\text{codim } B = k$ . Let  $\mathcal{P}$  be a nonempty subset of  $\mathcal{G}_k$  such that  $\mathcal{P} \subset \text{int } \overline{\mathcal{P}}$ . In view of Theorem 19 we need to assume that  $B^\circ \neq \emptyset$ . It is easily seen that now  $\mathcal{E}^k(B, \mathcal{P}) = \partial B$ . Select a coordinate system such that  $\mathbf{0} \in B^\circ$ . Let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$  such that  $C \subset \text{aff } B$ . Since  $\mathcal{P}$  is somewhere dense in  $\mathcal{G}_k$  and  $\text{codim}(\text{aff } B) = k$  we can find a  $P \in \mathcal{P}$  such that  $P \cap \text{aff } B = \{\mathbf{0}\}$ . Then  $\psi_P \upharpoonright \text{aff } B : \text{aff } B \rightarrow P^\perp$  is a homeomorphism and hence  $\overline{\psi_P(C)} = \overline{\psi_P(B)}$  implies  $C = B$ . We have that  $C \setminus \mathcal{E}^k(B, \mathcal{P}) = B^\circ$  contains a topological copy of  $\ell^2$  and hence Theorem 2 is false whenever  $\text{codim } B = k$  and  $B^\circ \neq \emptyset$ .

Consider now Theorems 3 and 2', that is, Theorem 2 without the requirement that  $C \subset B$ . Then we have two cases.

**Case I.** There is an  $x \notin \text{aff } B$  such that  $\psi_P(x) \in \psi_P(B)$  for every  $P \in \mathcal{P}$ . In this case the conclusions of Theorems 2' and 3 are valid for  $B$ . Put  $B' = \overline{B \cup \{x\}}$  and note that  $B'$  is a closed and convex weak  $\mathcal{P}$ -imitation of  $B$  with  $\text{codim } B' = k - 1$ . By Lemma 24, we have  $\mathcal{E}^k(B, \mathcal{P}) = \mathcal{E}^k(B', \mathcal{P})$ . We can apply both Theorems 2 and 3 to  $B'$  and reach the desired conclusion for  $B$ .

**Case II.** Otherwise. In this case the conclusions of Theorems 2' and 3 are always false. Let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$  and assume that there is an  $x \in C \setminus \text{aff } B$ . Then there is a  $P_1 \in \mathcal{P}$  such that  $\psi_{P_1}(\frac{1}{2}x) \notin \psi_{P_1}(B)$ . Since  $\mathbf{0} \in B^\circ$ , it is easily verified that  $\psi_{P_1}(x) \notin \overline{\psi_{P_1}(B)}$  and hence  $C$  is not a weak  $\mathcal{P}$ -imitation of  $B$ . Thus we may conclude that  $C \subset \text{aff } B$  and by the argument above we have  $C = B$ . In this case, just as in the case  $\text{codim } B > k$ , the set  $B$  has only itself as closed weak  $\mathcal{P}$ -imitation and Theorem 1 is essentially void. Moreover,  $C \setminus \mathcal{E}^k(B, \mathcal{P}) = B^\circ$  and the conclusions of Theorems 2' and 3 are invalid.

We show that the two assumptions in Theorem 1 that  $B$  contains no  $k$ -hyperplane and that  $B$  is not an  $(\text{int } \overline{\mathcal{P}})$ -imitation of  $\ell^2$  are necessary conditions.

**Proposition 31.** *Let  $k \in \mathbb{N}$ , let  $B$  be a closed convex set in  $\ell^2$ , and let  $\mathcal{P}$  be a nonempty subset of  $\mathcal{G}_k$  such that  $\mathcal{P} \subset \text{int } \overline{\mathcal{P}}$ .*

- (a) *If  $B$  is a  $\mathcal{P}$ -imitation of  $\ell^2$ , then  $B$  has a closed  $\mathcal{P}$ -imitation that is zero-dimensional.*
- (b) *If  $B$  contains a  $k$ -hyperplane, then either  $B$  has a closed  $\mathcal{P}$ -imitation that is zero-dimensional or  $B$  is a  $k$ -hyperplane that admits no closed weak  $\mathcal{P}$ -imitation other than itself.*

**Proof.** (a) Assume that  $B$  is a  $\mathcal{P}$ -imitation of  $\ell^2$ . Then by Theorem 4 we get that  $B^\circ \neq \emptyset$  and by Lemma 24 we have  $\mathcal{E}^k(B, \mathcal{P}) = \mathcal{E}^k(\ell^2, \mathcal{P}) = \emptyset$ . Let  $P \in \mathcal{P}$  and note that  $\text{codim } B \leq \text{codim } \psi_P(B) = \text{codim } P^\perp = \dim P = k$ . If  $\text{codim } B < k$ , then  $B$  has a zero-dimensional and closed  $\mathcal{P}$ -imitation by Theorem 2. If  $\text{codim } B = k$ , then since  $B$  is a  $\mathcal{P}$ -imitation of  $\ell^2$  and  $B^\circ \neq \emptyset$  we are in Case I of Remark 6 and  $B$  also has a zero-dimensional and closed  $\mathcal{P}$ -imitation.

(b) Assume that  $B$  contains a  $k$ -hyperplane. Then, by Lemma 15,  $B^\circ \neq \emptyset$ . Next, by Lemma 12, we get that every derived face of  $B$  has  $\text{codim} \leq k$  and hence  $\mathcal{E}^k(B, \mathcal{P}) = \emptyset$ . If  $\text{codim } B < k$ , then  $B$  has a zero-dimensional and closed  $\mathcal{P}$ -imitation by Theorem 2. So we may assume that  $B$  is a  $k$ -hyperplane. If  $B' = \bigcap \{B + P : P \in \mathcal{P}\}$  is not equal to  $B$ , then we are in Case I of Remark 6 and  $B$  admits a zero-dimensional and closed  $\mathcal{P}$ -imitation. If  $B' = B$  then we are in Case II and we may conclude that  $B$  has only itself as a closed weak  $\mathcal{P}$ -imitation.  $\square$

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