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SOME PROPERTIES OF STAR-COUNTABLE COVERS

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1. Introduction. This paper deals with star-countable covers. A family $\mathcal{F}$ of ubsets of a topological space X is called star-countable if for each $V \in \mathcal{F}$, $\mathcal{F}_V = \{U :$ $\in \mathcal{F}, U \bigcap V \neq \emptyset\}$ is countable. Our interest to star-countable covers comes from wo directions. The first one concerns an open problem of MICHAEL and NAGAMI ([5], roblem 1.5). That question could be restated in terms of coverings (see Section 2). Iere we give a positive answer to this question for a star-countable cover instead of oint-countable. The second direction is related to various assumptions made on star-ountable covers. Those assumptions for point-countable covers are discussed in [3]. et us note that some of the equivalencies that we show here are not valid or it is not nown whether they are valid for point-countable covers. We will consider topological paces with the following conditions (they are stated under the same labels in [3]):

.1) X has a base $\mathcal{F}$.

.2) X has an open cover $\mathcal{F}$ that separates points.

.3) X has a cover $\mathcal{F}$ such that, if $x \in W$ with W open in X, then there is a finite ubcollection $\mathcal{V}$ of $\mathcal{F}$ such that $x \in \operatorname{Int}(\bigcup \mathcal{V})$, $\bigcup \mathcal{V} \subset W$, and $x \in \bigcap \mathcal{V}$.

.4) Same as (1.3), but without requiring $x \in \bigcap \mathcal{V}$.

.5) X has a cover $\mathcal{F}$ such that if $x, y \in X$ with $x \neq y$, there is a finite subcollection of $\mathcal{F}$ such that $x \in \operatorname{Int}(\bigcup \mathcal{V})$ and $y \notin \bigcup \mathcal{V}$.

.5)$^-$ X has a cover $\mathcal{F}$ such that if $x, y \in X$ with $x \neq y$, there is a finite subcollection of $\mathcal{F}$ such that $x \in \operatorname{Int}(\bigcup \mathcal{V})$ and $y \notin \operatorname{Int}(\bigcup \mathcal{V})$.

.5)$^+$ X has a cover $\mathcal{F}$ such that if $x, y \in X$ with $x \neq y$, there is a finite subcollection of $\mathcal{F}$ such that $x \in \operatorname{Int}(\bigcup \mathcal{V}) = \bigcup \mathcal{V}$ and $y \notin \bigcup \mathcal{V}$.

The paper is arranged as follows. In Section 2 we prove our main results. Section deals with spaces having a star-countable base and the last section is devoted to xamples.

2. Main results. Let us begin this section with a notation that we will need in e sequel. A map $f : X \mapsto Y$ is called an sc-map if there is a base $\mathcal{B}$ for X such that $(\mathcal{B}) = \{f(B) : B \in \mathcal{B}\}$ is star-countable cover of Y, equivalently, for each $V \in \mathcal{B}$, $_V = \{U : U \in \mathcal{B}, U \bigcap f^{-1}(f(V)) \neq \emptyset\}$ is countable. For a topological space X we onsider the following two conditions:

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*) (see [4], 1.5) X has a cover $\mathcal{F}$ such that if $x \in K \bigcap U$ with K compact and U pen in X, then there is a finite $\mathcal{V} \subset \mathcal{F}$ such that $\bigcup \mathcal{V} \subset U$, $x \in \bigcap \mathcal{V}$ and $\mathcal{V}$ covers a eighbourhood of x in K.

**) ([1]) X has a cover $\mathcal{F}$ such that for every compact K and open cover $\mathcal{U}$ of K there s a finite $\mathcal{V} \subset \mathcal{F}$ that refines $\mathcal{U}$ and admits a finite closed refinement that covers K.

Having in mind ([4], Theorem 6.1) and ([1], Theorem 1.1) the question of Michael nd Nagami, mentioned above, could be formulated in the following way: Are (*) and **) with a point-countable $\mathcal{F}$ ($\mathcal{F}$ in (*) and (**) could be different) equivalent for a ausdorff k-space X? While this problem is still open we have the following result:

Theorem 2.1. For a Hausdorff $k-$space X the following are equivalent:

) X satisfies (*) with a star-countable $\mathcal{F}$;

) X satisfies (**) with a star-countable $\mathcal{F}$.

Proof. (a)$\Rightarrow$(b). Denote by $\mathcal{S}$ all finite subcollections of $\mathcal{F}$ and let

$$\mathcal{P} = \{\bigcup \mathcal{V} : \mathcal{V} \in \mathcal{S}, \bigcap \mathcal{V} \neq \emptyset\}.$$

et us see that $\mathcal{P}$ is star-countable. Indeed, for $F \in \mathcal{F}$ write

$$\mathcal{P}_F = \{\bigcup \mathcal{V} \in \mathcal{P} : F \in \mathcal{V}\}.$$

early, $\mathcal{P}_F$ is countable. Get an arbitrary $\bigcup \mathcal{V} \in \mathcal{P}$. If $\bigcup \mathcal{W} \in \mathcal{P}$ and $(\bigcup \mathcal{W}) \bigcap (\bigcup \mathcal{V}) \neq \emptyset$ en $\bigcup \mathcal{W} \in \bigcup \{\mathcal{P}_F : F \bigcap (\bigcup \mathcal{V}) \neq \emptyset\}$. Hence, $\mathcal{P}$ is star-countable. Further, let K is mpact in X and $\mathcal{U}$ is an open cover of it. We can take finitely many elements of $\mathcal{P}$, $P_1, P_2, \ldots, P_k$, such that $\mathcal{U}' = \{\mathrm{Int}\ P_i : i = \overline{1,n}\}$ is a cover of K and $\{P_i : i = \overline{1,k}\}$ ines $\mathcal{U}$. Obviously, $\mathcal{U}'$ admits a finite closed refinement and thus (b) is established.

(b)$\Rightarrow$(a). Let $\mathcal{F}$ satisfy (**). We show that $\mathcal{F}$ satisfies (*) as well. Get a compact and let $x \in K \bigcap U$ with U open in X. We construct an open cover $\mathcal{U}$ of K such at $U \in \mathcal{U}$ and U is the only element containing x. Let $\mathcal{P} \subset \mathcal{F}$ is finite and refines $\mathcal{U}$ d admits a finite closed refinement $\mathcal{W}$. Then $\mathcal{V}' = \{W : W \in \mathcal{W}, x \in W\}$ covers a ghbourhood of x in K. Let $\mathcal{V} = \{V : V \in \mathcal{P}, V$ contains some $W \in \mathcal{V}'\}$. Now, we e that $\mathcal{V}$ covers a neighborhood of x in $K, \bigcup \mathcal{V} \subset U$ and $x \in \bigcap \mathcal{V}$. That completes proof.

As a consequence we obtain the following assertion.

Corollary 2.2. The following properties of a Hausdorff space X are equivalent:

(a) X is an image of a metric space under a quotient $sc-$map;

(b) X is an image of a metric space under a quotient compact-covering sc-map.

Proof. (a) $\Rightarrow$(b). Let $f : M \mapsto X$ be a quotient sc-map from a metric space X. Pick a base $\mathcal{B}$ of M such that $f(\mathcal{B})$ is star-countable. Since f is quotient and an open cover of M, by ([4], Lemma 1.7) and [4], Prop. 2.7), we obtain that X sfies (*) with $\mathcal{F} = f(\mathcal{B})$. Now, applying Theorem 2.1, without loss of generality, can assume that X satisfies (**) with some $\mathcal{F}$. Giving $\mathcal{F}$ the discrete topology the ntable product $\mathcal{F}^\omega$ is metrizable. Denote by $M(\mathcal{F})$ the set of all $(V_n) \in \mathcal{F}^\omega$ such , for some $x \in \bigcap_n V_n$, every neighbourhood of x contains some V_k. We define a $\varphi(\mathcal{F}, X) : M(\mathcal{F}) \mapsto X$ by $\varphi(\mathcal{F}, X)((V_n)) = \bigcap_n V_n$. Also, $\varphi(\mathcal{F}, X)$ is continuous onto (see [5] or [6]). In the proof of ([1], Theorem 1.1) it is shown that $\varphi(\mathcal{F}, X)$ is ient compact-covering. To conclude the proof it suffices to observe that $\varphi(\mathcal{F}, X)$ sc-map ($\varphi^{-1}(\varphi(V))$ is separable if $V = M(\mathcal{F}) \bigcap (V_1 \times V_2 \times \ldots V_k \times \prod_{i=k+1}^{\infty} \mathcal{F}_i)$.

(b) $\Rightarrow$(a) is clear. That completes the proof.

From now on we will consider (1.1)–(1.5)$^+$ with respect to $\mathcal{F}$ being star-countable. erve that as Example 4.2 shows if $\mathcal{F}$ is point-countable (1.5)$^-$ does not imply (1.5). ontrast, for star-countable $\mathcal{F}$ we have the following principle lemma:

Lemma 2.3. Let X be a topological space and (1.1), (1.3), (1.5), $(1.5)^-$, $(1.5)^+$ be considered with a star-countable $\mathcal{F}$. Then

(a) $(1.5)\Leftrightarrow(1.5)^- \Leftrightarrow(1.5)^+$;

(b) $(1.1)\Leftrightarrow(1.3)$.

Proof. Here as well as in the remaining assertions of this section we essentially use some ideas from [3]. The implication of (a) should be compared to ([3], Theorem 8.1).

(a) Let $\mathcal{F}$ be a star-countable cover of X satisfying $(1.5)^-$. Denote by $\mathcal{S}$ all finite subcollections of $\mathcal{F}$. For $\mathcal{V} \in \mathcal{F}$, let

$$M(\mathcal{V}) = \{x \in \text{Int } (\bigcup \mathcal{V}) : x \notin \text{Int } (\bigcup \mathcal{W}) \text{ if } \mathcal{W} \subset \mathcal{V}, \mathcal{W} \neq \mathcal{V}\}.$$

We show that $\mathcal{F}' = \{M(\mathcal{V}) : \mathcal{V} \in \mathcal{S}\}$ is star-countable. Get an arbitrary $M(\mathcal{V}) \in \mathcal{F}'$ and let $M(\mathcal{W}) \in \mathcal{F}'$ such that $M(\mathcal{V}) \bigcap M(\mathcal{W}) \neq \emptyset$. So, we can find $x \in M(\mathcal{V}) \bigcap M(\mathcal{W})$. Let us observe that if $W \in \mathcal{W}$ then $W \bigcap V \neq \emptyset$ for some $V \in \mathcal{V}$, because $x \in \text{Int } (\bigcup \mathcal{W})$, but $x \notin \text{Int } (\bigcup(\mathcal{W} - \{W\})$. Consequently, $\mathcal{W} \subset \{W : W \bigcap (\bigcup \mathcal{V}) \neq \emptyset\}$. Since $\mathcal{F}$ is star-countable we obtain that only countably many $M(\mathcal{W})$ intersect $M(\mathcal{V})$. So $\mathcal{F}'$ is also star-countable. Now, since for $\mathcal{V} \in \mathcal{F}$ $\text{Int } (\bigcup \mathcal{V}) = \bigcup\{M(\mathcal{W}) : \mathcal{W} \subset \mathcal{V}\}$ we obtain that $\mathcal{F}'$ satisfies $(1.5)^+$.

(b) Suppose that $\mathcal{F}$ satisfies (1.3) and $\mathcal{S}$ is the set of all finite subcollections of $\mathcal{F}$. Let

$$\mathcal{P} = \{\bigcup \mathcal{V} : \mathcal{V} \in \mathcal{S}, \bigcap \mathcal{V} \neq \emptyset\}.$$

Then $\mathcal{P}$ is star-countable for $\mathcal{F}$ is star-countable. Then $\{\text{Int } P : P \in \mathcal{P}\}$ is star-countable satisfying (1.1).

Theorem 2.4. The following are equivalent for a countably compact Hausdorff space X:

(a) X is metrizable;

(b) X satisfies $(1.5)^-$ with a star-countable $\mathcal{F}$.

Proof. It follows immediately from ([2], Theorem 2.1) (see also [4]) and Lemma 2.3 (a).

Analogously to [3] we have the following theorem.

Theorem 2.5. Let $f : X \mapsto Y$ be a perfect map. Then if X satisfies (1.5) with a star-countable $\mathcal{F}$ then so does Y.

Proof. Write

$$\mathcal{S} = \{\mathcal{V} : \mathcal{V} \subset \mathcal{F}, \mathcal{V} \text{ is finite }\}.$$

For $\mathcal{V} \in \mathcal{S}$, let

$$M(\mathcal{V}) = \{y \in Y : \mathcal{V} \text{ is a minimal cover of } f^{-1}(y)\}$$

and let

$$\mathcal{F}' = \{M(\mathcal{V}) : \mathcal{V} \in \mathcal{S}\}.$$

Let us show that $\mathcal{F}'$ is star-countable. Indeed, pick an arbitrary $M(\mathcal{V}) \in \mathcal{F}'$ and let $M(\mathcal{W}) \in \mathcal{F}'$ and $M(\mathcal{V}) \bigcap M(\mathcal{W}) \neq \emptyset$. Then, $W \in \mathcal{W}$ implies that $W \bigcap V \neq \emptyset$ for some $V \in \mathcal{V}$. Since $\mathcal{F}$ is star-countable it follows that $\mathcal{P}_\mathcal{V} = \{F : F \bigcap (\bigcup \mathcal{V}) \neq \emptyset\}$ is countable. $\mathcal{F}'$ is star-countable because $\mathcal{W} \subset \mathcal{P}_\mathcal{V}$. Further, as in the proof of [3] we show that $\mathcal{F}'$ satisfies (1.5). Suppose that $x, y \in Y, x \neq y$. Find a finite $\mathcal{V} \in \mathcal{S}$ such that $f^{-1}(x) \subset \text{Int } (\bigcup \mathcal{V})$ and $\bigcup \mathcal{V}$ does not cover $f^{-1}(y)$. Since f is closed there is a neighbourhood U of x with $f^{-1}(U) \subset \text{Int } (\bigcup \mathcal{V})$. Now, let

$$\mathcal{P} = \{M(\mathcal{W}) : \mathcal{W} \subset \mathcal{V}\}.$$

Then $\mathcal{P}$ is a finite subcollection of $\mathcal{F}'$, $y \notin \bigcup \mathcal{P}$ and one can easily see that $U \subset \bigcup \mathcal{P}$.
s, we finish the proof.

3. Spaces with star-countable bases. Let us give some examples of spaces
ng star-countable bases. Each of the following implies that X has a star-countable
:

X is a second countable space;

X is strongly paracompact and every $x \in X$ has an neighbourhood with countable
e.

Also, if X is a space with a point-countable base then X has a star-countable base
nd only if X is locally separable, i.e. each $x \in X$ has a separable neighbourhood.

Lemma 2.3 (b) gives us a characteristic for spaces with star-countable bases. Anal-
usly to [[5], Theorem 1.4) we state the following theorem. Its proof is quite similar
he theorem we have just mentioned.

Theorem 3.1. For a topological space X the following are equivalent:

(a) X has a star-countable base;

(b) X is an image of a space with star-countable base under an open sc-map;

(c) X is an image of a metric space under an open compact-covering sc-map.

4. Examples. Example 4.1. We construct a space X that

(a) is neither $k-$space nor $c-$space;

(b) has a star-countable $\mathcal{F}$ satisfying (1.2), but does not have a star-countable $\mathcal{F}$
isfying (1.1).

Let $X = [0,1]$. $(0,1]$ is with the standard topology and a base at $\{0\}$ is of the
m

$$[0, \frac{1}{n}] \backslash C, \text{ where } C \text{ is countable.}$$

early, X is not first countable, so it does not satisfy (1.1) with some $\mathcal{F}$. Now, let us
fine a star-countable $\mathcal{F}$ that satisfies (1.2). We define that $\mathcal{F}$ as follows.

(a) all elements of a countable base for $(0,1]$ are in $\mathcal{F}$;

(b) if $\{0\} \in F \in \mathcal{F}$ then F is of the form $[0, \frac{1}{n})$.

Example 4.2. ([2], Example 3.1) This example shows that in general $(1.5)^{-}$ does
ot imply (1.5).

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