AN APPROACH FOR PRICING AMERICAN-STYLE DERIVATIVES

DISSERTATION

for awarding of the scientific degree

DOCTOR OF SCIENCE

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Published articles related to the dissertation

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2. Zaevski (2021a) Laplace transforms of the Brownian motion's first exit from a strip. *Comptes rendus de l'Académie bulgare des Sciences*, 74(5):669–676, 2021a. ISSN 2367-6248 (print),2603-4832 (online).
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3. Zaevski (2024a) Some limits for the Laplace transform of the Brownian motion's first hit to a linear function. *Serdica Mathematical Journal*, 50(2):183–202, 2024a. ISSN 1310-6600 (print), 2815-5297 (online).
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12. Zaevski (2023b) Perpetual cancellable American options with convertible features. *Modern Stochastics: Theory and Applications*, 10(4):367–395, 2023b. ISSN 2351-6046 (print), 2351-6054 (online).
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13. Zaevski (2022b) Pricing cancellable American put options on the finite time horizon. *Journal of Futures Markets*, 42(7):1284–1303, 2022b. ISSN 0270-7314 (print), 1096-9934 (online).
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Preface

Derivatives are one of the most important instruments against the financial risks. They are based on some underlying object which can be a single asset, portfolio of assets, financial index, commodities, bonds, debt, other derivatives, volatility (Cboe Volatility Index), etc. All they are constructing to hedge the underlying against different market scenarios giving different rights and obligations for both their writer and holder. Furthermore, the more complicated instruments can be viewed as a transmission between different financial factors – assets, returns, risk, market uncertainty, etc – as well as a regulator between them. The evaluation of these constructions is one of the main tasks in financial mathematics. It is important to mention that the derivative's price is closely related to the value of the underlying object due to the strong dependence between them. The large class of the derivatives includes many different kinds – futures, forwards, options, swaps, some debt instruments, etc.

The options are some of the most tradable derivatives in the modern financial markets. In essence, they are contracts between two participants – writer and holder – that give some rights and obligations to both of them. Particularly, the options depend on an underlying asset and give their holder the right without obligation to sell or buy it at a pre-agreed price, named strike price, until or at some maturity date. The amount that the holder has to pay for this right forms the option's price and the related premium. There are two main types of options – European and American – this distinction is not geographically, but determined by their different characteristics. The main one is the moment they expire. The European contracts can be exercised only at the maturity date. Alternatively, an American option gives its holder the right to choose the expiration date. In such a way, he can capture immediately the payoff when the underlying asset reaches the desirable level. This additional right is very important for the investors and it explains the larger part that the American options have amongst all traded derivatives at the global markets. In addition, many exotic modifications exist – Asian, Bermuda, barriers, capped, straddle, strangle, cancellable, etc.

In modern financial theory, the asset prices are driven by stochastic processes – diffusions, Lévy processes, etc. We work under the log-normal assumptions staying in the base of the Black and Scholes (1973) model. Nonetheless, it does not capture some important practical phenomena observed at all financial markets – heavy tails, sudden jumps, volatility clustering, long-range dependence, leverage effects, etc. – we chose it because it is very intuitive and allows to be generalized. Something more, many of the derived results are true under quite complicated assumptions. Also, the Black-Scholes model is so fundamental, that every new approach has to be applied firstly to it for checking its scientific significance.

There are two main methods for pricing European-style derivatives. A main theorem in mathematical finance states that all discounted by the money-account price processes are martingales under the so-called risk-neutral measure. In a contrast to the natural probability measure that describes the market uncertainty (the so-called real-world measure), the risk-neutral one is a mathematical construction based on some non-arbitrage arguments. It allows the fair derivative price to be obtained as the mathematical expectation of its discounted payoffs. Alternatively, if the derivative price is considered as a function of the time and the current price of the underlying asset, then this function solves a partial differential equation with boundary conditions determined by the payoffs. The form of the equation depends on the stochastic process that describes the underlying asset via its infinitesimal generator. For example, if the price process is log-normal, then we have a parabolic heat-style equation. Alternatively, if it is Lévy one, then we reach a pseudo(integro)-differential equation. In fact, this duality is an application of the well-known relation stated by Kolmogorov – the solution of a differential equation composed by the infinitesimal generator of some Feller-Markov process can be derived as the mathematical expectation of the boundary conditions taken for this process.

On the other hand, the American option pricing problem is more complicated. Obviously, the early exercise feature leads to an optimal stopping task. We need to divide the state space into two subsets. In the first one, the optimal set, the immediate exercise is the best holder's strategy. On the contrary, keeping the option alive leads to a better financial result if the spot price falls in the so-called continuation set. This way the holder exercises when the underlying asset reaches it. The boundary between these sets is known as the optimal or early exercise boundary. Usually, the optimal stopping problems are viewed as free boundary differential tasks through the related variational inequalities. Thus the American options are evaluated by a two-dimensional dynamical system of integral equations which has to be solved numerically. Alternatively, using some first-hitting properties of the Brownian motion, we approximate the optimal boundary maximizing the holder's financial result. As a consequence, we can derive the option price with a high-enough accuracy in a real time. In addition, once the optimal boundary is known, the free boundary differential task turns into a boundary value problem in a known region for which many numerical methods are applicable. The derivatives without maturity restrictions, known as perpetuals, are specific. Their optimal boundaries are time-independent since the holder and writer are not threatened by a forced exercise at maturity. This feature allows solving the pricing problem in a closed form. Although the perpetual derivatives are rarely traded, mainly on some unofficial markets, they provide very important information. The asymptotic characteristics they describe, destine the whole behavior of the optimal boundaries as well as the prices under the finite maturities. .

The plan of the dissertation is as follows. The motivation of this study is presented in Chapter 1. We discuss the current state of the derivatives market as well as the related scientific literature. We pay special attention to the financial instruments considered later in the dissertation. Some propositions about the Laplace transforms of the first hit and exit of the Brownian motion are proved in Chapter 2. Also, some limits are derived. Later, these results are applied to the evaluation of the considered financial instruments. The theoretical base we use later is presented in Chapter 3. The classical American options with additional discount factor are considered in Chapter 4. The so-called caped options are investigated in Chapter 5. Some financial instruments with payoffs that lead to one-sided optimal stopping problems are discussed in 6. The strangle strategies are examined in Chapter 7. The quadratic strangles are investigated in the light of the two-sided optimal exit problems in Chapter 8. The results for the perpetual options (call and put) with a cancellable feature are provided in Chapters 9 and 10. The cancellable options for which the penalty that the writer owes for his early cancelling right is a proportion of the usual option payoff are examined in Chapter 11. Cancellable options with penalties consisting of three parts – a proportion of the usual payoff, some number of shares of the underlying asset, and a fixed amount – are defined and studied in Chapter 12. The pricing task for the finite maturity cancellable options is solved in Chapter 13. Some selected MATLAB codes that implement and validate the theoretical results and the proposed pricing method are given in Chapter 14. We conclude by Chapter 15. The scientific contributions of the dissertation are summarized in Chapter 16.

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Chapter 1

Introduction

1.1 Aim of the dissertation

The main purpose of this work is to investigate the American-stye financial instruments and to construct a fast and accurate method for their analysis and evaluation. Our research is stated in the framework of the famous Black and Scholes (1973) model assuming that the underlying asset is driven by a log-normal stochastic process. The main feature that distinguishes the American from the other derivatives is the early exercise right that the holder may use at an arbitrary moment until maturity. However, namely this characteristic makes the analysys of such instruments difficult due to the decision-making task they lead.

We first aim to examine some classical instruments such as the usual American options as well as their capped versions working with the suggested novel approach. Based on the derived results, we shall systematize mathematically these derivatives w.r.t. to the kind of the stopping problem they lead – one-sided (hitting) task or two-sided (exit) task. We will examine in detail the properties of the so-separated groups. Thus we can fulfill the next purpose, namely to suggest and explore a rigorous mathematical method for defining and investigating new financial derivatives that meet the various needs of the investors. They include strangle strategies, quadratic strangles, powered instruments, cancellable options, and other instruments with generalized payoffs. We shall provide several such examples. Last but not least, we strive to implement the theoretical result via the platform for mathematical computing MATLAB.

1.2 Actuality of the topic

The early exercise right that characterizes the American-style derivatives implies their significance for the financial industry. This fact is supported by the largest part of the traded assets they have in the modern markets. They are important not only as traded sources but also as a regulator of market uncertainty as well as an anti-risk factor. On the other hand, the financial crisis of 2008 led to the appearance of different novel instruments against the arising new risks. Furthermore, the investors need both fast and accurate methods for evaluation. This is even more true in light of the increasing high-frequency trading in the recent decade. All these destine the outstanding importance of contemporary scientific investigations in this field – a fact supported by the increasing literature devoted to the topic.

1.3 Original theoretical and practical findings

The presented dissertation provides several theoretical generalizations as well as solutions of the problems arising from the financial practice.

Two main questions stand for the holder of an American-style financial instrument – is it optimal to exercise immediately and, if not, what is the fair price of the derivative. The main existing methods for evaluating such instruments are based on a dynamic approximation of the pair price-optimal boundary. They are relatively high time consuming. We solve this problem by building a fast and efficient method for approximating the optimal boundary at a rare time grid. As a consequence, we price the option with high precision (fourth sign after the decimal point) in real-time. Thus, we can decide almost immediately whether the moment is suitable for exercising and what the option price is if keeping is preferable. This is of outstanding importance in light of the increasing part of high-frequency trading in modern financial markets.

On the other hand, if we need a denser grid, we construct several methods for solving the pricing problem, which are based on Monte Carlo simulations as well as on various finite difference schemes.

Our methodology is based on several steps. We first solve the pricing task without maturity constraints. Then we approximate the optimal boundaries under a finite maturity horizon maximizing the financial result of the option's holder. As a consequence, we deal with the pricing problem. Later, we discuss this scheme in detail.

The so-establish method is applied to several classical instruments including the options and their capped versions. In addition, we generalize the theoretic results by defining and investigating several new classes of financial derivatives.

Another theoretical contribution of the dissertation is an approach for distinguishing the kinds of the financial derivatives w.r.t. their optimal regions. We differentiate three major types. For the call-style instruments, the optimal points are above some boundary, whereas for the put ones, they are below. In addition, we provide a condition that leads to two optimal sets – above and below the related boundaries. This leads to optimal strategies that are an exit from a strip.

To develop all the above-mentioned constructions, we need several results related to the Laplace transform of the first hitting time of a Brownian motion, or more precisely of its density. In probabilistic terms, this is the moment generating function of the stopping time. We prove the necessary statements in a separate chapter.

The practical findings of the dissertation are in several directions. First, we apply the theoretical results to some existing American-style instruments – classical options, capped options, strangle strategies, cancellable options, etc. Second, we build a method for investigating derivatives with generalized payoffs distinguishing put- and call-style instruments as well as hybrid two-sided ones. Particularly, we suggest several new American-style derivatives such as strangles with arbitrary strikes and different weights for the call and put legs, quadratic strangles, power futures contracts, cancellable options with a convertible penalty, etc. These instruments can address many requirements of the investors. The arbitrary strikes as well as the different weights in the strangles allow the construction of many novel strategies combining the put and call features. The proposed quadratic strangles allow stronger hedging of the risky positions that are far-from-the-money. The opposite is true for the near-the-money positions. This is achieved by a payoff changed from $|x - K|$ to $(x - K)^2$. These conclusions hold for the power futures contracts too. Something more, the investigated generalized payoffs can be used for the specific needs of an investor. On the other hand, the cancellable options give their writer a possibility for early canceling. The price of this right is a penalty amount above the usual option payoff. We enlarge the flexibility of these instruments by considering three component penalties – a fixed amount, some share of the underlying asset, and a proportion of the payoff.

The so-established theoretical methods are practically validated via software – we use the platform for mathematical computing MATLAB. The closed-form formulas for the perpetual instruments are implemented. On the other hand, we approximate the optimal boundaries for the finite maturities and thus provide a relatively fast pricing method. In addition, we construct different numerical approaches based on Monte Carlo simulations or different finite difference schemes based on significantly denser grids. These tools are prepared for all studied options – classical ones, capped options, powered instruments, strangles and quadratic strangles, cancellable options, etc. Something more, the proposed methods can be applied to different new instruments.

1.4 Motivation and classical methods

In recent years and even more so after the financial crisis of 2008, there has been an increased interest in the international financial markets to the financial derivatives since they are one of the major instruments against financial risk. They exhibit a very large variety – most popular are options, futures, bonds, swaps, etc. Conventionally speaking, we can recognize two types – European and American. The European derivatives have a previously fixed date at which the transaction is executed. Alternatively, the American-style instruments give its holder the right to choose when to exercise until maturity. This right makes the American derivatives preferable for the investors and determines the largest segment of the traded assets they have in the modern financial markets. Many kinds of such instruments are available, a fact which leads to a growing scientific literature devoted to the topic. Since the holder has the right to choose the exercise moment, it is natural to assume that he will follow a strategy that maximizes his profit. Thus we reach to a problem for optimal stopping – see Lamberton and Lapeyre (1996) or Wong (1996). These problems are solutions of the so-called free boundary or obstacle problems – see Bensoussan (1984), Jaillet et al. (1990), Kim (1990), Jacka (1991), Peskir and Shiryaev (2006), Pascucci (2008), or Magirou et al. (2020). For these tasks, we know the differential equation and we have to find its solution as well as the region in which it holds. The theory for this kind of equations can be found in Bather (1970), Moerbeke (1973), Friedman (1975), Friedman (2010), and Shiryaev (2009). Deriving a closed-form solution is hard and often impossible except when there are no maturity constraints. For this many authors propose different numerical solutions. Cox et al. (1979) suggest a very useful one based on the binomial trees. Other interesting numerical methods are proposed by Johnson (1983), Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Bjerksund and Stensland (1993), Ho et al. (1994), Ju (1998), and Longstaff and Schwartz (2001). We mention also the works of Brennan and Schwartz (1977a), Myneni (1992), Karatzas (1988), Rogers (2002), and Huang et al. (1996). They use a recursive method based on the classical works of Kim (1990), Jacka (1991), and Carr et al. (1992). Some comparisons are presented in Zhao (2018). On the other hand, the optimal boundary is time-independent for the perpetual options, which allows deriving the closed-form formulas – see Shiryaev et al. (1995a) or Shiryaev et al. (1994a), in addition to the mentioned above works. Some formulas for Lévy driven models can be found in Gerber and Shiu (1994), Pham (1997), Gerber and Shiu (1996), Mordecki (1999, 2002), Boyarchenko and Levendorskii (2002), Levendorskiĭ (2004), Alili and Kyprianou (2005), and Ivanov (2007). An overview of the credit derivatives is provided in Popchev and Radeva (2008).

1.5 American options in the contemporary scientific literature

Many authors suggest new approaches for pricing American derivatives in the recent years. Various assumptions for the underlying asset are considered – Gaussian, Lévy, stochastic volatility, fractional Brownian, etc. Rad et al. (2015) use a radial basis point interpolation method. Yoon (2015) examines perpetual options with a stochastic elasticity of the variance. Yu and Xie (2015) present an entropy based model with incorporated risk-neutral moments. Backward stochastic differential equations are used in Klimsiak et al. (2016). An approach, based on integral equations, is applied to American style Parisian down-and-out options in Zhu et al. (2018a) and Le and Dang (2017). Some drawdown options without maturity restrictions are studied in Zhang et al. (2021a). The Laplace-Carson transform is used in Park and Jeon (2017) and Kang et al. (2017a). An interpolation method based on radial basis functions is applied in Heidari and Azari (2018). Some minmax theorems for American options related to the lower Snell envelope minimax theorem can be found in Belomestny et al. (2019). A viscosity solution method is applied to the passport options in Wang et al. (2019). An independent from the model pricing approach is suggested in Hobson and Neuberger (2017) and Hobson and Norgilas (2019). Three methods for pricing American options in the presence of stochastic volatility are presented in Balajewicz and Toivanen (2017), Gong and Zhuang (2017), and Kozpınar et al. (2020) – there are introduced in addition jumps in the first two papers. Other models that exhibit jump behavior are presented in Deng (2015a), Haghi et al. (2018), Gan et al. (2020), Boen and in 't Hout (2020), Company et al. (2020), Thakoor et al. (2020), and Yang (2020). A numerical method for solving the arising partial integro-differential equations in another jump model, namely Kou's jump-diffusion model, can be found in Gan et al. (2020). A two dimensional Lévy framework is used in Chen and Wang (2017), whereas Chen et al. (2018) use stable processes in the Black-Scholes framework. Some Monte-Carlo methods can be found in Broadie and Glasserman (1997), Longstaff and Schwartz (2001), Rogers (2002), Cortazar et al. (2008), and Zaevski (2021b). Some other software algorithms and Monte Carlo simulations for different kind of options – European, American, basket, and others – are presented in Popchev and Velinova (2001), Popchev and Velinova (2003a), and Popchev and Velinova (2003b). Several results about the put-call duality are provided in Peskir and Shiryaev (2002), Fajardo and Mordecki (2006), Eberlein et al. (2008), and Nunes et al. (2020). The pricing-hedging duality in a discrete market is considered in Aksamit et al. (2019). The valuation of American options is considered as an optimal stopping problem in Armerin (2019a) and Dautov et al. (2020). A specific style lookback options, named watermark, are examined in Rodosthenous and Zervos (2017). In addition to the running maximum dependence Egami and Oryu (2017) introduces an Asian features. Perpetual lookback American options under a stochastic volatility are examined in Deng (2020); other lookback examples can be found in Woo and Choe (2020) and Zhang et al. (2021b). Some stochastic volatility models based on the Heston framework are presented in Deng (2015b), Mehrdoust et al. (2018), Safaei et al. (2018), Mollapourasl et al. (2019), Jeon et al. (2020), Mohamed (2021), Ma et al. (2021), Mehrdoust et al. (2021), Zhang et al. (2022), and Yan et al. (2022). Alternatively, Simonato (2019) introduces non-normal innovations in a GARCH model. A possibility of a credit risk is assumed in Chen and Wang (2017), Dumitrescu et al. (2018), Company et al. (2024), and Li and Wu (2024). Options with a random start are considered in Armerin (2019b) and Pereira and Rodrigues (2019). In

Zhao and Yang (2018) are presented semi-smooth Newton methods for pricing American options. Systems of partial integro-differential equations are used in Yousuf et al. (2018). The Brennan-Schwartz algorithm is applied to American option pricing in Madi et al. (2018). A Fourier-Padé method for pricing and hedging early exercise options – American, Bermuda, etc – is applied in Chan (2020). Another Fourier approach is presented in Chan (2019). An asymptotic expansion approach is presented in Li and Ye (2019). Gao et al. (2020) and Jeon and Kim (2022a) examine American better-of options on two assets. Options with asset-dependent discounting are considered in Al-Hadad and Palmowski (2024). The so-called variable annuity contracts are studied via the arising variational inequalities in Jeon and Kim (2023). Some first-touch digitals are considered in Lee et al. (2024). The pricing problem for American power options written on a non-dividend underlying asset is examined in Lee (2020). The Mellin transform method is applied for pricing American power put options in Nwozo and Fadugba (2014), Fadugba and Nwozo (2015), and Fadugba and Nwozo (2020).[1] Also, some authors examine the power version of the exchange options – see Blenman and Clark (2005) and Miao et al. (2016). American barrier options are considered in Deng (2014), Park and Jeon (2017), Nunes et al. (2020), Gao et al. (2021), and Lin and He (2021). Some CEV (constant elasticity of variance) models can be found in Ballestra and Cecere (2016), Cruz and Dias (2020) and Lee (2021). Zaevski (2019a) and Michael (2020) focus on the so-called early exercise premium which is the price of the option's holder early exercise right. Several regime-switching models can be found in S. Heidari and Azari (2018), Lu and Putri (2020), Jeon et al. (2020), Song et al. (2022), Shirzadi et al. (2023), Huang et al. (2024), and Zheng and Zhu (2025). The quanto options are examined in Battauz et al. (2022a). The American option valuation problem in the fractional Black-Scholes market is considered in Chen et al. (2019) and Nuugulu et al. (2024). A deep neural network for deriving the American option prices as well as the corresponding deltas is applied in Chen and Wan (2021). A model with stochastic spreads is examined in Arregui et al. (2020). The optimality of the early exercise near the maturity is discussed in Battauz et al. (2022b). A model under the stochastic interest rates is presented in Battauz and Rotondi (2022). Some finite difference methods for American option valuation are presented in Floc'h

[1] A little clarification is needed for the derived optimal boundary for the perpetual option written on a non-dividend asset.

(2014), Chernogorova et al. (2018), Soleymani et al. (2018), Cen and Chen (2019), and Krzyżanowski and Magdziarz (2021). Other numerical methods are provided in Abdi-Mazraeh and Khani (2018), Moradipour and Yousefi (2018), Zhang et al. (2020), Battauz and Rotondi (2024) Han and Zheng (2024), Song et al. (2024),Huang et al. (2024), Nwankwo et al. (2024), and Shen et al. (2024). The errors of the Euler finite element approximation of American calls are studied in Dautov and Lapin (2020). The option pricing problem under negative interest rates is studied in Healy (2014) and Battauz et al. (2015). The cancellable American options are discussed in Guo et al. (2020) and Palmowski and Stępniak (2023). Multi-person game derivatives are introduced and examined in Guo and Rutkowski (2017). Other options leading to two sided optimal stopping problems are studied in Ma et al. (2018), Jeon and Oh (2019), Donno et al. (2020), Qiu (2020), Xu and Ye (2021), Palmowski et al. (2021), Goard and AbaOud (2022), and Jeon and Kim (2022b). The hedging task for the American options in the presence of transaction costs is studied in Goudenège et al. (2024). Some acceleration clauses that reduce the option life are considered in Battauz and Staffolani (2024). Some Monte Carlo simulations are discussed in Reesor et al. (2024). American put options written on zero-coupon bonds are studied in Zhang et al. (2024a). The Greeks of the American options are discussed in in 't Hout (2024). Some stochastic liquidity and jump risks for American options are investigated in Zhang et al. (2024b). The American strangles are studied under a stochastic volatility and fast mean reversion in Ha et al. (2025).

Another important class of early exercise derivatives are the so-called convertible bonds. Their importance is due to the mixed nature they exhibit — a debt which can be converted to stocks. For some classical works in this area, we refer to Brennan and Schwartz (1977b), Tsiveriotis and Fernandes (1998), Ayache et al. (2003), and Spiegeleer et al. (2011). A historical overview of the CoCo market trends is provided in Verwijmeren and Yang (2020). A general evaluation method is presented in Batten et al. (2018). Some numerical – binomial trees and finite difference – methods for pricing CoCo bonds (abbreviated from *contingent convertible*) with and without credit risk are presented in Milanov et al. (2012) and later in Milanov et al. (2019). A modification of the CoCo bonds, named CoCoCat, is considered in Burnecki et al. (2019). They depend on some catastrophe stochastic event, not related to the financial market risks. Also, a financial decision technique is applied in Del Viva and El Hefnawy (2020) under an assumption of a possible default. An integral approach for pricing convertible bonds with puttable features is presented in Zhu et al. (2018b). A model in accordance with the Chinese specifics is presented in Ma et al. (2020). Resettable convertible bonds are considered in Lin and Zhu (2019). The Heston stochastic volatility framework together with the Cox–Ingersoll–Ross interest rate term structure is used in Lin and Zhu (2020) for a numerical evaluation. Some specifics of the Western European markets are studied in Adoukonou et al. (2018) and Adoukonou et al. (2021). The relations of the CoCo bonds with different market characteristics as the underlying asset, credit default swap spreads, interest rates, implied volatilities and etc. are studied in Zeitsch and Davis (2021). The option featues of the convertible bonds are considered in Jin et al. (2023). The relation between the convertible bonds and stock returns is examined in Chen et al. (2023).

1.6 American financial instruments

In fact, the American-style derivatives are a specific extension of the European derivatives, where the latter gives the holder the right to receive some previously defined amount $N(x)$ if at the maturity date T, the underlying asset has value $S_T = x$. In that way, the set of moments at which the exercise is permitted is simply $\mathcal{U} = \{T\}$. The generalization that can be made is to enlarge the set $\mathcal{U}$. If the set $\mathcal{U}$ is discrete we have a Bermuda derivative, whereas if the set is $\mathcal{U} = [0, T]$ we have an American derivative. Since the holder can exercise at any moment we have to define the payment structure $N(t, x)$. It represents the amount which the holder will receive if he decides to exercise at moment t given the underlying asset has value $S_t = x$. If the derivative is an American call option the holder has the right to buy the underlying asset at a previously defined price of K in an arbitrary moment before the maturity. The American put gives the holder the right to sell. Thus the payment structures are $N(t, x) = (x - K)^+$ and $N(t, x) = (K - x)^+$, respectively. A time dependence is introduced through a new discount factor. This way the payment structures turn into $N(t, x) = e^{-\lambda t}(x - K)^+$ and $N(t, x) = e^{-\lambda t}(K - x)^+$, i.e. the holder has a benefit to exercise earlier. In such a way these options are related to the financial instruments which lose some value over time. Also, in Zaevski (2020b), proposition 2.3, is proven that a model with continuous dividend payments can be written as a non-dividend model under a new parametrization. Hence, we can assume that the dividend rate is zero without loss of generality. Note also that the mentioned above change of parameters may lead to a negative risk-free rate. The set $[0, T] \times \mathbb{R}^+$ which consists of all admissible values (t, S_t), can be divided into two parts. In the first part, the immediate exercise is optimal, whereas in the second one keeping the derivative is preferable. The boundary between them is called early exercise boundary or, for the sake of simplicity, only exercise boundary.

To meet the financial investors' expectations for additional protection, several instruments are listed in the markets. They preserve the derivative's writer against the extremely high movements of the underlying asset. The so-called cancellable American options protect the writer's interest giving him an early exercise right. Alternatively, other derivatives preserve the writer by adding a cap level limiting the price of the underlying asset at which the option can be exercised. A seminal work for these options based on the Black-Scholes framework is Broadie and Detemple (1995). Later Detemple and Tian (2002) examines capped options under a general diffusion process. Several researchers use the capped style options to evaluate the ordinary American options – see Broadie and Detemple (1996) and Deng and Peng (2014).

As we mentioned above, the cancellable options give the writer the right to cancel the derivative prematurely. These options, also known as game or Israeli, are introduced by Kifer (2000). They are a particular case of the so called Dynkin games, see Dynkin (1969) or Dynkin (1969). Along with Kifer (2000) and Kifer (2013), there are several seminal works in the area. Kunita and Seko (2004) examine the exercise regions for the game call options. In Baurdoux and Kyprianou (2004) and Meyer (2016) game options are discussed from the point of view of the free boundary problems. This method is a particular case of the stochastic differential games examined in Bensoussan and Friedman (1977). The variational inequalities are applied in Kyprianou (2004) and Kühn and Kyprianou (2007). Kallsen and Kühn (2004) and Kühn (2004) examine incomplete markets. The so-called excessive functions are used in Ekström and Villeneuve (2006) and Ekström (2006). A path-wise approach can be found in Kühn et al. (2007). Hedging strategies for evaluating the game options are used in Hamadène (2006) and Y. Dolinsky and Kifer (2007). Some other examples of game put options are presented in Kyprianou (2004), and Suzuki and Sawaki (2007). Emmerling (2012) and Yam et al. (2014) study options whose underlying asset pays dividends. The relation between the game options and the backward stochastic differential equations can be found in Matoussi et al. (2014). Path dependent options, namely look-back and Asian, are considered in Guo et al. (2014), Guo (2020), and Guo et al. (2020). Cancellable options in the presence of a credit risk are studied in Dumitrescu et al. (2017). Transaction costs in a multi-asset model are admited in Roux (2016) and Roux and Zastawniak (2018). The hedging problem is studied in Dolinsky and Kifer (2016) and Dolinsky (2020). The cancellable options in a regime switching diffusion framework are examined in Lv et al. (2020). Some specific perpetual options are discussed in Gapeev et al. (2021).

Principally, the cancellable options lead to two-sided optimal stopping problems related to the first exit from a strip. The exit from one of the boundaries is the optimal strategy for the holder, whereas the exit from the other is optimal for the writer.

Another class of derivatives, that leads to two-sided optimal problems are the straddles and strangle strategies. These derivatives are hybrid strategies including both of a call and put option. They appear as a financial instrument that preserves against the financial risk when the investor expects some large deviations but he is not sure of the direction of these movements. Usually, the options without maturity restrictions are easier to research due to the absence of the coercive exercise at maturity. As a consequence, the optimal boundaries are flat. The first results for these perpetual options can be found in Beibel and Lerche (1997) and Shiryaev (1999), see also Gapeev and Lerche (2011) and Qiu (2020). Later, several authors turn to the strangle pricing problem under a finite maturity horizon. A major subclass of such options is the so-called straddles. Their main feature is the matching put and call strikes. A Laplace transform method is applied to the American straddles in Alobaidi and Mallier (2002). The same authors examine the behavior of these options near maturity – see Alobaidi and Mallier (2006). Chiarella and Ziogas (2005) overcome some problems with the Laplace transform using the Fourier one. Alternatively, Kang et al. (2017b) use the Laplace-Carlson transform. An approach based on deriving the limits for the boundaries by the use of capping is presented in Ma et al. (2018). The variational inequalities method is presented in Jeon and Oh (2019). This method is closely related to the corresponding two-sided free boundary differential problem that describes the strangle pricing task. Another such approach is presented in Qiu (2020) – the related integral equations are derived via the so-called early exercise premium (EEP). Also, Abdou and Moraux (2016) uses an EEP method for pricing and hedging. Recently, Jeon and Kim (2022b) derived an analytic valuation formula under mean-reversion assumptions.

At last but not least, we discuss financial derivatives with general payoff structures using the technique of the infinitesimal generators. We need to distinguish the payoffs that lead to one- or two-sided optimal stopping problems and, in addition, when an one-sided problem is related to the first hit of the asset to the boundary above or below. This characterization allows many novel instruments to be proposed and studied under our framework.

1.7 Methodology

The financial instruments considered in this dissertation are studied through the methodology summarized in the following steps:

1. We obtain the shape of the optimal regions and the related early exercise boundaries.

2. We solve the pricing problem in a closed form when there are no maturity constraints.

3. We derive the pricing function of a derivative that expires when the log-price of the underlying asset reaches a piecewise linear boundary or exits from a strip formed by such functions.

4. We approximate the optimal boundary in a rare time grid. Thus we recognize whether the immediate exercise is optimal or not.

5. We derive the option price based on this approximation. This is a fast and relatively accurate procedure – the error is in the fourth sign after the decimal point.

6. We approximate the whole boundary if we need a denser grid.

7. We solve the related Black-Scholes style equation numerically by a finite difference scheme or by a Monte Carlo simulation.

Chapter 2

First hitting time properties

based on the papers

Zaevski, T. (2020). Laplace transforms for the first hitting time of a Brownian motion. Comptes rendus de l'Académie bulgare des Sciences, 73(7), 934-941.

Zaevski, T. S. (2021). Laplace Transforms of the Brownian Motion's Fists Exit from a Strip. Comptes rendus de l'Académie bulgare des Science, 74(5), 669-676.

Zaevski, T. (2024) Some limits for the Laplace transform of the Brownian motion's first hit to a linear function, Serdica Mathematical Journal. Sofia, Bulgaria, 50(2), pp. 183–202. doi: 10.55630/serdica.2024.50.183-202.

This chapter contains some results about the stopping times related to the Brownian motion's first hit to a piece-wise linear function or its first exit from a strip. These results are used later for approximating the optimal boundaries and, as a consequence, for deriving the fair derivatives prices.

2.1 Motivation and main results

The problem of first hitting of a Brownian motion to some boundary is examined in many studies – see for example Wang and Pötzelberger (1997), Donchev (2007), and Jin and Wang (2017). We investigate now the Laplace transform of the first hitting time to some piecewise linear function. We introduce also a terminal date and examine the lower between the hitting time and the terminal moment. Regardless of whether the hitting time is before the terminal date or not, we shall only know either the stopping time value or the value of the Brownian motion. This requires the separate examination of both cases. There are many reasons that motivate us to do this research. First, the Laplace transform is just the moment generating function of the stopping time and therefore it is a powerful tool for examining its distribution, moments, and cumulants. Second, if we approximate a curve by some piecewise linear functions, we can numerically derive the corresponding results for first hitting to this curve. Third, the wide use of the Brownian motion in different real life areas implies the practical significance of the derived results.

For instance, if a financial market is described by the famous Black-Scholes framework, the asset price is presented by a geometric Brownian motion. Thus the Laplace transform appears in many path dependent financial derivatives. If the hitting time is before the terminal date, then we know the value of the Brownian motion and hence we have to derive the Laplace transform of the hitting time. Otherwise, if the hitting time is after the terminal moment, then the stopping time is equal to the terminal value. Thus we have to derive the Laplace transform of the Brownian motion provided that it does not hit the barrier before the terminal moment.

Also, we consider the first exit time of a Brownian motion from a strip – for some results, we refer to Anderson (1960), Pötzelberger and Wang (2001), and Donchev (2010). We assume now that the strip boundaries are continuous piecewise linear functions allowing some of them to vanish in some intervals.

We are interested also in some limits of the form $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T}\right]$ restricted on the sample paths at which the Brownian motion still does not hit a linear boundary. This term is related to the moment generating function of the normal distribution. Also, we consider an additional restriction only at the moment T by another linear function. We can proceed in two manners using the symmetry of the Brownian motion. First, we may examine only upper-hitting tasks and arbitrary constant θ. Alternatively, we may assume that $\theta > 0$ and consider both lower and upper hits. We chose the second scheme.

This chapter is organized as follows. We discuss some Laplace transforms of the first hitting moment of a Brownian motion to a piece-wise linear boundary in Sections 2.2 and 2.3. The first exit from similar strips is considered in Section 2.4. We provide some important limits in Section 2.5.

2.2 Laplace transforms for the first hitting time of a Brownian motion

2.2.1 Preliminaries

Let B_t be a Brownian motion and T be the terminal moment, $T \leq \infty$. Let $0 \equiv t_0 < t_1 < ... < t_n \equiv T$ be a time grid and $b(t)$ be a piecewise linear function on it

$$b(t) = \sum_{i=1}^{n} (b_{1,i}t + b_{2,i}) I_{t \in [t_{i-1}, t_i]}, \tag{2.1}$$

where I is the indicator function. Note that it is sufficient to examine the upper version of the problem due to the symmetry of the Brownian motion. Let $\beta_i = b(t_i)$ and therefore $\beta_0 > 0$. We shall assume also that function (2.1) is continuous, $b_i(t_i) = b_{i-1}(t_i)$, but the presented results can be easily generalized if it has discontinuities. We shall denote by τ the first hitting time to the function $b(\cdot)$. Let us notate by Λ_t the indicator process $\Lambda_t = I_{\tau \leq t}$ and by $N(\cdot)$ the cumulative distribution function of the standard normal distribution.

2.2.2 Linear case

Suppose that the function $b(t)$ is linear. Note that $b_2 \equiv \beta_0 > 0$. We shall use several times the following result, which is reported in Borodin and Salminen (2015), equation (2.0.2) on page 223.

Lemma 2.1. *The probability density function of τ is*

$$p(t) = \frac{b_2}{\sqrt{2\pi} t^{\frac{3}{2}}} \exp\left(-\frac{(b_1 t + b_2)^2}{2t}\right). \tag{2.2}$$

An immediate corollary is the form of the cumulative distribution function.

Proposition 2.1. *The cumulative distribution function of τ, which we shall notate by $g(\cdot)$, is given by the equation*

$$g(T; b_1, b_2) \equiv \mathbb{P}(\tau < T) = 1 - N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) + \exp(-2b_1 b_2) N\left(\frac{b_1 T - b_2}{\sqrt{T}}\right). \tag{2.3}$$

Proof: We have $g(0; b_1, b_2) = 0$ since $b_2 > 0$. To finish the proof, we need to check that the derivative $g_t(t; b_1, b_2)$ is given by density (2.2). Having in mind

$$\left(\frac{b_1 T \pm b_2}{\sqrt{T}}\right)' = \frac{1}{2\sqrt{T}}\left(b_1 \mp \frac{b_2}{T}\right),$$

we derive

$$\begin{aligned} g_t(T; b_1, b_2) &= -\exp\left(-\frac{(b_1 T + b_2)^2}{2T}\right) \frac{b_1 T - b_2}{2\sqrt{2\pi} T^{\frac{3}{2}}} \\ &\quad + \exp(-2b_1 b_2) \exp\left(-\frac{(b_1 T - b_2)^2}{2T}\right) \frac{b_1 T + b_2}{2\sqrt{2\pi} T^{\frac{3}{2}}} \\ &= \frac{b_2}{\sqrt{2\pi} T^{\frac{3}{2}}} \exp\left(-\frac{(b_1 T + b_2)^2}{2T}\right). \end{aligned}$$

This way we finish the proof. □

Our first result is established in the following theorem.

Theorem 2.1. *[Theorem 3.1 from Zaevski (2020a)] Let $\theta > 0$. The Laplace transform of τ before T is given by*

$$L(T,\theta;b_1,b_2) = \mathbb{E}\left[e^{-\theta\tau}\Lambda_T\right] = e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} g\left(T;\sqrt{b_1^2+2\theta},b_2\right), \tag{2.4}$$

where the function $g(\cdot)$ *is given by equation* (2.3).

Proof: Using Lemma 2.1 and Proposition 2.1 we derive

$$\begin{aligned}
\mathbb{E}\left[e^{-\theta\tau}\Lambda_T\right] &= \int_0^\infty e^{-\theta t} I_{t\le T} \frac{b_2}{\sqrt{2\pi}t^{\frac{3}{2}}} \exp\left(-\frac{(b_1 t + b_2)^2}{2t}\right) dt \\
&= e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} \int_0^T \frac{b_2}{\sqrt{2\pi}t^{\frac{3}{2}}} \exp\left(-\frac{\left(\sqrt{b_1^2+2\theta}t+b_2\right)^2}{2t}\right) dt \\
&= e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} g\left(T;\sqrt{b_1^2+2\theta},b_2\right).
\end{aligned} \tag{2.5}$$

□

The following corollary gives the results if there is not a terminal moment.

Corollary 2.1. *Suppose that* $T=\infty$. *The probability* τ *to be finite is*

$$g(\infty;b_1,b_2) \equiv \mathbb{P}(\tau<\infty) = \begin{cases} 1, & \text{if } b_1 \le 0 \\ \exp(-2b_1 b_2), & \text{if } b_1 > 0. \end{cases}$$

The corresponding Laplace transform is given by

$$L(\infty,\theta;b_1,b_2) = \mathbb{E}\left[e^{-\theta\tau} I_{\tau<\infty}\right] = e^{-b_2\left(\sqrt{b_1^2+2\theta}+b_1\right)}. \tag{2.6}$$

Although we need only the case $\theta > 0$ in Theorem 2.1, we present below the results without this restriction. To do this, we need to define the error function in the complex plane. If $z = x + iy$, then we change the variables as $v = \frac{u}{z} \Leftrightarrow u = vz$ to derive

$$
\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du = (x+iy)\int_0^1 e^{-(x+iy)^2v^2}dv \\
&= \frac{2}{\sqrt{\pi}}(x+iy)\int_0^1 e^{\left(y^2-x^2\right)v^2}\left[\cos\left(2xyv^2\right) - i\sin\left(2xyv^2\right)\right]dv \\
&= \frac{2}{\sqrt{\pi}}\int_0^1 e^{\left(y^2-x^2\right)v^2}\left[x\cos\left(2xyv^2\right) + y\sin\left(2xyv^2\right)\right]dv \\
&+ i\frac{2}{\sqrt{\pi}}\int_0^1 e^{\left(y^2-x^2\right)v^2}\left[y\cos\left(2xyv^2\right) - x\sin\left(2xyv^2\right)\right]dv.
\end{aligned}
\tag{2.7}
$$

Proposition 2.2. *The following statements hold.*

1. *If $\theta < -\frac{b_1^2}{2}$, then*

$$
L(T,\theta;b_1,b_2) = e^{-b_1b_2}\left[\begin{array}{l} \cos(cb_2) + \frac{e^{-icb_2}}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(ic\sqrt{T} - \frac{b_2}{\sqrt{T}}\right)\right) \\ -\frac{e^{icb_2}}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(ic\sqrt{T} + \frac{b_2}{\sqrt{T}}\right)\right) \end{array}\right],
\tag{2.8}
$$

   where $c = \sqrt{|b_1^2 + 2\theta|}$. Note that presentation (2.8) is indeed a real function – see Corollary 2.2 below.

2. *If $\theta = -\frac{b_1^2}{2}$, then $L(T,\theta;b_1,b_2) = 2e^{-b_1b_2}N\left(-\frac{b_2}{\sqrt{T}}\right)$.*

3. *If $\theta > -\frac{b_1^2}{2}$, then the Laplace transform is given by formula (2.4).*

Remark 2.1. *Note that function (2.8) is analytical since the expectation in (2.4) defines an entire function.*

Proof: The third statements is already proven in Theorem 2.1. Applying formula (2.5) to the case $\theta = -\frac{b_1^2}{2}$ and changing the variables $y = \frac{b_2}{\sqrt{t}}$, we derive

$$\begin{aligned}
L(T,\theta;b_1,b_2) &= e^{-b_1 b_2}\int\limits_0^T \frac{b_2}{\sqrt{2\pi}t^{\frac{3}{2}}}\exp\left(-\frac{\left(\sqrt{b_1^2+2\theta}t+b_2\right)^2}{2t}\right)dt \\
&= \frac{2e^{-b_1 b_2}}{\sqrt{2\pi}}\int\limits_{\frac{b_2}{\sqrt{T}}}^{+\infty} e^{-\frac{y^2}{2}}dy \\
&= 2e^{-b_1 b_2}\left(1-N\left(\frac{b_2}{\sqrt{T}}\right)\right) \\
&= 2e^{-b_1 b_2}N\left(-\frac{b_2}{\sqrt{T}}\right).
\end{aligned}$$

The last statement can also be derived by taking $b_1 = 0$ in function (2.3).

Assume now $\theta < -\frac{b_1^2}{2}$ and $c = \sqrt{|b_1^2 + 2\theta|}$. Lemma 2.1 and Proposition 2.1 leads to

$$\mathbb{E}\left[e^{-\theta\tau}\Lambda_T\right] = \int_0^T e^{-\theta t}\frac{b_2}{\sqrt{2\pi}t^{\frac{3}{2}}}\exp\left(-\frac{(b_1 t + b_2)^2}{2t}\right)dt$$
$$= e^{-b_1 b_2}\frac{b_2}{\sqrt{2\pi}}\int_0^T \frac{1}{t^{\frac{3}{2}}}\exp\left(-\frac{1}{2}\left(\left(b_1^2 - 2\theta\right)t + \frac{b_2}{t}\right)\right)dt$$
$$= e^{-b_1 b_2} b_2\sqrt{\frac{2}{\pi}}\int_0^{\sqrt{T}} \frac{1}{y^2}\exp\left(-\frac{1}{2}\left(\left(b_1^2 - 2\theta\right)y^2 + \frac{b_2}{y^2}\right)\right)dy$$
$$= \frac{e^{-b_1 b_2}}{\sqrt{2\pi}}\left[\begin{array}{c} e^{-icb_2}\int_0^{\sqrt{T}}\exp\left(-\frac{1}{2}\left(icy - \frac{b_2}{y}\right)^2\right)\left(c + \frac{b_2}{y^2}\right)dy \\ -e^{icb_2}\int_0^{\sqrt{T}}\exp\left(-\frac{1}{2}\left(icy + \frac{b_2}{y}\right)^2\right)\left(c - \frac{b_2}{y^2}\right)dy \end{array}\right]$$
$$= \frac{e^{-b_1 b_2}}{\sqrt{\pi}}\left[\begin{array}{cc} e^{-icb_2} & \int_{-\infty}^{\frac{1}{\sqrt{2}}\left(ic\sqrt{T} - \frac{b_2}{\sqrt{T}}\right)}\exp\left(-x^2\right)dx \\ +e^{icb_2} & \int_{\frac{1}{\sqrt{2}}\left(ic\sqrt{T} + \frac{b_2}{\sqrt{T}}\right)}^{+\infty}\exp\left(-x^2\right)dx \end{array}\right]$$
$$= e^{-b_1 b_2}\left[\begin{array}{c} e^{-icb_2}\left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(icT - \frac{b_2}{T}\right)\right)\right) \\ +e^{icb_2}\left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(icT + \frac{b_2}{T}\right)\right)\right) \end{array}\right]$$
$$= e^{-b_1 b_2}\left[\cos\left(cb_2\right) + \frac{e^{-icb_2}}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(ic\sqrt{T} - \frac{b_2}{\sqrt{T}}\right)\right) - \frac{e^{icb_2}}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(ic\sqrt{T} + \frac{b_2}{\sqrt{T}}\right)\right)\right].$$

We have used above the change of variables

$$y = \sqrt{t}$$
$$x = \frac{1}{\sqrt{2}}\left(icy - \frac{b_2}{y}\right)$$
$$x = \frac{1}{\sqrt{2}}\left(icy - \frac{b_2}{y}\right).$$

Also, the integration in the complex plain is made on the contours $\left(-\infty \to 0 \to \frac{1}{\sqrt{2}}\left(icT - \frac{b_2}{T}\right)\right)$

and $\left(\frac{1}{\sqrt{2}}\left(icT+\frac{b_2}{T}\right)\to 0\to\infty\right)$. □

Corollary 2.2. *If $\theta < -\frac{b_1^2}{2}$, then formula (2.8) can be written as*

$$L\left(T,\theta;b_1,b_2\right) = e^{-b_1 b_2}\left[\begin{array}{l}\cos\left(cb_2\right)\\ +\sqrt{\frac{2}{\pi}}\int\limits_0^1 \exp\left(\frac{1}{2}\left(Tc^2-\frac{b_2^2}{T}\right)v^2\right)\left[\begin{array}{l}\cos\left(b_2 c\right)\left[y\sin\left(b_2 cv^2\right)-x\cos\left(b_2 cv^2\right)\right]\\ -\sin\left(b_2 c\right)\left[x\sin\left(b_2 cv^2\right)+y\cos\left(b_2 cv^2\right)\right]\end{array}\right]dv\end{array}\right],$$

where

$$c=\sqrt{\left|b_1^2+2\theta\right|}$$
$$x=\frac{b_2}{\sqrt{T}}$$
$$y=\sqrt{T}c.$$

Proof: The proof is a consequence from formula (2.8) and the definition of the error function on the complex plane (2.7). □

Remark 2.2. *Let us consider the Gaussian cumulative distribution function on the complex plain. Formally, we can define it via the error function*

$$N\left(z\right)=\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right). \tag{2.9}$$

This presentation indeed holds because the integral $\int_{-\infty}^{z} e^{-\frac{x^2}{2}}dx$ can be taken on the contour $\{-\infty\to 0\to z\}$.

Note that under these assumptions, formulas (2.4) and (2.8) coincides.

Let us turn to the case when the Brownian motion does not hit the linear function before the moment T. We shall use the following lemma whose proof can be found in Siegmund (1986).

Lemma 2.2. *If $z < b\left(T\right)$, then the probability of $\tau > T$, conditioned on $B_T = z$ is*

$$\mathbb{P}\left(\tau > T \middle|\, B_T = z\right) = 1-\exp\left(-\frac{2b_2\left(b\left(T\right)-z\right)}{T}\right).$$

Proposition 2.3. *For all z and t such that $z < b(t)$ the following statement holds*

$$P(B_t < y, \tau > t) = \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{y} \left(1 - \exp\left(-\frac{2b_2(b(T) - u)}{T}\right)\right) \exp\left(-\frac{u^2}{2T}\right) du.$$

Proof: Using Lemma 2.2 we derive

$$\begin{aligned}\mathbb{P}(B_T < y, \tau > T) &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \mathbb{P}(B_T < y, \tau > T | B_T = u) \exp\left(-\frac{u^2}{2T}\right) du \\ &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{y} \left(1 - \exp\left(-\frac{2b_2(b(T) - u)}{T}\right)\right) \exp\left(-\frac{u^2}{2T}\right) du.\end{aligned}$$

□

Now we can prove our second result.

Theorem 2.2. *[Theorem 3.2 from Zaevski (2020a)] If $z < b(T)$, then*

$$\begin{aligned}&V(\theta, z, T; b_1, b_2) \equiv \mathbb{E}\left[e^{\theta B_T} I_{B_T > z, \tau > T}\right] = \\ &= \exp\left(\frac{T\theta^2}{2}\right) \left[\begin{array}{l} N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) - N\left(\frac{z - T\theta}{\sqrt{T}}\right) \\ + e^{2b_2(\theta - b_1)}\left(N\left(\frac{z - T\theta - 2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right)\end{array}\right].\end{aligned} \tag{2.10}$$

Proof: Taking in attention that $\mathbb{P}(B_T < y, \tau > T) = \mathbb{P}(B_T < b(T), \tau > T)$ when $y > b(T)$ and using Proposition 2.3, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{\theta B_T} I_{B_T > z, \tau > T}\right] &= \int_{z}^{b(T)} e^{\theta y} d\mathbb{P}(B_T < y, \tau > T) \\ &= \frac{1}{\sqrt{2\pi T}} \int_{z}^{b(T)} e^{\theta y} \left(1 - \exp\left(-\frac{2b_2(b(T) - y)}{T}\right)\right) \exp\left(-\frac{y^2}{2T}\right) dy.\end{aligned} \tag{2.11}$$

We derive formula (2.10) dividing integral (2.11) into two parts . □

2.2.3 Piecewise linear case

Let the function $b(\cdot)$ be piecewise linear. The density of the stopping time is given in the following lemma, whose proof can be found in Wang and Pötzelberger (1997) or Jin and Wang (2017).

Lemma 2.3. *Let t belongs to the m-th sub-interval, $t_{m-1} < t < t_m$. Then the τ-density in the point t is*

$$\int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ \frac{\beta_{m-1}-x_{m-1}}{\sqrt{2\pi}(t-t_{m-1})^{\frac{3}{2}}} \exp\left(-\frac{(b_{1,m}t+b_{2,m}-x_{m-1})^2}{2(t-t_{m-1})}\right) \end{array} \right) dx_1 \ldots dx_{m-1}. \tag{2.12}$$

The following theorem corresponds to Theorem 2.1 when the boundary is piecewise linear.

Theorem 2.3. *[Theorem 4.1 from Zaevski (2020a)] Let $\theta > 0$. The Laplace transform of the first hitting time in the m-th interval is given by*

$$\begin{aligned} &\mathbb{E}\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m]}\right] \\ &= \int_{-\infty}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\theta t_{m-1}} L\left(t_m - t_{m-1}, \theta; b_{1,m}, \beta_{m-1} - x_{m-1}\right) \end{array} \right) dx_1 \ldots dx_{m-1} \end{aligned} \tag{2.13}$$

where the function $L(\cdot)$ is given by equation (2.4). For the case $\theta < 0$, see Proposition 2.2.

Proof: Using the form of stopping time density (2.12), we derive

$$\mathbb{E}\left[e^{-\theta\tau}I_{\tau\in(t_{m-1},t_m]}\right]$$
$$= \int\limits_{t_{m-1}}^{t_m}\int\limits_{-\infty}^{\beta_1,\dots,\beta_{m-1}}\left(\begin{array}{l}\prod\limits_{i=1}^{m-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right)\\ \prod\limits_{i=1}^{m-1}\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ e^{-\theta t}\frac{\beta_{m-1}-x_{m-1}}{\sqrt{2\pi}(t-t_{m-1})^{\frac{3}{2}}}\exp\left(-\frac{(b_{1,m}t+b_{2,m}-x_{m-1})^2}{2(t-t_{m-1})}\right)\end{array}\right)dx_1...dx_{m-1}dt$$
$$= \int\limits_{-\infty}^{\beta_1,\dots,\beta_{m-1}}\left(\begin{array}{l}\prod\limits_{i=1}^{m-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right)\\ \prod\limits_{i=1}^{m-1}\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ e^{-\theta t_{m-1}}\int\limits_0^{t_m-t_{m-1}}\left(\begin{array}{l}e^{-\theta u}\frac{\beta_{m-1}-x_{m-1}}{\sqrt{2\pi}u^{\frac{3}{2}}}\\ \exp\left(-\frac{(b_{1,m}u+\beta_{m-1}-x_{m-1})^2}{2u}\right)\end{array}\right)du\end{array}\right)dx_1...dx_{m-1}.$$

We have changed the variable t to $u = t - t_{m-1}$ above, using $\beta_{m-1} = b_{1,m}t_{m-1} + b_{2,m}$. We finish the proof using Theorem 2.1. □

The corresponding to Theorem 2.2 result in the piecewise linear case is as follows.

Theorem 2.4. *[Theorem 4.2 from Zaevski (2020a)]*
If $z < b(T)$, then the Laplace transform when first hitting is after the terminal moment is

$$\mathbb{E}\left[e^{\theta B_T}I_{B_T>z,\tau>T}\right]$$
$$= \int\limits_{-\infty}^{\beta_1,\dots,\beta_{n-1}}\left(\begin{array}{l}\prod\limits_{i=1}^{n-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right)\\ \prod\limits_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}}\\ e^{\theta x_{n-1}}V\left(\theta, z-x_{n-1}, t_n-t_{n-1}; b_{1,n-1}, \beta_{n-1}-x_{n-1}\right)\end{array}\right)dx_1...dx_{n-1},$$

where the function $V(\cdot)$ is given by equation (2.10).

Proof: Note again that $\mathbb{P}(B_T < y, \tau > T) = \mathbb{P}(B_T < b(T), \tau > T)$ when $y > b(T)$. Using the Markovian property of the Brownian motion we derive

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] = \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\mathbb{P}\left(B_T < u, \tau > T\right)$$

$$= \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\left(\int\limits_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \mathbb{P}\left(B_T < u, \tau > T \middle| B_{t_1} = x_1, ..., B_{t_{n-1}} = x_{n-1}\right) \\ \prod\limits_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \end{pmatrix} dx_1...dx_{n-1} \right)$$

$$= \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\left(\int\limits_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \prod\limits_{i=1}^{n-1} \mathbb{V}^{t_{i-1},x_{i-1}}\left(\tau > t_i \middle| B_{t_i} = x_i\right) \\ \prod\limits_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ \mathbb{V}^{t_{n-1},x_{n-1}}\left(\tau > T, B_T < u\right) \end{pmatrix} dx_1...dx_{n-1} \right)$$

$$= \int\limits_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \prod\limits_{i=1}^{n-1} \mathbb{V}^{t_{i-1},x_{i-1}}\left(\tau > t_i \middle| B_{t_i} = x_i\right) \\ \prod\limits_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\mathbb{V}^{t_{n-1},x_{n-1}}\left(\tau > T, B_T < u\right) \end{pmatrix} dx_1...dx_{n-1}.$$

We change the variable u to $v = u - x_{n-1}$ and apply Theorem 2.2 to transform the inner integral to

$$\int\limits_{z}^{b_{n(T)}} e^{\theta u} d\mathbb{V}^{t_{n-1},x_{n-1}}\left(\tau > T, B_T < u\right) = \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\mathbb{P}\left(\tau > T - t_{n-1}, B_T < u - x_{n-1}\right)$$

$$= e^{\theta x_{n-1}} \int\limits_{z - x_{n-1}}^{b_{n(T)} - x_{n-1}} e^{\theta v} d\mathbb{P}\left(\tau > T - t_{n-1}, B_T < v\right)$$

$$= e^{\theta x_{n-1}} V\left(\theta, z - x_{n-1}, t_n - t_{n-1}; b_{1,n-1}, \beta_{n-1} - x_{n-1}\right).$$

Using Lemma 2.2 we finish the proof. □

2.3 Formulation w.r.t. to the lower boundary

We formulate without proofs the derived in Section 2.2 results but assuming a lower hitting problem – this is possible due to the symmetry of the Brownian motion. Suppose that the function $b(\cdot)$ starts from a negative value, $b(0) < 0$.

Proposition 2.4. *The probability that the Brownian motion hits the linear function before the moment T is given by the equation*

$$g(T; b_1, b_2) \equiv \mathbb{P}(\tau < T) = N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) + \exp(-2b_1 b_2) N\left(\frac{-b_1 T + b_2}{\sqrt{T}}\right). \tag{2.14}$$

Proposition 2.5. *The truncated Laplace transform of the first hitting time before T can be obtain through the following statements.*

- *If $\theta > -\frac{b_1^2}{2}$, then*

$$L(T, \theta; b_1, b_2) = \mathbb{E}\left[e^{-\theta\tau}\Lambda_T\right] = e^{-b_2\left(\sqrt{b_1^2+2\theta}+b_1\right)} g\left(T; -\sqrt{b_1^2+2\theta}, b_2\right), \tag{2.15}$$

  where $g(\cdot)$ is the function from equation (2.14).

- *If $\theta = -\frac{b_1^2}{2}$, then $L(T, \theta; b_1, b_2) = 2e^{-b_1 b_2} N\left(\frac{b_2}{\sqrt{T}}\right)$.*

- *If $\theta < -\frac{b_1^2}{2}$, then*

$$L(T, \theta; b_1, b_2) = e^{-b_1 b_2}\left[\begin{array}{l} \cos(cb_2) + \frac{e^{icb_2}}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(ic\sqrt{T} + \frac{b_2}{\sqrt{T}}\right)\right) \\ -\frac{e^{-icb_2}}{2}\operatorname{erf}\left(\frac{1}{\sqrt{2}}\left(ic\sqrt{T} - \frac{b_2}{\sqrt{T}}\right)\right) \end{array}\right].$$

Corollary 2.3. *Suppose that $T = \infty$. The probability τ to be finite is*

$$g(\infty; b_1, b_2) \equiv \mathbb{P}(\tau < \infty) = \begin{cases} 1, \text{ if } b_1 \geq 0 \\ \exp(-2b_1 b_2), \text{ if } b_1 < 0. \end{cases}$$

The Laplace transform is

$$L(\infty, \theta; b_1, b_2) = \mathbb{E}\left[e^{-\theta\tau} I_{\tau<\infty}\right] = e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)}. \tag{2.16}$$

Proposition 2.6. *The truncated Laplace transform in the interval (t_{m-1}, t_m) of the first hitting time to the piecewise linear function is given by the equation*

$$\mathbb{E}\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m)}\right] = \int\limits_{\beta_1,\ldots,\beta_{m-1}}^{\infty} \left(\begin{array}{l} \prod_{i=1}^{m-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ e^{-\theta t_{m-1}} L\left(t_m - t_{m-1}, \theta; a_m, \beta_{m-1} - x_{m-1}\right) \end{array} \right) dx_1 \ldots dx_{m-1} \tag{2.17}$$

where the function $L(\cdot)$ is provided in Proposition 2.5.

Theorem 2.5. *Let $z > b(T)$ and the function be linear. Then we have*

$$V(\theta, z, T; b(\cdot)) \equiv \mathbb{E}\left[e^{\theta B_T} I_{B_T < z, \Phi_T = 1}\right] = = \exp\left(\frac{T\theta^2}{2}\right) \left[\begin{array}{l} N\left(\frac{z - T\theta}{\sqrt{T}}\right) - N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) \\ -e^{2b_2(\theta - b_1)}\left(N\left(\frac{z - T\theta - 2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right) \end{array} \right]. \tag{2.18}$$

Corollary 2.4. *We have for $z > b(T)$*

$$\begin{aligned} U(z, T; c(\cdot)) \equiv \mathbb{P}(B_T < z, \Phi_T = 1) &= N\left(\frac{z}{\sqrt{T}}\right) - N\left(\frac{b(T)}{\sqrt{T}}\right) \\ &- e^{-2b_1 b_2}\left[N\left(\frac{z - 2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T) - 2b_2}{\sqrt{T}}\right)\right]. \end{aligned} \tag{2.19}$$

Proof: We have to rewrite formula (2.18) for $\theta = 0$. □

Proposition 2.7. *If the boundary function is piecewise linear, then the corresponding to (2.18) and (2.19) formulas are*

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T < z, \Phi_T = 1}\right] = \int\limits_{\beta_1,\ldots,\beta_{n-1}}^{\infty} \left(\begin{array}{l} \prod_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ e^{\theta x_{n-1}} V\left(\theta, z - x_{n-1}, t_n - t_{n-1}; b_{n-1}(\cdot) - x_{n-1}\right) \end{array} \right) dx_1 \ldots dx_{n-1} \tag{2.20}$$

and

$$\mathbb{P}(B_T < z, \Phi_T = 1)$$
$$= \int\limits_{\beta_1,\ldots,\beta_{n-1}}^{\infty} \left(\begin{array}{c} \prod\limits_{i=1}^{n-1} \left(1 - \exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}}\right)\right) \\ \prod\limits_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ U(z - x_{n-1}, t_n - t_{n-1}; b(\cdot) - x_{n-1}) \end{array} \right) dx_1 \ldots dx_{n-1}, \quad (2.21)$$

where the functions $V(\cdot)$ *and* $U(\cdot)$ *are given by equations* (2.18) *and* (2.19).

2.4 Laplace transforms of the Brownian motion's first exit from a strip

2.4.1 Preliminaries

Let B_t be a Brownian motion; T be the terminal moment, $T \leq \infty$; and $0 \equiv t_0 < t_1 < \ldots < t_n \equiv T$ be a division of the time interval. Let α_i and β_i, $i = 0, 1, \ldots, n$, be some values such that $\alpha_0 < 0 < \beta_0$, $\alpha_i < \beta_i$, and $\gamma_i = \beta_i - \alpha_i$. The case when some last values of α_i or β_i are infinity is also admissible. Let the functions $a_i(t) = a_{i,1}t + a_{i,2}$ and $b_i(t) = b_{i,1}t + b_{i,2}$ be the linear functions, that connect the points α_i with α_{i+1} and β_i with β_{i+1}, respectively. Let the piecewise linear functions $a(t) < b(t)$ are composed by these functions. We shall denote by τ_1 and τ_2 the first hitting times to the functions $a(\cdot)$ and $b(\cdot)$, respectively, and by τ the lower one – $\tau = \tau_1 \wedge \tau_2$. Let us notate by Λ_t the indicator processes $\Lambda_t = I_{\tau \leq t}$ and by $N(\cdot)$ the cumulative distribution function of the standard normal distribution.

2.4.2 Linear boundaries

We shall use several times the following lemma. The proof of its first statement is an immediate consequence of theorem 4.2 from Anderson (1960). See also the proof of theorem 3 from Pötzelberger and Wang (2001). The second statement is proven in Siegmund (1986).

Lemma 2.4. *Let* $a(T) < z < b(T)$. *We have*

$$\mathbb{P}\left(\tau > T \middle| B_T = z\right) = 1 - \sum_{j=1}^{\infty} q_j\left(0, z; 1\right), \tag{2.22}$$

where

$$\begin{aligned} q_j\left(y, z; i\right) = &\exp\left(-\frac{2\left[j\gamma_{i-1} + \alpha_{i-1} - y\right]\left[j\gamma_i + \alpha_i - z\right]}{t_i - t_{i-1}}\right) \\ &- \exp\left(-\frac{2j\left[j\gamma_{i-1}\gamma_i + \gamma_{i-1}\left(\alpha_i - z\right) - \gamma_i\left(\alpha_{i-1} - y\right)\right]}{t_i - t_{i-1}}\right) \\ &+ \exp\left(-\frac{2\left[j\gamma_{i-1} - \left(\beta_{i-1} - y\right)\right]\left[j\gamma_i - \left(\beta_i - z\right)\right]}{t_i - t_{i-1}}\right) \\ &- \exp\left(-\frac{2j\left[j\gamma_{i-1}\gamma_i - \gamma_{i-1}\left(\beta_i - z\right) + \gamma_i\left(\beta_{i-1} - y\right)\right]}{t_i - t_{i-1}}\right). \end{aligned} \tag{2.23}$$

If $a(T) = -\infty$, then formula (2.22) turns into

$$\mathbb{P}\left(\tau > T \middle| B_T = z\right) = 1 - \exp\left(-\frac{2b_2\left(b\left(T\right) - z\right)}{T}\right).$$

Theorems 4.3 and 5.1 from Anderson (1960) give the probabilities and the corresponding densities the hitting time to be before T. We provide them in the following lemma.

Lemma 2.5.

$$
\begin{aligned}
&P_1^l\left(T;a_1,a_2,b_1,b_2\right)\equiv\mathbb{P}\left(\left(\tau_1\wedge\tau_2\right)<T,\tau_1<\tau_2\right)\\
&=N\left(\frac{a_1T+a_2}{\sqrt{T}}\right)\\
&+\sum_{j=1}^{\infty}\left\{\begin{array}{l}
e^{-2[-ja_2+(j-1)b_2][-ja_1+(j-1)b_1]}N\left(\frac{-a_1T-2(j-1)b_2+(2j-1)a_2}{\sqrt{T}}\right)\\
-e^{-2\left[j^2(a_1a_2+b_1b_2)-j(j-1)a_2b_1-j(j+1)b_2a_1\right]}N\left(\frac{-a_1T-2jb_2+(2j-1)a_2}{\sqrt{T}}\right)\\
-e^{-2[-(j-1)a_2+jb_2][-(j-1)a_1+jb_1]}N\left(\frac{a_1T-2jb_2+(2j-1)a_2}{\sqrt{T}}\right)\\
+e^{-2\left[j^2(a_1a_2+b_1b_2)-j(j-1)b_2a_1-j(j+1)a_2b_1\right]}N\left(\frac{a_1T+(2j+1)a_2-2jb_2}{\sqrt{T}}\right)
\end{array}\right\}\\
&P_2^l\left(T;a_1,a_2,b_1,b_2\right)\equiv\mathbb{P}\left(\left(\tau_1\wedge\tau_2\right)<T,\tau_2<\tau_1\right) \qquad (2.24)\\
&=1-N\left(\frac{b_1T+b_2}{\sqrt{T}}\right)\\
&+\sum_{j=1}^{\infty}\left\{\begin{array}{l}
e^{-2[jb_2-(j-1)a_2][jb_1-(j-1)a_1]}N\left(\frac{b_1T+2(j-1)a_2-(2j-1)b_2}{\sqrt{T}}\right)\\
-e^{-2\left[j^2(b_1b_2+a_1a_2)-j(j-1)b_2a_1-j(j+1)a_2b_1\right]}N\left(\frac{b_1T+2ja_2-(2j-1)b_2}{\sqrt{T}}\right)\\
-e^{-2[(j-1)b_2-ja_2][(j-1)b_1-ja_1]}N\left(\frac{-b_1T+2ja_2-(2j-1)b_2}{\sqrt{T}}\right)\\
+e^{-2\left[j^2(b_1b_2+a_1a_2)-j(j-1)a_2b_1-j(j+1)b_2a_1\right]}N\left(\frac{-b_1T-(2j+1)b_2+2ja_2}{\sqrt{T}}\right)
\end{array}\right\}.
\end{aligned}
$$

$$
\begin{aligned}
&p_1^l\left(t;a_1,a_2,b_1,b_2\right)\equiv\frac{d\mathbb{P}\left(\left(\tau_1\wedge\tau_2\right)\in dt,\tau_1<\tau_2\right)}{dt}\\
&=\frac{1}{\sqrt{2\pi}t^{\frac{3}{2}}}e^{-\frac{(a_1t+a_2)^2}{2t}}\sum_{j=0}^{\infty}\left\{\begin{array}{l}
e^{-\frac{2j[jb_2-(j+1)a_2][b(t)-a(t)]}{t}}\left[2jb_2-\left(2j+1\right)a_2\right]\\
-e^{-\frac{2(j+1)[(j+1)b_2-ja_2][b(t)-a(t)]}{t}}\left[2\left(j+1\right)b_2-\left(2j+1\right)a_2\right]
\end{array}\right\}\\
&p_2^l\left(t;a_1,a_2,b_1,b_2\right)\equiv\frac{d\mathbb{P}\left(\left(\tau_1\wedge\tau_2\right)\in dt,\tau_2<\tau_1\right)}{dt} \qquad (2.25)\\
&=\frac{1}{\sqrt{2\pi}t^{\frac{3}{2}}}e^{-\frac{(b_1t+b_2)^2}{2t}}\sum_{j=0}^{\infty}\left\{\begin{array}{l}
e^{-\frac{2j[(j+1)b_2-ja_2][b(t)-a(t)]}{t}}\left[\left(2j+1\right)b_2-\left(2j+1\right)a_2\right]\\
-e^{-\frac{2(j+1)[jb_2-(j+1)a_2][b(t)-a(t)]}{t}}\left[\left(2j+1\right)b_2-2ja_2\right]
\end{array}\right\}.
\end{aligned}
$$

Our first result is presented in the following theorem.

Theorem 2.6. *[Theorem 3.1 from Zaevski (2021a)] Let $\theta>0$. The Laplace transforms of τ, if it is before T, are*

$$
\begin{aligned}
&L_1(t,\theta;a_1,a_2,b_1,b_2) = \mathbb{E}\left[e^{-\theta\tau}\Lambda_T I_{\tau=\tau_1}\right] \\
&= e^{a_2\left(\sqrt{a_1^2+2\theta}-a_1\right)} P_1^l\left(T;\sqrt{a_1^2+2\theta},a_2,b_1+\sqrt{a_1^2+2\theta}-a_1,b_2\right) \\
&L_2(t,\theta;a_1,a_2,b_1,b_2) = \mathbb{E}\left[e^{-\theta\tau}\Lambda_T I_{\tau=\tau_2}\right] \\
&= e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} P_2^l\left(T;a_1+\sqrt{b_1^2+2\theta}-b_1,a_2,\sqrt{b_1^2+2\theta},b_2\right)
\end{aligned}
\tag{2.26}
$$

where $P_1^l(\cdot)$ and $P_2^l(\cdot)$ are given by equations (2.24).

Proof: Using equation (2.25) and the form of the normal distribution density we can easily obtain

$$
\begin{aligned}
\mathbb{E}\left[e^{-\theta\tau}\Lambda_T I_{\tau=\tau_1}\right] &= \int_0^\infty e^{-\theta t} I_{t<T} p_1^l(t;a_1,a_2,b_1,b_2)\,dt \\
&= e^{a_2\left(\sqrt{a_1^2+2\theta}-a_1\right)} P_1^l\left(T;\sqrt{a_1^2+2\theta},a_2,b_1+\sqrt{a_1^2+2\theta}-a_1,b_2\right).
\end{aligned}
$$

The Laplace transform L_2 can be derived analogously. □

Remark 2.3. *The case $\theta \le 0$ can be considered in a way similar to those used in Proposition 2.2. For example, the results w.r.t. the coefficient a_1 are as follows:*

- *If $\theta > -\frac{a_1^2}{2}$, then the Laplace transform is given by formulas (2.26).*
- *If $\theta = -\frac{a_1^2}{2}$, then*

$$
L_1(t,\theta;a_1,a_2,b_1,b_2) = e^{-a_1 a_2} P_1^l(T;0,a_2,b_1-a_1,b_2).
$$

- *If $\theta < -\frac{a_1^2}{2}$, then we can use again formulas (2.26) but extending the normal CDF in the complex plain via formula (2.9) from Remark 2.2.*

The results w.r.t. the coefficient b_1 can be formulated in the same way.

Note that the four terms of q_j in formula (2.23) (for $i = 1$) can be presented as exponent of linear functions $\lambda_{j,k}z + \xi_{j,k}$, $k = \{1, 2, 3, 4\}$, divided by T. We shall use also the notations $s_2 = s_4 = 1$ and $s_1 = s_3 = -1$. Using this presentation we can formulate our second result in the following way.

Theorem 2.7. *[Theorem 3.2 from Zaevski (2021a)] If $a(T) < z < b(T)$, then the Laplace transform of the Brownian motion if τ is after T is*

$$V(\theta, z, T, a(\cdot), b(\cdot)) = \mathbb{E}\left[e^{\theta B_T} I_{\tau>T, B_T>z}\right]$$
$$= \exp\left(\frac{\theta^2 T}{2}\right)\left\{\begin{array}{l}\left(N\left(\frac{b(T)-\theta T}{\sqrt{T}}\right) - N\left(\frac{z-\theta T}{\sqrt{T}}\right)\right) + \\ + \sum_{j=1}^{\infty}\sum_{k=1}^{4}\left[\begin{array}{l} s_k \exp\left(\lambda_{j,k}\theta + \left(\frac{\lambda_{j,k}^2 + 2\xi_{j,k}}{2T}\right)\right) \times \\ \left(N\left(\frac{b(T)-\left(\theta T + \lambda_{j,k}\right)}{\sqrt{T}}\right) - N\left(\frac{z-\left(\theta T+\lambda_{j,k}\right)}{\sqrt{T}}\right)\right)\end{array}\right]\end{array}\right\}. \quad (2.27)$$

Proof: Using equation (2.22) we obtain

$$\mathbb{P}(B_T < u, \tau > T) = \frac{1}{\sqrt{2\pi T}} \int_{a(T)}^{u} \sum_{j=1}^{\infty}\sum_{k=0}^{4} s_k \exp\left(\frac{\lambda_{j,k}v + \xi_{j,k}}{T}\right)\exp\left(-\frac{v^2}{2T}\right)dv.$$

Therefore

$$\mathbb{E}\left[e^{\theta B_T} I_{\tau>T, B_T<z}\right] = \int_{z}^{b(T)} e^{\theta u} d\mathbb{P}(B_T < u, \tau > T)$$
$$= \frac{1}{\sqrt{2\pi T}} \sum_{j=1}^{\infty}\sum_{k=0}^{4} s_k \int_{z}^{b(T)} e^{\theta u}\exp\left(\frac{\lambda_{j,k}u + \xi_{j,k}}{T}\right)\exp\left(-\frac{u^2}{2T}\right)dy,$$

which after rearranging leads to formula (2.27). □

2.4.3 Piecewise linear boundaries

The following theorem holds if the boundaries are piecewise linear.

Theorem 2.8. *[Theorem 4.1 from Zaevski (2021a)] Let $\theta > 0$. If the functions $L_{1,2}(\cdot)$ and $V(\cdot)$ are given by equations (2.26) and (2.27), then the Laplace transforms are*

$$\mathbb{E}\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m),\tau=\tau_{1,2}}\right] = \int_{\alpha_1,\ldots,\alpha_{m-1}}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{c} \prod_{i=1}^{m-1}\left(1-\sum_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right)\frac{\exp\left(-\frac{\left(x_i-x_{i-1}\right)^2}{2\left(t_i-t_{i-1}\right)}\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\theta t_{m-1}} L_{1,2}\left(\begin{array}{c} t_m-t_{m-1},\theta; \\ a_{m,1},\alpha_{m-1}-x_{m-1}, \\ b_{m,1},\beta_{m-1}-x_{m-1} \end{array}\right) \end{array} \right) dx_1 \ldots dx_{m-1} \tag{2.28}$$

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] = \int_{\alpha_1,\ldots,\alpha_{n-1}}^{\beta_1,\ldots,\beta_{n-1}} \left(\begin{array}{c} \prod_{i=1}^{n-1}\left(1-\sum_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right)\frac{\left(\exp\left(-\frac{\left(x_i-x_{i-1}\right)^2}{2\left(t_i-t_{i-1}\right)}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\theta x_{n-1}} V\left(\begin{array}{c} \theta, z-x_{n-1}, t_n-t_{n-1}; \\ a_{n,1},\alpha_{n-1}-x_{n-1}, \\ b_{n,1},\beta_{n-1}-x_{n-1} \end{array}\right) \end{array} \right) dx_1 \ldots dx_{n-1} \tag{2.29}$$

For negative values of θ, see Remark 2.3.

Proof: We have

$$
\begin{aligned}
L_1 &= \int\limits_{t_{m-1}}^{t_m} e^{-\theta u} d\mathbb{P}\left(\tau < u, \tau = \tau_1\right) \\
&= \int\limits_{t_{m-1}}^{t_m} \left(\begin{array}{l} e^{-\theta u} \prod\limits_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ d\mathbb{P}\left(\tau < u, \tau = \tau_1 \middle| B_{t_1 = x_1}, \dots, B_{t_{m-1} = x_{m-1}}\right) \end{array} \right) dx_1 \dots dx_{m-1} \\
&= \int\limits_{t_{m-1}}^{t_m} e^{-\theta u} d \left(\int\limits_{\alpha_1, \dots, \alpha_{m-1}}^{\beta_1, \dots, \beta_{m-1}} \left(\begin{array}{c} \prod\limits_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ \prod\limits_{i=1}^{m-1} \mathbb{V}^{t_{i-1}, x_{i-1}}\left(t_i < \tau \middle| B_{t_i = x_i}\right) \\ \mathbb{V}^{t_{m-1}, x_{m-1}}\left(\tau < u, \tau = \tau_1\right) \end{array} \right) dx_1 \dots dx_{m-1} \right) \\
&= \int\limits_{\alpha_1, \dots, \alpha_{m-1}}^{\beta_1, \dots, \beta_{m-1}} \left(\begin{array}{l} \prod\limits_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ \prod\limits_{i=1}^{m-1} \left(1 - \sum\limits_{j=1}^{\infty} q_j\left(x_{i-1}, x_i; i\right)\right) \\ e^{-\theta t_{m-1}} \int\limits_0^{t_m - t_{m-1}} e^{-\theta v} p_1^l \left(\begin{array}{l} v; a_{m,1}, \alpha_{m-1} - x_{m-1}, \\ b_{m,1}, \beta_{m-1} - x_{m-1} \end{array} \right) dv \end{array} \right) dx_1 \dots dx_{m-1}.
\end{aligned}
$$

We have used above Lemma 2.4 and the Markovian property of the Brownian motion. It remains to use Theorem 2.6. The form of L_2 can be obtained analogously. We prove formula (2.29) in a similar way

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] = \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\mathbb{P}\left(B_T < u, \tau > T\right)$$

$$= \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\left(\int\limits_{\alpha_1,\dots,\alpha_{n-1}}^{\beta_1,\dots,\beta_{n-1}} \left(\begin{array}{l} \mathbb{P}\left(B_T < u, \tau > T \middle| B_{t_1} = x_1, \dots, B_{t_{n-1}} = x_{n-1}\right) \\ \prod\limits_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \end{array}\right) dx_1 \dots dx_{n-1}\right)$$

$$= \int\limits_{\alpha_1,\dots,\alpha_{n-1}}^{\beta_1,\dots,\beta_{n-1}} \left(\begin{array}{l} \prod\limits_{i=1}^{n-1} \left(1 - \sum\limits_{j=1}^{\infty} q_j\left(x_{i-1}, x_i; i\right)\right) \\ \prod\limits_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ \int\limits_{z}^{b_{n(T)}} e^{\theta u} d\mathbb{V}^{t_{n-1}, x_{n-1}}\left(\tau > T, B_T < u\right) \end{array}\right) dx_1 \dots dx_{n-1}.$$

We finish the proof using Theorem 2.7. □

Finally, we present a modification of Theorem 2.8 concerning the case when one of the boundaries is infinitely large(small) after some moment. We shall present only the case when the lower boundary does not exist.

Theorem 2.9. *[Theorem 4.2 from Zaevski (2020a)] Let* $\alpha_i = -\infty$ *for all* $i \geq k$, $\theta > 0$, *and* $m > k$. *Then*

$$L_2 = \mathbb{E}\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m),\tau=\tau_{1,2}}\right]$$
$$= \int_{\alpha_1,\dots,\alpha_{k-1},-\infty}^{\beta_1,\dots,\beta_{m-1}} \begin{pmatrix} \prod_{i=1}^{k-1}\left(1-\sum_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right) \\ \prod_{i=k}^{m-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1}\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\theta t_{m-1}}\overline{L}\left(t_m-t_{m-1},\theta;b_{m,1},\beta_{m-1}-x_{m-1}\right) \end{pmatrix} dx_1 \dots dx_{m-1}$$

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right]$$
$$= \int_{\alpha_1,\dots,\alpha_{k-1},-\infty}^{\beta_1,\dots,\beta_{n-1}} \begin{pmatrix} \prod_{i=1}^{k-1}\left(1-\sum_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right) \\ \prod_{i=k}^{n-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\theta x_{n-1}}\overline{V}\left(\theta,z-x_{n-1},t_n-t_{n-1};b_{1,n-1},\beta_{n-1}-x_{n-1}\right) \end{pmatrix} dx_1 \dots dx_{n-1}.$$

The functions $\overline{L}(\cdot)$ and $\overline{V}(\cdot)$ are given by formulas (2.4) and (2.10).

If the upper boundary vanishes, then

$$E\left[e^{-\xi\tau} I_{\tau\in(t_{m-1},t_m)}, \tau=\tau_1\right]$$
$$= \int_{\alpha_1,\dots,\alpha_{m-1}}^{\beta_1,\dots,\beta_{k-1},\infty} \begin{pmatrix} \prod_{i=1}^{p-1}\left(1-\sum_{j=1}^{\infty} q_{i,j}\left(x_{i-1},x_i\right)\right)\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ \prod_{i=p}^{m-1}\left(1-\exp\left(-\frac{2(\alpha_{i-1}-x_{i-1})(\alpha_i-x_i)}{t_i-t_{i-1}}\right)\right)\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{-\xi t_{m-1}}L\left(t_m-t_{m-1},\xi;a_{m,1},\alpha_{m-1}-x_{m-1}\right) \end{pmatrix} dx_1 \dots dx_{m-1}, \quad (2.30)$$

$$E\left[e^{\xi B_T} I_{B_T<z,\tau>T}\right] = \int\limits_{\alpha_1,\ldots,\alpha_{n-1}}^{\beta_1,\ldots,\beta_{k-1},\infty} \left(\begin{array}{l} \prod\limits_{i=1}^{p-1}\left(1-\sum\limits_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right)\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ \prod\limits_{i=p}^{n-1}\left(1-\exp\left(-\frac{2(\beta_{i-1}-x_{i-1})(\beta_i-x_i)}{t_i-t_{i-1}}\right)\right)\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi(t_i-t_{i-1})}} \\ e^{\xi x_{n-1}} V\left(\xi, z-x_{n-1}, t_n-t_{n-1}; a_{1,n-1}, \alpha_{n-1}-x_{n-1}\right) \end{array} \right) dx_1 \ldots dx_{n-1}. \quad (2.31)$$

where $L(\cdot)$ and $V(\cdot)$ are given by formulas (2.15) and (2.18).

Proof: The proof is similar to the proof of Theorem 2.8. Note that when the lower boundary is infinity we have to use the second statement of Lemma 2.4. □

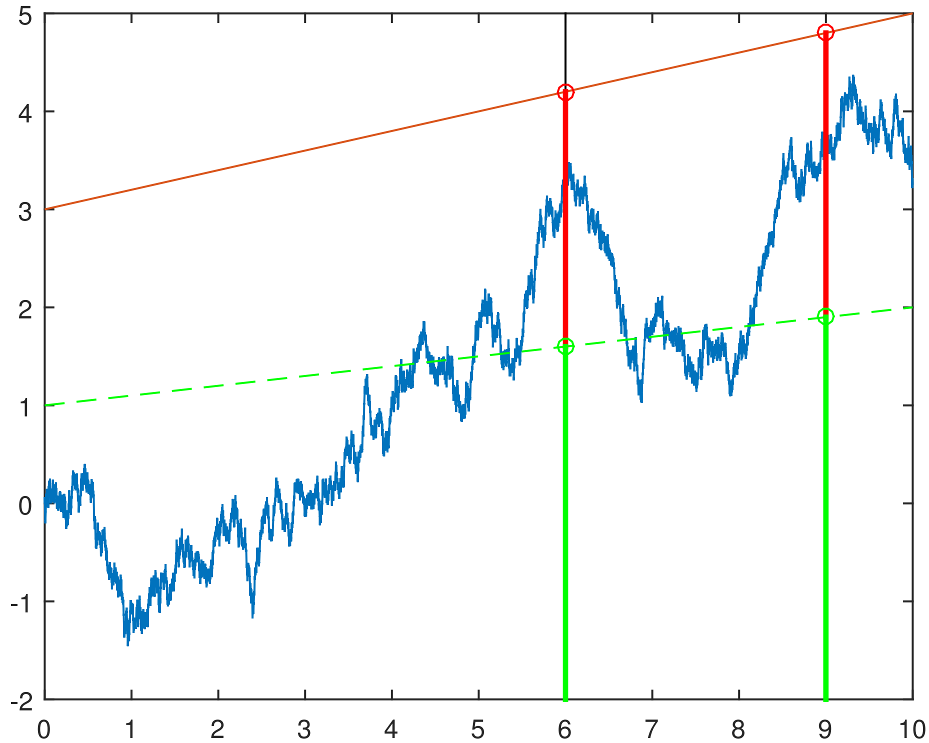
2.5 Some limits

We shall derive several limits related to the problem without terminal constraints, $T \to \infty$. Let ζ be the first hitting moment of the Brownian motion to the linear function $b(t) = b_1 t + b_2$. We illustrate our task for an upper-hitting problem by Figure 2.1. A trajectory that stays below the boundary $b(t) = b_1 t + b_2$ (red line) till moment $T = 10$ is plotted by blue color. Thus, we take the expectation over all such sample paths and search for the limit $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{B_t<b(t),\ t\in[0,T]}\right]$. In addition, let $z(t)$ be another linear function that is below $b(t)$ for large enough values of t – we plot it by a green dashed line. The second our task is to obtain the limit imposing the additional condition that the value of B_T is larger than $z(T)$, i.e. to fall in the red part. Thus the limit we search turns into $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{B_t<b(t)\ \forall\ t\in[0,T],\ B_T>z(T)}\right]$. Note that we do not restrict the sample paths to stay in the strip between the green and red lines but only the Brownian motion's value at the terminal moment T to be above $z(T)$ (of course below $b(T)$ too).

2.5.1 Some lemmas

Let B_t be a Brownian motion under the natural filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. Let ζ be the first hitting moment of the Brownian motion to

Figure 2.1: Illustration of the problem



the linear function $b(t) = b_1 t + b_2$.

Corollary 2.5. *Let $z(t) = z_1 t + z_2$ be another linear function, that satisfies the conditions:*

1. *If $b_2 > 0$, then $z_1 \leq b_1$. In addition, if $z_1 = b_1$, then $z_2 < b_2$.*
2. *If $b_2 < 0$, then $z_1 \geq b_1$. In addition, if $z_1 = b_1$, then $z_2 > b_2$.*

Expectation (2.10) for $z = z(T)$ can be rewritten as

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T > z(T), \zeta > T}\right] = \frac{e^{\left(-\frac{b_1^2}{2} + b_1 \theta\right) T}}{\sqrt{2\pi T}} \int\limits_{(z_1 - b_1)T + z_2}^{b_2} e^{-\frac{y^2}{2T}} e^{-y(b_1 - \theta)} \left(1 - e^{\frac{2b_2(y - b_2)}{T}}\right) dy. \tag{2.32}$$

Note that if $z_1 = b_1$, then

$$0 < \lim_{T\to\infty} T \int_{z_2}^{b_2} e^{-\frac{y^2}{2T}} e^{-y(b_1-\theta)} \left(1 - e^{\frac{2b_2(y-b_2)}{T}}\right) dy < \infty. \tag{2.33}$$

Proof: We have

$$\begin{aligned} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] &= e^{\left(k+\frac{\theta^2}{2}\right)T}\left[N\left((b_1-\theta)\sqrt{T}+\frac{b_2}{\sqrt{T}}\right) - N\left((z_1-\theta)\sqrt{T}+\frac{z_2}{\sqrt{T}}\right)\right] \\ &- e^{\left(k+\frac{\theta^2}{2}\right)T} e^{2b_2(\theta-b_1)}\left[N\left((b_1-\theta)\sqrt{T}-\frac{b_2}{\sqrt{T}}\right) - N\left((z_1-\theta)\sqrt{T}+\frac{z_2-2b_2}{\sqrt{T}}\right)\right]. \end{aligned} \tag{2.34}$$

Using the change of variables

$$y = x\sqrt{T} - m_1 T \Leftrightarrow x = m_1\sqrt{T} + \frac{y}{\sqrt{T}}, \tag{2.35}$$

for $m_1 = b_1 - \theta$, we derive for the first term

$$\begin{aligned} e^{\left(k+\frac{\theta^2}{2}\right)T}\left[N\left((b_1-\theta)\sqrt{T}+\frac{b_2}{\sqrt{T}}\right) - N\left((z_1-\theta)\sqrt{T}+\frac{z_2}{\sqrt{T}}\right)\right] &= \frac{e^{\left(k+\frac{\theta^2}{2}\right)T}}{\sqrt{2\pi}} \int_{(z_1-\theta)\sqrt{T}+\frac{z_2}{\sqrt{T}}}^{(b_1-\theta)\sqrt{T}+\frac{b_2}{\sqrt{T}}} e^{-\frac{x^2}{2}} dx \\ = \frac{e^{\left(k+\frac{\theta^2}{2}\right)T}}{\sqrt{2\pi T}} \int_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{1}{2}\left((b_1-\theta)\sqrt{T}+\frac{y}{\sqrt{T}}\right)^2} dy \\ = \frac{e^{\left(k-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}} \int_{(z_1-b_1)T+z_2}^{b_2} e^{-\frac{y^2}{2T}} e^{-y(b_1-\theta)} dy. \end{aligned} \tag{2.36}$$

Let us turn to the second part of formula (2.34). The change of variables $u = y + 2b_2$ in addition to the one used above leads to

$$
\begin{aligned}
&e^{\left(k+\frac{\theta^2}{2}\right)T}e^{2b_2(\theta-b_1)}\left[N\left((b_1-\theta)\sqrt{T}-\frac{b_2}{\sqrt{T}}\right)-N\left((z_1-\theta)\sqrt{T}+\frac{z_2-2b_2}{\sqrt{T}}\right)\right]\\
&=\frac{e^{\left(k+\frac{\theta^2}{2}\right)T}e^{2b_2(\theta-b_1)}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2-2b_2}^{-b_2}e^{-\frac{1}{2}\left((b_1-\theta)\sqrt{T}+\frac{y}{\sqrt{T}}\right)^2}dy\\
&=\frac{e^{\left(k-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2}^{b_2}e^{-\frac{u^2}{2T}}e^{-u(b_1-\theta)}e^{\frac{2b_2(u-b_2)}{T}}du.
\end{aligned}
\tag{2.37}
$$

Combining equations (2.36) and (2.37), we derive

$$
e^{kT}\mathbb{E}\left[e^{\theta B_T}I_{T<\zeta,B_T>z(T)}\right]=\frac{e^{\left(k-\frac{b_1^2}{2}+b_1\theta\right)T}}{\sqrt{2\pi T}}\int\limits_{(z_1-b_1)T+z_2}^{b_2}e^{-\frac{y^2}{2T}}e^{-y(b_1-\theta)}\left(1-e^{\frac{2b_2(y-b_2)}{T}}\right)dy.
$$

Restrictions (2.33) hold because $\left(1-e^{\frac{2b_2(y-b_2)}{T}}\right)$ tends to zero as $\frac{2b_2(b_2-y)}{T}$ for $T\to\infty$ and the term $e^{-\frac{y}{2T}}\to 1$ in the integral domain which is finite when $z_1=b_1$. $\square$

Suppose now that the conditions of Corollary 2.5 hold, $z_1<b_1$, and $z_2>b_2$. Let us denote by $\bar{t}$ the root of $b(t)=z(t)$, i.e. $\bar{t}=\frac{z_2-b_2}{b_1-z_1}$. We can rewrite expectation (2.32) for $T>\bar{t}$ as

$$
\begin{aligned}
\mathbb{E}\left[e^{\theta B_T}I_{B_T>z(T),\zeta>T}\right]&=\mathbb{E}\left[\mathbb{E}\left[e^{\theta B_T}I_{B_T>z(T),\zeta>T}\middle|\mathcal{F}_{\bar{t}}\right]\right]\\
&=\int\limits_{-\infty}^{b(\bar{t})}\mathbb{E}^{\bar{t},u}\left[e^{\theta B_T}I_{B_T>z(T),\zeta>T}\right]d\mathbb{P}\left(B_{\bar{t}}<y\right).
\end{aligned}
\tag{2.38}
$$

The notation $\mathbb{E}^{\bar{t},u}$ above means the expectation under the assumption that the Brownian motion has a value u at the moment $\bar{t}$.

We shall prove now several lemmas – later we shall use them to prove our main results.

Lemma 2.6. *If the constant* m *is positive, then*

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{m^2}{2}} \frac{m}{m^2+1} < N(-m) < e^{\frac{-m^2}{2}} \frac{1}{m\sqrt{2\pi}}.$$

Proof: The lemma follows from the well-known result of Robert Gordon for the Mills ratio for $x > 0$:

$$\frac{x}{x^2+1} \leq e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{u^2}{2}} du \leq \frac{1}{x}.$$

□

Lemma 2.7. *If the constants* m_1 *and* m_2 *are positive and* $T > \frac{m_2}{m_1}$*, then the following inequalities hold*

$$\begin{aligned} &\sqrt{\frac{2T}{\pi}} \frac{m_2 e^{m_1 m_2}}{m_1^2 T^2 - m_2^2} \left[1 - T\frac{3m_1^2 T^2 + m_2^2}{(m_1^2T^2 - m_2^2)^2}\right] \exp\left(-\frac{1}{2}\left(m_1^2 T + \frac{m_2^2}{T}\right)\right) \\ &< N\left(-m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) - e^{2m_1 m_2} N\left(-m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right) \\ &< \sqrt{\frac{2T}{\pi}} \frac{m_2 e^{m_1 m_2}}{m_1^2 T^2 - m_2^2} \exp\left(-\frac{1}{2}\left(m_1^2 T + \frac{m_2^2}{T}\right)\right). \end{aligned} \tag{2.39}$$

Note that the first term is positive for large enough values of T*.*

Proof: We have

$$
\begin{aligned}
&N\left(-m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-e^{2m_1m_2}N\left(-m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}}^{\infty} e^{-\frac{x^2}{2}}dx-e^{2m_1m_2}\int_{m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}}^{\infty} e^{-\frac{x^2}{2}}dx\right]\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{\sqrt{T}}^{\infty} e^{-\frac{\left(m_1y-\frac{m_2}{y}\right)^2}{2}}\left(m_1+\frac{m_2}{y^2}\right)dy-e^{2m_1m_2}\int_{\sqrt{T}}^{\infty} e^{-\frac{\left(m_1y+\frac{m_2}{y}\right)^2}{2}}\left(m_1-\frac{m_2}{y^2}\right)dy\right] \qquad (2.40)\\
&=\sqrt{\frac{2}{\pi}}e^{m_1m_2}m_2\int_{\sqrt{T}}^{\infty}\frac{e^{-\frac{m_1^2y^2+m_2^2y^{-2}}{2}}}{y^2}dy\\
&=\sqrt{\frac{2}{\pi}}e^{m_1m_2}m_2\int_{0}^{\frac{1}{\sqrt{T}}} e^{-\frac{m_1^2z^{-2}+m_2^2z^2}{2}}dz.
\end{aligned}
$$

We have used above the changes of variables

$$
\begin{aligned}
x&=m_1y-\frac{m_2}{y}\\
x&=m_1y+\frac{m_2}{y}\\
y&=\frac{1}{z}.
\end{aligned}
$$

Let M_1 and M_2 be defined as $M_1=m_1^2$ and $M_2=m_2^2$. We need the following derivatives before to continue

$$
\begin{aligned}
&\left(-\frac{M_1z^{-2}+M_2z^2}{2}\right)'=\frac{M_1}{z^3}-M_2z\\
&\left(\frac{z^3}{M_1-M_2z^4}\right)'=z^2\frac{M_2z^4+3M_1}{\left(M_1-M_2z^4\right)^2}\\
&\left(\left(3M_1z^{-4}+M_2\right)\left(\frac{z^3}{M_1-M_2z^4}\right)^3\right)'=3\left(\frac{z}{M_1+M_2z^4}\right)^4\left(M_2^2z^8+10M_1M_2z^4+5M_1^2\right). \qquad (2.41)
\end{aligned}
$$

Applying twice integration by parts in formula (2.40) having in mind the restriction $T > \frac{m_2}{m_1}$, and using derivatives (2.41), we derive

$$\int_0^{\frac{1}{\sqrt{T}}} e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}} dz = \int_0^{\frac{1}{\sqrt{T}}} \frac{z^3}{M_1 - M_2 z^4} d\left(e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}}\right)$$

$$= \frac{z^3}{M_1 - M_2 z^4} e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}}\Bigg|_0^{\frac{1}{\sqrt{T}}} - \int_0^{\frac{1}{\sqrt{T}}} z^2 \frac{M_2 z^4 + 3M_1}{\left(M_1 - M_2 z^4\right)^2} e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}} dz$$

$$= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}} - \int_0^{\frac{1}{\sqrt{T}}} z^2 \frac{M_2 z^4 + 3M_1}{\left(M_1 - M_2 z^4\right)^2} e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}} dz \qquad (2.42)$$

$$= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}} - \int_0^{\frac{1}{\sqrt{T}}} z^2 \frac{M_2 z^4 + 3M_1}{\left(M_1 - M_2 z^4\right)^2} \frac{z^3}{M_1 - M_2 z^4} d\left(e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}}\right)$$

$$= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}}$$
$$- \int_0^{\frac{1}{\sqrt{T}}} \left(3M_1 z^{-4} + M_2\right) \left(\frac{z^3}{M_1 - M_2 z^4}\right)^3 d\left(e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}}\right)$$
$$= \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}}$$
$$- \left(3M_1 z^{-4} + M_2\right) \left(\frac{z^3}{M_1 - M_2 z^4}\right)^3 e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}}\Bigg|_0^{\frac{1}{\sqrt{T}}}$$
$$+ 3 \int_0^{\frac{1}{\sqrt{T}}} e^{-\frac{M_1 z^{-2}+M_2 z^2}{2}} \left(\frac{z}{M_1 + M_2 z^4}\right)^4 \left(M_2^2 z^8 + 10 M_1 M_2 z^4 + 5M_1^2\right) dz$$
$$> \frac{\sqrt{T}}{M_1 T^2 - M_2} e^{-\frac{M_1 T + M_2 T^{-1}}{2}} \left[1 - T \frac{3M_1 T^2 + M_2}{\left(M_1 T^2 - M_2\right)^2}\right]. \qquad (2.43)$$

We prove the second inequality in (2.39) combining equations (2.40) and

(2.42). The first one is a consequence of inequality (2.43). □

Lemma 2.8. *The following statements hold.*

1. *If* $k > 0$*,* $m < 0$*, and* $k > \frac{m^2}{2}$*, then* $\lim_{T\to\infty} e^{kT} N\left(m\sqrt{T}\right) = \infty$*.*

2. *If* $k > 0$*,* $m < 0$*, and* $k \le \frac{m^2}{2}$*, then* $\lim_{T\to\infty} e^{kT} N\left(m\sqrt{T}\right) = 0$*.*

Proof: These statements follow from Lemma 2.6. □

Lemma 2.9. *If* $m_1 < 0$ *and* $m_2 > 0$*, then*

1. *If* $k \le \frac{m_1^2}{2}$*, then* $\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) - e^{2m_1m_2} N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right] = 0$*.*

2. *If* $k > \frac{m_1^2}{2}$*, then* $\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) - e^{2m_1m_2} N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right] = \infty$*.*

Proof: The results follow from Lemma 2.7. □

Lemma 2.10. *If* $m_1 < 0$*,* $m_2 > 0$*,* $k > \frac{m_1^2}{2}$*,* $C > 1$*, and* $D \ge 0$ *then*

$$\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T} + \frac{m_2 + D}{\sqrt{T}}\right) - Ce^{2m_1m_2} N\left(m_1\sqrt{T} - \frac{m_2 - D}{\sqrt{T}}\right)\right] = -\infty. \tag{2.44}$$

Proof: We shall establish first the result for $D = 0$. We can rewrite equation (2.44) as

$$\begin{aligned}
&\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) - Ce^{2m_1m_2} N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right] \\
&= \lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) - e^{2m_1m_2} N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right] \\
&- \lim_{T\to\infty} e^{kT}\left[(C-1)\, e^{2m_1m_2} N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right].
\end{aligned}$$

Lemmas 2.6 and 2.7 show that the terms above are constrained by terms of type $e^{\left(k-\frac{m_1^2}{2}\right)T}\frac{A_1}{T\sqrt{T}}$ and $e^{\left(k-\frac{m_1^2}{2}\right)T}\frac{A_2}{\sqrt{T}}$, respectively, for large enough values of T and some positive constants A_1 and A_2. Thus the whole limit behavior is

$$\frac{e^{\left(k-\frac{m_1^2}{2}\right)T}}{T\sqrt{T}}\left[A_1-(C-1)\,A_2T\right],$$

which tends to minus infinity since $C>1$.

Suppose now that $D>0$. We can rewrite equation (2.44) as

$$\begin{aligned}
&\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2-D}{\sqrt{T}}\right)\right]\\
&=\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2-D}{\sqrt{T}}\right)\right]\\
&\pm\left\{\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]\right\}\\
&=\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)-Ce^{2m_1m_2}N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right]\\
&+\lim_{T\to\infty} e^{kT}\left[N\left(m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}\right)-N\left(m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}\right)\right]\\
&-\lim_{T\to\infty} Ce^{kT}e^{2m_1m_2}\left[N\left(m_1\sqrt{T}+\frac{D-m_2}{\sqrt{T}}\right)-N\left(m_1\sqrt{T}-\frac{m_2}{\sqrt{T}}\right)\right].
\end{aligned}$$

We shall denote by I_1, I_2, and I_3 the three components above. The limit of I_1 is minus infinity because we have already proved the lemma for $D=0$. Let us consider the term I_2. Changing the variables as in (2.35), we derive

$$
\begin{aligned}
I_2 &:= N\left(m_1\sqrt{T} + \frac{m_2+D}{\sqrt{T}}\right) - N\left(m_1\sqrt{T} + \frac{m_2}{\sqrt{T}}\right) \\
&= \int\limits_{m_1\sqrt{T}+\frac{m_2}{\sqrt{T}}}^{m_1\sqrt{T}+\frac{m_2+D}{\sqrt{T}}} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{T}} \int\limits_{m_2}^{m_2+D} e^{-\frac{\left(m_1\sqrt{T}+\frac{y}{\sqrt{T}}\right)^2}{2}} dy \\
&= \frac{e^{-\frac{m_1^2}{2}T}}{\sqrt{T}} \int\limits_{m_2}^{m_2+D} e^{-\frac{y^2}{2T}} e^{-m_1 y} dy.
\end{aligned}
$$

Analogously, using in addition the change $u = y + 2m_2 \Leftrightarrow y = u - 2m_2$, we obtain for I_3

$$
\begin{aligned}
I_3 &:= e^{2m_1m_2}\left[N\left(m_1\sqrt{T} + \frac{D-m_2}{\sqrt{T}}\right) - N\left(m_1\sqrt{T} - \frac{m_2}{\sqrt{T}}\right)\right] \\
&= e^{2m_1m_2}\frac{e^{-\frac{m_1^2}{2}T}}{\sqrt{T}} \int\limits_{-m_2}^{-m_2+D} e^{-\frac{y^2}{2T}} e^{-m_1 y} dy \\
&= \frac{e^{-\frac{m_1^2}{2}T}}{\sqrt{T}} \int\limits_{m_2}^{m_2+D} e^{-\frac{u^2}{2T}} e^{-m_1 y} e^{-2\frac{m_2^2-um_2}{T}} du.
\end{aligned}
$$

We can observe that $I_3 > I_2$ because $m_2 > 0$ and $u > m_2$. Having in mind that $C > 1$ and $I_3 > 0$, we conclude $I_2 - CI_3 < I_2 - I_3 < 0$. We finish the proof combining this inequality and the limit $\lim_{T\to\infty} I_1 = -\infty$. □

2.5.2 Main results

We shall obtain now the results for the expectation $\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right]$.

Theorem 2.10. *[Theorem 3.1 from Zaevski (2024a)]*

Let θ be a positive number, $b(\cdot)$ be the linear function $b(t) = b_1 t + b_2$, ζ be the Brownian motion's first hitting time to it, and k be an arbitrary constant. The following statements hold.

1. *If* $\{b_2 = 0\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
2. *If* $\left\{b_2 \neq 0, k < -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
3. *If* $\left\{b_2 \neq 0, b_1 = \theta, k = -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
4. *If* $\left\{b_2 \neq 0, b_1 = \theta, k > -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.
5. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
6. *If* $\left\{b_2 > 0,\ b_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2b_2(\theta - b_1)}$.
7. *If* $\left\{b_2 > 0,\ b_1 > \theta,\ k > -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.
8. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ -\frac{\theta^2}{2} < k \leq \frac{b_1^2}{2} - \theta b_1\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
9. *If* $\left\{b_2 > 0,\ b_1 < \theta,\ \frac{b_1^2}{2} - \theta b_1 < k\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.
10. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ k = -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
11. *If* $\left\{b_2 < 0,\ b_1 < \theta,\ k = -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 1 - e^{2b_2(\theta - b_1)}$.
12. *If* $\left\{b_2 < 0,\ b_1 < \theta,\ k > -\frac{\theta^2}{2}\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.
13. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ -\frac{\theta^2}{2} < k \leq \frac{b_1^2}{2} - \theta b_1\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = 0$.
14. *If* $\left\{b_2 < 0,\ b_1 > \theta,\ \frac{b_1^2}{2} - \theta b_1 < k\right\}$*, then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \infty$.

Proof: We shall discuss point by point all cases, paying special attention to a potential discontinuity at zero. If $b_2 > 0$, then formula (2.10), applied for $z = -\infty$, leads to

$$\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)} N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right]. \tag{2.45}$$

Otherwise, if $b_2 < 0$, then formula (2.18), written for $z = \infty$, leads to

$$\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[N\left(-\frac{b(T) - T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)} N\left(-\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right)\right]. \tag{2.46}$$

1. The first statement holds because $\zeta = 0$ at every sample path when $b_2 = 0$. Note that the arguments of the normal CDFs in formulas (2.45) and (2.46) are equal, which confirms that the limit is zero.

2. If $\left\{b_2 \neq 0, k < -\frac{\theta^2}{2}\right\}$, then the limit is zero because the exponent in formulas (2.45) or (2.46) tends to zero and the other term is finite.

3. We see that both exponents in formulas (2.45) or (2.46) vanish, because $k = -\frac{\theta^2}{2}$ and $b_1 = \theta$. Also, the arguments of the normal CDFs tend to zero because $b_1 = \theta$.

4. Suppose that $\left\{b_1 = \theta, k > -\frac{\theta^2}{2}\right\}$ and $b_2 > 0$. We have indeterminacy of the kind infinity multiplied by zero. Notating $d = k + \frac{\theta^2}{2}$, using formula (2.45), and changing the variable $y = x\frac{\sqrt{T}}{b_2} \Leftrightarrow x = y\frac{b_2}{\sqrt{T}}$, we derive

$$
\begin{aligned}
&e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right] = \\
&= e^{T\left(k+\frac{\theta^2}{2}\right)}\left[N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)}N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right] \\
&= e^{Td}\left[2N\left(\frac{b_2}{\sqrt{T}}\right)-1\right] \\
&= \frac{2e^{Td}}{\sqrt{2\pi}}\int_0^{\frac{b_2}{\sqrt{T}}} e^{-\frac{x^2}{2}}dx \\
&= \frac{2e^{Td}}{\sqrt{2\pi T}}\int_0^{1} e^{-\frac{x^2 b_2^2}{2T}}dx.
\end{aligned}
$$

We can see that the integrand tends to one when $T \to \infty$ and thus the whole integral tends to one. This proves that the limit is infinite since $d > 0$. The statement can be proven in the same way when $b_2 < 0$ by the use of formula (2.46).

5. The limit is zero because the exponent in formula (2.45) vanishes and the arguments of the normal CDF tend to minus infinity.

6. This statement holds because the arguments of the normal CDF now tend to plus infinity.

7. The arguments of the normal CDF tend again to plus infinity and thus

$$
N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right) - e^{2b_2(\theta-b_1)}N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right) \to 1 - e^{2b_2(\theta-b_1)}, \tag{2.47}
$$

which is strictly positive. Therefore, the whole limit is infinity since $k > -\frac{\theta^2}{2}$. Note that limit (2.47) is zero when $b_2 = 0$ or $b_1 = \theta$ – we have a discontinuity in these points.

8. Note that the arguments in the normal CDFs tend to $(b_1 - \theta)\sqrt{T}$. We derive the desired result using the second statement of Lemma 2.8.

9. The result follows from the second statement of Lemma 2.9.

The rest of the limits can be derived in the same manner using formula (2.46) instead (2.45). □

Theorem 2.11. *[Theorem 3.2 from Zaevski (2024a)]*
Let us have another linear function $z(t) = z_1 t + z_2$ in addition to the assumptions of Theorem 2.10. It has to satisfy the following conditions:

1. *If $b_2 > 0$, then $z_1 \leq b_1$. In addition, if $z_1 = b_1$, then $z_2 < b_2$.*
2. *If $b_2 < 0$, then $z_1 \geq b_1$. In addition, if $z_1 = b_1$, then $z_2 > b_2$.*

Under these assumptions, the statements 1-3, 5 and 10 of Theorem 2.10 applied to the limit $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right]$ hold. The statements 8, 9, 13, and 14 are similar, but the critical value at which the limit changes from zero to infinity is $\frac{z_1^2}{2} - z_1\theta$ instead $\frac{b_1^2}{2} - b_1\theta$. The fourth, sixth, seventh, eleventh, and twelfth cases are devised into the following sub-cases:

4.1 *If $b_2 \neq 0$, $z_1 < \theta = b_1$ and $-\frac{\theta}{2} < k$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = \infty$.*

4.2 *If $b_2 \neq 0$, $z_1 = \theta = b_1$ and $-\frac{\theta}{2} < k \leq \frac{b_1^2}{2} - b_1\theta$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = 0$.*

4.3 *If $b_2 \neq 0$, $z_1 = \theta = b_1$ and $\frac{b_1^2}{2} - b_1\theta < k$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = \infty$.*

6.1 *If $b_2 > 0$, $z_1 < \theta < b_1$, and $k = -\frac{\theta^2}{2}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = 1 - e^{2b_2(\theta - b_1)}$.*

6.2 *If $b_2 > 0$, $z_1 = \theta < b_1$, and $k = -\frac{\theta^2}{2}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = \frac{1}{2}\left(1 - e^{2b_2(\theta - b_1)}\right)$.*

6.3 *If $b_2 > 0$, $\theta < z_1 \leq b_1$, and $k = -\frac{\theta^2}{2}$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = 0$.*

7.1 *If $b_2 > 0$, $z_1 \leq \theta < b_1$, and $-\frac{\theta}{2} < k$, then $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T > z(T)}\right] = \infty$.*

7.2 *If* $b_2 > 0$, $\theta < z_1 \le b_1$ *and* $-\frac{\theta}{2} < k \le \frac{z_1^2}{2} - z_1\theta$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 0$.

7.3 *If* $b_2 > 0$, $\theta < z_1 \le b_1$ *and* $\frac{z_1^2}{2} - z_1\theta < k$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty$.

11.1 *If* $b_2 < 0$, $z_1 > \theta > b_1$, *and* $k = -\frac{\theta^2}{2}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T<z(T)}\right] = 1 - e^{2b_2(\theta-b_1)}$.

11.2 *If* $b_2 < 0$, $z_1 = \theta > b_1$, *and* $k = -\frac{\theta^2}{2}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T<z(T)}\right] = \frac{1}{2}\left(1 - e^{2b_2(\theta-b_1)}\right)$.

11.3 *If* $b_2 < 0$, $\theta > z_1 \ge b_1$, *and* $k = -\frac{\theta^2}{2}$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T<z(T)}\right] = 0$.

12.1 *If* $b_2 < 0$, $z_1 \ge \theta > b_1$, *and* $-\frac{\theta}{2} < k$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty$.

12.2 *If* $b_2 < 0$, $\theta > z_1 \ge b_1$ *and* $-\frac{\theta}{2} < k \le \frac{z_1^2}{2} - z_1\theta$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = 0$.

12.3 *If* $b_2 < 0$, $\theta > z_1 \ge b_1$ *and* $\frac{z_1^2}{2} - z_1\theta < k$, *then* $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right] = \infty$.

Proof: Having in mind the assumptions for the coefficients z_1 and z_2, we prove the statements 1-3 and 5 using similar arguments as in Theorem 2.10.

Let us consider the fourth case. The sub-case 4.1 is obvious because the second part in the limit vanishes when $z_1 < \theta$. The results in cases 4.2 and 4.3 follow from inequalities (2.33) and the whole Corollary 2.5.

The difference in the sixth statement arises from the inequalities $\theta < b_1$ and $z_1 \le b_1$ – thus we need to position z_1 w.r.t. θ.

Let us consider the seventh case. The result for $z_1 \le \theta$ is obvious. Suppose now that $\theta < z_1 < b_1$ and $z_2 < b_2$. We rewrite expectation (2.10) as

$$\mathbb{E}\left[e^{\theta B_T} I_{B_T>z(T),\zeta>T}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right) - N\left(\frac{z-T\theta}{\sqrt{T}}\right) \\ +e^{2b_2(\theta-b_1)}\left(N\left(\frac{z-T\theta-2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right)\end{array}\right]$$
$$= e^{2b_2(\theta-b_1)}\exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} N\left((\theta-b_1)\sqrt{T} + \frac{b_2}{\sqrt{T}}\right) - e^{2b_2(b_1-\theta)}N\left((\theta-b_1)\sqrt{T} - \frac{b_2}{\sqrt{T}}\right) \\ -N\left((\theta-z_1)\sqrt{T} + \frac{2b_2-z_2}{\sqrt{T}}\right) + e^{2b_2(b_1-\theta)}N\left((\theta-z_1)\sqrt{T} - \frac{z_2}{\sqrt{T}}\right)\end{array}\right] \tag{2.48}$$

If $z_1 = b_1$, then the desired result follows from Corollary 2.5. If $z_1 < b_1$ and $-\frac{\theta}{2} < k \le \frac{z_1^2}{2} - z_1\theta$, then we obtain the limits using the second inequality of Lemma 2.7. If $\frac{z_1^2}{2} - z_1\theta < k$, we apply Lemma 2.10 for $m_1 = \theta - z_1 < 0$, $m_2 = b_2 > 0$, $C = e^{2b_2(b_1 - z_1)} > 1$, and $D = b_2 - z_2 > 0$. If $z_2 > b_2$, we use presentation (2.38).

For cases 8 and 9 we use the original presentation (2.10) instead (2.48) since $b_1 < \theta$. The results can be obtained in a similar to the case 7 way via Corollary 2.5 and Lemma 2.10.

Some symmetrical arguments prove the desired results when $b_1 < 0$. □

Chapter 3

Preliminaries.

3.1 Some general remarks

We can summarize the main approach for the asset evaluation in the following steps. The underlying asset S_t is modeled as a function of some stochastic process and the time – $S_t = f(t, X_t)$. Note that this process is defined under the so-called real-world measure. Usually, this function is the exponent. Also, we have a risk-free asset whose rate of return often is assumed to be a constant r. The main pricing theorem states that all traded assets are martingales after discounting with the risk-free rate under some new measure $\mathbb{Q}$ – the risk-neutral measure. Using this theorem, we see that $e^{-rt}S_t = e^{-rt}f(t, X_t)$ has to be a $\mathbb{Q}$-martingale. Assuming that X_t is a Feller-Markov process with infinitesimal generator $\mathcal{A}^{\mathbb{Q}}$ w.r.t. the measure $\mathbb{Q}$, we conclude that the martingality is equivalent to

$$f_t + \mathcal{A}^{\mathbb{Q}}F - rf = 0.$$

This way we derive the dynamics of the process X_t under the measure $\mathbb{Q}$. For example, in the Black-Scholes model the asset is presented by a log-normal diffusion and the risk-neutral condition appears as an equivalence between its drift and the risk-free rate. On the other hand, the Black-Scholes model can be viewed as a particular case of the exponential Lévy models, i.e. the underlying asset is presented as an exponent of a Lévy process – $f(t, x) = e^x$ and X_t is a Lévy process. If its characteristic exponent under the risk-neutral measure is denoted by $\psi^{\mathbb{Q}}(x)$, then the risk-neutral condition appears as

$$r + \psi^{\mathbb{Q}}(-i) = 0. \tag{3.1}$$

Thus, we have first to identify the set of the equivalent measures $\mathbb{Q}$ and then to find a measure that satisfies condition (3.1). Note that the choice of this measure may be not unique. Something more, the Black-Scholes model is one of the few that leads to a unique risk-neutral measure – condition (3.1) leads to $\mu^{\mathbb{Q}} = r$, where $\mu^{\mathbb{Q}}$ is the drift of the underlying asset under the measure $\mathbb{Q}$.

Let us discuss briefly the pricing task for European-style derivatives. They maturate at a pre-determined date. In addition, they may expire also if the underlying asset reaches some pre-defined level – for example, the barrier options. Thus we have a set $D \subset \mathbb{R}^+ \times \mathbb{R}^+$ in which the option is alive. Let us denote by ∂D the boundary of D without the part $t = 0$. The holder receives some amount when the option expires. Usually, this amount is a function of the underlying asset and the time, say $g(\cdot,\cdot)$. Thus the option pays to its holder the amount of $g(\tau, x)$ if $(t, x) \in \partial D$ and $S_t = x$. If the underlying asset is modeled by a jump-process, then this boundary condition has to be imposed not on the boundary ∂D but on the complement $\{\mathbb{R}^+ \times \mathbb{R}^+\} \backslash D$. Let τ be the first exit of the underlying asset from the set D. Let us denote by $V(t, x)$, $(t, x) \in D$, the derivative's price at the moment t if the spot price is $S_t = x$. Thus $V(\tau, S_\tau) = g(\tau, S_\tau)$. The above-mentioned pricing theorem states that the process $e^{-rt}V(t, S_t)$ is a $\mathbb{Q}$ – martingale. Therefore,

$$V(t, x) = \mathbb{E}^{\mathbb{Q},(t,x)}\left[e^{-r(\tau-t)}g(\tau, S_\tau)\right]. \tag{3.2}$$

The superscript above means the expectation under the measure $\mathbb{Q}$ assuming that $S_t = x$ – note that $x < \tau$. Let $\mathcal{A}^{\mathbb{Q}}$ be the infinitesimal generator of the underlying asset under the measure $\mathbb{Q}$. Thus Kolmogorov backward equation shows that $V(t, x)$ solves the following terminal value problem

$$\begin{aligned} &V_t(t, x) + \left(\mathcal{A}^{\mathbb{Q}}V\right)(t, x) - rV(t, x) = 0, \quad (t, x) \in D \\ &V(t, x) = g(t, x), \quad (t, x) \in \partial D. \end{aligned} \tag{3.3}$$

Equation (3.3) is the famous Black-Scholes equation. Note again that if the underlying asset is modeled by a jump process, then the boundary constraint has to be stated on the set $\{\mathbb{R}^+ \times \mathbb{R}^+\} \backslash D$. A European derivative can be evaluated using formula (3.2) or by solving equation (3.3).

The American style derivatives provide in addition a holder's right to choose the expiration date. This way an optimal stopping problem arises. Thus equation (3.3) turns into a free boundary differential task – we need to derive its solution as well as the region in which it holds. Roughly said, the approach we use in this dissertation is to maximize the holder's utility and thus approximate the so-called optimal boundary. Thereby, the free boundary task will be converted into the boundary value problem (3.3).

3.2 Black-Scholes model

Let B_t be a Brownian motion under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ which satisfies the usual conditions – the filtration is right continuous and complete, see Protter (2005). The measure $\mathbb{Q}$ is assumed to be risk neutral, i.e. all discounted cash flows are martingales. Let the risk-free rate be a positive constant during the option life, we denote it by $r \geq 0$, and the underlying asset pays continuously dividends at the rate $\delta \geq 0$ – note that it is non-negative. Let the asset be driven by the log-normal process

$$dS_t = (r - \delta)\, S_t dt + \sigma S_t dB_t \tag{3.4}$$

or equivalently

$$S_t = S_0 e^{\left(r - \delta - \frac{\sigma^2}{2}\right) + \sigma B_t}. \tag{3.5}$$

The drift is just $r - \delta$ due to the above-mentioned risk-neutral condition. Let us denote by $\mathcal{A}$ the infinitesimal generator of process (2.1), i.e. for the twice differentiable function $f(\cdot)$, $(\mathcal{A}f)(x)$ is

$$(\mathcal{A}f)(x) = (r - \delta)\, x f'(x) + \frac{\sigma^2 x^2}{2} f''(x)\,.$$

Note that the following simple statement holds.

Lemma 3.1. *Let us mark the initial value of the process S_t by a superscript, i.e. S_t^x means the value of the process at time t conditioned on $S_0 = x$. We have $S_t^y = \frac{y}{x} S_t^x$ for every positive values x and y.*

Let the strike price be the constant K and the option maturates at the moment T. Thus the put/call payoffs are

$$
\begin{aligned}
G(x) &= (x-K)^{+} \quad \text{call} \\
G(x) &= (K-x)^{+} \quad \text{put.}
\end{aligned} \tag{3.6}
$$

The prices of the related European options are derived in Black and Scholes (1973) as

$$
\begin{aligned}
V_{call} &= e^{-rT}\mathbb{E}\left[(S_T-K)^{+}\right] = S_0e^{-\delta T}N(d_1) - Ke^{-rT}N(d_2) \\
V_{put} &= e^{-rT}\mathbb{E}\left[(K-S_T)^{+}\right] = Ke^{-rT}N(d_2) - S_0e^{-\delta T}N(d_1),
\end{aligned} \tag{3.7}
$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution and the constants d_1 and d_2 are defined as

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S_0}{K} + \left(r-\delta+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\
d_2 &= \frac{\ln\frac{S_0}{K} + \left(r-\delta-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \equiv d_1 - \sigma\sqrt{T}.
\end{aligned}
$$

Formulas (3.7) are in fact solutions of the famous Black-Scholes differential equation. Its dynamics is obtained via the risk-neutrality, whereas the payoff (not necessary given by formulas (3.6)) influences the terminal condition. Thus, if the derivative price can be presented as $V(t,S_t)$ for some twice-differential function $V(t,x)$, then it satisfies the following heat-style terminal value problem

$$
\begin{aligned}
&V_t(t,x) + \mathcal{A}V(t,x) - rV(t,x) = 0. \\
&V(T,x) = G(T,x).
\end{aligned}
$$

The infinitesimal generators is applied to the second variable of the function $V(t,x)$.

Remark 3.1. *In fact, the Black-Scholes equation is a particular case of the Kolmogorov backward equation and its Feynman–Kac extension. Indeed, the risk-neutral property shows that the derivative price processes can be presented as*

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}\left[V(T, S_T)|\,\mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}\left[G(S_T)|\,\mathcal{F}_t\right].$$

Hence, the Markov property of geometric Brownian motion (3.4) shows that the function $V(t,x)$ has to satisfy the related boundary value problem.

3.3 Alternative parametrization

We shall present now an alternative parametrization suggested by Shiryaev et al. (1995a), see also Shiryaev et al. (1994a), Shiryaev et al. (1995b) or Shiryaev et al. (1994b). Suppose that a derivative maturates at a stopping time ζ paying the amount of

$$N(\zeta, S_\zeta) = e^{-\lambda\zeta}G(\zeta, S_\zeta)$$

for some functions $N(t,x)$ and $G(t,x)$ and a non-negative constant λ that we shall view as an additional discount factor. If we have an option, then payoffs (3.6) turn into

$$\begin{aligned} N(t,x) &= e^{-\lambda t}(x-K)^+ \quad \text{call} \\ N(t,x) &= e^{-\lambda t}(K-x)^+ \quad \text{put.} \end{aligned} \tag{3.8}$$

This time-dependent structure influences mainly the American-style derivatives. If the option is European, then its payoff and price are simply multiplied by the constant $e^{-\lambda T}$. We shall prove now the following theorem.

Theorem 3.1. *[See Proposition 2.3 from Zaevski (2020b)]*
A (r,λ,δ)-model is equivalent to a $(r-\delta,\lambda+\delta,0)$-one in the sense that the derivative has the same price under both models at the initial moment.

Proof: The derivative price should be

$$V(0,x;r,\lambda,\delta) = \mathbb{E}^x\left[e^{-r\zeta}\left(e^{-\lambda\zeta}G(\zeta, S_\zeta)\right)\right].$$

Let us consider the model for which the risk-free and discount rates are $\overline{r} = r-\delta$ and $\overline{\lambda} = \lambda+\delta$. The dividend rate has to be zero under this model since $e^{-(r-\delta)t}S_t$ is a $\mathbb{Q}$-martingale due to the risk-neutral condition. Therefore

$$\begin{aligned} V\left(0,x;\overline{r},\overline{\lambda},0\right) &= \mathbb{E}^x\left[e^{-\overline{r}\zeta}\left(e^{-\overline{\lambda}\zeta}G\left(\zeta,S_\zeta\right)\right)\right] \\ &= \mathbb{E}^x\left[e^{-r\zeta}\left(e^{-\lambda\zeta}G\left(\zeta,S_\zeta\right)\right)\right] \\ &= V\left(0,x;r,\lambda,\delta\right). \end{aligned}$$

This equality finishes the proof. □

The result above can be viewed as a consequence of the more general result presented below.

Proposition 3.1. *Let the constant α be such that $0 \leq \alpha \leq \delta$. Then the following relation between the prices holds*

$$V\left(t,x;r,\lambda,\delta\right) = e^{\alpha t}V\left(t,x;r-\alpha,\lambda+\alpha,\delta-\alpha\right) \tag{3.9}$$

if the derivative is not expired at the moment t.

Proof: We have

$$\begin{aligned} V\left(t,x;r,\lambda,\delta\right) &= \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}\left(e^{-\lambda\zeta}G\left(\zeta,S_\zeta\right)\right)\right] \\ &= e^{\alpha t}\mathbb{E}^x\left[e^{-(r-\alpha)(\zeta-t)}\left(e^{-(\lambda+\alpha)\zeta}G\left(\zeta,S_\zeta\right)\right)\right] \\ &= e^{\alpha t}V\left(t,x;r-\alpha,\lambda+\alpha,\delta-\alpha\right). \end{aligned}$$

This finishes the proposition. □

Remark 3.2. *Theorem 3.1 shows that we can work under non-dividend assumptions in the presence of additional discounting. It is important to note that the restrictions now have to be $\lambda \geq 0$ and $r+\lambda > 0$, but negative risk-free interest rates are possible. In Shiryaev et al. (1995a) this case is not considered. Note that the case $\lambda = 0$ is related to a model without dividends – it is well-known that this leads to qualitative differences.*

Let the stopping time ζ be the constant T. Having in mind Theorem 3.1 we can rewrite the Black-Scholes pricing formula under the assumptions of Remark 3.2 as the following theorem.

Theorem 3.2. *Let the risk-free and discount factors r and λ satisfy the conditions $\lambda \geq 0$ and $r + \lambda > 0$. Then the prices of the European-style options are*

$$V_{call} = S_0 e^{-\lambda T} N(d_1) - K e^{-(r+\lambda)T} N(d_2)$$
$$V_{put} = K e^{-(r+\lambda)T} N(d_2) - S_0 e^{-\lambda T} N(d_1),$$

where the constants d_1 and d_2 are

$$d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \equiv d_1 - \sigma\sqrt{T}.$$

Finally, we consider the Black-Scholes equation under the new parametrization.

Proposition 3.2. *Under a (r, λ, δ)-model the price function $V(t, x)$ solves the following terminal value problem*

$$\begin{aligned} &V_t(t,x) + (r-\delta) V_x(t,x) + \frac{\sigma^2 x^2}{2} V_{xx}(t,x) - rV(t,x) = 0. \\ &V(T,x) = e^{-\lambda T} G(T). \end{aligned} \tag{3.10}$$

Proof: The proof is a consequence of the fact that the underlying asset follows dynamics (3.4) under the (r, λ, δ)-model. □

Remark 3.3. *Let us discuss equation (3.9) in the light of Proposition 3.2. Note that the underlying asset follows dynamics (3.4) under both of the (r, λ, δ)- and $(r - \alpha, \lambda + \alpha, \delta - \alpha)$-models because the difference between the risk-free and dividend rates is $r - \delta$ for both models. We shall use the notation*

$$f(t,x) = V(t,x; r, \lambda, \delta)$$
$$g(t,x) = V(t,x; r-\alpha, \lambda+\alpha, \delta-\alpha).$$

Suppose that BVP (3.10) holds under the (r, λ, δ)-model. Using $g(t,x) = e^{-\alpha t} f(t,x)$ and having in mind that the function $f(t,x)$ solves BVP (3.10), we obtain

$$\begin{aligned} g_t &= e^{-\alpha t}\left(-\alpha f + f_t\right) \\ &= e^{-\alpha t}\left(-\alpha f - (r-\delta) f_x - \frac{\sigma^2 x^2}{2} f_{xx} + r f\right) \\ &= -\left(r-\delta\right) g_x - \frac{\sigma^2 x^2}{2} g_{xx} + (r-\alpha) g. \end{aligned}$$

Hence, the differential equation for $g(t,x)$ is satisfied. We can easily check that the terminal condition also holds.

We present below the relation between option prices at different moments.

Proposition 3.3. *The price of a live American-style derivative at a moment $t > 0$ is $V(t,x) = e^{-\lambda t} V(0,x)$.*

Proof: Since the asset price is driven by a Markov process, we have for the price of a live option at time $t > 0$

$$\begin{aligned} V(t,x) &= \mathbb{E}^{t,x}\left[e^{-r(\tau-t)} e^{-\lambda \tau} G(S_\tau)\right] \\ &= e^{-\lambda t}\mathbb{E}^{t,x}\left[e^{-(\lambda+r)(\tau-t)} G(S_\tau)\right] \\ &= e^{-\lambda t} V(0,x). \end{aligned}$$

We can prove the proposition for the put option in the same way. □

We shall work hereafter under the new parametrization $(r, \lambda, 0)$. It leads to some calculation facilities. Thus the underlying dynamic turns from (3.4)-(3.5) to

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dB_t \\ S_t &= S_0 e^{\left(r - \frac{\sigma^2}{2}\right) + \sigma B_t}. \end{aligned} \tag{3.11}$$

Thus the generator turns into

$$(\mathcal{A} f)(x) = r x f'(x) + \frac{\sigma^2 x^2}{2} f''(x). \tag{3.12}$$

We shall use several times the following well-known result known as Dynkin's formula.

Proposition 3.4 (Dynkin's formula)**.** *Let τ be a finite stopping time with values not less t. Then we have the following integral representation for the expectation $E^{t,x}\left[f\left(\tau, S_\tau\right)\right]$*

$$E^{t,x}\left[f\left(\tau, S_\tau\right)\right] = f\left(t,x\right) + \int_t^\tau f_t\left(u, S_u\right) + \left(\mathcal{A}f\right)\left(u, S_u\right) du. \tag{3.13}$$

In some cases the parametrization w.r.t. the time to maturity $\tau = T - t$ instead of the original (t,T)-one is more convenient. This is possible due to the Markov feature of the log-normal process. We shall write $(x;\tau)$ instead (t,x) to distinguish the notations.

3.4 First hitting properties

The following formulas can be found in Borodin and Salminen (2015), page 223, (2.0.1). They are consequences of formulas (2.6) and (2.16). We provide below a different proof – this way we introduce two very important constants that we use during the whole later work.

Proposition 3.5. *Let ζ be the first hitting moment of a Brownian motion with drift μ to the positive level a and $y \geq -\frac{\mu^2}{2}$. Then the Laplace transform of its distribution is*

$$\mathbb{E}\left[e^{-y\zeta} I_{\zeta<\infty}\right] = e^{-\left(\sqrt{\mu^2+2y}-\mu\right)a}. \tag{3.14}$$

If the value a is negative, then the Laplace transform is

$$\mathbb{E}\left[e^{-y\zeta} I_{\zeta<\infty}\right] = e^{\left(\sqrt{\mu^2+2y}+\mu\right)a}. \tag{3.15}$$

Proof: Let x be an arbitrary positive constant. We know that $e^{xB_t - \frac{x^2 t}{2}}$ is a martingale. Therefore

$$\begin{aligned}1 &= \lim_{T\to\infty} \mathbb{E}\left[e^{xB_{\zeta\wedge T}-\frac{x^2}{2}\zeta\wedge T}\right]\\ &= \mathbb{E}\left[e^{xB_\zeta \pm x\mu\zeta - \frac{x^2}{2}\zeta} I_{\zeta<\infty}\right]\\ &= \mathbb{E}\left[e^{xa-\left(\frac{x^2}{2}+\mu x\right)\zeta} I_{\zeta<\infty}\right].\end{aligned}$$

Let the constant y be defined as

$$y = \frac{x^2}{2} + \mu x.$$

or equivalently

$$x^2 + 2\mu x - 2y = 0. \tag{3.16}$$

Equation (3.16) has two roots

$$x_{1,2} = -\mu \pm \sqrt{\mu^2 + 2y}.$$

Note that $y \geq -\frac{\mu^2}{2}$. Therefore,

$$\mathbb{E}\left[e^{-y\zeta} I_{\zeta<\infty}\right] = e^{-ax_1} \quad \text{or} \quad \mathbb{E}\left[e^{-y\zeta} I_{\zeta<\infty}\right] = e^{-ax_2}.$$

We need to recognize which root is related to the positive and negative value of the boundary a. If $y = -\frac{\mu^2}{2}$, then both expectations (3.14) and (3.15) coincide since $x_1 = x_2$. Having in mind that the expectations decrease when y increases, we conclude that the root x_1 is for the positive values of a, whereas x_2 is for negatives. □

Assume now that the variable y is the total discount rate $y = r + \lambda$. We set the drift μ to be

$$\mu = \frac{r}{\sigma} - \frac{\sigma}{2}$$

because the underlying asset follows process (3.11). Under these assumptions, the roots $x_{1,2}$ turn into

$$x_{1,2} = -\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) \pm \sqrt{\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)^2 + 2\left(r + \lambda\right)}.$$

Definition 3.1. *The constants p and q are*

$$\begin{aligned} p &:= \frac{x_1 - x_2}{\sigma} = 2\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} \\ q &:= -\frac{x_2}{\sigma} = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \frac{r}{\sigma^2} - \frac{1}{2}. \end{aligned} \tag{3.17}$$

Corollary 3.1. *We have that $p \geq q + 1$. The equality holds in the undiscounted case $\lambda = 0$ – note that this means that the asset does not pay dividends.*

Proof: We can rewrite the difference $p - q - 1$ as

$$\begin{aligned} p - q - 1 &= \sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + 2\frac{\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2} + \frac{1}{2}\right) \\ &\geq \left|\frac{r}{\sigma^2} + \frac{1}{2}\right| - \left(\frac{r}{\sigma^2} + \frac{1}{2}\right) \geq 0. \end{aligned}$$

The first inequality is strong when $\lambda > 0$. Otherwise, both inequalities turn to equalities when $\lambda = 0$ because $r > 0$ in this case. □

The following lemma holds:

Lemma 3.2. *The inequality $2q + 1 > p$ is equivalent to $r > 0$.*

We shall use also the following lemma reported in Borodin and Salminen (2015) as equations (3.0.5 a & b).

Lemma 3.3. *Let ζ be the first exit of a Brownian motion with drift μ from a strip (a, b). Then we have*

$$\begin{aligned} \mathbb{E}\left[e^{-y\zeta} I_{\zeta=\zeta^a}\right] &= e^{\mu a} \frac{\sinh\left(b\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)} \\ \mathbb{E}\left[e^{-y\zeta} I_{\zeta=\zeta^b}\right] &= e^{\mu b} \frac{\sinh\left(-a\sqrt{2y+\mu^2}\right)}{\sinh\left((b-a)\sqrt{2y+\mu^2}\right)}. \end{aligned} \tag{3.18}$$

3.5 Crank–Nicolson finite difference method for American-style derivatives

We give now an efficient Crank–Nicolson finite difference approach for solving the Black-Scholes style equation in a known region:

$$\begin{aligned} &F_t(t,x) + rxF_x(t,x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t,x) - rF(t,x) = 0 \\ &F(t,c(t)) = e^{-\lambda t} G_1(x), \quad t \in (0,T) \\ &F(t,d(t)) = e^{-\lambda t} G_2(x), \quad t \in (0,T) \\ &F(T,x) = e^{-\lambda T} G(x), \quad x \in (C,D). \end{aligned} \tag{3.19}$$

The equation holds in the region $(t,x) \in \{(0,T) \times (c(t), d(t))\}$ and the boundary constraints are imposed on the lower, upper, and the right boundaries.

1. We divide the time-state space as $T \equiv t_1 > t_2 > ... > t_M \equiv 0$ and $0 \equiv x_1 < x_2 < ... < x_N \equiv c(0)$. We denote the price at the (m,n)-th node by $F(m,n)$. The values $F(1,n)$ and $F(M,n)$ are the prices at the maturity and in the initial moment, respectively, since we work backwards. We denote by C_m and D_m the values of the boundaries $c(t)$ and $d(t)$ at the nodes.

2. The terminal condition can be written as

$$F(1,n) = e^{-\lambda T} G(x_n).$$

3. Let us denote by l_m the highest n such that $x_{l_m} < C_m$. Analogously, let k_m be the lowest n such that $x_{k_m} > D_m$.

4. The lower and upper boundary conditions appear as

$$\begin{aligned} F(m,n) &= e^{-\lambda t_m} G_1(x_n) \ \forall\ m \text{ and } n \le l_m \\ F(m,n) &= e^{-\lambda t_m} G_2(x_n) \ \forall\ m \text{ and } n \ge k_m. \end{aligned}$$

5. We obtain the values of $F(m,n)$ by the Crank–Nicolson scheme using iteratively the already derived $F(i,n)$ for all n and $i < m$. The derivatives can be approximated via formulas (3.20). Thus the BVP (3.19) can be written as equation (3.21). Rearranging w.r.t. $n \in \{l_m+1, ..., k_m-1\}$ we find a linear system for $F(m,n)$, namely equations (3.22), (3.23), and (3.24).

3.5.1 Finite difference terms

$$
\begin{aligned}
F_t &= \frac{F(m-1,n) - F(m,n)}{\Delta t} \\
F &= \frac{F(m-1,n) + F(m,n)}{2} \\
F_x &= \frac{F(m-1,n) - F(m-1,n-1) + F(m,n) - F(m,n-1)}{2\Delta x} \\
F_{xx} &= \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{2(\Delta x)^2} \\
&\quad + \frac{F(m,n+1) - 2F(m,n) + F(m,n-1)}{2(\Delta x)^2}.
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
0 =& \frac{F(m-1,n) - F(m,n)}{\Delta t} + \\
&+ \frac{1}{2} r x_n \frac{F(m-1,n) - F(m-1,n-1) + F(m,n) - F(m,n-1)}{\Delta x} \\
&+ \frac{1}{4}\sigma^2 x_n^2 \left(\begin{array}{l} \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2} \\ + \frac{F(m,n+1) - 2F(m,n) + F(m,n-1)}{(\Delta x)^2} \end{array} \right) \\
&- \frac{1}{2} r \left(F(m-1,n) + F(m,n) \right).
\end{aligned} \tag{3.21}
$$

- If $n = l_m + 1$, then

$$\begin{aligned}
&F\left(m,n\right)\left(\frac{1}{\Delta t}-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{2}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}+\frac{1}{2}r\right)\\
&-F\left(m,n+1\right)\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}\\
&=F\left(m-1,n-1\right)\left(-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}\right)\\
&+F\left(m-1,n\right)\left(\frac{1}{\Delta t}+\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{2}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}-\frac{1}{2}r\right)\\
&+F\left(m-1,n+1\right)\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}\\
&-F\left(m,l_m\right)\left(\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}\right)\\
&.
\end{aligned}\tag{3.22}$$

- If $l_m+1<n<k_m-1$, then

$$\begin{aligned}
&F\left(m,n-1\right)\left(\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}\right)\\
&+F\left(m,n\right)\left(\frac{1}{\Delta t}-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{2}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}+\frac{1}{2}r\right)\\
&-F\left(m,n+1\right)\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}\\
&=F\left(m-1,n-1\right)\left(-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)2}\right)\\
&+F\left(m-1,n\right)\left(\frac{1}{\Delta t}+\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{2}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}-\frac{1}{2}r\right)\\
&+F\left(m-1,n+1\right)\frac{1}{4}\frac{\sigma^2x_n^2}{\left(\Delta x\right)^2}.
\end{aligned}\tag{3.23}$$

- If $n=k_m-1$, then

$$\begin{aligned}
&F(m,n-1)\left(\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{4}\frac{\sigma^2x_n^2}{(\Delta x)^2}\right)\\
&+F(m,n)\left(\frac{1}{\Delta t}-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{2}\frac{\sigma^2x_n^2}{(\Delta x)^2}+\frac{1}{2}r\right)\\
&=F(m-1,n-1)\left(-\frac{1}{2}\frac{rx_n}{\Delta x}+\frac{1}{4}\frac{\sigma^2x_n^2}{(\Delta x)^2}\right)\\
&+F(m-1,n)\left(\frac{1}{\Delta t}+\frac{1}{2}\frac{rx_n}{\Delta x}-\frac{1}{2}\frac{\sigma^2x_n^2}{(\Delta x)^2}-\frac{1}{2}r\right)\\
&+F(m-1,n+1)\frac{1}{4}\frac{\sigma^2x_n^2}{(\Delta x)^2}\\
&+F(m,n+1)\frac{1}{4}\frac{\sigma^2x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{3.24}$$

Chapter 4

A new approach for pricing discounted American options.

based on the paper

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Abstract: The purpose of this chapter is to present a new numerical approach for finding the early exercise boundary of discounted American options, whose payment structure is generalized by adding a new discount factor. We approximate this boundary by the exponent of piecewise linear functions maximizing the option's holder utility. Once we have derived the exercise boundary, the free boundary problem, that describes the American derivative price, turns into a partial differential equation (PDE) with known boundaries. This PDE is an extension of Kolmogorov's equation and can be converted to the heat equation, for which many numerical solvers are available. We present two different Monte Carlo methods and a finite difference approach to obtain fair option prices and give some numerical examples. In the perpetual case, we derive closed-form formulas.

4.1 Motivation and main results

As we mentioned above, the American-style derivatives are a specific extension of the European derivatives, that gives the holder the right to exercise prematurely receiving some predefined amount $N(t, x)$ at the spot price $S_t = x$. If the derivative is an American call option the holder has the right to buy the underlying asset at a predefined (strike) price of K in an arbitrary moment before the maturity. The American put gives the holder the right to sell. Thus the payment structures are $(x - K)^+$ and $(K - x)^+$, respectively. We introduce a time dependence by a new discount factor. This way the payoff functions turn into $N(t, x) = e^{-\lambda t}(x - K)^+$ and $N(t, x) = e^{-\lambda t}(K - x)^+$, i.e. the holder has a benefit to exercise earlier. In such a way these options are related to the financial instruments which lose some value over time. We prove that a model with a continuous dividend payment can be written as a non-dividend model, but with new parameters. Hence, we can assume that the dividend rate is zero without loss of generality. Note also that the above-mentioned change of parameters may lead to a negative risk-free rate.

The set $[0, T] \times \mathbb{R}^+$ which consists of all admissible values (t, S_t), can be divided into two parts. In the first part, the immediate exercise is optimal, whereas in the second one keeping the derivative is preferable. The boundary between them is called early exercise boundary or for the sake of simplicity only exercise boundary. We prove that similarly to the undiscounted case (i.e. $\lambda = 0$), the points below the exercise boundary are optimal for the put options, whereas these above the boundary are optimal for the call options. The undiscounted case of an American call is specific – we give a brief proof of the well-known fact that early exercising is never optimal and thus the early exercise boundary is infinity. On the contrary, early exercising is possible when the discount factor is positive. Both facts are confirmed by the presented numerical results.

Usually, the American option pricing task is considered as an optimal stopping problem or as a free boundary differential problem. We propose a different approach based on the holder's utility. Two main questions interest the holder of an American derivative – what is the fair price of the derivative and is the immediate exercise optimal or not? We first solve the second problem approximating the early exercise boundary. We use an exponent of piecewise linear functions imposing continuity at the nodes. We derive these functions maximizing the expected payoffs of specific derivatives. They have the same payment structure as the corresponding option, but the exercise moment is the first hit to a previously defined boundary. We obtain the desired statements applying the results of Chapter 2.2. This way we can recognize whether the immediate exercise is optimal as well as price the option accurately in real time.

If we need a denser grid, we can view the free boundary problem (BVP) as a boundary value problem for a partial differential equation with known boundary constraints. This BVP can be converted to a heat equation for which there exist many numerical solvers. We give two Monte Carlo methods that lead to the prices of the discounted American options. We compare some prices of undiscounted American put options derived by our approach with the corresponding prices derived by other existing methods. In such a way we validate our results. In addition, we present an approach based on a finite differences approximation of the BVP. Applying it to some call style options we ascertain that it works very fast and accurately.

We also examine the perpetual case in which there is no maturity date, i.e. it is infinity. Since the asset price is presented by a Markov process, the future behavior of the asset is affected only by its current value. Also, the option holder is not threatened by a forced exchange at maturity because there is not a maturity horizon. This means that he can wait for the asset price to reach the optimal value which is independent of the time. Hence, the exercise boundary is flat. Using our approach for maximizing the future payoffs, we derive closed-form formulas for the early exercise boundary as well as for the fair option price.

The chapter is organized as follows. The classical results of the Black-Scholes model are presented in Section 3.2. In Section 4.2 we define the optimal and continuation regions for the American-style derivatives. In Sections 4.4 and 4.5 are given the algorithms for deriving the early exercise boundary and the price of American put and call options, respectively. Some numerical examples are presented too. In Section 4.6 we obtain the closed-form formulas for the perpetual discounted American options.

4.2 Optimal and continuation regions for American-style derivatives

We shall work hereafter under the alternative parametrization presented in Section 3.3. Note that $\lambda \geq 0$ and $r + \lambda > 0$, but negative values for r

are possible – see Remark 3.2. Let $\mathcal{T}_{[t,T]}$ be the set of all stopping times with values between t and T, $t < T$. The set $\{(t,x) : 0 \le t \le T, x \ge 0\}$ can be divided into two parts – continuation and exercise regions. In the first one keeping the derivative is preferable, whereas in the second one, the immediate exercise is optimal. We shall denote by Υ the exercise region, by $\overline{\Upsilon}$ the continuation region, and by $c(t)$ the exercise boundary. Let τ be the exercise moment and Λ_t and Φ_t be the indicator processes $\Lambda_t = I_{\{\tau \le t\}}$ and $\Phi_t = I_{\{t<\tau\}} \equiv 1 - \Lambda_t$. Let $V(t,x)$ be the function that determines the price of the American derivative, namely $V(t, S_t)$.

We define now the exercise and continuation regions of an arbitrary American-style derivative:

Definition 4.1. *Let $0 \le t \le T$ and $x > 0$. Let $N(t,x)$ be the payoff structure, not necessarily given by formulas (3.8).*

1. *The point $(t,x) \in \Upsilon$ if for all stopping times $\zeta \in \mathcal{T}_{[t,T]}$,*

$$N(t,x) \ge \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right]. \tag{4.1}$$

2. *The point $(t,x) \in \overline{\Upsilon}$ if there exists a stopping time $\zeta \in \mathcal{T}_{[t,T]}$, such that*

$$N(t,x) < \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right].$$

Below we present a more usual way of defining both regions. If the spot price at a moment t is $S_t = x$ and the option's holder chooses a strategy ζ, then its financial result is given by the function

$$M(t,x;\zeta) = \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right].$$

The value function of the optimal stopping problem is

$$V(t,x) = \sup_{\zeta \in \mathcal{T}_{[t,T]}} M(t,x;\zeta).$$

Thus the optimal set Υ, formed by the points (t,x) at which the immediate exercise is the best holder's strategy, can be defined as

$$\Upsilon = \{(t,x) : V(t,x) = N(t,x)\}.$$

We shall use both of these definitions, but more often the first one.

4.3 Exercise region

Now we turn to the defined above discounted American options. The corresponding payoffs for the call and put options are given by formulas (3.8). We shall prove some propositions for the shape of the exercise region.

Proposition 4.1. *In the case of undiscounted American call options, $\lambda = 0$, we have that $\Upsilon = \emptyset$, i.e. early exercising is never optimal.*

Proof: We have $r > 0$ because the total discount rate is positive, $r + \lambda > 0$, and $\lambda = 0$. Suppose that $(t, x) \in \Upsilon$ and the stopping time $\zeta \in \mathcal{T}_{[t,T]}$ is arbitrary. Obviously $x = S_t > K$ because in the opposite case, the holder receives nothing. Using that $e^{-rt}S_t$ is a martingale, we derive

$$\begin{aligned}\mathbb{E}^{t,x}\left[e^{-r\zeta}N\left(\zeta, S_\zeta\right)\right] &= E^{t,x}\left[e^{-r\zeta}(S_\zeta - K)^+\right] \\ &\leq e^{-rt}\left(x-K\right) \\ &= \mathbb{E}^{t,x}\left[e^{-r\zeta}S_\zeta\right] - Ke^{-rt} \\ &< \mathbb{E}^{t,x}\left[e^{-r\zeta}\left(S_\zeta - K\right)\right] \\ &\leq \mathbb{E}^{t,x}\left[e^{-r\zeta}(S_\zeta - K)^+\right].\end{aligned}$$

The contradiction finishes the proof. □

Differently from the undiscounted case, the exercise region for an American call is not empty in the presence of a discount factor. This can be seen from the following proposition.

Proposition 4.2. *If $\lambda > 0$, the early exercise region Υ is not empty. Something more, if $(t, x) \in \Upsilon$ and $y > x$, then (t, y) is also in Υ.*

Proof: Lemma 3.1 shows that for a large enough x the inequality

$$\begin{aligned}&\mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}(S_\zeta - K)^+\right] - e^{-(r+\lambda)t}\left(x-K\right) \\ &= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(-K - S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), -e^{\lambda(\zeta-t)}S_\zeta\right)\right] + e^{-(r+\lambda)t}K \\ &= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\begin{pmatrix} K\left(e^{(r+\lambda)(\zeta-t)}-1\right) - S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), \\ -e^{\lambda(\zeta-t)}S_\zeta + e^{(r+\lambda)(\zeta-t)}K \end{pmatrix}\right] \leq 0\end{aligned}$$

is true for every $\zeta \in \mathcal{T}_{[t,T]}$ and therefore $(t, x) \in \Upsilon$. Note that S_ζ appears with a negative sign. Hence, the set Υ is not empty. Now suppose that $y > x$

and $(t,x) \in \Upsilon$. First, note that $y > x > K$. Let $\zeta \in \mathcal{T}_{[t,T]}$ be arbitrary. Using Lemma 3.1, we derive

$$\begin{aligned}
&\mathbb{E}^{t,y}\left[e^{-(r+\lambda)\zeta}(S_\zeta - K)^+\right] - e^{-(r+\lambda)t}(y-K) \\
&= \mathbb{E}^{t,y}\left[e^{-(r+\lambda)\zeta}(S_\zeta - K)^+\right] - e^{-\lambda t}E^{t,y}\left[e^{-r\zeta}S_\zeta\right] + e^{-(r+\lambda)t}K \\
&= \mathbb{E}^{t,y}\left[e^{-(r+\lambda)\zeta}\max\left(-K - S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), -e^{\lambda(\zeta-t)}S_\zeta\right)\right] + e^{-(r+\lambda)t}K \\
&\le \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(-K - S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), -e^{\lambda(\zeta-t)}S_\zeta\right)\right] + e^{-(r+\lambda)t}K \\
&= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}(S_\zeta - K)^+\right] - e^{-(r+\lambda)t}(x-K) \le 0.
\end{aligned}$$

Therefore $(t,y) \in \Upsilon$ too. □

When the option is put style, the following proposition determines its exercise region.

Proposition 4.3. *The set Υ is not empty. Let $(t,x) \in \Upsilon$ and $y < x$. Then (t,y) is also in Υ.*

Proof: We have for an arbitrary stopping time $\zeta \in \mathcal{T}_{[t,T]}$

$$\begin{aligned}
&\mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}(K - S_\zeta)^+\right] - e^{-(r+\lambda)t}(K-x) \\
&= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}(K - S_\zeta)^+\right] + e^{-\lambda t}E^{t,x}\left[e^{-r\zeta}S_\zeta\right] - e^{-(r+\lambda)t}K \\
&= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(K + \left(e^{\lambda(\zeta-t)}-1\right)S_\zeta, e^{\lambda(\zeta-t)}S_\zeta\right)\right] - e^{-(r+\lambda)t}K \qquad (4.2) \\
&= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\begin{pmatrix} -\left(e^{(r+\lambda)(\zeta-t)}-1\right)K + \left(e^{\lambda(\zeta-t)}-1\right)S_\zeta, \\ -e^{(r+\lambda)(\zeta-t)}K + e^{\lambda(\zeta-t)}S_\zeta \end{pmatrix}\right].
\end{aligned}$$

We use Lemma 3.1 to conclude that for a small enough x expression (4.2) is not positive and therefore the set Υ is not empty. Note that S_ζ appears with a positive sign. Hence, the set Υ is not empty.

Suppose now that $(t,x) \in \Upsilon$ and $y < x < K$. Let $\zeta \in \mathcal{T}_{[t,T]}$ be arbitrary. We shall use again Lemma 3.1. We have

$$\begin{aligned}
&\mathbb{E}^{t,y}\left[e^{-(r+\lambda)\zeta}(K - S_\zeta)^+\right] - e^{-(r+\lambda)t}(K-y) \\
&= \mathbb{E}^{t,y}\left[e^{-(r+\lambda)\zeta}(K - S_\zeta)^+\right] + e^{-\lambda t}E^{t,y}\left[e^{-r\zeta}S_\zeta\right] - e^{-(r+\lambda)t}K \\
&= \mathbb{E}^{t,y}\left[e^{-(r+\lambda)\zeta}\max\left(K + S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), e^{\lambda(\zeta-t)}S_\zeta\right)\right] - e^{-(r+\lambda)t}K \\
&\le \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\left(K + S_\zeta\left(e^{\lambda(\zeta-t)}-1\right), e^{\lambda(\zeta-t)}S_\zeta\right)\right] - e^{-(r+\lambda)t}K \\
&= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}(K - S_\zeta)^+\right] - e^{-(r+\lambda)t}(K-x) \le 0.
\end{aligned}$$

Therefore $(t, y) \in \Upsilon$. □

Remark 4.1. *Propositions 4.1, 4.2, and 4.3 can be proved in the perpetual case by letting T to tend to infinity.*

Propositions 4.2 and 4.3 indicate that the time-state space can be divided into two connected parts – the optimal or early exercise set and the continuation region. We shall notate the boundary between them as $c(t)$ – it is known as the early exercise or optimal boundary. The optimal points are above it for the call options whereas they are below it for the puts. The next step is to determine its value at the maturity. We shall use a similar approach to one presented in Kwok (1998), page 257.

Proposition 4.4. *The value of the exercise boundary at the maturity for a call option is given by*

$$c(T) = \max\left(\frac{r+\lambda}{\lambda}, 1\right) K. \tag{4.3}$$

Proof: We have to derive the lower value of x that belongs to the exercise region near the maturity. Note that it is not below the strike. Suppose that $(t, x) \in \Upsilon$ and therefore the option value is given by the payoff function

$$V(t, x) = N(t, x) = e^{-\lambda t}(x - K).$$

Also, as a consequence of the definition of the exercise region and inequality (4.1) we derive the variational inequality

$$\lim_{\epsilon \to 0} \frac{E^{t,x}\left[N(t+\epsilon, S_{t+\epsilon})\right] - N(t, x)}{\epsilon} = N_t(t, x) + \mathcal{A}N(t, x) - rN(t, x) \leq 0, \tag{4.4}$$

where $\mathcal{A}$ the infinitesimal generator of the log-normal process (3.11) given by formula (3.12). Note that the dividend rate is zero since we work under the $(r, \lambda, 0)$-model. Inequality (4.4) is equivalent to

$$\overline{K} \equiv \frac{r+\lambda}{\lambda} K \leq x. \tag{4.5}$$

Suppose that the risk-free rate is positive, $r > 0$, or equivalently $\overline{K} > K$. We shall prove that the exercise boundary at the maturity is just $\overline{K}$. Suppose that this is not true, i.e. the exercise boundary at the maturity is larger than $\overline{K}$. This means that there exists $x > \overline{K}$ that belongs to the continuation region near the strike. We shall use that $V(t,x) > N(t,x)$ in this set and in addition, $V(t,x)$ solves the Black-Scholes equation. Having in mind that for $x > \overline{K} > K$, the function $N(T,x)$ is differentiable, we derive

$$\begin{aligned}
0 &< \lim_{t\to T} \frac{V(t,x) - N(t,x)}{T-t} \\
&= -\lim_{t\to T} \frac{V(T,x) - V(t,x)}{T-t} + \lim_{t\to T} \frac{N(T,x) - N(t,x)}{T-t} \\
&= \mathcal{A}V(T,x) - rV(T,x) + N_t(T,x) \\
&= rxe^{-\lambda t} - re^{-\lambda t}(x-K) - \lambda e^{-\lambda t}(x-K) \\
&= e^{-\lambda t}\left[(r+\lambda)K - \lambda x\right] < 0.
\end{aligned}$$

The contradiction shows that the exercise boundary can not be higher than the critical value $\overline{K}$.

If the risk-free rate is negative, $r < 0$ (equivalent to $\overline{K} < K$) the same reasons as above show that the exercise boundary can not be larger than the strike. This finishes the proof. □

Remark 4.2. *If the discount rate is zero (and therefore $r > 0$), exercise boundary (4.3) is infinity. This corresponds to the fact that early exercising is never optimal (Proposition 4.1).*

We can prove the analogous proposition for the put options.

Proposition 4.5. *The value of the exercise boundary at the maturity for a put option is given by*

$$c(T) = \min\left(\frac{r+\lambda}{\lambda}, 1\right) K. \tag{4.6}$$

Proof: The difference with Proposition 4.4 is that we have to derive the higher value of x, for which variational inequality (4.4) holds. The payoff function of the put option is

$$N(t,x) = e^{-\lambda t}(K-x).$$

Therefore inequality (4.5) turns into

$$\frac{r+\lambda}{\lambda}K > x.$$

We finish the proof analogously to the proof of Proposition 4.4. □

Remark 4.3. *Note that if the risk-free rate is positive, then the exercise boundary value at the maturity is the strike.*

4.4 The algorithm for pricing discounted American put options

Note that Proposition 4.3 means that the form of the exercise region is $\Upsilon = \{(t,x) : t \in [0,T], x \in (0, c(t)]\}$, where $c(t)$ is the exercise boundary. We shall approximate it by the exponent of piecewise linear functions. Let the time interval $[0,T]$ be divided into n sub-intervals $0 \equiv t_0 < t_1 < ... < t_n \equiv T$. Suppose that the holder's strategy ζ is to exercise when the asset reaches the level $\exp(a_i t + b_i)$ if this happens in the interval $[t_{i-1}, t_i)$, $i = 1, 2, ..., n$. We also assume continuity at the nodes $\exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1}) \equiv C_i$. Assume that the underlying asset starts from the value x, i.e. $S_0 = x$. Therefore the exercise happens when the Brownian motion touches the level

$$\frac{1}{\sigma}\left(\left(a_i - r + \frac{\sigma^2}{2}\right)t + b_i - \log(x)\right) = A_i t_i + B_i$$

for

$$A_i = \frac{1}{\sigma}\left(a_i - r + \frac{\sigma^2}{2}\right)$$
$$B_i = \frac{b_i - \log(x)}{\sigma}.$$

Let us define a derivative which pays amount of $\exp(-\lambda(\zeta \wedge T))(K - S_{\zeta\wedge T})^+$ at the moment $\zeta \wedge T$. We denote its price by

$$
\begin{aligned}
&V\left(x;\{t_0,...,t_n\};\{C_0,...,C_n\}\right)=\mathbb{E}^x\left[e^{-(r+\lambda)(\tau\wedge T)}(K-S_{\tau\wedge T})^+\right]\\
&=\mathbb{E}^x\left[e^{-(r+\lambda)\tau}\left(K-S_\tau\right)\Lambda_T\right]+E^x\left[e^{-(r+\lambda)T}(K-S_T)^+\Phi_T\right]\\
&=K\mathbb{E}\left[e^{-\alpha_1\tau}\Lambda_T\right]-x\sum_{m=1}^{n}e^{\sigma B_m}E\left[e^{-\alpha_{2,m}\tau}I_{t_{m-1}<\tau\le t_m}\right]\\
&+Ke^{-\alpha_1 T}\mathbb{Q}\left(B_T<k,\Phi_T=1\right)-xe^{-\alpha_3 T}\mathbb{E}\left[e^{\sigma B_T}I_{B_T<k,\Phi_T=1}\right]
\end{aligned}
\tag{4.7}
$$

for

$$
\begin{aligned}
\alpha_1&=r+\lambda\\
\alpha_{2,m}&=(r+\lambda)-\left(r-\frac{\sigma^2}{2}\right)-A_m\sigma=\frac{\sigma^2}{2}-A_m\sigma+\lambda\\
\alpha_3&=\lambda+\frac{\sigma^2}{2}\\
k&=\frac{1}{\sigma}\log\left(\frac{K}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T.
\end{aligned}
\tag{4.8}
$$

Our algorithm is based on equations (4.7) and (4.8) and Propositions 2.6 and 2.7 and is as follows.

1. The value of the exercise boundary at the maturity, C_n, is given by equation (4.6).

2. We shall find the value of the exercise boundary at the previous point t_{n-1}. Let us define $C(x)$ for a fixed $x\le K$ as

$$C(x)=\arg\max\left\{C:V\left(x;\{0,t_n-t_{n-1}\};\{C,C_n\}\right)\right\}.$$

Proposition 4.3 shows that the exercise region at moment t_{n-1}, Υ_{n-1}, has the form $[0,C_{n-1}]$. Therefore

$$C_{n-1}=\max\left\{x:C(x)=x\right\} \tag{4.9}$$

Using the Markov property of S_t, we see that presentation (4.9) is equivalent to

$$C_{n-1} = \max\left\{x : V\left(x; \{0, t_n - t_{n-1}\}; \{C(x), C_n\}\right) = K - x\right\}. \quad (4.10)$$

Functions $V(x; \{0, t_n - t_{n-1}\}; \{C, C_n\})$ are calculated by the use of equations (4.7) and (4.8), Propositions 2.5 and 2.5, and Corollary 2.4. Note that if $r > 0$, the last two terms of formula (4.7) vanish because $C_n = K$ in this case and thus $(K - S_T)^+ = 0$ if $\Phi_T = 1$.

3. Suppose that we have found values C_m, C_{m+1}, ..., C_n for some $m < n$. We proceed in the same way as above to find the value of C_{m-1}. Let us fix some $x \leq K$ and denote by $C(x)$

$$C(x) = \arg\max\left\{C : V\left(x; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\}; \{C, C_m, ..., C_n\}\right)\right\}.$$

Analogously to equations (4.9) or (4.10) we can find C_{m-1} through one of the formulas

$$C_{m-1} = \max\{x : C(x) = x\}$$
$$C_{m-1} = \max\left\{x : V\left(x; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\}; \{C(x), C_m, ..., C_n\}\right) = K - x\right\}.$$

Functions $V(x; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\}; \{C, C_m, ..., C_n\})$ are again calculated using equations (4.7) and (4.8) and Propositions 2.6 and 2.7.

Typical behavior of the function $C(x) - x$ is presented in Figure 4.1a. The value of the early exercise boundary is marked by a red point. Thus we can formulate our fast approach for pricing American puts.

Fast Pricing Approach 4.1. *Once we approximate the optimal boundary through the algorithm above, we derive the option price via formulas (4.7) and (4.8) taken at the point $x = S_0$.*

Remark 4.4. *It turns out that the algorithm with four nodes produces a very precise prices – the error is in the fourth digit after the decimal point.*

Alternatively, we can approximate the optimal boundary at a much denser grid. Thus the free boundary problem for a discounted American put turns into the boundary value problem

$$\begin{aligned} &V_t\left(t,x\right)+rxV_x\left(t,x\right)+\frac{1}{2}\sigma^2x^2V_{xx}\left(t,x\right)-rV\left(t,x\right)=0 \\ &V\left(t,c\left(t\right)\right)=\exp\left(-\lambda t\right)\left(K-c\left(t\right)\right),\ t\in\left[0,T\right] \\ &V\left(T,x\right)=\exp\left(-\lambda T\right)\left(K-x\right)^+,\ x>c\left(T\right). \end{aligned} \tag{4.11}$$

The equation is satisfied in the strip $(t,x) \in \{(0,T) \times (c(t),\infty)\}$ and the boundary constraints are imposed on the lower and the right boundaries. This equation can be inverted to a heat equation for which there exist many numerical solvers. On the other hand, it is an extension of the Kolmogorov backward equation and its solution is given by

$$\begin{aligned} V\left(t,x\right)=&\,e^{-\lambda t}\mathbb{E}^{t,x}\left[e^{-(\lambda+r)(\tau-t)}\left(K-c\left(\tau\right)\right)\Lambda_T\right] \\ &+e^{-\lambda t}\mathbb{E}^{t,x}\left[e^{-(\lambda+r)(T-t)}(K-S_T)^+\Phi_T\right]. \end{aligned} \tag{4.12}$$

Since the process is Markov, it is enough to examine the case $t=0$. We shall give two Monte Carlo methods for numerical deriving of the expectations in equation (4.12). Some other methods based on Monte Carlo simulations can be found in Broadie and Glasserman (1997), Longstaff and Schwartz (2001), Rogers (2002), and Cortazar et al. (2008).

4.4.1 First Monte Carlo method for pricing discounted American put options

We simulate the Brownian motion paths dividing the time interval into N sub-intervals. We generate N normally distributed with zero expectation and standard deviation $\sqrt{T/N}$ random numbers. A Brownian path $\{B_{t_i}\},\ i=1,...,N$ is represented by the cumulative sum. We repeat this H times. If the sample path falls below the boundary before the maturity, we calculate the term p_i as

$$p_i=\exp\left(-\left(r+\lambda\right)\tau_i\right)\left(K-c\left(\tau_i\right)\right).$$

We set it to be

$$p_i = \exp\left(-\left(r+\lambda\right)T\right)\left(K - S_0\exp\left(\left(r-\frac{\sigma^2}{2}\right)T + \sigma B_T\right)\right)^+$$

in the opposite case. We calculate price (4.12) as $\left(\sum_{i=1}^{H} p_i\right)\Big/H$.

4.4.2 Second Monte Carlo method

The second Monte Carlo method is based on an approach suggested by Wang and Pötzelberger (1997), section 5. We use it to calculate the integrals in equations (2.6), (2.20), and (2.21). In the original work of Wang and Pötzelberger (1997), the hitting time is assumed to be above. To follow the same scheme we have to use the corresponding negative values $\bar{c}_m = -c_m$. This is possible since the Brownian motion is symmetric. To calculate the option price we use formula (4.7). To derive the first two terms we use Proposition 2.6. The m'th Laplace transform is obtained by the following steps

1. We generate $m-1$ normal random numbers with zero expectation and standard deviation one. They form the vector u.

2. Let D be the $(m-1)\times(m-1)$ diagonal matrix composed by values $\sqrt{T/N}$ and M be a $(m-1)\times(m-1)$ lower triangle matrix with values ones. We define the vector x as $x = MDu$.

3. We calculate the values of the function

$$w_j = e^{-\alpha t_{m-1}} L\left(t_m - t_{m-1}, \alpha; a_m, b_m - x_{m-1}\right)$$

   that appears in integral (2.17).

4. We derive the values of the function

$$v_j = v\left(x_1, ..., x_{m-1}\right) = \prod_{i=1}^{m-1} I_{x_i<\bar{c}_i}\left(1 - \exp\left(-\frac{2\left(\bar{c}_{i-1} - x_{i-1}\right)\left(\bar{c}_i - x_i\right)}{\frac{T}{N}}\right)\right).$$

5. We calculate the term $p_j = w_j v_j$.

6. We calculate the truncated Laplace transform repeating the above procedure H times and averaging $\left(\sum_{i=1}^{H} p_i\right) \Big/ H$.

The last two terms in equation (4.7) are obtained in the same way, by taking $m = n$ and changing the term $e^{-\alpha t_{m-1}} L(\cdot)$ in w (step 3) by $e^{-\alpha x_{m-1}} U(\cdot)$ and $e^{-\alpha x_{m-1}} V(\cdot)$, respectively. Note that the second product in integral (2.17) is incorporated when we generate normal random numbers and calculate the corresponding expectation.

4.4.3 Numerical results

We consider first the undiscounted case which allows us to compare the results obtained by our approach with the results derived by several existing methods. After that, we shall give some examples with positive values of the discount rate λ. We devise the time interval into 128 sub-intervals. For each node, we use our algorithm with three, four, and five steps. In such a way, by comparing the computational times and price deviations, we can obtain the optimal algorithm. The strike price is assumed to be $K = 20\$$, the risk-free rate is $r = 0.05$,[1] the volatility is $\sigma = 0.3$, and the time to maturity is $T = 1$. We present in Figure 4.1b the early exercise boundary for a non-discounted American put calculated by several methodologies. The benchmark is obtained using the Cox et al. (1979) tree method with $N = 10\ 000$ steps. We present also the boundaries obtained by the approaches of Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993). The corresponding boundary values are calculated as the largest value of x, for which the option price is equal to $K - x$. It can be seen that the boundary obtained by our five-step algorithm is very close to the binomial tree boundary. Also, the boundaries obtained by the three- and four-step algorithms are admissible and they are much closer to the binomial tree one in comparison with the boundaries derived by the methods of Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993).

Some option prices are presented in Table 4.1 – the first reported values are derived by the first Monte Carlo method; right to them we place the prices

[1] Note that Remark 4.3 says that this risk-free value leads to $c(T) = K$ and thus the last term in formula (4.12) vanishes.

obtained using the second one.[2] We use values $N = 2\ 000$ and $H = 200\ 000$. The time interval is divided to 16, 32, 64, or 128 sub-intervals. We also take the option to be at-the-money, i.e. $S_0 = K = 20$. It is important to note that the second method is significantly faster as well as converges faster. To minimize the pricing error, we calculate the prices 100 times and averaging. In addition to the results obtained by the methods of Barone-Adesi and Whaley (1987) and Bjerksund and Stensland (1993), we add the prices derived by the numerical approach of Longstaff and Schwartz (2001). The corresponding prices are calculated again 100 times and averaged. We see that our method based on the 128 sub-intervals produces values very close to the binomial tree prices. The difference between the results derived by three-, four-, and five-step algorithms is insignificant. Something more, if we use the same Monte Carlo simulation for the algorithm with different steps, the produced results are practically identical. The 16, 32, and 64 time interval divisions produce very good results too. Also, note that they are significantly better than the results obtained by methods of Barone-Adesi and Whaley (1987), Bjerksund and Stensland (1993), and Longstaff and Schwartz (2001).

All calculations are performed by the use of MATLAB with Intel i7-10510U (4.9G)/ 16GB RAM/ 1000GB SSD. All computational times are reported in Table 4.5. We can see that they increase significantly for the algorithms with more steps. On the other hand, this does not lead to distinctly more accurate prices. Hence, we can use the algorithm with three steps and 16 grid nodes.

The behavior of the early exercise boundary for the discounted American puts is presented in Figures 4.2a and 4.3a. In the first one, the time varies between zero and one, whereas for the second one $t \in [1, 20]$. The values of the discount factor λ are between zero and one. In Figure 4.3a we plotted the boundary values for the corresponding perpetual options by red points. They are calculated using the closed form formula derived in Section 4.6, equation (4.24). It can be seen that for large enough t the boundary surface tends to the perpetual values. We also present the behavior of the option prices w.r.t. the discount factor and the time in Figure 4.4a. These prices are calculated using the second Monte Carlo method. We mark by red points the prices of the corresponding perpetual puts – we use the formula given in

[2]Other numerical results with different values of the parameters are presented in Zaevski (2019b).

Theorem 4.2. It can be seen that for large t the option prices tend to the perpetual ones.

In Table 4.2 we present some prices for discounted put options. We assume that the risk-free rate is negative, $r = -0.03$. This leads to a value of the exercise boundary at the maturity below the strike – see formula (4.6). We vary the time to maturity among $\tau \in \{0.5,\ 1,\ 2,\ 3\}$, the penalty among $\lambda \in \{0.031,\ 0.04,\ 0.05,\ 0.06,\ 0.07,\ 0.08,\ 0.09,\ 0.1\}$. These values guarantee that the total discount rate is positive, $r + \lambda > 0$. All other parameters are the same – $\sigma = 0.03$, $K = 20$, and the initial asset price is amongst $S_0 \in \{13,\ 15,\ 17,\ 20,\ 22,\ 25\}$. We can observe that the option prices decrease when the discount factor increases. Something more – this relation is stronger for the longer maturities.

4.5 Pricing discounted American call options

As we can see in Proposition 4.1, the early exercise is never optimal if the discount factor is zero, $\lambda = 0$. Therefore the price of an American call coincides with the price of the corresponding European call. Suppose now that $\lambda > 0$. The algorithm described in Section 4.4 is appropriate for call options too. We shall present it briefly emphasizing the differences with the put case. Proposition 4.2 gives that the exercise region at time t has the form $[c(t), \infty)$. We use the same intervals as in the put case. The difference is that in the call case, the piecewise linear functions for the Brownian motion are above. The corresponding to (4.7) derivative prices, which we shall maximize to approximate the exercise boundary, now have the form

$$\begin{aligned} &V\left(x;\{t_0,...,t_n\};\{C_0,...,C_n\}\right) = \mathbb{E}^x\left[e^{-(r+\lambda)(\tau\wedge T)}(S_{\tau\wedge T}-K)^+\right] \\ &= \mathbb{E}^x\left[e^{-(r+\lambda)\tau}\left(S_\tau-K\right)\Lambda_T\right] + e^{-(r+\lambda)T}\mathbb{E}^x\left[(S_T-K)^+\Phi_T\right] \\ &= x\sum_{m=1}^{n}\mathbb{E}\left[e^{-\alpha_{2,m}\tau}I_{t_{m-1}<\tau\le t_m}\right] - K\mathbb{E}\left[e^{-\alpha_1\tau}\Lambda_T\right] \\ &+ xe^{-\alpha_3 T}\mathbb{E}\left[e^{\sigma B_T}I_{B_T>k,\Phi_T=1}\right] - Ke^{-\alpha_1 T}\mathbb{Q}\left(B_T>k,\Phi_T=1\right). \end{aligned} \tag{4.13}$$

Hence, the algorithm turns into

1. The value of the exercise boundary at the maturity is given by equation (4.3).

2. Suppose that we have found the values of C_m, C_{m+1},..., C_n for some $m < n$. For a fixed x we define $C(x)$ by

$$C(x) = \arg\max \{C : V(x; \{0, t_m - t_{m-1}..., t_n - t_{m-1}\}; \{C, C_m, ..., C_n\})\}.$$

Note that to calculate the Laplace transforms in equation (4.13), we use the results presented in Section 2.2. The value of C_{m-1} is derived using one of the formulas

$$C_{m-1} = \min\{x : C(x) = x\}$$
$$C_{m-1} = \min\{x : V(x; \{0, t_m - t_{m-1}..., t_n - t_{m-1}\}; \{C(x), C_m, ..., C_n\}) = x - K\}.$$

We can formulate our fast method for call option pricing:

Fast Pricing Approach 4.2. *We derive the option price via formula* (4.13) *taken at the point $x = S_0$. Note that Remark 4.4 still holds – four nodes are enough for a very precise option price.*

In Figure 4.2b we present the early exercise boundary varying the discount factor λ between 0.01 and 1 and the time between zero and one. All other parameters have the same values as in Section 4.4.3 – $r = 0.05$, $\sigma = 0.3$, and $K = 20$. We can see that for small λ's, the exercise boundary has significantly large values. This is in accordance with the fact that the early exercise is never optimal when the discount factor is zero. In Figure 4.3b we present the behavior of the early exercise boundary for longer maturities. The discount factor is taken in interval $[0.1, 1]$ whereas the time is between 1 and 20.

The above-presented Monte Carlo methods can be applied to call options too when we need to price at a much more dense grid. The derived prices are presented in Figure 4.4b. We use the second Monte Carlo method since we have seen that it is faster and produces more accurate results. The perpetual values, obtained by the use of explicit formulas (4.17) and (4.18), are plotted by red points. It can be seen again that for a large enough t the surfaces tend to the corresponding perpetual values.

In addition to these Monte Carlo methods, we present a finite difference approach for solving the Black-Scholes style equation (4.11), which in the call case turns into

$$
\begin{aligned}
&V_t(t,x) + rxV_x(t,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t,x) - rV(t,x) = 0,\ (t,x) \in (0,T) \times (0,c(t)) \\
&V(t,c(t)) = \exp(-\lambda t)(c(t) - K),\ t \in [0,T] \\
&V(T,x) = \exp(-\lambda T)(x-K)^+,\ x < c(T).
\end{aligned} \tag{4.14}
$$

We solve equation (4.14) backwards in the region $[0,T]\times[0,c(0)]$. We present below our finite difference algorithm based on the explicit method.

1. We divide the region $[0,T]\times[0,c(0)]$ into $M\times N$ nodes. We shall denote by x_n and t_m the values of x and t in the nodes, and by $F(m,n)$ our approximation for the equation solution. Note that $F(1,\cdot)$ are the solutions at maturity since we work backwards. The length of the divisions will be denoted by Δt and Δx.

2. We calculate $q < M$ uniformly taken points from the optimal exercise boundary.

3. We use a cubic spline interpolation to derive the exercise boundary values at all M points. We shall denote them by c_m, $m = 1,2,...,M$. A similar suggestion can be found in Lee (2020).

4. For every $m = 1,2,...,M$ we found the lowest node (m,n) which is above the exercise boundary c_m. We shall denote it by L_m.

5. Lower boundary condition: for every $m = 1,2,...,M$ we state $F(m,1) = 0$.

6. Upper boundary condition: for every $m = 1,2,...,M$ and $n = L_m, L_{m+1},...,N$ we state $F(m,n) = e^{-\lambda t_m}(x_n - K)$.

7. Right boundary condition: for every $n = 1,2,...,N$ we state $F(1,n) = e^{-\lambda T}(x_n - K)^+$.

8. Suppose that for some m we have derived the solution values $F(j,n)$ for every $n = 1,2,...,N$ and $j = 1,2,...,m-1$. Let $1 < n < L_m$. We discretize equation (4.14) in the point (m,n), as

$$\begin{aligned}0 =& \frac{F(m-1,n) - F(m,n)}{\Delta t} + rx_n \frac{F(m-1,n) - F(m-1,n-1)}{\Delta x} \\ &+ \frac{1}{2}\sigma^2 x_n^2 \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2} \\ &- rF(m,n)\,.\end{aligned} \tag{4.15}$$

The derivatives F_t, F_x, and F_{xx} are approximated by

$$\begin{aligned}F_t &= \frac{F(m-1,n) - F(m,n)}{\Delta t} \\ F_x &= \frac{F(m-1,n) - F(m-1,n-1)}{\Delta x} \\ F_{xx} &= \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2},\end{aligned}$$

respectively. We derive the value $F(m,n)$ rearranging equation (4.15)

$$\begin{aligned}F(m,n) =& \frac{F(m-1,n)}{1 + r\Delta t} \\ &+ \frac{\Delta t}{1 + r\Delta t}\left[\begin{array}{l} rx_n \frac{F(m-1,n) - F(m-1,n-1)}{\Delta x} \\ + \frac{1}{2}\sigma^2 x_n^2 \frac{F(m-1,n+1) - 2F(m-1,n) + F(m-1,n-1)}{(\Delta x)^2} \end{array}\right].\end{aligned}$$

We apply this algorithm to price call options with the following parameters – the risk-free rate is $r = 0.05$, the volatility is $\sigma = 0.3$, and the strike is $K = \$20$. We vary the maturity amongst $T \in \{1,\ 2,\ 3\}$, the discount factor amongst $\lambda \in \{0.01,\ 0.04,\ 0.07,\ 0.1\}$, and the initial asset value amongst $S_0 \in \{\$25,\ \$30,\ \$35,\ \$40,\ \$45,\ \$50\}$. The presented for put options results show that $q = 16$ points are quite enough to approximate the exercise boundary. We derive its values using the four-step algorithm. We made the same observation as in the put case – the three step-algorithm produces practically the same prices, but the computational time is significantly lower. We divide the time interval into $M = 500\ 000$ whereas we use a $N = 1\ 000$-division of the x-interval. We compare the derived results with the results obtained by the second Monte Carlo method, which is iterated 100 times and then averaged. Also, both results are compared with the prices obtained via the Cox-Ross-Rubinstein binomial trees with 10 000 steps. All results are presented in Table 4.3. The first values are obtained using the binomial tree method. The second reported values are calculated through the second Monte Carlo method, whereas the third values are derived by the use of the finite difference approach, presented above. We can see that the prices that the Monte Carlo method produces, are very near to the real ones, but the finite difference approach gives values that are almost identical to the binomial tree prices. We use a very frequent discretization above to obtain as accurate results as possible. This leads to some increase in computational time. In practice, it is enough to use values $M = 20\ 000$ and $N = 250$ to obtain sufficiently accurate results – we report them in Table 4.4. Also, we present in this table the Monte Carlo results with only one iteration. We can observe that the finite difference approach derives very accurate results, whereas the Monte Carlo method exhibits some fluctuations.

The computational times for all algorithms are presented in Table 4.5. The necessary time for deriving the exercise boundaries for the Monte Carlo method is the same as for the put style options. We can see that the finite difference method works very fast and produces extremely accurate prices. Something more, the finite difference approach allows deriving the whole price structure w.r.t. the time as well as w.r.t. the initial asset price by one iteration.

4.6 Perpetual American options

Now we shall examine options without maturity restrictions, i.e. $T = \infty$. As we already mentioned, the early exercise boundary is flat in the perpetual case. Hence, we can suppose that it is a constant denoted by c. Let the option be a call and suppose that $t = 0$. We shall denote its price by $V(x)$ assuming that the initial asset value is x. If the discount rate is zero, Proposition 4.1 together with Remark 4.1 shows that the early exercise is never optimal. Therefore, the price of a perpetual American call can be derived by getting the limit $T \to \infty$ in Black-Scholes formula (3.2). Therefore $V(x) = x$. Suppose now that $\lambda > 0$. We need also the following constant

$$d = \frac{\ln K - \ln x}{\sigma}. \tag{4.16}$$

The following theorem holds.

Theorem 4.1. *The price of a discounted American perpetual call option written on the asset with an initial price below the exercise boundary* $S_0 = x < c$ *and strike price* K *is*

$$V(x) = \left(\frac{x}{p-q}\right)^{p-q}\left(\frac{p-q-1}{K}\right)^{p-q-1}, \tag{4.17}$$

where p *and* q *are defined by formulas* (3.17)*. Note that* $p > q+1$ *when* $\lambda > 0$ *due to Corollary 3.1.*

If the initial asset value is above the exercise boundary, then the option price is $V(x) = x - K$*. The exercise boundary is*

$$c = \frac{p-q}{p-q-1}K. \tag{4.18}$$

The optimal stopping moment is the first hit to the interval $[c, \infty)$*.*

Proof: Let the optimal boundary be the constant c and the stopping time ζ be the first hit of the underlying asset to this level. We can assume without any restrictions that the boundary c is above the strike, $c > K$, since otherwise, the option pays nothing. Having in mind that the asset price can be written as

$$S_t = e^{\ln x + \left(r - \frac{\sigma^2}{2}\right)t + \sigma B_t},$$

The stopping time ζ can be viewed also as the first hitting time of the Brownian motion with drift

$$\mu = \frac{r}{\sigma} - \frac{\sigma}{2} \tag{4.19}$$

to the value

$$a = \frac{\ln c - \ln x}{\sigma} > 0. \tag{4.20}$$

Note that $a > d$, d is given by formula (4.16), because $c > K$. We can see that ζ can be viewed also as the first hit of the Brownian motion to the linear function $b(t) = b_1 t + b_2$ for $b_1 = -\mu$ and b_2 is given by equation (4.20). Using equation (3.14) from Proposition 3.5 we derive

$$\begin{aligned}
V(x;c) &= \mathbb{E}^x\left[e^{-(r+\lambda)\zeta}(S_\zeta - K) I_{\zeta<\infty}\right] + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}(S_T-K)^+ I_{T<\zeta}\right] \\
&= (c-K)\,\mathbb{E}^x\left[e^{-(r+\lambda)\zeta} I_{\tau<\infty}\right] + \lim_{T\to\infty} e^{-\left(\lambda+\frac{\sigma^2}{2}\right)T}\mathbb{E}^x\left[e^{\sigma B_t} I_{T<\zeta, B_T>b_1 T+d}\right] \\
&- K\lim_{T\to\infty} e^{-(r+\lambda)T}\mathbb{Q}(T<\zeta, B_T>b_1T+d).
\end{aligned} \tag{4.21}$$

Obviously, the second limit is zero. The second statement of Theorem 2.11, see also Theorem 2.10, shows that the first limit is zero too. Hence,

$$\begin{aligned}
V(x;c) &= (c-K)\,\mathbb{E}^x\left[e^{-(r+\lambda)\zeta} I_{\tau<\infty}\right] \\
&= (c-K)\exp\left\{-\left(\sqrt{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^2 + 2(r+\lambda)} - \left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right)\frac{\ln c - \ln x}{\sigma}\right\} \\
&= (c-K)\exp\left\{-\left(\sqrt{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\right)(\ln c - \ln x)\right\} \\
&= (c-K)\left(\frac{c}{x}\right)^{-(p-q)}.
\end{aligned} \tag{4.22}$$

Elementary calculations show that function (4.22) has maximum w.r.t. the variable c when its value is just (4.18). Therefore, equation (4.22) leads to option price (4.21).

$$\begin{aligned}
c(x) &= (c-K)\left(\frac{c}{x}\right)^{-(p-q)} \\
&= \left(\frac{x}{p-q}\right)^{p-q}\left(\frac{p-q-1}{K}\right)^{p-q-1}
\end{aligned}$$

that confirms equation (4.17). □

Remark 4.5. *Let us see what happens when $\lambda = 0$. Corollary 3.1 leads to $p = q+1$ and therefore equation (4.18) turns into $c = \infty$. Hence, the optimal stopping time does not exist as we already proved in Proposition 4.1. Also, formula (4.17) turns into $V(x) = x$. The second limit in formula (4.21) is again zero, but we need to use the fifth statement of Theorem 2.11 instead of the second one. Note that $b_1 < \sigma$ because $r > 0$ when $\lambda = 0$.*

Analogously, we can prove a theorem for the price of a discounted perpetual American put.

Theorem 4.2. *If the initial asset price is above the early exercise boundary, $S_0 = x > c$, then the price of the perpetual American put is*

$$V(x) = \left(\frac{K}{q+1}\right)^{q+1}\left(\frac{q}{x}\right)^{q}, \tag{4.23}$$

where q is defined in formulas (3.17). If the initial asset value is below the exercise boundary, then the option price is

$$V(x) = K - x.$$

The exercise boundary is

$$c = \frac{q}{q+1}K. \tag{4.24}$$

The optimal stopping moment is the first hit to the interval $[0, c]$.

Proof: Analogously to the proof of Theorem 4.1, we see that the stopping time is the first hitting time of the Brownian motion with drift (4.19) to the value (4.20), which in this case is negative. Note again that ζ is the first hit of the Brownian motion to the linear function $b(t) = b_1 t + b_2$ for $b_1 = -\mu$ and b_2 is given by equation (4.20). Note also that $d > b_2$ for the put options since $K > c$. We have to use equation (3.15) to derive

$$\begin{aligned} V(x;c) &= E^x\left[e^{-(r+\lambda)\tau}(K - S_\tau)I_{\tau<\infty}\right] + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}(K - S_T)^+ I_{T<\zeta}\right] \\ &= (K - c)E^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right] - \lim_{T\to\infty} e^{-\left(\lambda+\frac{\sigma^2}{2}\right)T}\mathbb{E}^x\left[e^{\sigma B_t}I_{T<\zeta, B_T<b_1T+d}\right] \\ &+ K\lim_{T\to\infty} e^{-(r+\lambda)T}\mathbb{Q}(T<\zeta, B_T < b_1T + d). \end{aligned}$$

The second limit is zero. If $\lambda > 0$, then the second statement of Theorem 2.10 shows that the first limit is zero too. If $\lambda = 0$, then point 11.3 from the same theorem shows that the limit is again zero. Note that $b_1 < \sigma$ because $r > 0$ when $\lambda = 0$. Hence,

$$V(x;c) = (K-c)\exp\left\{\left(\sqrt{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^2+2(r+\lambda)}+\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right)\frac{\ln c-\ln x}{\sigma}\right\}$$
$$= (K-c)\left(\frac{c}{x}\right)^{\gamma}. \tag{4.25}$$

Formula (4.25) has a maximum when c is as in equation (4.24). Thus price (4.25) turns into formula (4.23). □

The following proposition is a corollary from Theorems 4.1 and 4.2 and gives the option prices at some moment $t > 0$.

Proposition 4.6. *The price of a live perpetual American option in moment $t > 0$ is $V(t,x) = e^{-\lambda t}V(x;c)(x)$, where the function $V(x)$ is given by formula (4.17) for a call option and by (4.23) for a put.*

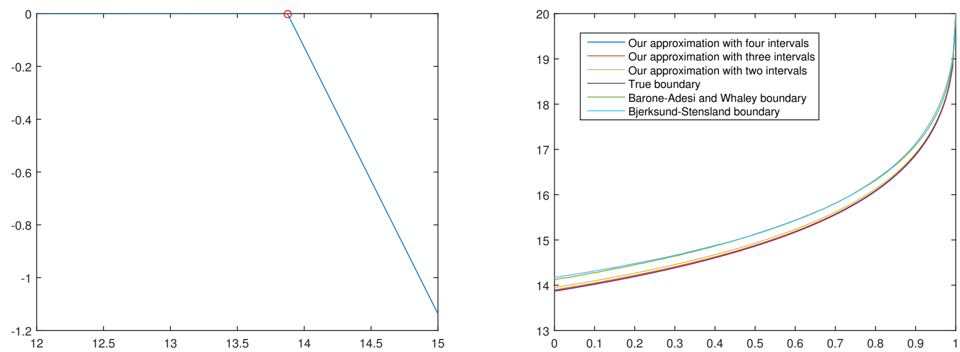
Proof: We have to apply Proposition 3.3. □

Remark 4.6. *We can easily check that the function $V(t,x)$ is the solution of the boundary value problems*

$$f_t(t,x) + \mathcal{A}f(t,x) - rf(t,x) = 0$$
$$f(t,c) = e^{-\lambda t}\frac{K}{\gamma \mp 1} \equiv e^{-\lambda t}\begin{cases} c-K & \text{call} \\ \textit{or} & \\ K-c & \text{put.}\end{cases}$$

The constant γ is equal to $p-q$ for a call option and $\gamma = q$ for a put – p and q are defined via formulas (3.17). The equations for the call and put options hold in the regions $(0,c)$ and (c,∞), respectively. Also, we can easily check that the function $V(t,x)$ is continuously differentiable in the point c. This is not accidental – it is a manifestation of the so-called smooth fit principle, that gives the relation between optimal stopping and variational inequalities – see McKean (1965), Brekke and Øksendal (1990), Shiryaev (2009), or Bensoussan and Lions (2011).

Figure 4.1: Early exercise boundary of a put option. The parameters are $r = 0.05$, $\lambda = 0$, $\sigma = 0.3$, $K = 20$, and $T = 1$.



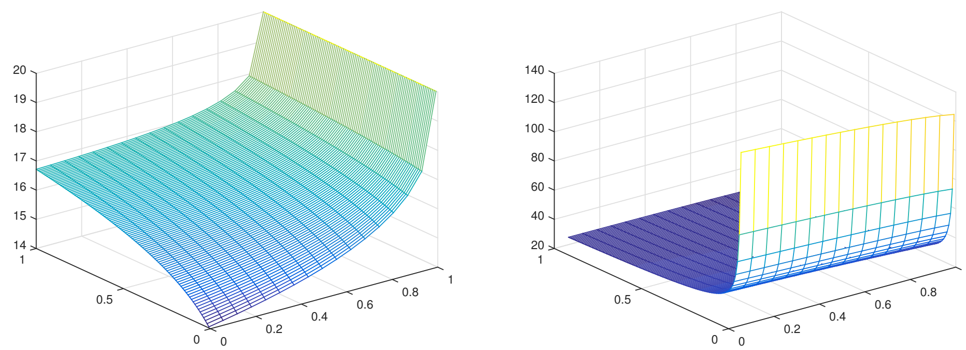
(a) Deriving the boundary

(b) Put boundary

4.7 Conclusions

In this chapter, we have examined the problem of pricing discounted American options. We first approximate the early exercise boundary by the exponent of piecewise linear functions. This turns the free boundary problem for American option pricing into a boundary value problem with known boundaries. We presented two Monte Carlo methods for deriving the option prices as well as a finite difference approach. The reported results confirm the consistency of our approach. We also conclude that the second Monte Carlo method works and converges faster than the first one, but the finite difference method produces very accurate results for a very small computational time. Also, we conclude that relatively small numbers of the grid nodes and the algorithm steps produce very good results. This diminishes significantly the computation time. We also derived the closed-form formulas for the perpetual options.

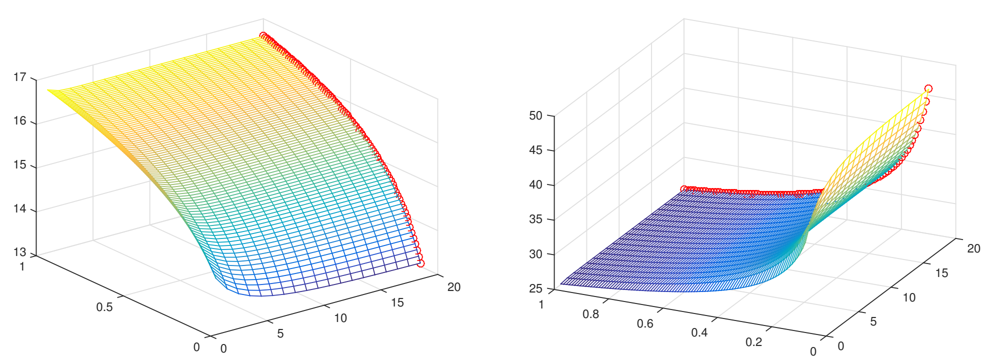
Figure 4.2: Early exercise boundary – short time range. The parameters for the put options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0, 1)$, and $T \in (0, 1)$. The values for the call style options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0.01, 1)$, and $T \in (0, 1)$. The algorithm with three steps is used.



(a) The put exercise boundary

(b) The call exercise boundary

Figure 4.3: Early exercise boundary – long time range. The parameters for the put options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0, 1)$, and $T \in (0, 20)$. The values for the call style options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0.1, 1)$, and $T \in (0, 20)$. The algorithm with three steps is used.



(a) The put exercise boundary

(b) The call exercise boundary

Table 4.1: American put option prices. The parameters are $r = 0.05$, $\lambda = 0$, $\sigma = 0.3$, $K = 20$, and $T = 1$. Our approach is presented by three, four, and five steps for deriving the exercise boundary. Versions with 16, 32, 64, and 128 grid nodes are provided. The first and the second values are the prices obtained by both of Monte Carlo methods. We present also the binomial tree prices, as well as the prices obtained by the methods of Bjerksund and Stensland (1993), Longstaff and Schwartz (2001), and Barone-Adesi and Whaley (1987).

	Binomial tree	Our model – 16 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0774/5.0775	5.0776/5.0783	5.0776/5.0769	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5520/3.5516	3.5521/3.5514	3.5522/3.5522	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9707/1.9691	1.9708/1.9693	1.9701/1.9695	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2910/1.2888	1.2911/1.2891	1.2909/1.2893	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6613/0.6600	0.6614/0.6606	0.6613/0.6601	0.6563	0.6803	0.6758
	Binomial tree	Our model – 32 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0779/5.0778	5.0779/5.0778	5.0778/5.0795	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5534/3.5530	3.5536/3.5528	3.5536/3.5533	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9721/1.9713	1.9725/1.9717	1.9725/1.9706	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2929/1.2922	1.2931/1.2927	1.2929/1.2923	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6629/0.6625	0.6630/0.6627	0.6630/0.6627	0.6563	0.6803	0.6758
	Binomial tree	Our model – 64 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0778/5.0782	5.0783/5.0782	5.0783/5.0789	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5541/3.5533	3.5540/3.5546	3.5540/3.5540	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9733/1.9724	1.9735/1.9729	1.9736/1.9740	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2938/1.2937	1.2939/1.2933	1.2940/1.2940	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6637/0.6630	0.6640/0.6635	0.6637/0.6639	0.6563	0.6803	0.6758
	Binomial tree	Our model – 128 nodes grid			B-S	L-S	BA-W
initial price		3 steps	4 steps	5 steps			
$S_0 = 13$	7.0000	7.0000/7.0000	7.0000/7.0000	7.0000/7.0000	7.0000	7.0163	7.0000
$S_0 = 15$	5.0786	5.0782/5.0768	5.0785/5.0778	5.0786/5.0772	5.0703	5.1133	5.0592
$S_0 = 17$	3.5546	3.5541/3.5549	3.5545/3.5544	3.5545/3.5547	3.5399	3.5706	3.5381
$S_0 = 20$	1.9740	1.9739/1.9729	1.9737/1.9748	1.9737/1.9731	1.9582	1.9841	1.9758
$S_0 = 22$	1.2945	1.2943/1.2945	1.2941/1.2929	1.2941/1.2940	1.2816	1.3142	1.3030
$S_0 = 25$	0.6643	0.6640/0.6638	0.6640/0.6639	0.6640/ 0.6642	0.6563	0.6803	0.6758

Table 4.2: American put option prices – discounted case. The used parameters are $r = -0.03$, $\sigma = 0.3$, $K = 20$, $\lambda \in \{0.031, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1\}$, and $T \in \{0.5, 1, 2, 3\}$. The time grid has 16 nodes; the algorithm is with three steps. The second Monte Carlo method is used.

	$\lambda = 0.031$	$\lambda = 0.04$	$\lambda = 0.05$	$\lambda = 0.06$	$\lambda = 0.07$	$\lambda = 0.08$	$\lambda = 0.09$	$\lambda = 0.1$
$T = 0.5$								
$S_0 = 13$	7.2144	7.1783	7.1421	7.1113	7.0753	7.0518	7.0352	7.0239
$S_0 = 15$	5.3459	5.3207	5.2933	5.2728	5.2435	5.2219	5.2026	5.1820
$S_0 = 17$	3.6913	3.6723	3.6551	3.6349	3.6167	3.5976	3.5785	3.5689
$S_0 = 20$	1.8296	1.8219	1.8115	1.8053	1.7917	1.7889	1.7781	1.7696
$S_0 = 22$	1.0484	1.0424	1.0386	1.0302	1.0263	1.0222	1.0158	1.0120
$S_0 = 25$	0.4039	0.4029	0.4008	0.3987	0.3970	0.3955	0.3934	0.3917
$T = 1$								
$S_0 = 13$	7.4999	7.4357	7.3652	7.3024	7.2419	7.1974	7.1660	7.1251
$S_0 = 15$	5.8151	5.7651	5.7085	5.6562	5.6036	5.5553	5.5217	5.4824
$S_0 = 17$	4.3518	4.3081	4.2709	4.2269	4.1898	4.1534	4.1186	4.0907
$S_0 = 20$	2.6553	2.6295	2.6015	2.5767	2.5516	2.5274	2.5086	2.4870
$S_0 = 22$	1.8442	1.8318	1.8123	1.7921	1.7737	1.7611	1.7391	1.7264
$S_0 = 25$	1.0286	1.0186	1.0082	1.0025	0.9903	0.9792	0.9723	0.9628
$T = 2$								
$S_0 = 13$	8.1410	8.0041	7.8523	7.7230	7.6223	7.5280	7.4635	7.4024
$S_0 = 15$	6.6731	6.5589	6.4308	6.3218	6.2252	6.1371	6.0676	5.9986
$S_0 = 17$	5.4104	5.3055	5.2042	5.1168	5.0343	4.9526	4.8748	4.8106
$S_0 = 20$	3.8666	3.7982	3.7256	3.6529	3.5859	3.5336	3.4735	3.4185
$S_0 = 22$	3.0658	3.0102	2.9488	2.8976	2.8444	2.7933	2.7484	2.7049
$S_0 = 25$	2.1439	2.1075	2.0634	2.0265	1.9838	1.9503	1.9171	1.8804
$T = 3$								
$S_0 = 13$	8.7434	8.5121	8.2875	8.1108	7.9558	7.8369	7.7318	7.6422
$S_0 = 15$	7.4200	7.2249	7.0267	6.8641	6.7122	6.5916	6.4825	6.3747
$S_0 = 17$	6.2585	6.0947	5.9232	5.7791	5.6455	5.5291	5.4201	5.3231
$S_0 = 20$	4.8275	4.6914	4.5649	4.4392	4.3335	4.2328	4.1410	4.0585
$S_0 = 22$	4.0485	3.9348	3.8260	3.7209	3.6255	3.5379	3.4556	3.3781
$S_0 = 25$	3.0941	3.0136	2.9293	2.8432	2.7681	2.6962	2.6376	2.5721

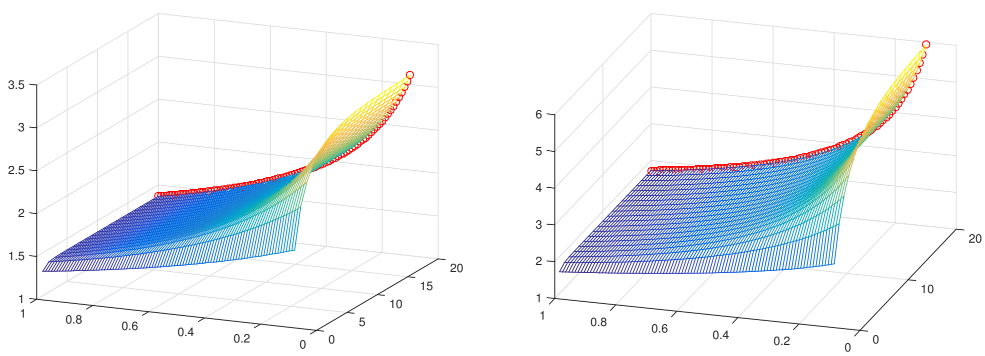
Table 4.3: American call option prices – high frequency grid. The used parameters are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in \{0.01, 0.04, 0.07, 0.1\}$, and $T \in \{1, 2, 3\}$. The grid for the finite difference approach has $M = 500\ 000$ time nodes, and $N = 1\ 000$ x-nodes. The first values are obtained by the binomial tree method, the second values by the second Monte Carlo method, and the third values by the finite difference approach.

	T=1	T=2	T=3
$\lambda = 0.01$			
$S_0 = 25$	6.5497/6.5492/6.5467	7.8889 /7.8869/7.8839	8.9776/8.9760/8.9713
$S_0 = 30$	11.0641/11.0634/11.0626	12.2126/12.2112/12.2093	13.2259/13.2247/13.2213
$S_0 = 35$	15.8756/15.8770/15.8750	16.8346/16.8355/16.8326	17.7389/17.7385/17.7357
$S_0 = 40$	20.7842/20.7808/20.7840	21.6013/21.5991/21.6001	22.3989/22.4012/22.3966
$S_0 = 45$	25.7221/25.7212/25.7221	26.4373/26.4371/26.4366	27.1413/27.1443/27.1398
$S_0 = 50$	30.6688/30.6716/30.6688	31.3068/31.3110/31.3064	31.9314/31.9350/31.9303
$\lambda = 0.04$			
$S_0 = 25$	6.3571/6.3541/6.3560	7.4446/7.4439/7.4427	8.2591/8.2628/8.2567
$S_0 = 30$	10.7429/10.7413/10.7424	11.5481/11.5468/11.5469	12.2181/12.2233/12.2164
$S_0 = 35$	15.4283/15.4284/15.4281	15.9655/15.9576/15.9648	16.4717/16.4748/16.4705
$S_0 = 40$	20.2301/20.2361/20.2300	20.5649/20.5630/20.5644	20.9234/20.9244/20.9225
$S_0 = 45$	25.0958/25.0919/25.0958	25.2866/25.2880/25.2863	25.5227/25.5156/25.5221
$S_0 = 50$	30.0181/30.0179/30.0181	30.1060/30.1047/30.1058	30.2437/30.2444/30.2432
$\lambda = 0.07$			
$S_0 = 25$	6.1850/6.1851/6.1841	7.0945/7.0920/7.0931	7.7374/7.7360/7.7356
$S_0 = 30$	10.4893/10.4889/10.4889	11.0874/11.0823/11.0864	11.5667/11.5712/11.5654
$S_0 = 35$	15.1481/15.1496/15.1479	15.4653/15.4639/15.4647	15.7714/15.7697/15.7705
$S_0 = 40$	20.0089/20.0080/20.0088	20.1186/20.1243/20.1183	20.2732/20.2680/20.2726
$S_0 = 45$	25.0000/25.0000/25.0000	25.0003/25.0001/25.0003	25.0316/25.0347/25.0314
$S_0 = 50$	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000
$\lambda = 0.1$			
$S_0 = 25$	6.0440/6.0423/6.0432	6.8218/6.8198/6.8206	7.3406/7.3413/7.3391
$S_0 = 30$	10.3176/10.3187/10.3172	10.7693/10.7720/10.7685	11.1170/11.1151/11.1160
$S_0 = 35$	15.0274/15.0278/15.0273	15.1914/15.1923/15.1910	15.3658/15.3652/15.3651
$S_0 = 40$	20.0000/20.0000/ 20.0000	20.0002/20.0001/20.0002	20.0271/20.0279/20.0269
$S_0 = 45$	25.0000/25.0000/ 25.0000	25.0000/25.0000/25.0000	25.0000/25.0000/25.0000
$S_0 = 50$	30.0000/30.0000/ 30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000

Table 4.4: American call option prices – low frequency grid. The used parameters are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in \{0.01, 0.04, 0.07, 0.1\}$, and $T \in \{1, 2, 3\}$. The grid for the finite difference approach has $M = 20\ 000$ time nodes, and $N = 250$ x-nodes. First are reported binomial tree values, after them are the values derived by the second Monte Carlo method, and the third values are obtained by the finite difference approach.

	T=1	T=2	T=3
$\lambda = 0.01$			
$S_0 = 25$	6.5497/6.5374/6.5365	7.8889 /7.9319/7.8685	8.9776/8.9786/8.9522
$S_0 = 30$	11.0641/11.0810/11.0580	12.2126/12.2822/12.1993	13.2259/13.2309/13.2075
$S_0 = 35$	15.8756/15.8597/15.8733	16.8346/16.8704/16.8267	17.7389/17.6860/17.7261
$S_0 = 40$	20.7842/20.7670/20.7833	21.6013/21.5840/21.5967	22.3989/22.4813/22.3901
$S_0 = 45$	25.7221/25.7044/25.7218	26.4373/26.4136/26.4346	27.1413/27.0754/27.1354
$S_0 = 50$	30.6688/30.6498/30.6686	31.3068/31.3738/31.3052	31.9314/31.9680/31.9273
$\lambda = 0.04$			
$S_0 = 25$	6.3571/6.3715/6.3525	7.4446/7.4297/7.4372	8.2591/8.3046/8.2496
$S_0 = 30$	10.7429/10.8055/10.7407	11.5481/11.5695/11.5432	12.2181/12.1778/12.2109
$S_0 = 35$	15.4283/15.4322/15.4273	15.9655/15.9233/15.9623	16.4717/16.4139/16.4663
$S_0 = 40$	20.2301/20.2321/20.2296	20.5649/20.5862/20.5628	20.9234/20.9299/20.9194
$S_0 = 45$	25.0958/25.1078/25.0955	25.2866/25.2922/25.2852	25.5227/25.5169/25.5198
$S_0 = 50$	30.0181/30.0024/30.0179	30.1060/30.1112/30.1051	30.2437/30.2085/30.2416
$\lambda = 0.07$			
$S_0 = 25$	6.1850/6.1967/6.1814	7.0945/7.1031/7.0888	7.7374/7.7462/7.7300
$S_0 = 30$	10.4893/10.5242/10.4875	11.0874/11.0556/11.0834	11.5667/11.5803/11.5610
$S_0 = 35$	15.1481/15.1647/15.1472	15.4653/15.4483/15.4627	15.7714/15.7806/15.7673
$S_0 = 40$	20.0089/20.0005/20.0086	20.1186/20.1092/20.1172	20.2732/20.2294/20.2705
$S_0 = 45$	25.0000/25.0000/25.0000	25.0003/25.0012/25.0001	25.0316/25.0000/25.0304
$S_0 = 50$	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000
$\lambda = 0.1$			
$S_0 = 25$	6.0440/6.0455/6.0409	6.8218/6.8255/6.8168	7.3406/7.3565/7.3344
$S_0 = 30$	10.3176/10.2889/10.3160	10.7693/10.7896/10.7658	11.1170/11.1333/11.1123
$S_0 = 35$	15.0274/15.0205/15.0269	15.1914/15.2057/15.1895	15.3658/15.3476/15.3627
$S_0 = 40$	20.0000/20.0000/ 20.0000	20.0002/20.0000/20.0000	20.0271/20.0267/20.0259
$S_0 = 45$	25.0000/25.0000/ 25.0000	25.0000/25.0000/25.0000	25.0000/25.0000/25.0000
$S_0 = 50$	30.0000/30.0000/ 30.0000	30.0000/30.0000/30.0000	30.0000/30.0000/30.0000

Figure 4.4: American option prices. The used parameters for the put options are $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0, 1)$, and $T \in (0, 20)$. For the call style options we use values $r = 0.05$, $\sigma = 0.3$, $K = 20$, $\lambda \in (0.1, 1)$, and $T \in (0, 20)$. The second Monte Carlo method is used for option pricing.



(a) The put option prices (b) The call option prices

Table 4.5: Computational times in seconds

	Time
Deriving a point from the exercise boundary – three points algorithm	0.8698
Deriving a point from the exercise boundary – four points algorithm	23.5644
Deriving a point from the exercise boundary – five points algorithm	437.8964
Pricing – first Monte Carlo method, 100 iterations	182.0521
Pricing – first Monte Carlo method, 1 iterations	1.5644
Pricing – second Monte Carlo method, 100 iterations	143.0678
Pricing – second Monte Carlo method, 1 iterations	1.3534
Pricing – finite difference method, high frequency	104.4804
Pricing – finite difference method, low frequency	0.9852

Chapter 5

Pricing discounted American capped options

based on the paper

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Abstract: The purpose of this chapter is to present an efficient pricing method for discounted American capped options. They differ from the corresponding uncapped ones by the existing trigger level for the underlying asset. In such a way the option's writer is preserved from the possible large movements of the underlying asset. We first obtain the optimal exercise region and by the use of some hitting properties, we derive the fair option price. We use the Crank–Nicolson finite difference approach together with a Monte Carlo method to implement the obtained formulas. This method applied for the pricing problem of the ordinary American options has its own significance. Finally, we present some numerical results.

5.1 Motivation and main results

To meet the financial investors' expectations for additional protection, several instruments are listed in the markets. They preserve the derivative's writer against the extremely high movements of the underlying asset. Such instruments are the so-called capped options for which the price of the underlying asset at which the option can be exercised is capped by a predetermined value. A seminal work for these options based on the Black-Scholes framework is Broadie and Detemple (1995). Later Detemple and Tian (2002) examines capped options under a general diffusion process. Several researchers use the capped style options to evaluate the ordinary American options – see Broadie and Detemple (1996) and Deng and Peng (2014).

Our approach for pricing caped options is based on deriving the option's holder optimal boundary. We prove two theorems for the call and put options. The first one says that the exercise boundary of an American capped call option is the minimum between the optimal boundary of the uncapped option and the cap. Analogously, the second theorem states that the optimal boundary of the capped put option is the larger between the cap value and the optimal boundary of the uncapped option. A similar result is obtained in Broadie and Detemple (1995). We have to mention that we prove these two theorems using an approach based on the infinitesimal generators. In such a way the theorems can be easily proven under more general assumptions – for example for models driven by Feller-Markov processes.

Using the approach presented in Chapter 4 we can approximate the exercise boundary of the uncapped option. Hence, we can obtain the early exercise boundary of the capped option using the above-mentioned theorems. It divides the state space of the underlying asset into two parts – the continuation region and the early exercise region. In the first one, it is optimal for the holder to keep the option, whereas in the second one, the immediate exercise leads to a better financial result. It is well known that the pricing of an American-style derivative is closely related to a free boundary problem arising from the necessity of identifying the optimal stopping moment. Since we have approximated the exercise boundary, we arrive to a partial differential equation (PDE, hereafter) satisfied in a known region – namely the continuation region. The form of the exercise boundary of the capped option leads to a (semi-)closed form formula for the price. They consist of a part related to the first hitting moment of the underlying asset to the cap and conceivably of the averaged prices of some uncapped American options. In some cases, the capped and uncapped options may coincide. We use a finite difference approach to solve the PDE for the uncapped American options. This method is chosen because it obtains all prices at once which reduces computational time significantly. More precisely, the results of our algorithm are the prices for all maturities below some moment and different cap values. Particularly, we use the version of Crank and Nicolson (1947). It has some major advantages. First, it has been developed namely for the heat style equations. Second, this method has a very fast convergence avoiding some possible oscillations in addition. A shortcoming to note is a larger time consumption compared to the standard explicit method due to the emerging system of linear equations that needs to be solved. The available boundary constraints for the put options, whether capped or uncapped, are only at the lower and the right boundaries since the continuation region is open above. Usually, the put option price is set to be zero for some large value of the underlying asset when a finite difference scheme is used. Unfortunately, this assumption has a major disadvantage – this value has to be extremely large for longer maturities. To avoid this inconvenience we obtain the option prices for a suitable value of the underlying asset using a Monte Carlo method presented in Chapter 4 (the second one). This method is relatively fast and accurate. The produced prices are used for the upper boundary constraint. The same modification of the finite difference approach is helpful for time reduction when we consider call options with large strikes or large exercise boundaries. Last but not least, some numerical results are presented. We compare the capped and uncapped prices as well as the so-called premium for capping. It presents the price that the option's writer pays for the cap feature, i.e. the difference between the prices of the American uncapped and capped options. Here is the place to note another importance of the above-mentioned Crank–Nicolson numerical method. Using it, we obtain jointly the prices of the capped and uncapped options which significantly reduces the time consumption.

The chapter is structured in the following way. In Section 5.2 we present the base we shall use later. In Sections 5.3 and 5.4 we present the derived results for the put and call options, respectively.

5.2 Preliminaries

We assume again that the underlying asset follows the dynamics (3.11). Let the maturity date be denoted again by T, and the option's strike and the cap level be K and L, respectively. If we have a put option then $L < K$ and $L > K$ for call options. We shall use the symbols "$\wedge$" and "$\vee$" for minimum and maximum, i.e. $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. The functions that describe the payment structure for the discounted American capped call and put options can be written as

$$N(t,x) = e^{-\lambda t}(S_t \wedge L - K)^+$$
$$N(t,x) = e^{-\lambda t}(K - S_t \vee L)^+,$$

respectively.

5.3 Put capped options

Let us consider a put capped option. We shall derive its optimal exercise region after which we shall obtain the option pricing formulas. We shall also present a finite difference approach based on the Crank–Nicolson procedure and we shall apply it for some particular parameter values.

5.3.1 Optimal region. Pricing.

Let us denote by Υ^A and $c^A(t)$ the exercise region and the corresponding boundary of the uncapped American option. We can approximate this boundary using the approach presented in Chapter 4. Let us consider $c^A(t)$ w.r.t. time to maturity. Proposition 4.5 shows that it starts from the value

$$D_1 = \min\left(\frac{r+\lambda}{\lambda}, 1\right) K$$

and decreases to its perpetual value given in Theorem (4.2)

$$D_2 = \frac{q}{q+1} K$$

for q defined in formulas (3.17).

Remark 5.1. *We shall show that $D_2 < D_1$. The inequality $r + \lambda > 0$ leads to*

$$q = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) > \left|\frac{r}{\sigma^2} - \frac{1}{2}\right| + \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \geq 0.$$

Hence $D_2 < K$. So if $r > 0$, then $D_1 = K$ and therefore $D_2 < D_1$. Suppose now that the risk-free rate is fixed at some negative value, $r < 0$. We have to prove $q/(q+1) < (r+\lambda)/\lambda$ which is equivalent to $\lambda + r(q+1) > 0$. Let us consider the left hand side as a function of λ

$$f(\lambda) = \lambda + r\left(\sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} + \frac{r}{\sigma^2} + \frac{1}{2}\right)$$
$$= \lambda + r\left(\sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2\lambda}{\sigma^2}} + \frac{r}{\sigma^2} + \frac{1}{2}\right).$$

Its derivative is

$$f'(\lambda) = 1 + \frac{r}{\sigma^2\sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2\lambda}{\sigma^2}}}.$$

We can easily prove that $f'(\lambda) > 0$ for all $\lambda > -r$. Therefore the function $f(\lambda)$ is increasing and it has a minimum for $\lambda = -r$

$$f(\lambda) > f(-r) = r\left(\left|\frac{r}{\sigma^2} - \frac{1}{2}\right| + \frac{r}{\sigma^2} - \frac{1}{2}\right) = 0.$$

Note that the risk free rate is negative. The fact that $\lambda + r(q+1) > 0$ finishes the proof.

We shall prove the following theorem that determines the exercise boundary of the capped option.

Theorem 5.1. *[Theorem 3.1 of Zaevski (2022a)] The exercise boundary of the discounted American capped put option is*

$$c(t) = c^A(t) \vee L.$$

Proof: Suppose first that for some t and $x > L$, the point $(t, x) \in \Upsilon^A$. Note that x has to be less than the strike, $x < K$. Hence, for every stopping time ζ from the set $\mathcal{T}_{[t,T]}$

$$\begin{aligned}(K - x \vee L) \equiv (K - x) &\geq \mathbb{E}^{t,x}\left[e^{-(r+\lambda)(\zeta - t)}(K - S_\zeta)^+\right] \\ &\geq \mathbb{E}^{t,x}\left[e^{-(r+\lambda)(\zeta - t)}(K - S_\zeta \vee L)^+\right].\end{aligned}$$

Therefore the point $(t, x) \in \Upsilon$ too. The next step is to prove that all points below the cap are optimal. Suppose that $(t, x) \in \overline{\Upsilon}$ for some t and $x < L$. Therefore there exists a stopping time $\zeta \in \mathcal{T}_{[t,T]}$ which gives a better result for the option's holder than the immediate exercise. We have $S_\zeta < K$, because in the opposite case, the holder receives nothing. Hence,

$$\begin{aligned}K - L \equiv K - x \vee L &< \mathbb{E}^{t,x}\left[e^{-(r+\lambda)(\zeta - t)}(K - S_\zeta \vee L)^+\right] \\ &= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)(\zeta - t)}(K - S_\zeta \vee L)\right] \\ &< \mathbb{E}^{t,x}\left[e^{-(r+\lambda)(\zeta - t)}(K - L)\right] \\ &< K - L.\end{aligned}$$

We have used above the restriction $r + \lambda > 0$. The contradiction shows that the points below the cap level L are optimal. It remains to prove that all points (t, x), such that $x > L \vee c^A(t)$, are not optimal. We shall use an approach similar to one presented in Broadie and Detemple (1995).

Suppose $(t, x) \in \Upsilon$ and $x > L \vee c^A(t)$. Note that if $L \leq D_2$, then $L < c(t)$ for every t. This is true because D_2 is just $c^A(\infty)$ and the optimal boundary for an American put decreases w.r.t. the time to maturity. Hence, the capped option is an ordinary American option and the theorem is proven. Suppose now that $D_2 < L < D_1$. Therefore there exists an ordinary American option with shorter maturity, say $\overline{T} < T$, such that its optimal boundary is above the cap level L at moment t, $L < \overline{c}^A(t)$. Thus the point (t, x) is not optimal for the new option. Something more if ζ is the first hitting time to $\overline{c}^A(t)$ if it happens before $\overline{T}$ and $\overline{T}$ otherwise, then

$$K - x < \mathbb{E}^{t,x}\left[e^{-(r+\lambda)(\zeta - t)}(K - S_\zeta)^+\right]. \tag{5.1}$$

Having in mind that $S_\zeta > L$, we conclude that inequality (5.1) means that the strategy ζ gives a better financial result for the holder of the original capped option than the immediate exercise and therefore the point $(t, x) \notin \Upsilon$.

Suppose now that $L \geq D_1$. Note that this is possible only when $r < 0$ which leads to $D_1 < K$. Let ζ be the lower between the first hitting moment of the underlying asset to the value L and the maturity date T. We shall use the equality $(K-x)^{+}=(K-x)+(x-K)\,I_{x\geq K}$ as well as Dynkin's formula (3.13) to obtain the following statement:

$$
\begin{aligned}
&\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}N\left(\zeta,S_{\zeta}\right)\right]=\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(K-S_{\zeta}\right)^{+}\right]\\
&=\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(K-S_{\zeta}\right)\right]+\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(S_{\zeta}-K\right)I_{S_{\zeta}\geq K}\right]\\
&\pm e^{-\lambda t}\left(K-x\right)\\
&=e^{-\lambda t}\left(K-x\right)+\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(S_{\zeta}-K\right)I_{S_{\zeta}\geq K}\right]\\
&+\mathbb{E}^{t,x}\left[K\left(e^{-r(\zeta-t)-\lambda\zeta}-e^{-\lambda t}\right)\right]-\mathbb{E}^{t,x}\left[S_{\zeta}e^{-r(\zeta-t)-\lambda\zeta}\right]+xe^{-\lambda t}\\
&=e^{-\lambda t}\left(K-x\right)+\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(S_{\zeta}-K\right)I_{S_{\zeta}\geq K}\right]\\
&+\mathbb{E}^{t,x}\left[K\int_{t}^{\zeta}-\left(r+\lambda\right)e^{-\lambda u-r(u-t)}du\right]-xe^{-\lambda t}\\
&-\mathbb{E}^{t,x}\left[\int_{t}^{\zeta}-\lambda e^{-r(u-t)-\lambda u}S_{u}du\right]+xe^{-\lambda t}\\
&=e^{-\lambda t}\left(K-x\right)+\mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(S_{\zeta}-K\right)I_{S_{\zeta}\geq K}\right]\\
&+\mathbb{E}^{t,x}\left[\int_{t}^{\zeta}e^{-r(u-t)-\lambda u}\left(\lambda S_{u}-\left(r+\lambda\right)K\right)du\right]\\
&>e^{-\lambda t}\left(K-x\right).
\end{aligned}
\tag{5.2}
$$

The last inequality is true because (A) the first expectation is always positive and (B), the integral is positive because $S_u > L > D_1$ for $u < \zeta$ equivalently to $\lambda S_u > (r+\lambda)\,K$. The contradiction between inequality (5.2) and Definition 4.1 is due to the assumption $(t,x)\in\Upsilon$. This finishes the proof. □

Remark 5.2. *Let us mention that a similar approach as above can be used if the model is stated under significantly general assumptions. First, if we consider a general diffusion model, then its drift under the risk-neutral measure has to be again rS_t. We have applied Dynkin's formula (3.13) to the function*

$$f\left(u,x\right)=xe^{-r(u-t)-\lambda u}.$$

Since it is linear w.r.t. the variable x, only the first derivative influences its infinitesimal generator. Hence, conclusions (5.2) are true for the general diffusion models too and therefore Theorem 5.1 still holds.

More generally, if the underlying asset is driven by some Feller-Markov process, we can use an analogue of Proposition 3.4 . Hence, scheme (5.2) can be used again with the corresponding form of the infinitesimal generator.

Hereafter, we assume $t = 0$ referring to Proposition 3.3. If $L \in [D_1, K)$ (this is possible only when $r < 0$) then the exercise boundary is the cap value L. If the asset starts below L then the option price is simply $V = K - L$. Suppose now that $S_0 > L$. We shall use the results of Section 2.3 for the Laplace transforms of the first hitting time. Let ζ be the first hitting moment of the underlying asset to L or equivalently, the hitting of the Brownian motion to the linear function $b(t) = -(b_1 t + b_2)$ for

$$\begin{aligned} b_1 &= \frac{r}{\sigma} - \frac{\sigma}{2} \\ b_2 &= \frac{\ln S_0 - \ln L}{\sigma}. \end{aligned} \tag{5.3}$$

Therefore, the option price is

$$\begin{aligned} V &= \mathbb{E}\left[e^{-(r+\lambda)\zeta}(K-L) I_{\zeta \le T}\right] + \mathbb{E}\left[e^{-(r+\lambda)T}(K-S_T)^+ I_{\zeta > T}\right] \\ &= (K-L)\,\mathbb{E}\left[e^{-(r+\lambda)\zeta} I_{\zeta \le T}\right] + K e^{-(r+\lambda)T}\mathbb{E}\left[I_{\zeta > T, S_T < K}\right] \\ &\quad - S_0 e^{-\left(\lambda + \frac{\sigma^2}{2}\right)T}\mathbb{E}\left[e^{\sigma B_T} I_{\zeta > T, S_T < K}\right]. \end{aligned}$$

Using Proposition 2.5, Theorem 2.5, and Corollary 2.4 (see also the results of Section 2.2) we conclude

$$\begin{aligned} V &= (K-L)\, e^{b_2\left(\sqrt{b_1^2 + 2(r+\lambda)} - b_1\right)} g\left(T, \sqrt{b_1^2 + 2(r+\lambda)}, b_2\right) \\ &\quad + K e^{-(r+\lambda)T} W\left(0, d(T,K), T; b_1, b_2\right) - S_0 e^{-\left(\lambda + \frac{\sigma^2}{2}\right)T} W\left(-\sigma, d(T,K), T; b_1, b_2\right), \end{aligned} \tag{5.4}$$

where the functions $g(\cdot)$, $d(\cdot)$ and $W(\cdot)$ are

$$
\begin{aligned}
&g\left(T;b_1,b_2\right)=1-N\left(\frac{b_1T+b_2}{\sqrt{T}}\right)+\exp\left(-2b_1b_2\right)N\left(\frac{b_1T-b_2}{\sqrt{T}}\right)\\
&d\left(t,x\right)=\frac{\ln S_0-\ln x}{\sigma}+\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)t\\
&W\left(\theta,z,T;b_1,b_2\right)\\
&=\exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l}N\left(\frac{b(T)-T\theta}{\sqrt{T}}\right)-N\left(\frac{z-T\theta}{\sqrt{T}}\right)\\+e^{2b_2(\theta-b_1)}\left(N\left(\frac{z-T\theta-2b_2}{\sqrt{T}}\right)-N\left(\frac{b(T)-T\theta-2b_2}{\sqrt{T}}\right)\right)\end{array}\right].
\end{aligned}
\tag{5.5}
$$

Suppose now that $L\in(D_2,D_1)$. Let us denote by τ^* this time to maturity, for which $c(\tau^*)=L$. If $T\le\tau^*$, then the option turns into an ordinary American put. Suppose now that $T>\tau^*$. Therefore $c(t)=L$ for $t<T-\tau^*$ and $c(t)=c^A(t)$ for $t\ge T-\tau^*$. Let the function $f(\cdot)$ be defined as

$$
f\left(t,y\right)=\frac{1}{\sqrt{2\pi t}}\left(1-\exp\left(-\frac{2b_2\left(b\left(t\right)-y\right)}{t}\right)\right)\exp\left(-\frac{y^2}{2t}\right). \tag{5.6}
$$

Let us denote by $A(y,\tau)$ the price of an ordinary American put if the asset starts from the value y and the time to maturity is τ. Using Propositions 3.3 and 2.3, we conclude that the option price can be presented as

$$\begin{aligned}
V &= E\left[e^{-(r+\lambda)\zeta}\left(K-L\right)I_{\zeta\leq T-\tau^*}\right]\\
&+e^{-r(T-\tau^*)}\int\limits_{-d(T-\tau^*,L)}^{\infty} e^{-\lambda(T-\tau^*)}A\left(S_0e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau^*)+\sigma y},\tau^*\right)d\mathbb{Q}\left(B_{T-\tau^*}<y,\zeta>T-\tau^*\right)\\
&=E\left[e^{-(r+\lambda)\zeta}\left(K-L\right)I_{\zeta\leq T-\tau^*}\right]\\
&+e^{-(r+\lambda)(T-\tau^*)}\int\limits_{-d(T-\tau^*,L)}^{\infty} A\left(S_0e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau^*)+\sigma y},\tau^*\right)d\mathbb{Q}\left(\overline{B}_{T-\tau^*}>-y,\zeta>T-\tau^*\right)\\
&=\left(K-L\right)e^{b_2\left(\sqrt{b_1^2+2(r+\lambda)}-b_1\right)}g\left(T-\tau^*,\sqrt{b_1^2+2\left(r+\lambda\right)},b_2\right)\\
&+e^{-(r+\lambda)(T-\tau^*)}\int\limits_{-d(T-\tau^*,L)}^{\infty} A\left(S_0e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau^*)+\sigma y},\tau^*\right)f\left(T-\tau^*,-y\right)dy\\
&=\left(K-L\right)e^{b_2\left(\sqrt{b_1^2+2(r+\lambda)}-b_1\right)}g\left(T-\tau^*,\sqrt{b_1^2+2\left(r+\lambda\right)},b_2\right)\\
&+e^{-(r+\lambda)(T-\tau^*)}\int\limits_{-\infty}^{d(T-\tau^*,L)} A\left(S_0e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau^*)-\sigma y},\tau^*\right)f\left(T-\tau^*,y\right)dy.
\end{aligned}\tag{5.7}$$

Finally, note that if $L\leq D_2$, then the optimal exercise boundary of the ordinary American option is always above the cap level. Therefore the capped option turns into an ordinary one. We can summarize the derived results in the following theorem.

Theorem 5.2. *[Theorem 3.2 of Zaevski (2022a)] The price of an American capped put option can be derived through one of the following statements.*

1. *If $L\in[D_1,K)$, then the option price is given by equation (5.4) when $S_0>L$. Otherwise the price is $K-L$.*

2. *If $L\in(D_2,D_1)$ and $T>\tau^*$, then the price of the American capped option is given by equation (5.7) when $S_0>L$. Otherwise the price is $K-L$.*

3. *If $L\leq D_2$ or $L\in(D_2,D_1)\cap T\leq\tau^*$, then the option is an ordinary American put and its price can be found using the approach provided in Chapter 4.*

As a corollary of Theorem 5.2, we provide the result in the perpetual case, i.e. when $T = \infty$.

Corollary 5.1. *Suppose that $T = \infty$. Then the early exercise boundary is the line $C = L \vee D_2$. If the asset starts above it, $S_0 > C$, the option price is*

$$V = (K - C)\left(\frac{C}{S_0}\right)^q,$$

where q is given by equation (3.17). Otherwise, if $S_0 \leq C$, then the option price is $V = K - S_0$.

Proof: The proof is an immediate consequence from Theorems 5.2 and 4.2. □

Remark 5.3. *We shall comment on the result of Corollary 5.1 having in mind Theorem 5.2. If $L \geq D_1$, we use equation (5.4). Since $T = \infty$, the last two terms vanish and the first one leads to formula (4.23) – note that $C = L$.*

If $L \in (D_2, D_1)$, then we use equation (5.7). Note that always $T > \tau^*$. The second term vanishes due to the discount factor. The first term leads to formula (4.23).

Finally, if $L \leq D_2$, then the option is a perpetual American and therefore we have to use Theorem 4.2.

5.3.2 The Crank–Nicolson finite difference method for American option pricing

We give now an efficient Crank–Nicolson finite difference approach for pricing an American uncapped option based on the results of Section 3.5 and considering a dense grid. First, we use the three-step algorithm presented in Section 4 to approximate its early exercise boundary. We derive the values in j_1 main nodes. The algorithm produces meanwhile the values in some additional nodes – we denote by j_2 the number of the total different nodes at which we know the exercise boundary. After that, we apply a cubic spline interpolation to determine the boundary at the uniform grid with significantly more nodes – their number is denoted by j_3. It turns out that $j_1 = 5$ and $j_3 = 128$ are sufficient to approximate the optimal boundary. In this case, the value of j_2 is 15.

The next step is to apply the second Monte Carlo method presented in Section 4 to determine the option prices for some fixed initial asset value, denoted by H, and different maturities. The value of H will be specified later. We calculate the prices at the same j_1 nodes and interpolate the rest j_3 nodes. We shall denote the values at this upper boundary H by $h(t)$. Having in mind Proposition 3.3, we conclude that the PDE for the American uncapped option turns into

$$
\begin{aligned}
&A_t(t,x) + rxA_x(t,x) + \frac{1}{2}\sigma^2 x^2 A_{xx}(t,x) - rA(t,x) = 0 \\
&A\left(t, c^A(t)\right) = \exp(-\lambda t)\left(K - c^A(t)\right),\ t \in [0,T] \\
&A(t,H) = \exp(-\lambda t)\, h(t),\ t \in [0,T] \\
&A(T,x) = \exp(-\lambda T)(K-x)^+,\ c(T) < x < H.
\end{aligned}
$$

The equation holds in the region $(t,x) \in \{(0,T) \times (c(t), H)\}$. We continue using the method presented in Section 3.5.

5.3.3 Numerical method for pricing American capped put options

When we price a capped option, we have to modify the approach from Section 5.3.2. We present below an algorithm that allows us to calculate the price of the capped option as well as the value of its uncapped analogue. In such a way we evaluate the premium for capping too. As we have seen in Theorem 5.2 three cases are possible. If $L \leq D_2$ or $L \in (D_2, D_1) \cap T \leq \tau^*$, then capping has no influence, and therefore the capped option turns into a usual American one. On the other hand if $L \in [D_1, K)$, there exists a closed form formula for the capped option price – formula (5.4). The most complicated case is $L \in (D_2, D_1) \cap T > \tau^*$ – it combines the features of both previous cases. Having in mind the remarks above, we construct the following algorithm.

1. We used the presented in Section 5.3.2 approach to derive the exercise boundary of the uncapped option and its price structure w.r.t the time and state division.

2. If the exercise boundary at the initial moment is above the cap, $c(t_M) \equiv c(0) \geq L$, then both capped and uncapped options coincide and therefore the task is solved. Recall that we work backwards in the finite difference scheme.

3. If $L \in [D_1, K)$, we have to apply closed form formula (5.4) to obtain the price of the capped option.

4. We consider the case $L \in (c(0), D_1)$. We have to rely now on formula (5.7). Its first part can be derived explicitly. Hence, we have to evaluate the integral part. The rest of the algorithm is intended for this.

5. We derive the lowest m for which $c(t_m) > L$. Let us denote it by $\overline{m}$. It approximates the value of τ^*, $\tau^* \approx t_{\overline{m}}$.

6. If $\overline{m} = M$, then the optimal boundary is above the cap only at the maturity and hence the option price is approximately equal to formula (5.4).

7. Suppose now $\overline{m} < M$. Recall that $l_{\overline{m}}$ is the largest value of n, for which x_n is not above $c(t_{\overline{m}})$. We use the derived values of $V(n, \overline{m})$, $n = l_{\overline{m}} + 1, ..., N$ to approximate the price function for the uncapped option that appears in the integral part of formula (5.7) –

$$A\left(S_0 e^{\left(r - \frac{\sigma^2}{2}\right) t_{\overline{m}} - \sigma y}, t_{\overline{m}}\right). \tag{5.8}$$

   Note that in the finite difference scheme, the x-nodes are for the underlying asset, not for the Brownian motion.

8. Using a cubic spline interpolation we approximate function (5.8) at N_1 uniformly distributed on the interval $(c(t_{\overline{m}}), H)$ nodes.

9. We value function (5.6) at the obtained above x-nodes.

10. We approximate the integral from equation (5.7) averaging the product of functions (5.8) and (5.6).

11. We finish our algorithm discounting the integral by $e^{-(r+\lambda)t_{\overline{m}}}$ and summing both parts of formula (5.7).

12. The premium for capping can be obtained as the difference between the prices of the uncapped and capped options.

5.3.4 Numerical results

Both typical forms of the optimal boundary of an American capped put options are presented in Figures 5.1a and 5.1b. We consider a negative risk-free rate of return, $r = -0.03$ because in this case, the exercise boundary is below the strike at the maturity. We assume that the option expires after one year, i.e. $T = 1$, the strike price is $K = 20$, and the volatility is $\sigma = 0.3$. For Figure 5.1a we use value $\lambda = 0.05$. The terminal values D_1 and D_2 are $D_1 = 8.0000$ and $D_2 = 3.7906$. The critical value for the time to maturity is $\tau^* = 0.5235$ and it is marked by a red color in Figure 5.1a. If $\lambda = 0.06$, then $D_1 = 10.0000$, $D_2 = 5.0000$, and $\tau^* = 2.8290$. The option turns into an uncapped American put since $\tau^* > T$. The boundary can be seen in Figure 5.1b.

In Figures 5.1e and 5.1c we present the exercise boundaries of the capped and uncapped options varying the time to maturity between zero and three, $\tau \in (0, 3)$, and the discount rate among $\lambda \in (0.031, 0.1)$. Note that these values guarantee that $r + \lambda > 0$. We again mark by red color the values at which the boundary turns into the cap. The boundary's behavior for longer maturities can be seen in Figures 5.1f and 5.1d. The time to maturity belongs to the interval $(0, 200)$. The green points are the corresponding perpetual values calculated by the use of Corollary 5.1. We can see that for the longer maturities, the boundary's values are very close to the corresponding perpetual ones.

In Figure 5.2 we present the behavior of the option prices (with and without capping) and the premium for capping. We separately consider the cases with short and middle maturities as well as with large and extremely large maturities. For the first one, the time to maturity is assumed to be $\tau \in (0, 3)$, whereas for the second one, we use values larger than $20 - \tau \in (20, 200)$. For both cases we have to derive this value of H, which guarantees that the interval $(c(t_{\overline{m}}), H)$ comprises the significant mass of the distribution $f(\cdot)$ from equation (5.6) that appears in the integral part of the price formula (5.7). It turns out that for the shorter maturities, the sufficient value is $H = 2K = 40$, whereas for the longer ones we need a larger value of $H = 10K = 200$. The time interval is divided into $M = 2000$ steps. To use a relatively equal size of the state grid, we use $N = 200$ and $N = 1000$ point divisions, respectively, for the short and long maturities. Also, to evaluate correctly the integral in formula (5.7) we divide the interval $(c(t_{\overline{m}}), H)$ into

$N_1 = 500$[1] pieces for which we use the cubic spline interpolation. In Figures 5.2a, 5.2c, and 5.2e are presented the uncapped, capped option prices, and the premium for capping, respectively. We observe that for small values of the discount factor λ the cap is above the optimal boundary value at the maturity, $L > D_1$. Hence we price the capped options via formula (5.4). For larger values of λ the cap is below the terminal boundary value, but it is above its initial value, $c(0) < L < D_1$. Hence we have to use formula (5.7). For even greater values of λ the optimal boundary for the uncapped option is always above the cap and hence both of capped and uncapped options coincide. Particularly, the premium for capping turns into zero. We marked by red and yellow points the critical values of the discount factor at which the capped option changes its feature. Note that the first critical value for the discount factor λ can be calculated explicitly since it leads to $D_1 = L$. Hence,

$$\lambda_1 = -\frac{rK}{K - L}.$$

Recall that the risk-free rate is negative. Let us mention that if the time to maturity is zero, then $c(0) \equiv D_1$, and therefore both critical values coincide. This can be viewed in Figure 5.2. The assumed parameters lead to $\lambda_1 = 0.0462$. In Figures 5.2b and 5.2d we present the price behavior of the long maturing uncapped and capped options. The premium for capping can be seen in Figure 5.2f. The perpetual values, calculated explicitly via Corollary 5.1, are plotted again by a green color. We can see that both prices and the premium tend to the perpetual values for large times to maturity.

The option prices for some particular values of the parameters are presented in Table 5.1. We assume again $r = -0.03$, $\sigma = 0.3$, and $K = 20$. The considered options are at-the-money, i.e $S_0 \equiv K = 20$. We vary the time to maturity as $\tau \in \{0.5,\ 1,\ 2,\ 3\}$, the cap as $L \in \{10,\ 12,\ 14,\ 16,\ 18\}$, and the discount factor among $\lambda \in \{0.0031,\ 0.04,\ 0.05,\ 0.06,\ 0.07,\ 0.08,\ 0.09,\ 0.1\}$. The first column presents the prices of the uncapped American options. It turns out that due to the Monte Carlo modification, a relatively low number of the grid nodes is enough for very precise results – $M = 5000$ time and $N = 1000$ state nodes. We use a $N_1 = 500$-step division for the integral interpolation in formula (5.7). Right to the option prices, we mark which formula is valid – "1" if we use formula (5.4), "2" for formula (5.7), and "3"

[1] For the shorter maturities we can use $N_1 = 200$.

if the capped option coincides with the corresponding uncapped one. We can observe the following natural relations – (A) the price increases w.r.t. the time to maturity, (B) decreases w.r.t. the discount factor λ and the capped value, and (C) for small enough L the capped option turns into an ordinary American one.

5.4 Call style options

Since the call and put early exercise boundaries are symmetric in some sense, we shall present the results for the call capped options mentioning the differences. We shall use the same notations as above. The value of the exercise boundary of the corresponding uncapped option at the maturity D_1 and the perpetual value D_2 now are given by

$$D_1 = \max\left(\frac{r+\lambda}{\lambda}, 1\right) K$$
$$D_2 = \frac{p-q}{p-q-1} K.$$

The constants p and q are defined by formulas (3.17). Since the optimal exercise boundary of the uncapped call option is an increasing function w.r.t. the time to maturity, it has to intersect the capped level in the case $D_1 < L < D_2$. We shall denote this moment with τ^* again.

Remark 5.4. *Similarly to Remark 5.1, we shall show that $D_1 < D_2$. First, note that $p - q \geq 1$ since*

$$\begin{aligned} p - q &= \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \\ &\geq \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \\ &= \left|\frac{r}{\sigma^2} + \frac{1}{2}\right| - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \geq 1. \end{aligned}$$

Hence, if $r \leq 0$, then $D_2 > K = D_1$. We have to examine the case $r > 0$ and prove that

$$\frac{r+\lambda}{\lambda}K < \frac{p-q}{p-q-1}K,$$

which is equivalent to the positiveness of the function

$$\begin{aligned} f(\lambda) &= \lambda - r(p-q-1) \\ &= \lambda - r\left(\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2 + 2\frac{\lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2}+\frac{1}{2}\right)\right) \end{aligned}$$

for $\lambda \geq 0$. *Its derivative is*

$$f'(\lambda) = 1 - \frac{r}{\sigma^2\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2 + 2\frac{\lambda}{\sigma^2}}}.$$

It is always positive because $\lambda > 0$ *and*

$$\sqrt{\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)^2 + 2\frac{\lambda}{\sigma^2}} > \frac{r}{\sigma^2}.$$

Hence, $f(\lambda)$ *is an increasing function and therefore* $f(\lambda) > f(0) = 0$. *This finishes the proof.*

Now we shall establish the theorem that gives the form of the early exercise boundary for the capped call options.

Theorem 5.3. *[Theorem 4.1 of Zaevski (2022a)]*
The exercise boundary of an American call capped option is $c(t) = c^A(t) \wedge L$.

Proof: Suppose first that for some t and $x \in (K, L)$, we have $(t,x) \in \Upsilon^A$. Therefore, for an arbitrary strategy $\zeta \in \mathcal{T}_{[t,T]}$

$$\begin{aligned} (x \wedge L - K) \equiv (x-K) &\geq E^{t,x}\left[e^{-(r+\lambda)(\zeta-t)}(S_\zeta - K)^+\right] \\ &\geq E^{t,x}\left[e^{-(r+\lambda)(\zeta-t)}(S_\zeta \wedge L - K)^+\right]. \end{aligned}$$

Hence, $(t,x) \in \Upsilon$ too. Suppose that $(t,x) \in \overline{\Upsilon}$ for some t and $x > L$. Therefore there exists a strategy ζ which leads to a larger profit than the immediate exercise. Note that $S_\zeta > K$ because otherwise, the option's holder receives nothing. Thus

$$\begin{aligned} L - K \equiv x \wedge L - K &< E^{t,x}\left[e^{-(r+\lambda)(\zeta-t)}(S_\zeta \wedge L - K)^+\right] \\ &= E^{t,x}\left[e^{-(r+\lambda)(\zeta-t)}(S_\zeta \wedge L - K)\right] \\ &< E^{t,x}\left[e^{-(r+\lambda)(\zeta-t)}(L - K)\right] \\ &< L - K. \end{aligned}$$

Hence, $(t,x) \in \Upsilon$. It remains to prove that all points below the contour $c^A(t) \wedge L$ are not optimal. Suppose the contrary, i.e. there exists a point $(t,x) \in \Upsilon$ such that $x < c^A(t) \wedge L$. If $L \geq D_2 \equiv c^A(\infty)$, then $c^A(t) \wedge L \equiv c^A(t)$. In this case the capped option coincides with an ordinary American option, and therefore the theorem is true. Suppose now that $D_1 < L < D_2$. Hence, there exists an option with a shorter maturity $\overline{T}$, which optimal exercise boundary is below the capped level, $\overline{c}^A(t) < L$. Let us examine a strategy ζ which is the first hitting time to the contour $\overline{c}^A(t) \wedge \overline{T}$. We have

$$K - x < E^{t,x}\left[e^{-(r+\lambda)(\zeta-t)}\left(S_\zeta - K\right)^+\right]. \tag{5.9}$$

Obviously $\zeta \in \mathcal{T}_{[t,T]}$. Also, having in mind that $S_\zeta < L$ and inequality (5.9) we conclude that the strategy ζ is better for the holder of the original option than immediate exercising. Therefore the point (t,x) can not be optimal.

Finally, we have to examine the case $L \leq D_1$, which is possible only when $r > 0$. We shall examine a strategy ζ which is the first hitting time to the cap if it happens before T, and T otherwise. Using the equality $(x-K)^+ = (x-K) + (K-x)\,I_{x \leq K}$ and Dynkin's formula (3.13) we derive

$$
\begin{aligned}
&E^{t,x}\left[e^{-r(\zeta-t)}N\left(\zeta,S_\zeta\right)\right]=E^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(S_\zeta-K\right)^+\right]\\
&=E^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(S_\zeta-K\right)\right]+E^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(K-S_\zeta\right)I_{S_\zeta\le K}\right]\\
&\pm e^{-\lambda t}\left(x-K\right)\\
&=e^{-\lambda t}\left(x-K\right)+E^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(K-S_\zeta\right)I_{S_\zeta\le K}\right]\\
&-E^{t,x}\left[K\left(e^{-r(\zeta-t)-\lambda\zeta}-e^{-\lambda t}\right)\right]+E^{t,x}\left[S_\zeta e^{-r(\zeta-t)-\lambda\zeta}\right]-xe^{-\lambda t}\\
&=e^{-\lambda t}\left(x-K\right)+E^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(K-S_\zeta\right)I_{S_\zeta\le K}\right]\\
&+E^{t,x}\left[K\int_t^\zeta\left(r+\lambda\right)e^{-\lambda u-r(u-t)}du\right]+xe^{-\lambda t}\\
&+E^{t,x}\left[\int_t^\zeta-\lambda e^{-r(u-t)-\lambda u}S_u du\right]-xe^{-\lambda t}\\
&=e^{-\lambda t}\left(x-K\right)+E^{t,x}\left[e^{-r(\zeta-t)}e^{-\lambda\zeta}\left(K-S_\zeta\right)I_{S_\zeta\le K}\right]\\
&+E^{t,x}\left[\int_t^\zeta e^{-r(u-t)-\lambda u}\left(-\lambda S_u+\left(r+\lambda\right)K\right)du\right]\\
&>e^{-\lambda t}\left(x-K\right).
\end{aligned}
$$

The last inequality is true because

$$
S_u<L<D_1=\frac{r+\lambda}{\lambda}K.
$$

for every $u<\zeta$. We conclude that the point (t,x) can not be optimal since the strategy ζ leads to a better result. □

Remark 5.5. *Note that the conclusions of Remark 5.2 are true for call options too – Theorem 5.3 holds under significantly general assumptions.*

We shall prove now the analogue of theorem (5.2). Let the constants b_1 and b_2, and the functions $g\left(\cdot\right)$, $V\left(\cdot\right)$, and $d\left(\cdot\right)$ be defined again by the equalities (5.3) and (5.5). The price of the uncapped call option shall be denoted by $A\left(\cdot\right)$.

Theorem 5.4. *[Theorem 4.2 of Zaevski (2022a)]*
If $L\in(K,D_1]$ then $c\left(t\right)\equiv L$. If $x\ge L$, then $V=L-K$. Otherwise, if $x<L$, then

$$\begin{aligned} V &= (L-K)\, e^{-b_2\left(\sqrt{b_1^2+2(r+\lambda)}+b_1\right)} g\left(T, \sqrt{b_1^2+2\,(r+\lambda)}, -b_2\right) \\ &+ S_0 e^{-\left(\lambda+\frac{\sigma^2}{2}\right)} V\left(\sigma, -d\,(T,K)\,, T; -b_1, -b_2\right) \\ &- K e^{-(r+\lambda)T} V\left(0, -d\,(T,K)\,, T; -b_1, -b_2\right). \end{aligned} \tag{5.10}$$

If $L \geq D_2$ or $L \in (D_1, D_2) \cap T \leq \tau^*$, then the option turns into an ordinary American call. Finally, if $L \in (D_1, D_2)$ and $T > \tau^*$, then the option price can be presented as

$$\begin{aligned} V &= (L-K)\, e^{-b_2\left(\sqrt{b_1^2+2(r+\lambda)}+b_1\right)} g\left(T-\tau^*, \sqrt{b_1^2+2\,(r+\lambda)}, -b_2\right) \\ &+ e^{-(r+\lambda)(T-\tau^*)} \int_{-\infty}^{-d(\tau^*,L)} A\left(S_0 e^{\left(r-\frac{\sigma^2}{2}\right)\tau^*+\sigma y}, \tau^*\right) f\left(\tau^*, y\right) dy. \end{aligned} \tag{5.11}$$

Proof: The cases $L \geq D_2$ or $L \in (D_1, D_2)$ jointly with $\tau^* \geq T$ are obvious because both exercise boundaries coincide, $c\,(t) \equiv c^A\,(t)$ for $t \leq T$.

Suppose now that $K < L \leq D_1$. Note that this is possible only when $r > 0$. Hence, the exercise boundary of the corresponding uncapped option is always above the cap and therefore the optimal boundary for the capped option is simply the value L. Let us notate by ζ the first hitting moment to the cap level L. If the asset starts below the cap, then the option price is

$$\begin{aligned} V &= E\left[e^{-(r+\lambda)\zeta}\,(L-K)\, I_{\zeta\leq T}\right] + E\left[e^{-(r+\lambda)T}\,(S_T-K)^+\, I_{\zeta>T}\right] \\ &= (L-K)\, E\left[e^{-(r+\lambda)\zeta} I_{\zeta\leq T}\right] + S_0 e^{-\left(\lambda+\frac{\sigma^2}{2}\right)} E\left[e^{\sigma B_T} I_{\zeta>T, S_T>K}\right] \\ &- K e^{-(r+\lambda)T} E\left[I_{\zeta>T,S_T>K}\right]. \end{aligned}$$

It remains to use Propositions 2.1 and 2.2 to derive formula (5.10). We use the arguments presented in equation (5.7) to prove formula (5.11). □

Remark 5.6. *Note that when $\lambda = 0$, the early exercising is never optimal for the uncapped option. Thus the capped option boundary is the cap value and its price is given by formula (5.10).*

We present below the analogue of the Corollary 5.1 for the perpetual capped call.

Corollary 5.2. *When $T = \infty$, the early exercise boundary is the line $C = L \wedge D_2$. If the initial asset price is below it, $S_0 < C$, then the option price is*

$$V = (c - K)\left(\frac{x}{c}\right)^{p-q},$$

If $S_0 \geq C$, then the option price is $V = S_0 - K$.

Remark 5.7. *If the additional discount factor is zero, $\lambda = 0$, we have $D_1 = D_2 = \infty$ and therefore the first case of Theorem 5.4 holds.*

To implement Theorem 5.4 on a dense grid, we need an efficient method for pricing the uncapped American options. A similar to the presented in Section 5.3.2 finite difference approach can be used. The corresponding PDE turns into

$$\begin{aligned}
&A_t(t,x) + rxF_x(t,x) + \frac{1}{2}\sigma^2 x^2 F_{xx}(t,x) - rF(t,x) = 0 \\
&A\left(t, c^A(t)\right) = \exp(-\lambda t)\left(c^A(t) - K\right),\ t \in [0,T] \\
&A(0,x) = 0,\ t \in [0,T] \\
&A(T,x) = \exp(-\lambda T)(x-K)^+,\ 0 \leq x < c(T).
\end{aligned}$$

The equation is satisfied in the strip $(0,T) \times (0, c(t))$, and the boundary constraints are imposed on the lower, upper, and right boundaries. In the call case, we do not need to use a Monte Carlo modification since the state space is closed on both sides. On the other hand, if the strike or the exercise boundary is large, then the Monte Carlo modification can be very helpful. We have to change only the boundary constraints in the presented in Section 5.3.2 algorithm to obtain the prices of the American uncapped call options. Theorem 5.4 consists again of three parts – (A) the option can be an ordinary American call and thus it can be priced by the presented in Section (3.5) Crank–Nicolson finite difference method, (B) the optimal boundary may coincide with the cap and hence the closed form formula is available, or (C) the capped option price can be given by equation (5.11) – in this case we can use the approach presented in Section 5.3.3 to evaluate numerically the integral.

The typical forms of the early exercise boundary of the American capped call options can be seen in Figures 5.3a and 5.3b. The used parameters are $r = 0.02$, $\sigma = 0.3$, $K = 20$, $T = 1$, and $L = 35$. In Figure 5.3a, we present the optimal boundary assuming that the additional discount factor is $\lambda = 0.04$. The initial value is $D_1 = 30$, whereas the perpetual one is $D_2 = 62.9719$. The critical value for the time to maturity is $\tau^* = 0.4437$ and it is marked by a red point. For Figure 5.3b we use a value $\lambda = 0.08$, which leads to the corresponding terminal levels $D_1 = 25$, $D_2 = 45.1842$, and $\tau^* = 1.1569$. Since $\tau^* > T$, the capped and uncapped options coincide.

A reasonable hypothesis is that the speed of the convergence of the optimal boundary to the perpetual level is larger for the larger values of the discount factor λ. To confirm this we consider $\lambda \in (0.1, 1)$. The short maturity behavior – $\tau \in (0, 3)$ – for the uncapped and capped options is presented in Figures 5.3c and 5.3e, respectively. We mark again by red points the values at which the uncapped boundary rises above the cap L. In Figures 5.3d and 5.3f, we present the boundary surface for the longer maturities $\tau \in (0, 20)$. The perpetual values calculated via Corollary 5.2 are plotted by green points. We can see that the optimal boundaries are relatively close to them for $\tau = 20$. Hence, the boundaries indeed converge faster for the larger discount factors.

We presented the behavior of the option prices in Figure 5.4. The initial asset price is assumed to be $S_0 = 23$, whereas the cap level is $L = 27$. Figures 5.4a, 5.4c, and 5.4e illustrate the short time range – $\tau \in (0, 3)$. The discount factor is varied among $\lambda \in (0.01, 1)$. We again mark by red and yellow points the critical values λ_1 and λ_2, respectively, at which the capped option changes its features. More precisely, if $\lambda \leq \lambda_1$, then the optimal boundary of the capped option coincides by the cap L. Hence its price is calculated trough formula (5.10). We use formula (5.11) when $\lambda \in (\lambda_1, \lambda_2)$. If the discount factor is large enough, $\lambda \geq \lambda_2$, both options coincide. Note that λ_1 is the discount value that leads to $D_1 = L$, i.e.

$$\lambda_1 = \frac{rK}{L - K}.$$

Our parameters lead to $\lambda_1 = 0.0571$. Let us mention again that if $\tau = 0$ then both critical values coincide – Figures 5.4a, 5.4c, and 5.4e confirm this.

The Figures 5.4b, 5.4d, and 5.4f present the uncapped and capped prices and the corresponding premium for capping assuming that $\tau \in (0, 20)$. Differently for the short maturity case, we consider discount factors larger than

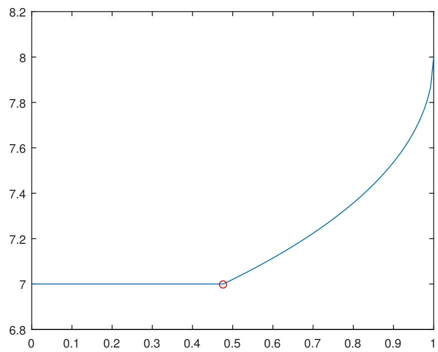
$0.1 - \lambda \in (0.1, 1)$. In such a way we confirm again that for larger discount values the convergence to the perpetual levels is faster. They are calculated by the use of Corollary 5.2 and are marked by green. We do not show the first critical value for the discount factor since it is out of our interval, $\lambda_1 \equiv 0.0571 < 0.1$. The second critical values which exceed 0.1 are marked by yellow.

Some particular call option prices are presented in Table 5.2. We assume again that the option is at-the-money, i.e. $S_0 \equiv K = 20$. We vary the cap value among $L \in \{21,\ 24,\ 27,\ 30,\ 33\}$, the time to maturity among $\{0.5,\ 1,\ 2,\ 3\}$, and the discount rate among $\{0,\ 0.02,\ 0.04,\ 0.06,\ 0.08,\ 0.1\}$. Note that early exercising of an uncapped American option is never optimal prematurely when $\lambda = 0$. Thus it turns into a European one. Accordingly, we use the Black-Scholes formula given in Theorem 3.2 to derive the American option prices presented in the first column of the first section of Table 5.2. Let us mention that the algorithm for call pricing does not involve a preliminary calculation of the option prices at the upper boundary. For this, we use a very precise grid considering $M = 20000$ and $N = 5000$. Right to the corresponding option price, we place the case of Theorem 5.4 which is actual. As expected, for large cap values both options coincide; formula (5.11) is used for middle cap values, whereas the option price is derived via formula (5.10) when the cap is small enough. Among various other observations, we can see that the capped option price increases to the uncapped value when the cap increases.

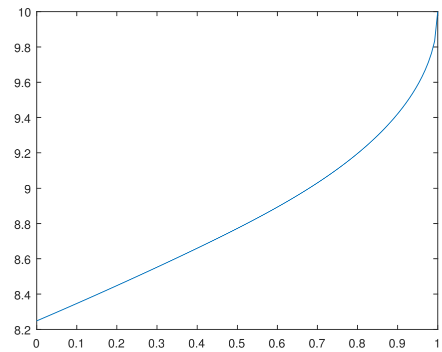
5.5 Conclusions

We have examined the discounted American capped options. The theorems for the form of the optimal regions have been obtained. It turns out that the early exercise boundary for the put/call capped option is the maximum/minimum between the boundary of the corresponding uncapped option and the cap value. The formulas for the option prices have been developed. There exist three possible cases – (A) the capped option price may coincide with the corresponding uncapped one, (B) the optimal boundary of the capped option may coincide with the cap value, and thus the pricing problem turns into a first hitting problem, and (C) a mixture between the previous two cases. We have derived a closed-form formula in case (B). We have presented a Crank–Nicolson finite difference approach for deriving the price of the American uncapped options. Using it we price the related uncapped option when cases (A) or (C) are actual. Last but not least, this numerical method allows us to calculate jointly both capped and uncapped option prices and the related premium for capping. This significantly reduces the computational time.

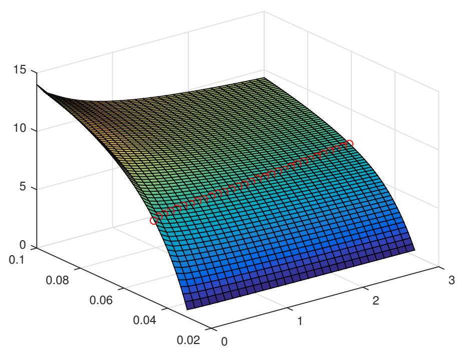
Figure 5.1: Put capped options



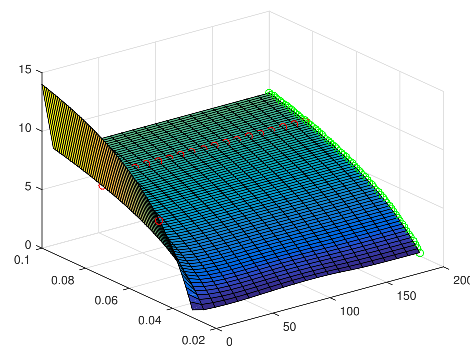
(a) The optimal boundary, $\lambda = 0.05$.



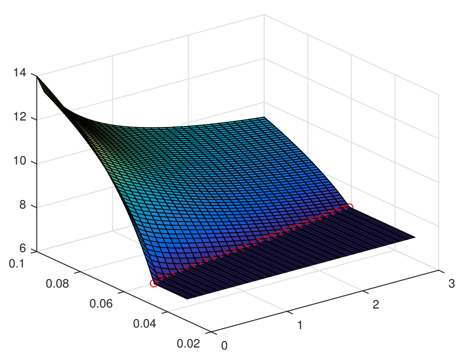
(b) The optimal boundary, $\lambda = 0.06$.



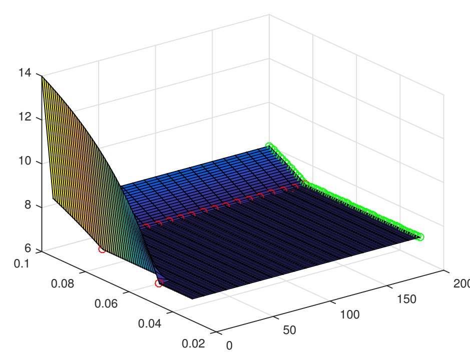
(c) The surface of the uncapped boundary – short maturities.



(d) The surface of the uncapped boundary – long maturities.

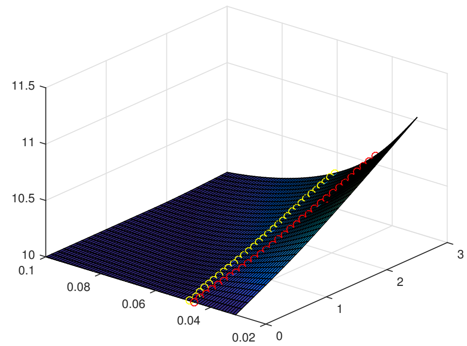


(e) The surface of the capped boundary – short maturities.

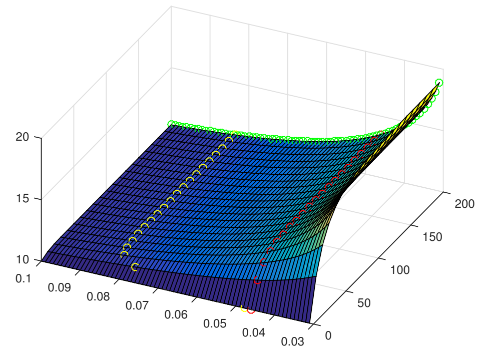


(f) The surface of the capped boundary – long maturities.

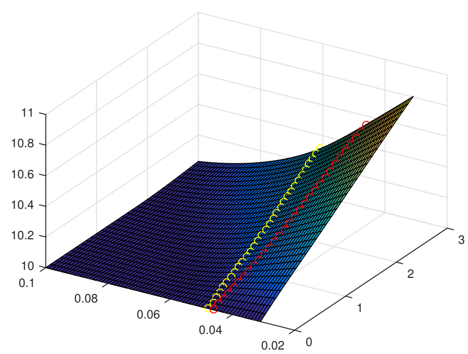
Figure 5.2: Put option prices



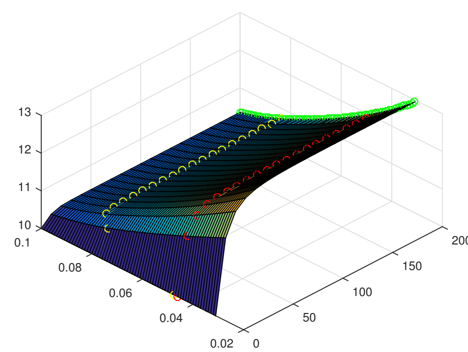
(a) Pure American put option prices, short maturities



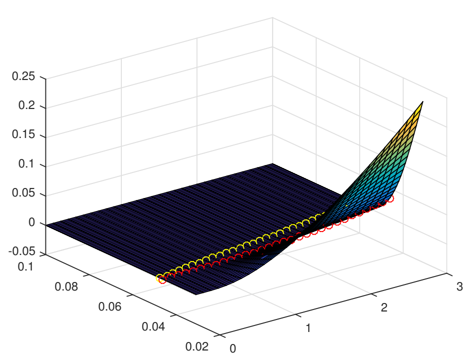
(b) Pure American put option prices, long maturities



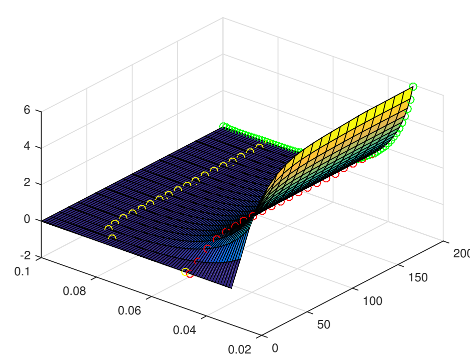
(c) American capped put option prices, short maturities



(d) American capped put option prices, long maturities

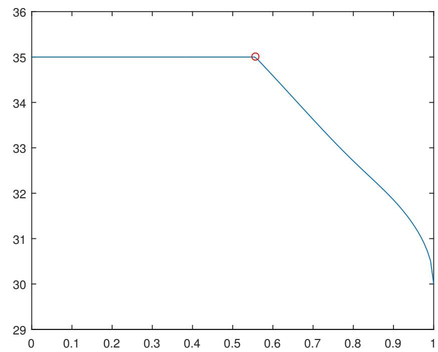


(e) Put premium, short maturities

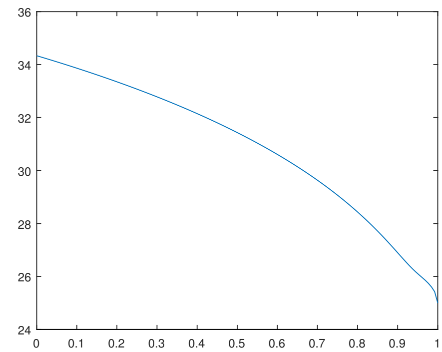


(f) Put premium, long maturities

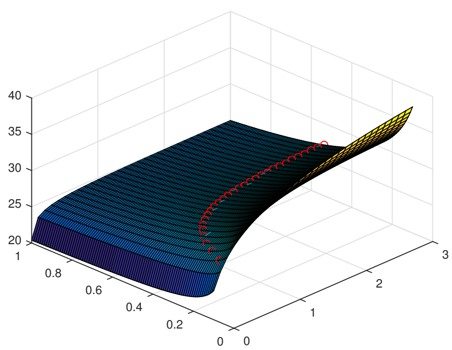
Figure 5.3: Call capped options



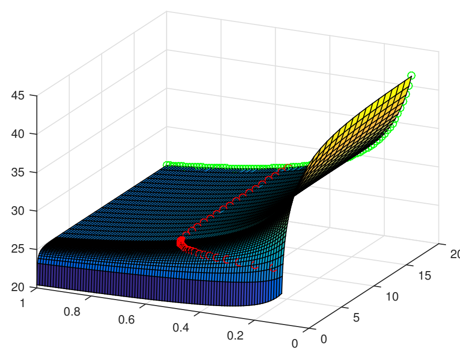
(a) The optimal boundary, $\lambda = 0.04$.



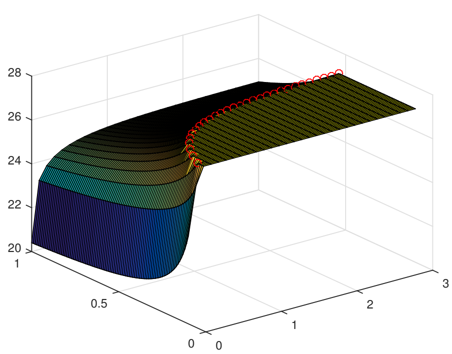
(b) The optimal boundary, $\lambda = 0.08$.



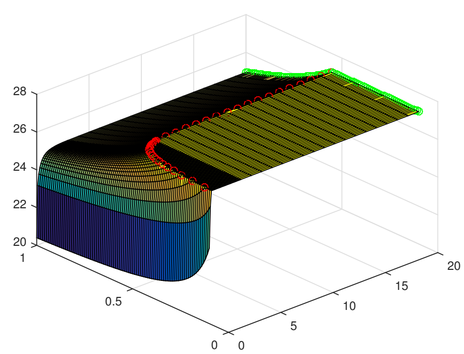
(c) The surface of the uncapped boundary – short maturities.



(d) The surface of the uncapped boundary – long maturities.

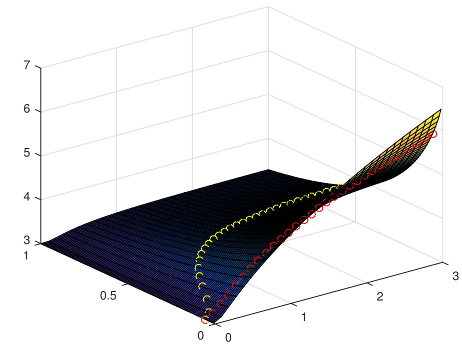


(e) The surface of the capped boundary – short maturities.

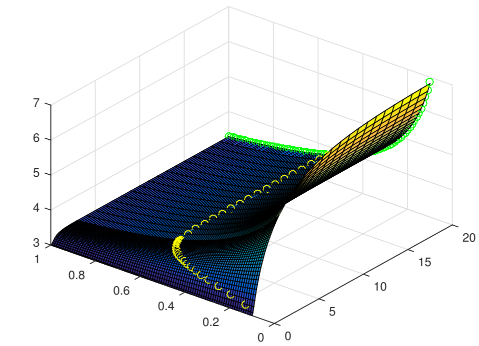


(f) The surface of the capped boundary – long maturities.

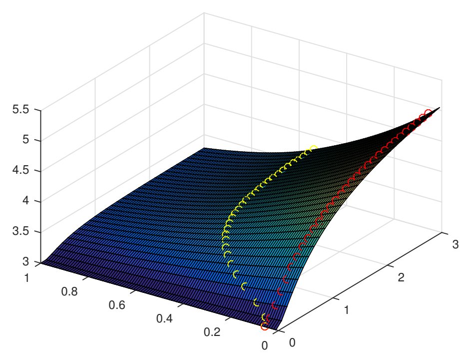
Figure 5.4: Call option prices



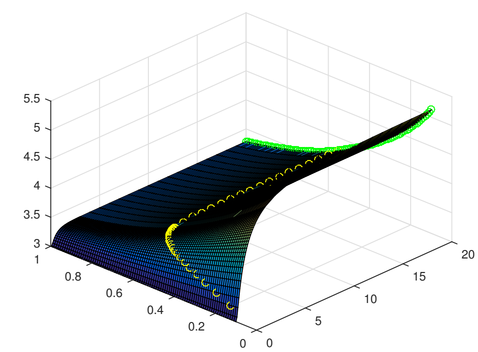
(a) Pure American call option prices, short maturities



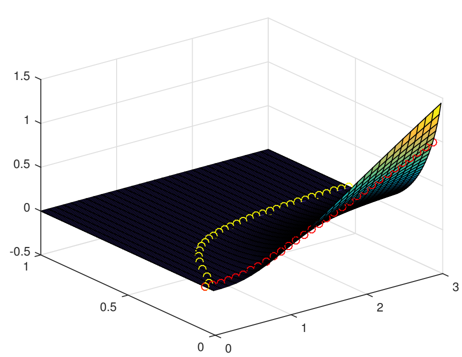
(b) Pure American call option prices, long maturities



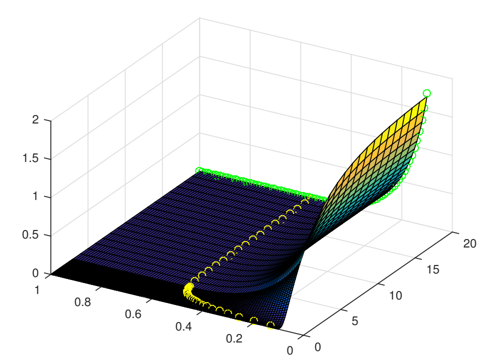
(c) American capped call option prices, short maturities



(d) American capped call option prices, long maturities



(e) Call premium, short maturities



(f) Call premium, long maturities

Table 5.1: Put option prices

$\lambda = 0.031$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.8291	1.8289/1	1.8281/1	1.8208/1	1.7654/1	1.3750/1
$\tau = 1$	2.6541	2.6518/1	2.6395/1	2.5847/1	2.3456/1	1.5829/1
$\tau = 2$	3.8695	3.8361/1	3.7521/1	3.5040/1	2.8948/1	1.7394/1
$\tau = 3$	4.8280	4.7169/1	4.5121/1	4.0457/1	3.1693/1	1.8091/1
$\lambda = 0.04$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.8209	1.8207/1	1.8201/1	1.8136/1	1.7603/1	1.3729/1
$\tau = 1$	2.6303	2.6287/1	2.6187/1	2.5685/1	2.3357/1	1.5795/1
$\tau = 2$	3.8004	3.7774/1	3.7051/1	3.4713/1	2.8769/1	1.7339/1
$\tau = 3$	4.6995	4.6224/1	4.4407/1	3.9986/1	3.1449/1	1.8021/1
$\lambda = 0.05$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.8117	1.8117/1	1.8112/1	1.8057/1	1.7548/1	1.3707/1
$\tau = 1$	2.6041	2.6033/1	2.5958/1	2.5507/1	2.3247/1	1.5756/1
$\tau = 2$	3.7258	3.7134/1	3.6537/1	3.4354/1	2.8571/1	1.7279/1
$\tau = 3$	4.5645	4.5198/1	4.3630/1	3.9472/1	3.1182/1	1.7943/1
$\lambda = 0.06$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.8027	1.8026/1	1.8023/1	1.7978/1	1.7492/1	1.3684/1
$\tau = 1$	2.5783	2.5782/1	2.5732/1	2.5330/1	2.3138/1	1.5718/1
$\tau = 2$	3.6554	3.6506/1	3.6032/1	3.4001/1	2.8376/1	1.7219/1
$\tau = 3$	4.4427	4.4198/1	4.2870/1	3.8968/1	3.0919/1	1.7867/1
$\lambda = 0.07$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.7937	1.7938/2	1.7936/1	1.7900/1	1.7437/1	1.3661/1
$\tau = 1$	2.5533	2.5531/2	2.5507/1	2.5154/1	2.3030/1	1.5680/1
$\tau = 2$	3.5905	3.5889/2	3.5534/1	3.3652/1	2.8183/1	1.7159/1
$\tau = 3$	4.3332	4.3225/2	4.2127/1	3.8473/1	3.0660/1	1.7791/1
$\lambda = 0.08$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.7849	1.7849/3	1.7845/2	1.7821/1	1.7381/1	1.3639/1
$\tau = 1$	2.5295	2.5295/2	2.5279/2	2.4980/1	2.2922/1	1.5643/1
$\tau = 2$	3.5306	3.5300/2	3.5043/2	3.3308/1	2.7992/1	1.7100/1
$\tau = 3$	4.2333	4.2286/2	4.1399/2	3.7986/1	3.0404/1	1.7717/1
$\lambda = 0.09$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.7762	1.7762/3	1.7760/2	1.7744/1	1.7326/1	1.3616/1
$\tau = 1$	2.5068	2.5068/3	2.5061/2	2.4807/1	2.2815/1	1.5605/1
$\tau = 2$	3.4746	3.4743/2	3.4562/2	3.2968/1	2.7803/1	1.7041/1
$\tau = 3$	4.1408	4.1385/2	4.0687/2	3.7508/1	3.0152/1	1.7643/1
$\lambda = 0.1$	American	$L = 10$	$L = 12$	$L = 14$	$L = 16$	$L = 18$
$\tau = 0.5$	1.7677	1.7677/3	1.7677/2	1.7663/2	1.7271/1	1.3594/1
$\tau = 1$	2.4852	2.4852/3	2.4848/2	2.4631/2	2.2709/1	1.5568/1
$\tau = 2$	3.4219	3.4218/2	3.4091/2	3.2632/2	2.7617/1	1.6983/1
$\tau = 3$	4.0544	4.0528/2	3.9992/2	3.7039/2	2.9904/1	1.7570/1

Table 5.2: Call option prices

$\lambda = 0$	American	$L = 21$	$L = 24$	$L = 27$	$L = 30$	$L = 33$
$\tau = 0.5$	1.7824	0.8070/1	1.6730/1	1.7692/1	1.7797/1	1.7817/1
$\tau = 1$	2.5643	0.8572/1	2.1342/1	2.4622/1	2.5351/1	2.5537/1
$\tau = 2$	3.7006	0.8928/1	2.5376/1	3.2320/1	3.5036/1	3.6097/1
$\tau = 3$	4.5886	0.9083/1	2.7320/1	3.6597/1	4.1130/1	4.3331/1
$\lambda = 0.02$	American	$L = 21$	$L = 24$	$L = 27$	$L = 30$	$L = 33$
$\tau = 0.5$	1.7644	0.8057/1	1.6640/1	1.7552/1	1.7633/1	1.7644/1
$\tau = 1$	2.5134	0.8551/1	2.1174/1	2.4323/1	2.4964/1	2.5095/1
$\tau = 2$	3.5577	0.8897/1	2.5084/1	3.1744/1	3.4229/1	3.5112/1
$\tau = 3$	4.3315	0.9045/1	2.6930/1	3.5787/1	3.9948/1	4.1841/1
$\lambda = 0.04$	American	$L = 21$	$L = 24$	$L = 27$	$L = 30$	$L = 33$
$\tau = 0.5$	1.7472	0.8044/1	1.6551/1	1.7413/1	1.7470/1	1.7472/2
$\tau = 1$	2.4666	0.8531/1	2.1008/1	2.4030/1	2.4584/1	2.4661/2
$\tau = 2$	3.4380	0.8867/1	2.4798/1	3.1183/1	3.3445/1	3.4157/2
$\tau = 3$	4.1287	0.9008/1	2.6551/1	3.5004/1	3.8810/1	4.0414/2
$\lambda = 0.06$	American	$L = 21$	$L = 24$	$L = 27$	$L = 30$	$L = 33$
$\tau = 0.5$	1.7310	0.8030/1	1.6463/1	1.7275/2	1.7310/2	1.7310/3
$\tau = 1$	2.4252	0.8511/1	2.0843/1	2.3739/2	2.4211/2	2.4252/2
$\tau = 2$	3.3350	0.8837/1	2.4518/1	3.0635/2	3.2684/2	3.3244/2
$\tau = 3$	3.9569	0.8971/1	2.6183/1	3.4246/2	3.7716/2	3.9055/2
$\lambda = 0.08$	American	$L = 21$	$L = 24$	$L = 27$	$L = 30$	$L = 33$
$\tau = 0.5$	1.7161	0.8017/1	1.6375/1	1.7139/2	1.7161/2	1.7161/3
$\tau = 1$	2.3874	0.8491/1	2.0681/1	2.3454/2	2.3853/2	2.3874/2
$\tau = 2$	3.2419	0.8808/1	2.4243/1	3.0101/2	3.1948/2	3.2376/2
$\tau = 3$	3.8044	0.8936/1	2.5825/1	3.3513/2	3.6666/2	3.7762/2
$\lambda = 0.1$	American	$L = 21$	$L = 24$	$L = 27$	$L = 30$	$L = 33$
$\tau = 0.5$	1.7021	0.8004/1	1.6288/1	1.7006/2	1.7021/2	1.7021/3
$\tau = 1$	2.3518	0.8471/1	2.0520/1	2.3175/2	2.3510/2	2.3518/2
$\tau = 2$	3.1562	0.8780/1	2.3974/1	2.9581/2	3.1237/2	3.1549/2
$\tau = 3$	3.6667	0.8901/1	2.5476/1	3.2805/2	3.5657/2	3.6530/2

Chapter 6

On some generalized American style derivatives.

based on the paper

Zaevski, Tsvetelin S. "On some generalized American style derivatives." Computational and Applied Mathematics 43.3 (2024): 1-29.

Abstract: This chapter aims to examine some American-style financial instruments with generalized payment structures, in particular for power functions. We first prove several propositions related to the optimal regions and the corresponding early exercise boundaries. Based on them we derive the boundary values when the time to maturity is zero or infinitely large. We present also a numerical algorithm for approximating the whole boundary. Thus we can view the arising free boundary equation as a boundary value problem in a known region. We apply to it the Crank-Nicolson finite difference approach to derive the fair prices. Some numerical experiments are provided.

6.1 Motivation and main results

The growing interest in American-style financial instruments motivated us to investigate them under a general framework. At the same time, the focus of this chapter is on the power payment structures. However, we impose much weaker conditions that guarantee that the pricing of such derivatives leads to one-sided optimal stopping problems. This means that the state space can be divided into two subsets of the forms $(0, c(t))$ and $(c(t), \infty)$ – in one of them the immediate exercise is the optimal strategy, whereas in the other keeping the asset leads to a better result. If the set $(0, c(t))$ consists of the optimal points, then we name the derivative put-style. Otherwise, if the optimal set is $(c(t), \infty)$, then the derivative is call-style. This distinction is made by analogy with the ordinary American options. The boundary between the sets, namely $c(t)$, is called early exercise or optimal boundary. We have two degenerate cases, $c(t) \equiv 0$ and $c(t) \equiv \infty$. Such derivatives exhibit a put feature as well as a call. However, more or less these cases are trivial – the immediate exercise is always or never optimal.

We solve the arising one-sided optimal stopping problem in several steps. First, we prove several propositions for the shape of the optimal boundary. It turns out that it is an increasing function w.r.t. the time to maturity for the call-style derivatives and, on the opposite, it decreases for the put-ones. The next step is to derive the endpoints of the boundary – the initial one corresponds to the maturity date whereas the infinity value is for the perpetual derivative. Thereafter we approximate the boundary using an exponent of piecewise linear function and applying the results of Chapter 2.2. This way we immediately recognize whether the exercise at the current spot price is optimal and evaluate the option accurately.

On the other hand, we can approximate the boundary at a denser grid. This way the free boundary equation that describes the derivative turns into a boundary value problem in a known region. We adapt the Crank-Nicolson finite difference approach to it and that way we derive the fair price.

The main restriction we impose is that the payoff is a twice differentiable function. Obviously, this condition is not satisfied by the most traded financial derivatives, namely the options – their payoffs are not differentiable at the strike. We can overcome this problem by approximating the payoffs by suitable twice differentiable functions. This is possible because the non-differentiability is only at a single point, namely the strike. We provide as an example an approach for evaluating classical options.

As we mentioned above, the power payoff functions of the type $Mx^n + K$ have a central place in our study. They generate derivatives which can be viewed as power futures contracts – long and short positioned. The choice of the power functions is motivated by several circumstances. The main of them is that these payoffs allow the investors to hedge different non-linear risks. For example, an investor will prefer derivatives with larger values of the power n when he needs to hedge stronger the deeply in-the-money positions. On the opposite, the lower values of n are preferable for the near-the-money positions. The terms *in-the-money, near-the-money, etc.* are used analogously to the meaning of the option positions, i.e. these terms indicate the payoff sign. In the usual case, i.e. $n = 1$, the values $M = 1$ and $K < 0$ lead to a long positioned futures related to a call option. Analogously, the short futures contracts ($M = -1$ and $K > 0$) are related to put options. It turns out that the derivative's style does not depend only on the signs of the constants M and K when $n \neq 1$, but it depends on the sign of another constant too. Some parameters' values lead again to the degenerated derivatives for which the immediate exercise is always or never optimal. We examine all cases in detail. After that, we present some numerical experiments which illustrate the derived results. As further work, we can consider power options whose payoff can be defined as $(Mx^n + K)^+$.

The chapter is organized as follows. We present the base we use later in Section 6.2. The shape of the optimal regions is obtained in Section 6.3. We examine the perpetual derivatives in Section 6.4. We present in Section 6.5 numerical algorithms for deriving the optimal boundary and the price under the finite maturity horizon. We examine power payoffs and the related derivatives in Section 6.6. The American call options are discussed in the light of our framework in Section 6.7.

6.2 Preliminaries

Let the derivative maturates at a moment $T \leq \infty$. Recall that the payoff of the derivative $N(t,x)$ means that if the holder exercises the instrument at the moment $t \leq T$ at the spot price $S_t = x$, then he receives the amount of $N(t,x)$. We also assume that the time dependence is presented only by the additional discount rate λ – the most real financial derivatives satisfy this condition. Thus we have the presentation

$$N(t,x) = e^{-\lambda t} G(x) \tag{6.1}$$

for some twice differentiable function $G(x)$. We define also the following related to the infinitesimal generator differential operator over the C^2 functions

$$(\mathcal{B}g)(x) = (\mathcal{A}g)(x) - (r+\lambda)\, g(x)\,. \tag{6.2}$$

The following definition presents the derivatives we investigate in this study.

Definition 6.1. *Let the derivative be characterized by payoff* (6.1).

1. *We name the instrument a call-style derivative if the following condition holds. If* $(\mathcal{B}G)(x) < 0$ *for some* x*, then* $(\mathcal{B}G)(y) < 0$ *for all* $y \geq x$.

2. *The condition for the put-style derivatives is: if* $(\mathcal{B}G)(x) < 0$ *for some* x*, then* $(\mathcal{B}G)(y) < 0$ *for all* $y \leq x$.

Although this definition seems artificial from a financial point of view, it has its significance – practical and theoretical. One of the main properties of the classical options is the fact that the immediate exercise is preferable for the holder if the asset price is large (for the calls) or small (for the puts) enough. The above-introduced conditions guarantee to keep this property. Also, they are not so restrictive.

We shall use the symbol V hereafter for the derivative price, i.e. $V^p(t,x)$ and $V^c(t,x)$ are the prices of put- and call-style derivatives assuming that the price of the underlying asset at the moment t is x. If we have a parametrization w.r.t. the time to maturity we shall use the notations $V^p(x;\tau)$ and $V^c(x;\tau)$.

Remark 6.1. *Let us explain the meaning of Definition 6.1. The payoff of an American call option with strike* K *is* $N(t,x) = e^{-\lambda t}(x-K)^+$ *or respectively* $G(x) = (x-K)^+$*. For the put options we have* $G(x) = (K-x)^+$*. Note that these functions are not differentiable at the point* $x = K$*. Nevertheless, we can examine functions* $\overline{G}(x) = x - K$ *and* $\overline{G}(x) = K - x$ *having in mind that early exercising for American options is never optimal when* $\overline{G}(x) < 0$*. Therefore, differential operator* (6.2) *turns into* $\left(\mathcal{B}\overline{G}\right)(x) = -\lambda x + (r+\lambda) K$ *for call options and* $(\mathcal{B}G)(x) = \lambda x - (r+\lambda) K$ *for put ones. Thus, the conditions of Definition 6.1 are satisfied.*

6.3 Exercise regions

The next step is to divide the state space that contains the points (t,x) for $t \in [0,T]$ and $x \in (0,+\infty)$. The points for which the immediate exercise is the best strategy for the derivative's owner form the so-called optimal set – we denote it by Υ. The remaining points form the continuation set, i.e. there exists a strategy that expected financial result is larger. We shall denote this set by $\overline{\Upsilon}$, i.e $\overline{\Upsilon} = \{[0,T] \times (0,+\infty)\} / \Upsilon$. We shall use also the notations $\Upsilon_{t,T}$ and $\overline{\Upsilon}_{t,T}$ for the corresponding sets at a fixed moment t assuming that the derivative maturates at the moment T. Alternatively, we shall use $\mathcal{T}_\tau$, Υ_τ, and $\overline{\Upsilon}_\tau$ instead $\mathcal{T}_{[t,T]}$, $\Upsilon_{t,T}$, and $\overline{\Upsilon}_{t,T}$ for the corresponding sets if we parametrize w.r.t. the time to maturity. All this is mathematically formalized in the following definition.

Definition 6.2. *Let the set $\mathcal{T}_{[t,T]}$ consists of all stopping times with values between t and T. Also, we denote by $\mathbb{E}^{t,x}$ and $\mathbb{E}^x$ the expectations under the conditions $S_t = x$ and $S_0 = x$.*

1. *The point (t,x) is optimal, $(t,x) \in \Upsilon$, if for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$,*

$$N(t,x) \geq \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right].$$

2. *Otherwise, $(t,x) \in \overline{\Upsilon}$ if there exists a stopping time $\zeta \in \mathcal{T}_{[t,T]}$, such that*

$$N(t,x) < \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)} N(\zeta, S_\zeta)\right]. \tag{6.3}$$

The following proposition characterizes the possible exercising moments for both of put- and call-style instruments.

Proposition 6.1. *A necessary condition a point (t,x) to be optimal, $(t,x) \in \Upsilon$, is $(\mathcal{B}G)(x) < 0$.*

Proof: Suppose that at the moment t the asset price is $S_t = x$ and $(\mathcal{B}G)(x) \geq 0$. Let the investor's strategy be the exercise after an infinitely small period ϵ. The related financial outcome, $\mathbb{E}^{t,x}\left[e^{-(r+\lambda)(t+\epsilon)} G(S_{t+\epsilon})\right]$, is larger than the result of immediate exercising since

$$\lim_{\epsilon \to 0} \frac{\mathbb{E}^{t,x}\left[e^{-(r+\lambda)(t+\epsilon)} G(S_{t+\epsilon})\right] - e^{-(r+\lambda)t} G(x)}{\epsilon} = e^{-(r+\lambda)t} (\mathcal{B}G)(x) \geq 0.$$

□

Next, we obtain the shape of the optimal regions proving several propositions.

Proposition 6.2. *If $\tau_1 > \tau_2$ and the point $(x;\tau_1)$ is optimal, then the point $(x;\tau_2)$ is optimal too.*

Proof: Suppose that $(x;\tau_2) \notin \Upsilon$ and thus there exists a strategy ζ such that inequality (6.3) holds. But this contradicts to $(x;\tau_1) \in \Upsilon$ because $\mathcal{T}_{\tau_1} \supset \mathcal{T}_{\tau_2}$ since $\tau_1 > \tau_2$. □

Proposition 6.3. *Let $(x;\tau) \in \Upsilon_\tau$ be an optimal point.*

1. *If the derivative is a call style and $y \geq x$, then $(y;\tau) \in \Upsilon_\tau$ too.*
2. *If the derivative is put style and $y \leq x$, then $(y;\tau) \in \Upsilon_\tau$.*

Proof: Suppose that for a call-style derivative $(x;\tau) \in \Upsilon_\tau$ and $y > x$. Let $\overline{\zeta}$ be an arbitrary stopping time, ζ^x be the first hitting moment of the underlying asset starting at the point y to the value x, and $\zeta = \overline{\zeta} \wedge \zeta^x \wedge \tau$. Note that the strategy ζ gives a not worse financial result than $\overline{\zeta}$ because the underlying asset is driven by a Markov process and the point $(x;\overline{\tau})$ is optimal for every $\overline{\tau}$ such that $\overline{\tau} < \tau$ due to Proposition 6.2.[1] Also let us mention that $(\mathcal{B}G)(S_u(\omega)) < 0$ for every u such that $u < \zeta(\omega)$ because for such sample paths we have $S_u(\omega) > x$ and thus we can use the definition of a call-style derivative – the first point of Definition 6.1. Using Dynkin's formula we derive

$$\begin{aligned} \mathbb{E}^y\left[e^{-(r+\lambda)\overline{\zeta}} G\left(S_{\overline{\zeta}}\right)\right] - G(y) &\leq \mathbb{E}^y\left[e^{-(r+\lambda)\zeta} G\left(S_\zeta\right)\right] - G(y) \\ &= \mathbb{E}^y\left[\int_0^\zeta (\mathcal{B}G)(S_u)\, du\right] < 0. \end{aligned} \tag{6.4}$$

Therefore the point $(y;\tau)$ is optimal because inequality (6.4) is true for an arbitrary stopping time $\overline{\zeta}$. The proof for a put-style derivative is analogous. □

The shapes of the optimal regions can be obtained through Proposition 6.3. We present them in the following corollary.

[1]If the underlying asset first reaches the value x, then it falls in the optimal set and thus at this trajectories the strategy ζ is better than $\overline{\zeta}$. At the rest sample paths both strategies have equal results.

Corollary 6.1. *The optimal region of a call-style derivative consists of all points above some boundary – the boundary is known as the optimal boundary. The region for a put-style derivative is set below the optimal boundary.*

Remark 6.2. *Note that these boundaries may be infinitely large or zero valued. A well-known example is the call options when the additional discounting is missing, $\lambda = 0$. This way the derivatives exhibit jointly put and call features.*

The behavior of the optimal boundaries can be obtained in a way similar to one presented in Jacka (1991), proposition 2.2. We shall only sketch the proof.

Proposition 6.4. *The following statements hold.*

1. *The optimal boundary for a call-style derivative is an increasing function w.r.t. the time to maturity.*

2. *The boundary of a put-style derivative decreases w.r.t. the time to maturity.*

Proof: Let us consider a call style derivative and a point at the optimal boundary $(\tau, c(\tau))$. Suppose that at this point the boundary is a decreasing function w.r.t. the time to maturity τ. This means that there exists $\epsilon > 0$ such that $(\epsilon, c(\tau)) \notin \Upsilon$ since the boundary is a continuous function.[2] This contradicts the Propositions 6.2 and 6.3. The rest of the proof can be made analogously. □

The next step is to determine the initial values of the optimal boundaries:

Proposition 6.5. *The following statements hold for the limits of the optimal boundaries when the time to maturity tends to zero.*

1. *Let us have a call-style derivative and B_1 be defined as*

$$B_1 = \inf\{x : \ (\mathcal{B}G)(x) < 0\}.$$

Note that this value exists, despite it can be infinity, due to the first statement of Definition 6.1. Namely B_1 is the value of the optimal boundary when the time to maturity is zero, i.e. $c(0) = B_1$.

[2] The last is true because the function $G(x)$ is twice differentiable.

2. *If the derivative is put-style and* A_1 *is defined as*

$$A_1 = \sup\{x : \ (\mathcal{B}G)(x) < 0\},$$

then $c(0) = A_1$.

Proof: Let us consider first a call-style derivative. We have that the initial boundary value is not less than B_1, $c(0) \geq B_1$, due to Proposition 6.1. It is left to be proven that all points above B_1 are optimal. Suppose the opposite, i.e. some point $x > B_1$ is not optimal near the initial moment $\tau = 0$. Using the inequality $V^c(t,x) > N(t,x)$ that holds in the continuation region together with the Black-Scholes equation that also is satisfied in this region, we obtain

$$\begin{aligned}
0 &< \lim_{t\to T} \frac{V^c(t,x) - N(t,x)}{T-t} \\
&= -\lim_{t\to T} \frac{V^c(T,x) - V^c(t,x)}{T-t} + \lim_{t\to T} \frac{N(T,x) - N(t,x)}{T-t} \\
&= \mathcal{A}V^c(T,x) - rV^c(T,x) + N_t(T,x) \\
&= e^{-\lambda t}\mathcal{A}G(x) - re^{-\lambda t}G(x) - \lambda e^{-\lambda t}G(x) \\
&= e^{-\lambda t}\mathcal{B}G(x) < 0.
\end{aligned}$$

The contradiction finishes the proof for the call-style derivatives. The result for put-style instruments can be obtained similarly. □

6.4 Perpetual derivatives

We now consider derivatives without maturity constraints, i.e. $\tau = T = \infty$. In this case, the optimal boundaries are flat – we denote them by A_2 and B_2 for put- and call-style derivatives, respectively. Let us denote by ζ^c the first hitting moment of the underlying asset to the value c. Note that ζ^c can also be viewed as the Brownian motion's first hitting to the linear function

$$d(t) = d_1 t + d_2,$$

where d_1 and d_2 are

$$\begin{aligned} d_1 &= \frac{\sigma}{2} - \frac{r}{\sigma} \\ d_2 &= \frac{1}{\sigma} \ln\left(\frac{c}{x}\right). \end{aligned} \tag{6.5}$$

The stopping time ζ^c is also the first hitting of the Brownian motion with drift $\mu = -d_1$ to the value d_2.

6.4.1 Call-style derivatives

Suppose first that the derivative is call style. We need the following lemmas.

Lemma 6.1. *Let the $g_c(c)$ be defined as*

$$g_c(c) = \frac{G(c)}{c^{p-q}}, \tag{6.6}$$

where the constants p and q are defined by formulas (3.17). The financial result of the strategy ζ^c, assuming that the asset starts from a point $x < c$, is

$$V^c(x;c) = g_c(c)\, x^{p-q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right]. \tag{6.7}$$

Proof: Equation (3.14) from Proposition 3.5 leads to

$$\begin{aligned} V^c(x;c) &= \mathbb{E}^x\left[e^{-(r+\lambda)\zeta^c} G(S_{\zeta^c})\, I_{\zeta^c<\infty}\right] + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] \\ &= G(c)\, \mathbb{E}^x\left[e^{-(r+\lambda)\zeta^c} I_{\zeta^c<\infty}\right] + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] \\ &= G(c)\exp\left\{-\frac{\sqrt{d_1^2 + 2(r+\lambda)} + d_1}{\sigma} \ln\left(\frac{c}{x}\right)\right\} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] \\ &= g_c(c)\, x^{p-q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right]. \end{aligned}$$

□

An immediate consequence of Corollary 6.1 is the following proposition.

Proposition 6.6. *If the maximum of the function $V^c(x;c)$, defined in equation (6.7), w.r.t. the variable c in the interval $[x,\infty)$ is at a point $c(x)$, then the maximum of the function $V^c(y;c)$ for every y, $x < y < c(x)$, is again at the point $c(x)$.*

Suppose now that x is small enough. The lemmas below characterize the behavior at the optimal point.

Lemma 6.2. *If the function $V^c(x;c)$ defined in equations (6.7) has a local maximum in a point $\overline{c} \in (x, \infty)$, then $(\mathcal{B}G)(\overline{c}) \le 0$.*

Proof: Suppose that $(\mathcal{B}G)(\overline{c}) > 0$. Therefore there exists a small enough but larger than $\overline{c}$ constant $\tilde{c}$ such that $(\mathcal{B}G)(c) > 0$ for all $c < \tilde{c}$. Dynkin's formula leads to

$$\begin{aligned} V^c(x;c) &= \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)\zeta^c\wedge T} G\left(S_{\zeta^c\wedge T}\right)\right] \\ &= G(x) + \lim_{T\to\infty} \mathbb{E}^x\left[\int_0^{\zeta^c\wedge T} (\mathcal{B}G)(S_u)\,du\right]. \end{aligned} \tag{6.8}$$

Using the fact that the function $V^c(x;c)$ has a local maximum in the point $\overline{c}$ we conclude that $V^c(x;\overline{c}) > V^c(x;\tilde{c})$ for all $x < \overline{c}$. Using equation (6.8) and taking the limit $x \uparrow \overline{c}$, we conclude

$$G(\overline{c}) = \lim_{x\uparrow\overline{c}} V^c(x;\overline{c}) > \lim_{x\uparrow\overline{c}} V^c(x;\tilde{c}) = V^c(\overline{c};\tilde{c}) = G(\overline{c}) + \lim_{T\to\infty} \mathbb{E}^{\overline{c}}\left[\int_0^{\zeta^{\tilde{c}}\wedge T} (\mathcal{B}G)(S_u)\,du\right].$$

We finish the proof having in mind that $(\mathcal{B}G)(S_u)$ is positive at all sample paths before the stopping time $\zeta^{\tilde{c}}$. □

Lemma 6.3. *The function $V^c(x;c)$ defined in equations (6.7) has no more than one local maximum in the interval (x,∞). This way its global maximum is either this local one or one of the interval endpoints.*

Proof: Suppose the opposite, i.e. there exist at least two local maximums – we denote them by c_1 and c_2. Note that $(\mathcal{B}G)(c_{1,2}) < 0$ due to Lemma 6.2. Also, the function $V^c(x;c)$ is increasing in some sub-interval $(C_1, C_2) \subset (c_1, c_2)$. Let us consider an initial point $x \in (C_1, C_2)$ and a strategy of exercising at the first hitting to $x + \delta$, where with δ we denote an infinitely small positive value. Having in mind the form of formula (6.8), the increasing behavior at the point x, and Dynkin's formula, we see that $(\mathcal{B}G)(x) > 0$ – the proof of this fact is similar to the proof of Lemma 6.2. But this contradicts to $(\mathcal{B}G)(c_1) < 0$, $c_1 < C_1 < x$, and the first statement of Definition 6.1. □

The proven above statements (Proposition 6.6 and Lemmas 6.2 and 6.3) lead to the next proposition.

Proposition 6.7. *The following three statements hold.*

1. *If there exists an initial point x, for which x is strictly less than $c(x)$, $x < c(x)$, then $c(x)$ is the optimal boundary.*
2. *If $x = c(x)$ for all x, then the optimal boundary is zero, i.e. all points are optimal.*
3. *If $x < c(x)$ for every x, then the optimal boundary is infinite, i.e. early exercising is never optimal.*

Proposition 6.7 shows how we derive the optimal boundary. We chose a small enough x, for which $x < c(x)$. Thus the optimal boundary is namely $c(x)$. If such x does not exist, then the optimal boundary is zero.

We can formulate now our main result for pricing perpetual call-style derivatives.

Theorem 6.1. *[Theorem 1 of Zaevski (2024b)]*

The optimal boundary of an American call-style derivatives under a perpetual assumptions is

$$B_2 = \lim_{x \to 0} \arg\max_{c \in [x, \infty)} \{V^c(x; c)\}.$$

If $x \geq B_2$, then the derivative price is $V^c(x; B_2) = G(x)$. Otherwise, if $x < B_2$, then the price is

$$V^c(x, B_2) = g_c(B_2)\, x^{p-q} + \lim_{T \to \infty} \mathbb{E}^x \left[e^{-(r+\lambda)T} G(S_T)\, I_{T < \zeta^{B_2}} \right], \tag{6.9}$$

where the function $g_c(\cdot)$ is defined in equation (6.6). Note that if $B_2 = \infty$, then formula (6.9) turns into

$$V^c(x; \infty) = \lim_{c \to +\infty} \left[\frac{G(c)}{c^{p-q}} x^{p-q} + \lim_{T \to \infty} \mathbb{E}^x \left[e^{-(r+\lambda)T} G(S_T)\, I_{T < \zeta^c} \right] \right].$$

6.4.2 Put-style derivatives

We present below the analogue of Theorem 6.1 when the derivative is put-style. The second statement of Corollary 6.1 shows that we have a lower first-hitting problem.

Theorem 6.2. *[Theorem 2 of Zaevski (2024b)] Let the constant q and d_1 be defined as in formulas (3.17) and (6.5). Note that*

$$q = \frac{\sqrt{d_1^2 + 2\left(r+\lambda\right)} - d_1}{\sigma}. \tag{6.10}$$

The price of the derivative associated with the first hitting a boundary below the starting point, $c < x$, is

$$V^p\left(x;c\right) = \frac{g_p\left(c\right)}{x^q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G\left(S_T\right) I_{T<\zeta^c}\right], \tag{6.11}$$

where the function $g_p\left(\cdot\right)$ is defined as

$$g_p\left(c\right) = G\left(c\right) c^q \tag{6.12}$$

The function (6.11) has no more than one local maximum in the interval $(0, x]$. Let us denote by $c\left(x\right)$ the optimal boundary if the asset's initial value is x. The optimal boundary A_2 is obtained through the following statements:

1. *If $c\left(x\right) < x$ for some x, then $A_2 = c\left(x\right)$.*

2. *If $c\left(x\right) = x$ for all x's, then all points are optimal, i.e. $A_2 = \infty$.*

3. *If $c\left(x\right) < x$ for all x, then early exercising is never optimal i.e. $A_2 = 0$.*

If $0 < x \leq A_2$, then the derivative price is just $V^p\left(x, A_2\right) = G\left(x\right)$. If $x > A_2 > 0$, then the price is

$$V^p\left(x;A_2\right) = \frac{g_p\left(A_2\right)}{x^q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G\left(S_T\right) I_{T<\zeta^{A_2}}\right].$$

If $A_2 = 0$, then

$$P\left(x;0\right) = \lim_{c\to 0}\left[\frac{g_p\left(c\right)}{x^q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G\left(S_T\right) I_{T<\zeta^c}\right]\right].$$

Proof: Suppose the initial asset value x is large enough and the derivative's holder exercises at the value $c < x$. Having in mind the second statement of Proposition 3.5, we derive for the put-price

$$\begin{aligned} V^p(x;c) &= \mathbb{E}^x\left[e^{-(r+\lambda)\zeta^c}G(S_{\zeta^c})\,I_{\zeta^c<\infty}\right] + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}G(S_T)\,I_{T<\zeta^c}\right] \\ &= G(c)\,\mathbb{E}^x\left[e^{-(r+\lambda)\zeta^c}I_{\zeta^c<\infty}\right] + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}G(S_T)\,I_{T<\zeta^c}\right] \\ &= G(c)\exp\left\{\frac{\sqrt{d_1^2+2(r+\lambda)}-d_1}{\sigma}\ln\left(\frac{c}{x}\right)\right\} + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}G(S_T)\,I_{T<\zeta^c}\right] \\ &= \frac{g_p(c)}{x^q} + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}G(S_T)\,I_{T<\zeta^c}\right]. \end{aligned}$$

The rest of the proof is identical to the call case. □

6.5 Finite maturities

Assume now that the maturity is finite, i.e. $T < \infty$. Let us exclude the combined put-call derivatives. We make this without loss of generality because the combined cases lead to derivatives for which early exercising is ever or never optimal. The following reasons confirm this. If the derivative exhibits a combined feature, then $\mathcal{B}(G)(x) \geq 0$ for all x's or $\mathcal{B}(G)(x) \leq 0$ ever. The put and call initial boundaries are $A_1 = 0$ and $B_1 = \infty$ in the first case. Proposition 6.4 shows that early exercising is never optimal. For the second case, we can examine the derivative with the opposite payoff.

6.5.1 Optimal boundary

The first task is to approximate the optimal boundary. Let us divide the time interval $[0, T]$ into l sub-intervals, namely $0 \equiv t_0 < t_1 < ... < t_l \equiv T$. Let the owner's strategy ζ be the first hitting moment to the level $\exp(a_i t + b_i)$ if it happens in the interval $[t_{i-1}, t_i)$, $i = 1, 2, ..., l$. We also impose a continuity at the nodes – $\exp(a_i t_i + b_i) = \exp(a_{i+1} t_i + b_{i+1}) \equiv c_i$. Let us consider a European style derivative that expires at the moment $\zeta \wedge T$ with payoff $\exp(-\lambda t)\,G(t)$. We denote its price by

$$V(x; \{t_0, ..., t_l\}; \{c_0, ..., c_l\}) = \mathbb{E}^x\left[e^{-(r+\lambda)(\zeta\wedge T)}G(S_{\zeta\wedge T})\right]. \qquad (6.13)$$

Our algorithm for deriving the optimal boundary is as follows.

1. The value of the exercise boundary at the maturity, c_l, is given in Proposition 6.5, the first statement for calls and second one for puts.

2. Suppose we have derived all boundary values after some index $m \leq l$, namely c_m, c_{m+1}, ..., c_l. The call boundary at the previous node is obtained as the lower value of x for which the payment $h(c) = V(x; \{0, t_m - t_{m-1}, ..., t_l - t_{m-1}\}; \{c, c_m, ..., c_l\})$ given in equation (6.13) achieves its maximum in the interval $[x, \infty)$ for $c = x$. This is the lower value that makes the immediate exercise optimal. For put-style derivatives, our approximation is the higher value of x for which the maximum in the interval $(0, x]$ is achieved for $c = x$.

6.5.2 Pricing

We provide first our fast pricing method:

Fast Pricing Approach 6.1. *Once we approximate the optimal boundary, we derive the option price via formula* (6.13) *taken at the point* $x = S_0$.

We can obtain the derivative price viewing it as the solution of a boundary value problem (BVP) if we need a denser grid. We begin with a call-style derivative. The region for the BVP is $(t, x) \in \{(0, T) \times (0, c(t))\}$. If the initial point is outside this range, then the price is simply $e^{-\lambda t} G(x)$. The BVP can be written as

$$\begin{aligned}
&V_t^c(t,x) + rxV_x^c(t,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}^c(t,x) - rV^c(t,x) = 0 \\
&V^c(t,0) = e^{-r(T-t)} e^{-\lambda T} G(0), \quad t \in (0,T) \\
&V^c(t, c(t)) = e^{-\lambda t} G(x), \quad t \in (0,T) \\
&V^c(T,x) = e^{-\lambda T} G(x), \quad x \in (0, B_1).
\end{aligned} \tag{6.14}$$

We use the Crank–Nicolson finite difference method presented in Section 3.5 to solve BVP (6.14).

We need a little adjustment of the pricing algorithm if we have a put-style derivative. The main difference is that the continuation region is infinite above. For this we set an auxiliary large enough boundary – we denote it by

$\overline{C}$. A relatively good approximation of the price function at this boundary is the value of the corresponding European-style derivative. Alternatively, we can use a Monte Carlo simulation. Let us denote this approximation by $V^E\left(t;\overline{C}\right)$. Thus the BVP (6.14) turns into

$$\begin{aligned}
&V_t^p(t,x) + rxV_x^p(t,x) + \frac{1}{2}\sigma^2x^2V_{xx}^p(t,x) - rV^p(t,x) = 0\\
&V^p\left(t,\overline{C}\right) = V^E\left(t;\overline{C}\right), \quad t\in(0,T)\\
&V^p(t,c(t)) = e^{-\lambda t}G(x), \quad t\in(0,T)\\
&V^p(T,x) = e^{-\lambda T}G(x), \quad x\in\left(A_1,\overline{C}\right).
\end{aligned} \tag{6.15}$$

It is left to apply again the presented in Section 3.5 Crank–Nicolson approach to BVP (6.15).

6.6 Power payoff functions

Suppose now that the payoff $G(\cdot)$ is given by $G(x) = Mx^n + K$, where $M \in \{-1,1\}$, n is a positive number, and K is an arbitrary real number. Note that the case $n < 0$ can be considered in our framework too because the following presentation holds

$$\frac{1}{S_t} = \frac{1}{x}e^{\left(\overline{r}-\frac{\sigma^2}{2}\right)t+\sigma\overline{B}_t},$$

where $\overline{r} = -r + \sigma^2$ and $\overline{B}_t = -B_t$ is again a Brownian motion.

We need first the following lemmas.

Lemma 6.4. *Let the constant L be defined as*

$$L = (n-1)\left(r + \frac{\sigma^2}{2}n\right) - \lambda. \tag{6.16}$$

If $L > 0$, then $p - q < n$ and $q < n + 2\frac{r}{\sigma^2} - 1$ and vice versa. If $L = 0$, then the inequalities turn into equalities.

Proof: Suppose that $L \geq 0$ or equivalently

$$\lambda \leq (n-1)\left(r + \frac{\sigma^2}{2}n\right). \tag{6.17}$$

Thus, inequality $r + \lambda > 0$ leads to

$$2\frac{r}{\sigma^2} + n - 1 > 0 \tag{6.18}$$

since

$$0 < r + \lambda \le n\frac{\sigma^2}{2}\left(2\frac{r}{\sigma^2} + n - 1\right).$$

Combining formulas (6.17) and (6.6), we derive

$$p - q \le \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2} + n\right)^2} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \tag{6.19}$$

Inequality (6.18) shows that

$$\frac{r}{\sigma^2} - \frac{1}{2} + n = \frac{1}{2}\left(2\frac{r}{\sigma^2} + n - 1\right) + \frac{n}{2} > 0. \tag{6.20}$$

Hence, equation (6.19) indeed leads to $p - q \le n$. Suppose now $L < 0$ or equivalently

$$\lambda > (n-1)\left(r + \frac{\sigma^2}{2}n\right).$$

Therefore,

$$\begin{aligned} p - q &> \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2} + n\right)^2} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) \\ &\ge \left(\frac{r}{\sigma^2} - \frac{1}{2} + n\right) - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right) = n. \end{aligned}$$

The statements for the constant q follow from definitions (6.6) and (6.10). □

Lemma 6.5. *The functions $g_c(\cdot)$ and $g_p(\cdot)$ defined in equations (6.6) and (6.12) are $g_c(c) = \frac{Mc^n + K}{c^{p-q}}$ and $g_p(c) = (Mc^n + K)\,c^q$. Their derivatives are*

$$\begin{aligned} g_c^{'}(c) &= \frac{Mc^n(n - p + q) - K(p - q)}{c^{p-q+1}} \\ g_V^{'}(c) &= c^{q-1}\left[Mc^n(n + q) + Kq\right]. \end{aligned}$$

Proof: The proof is an immediate consequence of the form of the functions $g_c(\cdot)$ and $g_p(\cdot)$. □

Lemma 6.6. *If $L > 0$, $\theta = n\sigma$, and $k = L - \frac{\theta^2}{2}$, then $-\frac{\theta^2}{2} < k < \frac{d_1^2}{2} - \theta d_1$ and $d_1 < n\sigma$.*

Proof: First, $k > -\frac{\theta^2}{2}$ since $L > 0$. Second, inequality $k < \frac{d_1^2}{2} - \theta d_1$ holds because it is equivalent to

$$-(r+\lambda) < \frac{1}{2}\left(\frac{\sigma}{2} - \frac{r}{\sigma}\right)^2.$$

It is left to check the inequality $d_1 < n\sigma$. Indeed, $L > 0$ leads to inequality (6.20) that is equivalent to $d_1 < n\sigma$. □

The next step is to obtain the initial value of the optimal boundary. We have for the operator $(\mathcal{B}G)(\cdot)$

$$\begin{aligned}(\mathcal{B}G)(x) &= Mx^n\left[(n-1)\left(r + \frac{\sigma^2}{2}n\right) - \lambda\right] - K(r+\lambda) \\ &= MLx^n - K(r+\lambda).\end{aligned} \tag{6.21}$$

We see from equation (6.21) that the derivative's style depends on the sign of the constants M, L, and K. Note that some cases lead to joint put-call features. For convenience, we shall assume that the derivative is call style if $LM < 0$ and put style otherwise. The case $L = 0$ is special – it corresponds to the undiscounted case ($\lambda = 0$) when the payoff is linear. We shall examine all these cases separately.

6.6.1 The limiting case $L = 0$

We have now a put-call derivative. Note that the equality in formula (6.17) holds since $L = 0$. Let us consider first the derivative as call-style. Suppose that $K \geq 0$. Formula (6.21) shows that $(\mathcal{B}G)(x) \leq 0$ and therefore $B_1 = 0$. Price function (6.7) consists of two parts. The second one can be derived as

$$\lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T} G(S_T)\, I_{T<\zeta^c}\right] = Mx^n \lim_{T\to\infty} e^{-n^2\frac{\sigma^2}{2}T}\mathbb{E}^x\left[e^{n\sigma B_T}\right]. \tag{6.22}$$

Note that inequality (6.20) is equivalent to $d_1 < n\sigma$. Hence, the fifth statement of Theorem 2.10 shows that limit (6.22) is zero. Therefore, price function (6.7) turns into

$$V^c(x;c) = g_c(c)\,x^n = \frac{Mc^n + K}{c^n}x^n \tag{6.23}$$

because $p - q = n$ due to Lemma 6.4. Having in mind Lemma 6.5 we see that $V^c(x;c)$ is a decreasing function w.r.t. the variable c. The second statement of Proposition 6.7 shows that $B_2 = 0$ too. Hence, the perpetual price is

$$V^c(x) = Mx^n + K. \tag{6.24}$$

Let us consider the derivative as put-style. We have now $A_1 = \infty$. Using the eleventh statement of Theorem 2.10 (for $\theta = n\sigma$), equation (6.17) (note again that the inequality turns into equality when $L = 0$), and Lemma 6.4 we derive for price (6.11)

$$\begin{aligned} V^p(x;c) &= \frac{(Mc^n + K)\,c^q}{x^q} + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-(r+\lambda)T}\left(MS_T^n + K\right)I_{T<\zeta^c}\right] \\ &= \frac{(Mc^n + K)\,c^q}{x^q} + \lim_{T\to\infty} e^{-\frac{n^2\sigma^2}{2}T}\mathbb{E}^x\left[e^{n\sigma B_T}I_{T<\zeta^c}\right] \\ &= \frac{(Mc^n + K)\,c^q}{x^q} + Mx^n\left(1 - \left(\frac{c}{x}\right)^{n+q}\right) \\ &= \frac{Kc^q}{x^q} + Mx^n \end{aligned} \tag{6.25}$$

Hence, the value $c = x$ maximizes price (6.25). The second statement of Theorem 6.2 shows that $A_2 = \infty$ too – this confirms that all points are optimal.

Let us consider now the case $K < 0$ and the derivative as a call one. Formula (6.21) leads $(\mathcal{B}G)(x) > 0$ and hence $B_1 = \infty$. Analogously, we derive formula (6.29) for the price, and using Lemma 6.5 we obtain $B_2 = \infty$ too. If we consider the derivative as put-style, then $A_1 = 0$. On the other hand, price (6.25) is a decreasing function w.r.t. the variable c and thus the third statement of Theorem 6.2 shows that $A_2 = 0$ or equivalently there are no optimal points. The perpetual price can be derived as the limit for $c \to 0$ in formula (6.25). Hence,

$$V^p(x) = Mx^n. \tag{6.26}$$

Note that the same result can be derived using call-price (6.23) for $c \to \infty$.

6.6.2 Call-style derivatives – $LM < 0$

Having in mind equation (6.21), we derive for the boundary value when time to maturity is zero

$$B_1 = \left(\max\left\{K\frac{r+\lambda}{ML}, 0\right\}\right)^{\frac{1}{n}}. \tag{6.27}$$

The next task is to derive the perpetual value. The second part of price function (6.7) now can be written as

$$\lim_{T\to\infty} \mathbb{E}^x \left[e^{-(r+\lambda)T} G(S_T) I_{T<\zeta^c}\right] = Mx^n \lim_{T\to\infty} e^{\left(L-n^2\frac{\sigma^2}{2}\right)T}\mathbb{E}^x\left[e^{n\sigma B_T}\right]. \tag{6.28}$$

Suppose first that $L < 0$ and $M = 1$. Thus, we are in the second statement of Theorem 2.10 and therefore limit (6.28) is zero. Hence, price function (6.7) is given by formula

$$V^c(x;c) = g_c(c)\, x^n = \frac{c^n + K}{c^{p-q}} x^{p-q}. \tag{6.29}$$

The inequality $L < 0$ and Lemma 6.4 leads to $p - q > n$. Using Lemma 6.5 we see that the numerator of the derivative $g_c^{'}(c)$ is a decreasing function. If $K \geq 0$, then $g_c^{'}(c) \leq 0$ for $c > 0$ and therefore price function (6.7) decreases w.r.t. the variable c. We conclude that $B_2 = 0$ due to the second statement of Proposition 6.7. Note that $B_1 = 0$ too due to equation (6.27). This means that we have a combined derivative whose perpetual price is given by formula (6.24).

If $K < 0$, then the derivative $g_c^{'}(c)\, 0$ has a root and $g_c^{'}(c) > 0$ before the root and $g_c^{'}(c) < 0$ after. Namely, this root is the perpetual value B_2. Hence,

$$\begin{aligned} B_1 &= \left(K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ B_2 &= \left(-K\frac{p-q}{p-q-n}\right)^{\frac{1}{n}}, \end{aligned} \tag{6.30}$$

and the perpetual price for $x < B_2$ is

$$V^c(x) = \frac{B_2^n + K}{B_2^{p-q}} x^{p-q}. \tag{6.31}$$

Suppose now that $L > 0$ and $M = -1$. Equation (6.28) shows that the constants k and θ from Theorem 2.10 are $k = L - \frac{\theta^2}{2}$ and $\theta = n\sigma$. Thus, Lemma 6.6 shows that the eighth statement of Theorem 2.10 is actual and therefore limit (6.28) is zero. Hence, the price function is given again by formula (6.29). Using Lemma 6.4, we see that inequality $L > 0$ leads to $p - q < n$. The same conclusions w.r.t. the sign of K are valid, because $M = -1$. The only difference is in the sign in the formulation of boundaries (6.30) that turn into

$$\begin{aligned} B_1 &= \left(-K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ B_2 &= \left(-K\frac{p-q}{n-p+q}\right)^{\frac{1}{n}}. \end{aligned} \tag{6.32}$$

Perpetual price (6.31) turns into

$$V^c(x) = \frac{-B_2^n + K}{B_2^{p-q}} x^{p-q}. \tag{6.33}$$

6.6.3 Put-style derivatives – $LM > 0$

Equation (6.21) shows that the initial boundary value A_1 is given again by formula (6.27). We have to derive the perpetual value. Suppose first that $L < 0$ and $M = -1$. The second part of price function (6.11) can be written as formula (6.28). The second statement of Theorem 2.10 shows that limit (6.28) is zero. Thus, price function (6.11) is given by

$$V^p(x;c) = \frac{g_p(c)}{x^q} = \frac{(-c^n + K)\,c^q}{x^q}. \tag{6.34}$$

Lemma 6.5 shows that the second part of the derivative $g_p{}'(c)$, namely $-c^n(n+q) + Kq$, is a decreasing function. If $K \le 0$, then $g_c^{'}(c) \le 0$ for all admissible values of c, and therefore price function (6.7) decreases w.r.t. the variable c. We conclude that $A_2 = 0$ due to the third statement of

Theorem 6.2. Note that $A_1 = 0$ too because of equation (6.27). Hence, the perpetual price is $V^p(x) = 0$.

If $K > 0$, then the derivative $g_V^{'}(c)\,0$ has a root that leads to the maximum of price function (6.34). Hence,

$$\begin{aligned} A_1 &= \left(-K\frac{r+\lambda}{L}\right)^{\frac{1}{n}} \\ A_2 &= \left(\frac{Kq}{q+n}\right)^{\frac{1}{n}}, \end{aligned} \tag{6.35}$$

and the perpetual price when $x > A_2$ is

$$V^p(x) = \frac{\left(-A_2^n + K\right) A_2^q}{x^q}. \tag{6.36}$$

If $L > 0$ and $M = 1$, then the twelfth statement of Theorem 2.10 holds due to Lemma 6.6. Therefore, limit (6.28) is infinitely large which means that the immediate exercise is never optimal, i.e. $A_2 = 0$. Hence, the perpetual price is $V^p(x) = \infty$.[3] If $K \leq 0$, then formula (6.27) leads to $A_1 = 0$. Otherwise, if $K > 0$, then

$$A_1 = \left(K\frac{r+\lambda}{L}\right)^{\frac{1}{n}}. \tag{6.37}$$

6.6.4 Results

We formulate the derived results in the following theorem.

Theorem 6.3. *[Theorem 3 of Zaevski (2024b)] Let the payoff be $G(x) = Mx^n + K$, where $n > 0$ and $M \in \{-1, 1\}$. Also, let the constant L be defined as in equation (6.16). The following statements hold.*

1. *If $\{L = 0,\ K \geq 0\}$, $\{L < 0,\ M = 1,\ K \geq 0\}$, or $\{L > 0,\ M = -1,\ K \geq 0\}$, then the derivative is combined put-call and the immediate exercise is optimal everywhere. The call boundaries are $B_1 = B_2 = 0$, and the put ones are $A_1 = A_2 = \infty$. The perpetual price is given by formula (6.24).*

[3] The same conclusion can be made if we consider the derivative as call-style – formula (6.29) tends to infinity when $c \to \infty$ because $p - q < n$ when $L > 0$ (Lemma 6.4).

2. *If* $\{L = 0,\ K < 0\}$, $\{L < 0,\ M = -1,\ K \leq 0\}$, *or* $\{L > 0,\ M = 1,\ K \leq 0\}$, *then the derivative is again put-call style but the immediate exercise is never optimal. The call boundaries are* $B_1 = B_2 = \infty$ *and the put ones are* $A_1 = A_2 = 0$. *The perpetual price is presented by equation* (6.26) *in the first case, it is* $V(x) = 0$ *in the second one, and* $V(x) = \infty$ *in the third case.*

3. *If* $\{L < 0,\ M = 1,\ K < 0\}$ *or* $\{L > 0,\ M = -1,\ K < 0\}$, *then the derivative is call-style. The values of the optimal boundaries are given by formulas* (6.30) *and* (6.32), *respectively. The perpetual price is given by equation* (6.24) *if* $x \geq B_2$ *and by equations* (6.31) *or* (6.33), *otherwise (first and second case, respectively).*

4. *If* $\{L < 0,\ M = -1,\ K > 0\}$ *or* $\{L > 0,\ M = 1,\ K > 0\}$, *then we have a put-derivative. The boundaries are given in formulas* (6.35) *for the first case; the perpetual price is* (6.24) *if* $x \leq A_2$ *and* (6.36) *otherwise. The initial optimal boundary* A_1 *for the second case is given by formula* (6.37) *whereas the perpetual one is zero,* $A_2 = 0$. *The perpetual price in this case is* $V^p(x) = \infty$.

Remark 6.3. *Let us mention that the price function has a left discontinuity w.r.t. the variable* L *in the point* $L = 0$. *This can be explained by the vanishing expectation in formula* (6.7).

6.6.5 Finite maturity

We shall apply now the algorithm presented in Section 6.5. Suppose that the asset starts from a value $S_0 = x$. Therefore the strategy ζ, recall that it is the first hitting moment of the underlying asset to an exponent of a piecewise linear function, can be viewed as the Brownian motion's first hitting to the level

$$\frac{1}{\sigma}\left(\left(a_i - r + \frac{\sigma^2}{2}\right)t + b_i - \log(x)\right) = C_i t_i + D_i$$

for

$$C_i = \frac{1}{\sigma}\left(a_i - r + \frac{\sigma^2}{2}\right)$$
$$D_i = \frac{b_i - \log(x)}{\sigma}.$$

We can derive payment (6.13) as

$$\begin{aligned} & V(x; \{t_0, ..., t_l\}; \{c_0, ..., c_l\}) = \mathbb{E}^x\left[e^{-(r+\lambda)(\zeta\wedge T)} G\left(S_{\zeta\wedge T}\right)\right] \\ & = \mathbb{E}^x\left[e^{-(r+\lambda)\zeta}\left(MS_\zeta^n + K\right) I_{\zeta<T}\right] + \mathbb{E}^x\left[e^{-(r+\lambda)T}(MS_T^n + K) I_{\zeta\geq T}\right] \\ & = K\mathbb{E}\left[e^{-\alpha_1\zeta} I_{\zeta<T}\right] + Mx^n \sum_{m=1}^{l} e^{n\sigma D_m}\mathbb{E}\left[e^{-\alpha_{2,m}\zeta} I_{t_{m-1}<\zeta\leq t_m}\right] \\ & + Ke^{-\alpha_1 T}\mathbb{Q}(\zeta \geq T) + Mx^n e^{-\alpha_3 T}\mathbb{E}\left[e^{n\sigma B_T} I_{\zeta\geq T}\right] \end{aligned} \tag{6.38}$$

for

$$\begin{aligned} \alpha_1 &= r + \lambda \\ \alpha_{2,m} &= (r+\lambda) - n\left(r - \frac{\sigma^2}{2}\right) - nC_m\sigma = n\frac{\sigma^2}{2} - (n-1)r - nC_m\sigma + \lambda \\ \alpha_3 &= \lambda + n\frac{\sigma^2}{2} - (n-1)r. \end{aligned} \tag{6.39}$$

If we have a put-style derivative, then hitting is below and thus expectations in formula (6.38) can be derived through Propositions 2.6 and 2.7. If the derivative is call-style, then we have a hitting above problem. The corresponding expectations can be derived by Theorems 2.3 and 2.4.

We can reformulate the fast pricing approach 6.1 as

Fast Pricing Approach 6.2. *The option price can be obtained through formulas* (6.38) *and* (6.39) *taken at the point* $x = S_0$.

6.6.6 Numerical experiments

Suppose that the risk-free rate is $r = -0.01$, the additional discount rate is $\lambda = 0.03$, and the volatility is $\sigma = 0.3$. We investigate first the square root, i.e. $n = 0.5$. Also, we consider $M = -1$ and $K = 20$. These values lead to $L = -0.0362$ – the parameter L is given in formula (6.16). This means that the derivative is put-style since $ML > 0$. We present the optimal boundary in Figure 6.1a. The initial and perpetual values derived via formulas (6.35) are $c(0) = 121.7598$ and $c(\infty) = 54.6700$. The prices are presented in Figure 6.1b – the solid line is for $S_0 = \$80$ and the dashed one is for $S_0 = \$100$. The perpetual prices are \$11.2743 and \$10.5602, respectively.

Suppose now that $M = -1$, $K = -20$, and $n = 2$. Using formula (6.16), we obtain $L = 0.0500$ and therefore we have a call-style derivative. The optimal boundary is presented in Figure 6.1c. The initial value is $c(0) = 2.8284$ and the perpetual one is $c(\infty) = 7.9093$. The prices for initial asset values $S_0 = \$3$ and $S_0 = \$5$ can be seen in Figure 6.1d. The corresponding perpetual prices are negative, $-\$18.9980$ and $-\$41.2019$, respectively.

Next, we investigate the behavior of the derivatives w.r.t. the power n. Let us consider the parameter L given in equation (6.16) as a function of n. Lemma 6.4 shows that $L(n) < 0$ for $n < p - q$, $L(p-q) = 0$, and $L(n) > 0$ for $n > p - q$. In Table 6.1, we summarize the results presented in Theorem 6.3. We give only one pair of boundaries when the derivative exhibits a combined feature. We present the initial (blue one) and perpetual (red one) boundaries w.r.t. the power n in the following figures – 6.1e for $\{M = -1,\ K = -20\}$, 6.1f for $\{M = -1,\ K = 20\}$, 6.2a for $\{M = 1,\ K = -20\}$, and 6.2b for $\{M = 1,\ K = 20\}$. We conclude that the derivative is call style for lower values of n when $M = 1$ and $K < 0$ and for the larger ones when $M = -1$ and $K < 0$. On the other hand, the derivative is put-style for lower n's if $M = -1$ and $K > 0$ and for the higher values if $M = 1$ and $K > 0$. Also, for the call derivatives we have a continuity of the boundaries including in the point $n = p - q$ (the value in this point is the infinity). The same is true for the put-derivatives but only for the initial values. For the perpetual ones we have a discontinuity in the point $n = p - q$ – see also Remark 6.3. If $M = -1$ and $K = 20$, then $\lim_{n\to\gamma_{c-}} c(\infty) = 2.1733$, but $c(\infty) = \infty$ for $n \geq p - q$. On the other hand, if $M = 1$ and $K = 20$, then $\lim_{n\to\gamma_{c+}} c(\infty) = 0$, but $c(\infty) = \infty$ for $n \leq p - q$.

6.7 American call options

We discuss in this section the American call options in the light of the above-presented before general scheme. The main problem is that the payoff function $G(x) = (x-K)^+$ is not differentiable at the strike. To overcome this problem, we approximate it by the twice differentiable functions $G_\epsilon(x)$ defined as

$$G_\epsilon(x) = \begin{cases} 0, & \text{if } x < K \\ \frac{(x-K)^2}{2\epsilon}, & \text{if } K \le x < K+\epsilon \\ x - K - \frac{\epsilon}{2}, & \text{if } K+\epsilon \le x. \end{cases} \tag{6.40}$$

These approximations can be seen in Figure 6.2c – the strike is assumed to be $K = 20$ and the values of ϵ are $\epsilon \in \{0.1, 0.5, 1\}$. Thus the operator $\mathcal{B}$ applied to the function $G_\epsilon(\cdot)$ can be written as

$$(\mathcal{B}G_\epsilon)(x) = \begin{cases} 0, & \text{if } x < K \\ \frac{1}{2\epsilon}\left[(r-\lambda+\sigma^2)x^2 + 2\lambda K x - (r+\lambda)K^2\right], & \text{if } K \le x < K+\epsilon \\ -\lambda x + (r+\lambda)\left(K + \frac{\epsilon}{2}\right), & \text{if } K+\epsilon \le x. \end{cases}$$

We can easily check that $(\mathcal{B}G_\epsilon)(K) = \frac{K^2\sigma^2}{2\epsilon}$ and therefore $(\mathcal{B}G_\epsilon)(x) > 0$ for $K \le x < K+\epsilon$ when ϵ is small enough. Hence, if $(\mathcal{B}G_\epsilon)(x) < 0$, then $x \ge K+\epsilon$. Having in mind the form of the function $(\mathcal{B}G_\epsilon)(x)$ when $x \ge K+\epsilon$, we see that the second condition of Definition 6.1 is satisfied, and therefore the payoff (6.40) leads to a call-style derivative. Something more, we conclude that the optimal boundary at the maturity is

$$B_{1,\epsilon} = \begin{cases} K+\epsilon, & \text{if } r < \lambda\frac{\epsilon}{2K+\epsilon} \\ \frac{r+\lambda}{\lambda}\left(K+\frac{\epsilon}{2}\right), & \text{if } r \ge \lambda\frac{\epsilon}{2K+\epsilon}. \end{cases}$$

Note that if $\lambda = 0$, then $B_1 = \infty$. The way we derive the optimal boundary at maturity is shown in Figures 6.2d and (6.2e). The parameters are $\lambda = 0.03$, $K = 20$, $\epsilon = 1$, $\sigma = 0.3$, and $r = \pm 0.01$ – the risk-free values are for the first and second case, respectively. The optimal boundary is $B_{1,\epsilon} = 27.3333$ when $r = 0.01$ and $B_{1,\epsilon} = 21$ when $r = -0.01$ – it is marked by a red circle.

Let us turn to the perpetual value B_2. We shall prove first that the expectation in formula (6.7) tends to zero. We can rewrite it as

$$\mathbb{E}^x\left[e^{-(r+\lambda)T}G_\epsilon(S_T)I_{T<\zeta^c}\right] = e^{-(r+\lambda)T}\mathbb{E}^x\left[\left(S_T - K - \frac{\epsilon}{2}\right)I_{T<\zeta^c, B_T > \frac{1}{\sigma}\ln\frac{K+\epsilon}{x} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T}\right]$$
$$+ \frac{e^{-(r+\lambda)T}}{2\epsilon}\mathbb{E}^x\left[(S_T - K)^2 I_{T<\zeta^c, B_T \in \left(\frac{1}{\sigma}\ln\frac{K}{x} - \left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T, \frac{1}{\sigma}\ln\frac{K+\epsilon}{x} - \left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T\right]}\right].$$

The second term above is zero since $r+\lambda>0$ and $\left(S_T-K\right)^2<\epsilon^2$. On the other hand, the first term can be divided into two parts – the first one for $\left(K+\frac{\epsilon}{2}\right)$ and the second one for S_T. The first term is zero and the second one is

$$
\begin{aligned}
&e^{-(r+\lambda)T}\mathbb{E}^x\left[S_T I_{T<\zeta^c, B_T>\frac{1}{\sigma}\ln\frac{K+\epsilon}{x}-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T}\right] \\
&= xe^{-\left(\lambda+\frac{\sigma^2}{2}\right)T}\mathbb{E}^x\left[e^{\sigma B_T} I_{T<\zeta^c, B_T>\frac{1}{\sigma}\ln\frac{K+\epsilon}{x}-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T}\right].
\end{aligned}
\tag{6.41}
$$

Having in mind equation (2.10) we see that the expectation above tends to infinity as $e^{\frac{\sigma^2}{2}T}$. Hence, if $\lambda>0$, then expectation (6.41) tends to zero. It is left to consider the case $\lambda=0$ and therefore $r>0$ since $r+\lambda>0$. Looking again at equation (2.10), we see that the coefficient before T in the normal distribution CDF is $d_1-\sigma=-\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)<0$, where d_1 is given by formulas (6.5). Hence expectation (6.41) again tends to zero.

We conclude that the derivative price under the assumption that the holder exercises when the asset reaches the values c, given by formula (6.7), turns into

$$
V^c\left(x;c\right)=x^{p-q}\frac{G_\epsilon\left(c\right)}{c^{p-q}}. \tag{6.42}
$$

We need to consider the behavior of the function $f\left(c\right)=\frac{G_\epsilon(c)}{c^{p-q}}$. Its maximum will lead to the optimal boundary $B_{2,\epsilon}$. Obviously $f\left(c\right)=0$ when $c<K$. Suppose that $K\le c<K+\epsilon$. The derivative of the function $f\left(\cdot\right)$ is

$$
f'\left(c\right)=\left(c-K\right)\frac{2c-p+q\left(c-K\right)}{2\epsilon c^{p-q+1}}.
$$

Therefore $f'\left(c\right)>0$ in the whole interval $\left[K,K+\epsilon\right)$ for small enough ϵ's and thus the function $f\left(c\right)$ increases in this interval. It is left to consider the case $K+\epsilon\le c$. The derivative $f'\left(\cdot\right)$ turns into

$$
f'\left(c\right)=\frac{\left(p-q\right)\left(K+\frac{\epsilon}{2}\right)-\left(p-q-1\right)c}{c^{p-q+1}}.
$$

Hence, the function $f\left(c\right)$ achieves its maximum for small enough values of ϵ at the point

$$
B_{2,\epsilon}=\frac{p-q}{p-q-1}\left(K+\frac{\epsilon}{2}\right). \tag{6.43}
$$

The way we derive the perpetual optimal boundary can be seen in Figure 6.2f. The parameters are as above, we choose $r = 0.01$ for the risk-free rate. The optimal boundary is presented by a red point and it is $B_{2,\epsilon} = 70.6525$. Estimating value (6.43) in formula (6.42) and having in mind that $B_{2,\epsilon} > K + \frac{\epsilon}{2}$, we derive for the price when $x < B_{2,\epsilon}$

$$V^c(x; B_{2,\epsilon}) = \left(\frac{x}{p-q}\right)^{p-q} \left(\frac{p-q-1}{K+\frac{\epsilon}{2}}\right)^{p-q-1}.$$

If we take the limit $\epsilon \to 0$, we derive the endpoints of the optimal boundaries as

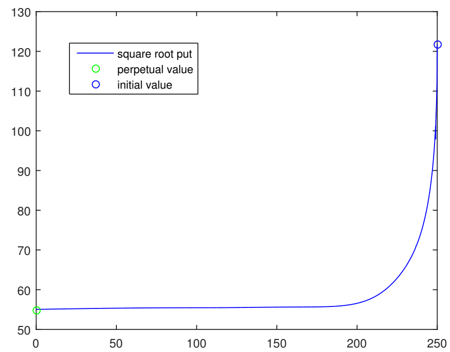
$$B_1 = \begin{cases} K, & \text{if } r < 0 \\ \frac{r+\lambda}{\lambda}K, & \text{if } r \geq 0. \end{cases}$$
$$B_2 = \frac{p-q}{p-q-1}K.$$

We can approximate now the whole optimal boundary – see Chapter 4. Having in mind Theorem 3.1, considering λ as a dividend rate, and changing the risk-free rate to $\overline{r} = r + \lambda$ we derive the well-known formulas for the optimal boundary endpoints – see for example Kim (1990).

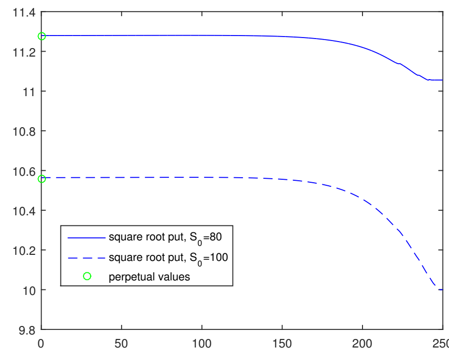
Similarly, we may examine the American put options approximating the put payoff $(K-x)^+$ by the functions

$$G_\epsilon(x) = \begin{cases} K + \frac{\epsilon}{2} - x, & \text{if } x < K - \epsilon \\ \frac{(K-x)^2}{2\epsilon}, & \text{if } K - \epsilon \leq x < K \\ 0, & \text{if } K \leq x. \end{cases}$$

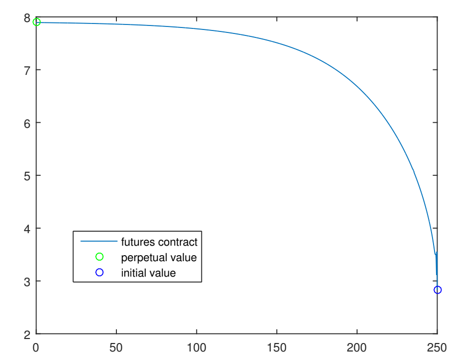
Figure 6.1: Put optimal boundaries and prices.



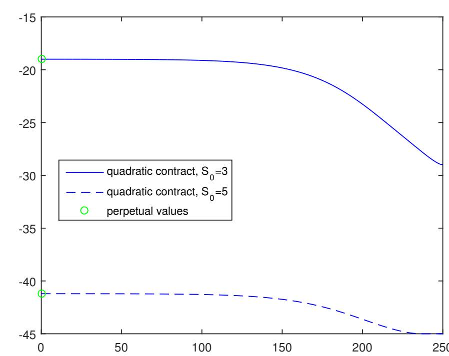
(a) Boundary, $n = \frac{1}{2}$,$M = -1$,$K = 20$, $r = -0.01$



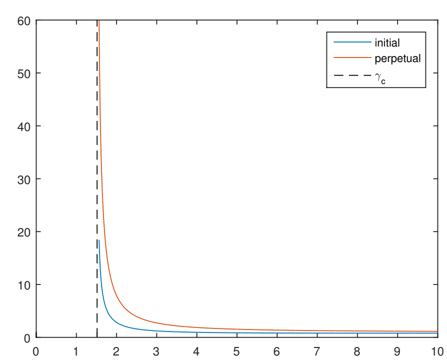
(b) Prices, $n = \frac{1}{2}$,$M = -1$,$K = 20$, $r = -0.01$



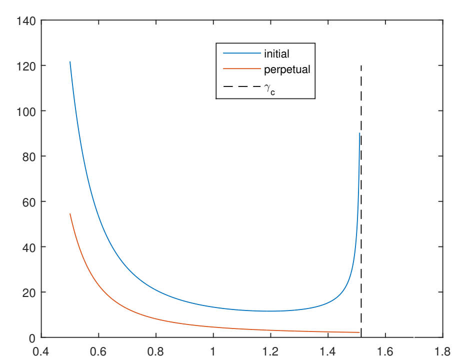
(c) Boundary, $n = 2$,$M = -1$,$K = -20$, $r = -0.01$



(d) Price, $n = 2$,$M = -1$,$K = -20$, $r = -0.01$

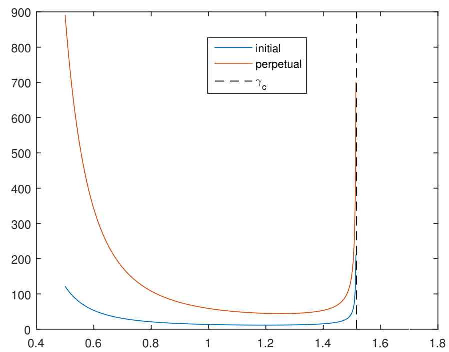


(e) call: $M = -1$,$K = -20$

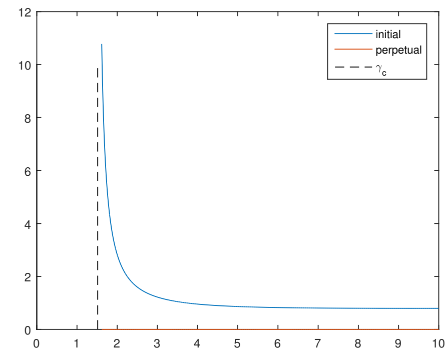


(f) put: $M = -1$,$K = 20$

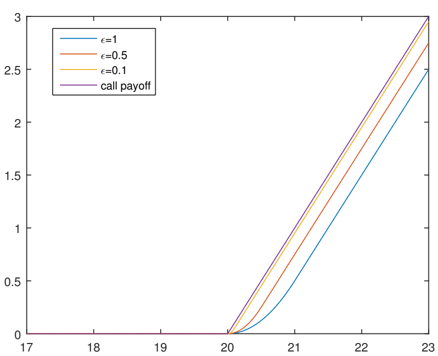
Figure 6.2: Call options



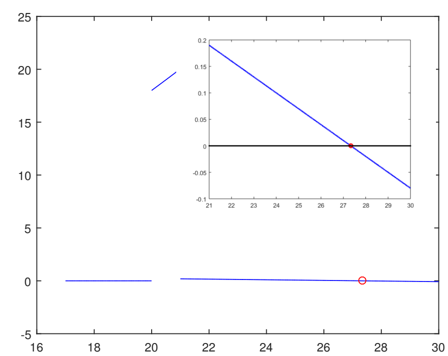
(a) call: $M = 1, K = -20$



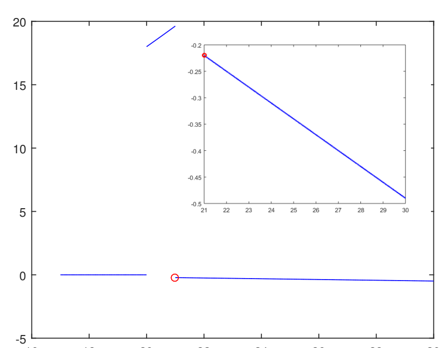
(b) put: $M = 1, K = 20$



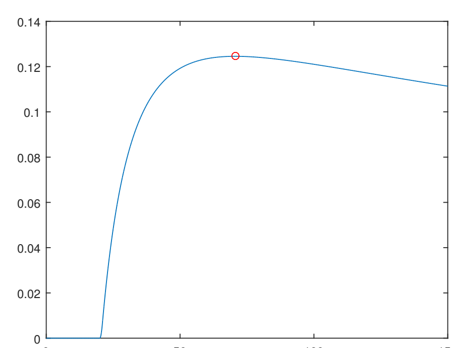
(c) Approximations of the call payoff



(d) Optimal boundary at the maturity, $r \geq \lambda\frac{\epsilon}{2K+\epsilon}$.



(e) Optimal boundary at the maturity, $r < \lambda\frac{\epsilon}{2K+\epsilon}$.



(f) Perpetual optimal boundary

Table 6.1: Derivative's characterization w.r.t. the power n

	style	initial boundary	perpetual boundary	perp. price $S_0 < c(\infty)$	perp. price $S_0 \geq c(\infty)$
parameters $M=-1,\ K<0$					
$n \leq p-q$	put-call	$B_1 \equiv c(0) = \infty$	$B_2 \equiv c(\infty) = \infty$	0	–
$n > p-q$	call	(6.32)	(6.32)	(6.33)	(6.24)
parameters $M=-1,\ K>0$					
$n < p-q$	put	(6.35)	(6.35)	(6.24)	(6.36)
$n \geq p-q$	put-call	$A_1 \equiv c(0) = \infty$	$A_2 \equiv c(\infty) = \infty$	(6.24)	–
parameters $M=1,\ K<0$					
$n < p-q$	call	(6.30)	(6.30)	(6.31)	(6.24)
$n \geq p-q$	put-call	$B_1 \equiv c(0) = \infty$	$B_2 \equiv c(\infty) = \infty$	∞	–
parameters $M=1,\ K>0$					
$n \leq p-q$	put-call	$A_1 \equiv c(0) = \infty$	$A_2 \equiv c(\infty) = \infty$	(6.24)	–
$n > p-q$	put	(6.37)	$A_2 \equiv c(\infty) = 0$	∞	–

Chapter 7

American strangle strategies with arbitrary strikes

based on the paper

Zaevski, Tsvetelin S. "American strangle options with arbitrary strikes." Journal of Futures Markets 43.7 (2023): 880-903.

Abstract: The so-called American strangles are examined in this chapter. Their main characteristic is the combined put and call feature. The holder has the right to exercise prematurely choosing the option's style – put or call. Thus these derivatives appear as hybrid strategies including both of a call and put option. We abandon the traditional assumption that the put strike is below the call one considering arbitrary values. We also assume that the put and call weights are different. The equations for the early exercise boundaries are derived in the perpetual case. After that, we approximate numerically these boundaries for the finite maturity options maximizing the option holder's utility. Based on them, we apply the Crank–Nicolson finite difference method to the corresponding Black-Scholes style partial differential equation to obtain the fair option price.

7.1 Motivation and main results

The strangle strategies appear as a financial instrument that preserves against the financial risk when the investor expects some large deviations but he is not sure of the direction of these movements. Usually, this characterizes the high volatility periods that happen relatively often – a fact supported by many market phenomena such as volatility clustering, leverage effect, long-range dependence, etc. Roughly said, the strangles are a combination of an American put and a call. Its holder has the right to choose how to exercise the option – as a put or as a call. This way the strangles keep the investor's interest from both upward and downward shocks. These options lead to a two-sided optimal stopping problem. The owner would exercise the option as a call if the underlying asset reaches some large enough value. On the contrary, if the asset falls enough, then the holder will exercise the option as a put.

A traditional assumption is that the put strike is less than the call one (only for the straddles they are equal). We remove this restriction considering arbitrary strikes. We also assume that the put and call features are presented with different weights. It turns out that the put-optimal region consists of all points below some function – it is known as the put-exercise boundary. Analogously, all points above another function – the call-optimal boundary – are call-optimal.

The importance of the asymptotic case motivates us to consider first the perpetual strangles. In this case, the optimal boundaries are flat due to the Markov property of the stochastic process that drives the underlying asset. We derive the equations that the boundaries have to solve and prove the uniqueness of the solutions. The approach we use is based on some exit properties of a Brownian motion from a strip – the strangle price is considered as a two-dimensional function w.r.t. its boundaries and we search for its maximum.

As for the ordinary American calls, the undiscounted case is specific – it is never optimal for a strangle's holder to exercise the option as a call earlier. This allows us to derive closed-form formulas for the put boundary as well as for the option price. It turns out that the existing call right influences the option although this right will be never used. Something more, this call impact appears only through the call weight, but not through the call strike.

Having in mind the results for the perpetual options, we turn to options written on a finite maturity horizon. It turns out that the put boundary is a decreasing function w.r.t. the time to maturity, whereas the call one increases. Using this and the derived terminal values (at maturity and infinity), we construct an algorithm to approximate the boundaries by applying the results of Chapter 2.4. This way we may immediately decide to exercise or not having in mind the current spot price. Also, we derive the option price accurately enough.

On the other hand, we can estimate the optimal boundaries at a denser grid turning the free boundary problem for strangle pricing into a boundary value problem in a given region. We use the Crank–Nicolson finite difference approach to solve numerically this task.

The chapter is organized as follows. In Section 7.2 we establish our model. The shape of the optimal regions is obtained in Section 7.3. The perpetual options are considered in Section 7.4, whereas the finite maturity case is examined in Section 7.5. Some numerical results are provided in Section 7.6. We conclude by Section 7.7.

7.2 Preliminaries

Recall that the asset price process is driven by the log-normal process

$$dS_t = rS_t dt + \sigma S_t dB_t$$

under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$, where B_t is a Brownian motion and the measure Q is risk neutral. We denote again by $T \leq \infty$ the maturity date, and by t and τ the current time and the time to maturity, i.e. $\tau = T - t$. As we mentioned above, the option's holder may choose the option feature when he exercises. If the investor prefers a put characteristic, he receives $C_1 > 0$ shares of a put option with strike K_1. Analogously, if the holder chooses a call feature, he receives the payoff of $C_2 > 0$ call options with strike K_2. Hence, the payoff of a strangle can be written as

$$N(t, x) = e^{-\lambda t} \max\left\{C_1 (K_1 - x)^+, C_2 (x - K_2)^+\right\}. \tag{7.1}$$

This means that the option's holder would receive the amount of $N(t, x)$ if he exercises the option in moment t at the spot value $S_t = x$. Let D_0 be the value for S_0 that makes the put and call payoffs equal

$$D_0 := \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}. \tag{7.2}$$

We name it the put-call barrier. We shall denote the strangle price by the function $F(t,x)$ assuming that in the moment t the underlying asset value is $S_t = x$. Obviously, if the immediate exercise is optimal in the point (t,x) then $V(t,x) \equiv N(t,x)$. We denote by $\mathcal{T}_{[t,T]}$, $t < T$, the set of all stopping times with values in the interval $[t,T]$. We view them as the possible strategies for the option's holder, i.e. the exercise happens when the stopping time occurs. We are interested in the optimal strategy, i.e. the stopping time that maximizes the financial result of the option. In the following definition, we formalize this.

Definition 7.1. *Suppose that the option is alive at the moment t and the current asset value is $S_t = x$ – we denote by $\mathbb{E}^{t,x}$ the expectation just under the assumption $S_t = x$. The strategy ζ is optimal if it maximizes the expected future option flows*

$$\mathbb{E}^{t,x}\left[e^{-r\zeta}N(\zeta, S_\zeta)\right].$$

The function $N(t,x)$ is given in equation (7.1).

The next step is to define the so-called early exercise and continuation regions,[1] and the related boundaries. We shall denote them by Υ and $\overline{\Upsilon}$, respectively.

Definition 7.2. *1. A point $(t,x) \in \Upsilon$ if for every $\zeta \in \mathcal{T}_{[t,T]}$*

$$N(t,x) \geq \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}N(\zeta, S_\zeta)\right].$$

2. *The region in which the option's holder exercises the option as a put/call shall be named the put/call optimal region and will be denoted by Υ^p and Υ^c, respectively. Note that $\Upsilon = \Upsilon^p \cup \Upsilon^c$. Let us state for convenience that if the call and put features lead to the same result, the option is exercised as a call. This is possible only when $x = D_0$. Under this assumption, we can think that $\Upsilon^p \cap \Upsilon^c = \emptyset$.*

3. *The continuation region is defined as $\overline{\Upsilon} = \{[t,T] \times (0,\infty) \setminus \Upsilon\}$.*

4. *We have two early exercise boundaries – between the continuation region and the put or call optimal regions. We shall denote these boundaries by $A(t)$ and $B(t)$, respectively.*

[1] The first one is also known as the optimal region.

Proposition 3.3 gives the time-dependence of the option price – we have $V(t,x) = e^{-\lambda t} V(0,x)$. We assume hereafter that the initial moment is zero.

7.3 Optimal regions

We shall provide a series of propositions for the form of the early exercise regions. Recall that we denote by $\mathbb{E}^{t,x}$ the expectation under the assumption $S_t = x$. The first one states that call early exercising is never optimal when $\lambda = 0$, a fact appearing for many different derivatives that exhibit the American call feature.

Proposition 7.1. *We have $\Upsilon^c \equiv \emptyset$ when $\lambda = 0$.*

Proof: We have $r > 0$ because $r + \lambda > 0$ and $\lambda = 0$. Suppose that the set Υ^c is not empty and it contains the point (t,x). Note that the value x has to be above the call strike, $x > K_2$. Let ζ be a stopping time from the set $\mathcal{T}_{[t,T]}$. We use that $e^{-rt}S_t$ is a martingale to obtain

$$
\begin{aligned}
\mathbb{E}^{t,x}\left[e^{-r\zeta} N(\zeta, S_\zeta)\right] &\le e^{-rt} C_2 (x - K_2) \\
&= \mathbb{E}^{t,x}\left[e^{-r\zeta} C_2 S_\zeta\right] - C_2 K_2 e^{-rt} \\
&< \mathbb{E}^{t,x}\left[e^{-r\zeta} C_2 (S_\zeta - K_2)\right] \\
&\le \mathbb{E}^{t,x}\left[e^{-r\zeta} C_2 (S_\zeta - K_2)^+\right] \\
&\le \mathbb{E}^{t,x}\left[e^{-r\zeta} \max\left\{C_1 (K_1 - S_\zeta)^+, C_2 (S_\zeta - K_2)^+\right\}\right] \\
&\equiv \mathbb{E}^{t,x}\left[e^{-r\zeta} N(\zeta, S_\zeta)\right].
\end{aligned}
$$

The contradiction leads to $\Upsilon^c \equiv \emptyset$. □

The next proposition shows that in the opposite case, $\lambda > 0$, the call optimal region Υ^c is not empty.

Proposition 7.2. *If $\lambda > 0$, then the call optimal region Υ^c is not empty. Also, if $(t,x) \in \Upsilon^c$ and $y > x$, then $(t,y) \in \Upsilon^c$*

Proof: Assume that $x > \max\{K_1, K_2\}$ and therefore $(t,x) \notin \Upsilon^p$. Hence, the immediate exercise leads to the result $e^{-\lambda t}(x - K_2)$. Let ζ be some stopping time from the set $\mathcal{T}_{[t,T]}$ and let us compare it with the immediate exercise. We shall denote by $f(t,x;\zeta)$ the difference between the outcomes of the strategy

ζ and the immediate exercise. Having in mind that the process $e^{-rt}S_t$ is a martingale and thus $x = \mathbb{E}^{t,x}\left[e^{-r(\zeta-t)}S_\zeta\right]$ (since), we write the function $f(t,x;\zeta)$ as

$$f(t,x;\zeta) := \mathbb{E}^{t,x}\left[e^{-r\zeta}N(\zeta,S_\zeta)\right] - e^{-(r+\lambda)t}C_2(x-K_2)$$
$$= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\begin{pmatrix} C_2K_2e^{(r+\lambda)(\zeta-t)} + C_1K_1 - S_\zeta\left(C_1 + C_2e^{\lambda(\zeta-t)}\right), \\ -e^{\lambda(\zeta-t)}C_2S_\zeta + e^{(r+\lambda)(\zeta-t)}C_2K_2, \\ C_2K_2\left(e^{(r+\lambda)(\zeta-t)} - 1\right) - S_\zeta C_2\left(e^{\lambda(\zeta-t)} - 1\right), \\ -e^{\lambda(\zeta-t)}C_2S_\zeta + e^{(r+\lambda)(\zeta-t)}C_2K_2 \end{pmatrix}\right] \quad (7.3)$$

Using Lemma 3.1, we conclude that for every stopping time ζ, function (7.3) is negative for a large enough initial value x. Note that S_ζ appears with a negative sign. Hence, the set Υ is not empty. Therefore, immediate exercising is optimal for these initial conditions. Hence, the call optimal region Υ^c is not empty.

Suppose now that $x \in \Upsilon^c$ and $y > x$. Therefore $f(t,x;\zeta) \leq 0$ for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$. The function $f(t,x;\zeta)$ is defined in equation (7.3). We observe that this function decreases w.r.t. the variable x due to Lemma 3.1. Therefore, $f(t,y;\zeta) \leq 0$ too. Since this is true for an arbitrary $\zeta \in \mathcal{T}_{[t,T]}$, the point (t,y) is call-optimal too. □

Below we present the analogue of Proposition 7.2 for the put region.

Proposition 7.3. *The put optimal region Υ^p is not empty. Something more, if $(t,x) \in \Upsilon^p$ and $y < x$, then $(t,y) \in \Upsilon^p$ too.*

Proof: We shall proceed similarly. Below we define the comparison function for an arbitrary stopping time $\zeta \in \mathcal{T}_{[t,T]}$ taking in attention that $N(t,x) = e^{-\lambda t}C_1(K_1 - x)$ for small enough initial conditions x. Let us denote again by $f(t,x;\zeta)$ the difference between the outcomes of the strategy ζ and the immediate exercise. Using the martingality of the process $e^{-rt}S_t$ we obtain the function $f(t,x;\zeta)$ as

$$f(t,x;\zeta) := \mathbb{E}^{t,x}\left[e^{-r\zeta}N(\zeta,S_\zeta)\right] - e^{-(r+\lambda)t}C_1(K_1-x)$$
$$= \mathbb{E}^{t,x}\left[e^{-(r+\lambda)\zeta}\max\begin{pmatrix} -C_1K_1\left(e^{(r+\lambda)(\zeta-t)} - 1\right) + C_1S_\zeta\left(e^{\lambda(\zeta-t)} - 1\right), \\ e^{\lambda(\zeta-t)}C_1S_\zeta - e^{(r+\lambda)(\zeta-t)}C_1K_1, \\ -C_2K_2 - e^{(r+\lambda)(\zeta-t)}C_1K_1 + S_\zeta\left(C_1e^{\lambda(\zeta-t)} + C_2\right), \\ -e^{(r+\lambda)(\zeta-t)}C_1K_1 + e^{\lambda(\zeta-t)}C_1S_\zeta \end{pmatrix}\right]. \quad (7.4)$$

Lemma 3.1 shows that for small enough initial values x and every stopping time ζ, function (7.4) is negative. Therefore, the immediate exercise is optimal for such initial conditions and thus the put optimal region Υ^p is not empty.

Let x be in the put optimal region, $x \in \Upsilon^p$, and $y < x$. Hence, $f(t, x; \zeta) \leq 0$ for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$. The function (7.4) is increasing w.r.t. the variable x due to Lemma 3.1. Therefore, $f(t, y; \zeta) \leq 0$ too – this leads to the put-optimality of the point (t, y). □

The following proposition establishes the behavior of the optimal boundaries $A(\tau)$ and $B(\tau)$.

Proposition 7.4. *The put boundary $A(\tau)$ is non-increasing w.r.t. the time to maturity, whereas the call one $B(\tau)$ is non-decreasing.*

Proof: Based on Propositions 7.2 and 7.3 we can use similar arguments to Jacka (1991), proposition 2.2. □

The next step is to obtain the values of these boundaries at the maturity.

Proposition 7.5. *The value of the put boundary at the maturity is*

$$D_1 \equiv A(0) = \min\left\{K_1, \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}, \frac{r + \lambda}{\lambda} K_1\right\}. \tag{7.5}$$

Proof: First, note that D_1 can not be above the put strike K_1 because the option's holder will receive nothing. Also, it can not be above put-call barrier (7.2) because in the opposite case, the holder will prefer to exercise the option as a call.

Suppose now that a point near to maturity (t, x) is put-optimal, $(t, x) \in \Upsilon^p$. In this region, the option price function $V(t, x)$ has to satisfy the variational inequality

$$V_t(t, x) + \mathcal{A}V(t, x) - rV(t, x) \leq 0. \tag{7.6}$$

We shall find the largest value of x for which inequality (7.6) holds for the limit function $N(t, x)$, which in this case turns into the put payoff $N(t, x) = e^{-\lambda t} C_1 (K_1 - x)$. Denoting by $\mathcal{A}\{x\}$, the infinitesimal generator applied to the function $f(x) = x$, we see

$$e^{-\lambda t}C_1\left[-\lambda\left(K_1-x\right)-\mathcal{A}\left\{x\right\}-r\left(K_1-x\right)\right]=C_1\left[\lambda x-\left(r+\lambda\right)K_1\right] \tag{7.7}$$

The largest value of x for which formula (7.7) is not positive is

$$d_1=\frac{r+\lambda}{\lambda}K_1.$$

Note that equation (7.5) can be written as $D_1=\min\left\{K_1,D_0,d_1\right\}$. Let us examine first the case $r<0$ that turns formula (7.5) into $D_1=\min\left\{D_0,d_1\right\}$. We shall prove that all points below D_1 are put-optimal near the maturity. Suppose that this is not true for some value $x<D_1$. Note that this point can not be call-optimal since the holder will prefer to exercise the option as a put. Hence the point (t,x) is in the continuation region, $(t,x)\in\overline{\Upsilon}$, and therefore the statement (7.6) turns into the equality in this point. Hence,

$$\begin{aligned}
0&<\lim_{t\to T}\frac{V\left(t,x\right)-N\left(t,x\right)}{T-t}\\
&=-\lim_{t\to T}\frac{V\left(T,x\right)-F\left(t,x\right)}{T-t}+\lim_{t\to T}\frac{N\left(T,x\right)-N\left(t,x\right)}{T-t}\\
&=\mathcal{A}V\left(T,x\right)-rV\left(T,x\right)+N_t\left(T,x\right)\\
&=C_1\left[-rxe^{-\lambda T}-re^{-\lambda T}\left(K_1-x\right)-\lambda e^{-\lambda T}\left(K_1-x\right)\right]\\
&=e^{-\lambda T}C_1\left[-\left(r+\lambda\right)K_1+\lambda x\right]<0.
\end{aligned} \tag{7.8}$$

The contradiction confirms that $(t,x)\in\Upsilon^p$. Suppose now that $r\geq 0$. Let us mention that in this case $d_1\geq K_1>x$ and therefore, using the analogous arguments as above, we reach the same contradiction (7.8). □

Proposition 7.6. *The value of the call boundary at the maturity is*

$$D_2\equiv B\left(0\right)=\max\left\{K_2,\frac{C_1K_1+C_2K_2}{C_1+C_2},\frac{r+\lambda}{\lambda}K_2\right\}. \tag{7.9}$$

Equation (7.9) can be written also as $D_2\equiv B\left(T\right)=\max\left\{K_2,D_0,d_2\right\}$ *for*

$$d_2=\frac{r+\lambda}{\lambda}K_2.$$

Proof: The proof is analogous to the proof of Proposition 7.5. The main difference is that we search for the lowest value of x for which inequality (7.6) holds, having in mind that the payoff function turns into $N(t,x) = e^{-\lambda t} C_2 (x - K_2)$. □

Remark 7.1. *We can see that $D_2 = \infty$ when $\lambda = 0$ – this is in accordance with Proposition 7.1.*

Corollary 7.1. *We have that $D_1 \leq D_2$. The equality holds in the following cases*

1. $r \geq 0$ *and*

$$K_2 \leq \frac{\lambda C_1 K_1}{(r + \lambda) C_1 + r C_2}. \tag{7.10}$$

2. $r < 0$ *and*

$$K_2 \leq \frac{(r + \lambda) C_2 + r C_1}{\lambda C_2} K_1. \tag{7.11}$$

Proof: If $K_1 < K_2$, then $D_1 < D_2$. Hence, the equality may hold only when $K_2 \leq K_1$. Suppose first that $r \geq 0$. Therefore $D_1 = D_0$ and

$$D_2 = \max \left\{ D_0, \frac{r + \lambda}{\lambda} K_2 \right\}.$$

Hence, $D_2 = D_0$ when $\frac{r+\lambda}{\lambda} K_2 \leq D_0$ which is equivalent to inequality (7.10). Analogously, if $r < 0$ we can prove that $D_1 = D_2 = D_0$ when inequality (7.11) holds. □

7.4 Perpetual options

The next step in our study is to obtain both strangle boundaries assuming that $T = \infty$. These boundaries have to be flat since (A) the option's holder has no time horizon, (B) the asset price is a Markov process, and (C) the specific payoff function $N(t,x)$.

7.4.1 Existence of discounting

Suppose first that $\lambda > 0$. The undiscounted case is considered later. Propositions 7.2 and 7.3 show that the exercise regions have to be of the form $\Upsilon^p = \left(0, \overline{A}\right]$ and $\Upsilon^c = \left(\overline{B}, \infty\right]$ for some constants $\overline{A} < \overline{B}$. They have to satisfy the following conditions

$$\overline{A} < K_1, \quad \overline{A} < D_0 \equiv \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}, \quad \overline{B} > K_2, \quad \overline{B} \geq D_0 \equiv \frac{C_1 K_1 + C_2 K_2}{C_1 + C_2}, \tag{7.12}$$

because (A) if $\overline{A} \geq K_1$ or $\overline{B} \leq K_2$, the option's holder will receive nothing, (B) if $\overline{A} \geq D_0$ the holder will exercise the option as a call, and (C) if $\overline{B} < D_0$ the holder will prefer the put feature.

Let us sketch first the approach we use to derive the optimal boundaries. We examine financial derivatives related to the first exit of the underlying asset from a strip (A, B), $A < B$. The constants A and B are chosen to satisfy conditions (7.12). We examine the price of such derivative as a function of its boundaries A and B. This function is obtained through some exit properties of a Brownian motion from a strip. We have to derive the maximum of this function. We do this by fixing one of the boundaries and finding the value for another that maximizes the price function. We prove the existence and uniqueness of these maxima. Note that we divide all prices (strikes, initial price, boundaries) to the fixed boundary – we do that to unify the optimization problems to the intervals $(0, 1)$ or $(1, \infty)$. Thus we obtain a two-dimensional system for the boundaries. We extract from this system one polynomial-style equation and prove that it has a unique solution. This way we derive the optimal boundaries and use them to find the fair option price.

Let us denote by ζ^A and ζ^B the first hitting moments to the levels A and B, and let $\zeta = \zeta^A \wedge \zeta^B$. Due to the log-normality of the asset price process, we can view ζ^A and ζ^B as the first hitting times of a Brownian motion with drift

$$\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$$

to the levels

$$\tilde{A} = \frac{\ln A - \ln S_0}{\sigma} < 0$$
$$\tilde{B} = \frac{\ln B - \ln S_0}{\sigma} > 0.$$

If the exit happens from the lower boundary, then the derivative's holder receives the put payoff $e^{-\lambda t} C_1 (K_1 - A)$. Analogously, if the asset exits from the upper boundary the derivative pays an amount of $e^{-\lambda t} C_2 (B - K_2)$. Hence, if the asset starts from a point x, $A < x < B$, then the price of such derivative has to be

$$f(A, B, x) = C_2 (B - K_2) \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^B} I_{\zeta^B \leq \zeta^A} \right] + C_1 (K_1 - A) \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A < \zeta^B} \right]. \tag{7.13}$$

Let the constants p and q be defined via formulas (3.17). Note that $p \geq q+1$ and the equality stands when $\lambda = 0$. The expectations in formula (7.13) can be obtained through Lemma 3.3. Estimating expectations (3.18), written for $y = r + \lambda$, in formula (7.13) we derive

$$f(A, B, x) = C_1 (K_1 - A) \left(\frac{A}{x} \right)^q \frac{B^p - x^p}{B^p - A^p} + C_2 (B - K_2) \left(\frac{B}{x} \right)^q \frac{x^p - A^p}{B^p - A^p}. \tag{7.14}$$

Let us fix the value B and change the variables as $a = \frac{A}{B}$, $k_1 = \frac{K_1}{B}$, $k_2 = \frac{K_2}{B}$ $y = \frac{x}{B}$. Note that $k_2 \leq 1$ and $C_1 + C_2 - C_1 k_1 - C_2 k_2 \geq 0$ due to restrictions (7.12). Price function (7.14) turns into

$$\begin{aligned} f(a) &= \frac{B}{y^q} \frac{C_1 (k_1 - a) a^q (1 - y^p) + C_2 (1 - k_2)(y^p - a^p)}{1 - a^p} \\ &= \frac{B}{y^q} \left(\frac{-a^p C_2 (1 - k_2) - a^{q+1} C_1 (1 - y^p) + a^q C_1 k_1 (1 - y^p) + C_2 (1 - k_2) y^p}{1 - a^p} \right). \end{aligned}$$

The next step is to recognize which value of a maximizes the function

$$g(a) = \frac{-a^p C_2 (1 - k_2) - a^{q+1} C_1 (1 - y^p) + a^q C_1 k_1 (1 - y^p) + C_2 (1 - k_2) y^p}{1 - a^p}. \tag{7.15}$$

in the interval $(0,1]$. Its derivative is

$$g_a(a) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \left[\begin{array}{l} -a^{p+1}C_1(p-q-1) + a^p C_1 k_1 (p-q) \\ -a^{p-q} p C_2 (1-k_2) - aC_1(q+1) + qC_1 k_1 \end{array} \right]. \tag{7.16}$$

Let us define the function $h(\cdot)$ as

$$h(a) = -a^{p+1}C_1(p-q-1) + a^p C_1 k_1 (p-q) - a^{p-q} p C_2 (1-k_2) - aC_1(q+1) + qC_1 k_1. \tag{7.17}$$

We show in Appendix 7.A.1 that derivative (7.16) has a unique root in the interval $(0,1]$. Note that this root is just $a=1$ when $C_1 + C_2 - C_1 k_1 - C_2 k_2 = 0$. Assume now $C_1 + C_2 - C_1 k_1 - C_2 k_2 > 0$ and let us consider the open interval $(0,1)$. The mentioned above root leads to a maximum for function (7.15) since

$$\begin{array}{l} h(0) = qC_1 k_1 > 0 \\ h(1) = -p(C_1 + C_2 - C_1 k_1 - C_2 k_2) < 0. \end{array} \tag{7.18}$$

We shall parametrize now w.r.t. the variable a. We can rewrite the equation $h(a) = 0$ as

$$-a^{p+1}C_1(p-q-1) + a^p C_1 \frac{K_1}{B}(p-q) - a^{p-q} p C_2 \left(1 - \frac{K_2}{B}\right) - aC_1(q+1) + qC_1 \frac{K_1}{B} = 0. \tag{7.19}$$

Hence, equation (7.19) leads to

$$B(a) = \frac{a^p C_1 K_1 (p-q) + a^{p-q} p C_2 K_2 + qC_1 K_1}{a^{p+1} C_1 (p-q-1) + a^{p-q} p C_2 + aC_1(q+1)}.$$

Let us denote by $A_1(a)$ the corresponding put boundary. Using $A_1(a) = aB(a)$, we derive

$$\begin{aligned} A_1(a) &= \frac{a^p C_1 K_1 (p-q) + a^{p-q} p C_2 K_2 + qC_1 K_1}{a^p C_1 (p-q-1) + a^{p-q-1} p C_2 + C_1(q+1)} \\ &= \frac{p-q}{p-q-1} a \frac{a^q C_1 K_1 + C_2 K_2 + \frac{q}{p-q}\left(C_2 K_2 + \frac{C_1 K_1}{a^{p-q}}\right)}{a^{q+1} C_1 + C_2 + \frac{q+1}{p-q-1}\left(C_2 + \frac{C_1}{a^{p-q-1}}\right)} \end{aligned} \tag{7.20}$$

Let us define the following functions

$$\begin{aligned}
X_1(a) &= a\left(a^q C_1 K_1 + C_2 K_2\right)\\
X_2(a) &= a\frac{q}{p-q}\left(C_2 K_2 + \frac{C_1 K_1}{a^{p-q}}\right)\\
X_3(a) &= a^{q+1} C_1 + C_2\\
X_4(a) &= \frac{q+1}{p-q-1}\left(C_2 + \frac{C_1}{a^{p-q-1}}\right).
\end{aligned}$$

In such a way, function (7.20) can be rewritten as

$$A_1(a) = \frac{p-q}{p-q-1}\frac{X_1(a) + X_2(a)}{X_3(a) + X_4(a)}. \tag{7.21}$$

Let us fix now the boundary A and change the variables for price function (7.14) as $b = \frac{B}{A}$, $k_1 = \frac{K_1}{A}$, $k_2 = \frac{K_2}{A}$, $y = \frac{x}{A}$. We have $k_1 > 1$ and $C_1 k_1 + C_2 k_2 - C_1 - C_2 > 0$ due to limitations (7.12). Now price function (7.14) turns into

$$\begin{aligned}
f(b) &= \frac{A}{y^q}\frac{C_1(k_1-1)(b^p - y^p) + C_2(b-k_2)b^q(y^p-1)}{b^p-1}\\
&= \frac{A}{y^q}\left[\frac{b^p C_1(k_1-1) + b^{q+1}C_2(y^p-1) - b^q C_2 k_2(y^p-1) - C_1(k_1-1)y^p}{b^p-1}\right].
\end{aligned}$$

We need to find the value of b that maximizes the function

$$g(b) = \frac{b^p C_1(k_1-1) + b^{q+1}C_2(y^p-1) - b^q C_2 k_2(y^p-1) - C_1(k_1-1)y^p}{b^p-1}.$$

Its derivative can be written as

$$\begin{aligned}
g_b(b) &= \frac{y^p-1}{(b^p-1)^2} b^{q-1}\left[\begin{array}{l} -b^{p+1}C_2(p-q-1) + b^p C_2 k_2(p-q)\\ +b^{p-q} p C_1(k_1-1) - b(q+1)C_2 + qC_2k_2\end{array}\right]\\
&= \frac{y^p-1}{(b^p-1)^2} b^{q-1} h(b)
\end{aligned} \tag{7.22}$$

for

$$h(b) = -b^{p+1}C_2(p-q-1)+b^pC_2k_2(p-q)+b^{p-q}pC_1(k_1-1)-b(q+1)C_2+qC_2k_2. \tag{7.23}$$

We prove in Appendix 7.A.2 that function (7.23) has a unique root larger than one. It leads to the maximum for function (7.22) since

$$h(1) = p(C_1k_1 + C_2k_2 - C_1 - C_2) > 0$$
$$h(\infty) = -\infty.$$

The root of function (7.23) leads to

$$A(b) = \frac{b^pC_2K_2(p-q) + b^{p-q}pC_1K_1 + qC_2k_2}{b^{p+1}C_2(p-q-1) + b^{p-q}pC_1 + b(q+1)C_2}.$$

Using $b = \frac{1}{a}$ and denoting by $A_2(a)$ the put boundary, we obtain

$$\begin{aligned} A_2(a) &= a\frac{a^pqC_2K_2 + a^qpC_1K_1 + (p-q)C_2K_2}{a^p(q+1)C_2 + a^{q+1}pC_1 + (p-q-1)C_2} \\ &= \frac{q}{q+1}\frac{a^{p-q}C_2K_2 + C_1K_1 + \frac{p-q}{q}\left(C_1K_1 + \frac{C_2K_2}{a^q}\right)}{a^{p-q-1}C_2 + C_1 + \frac{p-q-1}{q+1}\left(C_1 + \frac{C_2}{a^{q+1}}\right)} \\ &= \frac{p-q}{p-q-1}\frac{X_2(a)a^p + X_1(a)}{X_4(a)a^p + X_3(a)} \end{aligned} \tag{7.24}$$

Having in mind equations (7.20) and (7.24), we conclude that the equation $A_1(a) = A_2(a)$ has to be solved. It leads to

$$(1-a^p)(X_1(a)X_4(a) - X_2(a)X_3(a)) = 0$$

The inequality $a < 1$ leads to

$$X_1(a)X_4(a) - X_2(a)X_3(a) = 0. \tag{7.25}$$

In such a way we derive the value of a as the solution of equation (7.25) which also can be written as

$$\begin{aligned} H(a) := a^{p+1}C_1C_2K_2\alpha - a^pC_1C_2K_1\beta - a^{p-q}C_2^2K_2(\beta-\alpha) \\ - a^{q+1}C_1^2K_1(\beta-\alpha) - aC_1C_2K_2\beta + C_1C_2K_1\alpha = 0. \end{aligned} \tag{7.26}$$

The constants α and β are

$$\alpha = \frac{q}{q+1}$$
$$\beta = \frac{p-q}{p-q-1}.$$

We prove in Appendix 7.B that equation (7.26) has a unique solution in the interval $a \in (0,1)$. Something more, this solution leads to boundary values $\overline{A}$ and $\overline{B}$ such that $\overline{A} \leq D_1$ and $\overline{B} \geq D_2$, which validate their consistency.

We can formulate the results above in the following theorem.

Theorem 7.1. *[Theorem 4.1 of Zaevski (2023a)]*

Suppose that $\lambda > 0$ and let $\overline{a}$ be the unique solution of equation (7.26). Then the optimal boundaries can be derived as $\overline{A} = A_1(\overline{a}) \equiv A_2(\overline{a})$ and $\overline{B} = \frac{\overline{A}}{\overline{a}}$, where the functions $A_1(a)$ and $A_2(a)$ are given in equations (7.20) and (7.24). These boundaries lead to option price (7.14).

7.4.2 Non-discounted model

We shall derive now the closed-form formulas when the additional discount factor is zero, $\lambda = 0$. As a consequence, we have $r > 0$. It is proven in Appendix 7.A.2 that function (7.23) is positive for $b > 1$. This means that price function (7.14) is increasing. This corresponds to Proposition 7.1, which says that early exercising as a call is never optimal in the absence of discounting. We shall use an approach similar to the one presented in Section 7.4.1, keeping in mind that the call boundary does not exist. We need to consider a one-sided exit problem related to exercising the option as a put. This way we have a one-dimensional price function and we are looking for the boundary value that maximizes it.

Suppose now that the option's holder exercises when the underlying asset hits the value $A \in (0, \min\{K_1, D_0\})$ – we shall denote this moment by ζ^A. Thus the option price can be written as

$$\begin{aligned} Y(A) = {} & \mathbb{E}^x\left[e^{-r\zeta^A} C_1 \left(K_1 - S_{\zeta^A}\right)^+ I_{\zeta^A < \infty}\right] \\ & + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-rT} \max\left\{C_1 (K_1 - S_T)^+, C_2 (S_T - K_2)^+\right\} I_{T<\zeta^A}\right] \end{aligned} \tag{7.27}$$

Note that the function above depends on the variable A through the stopping time ζ^A. First, suppose that $K_1 \leq K_2$ – this leads to the domain $A \in (0, K_1)$. We have

$$\max\left\{C_1 (K_1 - x)^+, C_2 (x - K_2)^+\right\} = C_1 (K_1 - x)^+ + C_2 (x - K_2)^+$$

and therefore the option price (7.27) turns into

$$\begin{aligned} Y(A) &= C_1 (K_1 - A) \mathbb{E}^x \left[e^{-r\zeta^A} I_{\zeta^A < \infty}\right] \\ &+ C_1 \lim_{T\to\infty} \mathbb{E}^x \left[e^{-rT} (K_1 - S_T)^+ I_{T<\zeta^A}\right] + C_2 \lim_{T\to\infty} \mathbb{E}^x \left[e^{-rT} (S_T - K_2)^+ I_{T<\zeta^A}\right] \\ &= C_1 (K_1 - A) \mathbb{E}^x \left[e^{-r\zeta^A} I_{\zeta^A < \infty}\right] + C_1 \lim_{T\to\infty} P_{DO}(x, A, K_1) + C_2 \lim_{T\to\infty} C_{DO}(x, A, K_2). \end{aligned} \tag{7.28}$$

We denoted above by $P_{DO}(x, A, K)$ and $C_{DO}(x, A, K)$ the prices of the down-and-out barrier options (put and call, respectively) with strike K and barrier A. Taking the limit $T \to \infty$ for these prices – see for examples formulas (10.45) and (10.48) from Zhang (1997) – we obtain

$$\begin{aligned} &\lim_{T\to\infty} P_{DO}(x, A, K_1) = 0 \\ &\lim_{T\to\infty} C_{DO}(x, A, K_2) = x\left(1 - \left(\frac{A}{x}\right)^{1+\frac{2r}{\sigma^2}}\right). \end{aligned} \tag{7.29}$$

Remark 7.2. *Let us discuss the limits in formula (7.28) without using barrier options. We can rewrite them as*

$$\begin{aligned} \lim_{T\to\infty} \mathbb{E}^x \left[e^{-rT} (K_1 - S_T)^+ I_{T<\zeta^A}\right] &= \lim_{T\to\infty} e^{-rt} \mathbb{Q}\left(S_T < K_1, I_{T<\zeta^A}\right) \\ &+ \lim_{T\to\infty} e^{-\frac{\sigma^2}{2}} \mathbb{E}^x \left[e^{\sigma B_T} I_{T<\zeta^A, B_T < z(T)}\right] \\ \lim_{T\to\infty} \mathbb{E}^x \left[e^{-rT} (S_T - K_2)^+ I_{T<\zeta^A}\right] &= \lim_{T\to\infty} e^{-rt} \mathbb{Q}\left(S_T > K_2, I_{T<\zeta^A}\right) \\ &+ \lim_{T\to\infty} e^{-\frac{\sigma^2}{2}} \mathbb{E}^x \left[e^{\sigma B_T} I_{T<\zeta^A, B_T > y(T)}\right], \end{aligned} \tag{7.30}$$

where the functions $z(\cdot)$ and $y(\cdot)$ are linear. Also the stopping time ζ^A is the first hit of the Brownian motion to another linear function $b(\cdot)$. The coefficients of these functions are

$$\begin{aligned}b_1 = z_1 = y_1 &= \frac{\sigma}{2} - \frac{r}{\sigma}\\ b_2 &= \frac{1}{\sigma}\ln\frac{A}{S_0} < 0\\ z_2 &= \frac{1}{\sigma}\ln\frac{K_1}{S_0} > b_2\\ y_2 &= \frac{1}{\sigma}\ln\frac{K_2}{S_0}.\end{aligned}$$

Obviously, the parts of expectations (7.30) *related to the probabilities tend to zero. Note that* $\sigma > b_1$*. The limit of the expectation in the first statement is zero due to the point 11.3 of Theorem 2.11. The limit in the second statement can be derived via the point 11.3 of Theorem 2.11 together with the point 11 of Theorem 2.10 – it is*

$$1 - e^{2b_2(\sigma - b_1)} = 1 - \left(\frac{A}{x}\right)^{1+2\frac{r}{\sigma^2}}.$$

Having in mind that the stopping time ζ^A can be viewed as the first hit of a Brownian motion with drift $\mu = \frac{r}{\sigma} - \frac{\sigma}{2}$ to the value $\frac{1}{\sigma}\ln\left(\frac{A}{x}\right)$ and using the second statement of Lemma 3.5, we obtain

$$E^x\left[e^{-r\zeta^A} I_{\zeta^A<\infty}\right] = \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}. \tag{7.31}$$

Combining equations (7.29) and (7.31), we transform option price (7.28) to

$$Y(A) = C_2 x + \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}\left[-A(C_1 + C_2) + C_1 K_1\right]. \tag{7.32}$$

Its A-derivative is

$$Y'(A) = \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}\left[\frac{2rC_1K_1}{\sigma^2 A} - (C_1 + C_2)\left(\frac{2r}{\sigma^2} + 1\right)\right].$$

Considering the function

$$h(A) = \frac{2rC_1K_1}{\sigma^2 A} - (C_1 + C_2)\left(\frac{2r}{\sigma^2} + 1\right),$$

we see that it decreases starting from $h(0) = +\infty$ and ending at the negative value $h(K_1) < 0$. Hence it has the unique root

$$\overline{A} = \frac{2rC_1K_1}{(C_1 + C_2)(2r + \sigma^2)}, \tag{7.33}$$

that leads to the maximum of the option price. Hence, $\overline{A}$ is namely the optimal boundary.

Suppose now that $K_2 < K_1$ and thus $A \in (0, D_0)$. We have

$$\max\left\{C_1(K_1 - x)^+, C_2(x - K_2)^+\right\} = C_1(D_0 - x)^+ + C_2(x - D_0)^+ + C_1C_2\frac{K_1 - K_2}{C_1 + C_2}.$$

Hence, the second term of option price (7.27) turns into a sum of down-and-out call and put options and a bond. When we take the limit $T \to \infty$, the put and bond prices vanish due to the first equation of (7.29). Something more the second equation of (7.29) shows that the limit of the call does not depend on the strike K_2 and therefore formula (7.32) still holds. We have to check that $\overline{A} < D_0$ to conclude that the optimal boundary is again $\overline{A}$. Of course, we can use the arguments of Remark 7.2 too.

We can summarize the derived results in the following theorem.

Theorem 7.2. *[Theorem 4.2 of Zaevski (2023a)] If $\lambda = 0$, then the early exercising of a perpetual strangle is never optimal as a call. On the contrary, the option holder exercises as a put when the underlying asset reaches the level $\overline{A}$, where $\overline{A}$ is given by equation (7.33). If the starting point is below this value, $x \leq \overline{A}$, then the option price is $C_1(K_1 - x)$. Otherwise, if $x > \overline{A}$, then the price is $Y\left(\overline{A}\right)$, where the function $Y(\cdot)$ is given by equation (7.32).*

Remark 7.3. *Regardless that exercising as a call is never optimal, the put boundary and the option price depend on the call feature by the number of shares C_2, but not on the strike K_2.*

Remark 7.4. *Let us see what changes if $C_2 = 0$ in the light of Remark 7.3. This way boundary value (7.33) turns into $\frac{2r}{2r+\sigma^2}K_1 = \frac{q}{q+1}K_1$, because $q = \frac{2r}{\sigma^2}$ when $\lambda = 0$. In fact, when $C_2 = 0$, we have a classical put option and this value is namely its optimal boundary – see formula (4.24).*

7.5 Finite maturity horizon

Suppose now that the maturity is finite, $T < \infty$. For the functions $A(t)$ and $B(t)$, $A(t) < B(t)$, we define a European style derivative, that expires as a put if the asset falls below $A(t)$ and as a call if it rises above $B(t)$. We shall name these instruments $(A(t), B(t))$-European options. The corresponding stopping times shall be denoted by ζ^A and ζ^B, and the lower between them by ζ, $\zeta = \zeta^A \wedge \zeta^B$.

Let $0 \equiv t_0 < t_1 < t_2 < ... < t_n \equiv T$ be an increasing time sequence and $a(t)$ and $b(t)$ be two continuous piecewise linear functions w.r.t. it

$$
\begin{aligned}
a(t) &= \sum_{i=1}^{n} a_i(t) I_{t\in[t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (a_{1,i}t + a_{2,i}) I_{t\in[t_{i-1},t_i]} \\
b(t) &= \sum_{i=1}^{n} b_i(t) I_{t\in[t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (b_{1,i}t + b_{2,i}) I_{t\in[t_{i-1},t_i]},
\end{aligned}
$$

$a_i(t_i) = a_{i+1}(t_i)$ and $b_i(t_i) = b_{i+1}(t_i)$, $i = 1, 2, ..., n-1$. We impose the condition $a(t) < b(t)$ and additionally $a(0) < 0 < b(t)$. As we have mentioned above, we shall approximate the optimal boundaries as exponents of such functions – $A(t) = \exp(a(t))$ for the put boundary, and $B(t) = \exp(b(t))$ for the call one. The values of these functions at the grid nodes shall be denoted by $\alpha_i = a(t_i)$, $\beta_i = b(t_i)$, $A_i = A(t_i)$ and $B_i = B(t_i)$, $i = 0, 1, ..., n$. Let us introduce the functions

$$
\begin{aligned}
c(t) &= \sum_{i=1}^{n} c_i(t) I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (c_{1,i}t + c_{2,i}) I_{t\in(t_{i-1},t_i]} \\
d(t) &= \sum_{i=1}^{n} d_i(t) I_{t\in(t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (d_{1,i}t + d_{2,i}) I_{t\in(t_{i-1},t_i]}
\end{aligned}
$$

for

$$
\begin{aligned}
c_{1,i} &= \frac{a_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \;\; i = 1, ..., n \\
c_{2,i} &= \frac{a_{2,i} - \ln(x)}{\sigma}, \;\; i = 1, ..., n \\
d_{1,i} &= \frac{b_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \;\; i = 1, ..., n \\
d_{2,i} &= \frac{b_{2,i} - \ln(x)}{\sigma}, \;\; i = 1, ..., n.
\end{aligned}
$$

The stopping times ζ^A and ζ^B can be viewed as the first hitting moments of the Brownian motion to the functions $c(t)$ and $d(t)$, respectively. Using the notations above, we present the price of the $(A(t), B(t))$-European option as

$$
\begin{aligned}
&G(x, T; A(t), B(t)) = \mathbb{E}^x \left[C_1 e^{-(r+\lambda)\zeta^A} \left(K_1 - S_{\zeta^A}\right)^+ I_{\zeta^A = \zeta, \zeta < T} \right] \\
&+ \mathbb{E}^x \left[C_2 e^{-(r+\lambda)\zeta^B} \left(S_{\zeta^B} - K_2\right)^+ I_{\zeta^B = \zeta, \zeta < T} \right] \\
&+ \mathbb{E}^x \left[C_1 e^{-(r+\lambda)T} (K_1 - S_T)^+ I_{T \le \zeta, S_T \in (D_1, D_0)} \right] + \mathbb{E}^x \left[C_2 e^{-(r+\lambda)T} (S_T - K_2)^+ I_{T \le \zeta, S_T \in (D_0, D_2)} \right] \\
&= C_1 K_1 \sum_{i=1}^{n} \mathbb{E} \left[e^{-(r+\lambda)\zeta^A} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta^A} \right] - C_1 x \sum_{i=1}^{n} e^{\sigma c_{2,i}} \mathbb{E} \left[e^{-\psi_{1,i}\zeta^A} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta^A} \right] \\
&+ C_2 x \sum_{i=1}^{n} e^{\sigma d_{2,i}} \mathbb{E} \left[e^{-\psi_{2,i}\zeta^B} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta^B} \right] - C_2 K_2 \sum_{i=1}^{n} \mathbb{E} \left[e^{-(r+\lambda)\zeta^B} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta^B} \right] \\
&+ C_1 K_1 e^{-(r+\lambda)T} \mathbb{Q}\left(v_1 < B_T < l, T \le \zeta\right) - C_1 x e^{-\psi_3 T} \mathbb{E} \left[e^{\sigma B_T} I_{v_1 < B_T < l, T \le \zeta} \right] \\
&+ C_2 x e^{-\psi_3 T} \mathbb{E} \left[e^{\sigma B_T} I_{l < B_T < v_2, T \le \zeta} \right] - C_2 K_2 e^{-(r+\lambda)T} \mathbb{Q}\left(l < B_T < v_2, T \le \zeta\right),
\end{aligned}
\tag{7.34}
$$

where

$$\begin{aligned}
\psi_{1,i} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - \sigma c_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma c_{1,i} \\
\psi_{2,i} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - \sigma d_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma d_{1,i} \\
\psi_3 &= \lambda + \frac{\sigma^2}{2} \\
l &= \frac{1}{\sigma} \ln\left(\frac{D_0}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) T \\
v_1 &= \frac{1}{\sigma} \ln\left(\frac{D_1}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) T \\
v_2 &= \frac{1}{\sigma} \ln\left(\frac{D_2}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) T.
\end{aligned} \tag{7.35}$$

We derive the expectations in formula (7.34) using Theorems 2.6, 2.7, and 2.8. The probabilities can be obtained via equation (2.29) for $\theta = 0$. We shall use an alternative parametrization for the option price – $G(x; t_0, t_1, ...t_n; A_0, A_1, ..., A_n; B_1, B_2, ..., B_n)$ instead $G(x, T; A(t), B(t))$. Once we have a closed-form formula for $(A(t), B(t))$-European option, we apply the following algorithm for approximating the optimal boundaries

1. The boundaries at the maturity are given in Propositions 7.5 and 7.6 – $A_n = D_1$ and $B_n = D_2$.

2. Suppose that we have derived all values $A_m, A_{m+1}, ..., A_n$ and $B_m, B_{m+1}, ..., B_n$ for some $m < n$.

3. We derive the put boundary in the following way. For the constants $A < x$, let $B(x, A)$ be the value that maximizes

$$G(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A, A_m, ..., A_n; B, B_m, ..., B_n) \tag{7.36}$$

   amongst all $B > x$. Let us view equation (7.36) as a function of A, and $A(x)$ be the argument that maximizes

$$G(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A, A_m, ..., A_n; B(x, A), B_m, ..., B_n).$$

Our approximation for the put boundary value A_{m-1} is the largest x for which $x = A(x)$. In fact, this is the largest initial value of the underlying asset for which the immediate exercise as a put is optimal.

4. We obtain the call boundary analogously. Let for a fixed $x < B$, $A(x, B)$ be the value that maximizes function (7.36) w.r.t. the variable A. Also, let $B(x)$ maximizes

$$G\left(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A(x,B), A_m, ..., A_n; B, B_m, ..., B_n\right)$$

amongst all $B > x$.

5. We approximate the call boundary B_{m-1} as the smallest x for which $x = B(x)$. We illustrate the way we derive the boundaries in Figure 7.2a – we present there the difference $B(x) - x$. Our approximation is the smallest x for which this difference is zero – it is marked by a circle. The used parameters are reported in Section 7.6.

Once we approximate the exercise boundaries, we can formulate our fast approach as:

Fast Pricing Approach 7.1. *The option price can be obtained through formulas* (7.34) *and* (7.35) *taken at the point* $x = S_0$.

If we need a dense grid, we can view the option pricing task as the boundary value problem

$$\begin{aligned}
&V_t(t,x) + rxV_x(t,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t,x) - rF(t,x) = 0 \\
&V(t, A(t)) = e^{-\lambda t} C_1 (K_1 - x), \quad t \in (0,T) \\
&V(t, B(t)) = e^{-\lambda t} C_2 (x - K_2), \quad t \in (0,T) \\
&V(T,x) = e^{-\lambda T} \max\left\{C_1(K_1 - x), C_2(x - K_2)\right\}, \quad x \in (D_1, D_2).
\end{aligned}$$

The differential equation holds in the region $(t,x) \in \{(0,T) \times (A(t), B(t))\}$ and the boundary constraints are imposed on the lower, upper, and the right boundaries. We solve the BVP using the Crank–Nicolson finite difference scheme presented in Section 3.5.

7.6 Numerical results

We present now the results of some numerical experiments. The main values we use are – the initial asset value is $S_0 = 22$, the risk-free rate is $r = -0.02$, the additional discount factor is $\lambda = 0.05$, the volatility is $\sigma = 0.3$, the put and call strikes are $K_1 = \$25$ and $K_2 = \$20$, the corresponding weights are $C_1 = 3$ and $C_1 = 2$, the time to maturity is in the interval $\tau \in (0, 50)$. We shall mention expressly the used values if we vary some of them. The parameters we use for the Crank–Nicolson method are – $\overline{M} = 16$ points for the optimal boundaries obtained by the two steps algorithm, $M = 64$ points for the time grid, and $N = 1\ 000$ steps for the state space. Note that these points are enough due to the quadratic convergence.

In Figure 7.2b, we present the put and call boundaries w.r.t the actual time – the put boundary is the lower one. We mark by circles the perpetual values derived via Theorem 7.1. We can see that these values are the limits when maturity increases.

The rest of the experiments we provide are parametrized w.r.t. the time to maturity instead of the current time. We present in Figures 7.2c and 7.2d the surfaces of the boundaries and the option prices fixing the put strike as $K_1 = \$20$ and varying the call one among $K_2 = \$15,\ \$16, ...,\ \$25$. Figure 7.2e shows the behavior of the option prices varying the discount factor as $\lambda \in (0.02, 0.1)$, whereas Figure 7.2f presents the behavior w.r.t. the put and call weights. We vary the ratio $c = \frac{C_1}{C_2}$ as $c \in \{0.01, 0.2, 0.4, 0.6, 0.9, 1, 2, 4, 6, 8, 10\}$. The circles show again the perpetual values.

Some options' prices together with the corresponding boundaries are presented in Table 7.1. The varied parameters are both strikes – $K_{1,2} \in \{20,\ 22,\ 24,\ , 25\}$ – as well as the weight's ratio $\frac{C_1}{C_2} \in \{0.5, 1, 2, 3\}$. The time to maturity is fixed at $T = 3$. We report in the first row the optimal boundaries – the first is the put one, and the second is the call one. The second row gives the option price.

7.7 Conclusions

Specific financial instruments, namely American strangle strategies, are examined in this chapter. They exhibit joint put and call features giving the holder the right to choose how to exercise – as a call or as a put. Arbitrary values for the strikes have been considered – the call strike can be less than the put one. Also, the call and put options in the strangle derivative are presented by different weights. It turns out that the pricing of such derivatives leads to two-sided optimal stopping problems. Closed-form formulas for the optimal boundaries as well as for the fair price have been obtained for the perpetual versions of these instruments. It turns out that early exercising as a call is never optimal if the discount (dividend) rate misses. However, the call right has its impact – it appears via the call weight, but not through the call strike. On the other hand, a numerical approach is constructed to approximate the optimal boundaries when the maturity is finite. Based on them, the Crank-Nicolson finite difference scheme is adapted to the raising boundary value problem related to the option pricing task. Several numerical experiments are provided.

7.A Uniqueness of the solutions

7.A.1 Put boundary

We shall prove now that the function $h(\cdot)$ defined in equation (7.17) has a unique root in the interval $(0,1)$. First, we consider the undiscounted case, that is $\lambda = 0$. The function $h(\cdot)$ turns into

$$h(a) = a^{q+1} C_1 k_1 - a(q+1)(C_1 + C_2 - C_2 k_2) + q C_1 k_1$$

and its derivative is

$$h_a(a) = (q+1)\left[a^q C_1 k_1 - C_1 - C_2 + C_2 k_2\right].$$

We have $h_a(1) = -(q+1)\left[C_1 + C_2 - C_1 k_1 - C_2 k_2\right] < 0$. Therefore $h_a(a)$ is negative in the interval $(0,1)$ since it is an increasing function. Hence, the function $h(\cdot)$ is decreasing and thus it has a unique root due to inequalities (7.18).

Suppose now $\lambda > 0$ or equivalently $p > q+1$. We can present function (7.17) as

$$h(a) = C_1 \left[-\overline{h}(a) - a^{p-q} \frac{p}{C_1}(C_1 + C_2 - C_1 k_1 - C_2 k_2)\right] \tag{7.37}$$

for

$$\overline{h}\left(a\right)=a^{p+1}\left(p-q-1\right)-a^{p}k_{1}\left(p-q\right)-a^{p-q}p\left(1-k_{1}\right)+a\left(q+1\right)-qk_{1}.$$

The function $\overline{h}\left(a\right)$ is examined in Zaevski (2020b), appendix B.1 – there it is denoted by $h\left(a;0\right)$. It may behave in two ways – (A) starts from a negative value and increases to zero or (B) starts from a negative value, increases to a positive maximum, and decreases to zero. Hence, function (7.37) may be increasing only when it is negative (only in case (B)). Having in mind boundary values (7.18) we conclude that function (7.17) has just one root in the interval $(0,1)$.

7.A.2 Call boundary

First, we consider the case $\lambda=0$ or equivalently $p=q+1$. Function (7.23) turns into

$$h\left(b\right)=b^{q+1}C_{2}k_{2}+b\left(q+1\right)\left(C_{1}k_{1}-C_{1}-C_{2}\right)+qC_{2}k_{2}$$

and its derivative is

$$h_{b}\left(b\right)=\left(q+1\right)\left(b^{q}C_{2}k_{2}+C_{1}k_{1}-C_{1}-C_{2}\right).$$

This function starts from a positive value for $b=1$ and increases – this makes it positive for $b>1$. Hence, the function $h\left(b\right)$ is increasing and therefore it is positive for $b>1$ too since $g\left(1\right)=p\left(C_{1}k_{1}+C_{2}k_{2}-C_{1}-C_{2}\right)>0$.

Suppose now that $\lambda>0$. Function (7.23) can be presented as

$$h\left(b\right)=C_{2}\left[-\overline{h}\left(b\right)+b^{p-q}\frac{p}{C_{2}}\left(C_{1}k_{1}+C_{2}k_{2}-C_{1}-C_{2}\right)\right]$$

for

$$\overline{h}\left(b\right)=b^{p+1}\left(p-q-1\right)-b^{p}k_{2}\left(p-q\right)+b^{p-q}p\left(k_{2}-1\right)+b\left(q+1\right)-qk_{2}.$$

Let us change the variable b to $d=\frac{1}{b}$ turning in this way the interval $(1,\infty)$ into $(0,1)$. The function $\overline{h}\left(d\right)$ written as $\overline{h}\left(d;0\right)$ is examined in appendix B.1 from Zaevski (2020c). There are two cases for its behavior – starts from a positive value and decreases to zero or (B) starts from a positive value decreases to a negative minimum and increases to zero. We conclude that the function $h(d)$ starts from a negative value for $d = 0$ and finishes at a positive value for $d = 1$. Also, it may decrease only if it is positive – this is possible only when case (B) holds. Hence, it has a unique root for $d \in (0,1)$ and thus the same is true for $b > 1$.

7.B Existing and uniqueness of the optimal boundaries

First, we shall prove that equation (7.26) has a unique solution in the interval $(0,1)$. Note that $\beta > \alpha$. Hence, $H(0) = \alpha C_1 C_2 K_1 > 0$ and $H(1) = -(\beta-\alpha)\left(C_1C_2K_1 + C_1C_2K_2 + C_1^2K_1 + C_2^2K_2\right) < 0$. Also, function (7.26) can be decomposed into $H(a) = C_1C_2\alpha H_1(a) + H_2(a)$ for

$$
\begin{aligned}
&H_1(a) = (K_1 - aK_2)(1 - a^p)\\
&H_2(a) = -(\beta-\alpha)\, a\left(a^{p-1}C_1C_2K_1 + a^{p-q-1}C_2^2K_2 + a^qC_1^2K_1 + C_1C_2K_2\right).
\end{aligned}
$$

If $K_1 \geq K_2$, then both functions $H_1(a)$ and $H_2(a)$ are decreasing which leads to the existence and uniqueness of the solution. Suppose now that $K_1 < K_2$. The first and second derivatives of the function $H_1(a)$ are

$$
\begin{aligned}
H_1'(a) &= a^pK_2(p+1) - a^{p-1}K_1p - K_2\\
H_1''(a) &= a^{p-2}p\left[aK_2(p+1) - K_1(p-1)\right].
\end{aligned}
$$

Hence, the derivative $H_1'(a)$ starts from the negative value $H_1'(0) = -K_2$, decreases to a minimum and increases to the positive value $H_1'(1) = p(K_2 - K_1)$. Therefore, the function $H_1(a)$ starts from the positive value $H_1(0) = K_1$, decreases to a negative minimum and increases to zero. Hence, the solution of equation (7.26) exists and it is unique since the function $H_2(a)$ is decreasing and negative. We shall denote this solution by $\overline{a}$. Note that $\overline{a} < \frac{K_1}{K_2}$ since $H_1(\overline{a}) \leq 0$ for $a \geq \frac{K_1}{K_2}$.

To continue our discussion, we need to introduce the following two functions

$$
\begin{aligned}
G_1(a) &= \frac{p-q}{p-q-1}\, a\, \frac{a^qC_1K_1 + C_2K_2}{a^{q+1}C_1 + C_2}\\
G_2(a) &= \frac{q}{q+1}\,\frac{a^{p-q}C_2K_2 + C_1K_1}{a^{p-q-1}C_2 + C_1}.
\end{aligned}
\tag{7.38}
$$

Equations (7.21), (7.24), and (7.25) show that $\overline{A} = A_1(\overline{a}) = A_2(\overline{a}) = G_1(\overline{a}) = G_2(\overline{a})$. Note that the equations $G_1(a) = G_2(a)$ and $H(a) = 0$ are equivalent and therefore the first one has the same unique solution. Let us examine now the behavior of functions (7.38). The derivative of the function $G_1(a)$ is

$$G_1'(a) = C_2 \frac{p-q}{p-q-1} \frac{-a^{q+1}C_1K_2q + a^q(q+1)C_1K_1 + C_2K_2}{(a^{q+1}C_1 + C_2)^2}.$$

If we denote by $l(\cdot)$ the function

$$l(a) = -a^{q+1}C_1K_2q + a^q(q+1)C_1K_1 + C_2K_2,$$

then its derivative is

$$l'(a) = -a^{q-1}q(q+1)C_1(aK_2 - K_1).$$

It is always positive when $K_1 \geq K_2$ and it is positive for $a < \frac{K_1}{K_2}$ and negative, otherwise. We conclude that the function $G_1(a)$ is increasing when $K_1 \geq K_2$. Otherwise, if $K_1 < K_2$, then the function $G_1(a)$ is increasing or first increases and then decreases. The second case holds when $l(1) = qC_1(K_1 - K_2) + C_1K_1 + C_2K_2 < 0$.

Let us turn to the function $G_2(a)$. We have

$$G_2'(a) = C_2 \frac{q}{q+1} a^{p-q-2} \frac{a^{p-q}C_2K_2 + a(p-q)C_1K_2 - (p-q-1)C_1K_1}{(a^{p-q-1}C_2 + C)^2}$$

We see that the function $l(a) = a^{p-q}C_2K_2 + a(p-q)C_1K_2 - (p-q-1)C_1K_1$ starts from a negative value and increases. Hence, the function $G_2(a)$ is decreasing or first decreases and then increases. The second case is actual when $l(1) = (p-q)C_1(K_2 - K_1) + C_1K_1 + C_2K_2 < 0$.

We shall prove now that $\overline{A} < D_1$. First, we shall show that $\min\left\{K_1, \overline{K}\right\} > \overline{A}$ where $\overline{K} := K_1 \frac{r+\lambda}{\lambda}$. Formulas (3.17) lead to

$$\frac{r+\lambda}{\lambda} = \frac{q}{q+1} \frac{p-q}{p-q-1}$$

and therefore

$$\overline{K} = K_1 \frac{q}{q+1} \frac{p-q}{p-q-1}.$$

If $K_2 < K_1$ or equivalently $D_0 < K_1$, then we can easily check that $G_2(0) < \min\left\{K_1, \overline{K}\right\}$ and $G_2(1) < \min\left\{K_1, \overline{K}\right\}$, and thus $\overline{A} = G_2(\overline{a}) < \overline{K}$. If $K_1 \leq K_2$, we have $G_2\left(\frac{K_1}{K_2}\right) = K_1 \frac{q}{q+1} < \min\left\{K_1, \overline{K}\right\}$. The fact $\overline{a} < \frac{K_1}{K_2}$ establishes the result.

It is left to be proven that $\overline{A} < D_0$. We need to consider only the case $K_2 < K_1$ due to the result above. Let us examine the dependence on the discount factor λ. Its domain is $[\max\{0, -r\}, \infty)$. Hence the variable q increases in the interval $\left[\max\left\{0, \frac{2r}{\sigma^2}\right\}, \infty\right)$. We shall parametrize w.r.t. the variable q hereafter. Equations (3.17) gives

$$p(q) = 2q - 2\frac{r}{\sigma^2} + 1.$$

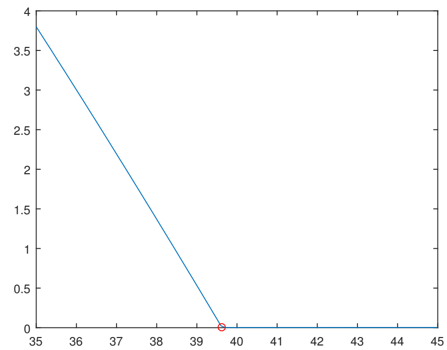
The functions (7.38) can be rewritten as

$$G_1(a, q) = \left(1 + \frac{1}{p(q) - q - 1}\right)\left(K_1 + C_2 \frac{aK_2 - K_1}{a^{q+1}C_1 + C_2}\right)$$
$$G_2(a, q) = \left(1 - \frac{1}{q+1}\right)\left(aK_2 - C_1 \frac{aK_2 - K_1}{a^{p(q)-q-1}C_2 + C_1}\right).$$

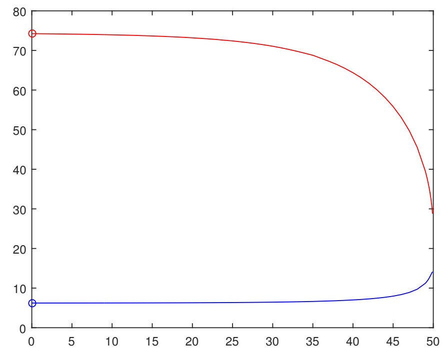
Having in mind that the function $p(q) - q - 1$ is increasing, we conclude that $G_1(a, q)$ decreases w.r.t. q, whereas $G_2(a, q)$ is a q-increasing function. Let us investigate the behavior of functions $G_1(a, q)$ and $G_2(a, q)$ when $q \to \infty$. We have $\lim_{q\to\infty} G_1(a, q) = aK_2$ for $a \in (0, 1)$ and $\lim_{q\to\infty} G_1(1, q) = D_0$. Analogously, $\lim_{q\to\infty} G_2(a, q) = K_1$ for $a \in (0, 1)$ and $\lim_{q\to\infty} G_2(1, q) = D_0$. We observe that $G_2^{-1}(D_0, q)$ is well-defined since $G_2(1, q) = \frac{q}{q+1} D_0 < D_0$. Also, $G_1^{-1}(D_0, q)$ is well defined because it is an a-increasing function. The function $L(q) = G_2^{-1}(D_0, q) - G_1^{-1}(D_0, q)$ is increasing. Let us consider the asymptotic case $q = \infty$. If we suppose that $\overline{a}(\infty) < 1$, then the equation $G_1(\overline{a}, \infty) = G_1(\overline{a}, \infty)$ leads to $\overline{a} = \frac{K_1}{K_2}$ – this is impossible due to $K_1 < K_2$. Hence $\overline{a}(\infty) = 1$ and therefore $\lim_{q\to\infty} G_1^{-1}(D_0, q) = \lim_{q\to\infty} G_2^{-1}(D_0, q) = 1$ due to the observed above shape of the the functions $G_1(a, q)$ and $G_2(a, q)$ for $q \to \infty$. This means that $L(\infty) = 0$ and thus $G_1^{-1}(D_0, q) \geq G_2^{-1}(D_0, q)$ for $q < \infty$ because the function $L(q)$ is increasing. Therefore $\overline{A} = G_1(\overline{a}) < D_0$ due to the a-behavior of the functions $G_1(a)$ and $G_2(a)$.

The inequality $\overline{B} > D_2$ can be proven analogously.

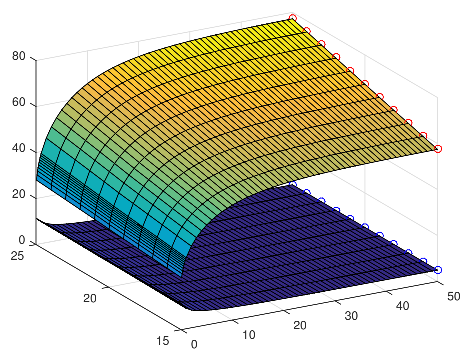
Figure 7.1: Optimal boundaries and option prices.



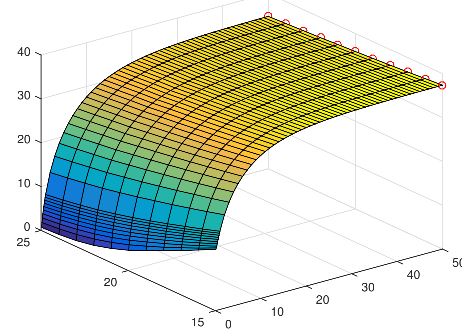
(a) Deriving the call boundary



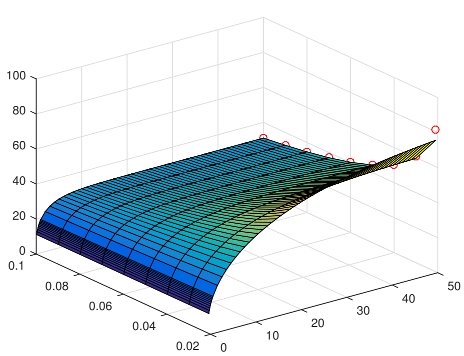
(b) Boundaries – $K_1 = 25, K_2 = 20$



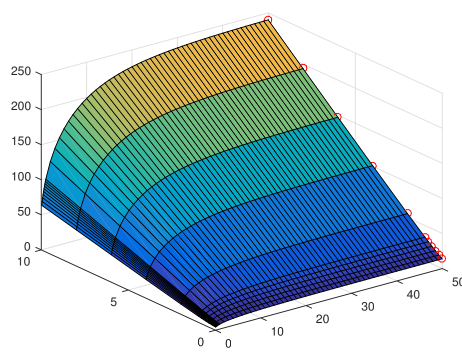
(c) Boundaries' surface



(d) Prices – $K_1 = 20,\ K_2 = 15, 16, ..., 25$



(e) Prices w.r.t. λ



(f) Prices w.r.t. the weights

Table 7.1: Option prices and optimal boundaries

K_1	20	22	24	25
		weight ratio $\frac{C_1}{C_2} = 0.5$		
$K_2 = 20$	{4.3621;52.5720} $12.4648	{4.7448;53.6082} $13.4766	{5.1220;54.6609} $14.5628	{5.3057;55.2218} $15.1438
$K_2 = 22$	{4.4103;56.8144} $10.8370	{4.7985;57.8294} $11.9038	{5.1814;58.8657} $13.1632	{5.3747;59.3756 } $13.7671
$K_2 = 24$	{4.4537;61.0725} $9.5939	{4.8466;62.0707} $10.6888	{5.2356;63.0865} $11.9389	{5.4255;63.6108} $12.5846
$K_2 = 25$	{4.4738;63.2076} $9.0320	{4.8693;64.1964} $10.1333	{5.2603;65.2015} $11.3953	{5.4532;65.7156} $12.0483
		weight ratio $\frac{C_1}{C_2} = 1$		
$K_2 = 20$	{4.8290;60.1743} $15.8613	{5.2781;62.1678} $18.0208	{5.7315;64.1334} $20.1833	{5.9502;65.1782} $21.5037
$K_2 = 22$	{4.8556;64.2656} $14.3319	{5.3119;66.1944} $16.4861	{5.7627;68.1735} $18.9870	{5.9877;69.1654} $20.2409
$K_2 = 24$	{4.8801;68.3783} $13.0994	{5.3386;70.2827} $15.2889	{5.7948;72.2092} $17.8092	{6.0219;73.1878} $19.1157
$K_2 = 25$	{4.8913;70.4431} $12.5413	{5.3514;72.3342} $14.7396	{5.8062;74.2744} $17.2781	{6.0362;75.2179} $18.5920
		weight ratio $\frac{C_1}{C_2} = 2$		
$K_2 = 20$	{5.1233;73.8414} $22.8514	{5.6225;77.3750} $27.2151	{6.1217;80.9273} $31.9439	{6.3690;82.7279} $34.4314
$K_2 = 22$	{5.1346;77.7337} $21.3369	{5.6360;81.2233} $25.6815	{6.1345;84.7604} $30.7194	{6.3847;86.5286} $33.2898
$K_2 = 24$	{5.1454;1.6612} $20.1202	{5.6470;85.1202} $24.4991	{6.1482;88.6067} $29.5812	{6.3979;90.3696} $32.2168
$K_2 = 25$	{5.1507;83.6339} $19.5654	{5.6533;87.0708} $23.9636	{6.1543;90.5508} $29.0628	{6.4042;92.2996} $31.7074
		weight ratio $\frac{C_1}{C_2} = 3$		
$K_2 = 20$	{5.2285;86.1083} $29.8512	{5.7437;91.0003} $36.4821	{6.2594;95.9010} $43.6647	{6.5167;98.3662} $47.4866
$K_2 = 22$	{5.2351;89.8781} &28.3389	{5.7513;94.7181} $34.9378	{6.2671;99.5959} $42.4953	{6.5247;102.0336} $46.3864
$K_2 = 24$	{5.2419;93.6623} &27.1472	{5.7576;98.5050} $33.7310	{6.2741;103.3287} $41.3697	{6.5322;105.7617} $45.3392
$K_2 = 25$	{5.2433;95.6113} $26.5958	{5.7595;100.4213} $33.2461	{6.2786;105.1839} $40.8602	{6.5350;107.6390} $44.8374

Chapter 8

Quadratic American strangles in the light of two-sided optimal stopping problems.

based on the paper

Zaevski, T.S. Quadratic American Strangle Options in Light of Two-Sided Optimal Stopping Problems. Mathematics 2024, 12, 1449. https://doi.org/10.3390/math12101449

Abstract: The aim of this chapter is to examine some American-style financial instruments that lead to two-sided optimal hitting problems. We pay particular attention to derivatives that are similar to the strangle strategies but have a quadratic payoff function. We consider these derivatives in light of much more general payoff structures under certain conditions which guarantee that the optimal strategy is an exit from a strip. Closed-form formulas for the optimal boundaries and the fair price are derived when the contract has no maturity constraints. We obtain the form of the optimal boundaries under the finite maturity horizon and approximate them by maximizing the financial utility of the derivative holder. The Crank–Nicolson finite difference method is applied to the pricing problem. The importance of these novel financial instruments is supported by several features that are very useful for financial practice. They combine the characteristics of the power options and the ordinary American straddles. Quadratic strangles are suitable for

investors who need to hedge strongly, far from the strike positions. In contrast, the near-the-money positions offer a relatively lower payoff than the ordinary straddles. Note that the usual options pay exactly the overprice; no more, no less. In addition, the quadratic strangles allow investors to hedge the positions below and above the strike together. This is very useful in periods of high volatility when large market movements are expected but their direction is unknown.

8.1 Motivation and main results

Derivatives are one of the major instruments against the financial risk. They exhibit a very large variety – most popular are options, futures, bonds, swaps, etc. Conventionally speaking, we can recognize two types – European and American. The European derivatives have a previously fixed date at which the transaction is executed. Alternatively, an American-style instrument gives its holder the right to choose when to exercise until maturity. This right makes American derivatives preferable for investors and determines the largest segment of the traded assets they have in the modern financial markets. The main exchanges at which the options are traded are Chicago Board Options Exchange (CBOE, https://www.cboe.com/), Philadelphia Stock Exchange (NASDAQ OMX PHLX, https://www.nasdaq.com/solutions/nasdaq-phlx), Frankfurt-based Eurex Exchange (https://www.eurex.com/ex-en/), Tokyo Stock Exchange (TSE), Taiwan Futures Exchange (TAIFEX), and two electronic platforms – Boston Options Exchange (BOX) and Miami International Securities Exchange (MIAX). We have to mention that not all markets offer both types of options.

Most of the available financial instruments preserve mainly from the so-called linear risks in the sense that the holder would receive the overprice (underprice) of the asset or a proportion of it. Namely, such protection provides the classical option and futures contracts. On the other hand, many risks exhibit non-linear essence – the holder needs a significantly larger payoff when the underlying asset is deeply above/below the strike. This motivates several authors to consider the so-called power options – see Heynen and Kat (1996), Macovschi and Quittard-Pinon (2006), Zhang et al. (2016), Nwozo and Fadugba (2014), Fadugba and Nwozo (2015), Fadugba and Nwozo (2020), and Lee (2020).

On the other hand, sometimes, especially in the high volatility periods, large movements in both directions are very likely. Therefore, the investors need protection jointly against the fall and rise of the assets. This kind of insurance is provided by the straddle and strangle strategies – we refer to Kang et al. (2017b), Jeon and Oh (2019) Qiu (2020), Jeon and Kim (2022b). See also the results derived in Chapter 7. In fact, these options give the right to choose the option's style – call or put. Note that they lead to two-sided optimal hitting problems. This way the optimal strategy is the first exit of the underlying asset from a strip. If the exit occurs from the lower boundary, then the holder exercises as a put, otherwise as a call.

To combine the non-linearity and the two-sided protection, we introduce and examine in this chapter a specific financial instrument – we name it a *quadratic strangle option*. Its payoff is defined as the square of the corresponding straddle payoff, that is $(S_T - K)^2$. Something more, we study these options in a quite general framework considering payment functions that lead to two-sided optimal hitting problems. We state our model under the assumptions of the Black and Scholes (1973) model – the underlying asset is driven by a geometric Brownian motion. Also, we consider continuously paying dividends assets using a method presented by Shiryaev et al. (1995a). It is based on an additional discount factor instead of the original dividend rate. This approach provides a computational convenience. It is well-known that the empirical stylized facts such as volatility clustering, leverage effect, long-range dependence, sudden market shocks, etc. cannot be captured by the Gaussian returns. Despite these limitations, the outstanding importance of the Black-Scholes model motivates us to work in its framework. On the other hand, our method can be adapted for models driven by other Feller-Markov processes such as exponential Lévy models, stochastic volatility models, regime-switching, etc. More precisely, our approach is based on the infinitesimal generators technique. We obtain a condition for the payoff that guarantees that the pricing problem leads namely to a two-sided optimal stopping task. Although the payoff of a quadratic strangle can be presented as a sum of two power payoffs, put and call, this does not mean that we can view it as a portfolio of two power options. This is because the possibility of the option to be exercised as a call influences the put optimal boundary, not only the call one. Of course, the put feature influences the call boundary too. This leads to the fact that the lower quadratic optimal boundary is below the optimal boundary of the related American put option. Also, the upper quadratic boundary is above the call one. As a consequence, the quadratic option has to be more expensive than the portfolio of the related two power options.

The scheme we use to evaluate such derivatives consists of several steps. First, we derive the values of the optimal boundaries at the maturity. We do this using the variational inequalities arising from the infinitesimal generator. Next, we consider the perpetual instruments. In this case, the optimal boundaries are time-independent because the underlying asset is driven by a Markov process. A system of two equations for the boundaries is obtained. Its solution leads to the fair price. The next task is to approximate the optimal boundaries. A natural assumption is that the holder's strategy maximizes his/her financial utility. So, we divide the time to maturity into several sub-intervals and approximate the boundaries at these nodes as the values that maximize the holder's result. To do this, we apply the results of Chapter 2.4. This way, we immediately tell the optimal points from these that provide better opportunities. Also, we derive the option price with a relatively high precision.

On the other hand, we can estimate the optimal boundaries at a denser grid. Thus, the free boundary problem that describes the option pricing task turns into a partial differential equation set in a known region. In fact, it is the continuation region – these points that make keeping the derivative preferable. We solve this problem numerically using the Crank-Nicolson finite difference approach.

We apply this general framework to price the defined above quadratic strangles. It turns out that there exists a critical value for the additional discount factor above which we have a real two-sided instrument. We consider separately finite and infinite maturities. Closed-form formulas for the optimal boundaries as well as for the quadratic strangle price are derived. Also, we obtain the boundaries' values at the maturity. Having the endpoints, we approximate the whole boundaries by exponents of piece-wise linear functions using some first exit properties of the Brownian motion from a strip. The prices are derived by the above-mentioned finite difference method.

On the other hand, if the discount factor is not above the critical value, then the quadratic strangle leads to a one-sided hitting problem. We apply the approach presented in Chapter 4 to evaluate the option. The closed-form formulas are derived for the perpetual modification. There exists a significant difference when the discount factor is equal to the critical value or below it. If it is below, then the early exercise is never optimal for the perpetual quadratic strangle. On the contrary, the immediate exercise can be the best strategy for the critical discount rate. In both cases, the finite maturities provide possibilities for early exercising. We validate and confirm all these results by several numerical tests.

The main contributions of the chapter are in several directions. First, we consider a relatively large class of novel American-style financial instruments. We establish general conditions for the payoffs which guarantee that the optimal holder's strategy is the first exit of the underlying asset price from a strip – thus we need to find the boundaries of this strip. Usually, the American style derivatives are considered as free boundary problems for which we need to obtain the set in which they hold as well as to solve the differential task. We provide a novel approach based on identifying the points in the time-price space for which the immediate exercise is the best holder's strategy and these for which keeping the derivative is preferable. Our approach is based on maximizing the holder's financial utility. We consider separately the options without maturity constraints and finite maturity ones. For the perpetual ones, we obtain a two-dimensional system that the boundaries have to solve. Alternatively, we approximate them under the finite maturity horizon. This way the free boundary differential problem turns into a boundary value problem in a known region. Note that the differential dynamics is driven by the infinitesimal generator of the stochastic process that presents the underlying asset price, whereas the payoff influences the boundary conditions.

Next, we turn to the second main purpose of the chapter, namely to examine the above-mentioned quadratic options. Their importance is determined by both features they combine. First, the quadratic feature allows the investor to hedge better the positions that are deeply far from the strike. Whereas the usual call option would provide only the overprice above the strike (underprice for a put), the quadratic one provides the square of this amount. On the other hand, the holder of the quadratic option will receive a relatively lower amount if he exercises near the strike. The second major importance is related to the possibility for the holder to hedge jointly the positions that are above or below the strike – a very important feature in high volatility periods. These conclusions are supported by the payoff structure – we compare the payoffs of an ordinary straddle to a quadratic one in Figure 8.3f.

We apply the derived theoretical results to these financial instruments. It turns out that two-sided exit problems arise for some values of the parameters, whereas other ones lead to one-sided hitting tasks. We examine in detail both cases – the optimal boundaries are approximated and the options are evaluated under and without maturity restrictions.

The chapter is organized as follows. The base we use is presented in Section 8.2. The shape of the optimal exercise regions is obtained in Section 8.3. The pricing problem is discussed in Section 8.4. We examine the quadratic strangles in Section 8.5. Some numerical experiments are provided in Section 8.6.

8.2 Preliminaries

Suppose again that the underlying asset, whose price we denote by S_t, follows the process

$$dS_t = rS_t dt + \sigma S_t dB_t, \tag{8.1}$$

Let $\mathcal{A}$ be the infinitesimal generator of process (8.1) and $\mathcal{B}$ be a related to it another differential operator:

$$\begin{aligned} \left(\mathcal{A}f\right)(x) &= rxf'(x) + \frac{\sigma^2 x^2}{2} f''(x) \\ \left(\mathcal{B}f\right)(x) &= \left(\mathcal{A}f\right)(x) - (r+\lambda) f(x). \end{aligned} \tag{8.2}$$

The operator $\mathcal{B}$ arises from discounting at the total rate $r + \lambda$. In fact, it is related to the famous Black-Scholes equation. Let us denote by t, T, and τ the current time, the maturity, and the time to maturity, respectively. Thus $t \le T \le \infty$ and $\tau = T - t$. We shall parametrize w.r.t. the present time as well as w.r.t. the time to maturity – the difference will be recognized by the used variables. Thus we shall denote the price of the studied instruments by $V(t, x)$ w.r.t. the current time and by $V(x; \tau)$ w.r.t. the time to maturity.

Let a derivative pay an amount of $N(t, x)$ if its holder exercises at the moment t at the spot price $S_t = x$. Suppose also that the time dependence is presented only by additional discounting at the rate λ. Assuming that the veritable payoff is presented by the twice differential function $G(x)$, we write

$$N(t, x) = e^{-\lambda t} G(x).$$

We restrict the choice of functions $G(x)$ in a way to guarantee that the pricing problem for the related derivative is a two-sided optimal stopping problem.

Condition 8.1. *There exist constants $C \leq D$, such that $(\mathcal{B}G)(x) < 0$ for $x < C$ and $x > D$, and $(\mathcal{B}G)(x) \geq 0$ for $x \in [C, D]$.*

Note that if $C = 0$ or $D = \infty$, then we have one-sided problems.

8.3 Exercise regions

Let the set at which the immediate exercise is the holder's optimal strategy be denoted by Υ – it is known as the optimal region. If $(t, x) \in \Upsilon$, then we shall call the point (t, x) optimal. Also, we use the symbol $\overline{\Upsilon}$ for the region in which keeping the derivative leads to a better financial result – the so-called continuation region. We shall mark by a subscript the dependence w.r.t. the current moment or w.r.t. the time to maturity – $\Upsilon_{[t,T]}$ and $\overline{\Upsilon}_{[t,T]}$ or Υ_τ and $\overline{\Upsilon}_\tau$, respectively.

We shall characterize now the shape of the optimal region as well as of the continuation set. It turns out that the optimal region consists of two disjoint subsets – the first one is below some boundary whereas the second is above another curve. We shall prove that the lower boundary is below the above-defined constant C, whereas the upper one is above D. These boundaries are known as early exercise or optimal boundaries. Thus all points between them form the continuation set. We shall investigate the features of the optimal boundaries deriving their endpoints and establishing their behavior. This allows us to approximate them later and, as a consequence, to construct an algorithm for deriving the fair derivative price.

8.3.1 The form of the regions

We shall investigate now the shape of the optimal region. The main goal is to prove that if a point below C is optimal, then all points below it are optimal too. Analogously, it turns out that if an optimal point is above the level D, then all points above it are optimal too. This way, we confirm the above-mentioned feature.

The following statements for the shape of the regions and boundaries hold.

Proposition 8.1. *If a point (t, x) is optimal, then $(\mathcal{B}G)(x) < 0$.*

Proof. The variational inequality

$$N_t(t,x) + \mathcal{A}N(t,x) - rN(t,x) < 0$$

has to be satisfied in the optimal region. It is equivalent to $(\mathcal{B}G)(x) < 0$. □

Proposition 8.2. *If $\tau_1 > \tau_2$ and the point $(x;\tau_1)$ is optimal, then the point $(x;\tau_2)$ is optimal too.*

Proof. The proposition holds because the set of possible strategies for a longer maturity contains all strategies for a shorter one. □

Proposition 8.3. *Let $(x;\tau)$ be an optimal point. The following two statements hold.*

1. *If $y > x > D$, then the point $(y;\tau)$ is optimal too.*
2. *If $y < x < C$, then the point $(y;\tau)$ is optimal.*

Proof. Note first that if $x \in [C,D]$, then it cannot be optimal, because $(\mathcal{B}G)(x) \geq 0$ and this contradicts Proposition 8.1. Suppose that $x < C$, $(x;\tau) \in \Upsilon_\tau$, and $y < x$. Let us denote by ζ^x the first hitting moment of the underlying asset to the value x if its initial point is y. Also, let us define ζ as the strategy $\zeta = \overline{\zeta} \wedge \zeta^x \wedge \tau$ for an arbitrary stopping time $\overline{\zeta}$. The strategy ζ is not worse than $\overline{\zeta}$ for the holder, because the points $(x;\overline{\tau})$ for all $\overline{\tau} < \tau$ are optimal due to Proposition 8.2.

We have that $(\mathcal{B}G)(S_u(\omega)) < 0$ for every $u < \zeta(\omega)$ because $S_u(\omega) < x < C$ and thus we can use Condition 8.1. Dynkin's formula leads to

$$\begin{aligned}\mathbb{E}^y\left[e^{-(r+\lambda)\overline{\zeta}}G\left(S_{\overline{\zeta}}\right)\right] - G(y) &\leq \mathbb{E}^y\left[e^{-(r+\lambda)\zeta}G(S_\zeta)\right] - G(y)\\ &= \mathbb{E}^y\left[\int_0^\zeta (\mathcal{B}G)(S_u)\right] < 0.\end{aligned}$$

Hence the point $(y;\tau)$ is optimal too. The second statement can be proven analogously using an upper construction. □

Proposition 8.3 shows that the continuation region is a strip in the time-state space. Also, the optimal set consists of two parts – one below the continuation region and another above it. We shall name them call- and put-optimal sets by an analogy with the usual options. We shall denote them by Υ^c and Υ^p, respectively. Thus there are two optimal boundaries – one between Υ^p and $\overline{\Upsilon}$ and another between Υ^c and $\overline{\Upsilon}$. We denote these curves by $c(t)$ and $d(t)$, respectively. Hence, we can write $\Upsilon^p = \{(t,x) : t \in \mathbb{R}^+, x \in (0, c(t))\}$ and $\Upsilon^c = \{(t,x) : t \in \mathbb{R}^+, x \in (d(t), \infty)\}$.

Another consequence of Proposition 8.2 describes the optimal boundaries.

Corollary 8.1. *The boundary $c(\tau)$ decreases w.r.t. the time to maturity, whereas $d(\tau)$ increases.*

8.3.2 Initial boundary values

Further, we derive the optimal boundary values when the time to maturity tends to zero.

Proposition 8.4. *The values of the optimal boundaries when time to maturity is zero are C and D.*

Proof. Let us consider the value of the lower boundary – we denote it by $\overline{C}$. We have that $\overline{C} \leq C$ due to Condition 8.1 and Proposition 8.1. Suppose that $\overline{C} < C$. Hence, there exists $x < C$ such that the points (t,x) belong to the continuation region near the maturity. Therefore,

$$\begin{aligned}
0 &< \lim_{t \to T} \frac{V(t,x) - N(t,x)}{T-t} \\
&= -\lim_{t \to T} \frac{V(T,x) - V(t,x)}{T-t} + \lim_{t \to T} \frac{N(T,x) - N(t,x)}{T-t} \\
&= \mathcal{A}V(T,x) - rV(T,x) + N_t(T,x) \\
&= (\mathcal{B}G)(x) < 0.
\end{aligned}$$

The contradiction leads to $\overline{C} = C$. Similar arguments show that the upper boundary is namely D at the maturity. □

8.3.3 Perpetual boundary values

After we establish that the initial values of the optimal boundaries are namely the constants C and D, we turn to the perpetual ones. To find them, we prove two propositions that characterize the price of derivatives that expire at a certain level. The first proposition is about levels below C, whereas the second one is for the levels above D. Once we have these statements, we can obtain a two-dimensional system that the optimal boundaries have to solve. Note that they are time-independent constants because the holder is not threatened by time expiring.

We shall denote by A and B the perpetual values of the lower and upper boundaries, respectively. We have $A < C \leq D < B$. Suppose that the initial asset value is x and it belongs to the continuation region, i.e. $A < x < B$. We search for these values of A and B that maximize the financial result of the strategy of the first exit from the strip (A, B). Let ζ_A and ζ_B be the first hitting moments of the underlying asset to the levels A and B and $\zeta = \zeta^A \wedge \zeta^B$. These stopping times can be viewed as the first hitting times of a Brownian motion with drift $\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$ to the values

$$\tilde{A} = \frac{\ln A - \ln x}{\sigma} < 0$$
$$\tilde{B} = \frac{\ln B - \ln x}{\sigma} > 0.$$

Let the constants p and q be defined via formulas (3.17). Using Lemma 3.3, we obtain the financial result of the strategy ζ:

$$\begin{aligned}
&f(A, B; x) = \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} G(S_\zeta)\right] \\
&= G(A)\,\mathbb{E}^x \left[e^{-(r+\lambda)\zeta^B} I_{\zeta^B \leq \zeta^A}\right] + G(B)\,\mathbb{E}^x \left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A < \zeta^B}\right] \\
&= G(A)\, e^{\psi \tilde{A}} \frac{\sinh\left(\frac{\sigma p \tilde{B}}{2}\right)}{\sinh\left(\frac{\sigma p\left(\tilde{B}-\tilde{A}\right)}{2}\right)} + G(B)\, e^{\psi \tilde{B}} \frac{\sinh\left(-\frac{\sigma p \tilde{A}}{2}\right)}{\sinh\left(\frac{\sigma p\left(\tilde{B}-\tilde{A}\right)}{2}\right)} \\
&= G(A)\, e^{(p-q)(\ln x - \ln A)} \frac{e^{p(\ln B - \ln x)} - 1}{e^{p(\ln B - \ln A)} - 1} + G(B)\, e^{(q)(\ln B - \ln x)} \frac{e^{p(\ln x - \ln A)} - 1}{e^{p(\ln B - \ln A)} - 1} \\
&= G(A) \left(\frac{x}{A}\right)^{p-q} \frac{\left(\frac{B}{x}\right)^p - 1}{\left(\frac{B}{A}\right)^p - 1} + G(B) \left(\frac{B}{x}\right)^q \frac{\left(\frac{x}{A}\right)^p - 1}{\left(\frac{B}{A}\right)^p - 1} \\
&= G(A) \left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + G(B) \left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}.
\end{aligned} \tag{8.3}$$

We can write the derivatives of function (8.3) w.r.t. A and B as

$$
\begin{aligned}
f_A(A,B;x) &= \frac{A^{q-1}(B^p - x^p)}{x^q (B^p - A^p)^2} \begin{bmatrix} (G'(A)A + G(A)q)(B^p - A^p) \\ -A^{p-q}p(G(B)B^q - G(A)A^q) \end{bmatrix} \\
f_B(A,B;x) &= \frac{B^{q-1}(x^p - A^p)}{x^q (B^p - A^p)^2} \begin{bmatrix} (G'(B)B + G(B)q)(B^p - A^p) \\ -B^{p-q}p(G(B)B^q - G(A)A^q) \end{bmatrix}.
\end{aligned}
\tag{8.4}
$$

Let us first fix the value of the boundary $A < C$. The following proposition for the price function $f(A,B;x)$ holds.

Proposition 8.5. *Let the boundary A be fixed and let us examine price (8.3) as a function of B – we write $f(B;x)$.*

1. *If for some $x > A$ the function $f(B;x)$ has a local maximum or minimum at a point $\overline{B} > x$, then it has the same local extremum at the point $\overline{B}$ for all $x \in (A, \overline{B})$.*
2. *If for some $x > A$ the function $f(B;x)$ has a local maximum at a point $\overline{B} > x$, then $(\mathcal{B}G)(\overline{B}) < 0$.*
3. *If for some $x > A$ the function $f(B;x)$ has a local maximum at a point $\overline{B} > x$, then $\overline{B} > D$.*
4. *If the function $f(B;x)$ has a local minimum at a point $\overline{B}$, then $(\mathcal{B}G)(\overline{B}) > 0$.*
5. *The function $f(B;x)$ has no more than one local maxima.*

Proof. Let us denote by $\xi^{A,B}$ the first exit of the underlying asset from the strip (A,B).

1. This statement holds due to the form of the derivative $f_B(B;x)$ given in equations (8.4) – its root w.r.t. B is independent of x.
2. Suppose that the function $f(B;x)$ has a local maximum at a point $\overline{B}$. Therefore there exist small enough but positive constants ϵ and δ such that $f(\overline{B};x) > f(\overline{B}+\epsilon;x) > f(\overline{B}-\delta;x)$. Using Dynkin's formula we derive

$$
\begin{aligned}
G\left(\overline{B}\right) &= \lim_{x\uparrow\overline{B}} f\left(\overline{B};x\right) > \lim_{x\uparrow\overline{B}} f\left(B+\epsilon;x\right) = f\left(B+\epsilon;\overline{B}\right) \\
&= \mathbb{E}^{\overline{B}}\left[e^{-(r+\lambda)\xi^{A,B+\epsilon}} G\left(S_{\xi^{A,B+\epsilon}}\right)\right] \\
&= G\left(\overline{B}\right) + \mathbb{E}^{\overline{B}}\left[\int_0^{\zeta^{A,B+\epsilon}} \left(\mathcal{B}G\right)\left(S_u\right) du\right]
\end{aligned}
$$

and therefore

$$
\mathbb{E}^{\overline{B}}\left[\int_0^{\zeta^{A,B+\epsilon}} \left(\mathcal{B}G\right)\left(S_u\right) du\right] < 0. \tag{8.5}
$$

Inequality (8.5) leads to

$$
\begin{aligned}
0 &> \mathbb{E}^{\overline{B}}\left[\mathbb{E}^{\overline{B}}\left[\int_0^{\zeta^{A,B+\epsilon}} \left(\mathcal{B}G\right)\left(S_u\right) du \middle| \mathcal{F}_{\xi^{\overline{B}-\delta,\overline{B}+\epsilon}}\right]\right] \\
&= \mathbb{E}^{\overline{B}}\left[\int_0^{\xi^{\overline{B}-\delta,\overline{B}+\epsilon}} \left(\mathcal{B}G\right)\left(S_u\right) du\right] + \mathbb{E}^{\overline{B}-\delta}\left[\int_0^{\xi^{A,\overline{B}+\epsilon}} \left(\mathcal{B}G\right)\left(S_u\right) du\right].
\end{aligned} \tag{8.6}
$$

On the other hand, similarly to inequality (8.5) we can prove

$$
\mathbb{E}^{\overline{B}-\delta}\left[\int_0^{\xi^{A,\overline{B}+\epsilon}} \left(\mathcal{B}G\right)\left(S_u\right) du\right] > 0. \tag{8.7}
$$

Combining inequalities (8.6) and (8.7) we conclude that $\left(\mathcal{B}G\right)(x)$ has to be negative in a small enough neighborhood of $\overline{B}$. Hence, $\left(\mathcal{B}G\right)\left(\overline{B}\right) < 0$.

3. Suppose the opposite, i.e. the function $f(B;x)$ has a local maximum at a point $\overline{B}$ such that $D > \overline{B} > x$. The previous statement shows that $\overline{B} \le C$. Hence, there exists a small enough constant ϵ such that $f\left(\overline{B};x\right) > G\left(\overline{B}-\epsilon\right)$. Using again Dynkin's formula we see

$$\begin{aligned} G\left(\overline{B}-\epsilon\right) < f\left(\overline{B};\overline{B}-\epsilon\right) &= \mathbb{E}^{\overline{B}-\epsilon}\left[e^{-(r+\lambda)\zeta^{A,\overline{B}}} G\left(S_{\zeta^{A,\overline{B}}}\right)\right] \\ &= G\left(\overline{B}-\epsilon\right) + \mathbb{E}^{\overline{B}-\epsilon}\left[\int_0^{\zeta^{A,\overline{B}}} (\mathcal{B}G)(S_u)\,du\right] < G\left(\overline{B}-\epsilon\right). \end{aligned}$$

The last equation is true due $\overline{B} \le C$ and Condition 8.1. The contradiction proves the third statement.

4. Some arguments similar to those presented for the second statement prove the desired result.

5. If there exist two local maxima in the interval (D,∞), then there exists a local minimum in this interval. The previous statement means that the value of $(\mathcal{B}G)(u)$ is positive in it, which is impossible due to Condition 8.1.

□

Similar reasons argue the analog of Proposition 8.5 when the upper boundary $B > D$ in function (8.3) is fixed. We consider now (8.3) as a function of the lower boundary A, $f(A;x)$.

Proposition 8.6. *The following statements hold.*

1. *If the function $f(A;x)$ has a local extremum at a point $\overline{A} < x$ for some $x < B$, then it has the same local extremum at the same point $\overline{A}$ for all $x \in \left(\overline{A}, B\right)$.*

2. *If the function $f(A;x)$ has a local maximum at a point $\overline{A}$, then $(\mathcal{B}G)\left(\overline{A}\right) < 0$.*

3. *If the function $f(A;x)$ has a local maximum at a point $\overline{A}$, then $\overline{A} < C$.*

4. *If the function $f(A;x)$ has a local minimum at a point $\overline{A}$, then $(\mathcal{B}G)\left(\overline{A}\right) > 0$.*

5. *The function $f(A;x)$ has no more than one local maxima.*

Propositions 8.5 and 8.6 and derivatives (8.4) show that we can derive the optimal boundaries as the solution of the following system

$$\begin{aligned} &(G'(A)A + G(A)q)(B^p - A^p) - A^{p-q}p(G(B)B^q - G(A)A^q) = 0 \\ &(G'(B)B + G(B)q)(B^p - A^p) - B^{p-q}p(G(B)B^q - G(A)A^q) = 0 \end{aligned} \tag{8.8}$$

in the domain $A < C$ and $B > D$. Note that if the solution exists, then it is unique. The system (8.8) can be rewritten as

$$\begin{aligned} \left(\frac{A}{B}\right)^q &= \frac{G'(B)B - (p-q)G(B)}{G'(A)A - (p-q)G(A)} \\ \left(\frac{A}{B}\right)^{p-q} &= \frac{G'(A)A + qG(A)}{G'(B)B + qG(B)}. \end{aligned} \tag{8.9}$$

Once we derive the solution of system (8.9) as the pair (A, B), we obtain the derivative's price estimating it in formula (8.4).

8.3.4 Finite maturities

Now we present the algorithm for approximating the optimal boundaries when the maturity is finite. Suppose that we have the time grid $0 \equiv t_0 < t_1 < t_2 < ... < t_n \equiv T$ and two continuous piecewise linear functions w.r.t. it $a(t) < b(t)$:

$$\begin{aligned} a(t) &= \sum_{i=1}^{n} a_i(t) I_{t\in[t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (a_{1,i}t + a_{2,i}) I_{t\in[t_{i-1},t_i]} \\ b(t) &= \sum_{i=1}^{n} b_i(t) I_{t\in[t_{i-1},t_i]} \equiv \sum_{i=1}^{n} (b_{1,i}t + b_{2,i}) I_{t\in[t_{i-1},t_i]}, \end{aligned}$$

$a_i(t_i) = a_{i+1}(t_i)$ and $b_i(t_i) = b_{i+1}(t_i)$, $i = 1, 2, ..., n-1$. Let also $a(0) < 0 < b(t)$. We shall approximate the optimal boundaries as exponents of such functions – $c(t) = \exp(a(t))$ and $d(t) = \exp(b(t))$. We denote the values at the grid nodes by $A_i = c(t_i)$ and $B_i = d(t_i)$, $i = 0, 1, ..., n$. The derivative price can be written as

$$V(x, t_1, ...t_n; A_1, ..., A_n; B_1, ..., B_n) = \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} G(S_{\zeta \wedge T}) \right]. \tag{8.10}$$

Remark 8.1. *Of course, the expectation in formula (8.10) cannot be derived in a closed form for every choice of the payoff $G(\cdot)$. Alternatively, it can be found via Kolmogorov backward equation or using some Monte Carlo simulations. On the other hand, a large part of the useful payoff functions admit closed-form formulas – see for example Section 8.5.*

We shall apply the following backward algorithm.

1. The boundaries at the maturities are the constants C and D. Thus $A_n = C$ and $B_n = D$.

2. Suppose that we have obtained all values $A_m, A_{m+1}, ..., A_n$ and $B_m, B_{m+1}, ..., B_n$ for some $m \le n$.

3. We approximate the lower boundary A_{m-1} in the following way. For fixed constants $A < x \le A_m$, we define $B(x, A)$ as the maximizer of

$$V(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A, A_m, ..., A_n; B, B_m, ..., B_n) \tag{8.11}$$

   amongst all $B > B_m$. We now consider (8.11) as a function of A and we denote by $A(x)$ the maximizer of

$$V(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A, A_m, ..., A_n; B(x, A), B_m, ..., B_n),$$

   that we find having in mind Remark 8.1. We approximate A_{m-1} as the largest x for which $x = A(x)$ – it is the largest optimal value in the lower segment for the underlying asset.

4. Analogously we obtain B_{m-1}. Let for fixed constants $x < B$, $A(x, B)$ be the maximizer of function (7.36) w.r.t. the variable A. Also, let $B(x)$ maximizes

$$V(x; 0, t_m - t_{m-1}, ...t_n - t_{m-1}; A(x, B), A_m, ..., A_n; B, B_m, ..., B_n)$$

amongst all $B > x$. Thus we approximate the boundary B_{m-1} as the smallest x for which $x = B(x)$.

8.4 Pricing

Once we approximate the optimal boundaries, we can formulate our fast pricing method as:

Fast Pricing Approach 8.1. *The option price can be obtained through formula (8.10) taken at the point $x = S_0$.*

If we need a dense grid, then we view the derivative's evaluation as a boundary value problem (BVP) in the region $(t, x) \in \{(0, T) \times (c(t), d(t))\}$. If the initial point is outside, then the price is $e^{-\lambda t} G(x)$. We can write the BVP as

$$\begin{aligned} &V_t(t, x) + rxV_x(t, x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t, x) - rV(t, x) = 0 \\ &V(t, c(t)) = e^{-\lambda t} G(x), \quad t \in (0, T) \\ &V(t, d(t)) = e^{-\lambda t} G(x), \quad t \in (0, T) \\ &V(T, x) = e^{-\lambda T} G(x), \quad x \in (C, D). \end{aligned} \tag{8.12}$$

We solve BVP (8.12) using the Crank–Nicolson finite difference approach presented in Section 3.5. The time dependence of the price function can be obtained via Proposition 3.3:

Proposition 8.7. *If we mark the dependence on the current time and the maturity, then the following relation holds*

$$V(t, T, x) = e^{-\lambda t} V(0, T - t, x).$$

Remark 8.2. *Note that Proposition 8.7 still holds if one of the boundaries vanishes or if the option is European style.*

8.5 Quadratic options

Before defining and investigating the quadratic options, we shall summarize the above-presented approach.

Algorithm 8.1. *Our approach is based on the following steps.*

1. *We apply the related to the infinitesimal generator differential operator $\mathcal{B}$, defined in formula (8.2), to the payoff $G(x)$. Thus we obtain the function $g(\cdot)$ as $g(x) = (\mathcal{B}G)(x)$.*

2. *We check whether the function $g(x)$ satisfies Condition 8.1 that guarantees that the American pricing task leads to a two-sided optimal stopping problem.*

3. *If the conditions of Condition 8.1 are satisfied, then we obtain the constants $C < D$ such that $g(x) < 0$ for $x \in (0, C) \cup (D, \infty)$ and $g(x) > 0$ for $x \in (C, D)$.*

4. *We set the initial values of the optimal boundary to C and D, i.e. $c(0) = C$ and $d(0) = D$.*

5. *We obtain the boundary values at infinity, $c(\infty)$ and $d(\infty)$, as the solution of the two-dimensional system established in (8.9).*

6. *The perpetual price is obtained via formula (8.3) using $A = c(\infty)$ and $B = d(\infty)$.*

7. *Once we have the endpoints of the functions $c(\cdot)$ and $d(\cdot)$, we approximate the whole boundaries applying the algorithm provided in Section 8.3.4.*

8. *When the optimal boundaries $c(\cdot)$ and $d(\cdot)$ are known, the free boundary differential task that describes the pricing problem turns into boundary value problem (8.12).*

9. *We solve it numerically, using the Crank-Nicolson finite difference approach presented in Section 8.4.*

We turn now to a specific class of American-style derivatives – we name them quadratic strangles. The payoff is defined by the function

$$G(x) = (x-K)^2. \tag{8.13}$$

We shall call the constant K strike and we assume that it is positive. At the end of this section, we briefly discuss what changes if $K \leq 0$. First, we derive the price of the corresponding European option.

Proposition 8.8. *The price of the European-style quadratic option at the initial moment is*

$$P(S_0) = S_0^2 e^{\left(r+\sigma^2-\lambda\right)T} - 2KS_0 e^{-\lambda T} + K^2 e^{-(r+\lambda)T}.$$

Proof. Using the risk-neutral pricing principle, we derive

$$\begin{aligned} P(S_0) &= \mathbb{E}\left[e^{-(r+\lambda)T}(S_T-K)^2\right] \\ &= e^{\left(r+\sigma^2-\lambda\right)T}\mathbb{E}\left[e^{-\left(2r+\sigma^2\right)T}S_T^2\right] - 2Ke^{-\lambda T}\mathbb{E}\left[e^{-rT}S_T\right] + K^2 e^{-(r+\lambda)T}. \end{aligned}$$

We finish the proof having in mind that $e^{-\left(2r+\sigma^2\right)T}S_T^2$ and $e^{-rT}S_T$ are martingales. ☐

The value of the operator $\mathcal{B}$ applied to function (8.13) is

$$(\mathcal{B}G)(x) = x^2\left(r+\sigma^2-\lambda\right) + 2\lambda Kx - (r+\lambda)K^2. \tag{8.14}$$

The discriminant of this quadratic function is positive since it can be written as $K^2\left(r^2+\sigma^2(r+\lambda)\right) > 0$. We have to examine now separately the cases w.r.t. the sign of $\left(r+\sigma^2-\lambda\right)$ or equivalently w.r.t. the position of the discount rate λ to the value of $r+\sigma^2$.

We need the following lemmas before continuing the financial analysis. They are necessary to distinguish what kind of optimal stopping problem describes the option pricing task – one or two-sided. The first one is related to the hit of the Brownian motion to a boundary, whereas the second one leads to an exit from a strip. Next, we examine these cases separately since they lead to different mathematical tasks.

Lemma 8.1. *The constants $\lambda - r - \sigma^2$ and $p - q - 2$ have the same signs (the constants p and q are defined by equations (3.17)). Something more, if $\lambda = r + \sigma^2$, then $q = 2\frac{r}{\sigma^2} + 1$.*

Proof. Suppose first that $\lambda > r + \sigma^2$. Inequality $p - q - 2 > 0$ is equivalent to

$$\sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + 2\frac{\lambda}{\sigma^2}} > \left(\frac{r}{\sigma^2} + \frac{1}{2}\right) + 1. \tag{8.15}$$

Statement (8.15) holds when the right hand-side is negative. If it is positive, then we reach the desired result by rising at the second power.

On the contrary if $r + \sigma^2 \geq \lambda$, then $\frac{r}{\sigma^2} + \frac{3}{2} > 0$ and thus we can raise on the second power.

The second part holds due to the presentation

$$\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r + \lambda}{\sigma^2} = \left(\frac{r}{\sigma^2} + \frac{3}{2}\right)^2$$

when $\lambda = r + \sigma^2$. We finish the proof having in mind $\frac{r}{\sigma^2} + \frac{3}{2} > \frac{r}{\sigma^2} + 1 = \frac{\lambda}{\sigma^2} > 0$. □

Lemma 8.2. *Let $0 < a < 1$, $0 < \epsilon < n$, and $0 < \delta < m$. The following inequality holds*

$$(m + \delta)\left(1 - a^{n-\epsilon}\right) + (m - \delta)\left(1 - a^{n+\epsilon}\right) < 2m\left(1 - a^n\right). \tag{8.16}$$

Proof. Inequality (8.16) is equivalent to

$$\delta a^{n-\epsilon}\left(1 - a^{2\epsilon}\right) + m a^{n-\epsilon}\left(1 - a^{\epsilon}\right)^2 > 0.$$

□

8.5.1 Negative quadratic coefficient

Suppose first that $\lambda > r + \sigma^2$. Condition 8.1 holds, because the vertex of quadratic function (8.14) is at the positive point $\frac{\lambda K}{\lambda - r - \sigma^2}$ and $(\mathcal{B}G)(0) = -(r + \lambda) K^2 < 0$. Under this assumption, we have a two-side optimal stopping problem related to the first exit of the Brownian motion from a strip. First, we shall consider quadratic options without maturity constraints. We transform the two-dimensional system (8.9), that the optimal boundaries solve, to a polynomial-style equation and then we prove the uniqueness of its solution. We can easily solve this equation numerically. As a consequence, we derive the option price. After that, we shall examine finite maturity options adapting the presented in Section 8.3.4 general approach to the quadratic options. This way the free boundary task turns into a boundary value problem in a known region to which we apply the Crank-Nicolson finite difference scheme, presented in Section 8.4.

The constants C and D that determine the initial values of the optimal boundaries are the roots of (8.14), namely

$$\{C, D\} = K\frac{\lambda \mp \sqrt{r^2 + \sigma^2 (r + \lambda)}}{\lambda - r - \sigma^2}. \tag{8.17}$$

Equations (8.8) that determine the perpetual boundaries turn into

$$\begin{aligned} \left(\frac{A}{B}\right)^q &= \frac{B-K}{A-K}\,\frac{(p-q)(B-K)-2B}{(p-q)(A-K)-2A} \\ \left(\frac{A}{B}\right)^{p-q} &= \frac{A-K}{B-K}\,\frac{2A+q(A-K)}{2B+q(B-K)}. \end{aligned} \tag{8.18}$$

We shall use the notations $a = \frac{A}{B}$ and $x = \frac{K}{B}$. Note that $a < x < 1$ and $\frac{x}{a} > 1$. System (8.18) can be rewritten as

$$\begin{aligned} g(x) &:= x^2 (p-q)(1-a^q) - 2x(p-q-1)\left(1-a^{q+1}\right) + (p-q-2)\left(1-a^{q+2}\right) = 0 \\ h\left(\frac{x}{a}\right) &:= \left(\frac{x}{a}\right)^2 q\left(1-a^{p-q}\right) - 2\left(\frac{x}{a}\right)(q+1)\left(1-a^{p-q-1}\right) + (q+2)\left(1-a^{p-q-2}\right) = 0. \end{aligned} \tag{8.19}$$

The following proposition stands.

Proposition 8.9. *System (8.19) has a unique solution such that $\{x, a\} \in (0,1)$ and $\frac{x}{a} > 1$.*

Proof. Let us use the notation $y = \frac{x}{a}$. We consider first the function $g(x)$. We have that $g(0) > 0$ and $g(1) < 0$ due to Lemmas 8.1 and 8.2. Hence, the equation $g(x) = 0$ has two roots such that $0 < x_1 < 1 < x_2$. Analogously, the roots of the equation $h(y) = 0$ satisfy the same order – $0 < y_1 < 1 < y_2$. Marking the dependence on the variable a we need to prove that there exists a unique value $a \in (0,1)$ such that $m(a) = 0$ for $m(a) = x_1(a) - a y_2(a)$. The existence follows from the inequalities

$$m(0) = 1 - \frac{2}{p-q} > 0$$
$$\lim_{a\to 1} m(a) = -2\frac{\sqrt{(p-q-1)^2(q+1)^2 - q(p-q)(p-q-2)(q+2)}}{q(p-q)} < 0.$$

To prove the uniqueness, we rewrite system (8.19) as

$$\begin{aligned} a^{q+2} &= \frac{f(ay)}{f(y)} \\ a^{p-q-2} &= \frac{F(y)}{F(ay)}. \end{aligned} \tag{8.20}$$

for

$$\begin{aligned} f(y) &= y^2(p-q) - 2y(p-q-1) + (p-q-2) \\ F(y) &= y^2 q - 2y(q+1) + q + 2. \end{aligned}$$

Let us consider the function $f(y)$. Its roots are $1 - \frac{2}{p-q}$ and 1. We use a notation w.r.t. the variable y, i.e. the pair $\{a_1(y), y\}$ solves the first equation from (8.20). We need the values $y > 1$ and $ay = x < 1$. Therefore $f(y) > 0$. Hence $f(ya_1(y)) > 0$ too or equivalently $a_1(y)y < 1 - \frac{2}{p-q}$. Hence, $f(y)$ increases w.r.t. y, but $f(ay)$ decreases for a fixed a in the domain $ay < 1$. Therefore $a_1(y)$ decreases. If we consider the second equation from (8.20) w.r.t. y, i.e. $\{a_2(y), y\}$ solves it, we conclude that $a_2(y)$ is an increasing function, because the roots of function $F(y)$ are 1 and $1 + \frac{2}{q}$. Hence, the equation $a_1(y) = a_2(y)$ has no more than one root. This finishes the proof. □

Proposition 8.9 shows that the equation $x_1(a) = ay_2(a)$ has a unique solution. Hence we derive the perpetual optimal boundaries via the following theorem.

Theorem 8.1. *[Theorem 1 of Zaevski (2024c)] Let $\overline{a}$ be the solution of*

$$\frac{(p-q-1)\left(1-a^{q+1}\right)-\sqrt{(p-q-1)^2\left(1-a^{q+1}\right)^2-(p-q)(p-q-2)\left(1-a^{q}\right)\left(1-a^{q+2}\right)}}{(p-q)\left(1-a^{q}\right)}$$
$$=a\frac{(q+1)\left(1-a^{p-q-1}\right)+\sqrt{(q+1)^2\left(1-a^{p-q-1}\right)^2-q(q+2)\left(1-a^{p-q}\right)\left(1-a^{p-q-2}\right)}}{q\left(1-a^{p-q}\right)}$$

in the interval $(0,1)$ *and* $\overline{x}$ *be defined as*

$$\overline{x}=\frac{(p-q-1)\left(1-\overline{a}^{q+1}\right)-\sqrt{(p-q-1)^2\left(1-\overline{a}^{q+1}\right)^2-(p-q)(p-q-2)\left(1-\overline{a}^{q}\right)\left(1-\overline{a}^{q+2}\right)}}{(p-q)\left(1-\overline{a}^{q}\right)}$$

The optimal boundaries of a perpetual quadratic strangle are $\overline{A}=\frac{\overline{a}}{\overline{x}}K$ *and* $\overline{B}=\frac{K}{\overline{x}}$. *The derivative price is*

$$f\left(\overline{A},\overline{B};x\right)=\left(x-\overline{A}\right)^2\left(\frac{\overline{A}}{x}\right)^q\frac{\overline{B}^p-x^p}{\overline{B}^p-\overline{A}^p}+\left(x-\overline{B}\right)^2\left(\frac{\overline{B}}{x}\right)^q\frac{x^p-\overline{A}^p}{\overline{B}^p-\overline{A}^p}.$$

Suppose now that the maturity is finite. We shall use the algorithm presented in Section 8.3.4. To do this, we need to derive the expectation in formula (8.10). We have

$$\begin{aligned}
&V(x,T;c(t),d(t))=\mathbb{E}^x\left[e^{-(r+\lambda)\zeta^A}\left(S_{\zeta^A}-K\right)^2 I_{\zeta^A=\zeta,\zeta<T}\right]\\
&+\mathbb{E}^x\left[e^{-(r+\lambda)\zeta^B}\left(S_{\zeta^B}-K\right)^2 I_{\zeta^B=\zeta,\zeta<T}\right]+\mathbb{E}^x\left[e^{-(r+\lambda)T}\left(S_T-K\right)^2 I_{T\le\zeta}\right]\\
&=K^2\sum_{i=1}^n\left(\mathbb{E}\left[e^{-(r+\lambda)\zeta^A}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^A}\right]+\mathbb{E}\left[e^{-(r+\lambda)\zeta^B}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^B}\right]\right)\\
&-2Kx\sum_{i=1}^n\left(e^{\sigma a_{2,i}}\mathbb{E}\left[e^{-\psi_{1,i}\zeta^A}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^A}\right]+e^{\sigma b_{2,i}}\mathbb{E}\left[e^{-\psi_{2,i}\zeta^B}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^B}\right]\right)\\
&+x^2\sum_{i=1}^n\left(e^{2\sigma a_{2,i}}\mathbb{E}\left[e^{-\eta_{1,i}\zeta^A}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^A}\right]+e^{2\sigma b_{2,i}}\mathbb{E}\left[e^{-\eta_{2,i}\zeta^B}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta^B}\right]\right)\\
&+K^2e^{-(r+\lambda)T}\mathbb{Q}\left(v_1<B_T<v_2,T\le\zeta\right)-2Kxe^{-\psi_3T}\mathbb{E}\left[e^{\sigma B_T}I_{v_1<B_T<v_2,T\le\zeta}\right]\\
&+x^2e^{-\psi_4T}\mathbb{E}\left[e^{2\sigma B_T}I_{v_1<B_T<v_2,T\le\zeta}\right],
\end{aligned}\tag{8.21}$$

where

$$\begin{aligned}
\psi_{1,i} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - \sigma a_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma a_{1,i} \\
\psi_{2,i} &= (r+\lambda) - \left(r - \frac{\sigma^2}{2}\right) - \sigma b_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma b_{1,i} \\
\eta_{1,i} &= (r+\lambda) - 2\left(\left(r - \frac{\sigma^2}{2}\right) - \sigma a_{1,i}\right) = \lambda + \sigma^2 - r - 2\sigma a_{1,i} \\
\eta_{2,i} &= (r+\lambda) - 2\left(\left(r - \frac{\sigma^2}{2}\right) - \sigma b_{1,i}\right) = \lambda + \sigma^2 - r - 2\sigma b_{1,i} \\
\psi_3 &= \lambda + \frac{\sigma^2}{2} \\
\psi_4 &= \lambda + \sigma^2 - r \\
v_1 &= \frac{1}{\sigma}\ln\left(\frac{C}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T \\
v_2 &= \frac{1}{\sigma}\ln\left(\frac{D}{x}\right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)T.
\end{aligned} \tag{8.22}$$

The expectations in statement (8.21) can be derived through the results of Section 2.4. Note that for some values of the parameters we need to use the analytic continuation of the erf-function – see for example Abramowitz and Stegun (2006) or Abramowitz and Stegun (1979). Based on formulas (8.21) and (8.22), we approximate the optimal boundaries using the algorithm established in Section 8.3.4. Once we have these approximations, we can use the following fast method for option pricing:

Fast Pricing Approach 8.2. *The option price can be obtained through formulas* (8.21) *and* (8.22) *taken at the point* $x = S_0$.

If we need a dense gird, then we apply the Crank-Nicolson finite difference approach presented in Section 8.4 to evaluate the derivative.

8.5.2 Non-negative quadratic coefficient

Suppose now that $\lambda \leq r + \sigma^2$. Let the constant C be defined as

$$C = \tfrac{r+\lambda}{2\lambda}K \qquad if\ \lambda = r+\sigma^2$$
$$C = \tfrac{\sqrt{r^2+\sigma^2(r+\lambda)}-\lambda}{r+\sigma^2-\lambda}K \qquad if\ \lambda < r+\sigma^2.$$

We have that $(\mathcal{B}G)(x) < 0$ for $x < C$ and $(\mathcal{B}G)(x) > 0$ when $x > C$, where $(\mathcal{B}G)$ is given by equation (8.14). Hence, we have a one-sided optimal stopping problem. It is put-style in the sense that the optimal points are below the exercise boundary. We shall use a method established in Zaevski (2024b) to examine such kind of financial instruments. First, we shall consider perpetual options deriving the optimal boundary and, as a consequence, the fair option price. After that, we shall approximate the optimal boundary for the finite maturity options and shall apply the presented in Section 8.4 finite-difference approach to the pricing task.

Propositions 6.4 and 6.5 show that the optimal boundary starts from the value C and decreases to its perpetual level. We shall derive it now. Suppose that $S_0 = x$ is a large enough initial point for the underlying asset. Suppose also that the holder exercises when the underlying asset reaches the value $A < x$. This stopping time, we denote it by ζ^A, is the first hitting moment of a Brownian motion with drift $\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$ to the value

$$\tilde{A} = \frac{\ln A - \ln x}{\sigma} < 0.$$

Using Proposition 3.5, we derive for the financial result of this strategy

$$\begin{aligned} V(x;A) &= \mathbb{E}^x\left[e^{-(r+\lambda)\zeta^c}\left(S_{\zeta^A}-K\right)^2 I_{\zeta^A<\infty}\right] + \lim_{T\to\infty}\mathbb{E}^x\left[e^{-(r+\lambda)T}\left(S_T-K\right)^2 I_{T<\zeta^A}\right] \\ &= (A-K)^2\,\mathbb{E}^x\left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A<\infty}\right] + \lim_{T\to\infty} e^{-(r+\lambda)T}\mathbb{E}^x\left[\left(S_T-K\right)^2 I_{T<\zeta^A}\right] \\ &= (A-K)^2\left(\frac{A}{x}\right)^q + K^2\lim_{T\to\infty} e^{-(r+\lambda)T}\mathbb{Q}\left(T<\zeta^A\right) \qquad (8.25) \\ &- 2Kx\lim_{T\to\infty} e^{-\left(\lambda+\frac{\sigma^2}{2}\right)T}\mathbb{E}^x\left[e^{\sigma B_T} I_{T<\zeta^A}\right] + x^2\lim_{T\to\infty} e^{-\left(\lambda+\sigma^2-r\right)T}\mathbb{E}^x\left[e^{2\sigma B_T} I_{T<\zeta^A}\right]. \end{aligned}$$

Using Theorem 2.5, we derive

$$\mathbb{E}^x\left[e^{\theta B_T} I_{T<\zeta}\right] = \exp\left(\frac{T\theta^2}{2}\right)\left[\begin{array}{l} 1 - N\left(\frac{-\psi T+\tilde{A}-T\theta}{\sqrt{T}}\right) \\ -e^{2\tilde{A}(\theta+\psi)}\left(1-N\left(\frac{-\psi T+\tilde{A}-T\theta-2\tilde{A}}{\sqrt{T}}\right)\right) \end{array}\right].$$

The first limit in price function (8.25) is zero. The second one is zero too when $\lambda > 0$ and finite when $\lambda = 0$. Also, the third limit is the infinity when $\lambda < r + \sigma^2$ and therefore $P(x; A) = \infty$. This means that optimal exercising is never optimal and hence $c(\infty) = 0$.

Suppose now that $\lambda = r + \sigma^2$. Using Lemma 8.1 we see that the third limit is

$$\left(1 - \left(\frac{A}{x}\right)^{q+2}\right).$$

Thus price function (8.25) turns into

$$\begin{aligned} V(x; A) &= (A - K)^2 \left(\frac{A}{x}\right)^q + x^2 \left(1 - \left(\frac{A}{x}\right)^{q+2}\right) \\ &= \frac{K^2 A^q - 2KA^{q+1}}{x^q} + x^2. \end{aligned} \tag{8.26}$$

Some calculations show that function (8.26) achieves its maximum for

$$\overline{A} = \frac{q}{2(q+1)} K. \tag{8.27}$$

We formulate these results in the following theorem.

Theorem 8.2. *[Theorem 2 of Zaevski (2024c)] If $\lambda < r + \sigma^2$, then the early exercise is never optimal for a perpetual quadratic strangle. Its price is infinitely large.*

If $\lambda = r + \sigma^2$, then all points below $\overline{A}$, given by formula (8.27), are optimal. The price is $(x - K)^2$ when the initial asset value $S_0 = x$ is below $\overline{A}$ and it is given by equation (8.26) for $A = \overline{A}$, otherwise.

Next, we discuss briefly an approach for approximating the optimal boundary when the maturity is finite – for more details see Chapter 6. Note that the early exercise can be optimal even if $\lambda < r + \sigma^2$ under the finite maturity horizon. We shall use again a similar but one-sided approximation to those presented in Section 8.5.1. Pricing formula (8.21) now turns into

$$
\begin{aligned}
V(x,T;c(t)) &= \mathbb{E}^x\left[e^{-(r+\lambda)\zeta^A}\left(S_{\zeta^A}-K\right)^2 I_{\zeta^A<T}\right]+\mathbb{E}^x\left[e^{-(r+\lambda)T}(S_T-K)^2 I_{T\le\zeta^A}\right] \\
&= K^2\sum_{i=1}^n \mathbb{E}\left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A\in(t_{i-1},t_i]}\right]-2Kx\sum_{i=1}^n e^{\sigma a_{2,i}}\mathbb{E}\left[e^{-\psi_{1,i}\zeta^A} I_{\zeta^A\in(t_{i-1},t_i]}\right] \\
&+ x^2\sum_{i=1}^n e^{2\sigma a_{2,i}}\mathbb{E}\left[e^{-\eta_{1,i}\zeta^A} I_{\zeta^A\in(t_{i-1},t_i]}\right]+K^2 e^{-(r+\lambda)T}\mathbb{Q}\left(v_1<B_T,T\le\zeta^A\right) \\
&- 2Kxe^{-\psi_3 T}\mathbb{E}\left[e^{\sigma B_T} I_{v_1<B_T,T\le\zeta^A}\right]+x^2 e^{-\psi_4 T}\mathbb{E}\left[e^{2\sigma B_T} I_{v_1<B_T,T\le\zeta^A}\right],
\end{aligned}
\tag{8.28}
$$

where the constants $\psi_{1,i}$, $\eta_{1,i}$, v_1, ψ_3, and ψ_4 are given again in (8.22). We derive the expectations in formula (8.28) using the results of Zaevski (2020a). The algorithm for approximating the boundary is again backward. The initial value is given by formula (8.23) or (8.24). Let us consider the price (8.28) as a function of the initial asset price and the boundary level at the current moment, namely $F(x,A)$. We denote by $A(x)$ the maximizer of the function $F(x,A)$ for a fixed x. Our approximation is the largest value of x for which $x = A(x)$, i.e. the largest initial value for which the immediate exercise is the optimal strategy.

Once we approximate the optimal boundary, we achieve our fast pricing method:

Fast Pricing Approach 8.3. *The option price can be obtained through formula* (8.28) *taken at the point* $x = S_0$.

If we need a denser grid, then we use again the Crank-Nicolson finite difference scheme to price the quadratic strangle. We need a little modification to the method presented in Section 8.4 since the continuation region is open above. We introduce a large enough axillary upper boundary at which we approximate the function as the price of the European-style option. To do this we use Proposition 8.8 together with Proposition 8.7. Some numerical tests show that an appropriate level for this axillary boundary is $10K$.

8.5.3 Negative strikes

Suppose now that $K \le 0$. If $\lambda < r + \sigma^2$, then function (8.14) has two roots, say $x_1 < x_2$, such that $(\mathcal{B}G)(x) < 0$ for $x \in (0, x_2)$ and $(\mathcal{B}G)(x) > 0$ when $x \in (x_2, \infty)$. Hence, we have a one-sided problem, and thus the results presented in Section 8.5.2 still hold.

If $\lambda > r + \sigma^2$, then function (8.14) has two negative roots. Also, if $\lambda = r + \sigma^2$, then function (8.14) has a unique, also negative, root. In both cases $(\mathcal{B}G)(x) < 0$ for all $x > 0$. Hence, the immediate exercise is ever optimal.

8.6 Numerical results

We provide further some numerical results related to the above-defined quadratic strangles. Let the risk-free rate be $r = -0.03$, the volatility be $\sigma = 0.3$, and the strike be $K = 5$. The critical value for the additional discount factor that determines the type of the quadratic option is $\overline{\lambda} = r + \sigma^2 = 0.06$. This way, we have a two-sided optimal stopping problem for $\lambda > 0.06$ and a one-sided task for $\lambda \leq 0.06$.

Let us first consider $\lambda < 0.06$. We present in Figure 8.2a the behavior of the optimal boundary w.r.t. the initial time. We use the discount rate $\lambda = 0.05$. We can see that for large maturities the optimal boundary tends to zero. This confirms the result derived in Section 8.5.2 – the early exercise is never optimal for the perpetual options when $\lambda < \overline{\lambda}$. This corresponds to $c(\infty) = 0$. The value at the maturity can be derived through formula (8.24) and it is $c(0) = 0.9808$ – we mark it by a green circle. The behavior of the optimal boundary w.r.t. the discount factor λ and the time to maturity τ is presented in Figure 8.2d.

If the discount factor takes its critical value, $\lambda = \overline{\lambda} = 0.06$, then we have again a one-sided optimal hitting problem. The main difference is that early exercising can be optimal even for the perpetual derivatives. The optimal boundary is presented in Figure 8.2f. The perpetual value is marked by a red circle and it is $c(\infty) = 0.6250$ due to equation (8.27). We can see that the optimal boundary tends namely to this level. Also, the maturity value is obtained via formula (8.23) and it is $c(0) = 1.2500$ – we mark it again by a green circle.

If $\lambda > \overline{\lambda}$, then we have a two-sided optimal hitting problem. Both optimal boundaries for $\lambda = 0.1$ can be seen in Figure 8.2c – the lower boundary is presented by a blue color whereas the upper one is in red. The parametrization is w.r.t. the initial time. The perpetual values are obtained via Theorem 8.1 and they are $c(\infty) = 1.1576$ and $d(\infty) = 35.4366$ – we mark them by circles (blue or red). The values at the maturity are obtained through statement (8.17) and they are $c(0) = 1.8934$ and $d(0) = 23.1066$ – we present them by green circles. The behavior of both boundaries w.r.t. the discount factor λ and the time to maturity τ can be seen in Figures 8.2d and 8.2e – the first for the lower boundary and the second for the upper one. The initial points are marked again by the green color and they are obtained via formulas (8.17). The perpetual boundaries are plotted in red – they are derived through Theorem 8.1. We can see that both boundaries tend to their perpetual levels when the time to maturity tends to infinity.

The price behavior w.r.t. the time to maturity τ and the discount factor λ can be seen in Figure 8.2. Particularly, in Figures 8.3a and 8.3b are presented the prices when the discount factor is less than the critical value, $\lambda < \overline{\lambda}$. The maturities are relatively short for Figure 8.3a, $\tau \in [0, 3]$, and long for Figure 8.3b, $\tau \in [0, 1000]$. The initial asset price is assumed to be $S_0 = 2$. Recall that we have a one-sided hitting problem. We can see that the price tends to infinity when $\tau \to \infty$ – this confirms the first result of Theorem 8.2. We can also observe that the lower the discount factor, the faster this convergence.

We consider discount rates larger than the critical value, $\lambda > \overline{\lambda}$, for Figures 8.3c (short maturities) and 8.3d (long maturities). We have now an exit problem from a strip. The initial value of the underlying asset is assumed to be $S_0 = 5$, i.e. at-the-money. We mark by red points the perpetual prices calculated through Theorem 8.1. We can see that for a fixed discount factor, the price increases from zero (since the quadratic strangle is at-the-money) to the perpetual value.

Next, we consider the critical discount factor, i.e. $\lambda = \overline{\lambda} = 0.06$. We have again a one-sided hitting problem. Differently to the case $\lambda < \overline{\lambda}$, Theorem 8.2 shows that the perpetual price is finite and it is given by formula (8.26) for $\overline{A} = c(\infty) = 0.6250$ derived via (8.27). The current parameters lead to a price 16.7238.

We provide a comparison between the optimal boundaries of the quadratic strangles and the usual straddles. We consider two values for the risk-free rate – one positive and one negative. Suppose first that $r = -0.03$. As we mentioned above, the critical value for the discount factor at which the quadratic coefficient in (8.14) changes its sign, is $\overline{\lambda} = r + \sigma^2 = 0.06$. We have a two-sided optimal stopping problem when $\lambda > \overline{\lambda}$ – we consider the discount rate to be amongst $\lambda \in \{0.07, 0.08, 0.09, 0.1\}$. The optimal boundaries are presented in Figures 8.4a and 8.4b – the lower ones in 8.4a and the upper ones in 8.4b. We can see that the lower boundary of a quadratic strangle is below the corresponding boundary of the related straddle. The opposite is true for the upper boundaries – the boundary of the quadratic strangle is above the related straddle one. We can see also that for both style options, as small is the discount factor, as large is the lower optimal boundary, and as small is the upper one. Note that in the asymptotic case $\lambda \to \infty$ both boundaries tend to the strike and thus the optimal set turns into the singleton $\{K\}$. Another observation we made is that the upper optimal boundary for the quadratic strangle increases relatively faster w.r.t. λ than the lower one decreases. Something more, the upper boundary tends to infinity for discount rates near the critical value $\overline{\lambda}$. This is by the fact that when $\lambda \leq \overline{\lambda}$, the quadratic strangle leads to a one-sided first hitting task. Note that it is put-style in the sense that the optimal points are below the optimal boundary. Thus the upper optimal boundary for the quadratic strangles vanishes. Next, we examine namely such tasks considering values $\lambda \in \{0.04, 0.05, 0.06\}$. Note that we need $\lambda > 0.03$ since $r + \lambda > 0$. We present in Figure 8.4e the lower boundary of the usual straddles as well as the unique quadratic one. We have to mention that it tends to zero for large maturities when $\lambda < \overline{\lambda}$. On the contrary, its limit is positive when $\lambda = \overline{\lambda}$. This is in accordance with Theorem 8.2. In addition, we present the optimal boundary of the related ordinary American put. We can observe that the boundaries of the ordinary put and the straddle are relatively near. However, the put boundary is above the straddle one due to the right of the straddle to be exercised as a call. On the contrary, the boundary of the quadratic option is significantly below the put and straddle ones. This means that the quadratic option is the most expensive, the usual American is the cheapest, and the straddle is between them.

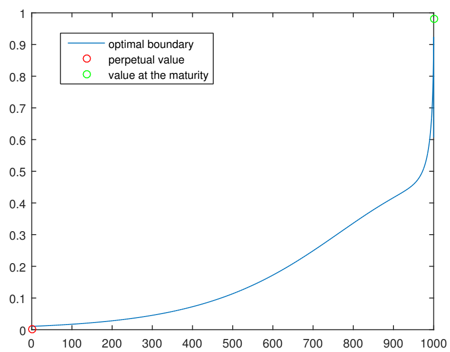
Let us consider now a positive value for the risk-free rate, $r = 0.03$. Thus the critical value for the discount factor λ is $\overline{\lambda} = r + \sigma^2 = 0.12$. We examine again separately values above and below it. Suppose that λ is amongst $\lambda \in \{0.13, 0.15, 0.2, 0.3\}$. Both boundaries are presented in Figures 8.4c and 8.4d. All conclusions made for the case $r = -0.03$ are valid for the positive interest rate too. Particularly, the upper boundary vanishes for $\lambda \leq \overline{\lambda}$. Thus we consider $\lambda \in \{0.01, 0.06, 0.12\}$. Note that when $r > 0$, the restriction for the additional discount factor is only $\lambda \geq 0$. In Figure 8.4f, we present the lower boundaries of the ordinary straddles, the unique ones of the quadratic strangles, and the boundaries for the usual American puts. We have to mention that the last one is the strike at maturity when $r \geq 0$, and it is below the strike for negative short rates. All conclusions made for the negative risk-free values are valid again. In addition, we can observe that the optimal quadratic boundaries converge faster to their perpetual values –

see again Theorem 8.2.

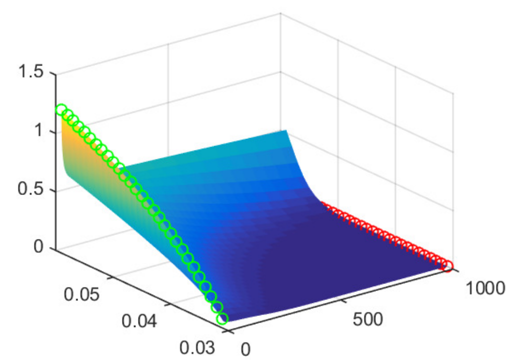
Some particular quadratic strangle prices are given in Table 8.1. The risk-free rate is varied amongst $r \in \{-0.04,\ -0.03,\ -0.02,\ -0.01, 0, 0.01\}$, the additional discount factor amongst $\lambda \in \{0.05,\ 0.06,\ 0.07,\ 0.08, 0.09, 0.1\}$, and the initial asset price amongst $S_0 \in \{\$1,\ \$2,\ \$3,\ \$4\}$. The time to maturity is assumed to be one year. The first part of the table presents the critical values for the additional discount factor, namely $\overline{\lambda}$, as well as the optimal boundaries. If we have a one-sided problem, equivalently to $\lambda \leq \overline{\lambda}$, then only one boundary (the lower one) exists. Otherwise, both boundaries are displayed. The rest of the table is devoted to the prices themselves. We approximate the boundaries at $\overline{M} = 10$ points using a two-step algorithm presented in Section 8.3.4 or its modification for one-sided problems given at the end of Section 8.5.2. We use $M = 500$ time- and $N = 400$ space-nodes for the finite difference grid.

Figures and Tables

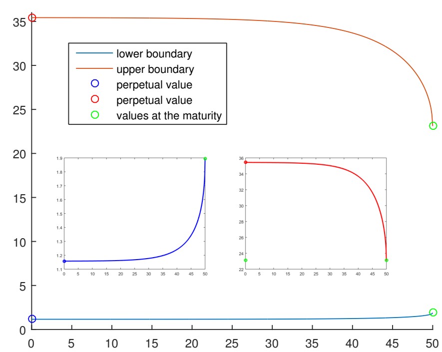
Figure 8.1: Exercise boundaries



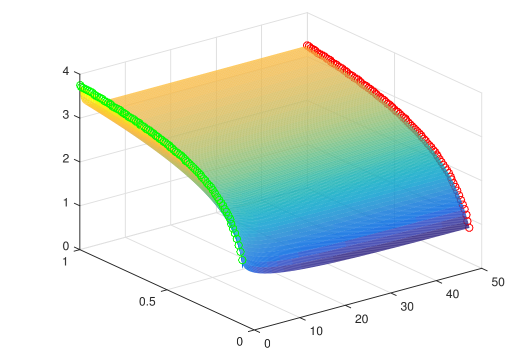
(a) Boundary $\lambda = 0.05$



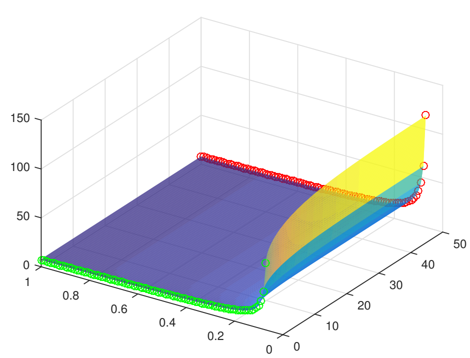
(b) Behavior of the boundary, $\lambda < 0.06$



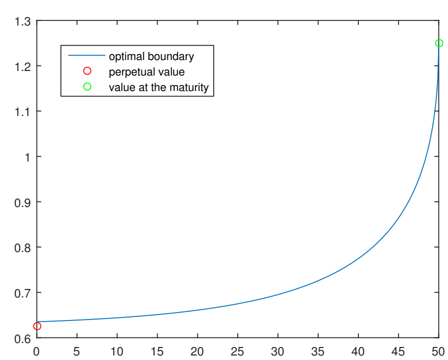
(c) Boundary, $\lambda = 0.1$



(d) Behavior of the lower boundary, $\lambda > 0.06$

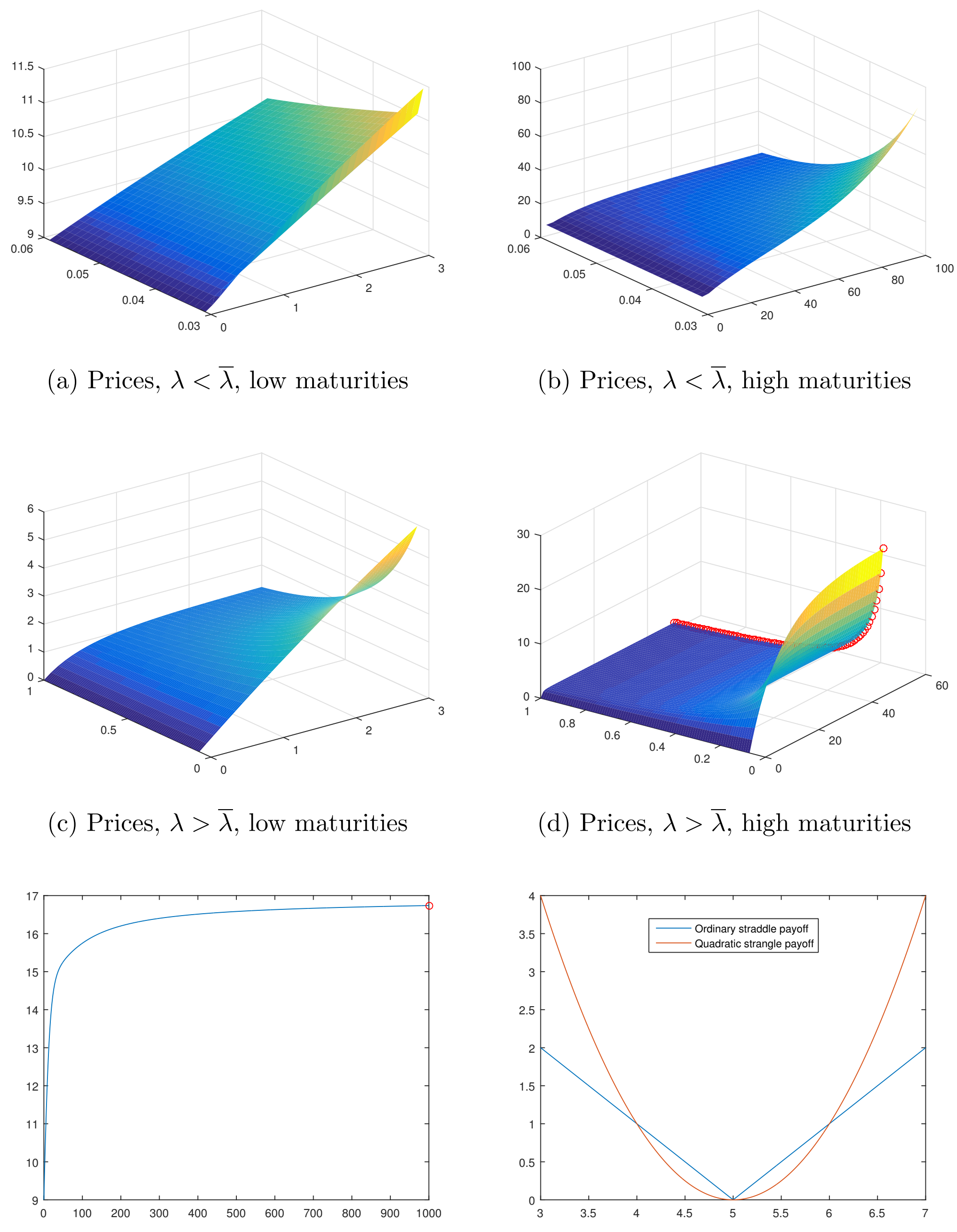


(e) Behavior of the upper boundary, $\lambda > 0.06$



(f) Critical value, $\lambda = 0.06$

Figure 8.2: Price behavior



(a) Prices, $\lambda < \overline{\lambda}$, low maturities

(b) Prices, $\lambda < \overline{\lambda}$, high maturities

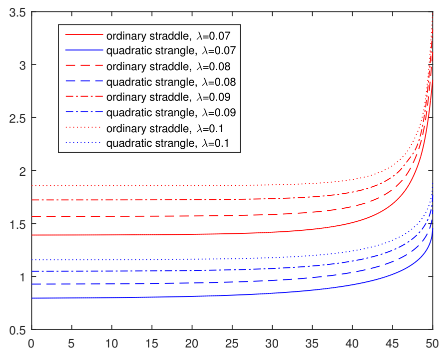
(c) Prices, $\lambda > \overline{\lambda}$, low maturities

(d) Prices, $\lambda > \overline{\lambda}$, high maturities

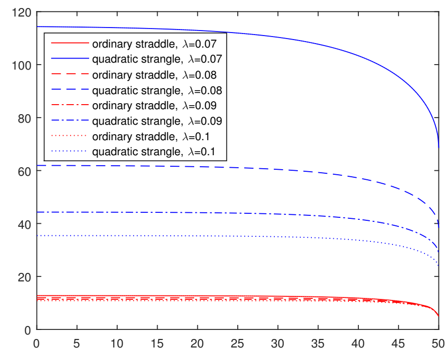
(e) Prices, $\lambda = \overline{\lambda}$

(f) Payoffs

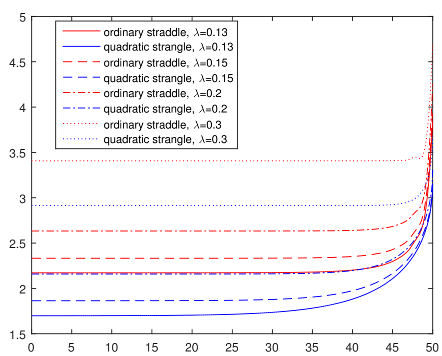
Figure 8.3: The lower boundary of the usual American straddle vs. quadratic strangle, $\sigma = 0.3$, $K = 5$



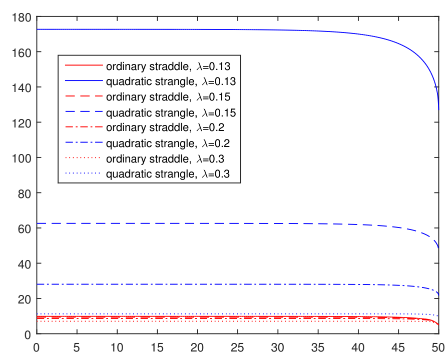
(a) Lower boundaries, $r = -0.03$



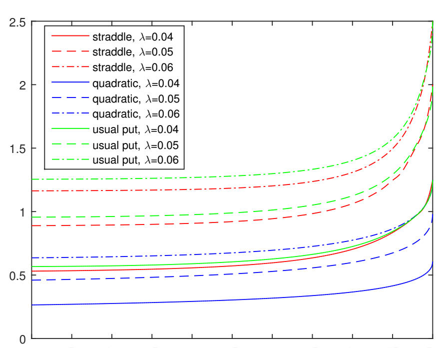
(b) Upper boundaries, $r = -0.03$



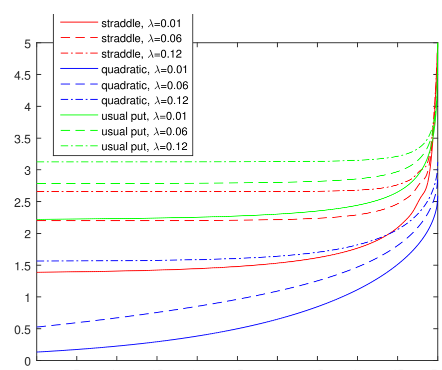
(c) Lower boundaries, $r = 0.03$



(d) Upper boundaries, $r = 0.03$



(e) Lower boundaries, $r = -0.03$



(f) Lower boundaries, $r = 0.03$

Table 8.1: Prices

boundaries	$r=-0.04$	$r=-0.03$	$r=-0.02$	$r=-0.01$	$r=0$	$r=0.01$
$\overline{\lambda}$	0.05	0.06	0.07	0.08	0.09	0.1
$\lambda=0.05$	0.4181	0.8175	1.1790	1.4933	1.7601	1.9841
$\lambda=0.06$	0.7072; 70.5159	1.0428	1.3493	1.6204	1.8551	2.0559
$\lambda=0.07$	0.9265; 40.3703	1.2182; 81.4564	1.4863	1.7261	1.9366	2.1191
$\lambda=0.08$	1.1020; 30.1705	1.3613; 45.5616	1.6007; 92.5220	1.8165	2.0079	2.1756
$\lambda=0.09$	1.2476; 24.9852	1.4817; 33.4892	1.6986; 50.8648	1.8953; 103.6539	2.0712	2.2266
$\lambda=0.1$	1.3715; 21.8199	1.5854; 27.3888	1.7841; 36.9010	1.9653; 56.2409	2.1281; 114.8156	2.2730
$S_0=1$	$r=-0.04$	$r=-0.03$	$r=-0.02$	$r=-0.01$	$r=0$	$r=0.01$
$\lambda=0.05$	16.2544	16.0533	16.0000	16.0000	16.0000	16.0000
$\lambda=0.06$	16.0933	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda=0.07$	16.0081	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda=0.08$	16.0000	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda=0.09$	16.0000	16.0000	16.0000	16.0000	16.0000	16.0000
$\lambda=0.1$	16.0000	16.0000	16.0000	16.0000	16.0000	16.0000
$S_0=2$	$r=-0.04$	$r=-0.03$	$r=-0.02$	$r=-0.01$	$r=0$	$r=0.01$
$\lambda=0.05$	9.7559	9.5432	9.3402	9.1639	9.0457	9.0005
$\lambda=0.06$	9.6353	9.4518	9.2609	9.1095	9.0210	9.0000
$\lambda=0.07$	9.5377	9.3451	9.1925	9.0680	9.0062	9.0000
$\lambda=0.08$	9.4453	9.2650	9.1218	9.0377	9.0000	9.0000
$\lambda=0.09$	9.3596	9.1966	9.0764	9.0106	9.0000	9.0000
$\lambda=0.1$	9.2825	9.1394	9.0429	9.0012	9.0000	9.0000
$S_0=3$	$r=-0.04$	$r=-0.03$	$r=-0.02$	$r=-0.01$	$r=0$	$r=0.01$
$\lambda=0.05$	5.2581	5.0915	4.9284	4.7704	4.6205	4.4833
$\lambda=0.06$	5.1706	5.0409	4.8802	4.7257	4.5809	4.4495
$\lambda=0.07$	5.1156	4.9655	4.8334	4.6829	4.5434	4.4179
$\lambda=0.08$	5.0636	4.9136	4.7704	4.6421	4.5080	4.3882
$\lambda=0.09$	5.0129	4.8650	4.7241	4.5931	4.4746	4.3603
$\lambda=0.1$	4.9634	4.8183	4.6813	4.5543	4.4406	4.3341
$S_0=4$	$r=-0.04$	$r=-0.03$	$r=-0.02$	$r=-0.01$	$r=0$	$r=0.01$
$\lambda=0.05$	2.7604	2.6605	2.5645	2.4727	2.3854	2.3030
$\lambda=0.06$	2.6860	2.6339	2.5390	2.4484	2.3623	2.2814
$\lambda=0.07$	2.6546	2.5740	2.5139	2.4244	2.3397	2.2603
$\lambda=0.08$	2.6266	2.5443	2.4675	2.4009	2.3175	2.2397
$\lambda=0.09$	2.5997	2.5177	2.4400	2.3670	2.2959	2.2195
$\lambda=0.1$	2.5734	2.4921	2.4151	2.3424	2.2740	2.1998

Chapter 9

Cancellable call options under perpetual assumptions.

based on the paper

Zaevski, Tsvetelin S. "Discounted perpetual game call options." Chaos, Solitons & Fractals 131 (2020): 109503.

Abstract: The purpose of this chapter is to examine the pricing task for cancellable call options discounted by an additional term and without maturity restrictions. In addition to the features of the American options, the game options give the writer the right to cancel the contract prematurely. As compensation, he has to pay some amount above the usual payoff. We assume that this penalty is a constant during the option life. We examine such derivatives without maturity constraints – the exercise can be made in every future moment. We first obtain the optimal exercise regions for the holder and the writer and then derive the fair option price. Our approach is based on some American-style derivatives with a stochastic maturity date.

9.1 Motivation and main results

The cancellable American options, also known as game or Israeli options, are introduced by Kifer (2000). They are a particular case of the so-called Dynkin games, see Dynkin (1969) or Dynkin (1969). These instruments arise as a natural extension of the American options whose main feature is the holder's right to exercise at every moment before maturity. In addition, the game options provide such rights for the writer. Its price is some amount above the usual payoff that has to be paid when the contract is canceled. We assume this penalty is a constant during the option life – we shall denote it by η. Thus both the holder and the writer have the right to stop the contract at every moment before the maturity.

In this chapter, we examine the perpetual variants of these instruments, i.e. contracts without maturity constraints. We shall denote by $N_1(t,x)$ and $N_2(t,x)$ the holder's and writer's payoff structures. Thus, the holder receives the amount of $N_1(t,x)$, if he exercises the option at the moment t at the spot price $S_t = x$. On the contrary, the amount is $N_2(t,x)$ if the writer cancels. For a game call option with strike price K, the payoffs are $N_1(t,x) = e^{-\lambda t}(x-K)^+$ and $N_2(t,x) = e^{-\lambda t}\left[(x-K)^+ + \eta\right]$, respectively.

The time-price space $\mathbb{R}^+ \times \mathbb{R}^+$ that consists of all possible states (t, S_t) can be divided into three parts – (A) the holder's exercise region, (B) the writer's canceling region, and (C) the continuation region. If the asset price falls in (A) or (B), then it is optimal for the corresponding participant to exercise immediately. If the price is in the continuation region, then keeping the option is preferable for both the writer and the holder. The boundaries between the regions are called optimal exercise boundaries. Since the asset price is driven by a Markov process, its future behavior depends only on the current value. The form of the payment structures leads to $N_1(t+s,x) = e^{-\lambda s}N_1(t,x)$ and $N_2(t+s,x) = e^{-\lambda s}N_2(t,x)$. Also, both participants are not threatened by a forced exchange since there are no maturity restrictions. All this means that the optimal strategies of the holder and writer depend on the current asset value, but are time-independent. Hence, the boundaries are constants and we can examine only the case $t = 0$.

The shape of the exercise regions and the corresponding boundaries varies significantly for different parameter values. We examine in detail all of the possible cases. The holder's exercise region has the $[B, \infty)$ for some constant B above the strike. In the undiscounted case, $\lambda = 0$, we prove a proposition with an analogue for the usual American options – the holder's exercise region is empty, i.e. $B = \infty$. We prove also that the writer's exercise region is $[K, \infty)$, whereas the continuation one is $(0, K)$. Otherwise, if $\lambda > 0$, then $B < \infty$. There are three possible cases for the writer's exercise region – $[K, A]$, $\{K\}$, and the empty set. In all cases, the points below the strike belong to the continuation region. In addition, if the constant A exists, then the interval (A, B) is optimal too.

The approach we use for deriving the optimal boundaries is based on a specific kind of American-style financial instruments with stochastic maturities. For a fixed stopping time ζ, we define a derivative that obligates the writer to pay an amount of $N_1(\zeta, S_\zeta)$ when the stopping time happens. The writer has the right to cancel the derivative paying a larger amount define by the function $N_2(t, x)$. We shall denote by $A(\zeta)$ its optimal strategy. Analogously, we define another financial derivative that obligates the writer to pay the amount of $N_2(\zeta, S_\zeta)$ when the stopping time occurs. Also, the holder has the right to exercise, receiving the amount of $N_1(t, S_t)$. We denote by $B(\zeta)$ its optimal strategy. We look for a stopping time ζ that satisfies $A(B(\zeta)) = \zeta$. Thus, the optimal strategies for the writer and holder will be ζ and $B(\zeta)$, respectively. Note that most of the derived results for the exercise regions are true when the maturity is finite too. Once we obtain the exercise regions, we can derive the option price using the first hitting properties of the Brownian motion presented in Chapter 2.

The chapter is organized as follows. The base we use later is provided in Section 9.2. In Section 9.3 we derive the exercise regions. The option pricing task is solved in Section 9.4. We discuss the undiscounted case in Section 9.5. Some numerical examples are provided in Section 9.6. The uniqueness of the solutions of the arising equations is proven in Appendices 9.A and 9.B.

9.2 Preliminaries

We shall work again in the framework of Section 3.3. Assume that if the holder exercises the option at the spot price $S_t = x$, then he receives the amount of $N_1(t, x)$, whereas if the writer cancels first, he has to pay the amount of $N_2(t, x)$.[1] It is natural to assume $N_1(t, x) < N_2(t, x)$. Let ζ_1 and ζ_2 be the holder's and writer's strategies (stopping times) and $\zeta = \zeta_1 \wedge \zeta_2$ be the end of the option life. Hence, the payment at this moment can be written as

$$N(\zeta, S_\zeta) = N_1(\zeta_1, S_{\zeta_1}) I_{\zeta_1 \leq \zeta_2} + N_2(\zeta_2, S_{\zeta_2}) I_{\zeta_2 < \zeta_1}. \tag{9.1}$$

Kifer (2013) examines separately $\zeta_1 = \zeta_2$ and another payoff $N_3(t, x)$ that

[1] Sawaki et al. (2012) introduce an additional term that can be treated as a coupon payment.

differs from $N_1(t,x)$ and $N_2(t,x)$. The necessary requirement is

$$N_1(t,x) \leq N_3(t,x) \leq N_2(t,x). \tag{9.2}$$

Thus, equation (9.1) can be written as

$$N(\zeta, S_\zeta) = N_1(\zeta_1, S_{\zeta_1}) I_{\zeta_1<\zeta_2} + N_2(\zeta_2, S_{\zeta_2}) I_{\zeta_2<\zeta_1} + N_3(\zeta, S_\zeta) I_{\zeta_1=\zeta_2}.$$

We have to impose another requirement

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left(|N_1(t,S_t)| + |N_2(t,S_t)| + |N_3(t,S_t)|\right)\right] < \infty. \tag{9.3}$$

With $\zeta = \zeta_1 \wedge \zeta_2$, let us define

$$\overline{Y}_t = \underset{\zeta_2\in\mathcal{T}_{[t,T]}}{ess\inf}\ \underset{\zeta_1\in\mathcal{T}_{[t,T]}}{ess\sup}\ \mathbb{E}\left[e^{-r(\zeta-t)}N(\zeta,S_\zeta)\middle|\mathcal{F}_t\right]$$
$$\underline{Y}_t = \underset{\zeta_1\in\mathcal{T}_{[t,T]}}{ess\sup}\ \underset{\zeta_2\in\mathcal{T}_{[t,T]}}{ess\inf}\ \mathbb{E}\left[e^{-r(\zeta-t)}N(\zeta,S_\zeta)\middle|\mathcal{F}_t\right].$$

The following theorem is proven in Kifer (2000) and Kifer (2013).

Theorem 9.1. *If conditions (9.2) and (9.3) hold, then*

$$Y_t = \overline{Y}_t \equiv \underline{Y}_t.$$

Moreover, the optimal stopping times are

$$\zeta_1 = \inf\{s : t \leq s,\ Y_s \leq N_1(s,S_s)\}$$
$$\zeta_2 = \inf\{s : t \leq s,\ Y_s \geq N_2(s,S_s)\}.$$

From now on we assume for simplicity $N_1(\tau_1, S_{\tau_1}) \equiv N_3(\tau_1, S_{\tau_1})$. Let the penalty be the constant η. Thus, the payoffs of a cancellable call option can be written as

$$N_1(t,x) = e^{-\lambda t}(x-K)^+$$
$$N_2(t,x) = e^{-\lambda t}\left((x-K)^+ + \eta\right).$$

The time dependence can be derived similarly as in Proposition 3.3. From now on we assume $t = 0$.

9.3 Exercise regions

The exercise region consists of two parts – one for the writer, Υ^s, and another for the holder, Υ^b. We need to define the following optimal stopping times.

Definition 9.1. *Let $\zeta \in \mathcal{T}_{[t,T]}$ be an arbitrary stopping time with values between t and T.*

1. *We define $A(\zeta; x)$ as the stopping time that minimizes*

$$\mathbb{E}^{t,x}\left[\begin{array}{l} e^{-r(\zeta-t)}N_1(\zeta, S_\zeta)\, I_{\zeta\le A(\zeta;\cdot)} \\ +e^{-r(A(\zeta;\cdot)-t)}N_2\left(A(\zeta;\cdot), S_{A(\zeta;\cdot)}\right) I_{A(\zeta;\cdot)<\zeta} \end{array}\right]. \tag{9.4}$$

   The stopping time $A(\zeta; x)$ can be viewed as the writer's optimal strategy if the holder follows the strategy ζ.

2. *Analogously, we define the holder's optimal strategy, $B(\zeta; x)$, as the stopping time that maximizes*

$$\mathbb{E}^{t,x}\left[\begin{array}{l} e^{-r(B(\zeta;\cdot)-t)}N_1\left(B(\zeta;\cdot), S_{B(\zeta;\cdot)}\right) I_{B(\zeta;\cdot)\le\zeta} \\ +e^{-r(\zeta-t)}N_2(\zeta, S_\zeta)\, I_{\zeta<B(\zeta;\cdot)} \end{array}\right]. \tag{9.5}$$

We are ready now to define the exercise regions.

Definition 9.2. *The optimal regions Υ^b and Υ^s are defined via the following statements.*

1. *The point $(t, x) \in \Upsilon^b$ if for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$,*

$$N_1(t,x) \ge \mathbb{E}^{t,x}\left[\begin{array}{l} e^{-r(\zeta_1-t)}N_1(\zeta, S_\zeta)\, I_{\zeta\le A(\zeta;\cdot)} \\ +e^{-r(A(\zeta;\cdot)-t)}N_2\left(A(\zeta;\cdot), S_{A(\zeta;\cdot)}\right) I_{A(\zeta;\cdot)<\zeta} \end{array}\right]. \tag{9.6}$$

2. *The point $(t, x) \in \Upsilon^s$ if for every stopping time $\zeta \in \mathcal{T}_{[t,T]}$,*

$$N_2(t,x) \le \mathbb{E}^{t,x}\left[\begin{array}{l} e^{-r(B(\zeta;\cdot)-t)}N_1\left(B(\zeta;\cdot), S_{B(\zeta;\cdot)}\right) I_{B(\zeta;\cdot)\le\zeta_2} \\ +e^{-r(\zeta-t)}N_2(\zeta, S_\zeta)\, I_{\zeta<B(\zeta;\cdot)} \end{array}\right]. \tag{9.7}$$

3. *The continuation region is $\overline{\Upsilon} = \{[0,T] \times \mathbb{R}^+\} / \left\{\Upsilon^b \cup \Upsilon^s\right\}$, i.e. the points that give the holder and the writer opportunities for a future larger profit (a smaller loss for the writer).*

Now we shall prove a series of propositions that characterize the shape of the early exercise regions.

9.3.1 Options without discounting

Note that $r > 0$ since $r + \lambda > 0$. We shall prove the following proposition that has an analogue for the ordinary American call options.

Proposition 9.1. *If the discount factor is zero, $\lambda = 0$ and thus $N_1(t,x) = (x-K)^+$, then the holder's exercise region is empty – $\Upsilon^b = \emptyset$.*

Proof: Suppose that $(t,x) \in \Upsilon^b$ and the stopping time ζ is finite. Obviously $x = S_t > K$. Using the martingality of $e^{-rt}S_t$, we derive

$$
\begin{aligned}
&\mathbb{E}^{t,x}\left[e^{-r\zeta}N_1(\zeta,S_\zeta)\,I_{\zeta\le A(\zeta;\cdot)} + e^{-rA(\zeta;\cdot)}N_2\left(A(\zeta;\cdot),S_{A(\zeta;\cdot)}\right)I_{A(\zeta;\cdot)<\zeta}\right]\\
&\le e^{-rt}N_1(t,x) = e^{-rt}(x-K)\\
&= \mathbb{E}^{t,x}\left[e^{-r(\zeta\wedge A(\zeta;\cdot))}S_{\zeta\wedge A(\zeta;\cdot)}\right] - Ke^{-rt}\\
&= \mathbb{E}^{t,x}\left[e^{-r\zeta}S_\zeta I_{\zeta\le A(\zeta;\cdot)}\right] + \mathbb{E}^{t,x}\left[e^{-rA(\zeta;\cdot)}S_{A(\zeta;\cdot)}I_{A(\zeta;\cdot)<\zeta}\right] - Ke^{-rt}\\
&< \mathbb{E}^{t,x}\left[e^{-r\zeta}(S_\zeta - K)\,I_{\zeta\le A(\zeta;\cdot)} + e^{-rA(\zeta;\cdot)}\left(S_{A(\zeta;\cdot)} - K\right)I_{A(\zeta;\cdot)<\zeta}\right]\\
&\le \mathbb{E}^{t,x}\left[e^{-r\zeta}N_1(\zeta,S_\zeta)\,I_{\zeta\le A(\zeta;\cdot)} + e^{-rA(\zeta;\cdot)}N_1\left(A(\zeta;\cdot),S_{A(\zeta;\cdot)}\right)I_{A(\zeta;\cdot)<\zeta}\right].
\end{aligned}
$$

We reach to a contradiction since $N_1(t,x) \le N_2(t,x)$. □

We shall prove later that the writer's optimal boundary is infinity too.

9.3.2 Discounted options

First, we shall show that if $\eta \ge K$, then the cancellable option turns into an ordinary American one.

Proposition 9.2. *If $\eta \ge K$, then $\Upsilon^s = \emptyset$.*

Proof: Suppose that $x \in \Upsilon^s$ and thus inequality (9.7) holds for every stopping time ζ. Let $\zeta = u > 0$ be deterministic. Note that when u tends to infinity, $B(\zeta;x)$ is finite and therefore (9.7) leads to

$$
\begin{aligned}
0 &\geq \lim_{u\to\infty} \left(\begin{array}{l} (x-K)^{+} + \eta\left(1 - e^{-(r+\lambda)u} P\left(u < B\left(u;\cdot\right)\right)\right) \\ -\mathbb{E}^{x}\left[{}^{-(r+\lambda)(u\wedge B(u;\cdot))}\left(S_{u\wedge B(u;\cdot)} - K\right)^{+}\right] \end{array} \right) \\
&= (x-K)^{+} + \eta - \mathbb{E}^{x}\left[e^{-(r+\lambda)B(u;\cdot)}\left(S_{B(u;\cdot)} - K\right)^{+}\right] \\
&= (x-K)^{+} + \eta \\
&- \mathbb{E}^{x}\left[e^{-(r+\lambda)B(u;\cdot)} S_{B(u;\cdot)} - e^{-(r+\lambda)B(u;\cdot)} K + e^{-(r+\lambda)B(u;\cdot)}\left(K - S_{B(u;\cdot)}\right)^{+}\right] \\
&\geq \max\left(\eta - K, \eta - x\right) + \mathbb{E}^{x}\left[e^{-(r+\lambda)B(u;\cdot)} \min\left(S_{B(u;\cdot)}, K\right)\right].
\end{aligned}
$$

The first term in the last equation is non-negative because $\eta \geq K$. The contradiction proves the proposition. □

It is clear that if $x \leq K$ then $x \notin \Upsilon^{b}$. We shall prove the analogous proposition for the writer's region.

Proposition 9.3. *If $x < K$ then $x \notin \Upsilon^{s}$.*

Proof: Obviously, if the initial point x is bellow the strike K, then the strategy of first hitting to K leads to a better result for the writer – he will pay amount $e^{-\lambda\tau}\eta$ in a future time τ. Its present value is $E\left[e^{-r\tau}e^{-\lambda\tau}\eta\right] < \eta$ since $r + \lambda > 0$. □

Next, we prove two propositions that characterize both optimal regions.

Proposition 9.4. *Suppose that $x \in \Upsilon^{b}$ and $y > x$. Then $y \in \Upsilon^{b}$ too.*

Proof: Let T be a fixed finite moment, $\zeta \in \mathcal{T}\left[0, T\right]$ be an arbitrary stopping time, and $A\left(\zeta; x\right)$ be the corresponding ζ-writer's optimal strategy. Condition (9.6) leads to

$$
\mathbb{E}^{x}\left[\begin{array}{l} e^{-(r+\lambda)\zeta}(S_{\zeta} - K)^{+} I_{\zeta \leq A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)}\left(\left(S_{A(\zeta;x)} - K\right)^{+} + \eta\right) I_{A(\zeta;x)<\zeta} \end{array} \right] \leq (x-K).
$$

Using Lemma 3.1 and the fact that $A\left(\zeta; y\right)$ minimizes (9.4), we derive

$$\begin{aligned}
&\mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(S_\zeta - K)^+ I_{\zeta \leq A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)} \left(\left(S_{A(\zeta;y)} - K\right)^+ + \eta \right) I_{A(\zeta;y)<\zeta} \end{array} \right] - (y-K) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(S_\zeta - K)^+ I_{\zeta \leq A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)} \left(\left(S_{A(\zeta;y)} - K\right)^+ + \eta \right) I_{A(\zeta;y)<\zeta} \end{array} \right] \\
&- \mathbb{E}^y \left[e^{-(r+\lambda)\zeta} e^{\lambda\zeta} S_\zeta I_{\zeta \leq A(\zeta;y)} + e^{-(r+\lambda)A(\zeta;y)} e^{\lambda A(\zeta;y)} S_{A(\zeta;y)} I_{A(\zeta;y)<\zeta} \right] + K \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(-\left(e^{\lambda\zeta}-1\right) S_\zeta - K, -e^{\lambda\zeta} S_\zeta \right) I_{\zeta \leq A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)} \max \left(\begin{array}{l} -\left(e^{\lambda A(\zeta;y)} - 1\right) S_{A(\zeta;y)} - K + \eta, \\ -e^{\lambda A(\zeta;y)} S_{A(\zeta;y)} + \eta \end{array} \right) I_{A(\zeta;y)<\zeta} \end{array} \right] + K \\
&\leq \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(-\left(e^{\lambda\zeta}-1\right) S_\zeta - K, -e^{\lambda\zeta} S_\zeta \right) I_{\zeta \leq A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \max \left(\begin{array}{l} -\left(e^{\lambda A(\zeta;x)} - 1\right) S_{A(\zeta;x)} - K + \eta, \\ -e^{\lambda A(\zeta;x)} S_{A(\zeta;x)} + \eta \end{array} \right) I_{A(\zeta;x)<\zeta} \end{array} \right] + K \\
&< \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(-\left(e^{\lambda\zeta}-1\right) S_\zeta - K, -e^{\lambda\zeta} S_\zeta \right) I_{\zeta \leq A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \max \left(\begin{array}{l} -\left(e^{\lambda A(\zeta;x)} - 1\right) S_{A(\zeta;x)} - K + \eta, \\ -e^{\lambda A(\zeta;x)} S_{A(\zeta;x)} + \eta \end{array} \right) I_{A(\zeta;x)<\zeta} \end{array} \right] + K \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(S_\zeta - K)^+ I_{\zeta \leq A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \left(\left(S_{A(\zeta;x)} - K\right)^+ + \eta \right) I_{A(\zeta;x)<\zeta} \end{array} \right] - (x-K) \leq 0.
\end{aligned}$$

We finish the proof by taking $T \to \infty$. □

Proposition 9.5. *Suppose that $x \in \Upsilon^s$ and $K < y < x$. Then $y \in \Upsilon^s$ too.*

Proof: Let again T be a fixed finite moment, $\zeta \in \mathcal{T}[0,T]$ be arbitrary, and $B(\zeta;x)$ be the ζ-holder's optimal strategy. Condition (9.7) leads to

$$\mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \left(S_{B(\zeta;x)} - K\right)^+ I_{B(\zeta;x) \leq \zeta} \\ +e^{-(r+\lambda)\zeta} \left((S_\zeta - K)^+ + \eta \right) I_{\zeta < B(\zeta;x)} \end{array} \right] \geq (x - K + \eta).$$

Analogously to the proof of Proposition 9.4, we derive

$$
\begin{aligned}
&\mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \left(S_{B(\zeta;y)} - K\right)^+ I_{B(\zeta;y) \leq \zeta} \\ + e^{-(r+\lambda)\zeta} \left((S_\zeta - K)^+ + \eta\right) I_{\zeta < B(\zeta;y)} \end{array} \right] - (y - K + \eta) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \left(S_{B(\zeta;y)} - K\right)^+ I_{B(\zeta;y) \leq \zeta} \\ + e^{-(r+\lambda)\zeta} \left((S_\zeta - K)^+ + \eta\right) I_{\zeta < B(\zeta;y)} \end{array} \right] \\
&- \mathbb{E}^y \left[e^{-(r+\lambda)B(\zeta;y)} e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} I_{B(\zeta;y) \leq \zeta} + e^{-(r+\lambda)\zeta} e^{\lambda\zeta} S_\zeta I_{\zeta < B(\zeta;y)} \right] + K - \eta \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \max \left(\begin{array}{c} -\left(e^{\lambda B(\zeta;y)} - 1\right) S_{B(\zeta;y)} - K, \\ -e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} \end{array} \right) I_{B(\zeta;y) \leq \zeta} \\ + e^{-(r+\lambda)\zeta} \max \left(\begin{array}{c} -\left(e^{\lambda\zeta} - 1\right) S_\zeta - K + \eta, \\ -e^{\lambda\zeta} S_\zeta + \eta \end{array} \right) I_{\zeta < B(\zeta;x)} \end{array} \right] + K - \eta \\
&\geq \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max \left(\begin{array}{c} -\left(e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} - K, \\ -e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{array} \right) I_{B(\zeta;x) \leq \zeta} \\ + e^{-(r+\lambda)\zeta} \max \left(\begin{array}{c} -\left(e^{\lambda\zeta} - 1\right) S_\zeta - K + \eta, \\ -e^{\lambda\zeta} S_\zeta + \eta \end{array} \right) I_{\zeta < B(\zeta;x)} \end{array} \right] + K - \eta \\
&> \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max \left(\begin{array}{c} -\left(e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} - K, \\ -e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{array} \right) I_{B(\zeta;x) \leq \zeta} \\ + e^{-(r+\lambda)\zeta} \max \left(\begin{array}{c} -\left(e^{\lambda\zeta} - 1\right) S_\zeta - K + \eta, \\ -e^{\lambda\zeta} S_\zeta + \eta \end{array} \right) I_{\zeta < B(\zeta;x)} \end{array} \right] + K - \eta \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \left(S_{B(\zeta;x)} - K\right)^+ I_{B(\zeta;x) \leq \zeta} \\ + e^{-(r+\lambda)\zeta} \left((S_\zeta - K)^+ + \eta\right) I_{\zeta < B(\zeta;x)} \end{array} \right] - (x - K + \eta) \geq 0.
\end{aligned}
$$

We finish the proof by taking $T \to \infty$. □

Proposition 9.4 prompts us to expect that the holder's exercise region is of the form $\Upsilon^b = [B, \infty)$. Proposition 9.1 leads to $B = \infty$ when $\lambda = 0$. Proposition 9.5 shows that the writer's exercise region has one of the forms $\Upsilon^s = [K, A]$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$.

Next, we prove a proposition for the writer's optimal region.

Proposition 9.6. *If the risk-free rate is negative, $r < 0$, then the writer's optimal region is either empty, $\Upsilon^s = \emptyset$, or it consists only of the strike, $\Upsilon^s = \{K\}$.*

Proof: We shall consider separately the positions of the initial asset value w.r.t. the strike.

1. Suppose first that $x < K$ and let ζ be the first hit to the strike. It is preferable for the writer to pay the amount of $e^{-\lambda\zeta}\eta$ in the future moment ζ instead to cancel immediately paying η because $\mathbb{E}\left[e^{-r\zeta}e^{-\lambda\zeta}\eta\right] < \eta$ since $r + \lambda > 0$. Hence, $(0, K) \cap \Upsilon^s = \emptyset$.

2. Let $x > K$ and suppose that $x \in \Upsilon^s$. Therefore $x \notin \Upsilon^b$. Proposition 9.4 gives us that there exists a constant $K_1 > x$, such that $K_1 \notin \Upsilon^b$ too. Let ζ be the first exit from the strip (K_1, K). Proposition 9.4 leads also to $B(\zeta; x) > \zeta$. Therefore,

$$\begin{aligned}
&\mathbb{E}^x\left[e^{-(r+\lambda)\zeta}\left(S_\zeta - K + \eta\right) I_{\zeta \leq B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}\left(S_\zeta - K\right) I_{B(\zeta;x)<\zeta}\right] \\
&\leq \mathbb{E}^x\left[e^{-r\zeta}\left(S_\zeta - K + \eta\right)\right] < x - K + \eta,
\end{aligned}$$

We have used above the martingality of the discounted asset price. Hence, $x \notin \Upsilon^s$.

□

Let us turn to the limiting case $\eta = 0$. Note that Propositions 9.4 and 9.5 still hold.

Proposition 9.7. *Let $\eta = 0$ and $x > K$. Then it is optimal for one of the holder or writer to exercise the option immediately.*

Proof: Let x be the initial asset value and the constants A and B follow the order $K \leq A < x < B$. Let us consider a financial derivative related to the first exit from the strip (A, B). Let it pay the amount of $B - K$ if the exit occurs from the upper boundary and $A - K$, otherwise. We denote by $f(A, B, x)$ the price function of this derivative. For a fixed A, we define $B(A, x)$ as the value that maximizes the derivative price in the interval $[x, \infty)$. Similarly, let $A(B, x)$ be the minimizer of $f(A, B, x)$ in the interval $[K, x]$ for a fixed B.

Suppose that $x \notin \Upsilon^b$. Using Proposition 9.4, we conclude that there exists $\overline{B} > x$ such that

$$f\left(A\left(\overline{B}, x\right), \overline{B}, x\right) > x - K.$$

The definition of the function $A(B, x)$ shows

$$f\left(A,\overline{B},x\right)\geq f\left(A\left(\overline{B},x\right),\overline{B},x\right)>x-K$$

for every $A\in[K,x)$. On the other hand, the definition of the function $B(A,x)$ leads to

$$f\left(A,B\left(A,x\right),x\right)\geq f\left(A,\overline{B},x\right)>x-K.$$

Hence, $x\in\Upsilon^s$. □

9.4 Pricing

It is proven in Section 9.3.2 that the holder's exercise region is of the form $\Upsilon^b=[B,\infty)$. Three cases are possible for the writer's one – $\Upsilon^s=[K,A)$, $\Upsilon^s=\{K\}$, or $\Upsilon^s=\emptyset$. Thus, we must to recognize which case holds and find the values of A and B.

Suppose the first case is actual– the writer's boundary is larger than the strike, $A>K$. If the initial asset price is between the barriers, $x\in[A,B]$, then the game option pricing task is related to the first exit of the Brownian motion from a strip. Let ζ_A and ζ_B be the first hitting moments of the asset to the values A and B, respectively. Therefore the exit from the continuation region is $\zeta=\zeta^A\wedge\zeta^B$. In the terms of the Brownian motion, ζ^A and ζ^B are the first hitting moments of the Brownian motion with drift

$$\psi=\frac{r}{\sigma}-\frac{\sigma}{2}$$

to the levels

$$\overline{A}=\frac{\ln A-\ln x}{\sigma}<0$$
$$\overline{B}=\frac{\ln B-\ln x}{\sigma}>0.$$

Hence, the price of the derivative related to the first exit of the strip $[A,B]$ is

$$
\begin{aligned}
f(A,B,x) &= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)\zeta^B} \left(S_{\zeta^B} - K\right) I_{\zeta^B \le \zeta^A} \\ +e^{-(r+\lambda)\zeta^A} \left(S_{\zeta^A} - K + \eta\right) I_{\tau^A < \zeta^B} \end{array} \right] \\
&= (B-K)\,\mathbb{E}^x \left[e^{-(r+\lambda)\zeta^B} I_{\zeta^B \le \zeta^A} \right] \\
&\quad + (A-K+\eta)\,\mathbb{E}^x \left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A < \zeta^B} \right].
\end{aligned} \tag{9.8}
$$

Let the constants p in q be defined by formulas (3.17). Recall that $p \ge q+1$ and the equality holds when $\lambda = 0$. Using Lemma 3.3, we derive for price function (9.8)

$$
\begin{aligned}
& f(A,B,x) = (A-K+\eta)\, e^{\psi \overline{A}} \frac{\sinh\left(\sigma \frac{p}{2} \overline{B}\right)}{\sinh\left(\sigma \frac{p}{2} \left(\overline{B} - \overline{A}\right)\right)} \\
& + (B-K)\, e^{\psi \overline{B}} \frac{\sinh\left(-\sigma \frac{p}{2} \overline{A}\right)}{\sinh\left(\sigma \frac{p}{2} \left(\overline{B} - \overline{A}\right)\right)} \\
& = (A-K+\eta)\, e^{(p-q)(\ln x - \ln A)} \frac{e^{p(\ln B - \ln x)} - 1}{e^{p(\ln B - \ln A)} - 1} \\
& + (B-K)\, e^{q(\ln B - \ln x)} \frac{e^{p(\ln x - \ln A)} - 1}{e^{p(\ln B - \ln A)} - 1} \\
& = (A-K+\eta) \left(\frac{x}{A}\right)^{p-q} \frac{\left(\frac{B}{x}\right)^p - 1}{\left(\frac{B}{A}\right)^p - 1} + (B-K) \left(\frac{B}{x}\right)^q \frac{\left(\frac{x}{A}\right)^p - 1}{\left(\frac{B}{A}\right)^{2c} - 1} \\
& = (A-K+\eta) \left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^{2c} - A^{2c}} + (B-K) \left(\frac{B}{x}\right)^q \frac{x^p - A^{2c}}{B^{2c} - A^p}.
\end{aligned} \tag{9.9}
$$

Suppose that B is fixed and let us change the variables as $a = \frac{A}{B}$, $k = \frac{K}{B}$, $y = \frac{x}{B}$, and $\xi = \frac{\eta}{B}$. Thus we have $0 < \xi < k < a < 1$. Therefore equation (9.9) turns into

$$
f(A,B,x) = \frac{B}{y^q} \frac{(a-k+\xi)\, a^q \left(1-y^p\right) + (1-k)\left(y^p - a^p\right)}{1-a^p}.
$$

We shall find which value of the variable a minimizes the function

$$
\begin{aligned}
g(a;y) &= \frac{(a-k+\xi)\, a^q \left(1-y^p\right) + (1-k)\left(y^p - a^p\right)}{1-a^p} \\
&= \frac{-a^p (1-k) + a^{q+1}\left(1-y^p\right) - a^q (k-\xi)\left(1-y^p\right) + y^p (1-k)}{1-a^p}.
\end{aligned}
$$

Its a-derivative is

$$g_a(a;y) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \left[\begin{array}{l} a^{p+1}(p-q-1) - a^p(k-\xi)(p-q) \\ -a^{p-q}p(1-k) + a(q+1) - q(k-\xi) \end{array} \right]. \quad (9.10)$$

It is proven in Appendix 9.A that the equation

$$a^{p+1}(p-q-1) - a^p(k-\xi)(p-q) - a^{p-q}p(1-k) + a(q+1) - q(k-\xi) = 0$$

has just one solution in the interval $(0,1)$ – we denote it by $a(B)$. Note that it depends on B through the variables k and ξ and is independent of the variable y. The function $g(a;y)$ has a minimum at this point because the derivative $g_a(a;y)$ is negative before and positive after it. Hence, if the holder's strategy is to exercise when the asset hits the value B, then the writer's optimal stopping time is the first hit to the value $A = a(B)B$.

Now we turn to the holder's exercise boundary. Suppose that A is fixed and let us change the variables as $b = \frac{B}{A}$, $k = \frac{K}{A}$, $y = \frac{x}{A}$, and $\xi = \frac{\eta}{A}$. Therefore $0 < \xi < k < 1 < b$. Thus, equation (9.9) turns into

$$f(A,B,x) = \frac{A}{y^q} \frac{(1-k+\xi)(b^p-y^p) + (b-k)b^q(y^p-1)}{b^p-1}.$$

Let the function $g(\cdot;\cdot)$ be defined as

$$g(b;y) = \frac{(1-k+\xi)(b^p-y^p) + (b-k)b^q(y^p-1)}{b^p-1}.$$

Its derivative is

$$g_b(b;y) = \frac{y^p-1}{(b^p-1)^2} b^{q-1} \left[\begin{array}{l} -b^{p+1}(p-q-1) + b^p k(p-q) \\ +b^{p-q}p(1-k+\xi) - b(q+1) + qk \end{array} \right]. \quad (9.11)$$

It is proven in Appendix 9.B that the equation

$$-b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q}p(1-k+\xi) - b(q+1) + qk = 0$$

has only one solution in the interval $(1,\infty)$ – we denote it by $b(A)$. The function $g(b;y)$ has a maximum at this value. Therefore, if the writer's strategy is the first hit to the value A, then the holder's optimal strategy is the first hitting moment to the value $B = b(A)A$. Thus, we can find the writer's boundary as the solution of the equation

$$b(y)\,a(yb(y)) = 1. \tag{9.12}$$

The holder's boundary is derived as $B = b(A)A$, where A is the solution of equation (9.12). Suppose first that $A > K$. Hence, if the initial asset value is between the boundaries, $x \in [A, B]$, then the option price is given by equation (9.9). If $x \in \Upsilon^s \equiv [K, A)$, then the price turns into $x - K + \eta$. Otherwise, if $x \in \Upsilon^b \equiv [B, \infty)$, then the price is $x - K$. If $x < K$, then the writer's strategy is the first hit to the strike. Hence, the option price can be derived using equation (3.14) from Proposition 3.5:

$$\mathbb{E}^x\left[e^{-(r+\lambda)\zeta}\left((S_\tau - K)^+ + \eta\right)I_{\zeta<\infty}\right] = \eta\mathbb{E}^x\left[e^{-(r+\lambda)\zeta}I_{\zeta<\infty}\right] = \eta\left(\frac{x}{K}\right)^{p-q}. \tag{9.13}$$

Note that to keep the differentiability of function (9.9), we have changed the payoff from $(A-K)^+ + \eta$ to $A - K + \eta$. This may lead to a result less than the strike, $A < K$. Hence, we have to proceed differently if this happens. Proposition 9.3 shows that the points below the strike are not optimal for the writer. Thus we need to find when the writer's exercise region consists only of the strike, $\Upsilon^s = \{K\}$. If $A < K$, then the writer has the alternatives (A) to cancel when the asset hits the strike or (B), to do nothing. In case (B), he waits for the holder to exercise. Thus, the game option turns into an ordinary American one. Theorem 4.1 shows that the optimal boundary is

$$B = \frac{p-q}{p-q-1}K. \tag{9.14}$$

Note that $p > q + 1$. If the initial asset value is $x = K$, then the option price turns into

$$\overline{\eta} = \frac{K}{p-q}\left(\frac{p-q-1}{p-q}\right)^{p-q-1}. \tag{9.15}$$

If the writer chooses case (A) he will pay the amount of η. Hence, the writer's exercise region is $\Upsilon^s = \{K\}$ if $\eta < \overline{\eta}$, and $\Upsilon = \emptyset$, otherwise. If $\eta \geq \overline{\eta}$, then the game option turns into a usual American one. If $\eta < \overline{\eta}$, we evaluate the option similarly to the case when $A > K$.

We summarize the obtained results in the following theorem.

Theorem 9.2. *[Theorem 4.1 of Zaevski (2020b)]*
Let $\overline{A}$ and $\overline{B}$ be the derived boundaries. The exercise regions and the option price (denoted by V) are derived via the following statements.

1. *if $\overline{A} > K$, then the exercise regions are $\Upsilon^s = \left[K, \overline{A}\right]$ and $\Upsilon^b = \left[\overline{B}, \infty\right)$. The option price can be derived through the alternatives:*

   (a) *If $x \leq K$, then*

$$V = \eta\left(\frac{x}{K}\right)^{p-q}. \tag{9.16}$$

   (b) *If $K < x < \overline{A}$, then*

$$V = x - K + \eta.$$

   (c) *If $\overline{A} \leq x \leq \overline{B}$, then*

$$V = \left(\overline{A} - K + \eta\right)\left(\frac{\overline{A}}{x}\right)^q \frac{\overline{B}^p - x^p}{\overline{B}^p - \overline{A}^p} + \left(\overline{B} - K\right)\left(\frac{\overline{B}}{x}\right)^q \frac{x^p - \overline{A}^p}{\overline{B}^p - \overline{A}^p}. \tag{9.17}$$

   (d) *If $\overline{B} < x$, then*

$$V = x - K. \tag{9.18}$$

2. *If $\overline{A} < K$, then*

   (a) *if $\eta \geq \overline{\eta}$, then the game option turns into an ordinary American one. The exercise regions are $\Upsilon^s = \emptyset$ and $\Upsilon^b = \left[\overline{B}, \infty\right)$. The value of $\overline{B}$ is given by equation (9.14). If the initial asset price is in the holder's exercise region, $x \in \Upsilon^b$, then the option price is given by equation (9.18). Otherwise, it is*

$$Y = \left(\frac{x}{p-q}\right)^{p-q}\left(\frac{p-q-1}{K}\right)^{p-q-1}.$$

(b) *If $\eta < \overline{\eta}$, then the exercise regions are $\Upsilon^s = \{K\}$ and $\Upsilon^b = \left[\overline{B}, \infty\right)$. The value of $\overline{B}$ is obtained as $\overline{B} = Kb(K)$ where $b(K)$ is the root of function (9.11).*

If $x \leq K$, then the option price is given by formula (9.16). If $x \in \left(K, \overline{B}\right]$, then the price is given by formula (9.17) for $\overline{A} = K$. If $x > \overline{B}$, then the price is given by formula (9.18).

Remark 9.1. *Note that Proposition 9.6 says that if the risk-free rate is negative, $r < 0$, then the second statement of Theorem 9.2 holds – the writer's exercise region is either empty or consists only of the strike.*

9.5 Perpetual options without discounting

We suppose now that $\lambda = 0$ and $r > 0$. Therefore, the payments are

$$N_1(t,x) = (x-K)^+$$
$$N_2(t,x) = (x-K)^+ + \eta.$$

Proposition 9.2 shows that the option turns into an ordinary American call when $\eta \geq K$. Suppose that $\eta < K$. Proposition 9.1 gives us that the holder's optimal region is empty and thus its boundary is infinity. Let B be fixed and suppose that the holder exercises when the asset hits it. We shall take the limit $B \to \infty$. We know that $p = q + 1$ and the equation that determines the optimal writer's boundary turns into $h(a;\xi) = 0$, where the function $h(a;\xi)$ is defined in (9.21). Its solution $\overline{a}$ is independent of B since $\xi = \frac{\eta}{B}$ and $k = \frac{K}{B}$. Therefore when $B \to \infty$, the writer's boundary $A = B\overline{a}$ also tends to infinity. This means that the writer's optimal region is $\Upsilon^s \equiv (K, \infty)$. We have to find the value of the option in the continuation region $\{x < K\}$. Using equation (3.14) from Proposition 3.5, we derive

$$V = \mathbb{E}^x\left[e^{-r\zeta}\left((S_\zeta - K)^+ + \eta\right)I_{\zeta<\infty}\right] = \eta\mathbb{E}^x\left[e^{-r\zeta}I_{\tau<\infty}\right] = \frac{x\eta}{K}, \qquad (9.19)$$

that corresponds to the results of Ekström (2006) and Ekström and Villeneuve (2006). Of course, equations (9.13) and (9.19) coincide because $p = q + 1$ when $\lambda = 0$.

9.6 Numerical results

We shall examine the behavior of game call options with different parameters. We assume that the risk-free rate is $r = 0.05$, the volatility is $\sigma = 0.3$, the strike price is $K = 5$, and the initial asset value is $x = 8\$$. We vary the penalty η between zero and 5$ and the discount factor λ between 0.0001 and 0.1. The results are presented in Figure 9.1. The optimal boundaries for the writer and the holder are plotted in Figures 9.1a and 9.1b, respectively. We can see that when λ tends to zero, both boundaries tend to infinity. This observation is in accordance with the results for the undiscounted options, presented in Section 9.5. Note that when the penalty η is smaller, the convergence is faster and vice versa. With the red points are marked the values for the penalty η after which the boundary A falls below the strike K. Thus the option changes its features from the second to the first statement of Theorem 9.2. The green points are the corresponding values of $\overline{\eta}$ from equation (9.15). If $\eta < \overline{\eta}$, the writer's exercise region is $\Upsilon^s = \{K\}$ and the boundary is $A = K$. The holder's optimal boundary is calculated as $\overline{B} = Kb(K)$ where $b(K)$ is the root of function (9.11). Otherwise, if $\overline{\eta} \leq \eta$, the game call option turns into an American one and the writer's optimal boundary does not exist.

In Figure 9.1c are commonly presented the writer's and holder's optimal boundaries with a fixed discount factor $\lambda = 0.01$. The value of the penalty η for which the writer's boundary turns into the strike is marked again with a red point and is 2.6968. The corresponding value of $\overline{\eta}$ is marked by a green point and is 3.5722. Proposition 9.7 has a confirmation here. When η tends to zero, both boundaries tend to the same value. .

Figure 9.1d presents the behavior of the game call option prices. Also, we provide in Table 9.1 some particular values. We vary the discount factor as $\lambda \in \{0.001, 0.01, 0.1\}$, the penalty as $\eta \in \{0.001, 1, 2, 3, 4\}$, and the initial asset value as $x \in \{6, 7, 8, 9, 10\}$. Right to the prices, we report the case of Theorem 9.2 that is actual. In the undiscounted case, examined in Section 9.5, the optimal boundaries are infinitely large. This explains why all initial values belong to the writer's exercise region when the discount factor is small, $\lambda = 0.001$. Also, for the large enough penalties η, A is below the strike. For the extremely large penalties, the game option turns into an ordinary American one.

9.A Uniqueness of the solutions: writer's boundary

We shall prove that derivative (9.10) has just one root in the interval $(0, 1)$. Let us denote by $h(\cdot;\cdot)$ the function

$$\begin{aligned} h(a;\xi) &= a^{p+1}(p-q-1) - a^{p}(k-\xi)(p-q) \\ &\quad - a^{p-q}p(1-k) + (q+1)a - q(k-\xi). \end{aligned} \tag{9.20}$$

First, we shall examine the case $p = q + 1$ equivalently to $\lambda = 0$. Function (9.20) that determines the behavior of derivative (9.10) turns into

$$h(a;\xi) = -a^{q+1}(k-\xi) + a(q+1)k - q(k-\xi). \tag{9.21}$$

Its derivative

$$h_a(a;\xi) = -a^{q}(q+1)(k-\xi) + (q+1)k$$

is a decreasing positive function since $h_a(1;\xi) = (q+1)\xi > 0$. Therefore, the function $h(a;\xi)$ is increasing. Hence, the solution of the equation $h(a;\xi) = 0$ is unique because $h(0;\xi) = -q(k-\xi) < 0$ and $h(1;\xi) = \xi(q+1) > 0$.

Suppose now that $p > q + 1$. The a-derivative of function (9.20) is

$$\begin{aligned} h_a(a;\xi) &= a^{p}(p+1)(p-q-1) - a^{p-1}p(k-\xi)(p-q) \\ &\quad - a^{p-q-1}(p-q)p(1-k) + (q+1). \end{aligned} \tag{9.22}$$

We shall examine firstly the case $\xi = 0$ and later we shall show that the general case $\xi > 0$ is its consequence.

Suppose that $\xi = 0$. Thus derivative (9.22) turns into

$$\begin{aligned} h_a(a;0) &= a^{p}(p+1)(p-q-1) - a^{p-1}pk(p-q) \\ &\quad - a^{p-q-1}(p-q)p(1-k) + (q+1). \end{aligned}$$

The second derivative is

$$\begin{aligned} h_{aa}(a;0) &= a^{p-1}p(p+1)(p-q-1) - a^{p-2}(p-1)pk(p-q) \\ &\quad - a^{p-q-2}(p-q-1)(p-q)p(1-k) \\ &= a^{p-q-2}\left[\begin{array}{l} a^{q+1}p(p+1)(p-q-1) - a^{q}(p-1)pk(p-q) \\ -(p-q-1)(p-q)p(1-k) \end{array}\right]. \end{aligned}$$

Let the function $l(\cdot)$ be defined as

$$\begin{aligned} l(a) &= a^{q+1}p(p+1)(p-q-1) - a^q(p-1)pk(p-q) \\ &\quad - (p-q-1)(p-q)p(1-k). \end{aligned}$$

Its a-derivative is

$$\begin{aligned} l_a(a) &= a^q(q+1)p(p+1)(p-q-1) - a^{q-1}q(p-1)pk(p-q) \\ &= a^{q-1}p\left[a(q+1)(p+1)(p-q-1) - q(p-1)k(p-q)\right]. \end{aligned}$$

Let $\overline{a}$ be the solution of the equation $l_a(a) = 0$:

$$\overline{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)} > 0.$$

We have $l_a(a) < 0$ for $a < \overline{a}$ and $l_a(a) > 0$ for $a > \overline{a}$. We shall examine separately the cases $\overline{a} \geq 1$ and $\overline{a} < 1$.

Suppose $\overline{a} \geq 1$. Therefore $l_a(a) < 0$ for all $a \in (0,1)$, which means that $l(a)$ is decreasing. Therefore, the function $l(a)$ is negative in the whole interval $(0,1)$ since $l(0) < 0$. The same is true for the function $h_{aa}(a;0)$ too. Hence, the function $h_a(a;0)$ is decreasing for $a \in (0,1)$. We conclude that $h_a(a;0) > 0$ for $a \in (0,1)$ because $h_a(1;0) = 0$. We have also $h_a(a;\xi) > h_a(a;0) > 0$. Therefore, the function $h(a;\xi)$ is increasing. Thus, the equation $h(a;\xi) = 0$ has just one solution because $h(0;\xi) = -q(k-\xi) < 0$ and $h(1;\xi) = p\xi \geq 0$.

Suppose now that $\overline{a} < 1$. Therefore, the function $l(a)$ has a minimum in the point $\overline{a}$. We have $l(\overline{a}) < 0$ since $l(0) < 0$. Suppose $l(1) \leq 0$. Therefore the function $l(a)$ is negative for $a \in (0,1)$ and we fall in the previous case.

Suppose now that $l(1) > 0$. Therefore, the function $l(a)$ decreases from 0 to $\overline{a}$ and then increases from $\overline{a}$ to 1. Hence, it crosses the abscissa at just one point. Also, it is negative before this point and positive after. Hence, the function $h_a(a;0)$ starts from the positive value $h_a(0;0) = q+1$, crosses the abscissa, has a minimum and increases to the value $h_a(1;0) = 0$ staying negative. Therefore, the function $h(a;0)$ starts from the negative value $h(0;0) = -qk$, crosses the zero, has a maximum and then decreases to the value $h(1;0) = 0$.

We have $h_a(a;\xi) > h_a(a;0)$. This means that the function $h(a;\xi)$ increases faster than the function $h(0;\xi)$ and decreases slower. Even more, it is possible $h(a;0)$ to decrease, whereas $h(a;\xi)$ to increase. Having in mind $h(0;0) = -qk < -q(k-\xi) = h(0;\xi)$, we conclude that the equation $h(a;\xi) = 0$ has only one solution, even though the function $h(a;\xi)$ can be not monotone.

Remark 9.2. *It is possible for the function $h(a;\xi)$ to have a local minimum near the point $a = 1$, but it has to be positive. This may happen for relatively low values of k and ξ.*

9.B Uniqueness of the solutions: holder's boundary

We shall prove that derivative (9.11) has a unique root in the interval $(1,\infty)$ except in the undiscounted case. Let $h(\cdot)$ be the function

$$h(b) = -b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q} p(1-k+\xi) - b(q+1) + qk. \tag{9.23}$$

First, we shall consider the undiscounted case $\lambda = 0$ equivalently to $p = q+1$. Function (9.23) turns into

$$h(b) = b^{q+1}k - b(q+1)(k-\xi) + qk.$$

Its derivative

$$h_b(b) = (q+1)\left[b^q k - (k-\xi)\right]$$

is positive for $b \in (1,\infty)$. Thus the function $h(b)$ is increasing. Therefore, the equation $h(b) = 0$ has no solutions larger than one since $h(1) = (q+1)\xi > 0$. This corresponds to Proposition 9.1 – the writer's exercise region is empty.

Suppose now that $p > q+1$. The derivatives of function (9.23) are

$$\begin{aligned} h_b(b) &= -b^p(p+1)(p-q-1) + b^{p-1}pk(p-q) \\ &\quad + b^{p-q-1}(p-q)p(1-k+\xi) - (q+1) \\ h_{bb}(b) &= b^{p-q-2}p\left[\begin{array}{l} -b^{q+1}(p+1)(p-q-1) + b^q(p-1)k(p-q) \\ +(p-q-1)(p-q)(1-k+\xi) \end{array}\right]. \end{aligned}$$

Let $l(b)$ be the function

$$l(b) = -b^{q+1}(p+1)(p-q-1) + b^{q}(p-1)k(p-q) + (p-q-1)(p-q)(1-k+\xi).$$

Its derivative is

$$l_b(b) = b^{q-1}\left[-b(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)\right]. \tag{9.24}$$

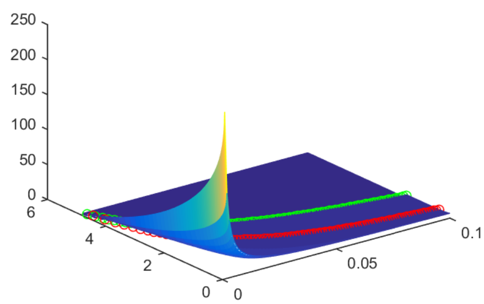
Let $\bar{b}$ be the root of derivative (9.24)

$$\bar{b} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}.$$

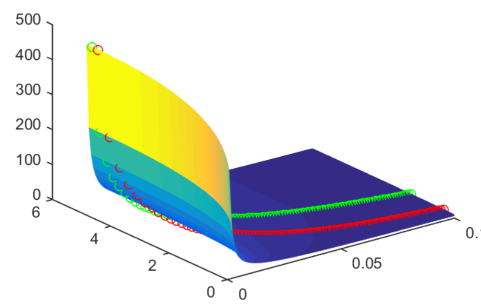
First, if $\bar{b} \leq 1$, then $l_b(b)$ is negative in the interval $b \in (1, \infty)$. Thus the function $l(b)$ is decreasing. Of course, this is true for the second derivative $h_{bb}(b)$ too, and thus it has at most one root larger than one. If it has one root, then the function $h_b(b)$ starts from the positive value $h_b(1) = p\xi(p-q) > 0$, increases to a maximum, and then decreases to minus infinity. If the second derivative $h_{bb}(b)$ has no roots, then the function $h_b(b)$ starts from the positive value $h_b(1) = p\xi(p-q) > 0$ and decreases to minus infinity. In both cases, the function $h_b(b)$ is first positive before its unique root and negative after it. Note that $h(1) = p\xi > 0$. Using the same arguments, we conclude that the equation $h(b) = 0$ has just one solution larger than one.

Suppose now that $\bar{b} > 1$. Therefore, the function $l(b)$ increases in the interval $\left(1, \bar{b}\right)$ and decreases to minus infinity for $b \in \left(\bar{b}, \infty\right)$. We have $l(1) > l(0) = (p-q-1)(p-q)(1-k+\xi) > 0$ because the function $l(b)$ increases in the interval $(0, 1)$. Therefore, $l(b)$ has a unique root larger than one, $l(b) > 0$ before it, and $l(b) < 0$ after. Similar arguments as above prove the uniqueness.

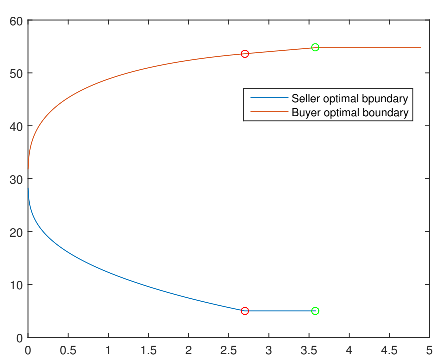
Figure 9.1: Call option boundaries



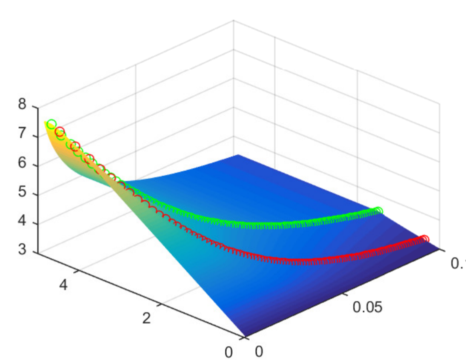
(a) writer's boundary



(b) holder's boundary



(c) Fixed discount factor



(d) Price of a game call option

Table 9.1: Call option prices

	initial price $S_0 = 6$				
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	1.0010/1.b	2.0000/1.b	3.0000/1.b	4.0000/1.b	5.0000/1.b
$\lambda = 0.01$	1.0010/1.b	2.0000/1.b	3.0000/1.b	3.9081/2.b.2	4.3659/2.a.2
$\lambda = 0.1$	1.0010/1.b	1.7262/2.b.2	2.0536/2.a.2	2.0536/2.a.2	2.0536/2.a.2
	initial price $S_0 = 7$				
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	2.0010/1.b	3.0000/1.b	4.0000/1.b	5.0000/1.b	6.0000/1.b
$\lambda = 0.01$	2.0010/1.b	3.0000/1.b	4.0000/1.b	4.7942/2.b.2	5.1731/2.a.2
$\lambda = 0.1$	2.0010/1.b	2.4743/2.b.2	2.6982/2.a.2	2.6982/2.a.2	2.6982/2.a.2
	initial price $S_0 = 8$				
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	3.0010/1.b	4.0000/1.b	5.0000/1.b	6.0000/1.b	6.0000/1.b
$\lambda = 0.01$	3.0010/1.b	4.0000/1.b	4.9938/1.c	5.6707/2.b.2	5.9919/2.a.2
$\lambda = 0.1$	3.0000/1.d	3.2678/2.b.2	3.4181/2.a.2	3.4181/2.a.2	3.4181/2.a.2
	initial price $S_0 = 9$				
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	4.0010/1.b	5.0000/1.b	6.0000/1.b	7.0000/1.b	8.0000/1.b
$\lambda = 0.01$	4.0010/1.b	5.0000/1.b	5.9591/1.c	6.5437/2.b.2	6.8212/2.a.2
$\lambda = 0.1$	4.0000/1.d	4.1172/2.b.2	4.2109/2.a.2	4.2109/2.a.2	4.2109/2.a.2
	initial price $S_0 = 10$				
	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
$\lambda = 0.001$	5.0010/1.b	6.0000/1.b	7.0000/1.b	8.0000/1.b	9.0000/1.b
$\lambda = 0.01$	5.0010/1.b	6.0000/1.b	6.9045/1.c	7.4166/2.b.2	7.6598/2.a.2
$\lambda = 0.1$	5.0000/1.d	5.0272/2.b.2	5.0748/2.a.2	5.0748/2.a.2	5.0748/2.a.2

Chapter 10

Cancellable put options without maturities.

based on the paper

Zaevski, Tsvetelin S. "Discounted perpetual game put options." Chaos, Solitons & Fractals 137 (2020): 109858.

Abstract: The aim of this chapter is to explore the behavior of the cancellable put options without maturity restrictions. Their main characteristic is the opportunity for the holder and the writer to exercise at an arbitrary moment. The writer is obliged to pay an amount above the normal option payoff if he terminates the contract. We include also a discount factor that provides an advantage for earlier exercising. The optimal strategies for both participants are obtained. Thus, the option pricing task turns into a first-exit problem. We base our examination on some financial instruments with random maturities that permit one of the writer or holder to maximize his expected future profit.

10.1 Motivation and main results

It is accepted in the present literature, that the cancellable puts are simpler for examining than the corresponding calls. This is because the writer's exercise region is either the empty set or consists only of the strike. This is right, but not entirely – it is true only when the risk-free rate is positive. But when it is negative, the writer's optimal region can be as complicated as the call style region. It may seem that the negative risk-free rates are unnatural, but they appear when we convert a model with dividend payments to a non-dividend one. More precisely, when the dividend rate is larger than the risk-free one.

The approach we use to evaluate such options is based on deriving the so-called early exercise (or optimal) boundaries. They are formed by these critical values that make keeping the option preferable than the immediate exercise. If the asset price falls enough, it can be optimal for the holder to exercise. Otherwise, if the asset is near the strike and the penalty is small enough, it can be optimal for the writer to cancel the option. These boundaries are constants due to the Markov property of the underlying asset and the absence of a terminal date at which the contract expires. We prove that the area where the immediate exercise is optimal for the holder is a strip, whereas the writer's optimal region can be a strip, a singleton, or even an empty set. The key to identifying the exercise boundaries is specific financial contracts that exhibit the features of the American-style derivatives but the expiration date is assumed to be random. Once we know the boundaries and the corresponding regions, we can use the first exit and hitting properties of the Brownian motion to derive the option price. Note that, most of the derived results are true for the finite-maturity options too.

The chapter is organized in the following way. In Section 10.2 we state our model and prove some statements related to the optimal regions. The pricing task is considered in Section 10.3. We validate our scheme by some numerical examples in Section 10.4. We prove in Appendices 10.A and 10.B the uniqueness of the solutions of the arising equations.

10.2 Exercise regions

We shall denote again by the function $N_1(t,x)$ the amount that the writer has to pay if the holder exercises at the moment t and the spot price $S_t = x$. Analogously, the function $N_2(t,x)$ defines the payoff if the writer cancels the contract. For put options, these functions are

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\ N_2(t,x) &= e^{-\lambda t}\left((K-x)^+ + \eta\right). \end{aligned} \tag{10.1}$$

Proposition 3.3 that gives the time-dependence of the option price allows us to examine only the case $t = 0$. The optimal strategies ($A(\zeta)$ and $B(\zeta)$) and regions (Υ^s and Υ^b) are specified by Definitions 9.1 and 9.2 having in mind that the payoffs are given by formulas (10.1). Recall that $A(\zeta)$ and $B(\zeta)$ are the optimal strategies for one of the option's participants if the other follows the strategy ζ. We shall prove now several propositions that characterize the exercise regions Υ^b and Υ^s.

Proposition 10.1. *If $x > K$, then $x \notin \Upsilon^s$.*

Proof: Suppose that $x > K$ and let ζ be the first hitting time to the strike. This strategy leads to the writer's income of $e^{-\lambda\zeta}\eta$ at the moment ζ. The present value of this strategy is $\mathbb{E}\left[e^{-r\zeta}e^{-\lambda\zeta}\eta\right] < \eta$ since $r+\lambda > 0$. Hence, the strategy ζ is more profitable for the writer than immediate canceling. $\square$

Proposition 10.2. *If $x \in \Upsilon^b$ and $0 < y < x$, then $y \in \Upsilon^b$ too.*

Proof: Note that x has to be below the strike, $x < K$. Let the option maturates at date T. Let $A(\zeta; x)$ be the ζ-writer's optimal strategy for a holder's strategy $\zeta \in \mathcal{T}[0, T]$. Using inequality (9.6), we derive

$$\mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(K - S_\zeta)^+ I_{\zeta \le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \left(\left(K - S_{A(\zeta;x)}\right)^+ + \eta\right) I_{A(\zeta;x)<\zeta} \end{array} \right] \le (K - x)\,.$$

Using that expectation (9.4) has a minimum for $A(\zeta; y)$ and Lemma 3.1, we obtain

$$
\begin{aligned}
&\mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(K-S_\zeta)^+ I_{\zeta \le A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)} \left(\left(K - S_{A(\zeta;y)}\right)^+ + \eta \right) I_{A(\zeta;y)<\zeta} \end{array} \right] - (K-y) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(K-S_\zeta)^+ I_{\zeta \le A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)} \left(\left(K - S_{A(\zeta;y)}\right)^+ + \eta \right) I_{A(\zeta;y)<\zeta} \end{array} \right] \\
&+ \mathbb{E}^y \left[e^{-(r+\lambda)\zeta} e^{\lambda\zeta} S_\zeta I_{\zeta \le A(\zeta;y)} + e^{-(r+\lambda)A(\zeta;y)} e^{\lambda A(\zeta;y)} S_{A(\zeta;y)} I_{A(\zeta;y)<\zeta} \right] - K \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(\left(e^{\lambda\zeta}-1\right) S_\zeta + K, e^{\lambda\zeta} S_\zeta \right) I_{\zeta \le A(\zeta;y)} \\ +e^{-(r+\lambda)A(\zeta;y)} \max \left(\begin{array}{l} \left(e^{\lambda A(\zeta;y)} - 1\right) S_{A(\zeta;y)} + K + \eta, \\ e^{\lambda A(\zeta;y)} S_{A(\zeta;y)} + \eta \end{array} \right) I_{A(\zeta;y)<\zeta} \end{array} \right] - K \\
&\le \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(\left(e^{\lambda\zeta}-1\right) S_\zeta + K, e^{\lambda\zeta} S_\zeta \right) I_{\zeta \le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \max \left(\begin{array}{l} \left(e^{\lambda A(\zeta;x)} - 1\right) S_{A(\zeta;x)} + K + \eta, \\ e^{\lambda A(\zeta;x)} S_{A(\zeta;x)} + \eta \end{array} \right) I_{A(\zeta;x)<\zeta} \end{array} \right] - K \\
&< \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)\zeta} \max\left(\left(e^{\lambda\zeta}-1\right) S_\zeta + K, e^{\lambda\zeta} S_\zeta \right) I_{\zeta \le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \max \left(\begin{array}{l} \left(e^{\lambda A(\zeta;x)} - 1\right) S_{A(\zeta;x)} + K + \eta, \\ e^{\lambda A(\zeta;x)} S_{A(\zeta;x)} + \eta \end{array} \right) I_{A(\zeta;x)<\zeta} \end{array} \right] - K \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)\zeta}(K-S_\zeta)^+ I_{\zeta \le A(\zeta;x)} \\ +e^{-(r+\lambda)A(\zeta;x)} \left(\left(K - S_{A(\zeta;x)}\right)^+ + \eta \right) I_{A(\zeta;x)<\zeta} \end{array} \right] - (K-x) \le 0.
\end{aligned}
$$

The last step is to take the limit $T \to \infty$. □

Proposition 10.3. *If $x \in \Upsilon^s$ and $x < y \le K$, then $y \in \Upsilon^s$.*

Proof: Let again the maturity T be finite, $\zeta \in \mathcal{T}[0,T]$ be a writer's strategy, and $B(\zeta;x)$ be the related ζ-optimal holder's strategy. Therefore, inequality (9.7) gives

$$
\mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \left(K - S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta} \left((K-S_\zeta)^+ + \eta\right) I_{\zeta<B(\zeta;x)} \end{array} \right] \ge (K - x + \eta).
$$

Similarly to Proposition 10.2, we conclude

$$
\begin{aligned}
&\mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K - S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\leq\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ + \eta\right) I_{\zeta<B(\zeta;y)} \end{array} \right] - (K - y + \eta) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K - S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\leq\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ + \eta\right) I_{\zeta<B(\zeta;y)} \end{array} \right] \\
&+ \mathbb{E}^y \left[e^{-(r+\lambda)B(\zeta;y)} e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} I_{B(\zeta;y)\leq\zeta} + e^{-(r+\lambda)\zeta} e^{\lambda\zeta} S_\zeta I_{\zeta<B(\zeta;y)}\right] - (K+\eta) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \max\left(\begin{array}{c} \left(e^{\lambda B(\zeta;y)} - 1\right) S_{B(\zeta;y)} + K, \\ e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} \end{array} \right) I_{B(\zeta;y)\leq\zeta} \\ +e^{-(r+\lambda)\zeta} \max\left(\begin{array}{c} \left(e^{\lambda\zeta} - 1\right) S_\zeta + K + \eta, \\ e^{\lambda\zeta} S_\zeta + \eta \end{array} \right) I_{\zeta<B(\zeta;x)} \end{array} \right] - (K+\eta) \\
&\geq \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max\left(\begin{array}{c} \left(e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} + K, \\ e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{array} \right) I_{B(\zeta;x)\leq\zeta} \\ +e^{-(r+\lambda)\zeta} \max\left(\begin{array}{c} \left(e^{\lambda\zeta} - 1\right) S_\zeta + K + \eta, \\ e^{\lambda\zeta} S_\zeta + \eta \end{array} \right) I_{\zeta<B(\zeta;x)} \end{array} \right] - (K+\eta) \\
&> \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max\left(\begin{array}{c} \left(e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} + K, \\ e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{array} \right) I_{B(\zeta;x)\leq\zeta} \\ +e^{-(r+\lambda)\zeta} \max\left(\begin{array}{c} \left(e^{\lambda\zeta} - 1\right) S_\zeta + K + \eta, \\ e^{\lambda\zeta} S_\zeta + \eta \end{array} \right) I_{\zeta<B(\zeta;x)} \end{array} \right] - (K+\eta) \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K - S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\leq\zeta} \\ +e^{-(r+\lambda)\zeta}\left(\left(K-S_\zeta\right)^+ + \eta\right) I_{\zeta<B(\zeta;x)} \end{array} \right] - (K - x + \eta) \geq 0.
\end{aligned}
$$

It remains to take the limit $T \to \infty$ to finish the proof. □

Proposition 10.4. *When the risk-free rate is positive, $r > 0$, the writer's optimal region is either empty, $\Upsilon^s = \emptyset$, or it is the singleton $\Upsilon^s = \{K\}$.*

Proof: It is proven in Proposition 10.1 that $(K, \infty) \cap \Upsilon^s = \emptyset$. Suppose now that $x < K$, $x \in \Upsilon^s$, and x is not the exercise boundary. Using Proposition 10.2, we conclude that there exists a constant $K_1 < x$ that is not optimal for the holder, $K_1 \notin \Upsilon^b$. Let ζ be the first exit of the underlying asset from the strip (K_1, K). Proposition 10.2 leads to $B(\zeta; x) > \zeta$. Hence,

$$
\begin{aligned}
&\mathbb{E}^x \left[e^{-(r+\lambda)\zeta}\left(K - S_\zeta + \eta\right) I_{\zeta\leq B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}\left(K - S_\zeta\right) I_{B(\zeta;x)<\zeta}\right] \\
&\leq \mathbb{E}^x \left[e^{-r\zeta}\left(K - S_\zeta + \eta\right)\right] < K + \eta - x,
\end{aligned}
$$

which means that $x \notin \Upsilon^s$. □

Note that in the terminal case $\eta = 0$, Propositions 10.2 and 10.3 still hold. The following statement gives the form of the exercise regions in this case.

Proposition 10.5. *If $\eta = 0$ and $x < K$, then the immediate exercise is optimal for one of the option participants.*

Proof: Let the initial asset value belong to the interval (A, B) for some constants, $A < x < B \leq K$. Let us define a financial instrument that pays the amount of $K - B$ if the asset leaves the strip (A, B) from the upper boundary. If the exit occurs from the lower one, then the derivative has to pay the amount of $K - A$. We shall denote its price by $f(A, B, x)$. Let $B(A, x)$ be the upper boundary that minimizes the derivative price in the interval $[x, K]$ assuming that the lower boundary is A. Analogously, let $A(B, x)$ be the boundary that maximizes the price in the interval $(0, x]$ if the upper boundary is B.

Suppose the value x is not optimal for the writer, $x \notin \Upsilon^s$. Proposition 10.2 leads to the existence of $\overline{B} > x$ that satisfies

$$f\left(A\left(\overline{B}, x\right), \overline{B}, x\right) < K - x.$$

Using that $A(B, x)$ maximizes the derivative price, we see

$$f\left(A, \overline{B}, x\right) \leq f\left(A\left(\overline{B}, x\right), \overline{B}, x\right) < x - K$$

for every $A \in (0, x]$. Having in mind that the function $B(A, x)$ minimizes the derivative price, we conclude

$$f(A, B(A, x), x) \leq f\left(A, \overline{B}, x\right) < x - K.$$

This means that $x \in \Upsilon^b$. □

10.3 Pricing cancellable puts

Propositions 10.2 and 10.3 show that the holder's exercise region is of the form $\Upsilon^b = (0, A]$, whereas the writer's one exhibits one of the following three forms – $\Upsilon^s = [B, K]$, $\Upsilon^s = \{K\}$, or $\Upsilon^s = \emptyset$.

Suppose that the writer's boundary is less than the strike, $B < K$ and the initial asset price is between the boundaries, $x \in [A, B]$. Thus the pricing task turns into a first exit problem of a Brownian motion from a strip. Using the same notations as in Section 9.4 as well as Lemma 3.3, we derive the price of the cancellable put

$$\begin{aligned} f(A,B,x) &= \mathbb{E}\left[\begin{array}{l} e^{-(r+\lambda)\zeta^B}\left(K - S_{\zeta^B} + \eta\right) I_{\zeta^B \le \zeta^A} \\ +e^{-(r+\lambda)\zeta^A}\left(K - S_{\zeta^A}\right) I_{\zeta^A < \zeta^B} \end{array}\right] \\ &= (K-A)\left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + (K - B + \eta)\left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}, \end{aligned} \tag{10.2}$$

where the constants p and q are defined by formulas (3.17). Recall that $p \ge q + 1$ and the equality holds only when $\lambda = 0$. Let us fix the upper boundary B. We shall use the change of variables $a = \frac{A}{B}$, $k = \frac{K}{B}$, $y = \frac{x}{B}$, and $\xi = \frac{\eta}{B}$. Hence $0 < a < 1 < k$. Thus equation (10.2) turns into

$$f(A,B,x) = \frac{B}{y^q}\frac{(k-a)\,a^q\,(1-y^p) + (k-1+\xi)\,(y^p - a^p)}{1-a^p}.$$

We have to derive the value of the variable a that maximizes the function

$$\begin{aligned} g(a;y) &= \frac{(k-a)\,a^q\,(1-y^p) + (k-1+\xi)\,(y^p - a^p)}{1-a^p} \\ &= \frac{-a^p\,(k-1+\xi) - a^{q+1}\,(1-y^p) + a^q k\,(1-y^p) + y^p\,(k-1+\xi)}{1-a^p}. \end{aligned}$$

We obtain its derivative:

$$g_a(a;y) = \frac{1-y^p}{(1-a^p)^2}a^{q-1}\left[\begin{array}{l} -a^{p+1}\,(p-q-1) + a^p k\,(p-q) \\ -a^{p-q}p\,(k-1+\xi) - a\,(q+1) + qk \end{array}\right]. \tag{10.3}$$

It is shown in Appendix 10.B that the equation

$$-a^{p+1}\,(p-q-1) + a^p k\,(p-q) - a^{p-q}p\,(k-1+\xi) - a\,(q+1) + qk = 0$$

has a unique root in the interval $(0,1)$ – we notate it by $a(B)$. Not that it is independent of the variable y and depends on B via k and ξ. The function

$g(a;y)$ has a maximum at this point since derivative (10.3) is positive before the root and negative after it. Hence, the holder has to follow the strategy of the first hit to the value $A = a(B)B$ if the writer waits for the asset to reach the level B.

The next step is to examine the writer's exercise boundary. Let us fix the lower boundary A and make an analogous change of the variables – $b = \frac{B}{A}$, $k = \frac{K}{A}$, $y = \frac{x}{A}$, and $\xi = \frac{\eta}{A}$. We have that $1 < b$. Hence, equation (10.2) turns into

$$f(A,B,x) = \frac{A}{y^q}\frac{(k-1)(b^p - y^p) + (k-b+\xi)b^q(y^p-1)}{b^p-1}. \tag{10.4}$$

The b-derivative of the function $g(\cdot;\cdot)$,

$$g(b;y) = \frac{(k-1)(b^p - y^p) + (k-b+\xi)b^q(y^p-1)}{b^p-1},$$

is

$$g_b(b;y) = \frac{y^p-1}{(b^p-1)^2}b^{q-1}\left[\begin{array}{c} b^{p+1}(p-q-1) - b^p(k+\xi)(p-q) \\ +b^{p-q}p(k-1) + b(q+1) - q(k+\xi) \end{array}\right]. \tag{10.5}$$

Suppose that the discount factor exists, i.e. $\lambda > 0$. It is shown in Appendix 10.A that the equation

$$b^{p+1}(p-q-1) - b^p(k+\xi)(p-q) + b^{p-q}p(k-1) + b(q+1) - q(k+\xi) = 0$$

has a unique root larger than 1 – we denote it by $b(A)$. Function (10.4) achieves its minimum namely at this point. Hence, if the holder exercises when the underlying asset hits the level A, then the writer's best strategy is to cancel the contract when the asset reaches the value $B = b(A)A$. Ergo, we derive the candidate for the writer's exercise boundary solving the equation

$$a(y)\,b(ya(y)) = 1. \tag{10.6}$$

Accordingly, we obtain the holder's optimal boundary as $A = a(B)B$, where B is the root of equation (10.6). Suppose first that $B \le K$. Hence, the optimal regions are $\Upsilon^b \equiv [0,A)$ and $\Upsilon^s \equiv [B,K]$ and therefore: (1) if the initial asset price is less than the holder's boundary, $x < A$, then the option price is $K - x$; (2) if $x \in [A, B]$, the option price is obtained by equation (10.2); (3) if $x \in [B, K]$, then the option price is $K - x + \eta$; and (4) if $K < x$, then the writer cancels when the asset reaches the strike and therefore the option price is

$$\mathbb{E}^x \left[e^{-(r+\lambda)\tau} \left((S_\tau - K)^+ + \eta \right) I_{\tau<\infty} \right] = \eta \mathbb{E}^x \left[e^{-(r+\lambda)\tau} I_{\tau<\infty} \right] = \eta \left(\frac{K}{x} \right)^q .$$

The last formula is obtained using equation (3.15) from Proposition 3.5.

Suppose now that the solution of equation (10.6) is larger than the strike. Having in mind that early canceling above the strike is never optimal for the writer (due to Proposition 10.1), we conclude that its true optimal region is either the singleton $\{K\}$ or the empty set. We have to recognize which case is actual. The writer has the alternatives to cancel the option when the asset reaches the strike or to do nothing. In the second case, the cancellable put turns into an ordinary American put – its optimal boundary is

$$A^* = \frac{q}{q+1} K. \tag{10.7}$$

due to Theorem 4.2. This theorem shows also that if the asset starts from the strike, i.e. $x = K$, then the American put price is

$$\overline{\eta} = \frac{K}{q+1} \left(\frac{q}{q+1} \right)^q . \tag{10.8}$$

If the writer chooses to cancel the option immediately, he has to pay the amount of η. Hence, its optimal region is the singleton $\{K\}$ when $\eta < \overline{\eta}$, and it is the empty set, otherwise. We can proceed similarly to the case $B < K$ if $\eta < \overline{\eta}$. Otherwise, the option turns into an ordinary American put when $\eta \geq \overline{\eta}$ – we can use the results of Theorem 4.2.

Suppose now that the option is without discounting, i.e. $\lambda = 0$ and therefore $r > 0$. It is proven in Appendix 10.A that derivative (10.5) is negative in the whole interval $(1, \infty)$. This means that price function (10.4) is minimized for $B = \infty$. Hence, the derived candidate for the writer's boundary is larger than the strike and thus we can use the arguments above. Note that this conclusion corresponds to Proposition 10.4.

We summarize the obtained results in the following theorem.

Theorem 10.1. *[Theorem 3.1 of Zaevski (2020c)]*

Suppose that the algorithm above produces values $\overline{A}$ and $\overline{B}$ for the boundaries. We shall denote by V the option price. We have

1. *When $\overline{B} < K$, the optimal regions are $\Upsilon^s = \left[\overline{B}, K\right]$ and $\Upsilon^b = \left(0, \overline{A}\right]$. The price of the cancellable put can be derived via one of the following statements.*

   (a) *If $x \geq K$, then*

$$V = \eta \left(\frac{K}{x} \right)^q . \tag{10.9}$$

   (b) *If $\overline{B} < x < K$, then*

$$V = K - x + \eta.$$

   (c) *If $\overline{A} \leq x \leq \overline{B}$, then*

$$V = \left(K - \overline{A}\right) \left(\frac{\overline{A}}{x} \right)^q \frac{\overline{B}^p - x^p}{\overline{B}^p - \overline{A}^p} + \left(K - \overline{B} + \eta\right) \left(\frac{\overline{B}}{x} \right)^q \frac{x^p - \overline{A}^p}{\overline{B}^p - \overline{A}^p} . \tag{10.10}$$

   (d) *If $x < \overline{A}$, then*

$$V = K - x. \tag{10.11}$$

2. *If $K \leq \overline{B}$, then*

   (a) *If $\eta \geq \overline{\eta}$, then the cancellable put turns into an ordinary American put and the exercise regions turn into $\Upsilon^s = \emptyset$ and $\Upsilon^b = (0, A^*]$. Equation (10.7) determines the holder's optimal boundary A^*. If the asset starts below this value, $x \in \Upsilon^b \equiv (0, A^*]$, then the option price is (10.11). Otherwise, we derive it as*

$$V = \left(\frac{q}{x} \right)^q \left(\frac{K}{q+1} \right)^{q+1} .$$

(b) When $\eta < \overline{\eta}$, the writer's optimal region is the singleton $\Upsilon^s = \{K\}$. The holder's exercise region is $\Upsilon^b = \left(0, \overline{A}\right]$, where the value of $\overline{A}$ is derived as $\overline{A} = Ka(K)$, where $a(K)$ is the root of function (10.5).

If the asset starts above the strike, then formula (10.9) determines the option price. If $x \in \left(\overline{A}, K\right)$, then formula (10.10) gives the option price. Finally, if the asset starts from the holder's exercise region, $x \leq \overline{A}$, then the price of the option is obtained by formula (10.11).

Remark 10.1. *It is proven in Proposition 10.4 that if the risk-free rate is positive, then the writer's exercise region is either the empty set or the singleton $\{K\}$. Hence, the second case of Theorem 10.1 holds.*

10.4 Numerical results

As we mentioned above, there is a significant difference when the risk-free rate is positive or negative. We consider first positive values, particularly $r = 0.05$. Hence, the writer's optimal region consists only of the strike or it is the empty set. We assume also that the volatility is $\sigma = 0.3$, the strike is $K = \$10$, and the initial asset value is $x = \$8$. The penalty η and the discount rate λ are considered in the intervals $\eta \in (0, 5)$ and $\lambda\,(0.0001, 0.1)$. We present the results in Figure 10.2. The holder's optimal boundary is presented in Figure 10.2a, whereas the price behavior can be seen in Figure 10.2b. We mark by red points the critical values $\overline{\eta}$ given by equation (10.8). When the penalty is above $\overline{\eta}$, the cancellable puts turn into ordinary American puts.

The option prices for some particular values are reported in Table 10.1. We place right to the prices the case of Theorem 10.1 that holds. We can see that the prices decrease w.r.t. the initial asset value and increase w.r.t. the penalty η. After the critical value $\overline{\eta}$, the option prices do not depend on the penalty since it turns into an ordinary American put.

Suppose now that $r = -0.05$. We shall vary the discount rate λ between 0.051 and 0.1 since $r + \lambda > 0$. We mark by red points the values of the penalty η above which the boundary B is larger than the strike K – in this point, the actual case of Theorem 10.1 changes from the first to the second one. The points marked by green present the values of $\overline{\eta}$ from equation (10.8). Hence, the writer's optimal region is the strip $[B, K]$ before the red points, it is the singleton $\{K\}$ between the red and green points, and it is the empty set after the green points – the game option turns into an ordinary American put. Figure 10.2e presents the writer's and holder's boundaries when $\lambda = 0.1$. The meaning of the red and green points is the same – their values are 1.0537 and 4.1404. We can also find a validation of Proposition 10.5 in this figure – when the penalty η tends to zero, both boundaries tend to the same value.

We give the price behavior of the cancellable puts in Figure 10.2f. In Table 10.2 are reported some particular prices for different parameter values. The initial asset price is taken to be $x \in \{7, 8, 9, 10, 11\}$, the penalty is among $\eta \in \{0.001, 1, 2, 3, 4\}$, and the discount rates are $\lambda \in \{0.051, 0.06, 0.15\}$. Note that $r + \lambda > 0$. We can see that for the large penalty values, the derived writer's boundary is above the strike – thus, the second case of Theorem 10.1 holds. Also, the cancellable puts turn into ordinary American puts when the penalty is sufficiently large.

10.5 Conclusions

We have examined cancellable puts in this chapter. In addition to their classical form, we have introduced a discount factor that allows us to price options written on a continuously paying dividends underlying asset. We have obtained the early exercise regions for both contract participants. It turns out that the shape of these regions strongly depends on the particular values of the parameters. It is proven that the arising non-linear equations for the optimal boundaries have unique roots. The results derived for the different cases are summarized in Theorem 10.1. Some numerical examples are presented. The parameter values are chosen in a way to capture all different cases – positive and negative risk-free rates, large and small penalties and discount rates. All of the reported results validate the consistency of the proposed pricing model.

10.A Uniqueness of the solutions: writer's boundary

We shall prove that if $\lambda > 0$, then derivative (10.5) has just one root larger than one. Let the function $h(\cdot;\cdot)$ be

$$h(b;\xi) = b^{p+1}(p-q-1) - b^{p}(k+\xi)(p-q) + b^{p-q}p(k-1) + b(q+1) - q(k+\xi). \tag{10.12}$$

Suppose first that $\lambda = 0$ or equivalently $p = q+1$. The function (10.12) turns into

$$h(b;\xi) = -b^{q+1}(k+\xi) + b(q+1)k - q(k+\xi).$$

Its first derivative is

$$h_b(b;\xi) = -b^{q}(q+1)(k+\xi) + (q+1)k.$$

It is negative in the whole interval $b \in (1,\infty)$, and therefore the function $h(b;\xi)$ is decreasing. Hence, the equation $h(b;\xi) = 0$ has no roots larger than one since $h(1;\xi) = -p\xi < 0$.

Assume now that $p > q+1$. Changing the variables as $d = \frac{1}{b}$, we derive for the function $h(b;\xi)$

$$h(d;\xi) = \frac{1}{d^{p+1}}\left[\begin{array}{l} -d^{p+1}q(k+\xi) + d^{p}(q+1) + d^{q+1}p(k-1) \\ -d(k+\xi)(p-q) + (p-q-1) \end{array}\right].$$

Let the function $\overline{h}(\cdot;\cdot)$ be

$$\overline{h}(d;\xi) = -d^{p+1}q(k+\xi) + d^{p}(q+1) + d^{q+1}p(k-1) - d(k+\xi)(p-q) + (p-q-1). \tag{10.13}$$

Its first derivative is

$$\overline{h}_d(d;\xi) = -d^{p}(p+1)q(k+\xi) + d^{p-1}p(q+1) + d^{q}(q+1)p(k-1) - (k+\xi)(p-q).$$

We consider first the case $\xi = 0$. Note that $\overline{h}_d(d;\xi) < \overline{h}_d(d;0)$. The second derivative of function (10.13) is

$$\overline{h}_{dd}(d;0) = d^{q-1}p\left[\begin{array}{l} -d^{p-q}(p+1)qk \\ +d^{p-q-1}(p-1)(q+1) + q(q+1)(k-1) \end{array}\right].$$

Let the function $l(\cdot)$ be

$$l(d) = -d^{p-q}(p+1)qk + d^{p-q-1}(p-1)(q+1) + q(q+1)(k-1).$$

Its derivative is

$$l_d(d) = d^{p-q-2}\left[-d(p-q)(p+1)qk + (p-q-1)(p-1)(q+1)\right]. \quad (10.14)$$

Let us denote by $\overline{d}$ the positive root of function (10.14):

$$\overline{d} = \frac{(p-q-1)(p-1)(q+1)}{(p-q)(p+1)qk}.$$

Suppose first that $\overline{d} \geq 1$. The derivative $l_d(d)$ is positive in the interval $(0,1)$, and thus the function $l(d)$ is increasing. Hence, the function $l(d)$ is positive since $l(0) > 0$. Therefore, the derivative $\overline{h}_d(d;0)$ is increasing and thus it is negative in the interval $(0,1)$ since $\overline{h}_d(1;0) = 0$. Therefore $\overline{h}_d(d;\xi) < \overline{h}_d(d;0) < 0$ and, as a consequence, the function $\overline{h}(d;\xi)$ is decreasing. Hence, the equation $\overline{h}(d;\xi) = 0$ has only one root in the interval $(0,1)$ because $\overline{h}(0;\xi) = p-q-1 > 0$ and $\overline{h}(1;\xi) = -\xi p < 0$.

Suppose now that $\overline{d} < 1$. Therefore, $l_d(d) > 0$ for $d < \overline{d}$ and $l_d(d) < 0$ for $d > \overline{d}$. Having in mind $l(0) > 0$, we conclude that the function $l(d)$ starts from the positive value, increases to a maximum and then decreases. If $l(1) \geq 0$, then the previous case holds. Suppose that $l(1) < 0$. Hence, the function $\overline{h}_{dd}(d;0)$ is first positive and then negative. Hence, the function $\overline{h}_d(d;0)$ starts from the negative value $\overline{h}_d(0;0) = -k(p-q)$ increases to a positive maximum, and decreases to zero staying positive. Therefore the function $\overline{h}(d;0)$ starts from the positive value $p-q-1$ decreases to a negative minimum, and increases to zero. Note that if the function $\overline{h}(d;0)$ decreases, then $\overline{h}(d;\xi)$ decreases faster since $\overline{h}_d(d;\xi) < \overline{h}_d(d;0)$. Also, if $\overline{h}(d;0)$ increases, then $\overline{h}(d;\xi)$ increases slower or decreases. Therefore, the equation $\overline{h}(d;\xi) = 0$ has a unique root in the interval $(0,1)$.

Hence, the equation $h(b;\xi) = 0$ has only one root larger than one since $b = \frac{1}{d}$.

10.B Uniqueness of the solutions: holder's boundary

We shall prove that derivative (10.3) has only root in the interval $(0,1)$. Let $h(\cdot;\cdot)$ be the function

$$\begin{aligned} h(a;\xi) &= -a^{p+1}(p-q-1) + a^p k (p-q) \\ &\quad - a^{p-q} p (k-1+\xi) - a(q+1) + qk. \end{aligned} \tag{10.15}$$

We investigate first the undiscounted case $\lambda = 0$ or equivalently $p = q+1$. The function (10.15) turns into

$$h(a;\xi) = a^p k - ap(k+\xi) + (p-1)k.$$

Its derivative

$$h_a(a;\xi) = p\left(a^{p-1}k - k - \xi\right)$$

is negative in the interval $(0,1)$, and therefore the function $h(a;\xi)$ is decreasing. Thus, the equation $h(a;\xi) = 0$ has a unique root because $h(0;\xi) = (p-1)k > 0$ and $h(1;\xi) = -\xi p < 0$.

Assume now that $p > q+1$ and thus the derivative of function (10.15) turns into

$$\begin{aligned} h_a(a;\xi) &= -a^p(p+1)(p-q-1) + a^{p-1}pk(p-q) \\ &\quad - a^{p-q-1}(p-q)p(k-1+\xi) - (q+1). \end{aligned}$$

We shall investigate the case $\xi = 0$. We have $h_a(a;\xi) < h_a(a;0)$. The second derivative of function (10.15) is

$$h_{aa}(a;0) = a^{p-q-2} p \left[\begin{array}{l} -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(k-1) \end{array} \right].$$

Let the function $l(a)$ be

$$\begin{aligned} l(a) &= -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ &\quad - (p-q-1)(p-q)(k-1). \end{aligned}$$

Its derivative

$$l_a(a) = a^{q-1}\left[-a(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)\right]$$

has a unique root between zero and one

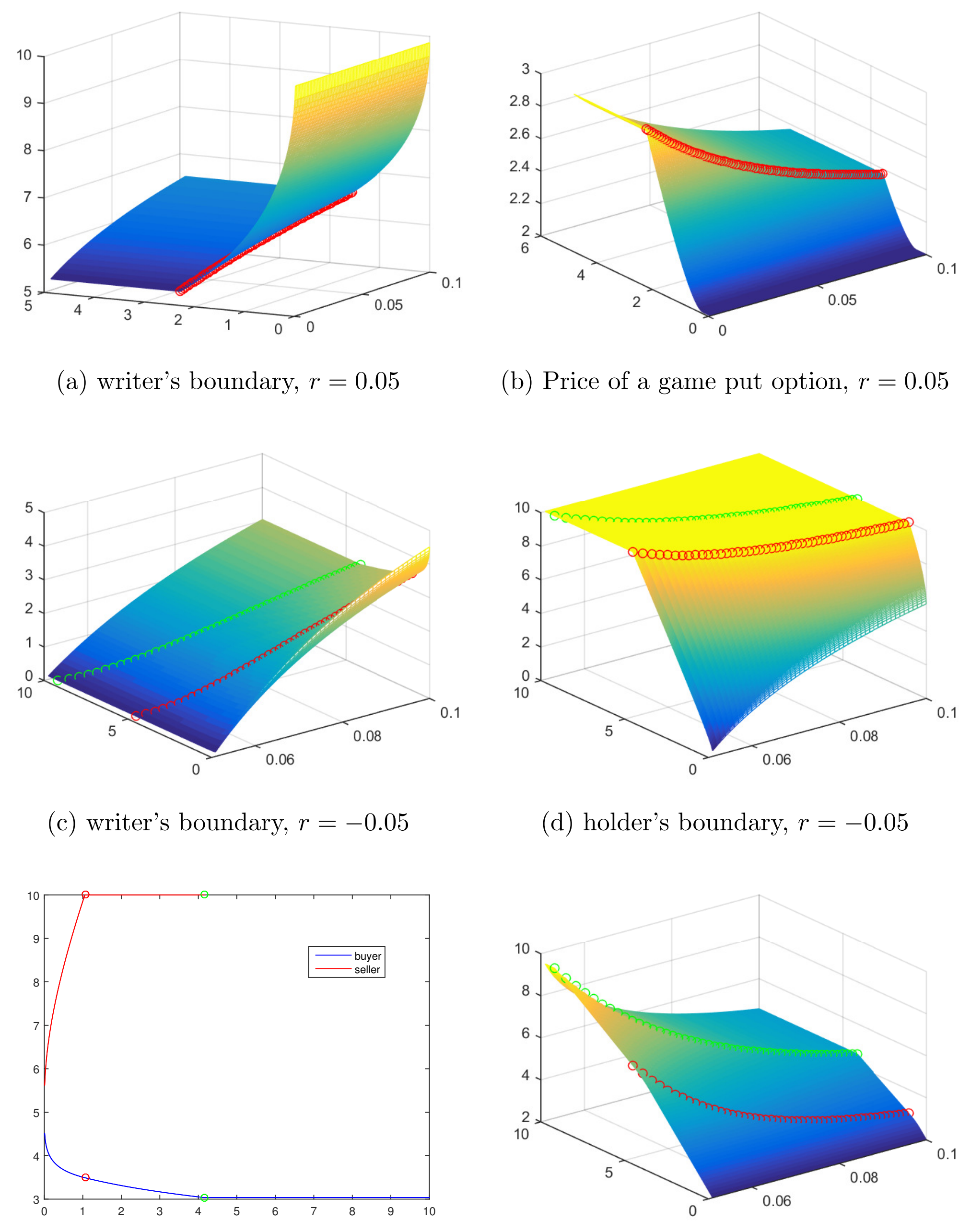
$$\overline{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}.$$

Suppose that $\overline{a} \geq 1$ and thus the derivative $l_a(a)$ is positive in the interval $(0,1)$. Hence, the function $l(a)$ is increasing. We have that $l(0) < 0$. If $l(1) < 0$, then $h_a(a;0)$ is decreasing and therefore it is negative since $h_a(0;0) = -(q+1) < 0$.[1] Otherwise, if $l(1) > 0$, then the function $h_a(a;0)$ starts from the negative value $h_a(0;0) = -(q+1)$, has a minimum, and increases to zero. Therefore, $h_a(a;0) < 0$ and thus the derivative $h_a(a;\xi)$ is negative too. Hence, the equation $h(a;\xi) = 0$ has only one root because $h(0;\xi) = qk > 0$ and $h(1;\xi) = -\xi p < 0$.

If $\overline{a} < 1$, then the function $l(a)$ starts from a negative value, increases to a maximum for $a = \overline{a}$, and then decreases. If $l(\overline{a}) < 0$ or $l(1) > 0$, then the previous case holds. Suppose that $l(\overline{a}) > 0$ and $l(1) < 0$. Therefore, the derivative $h_a(a;0)$ starts from the negative value $h_a(0;0) = -(q+1)$, decreases to a negative minimum, increases to a positive maximum, and decreases to zero. Hence, $h_a(a;0)$ is first negative and then positive. Therefore, the function $h(a;0)$ starts from the positive value $h(0;0) = qk$, decreases to a negative minimum, and increases to zero. The inequality $h_a(a;\xi) < h_a(a;0)$ means that when the function $h(a;0)$ decreases, the function $h(a;\xi)$ decreases faster. Also, if $h(a;0)$ increases, then $h(a;\xi)$ increases slower or decreases. Thus, we conclude that the function $h(a;\xi)$ has only one root.

[1] In fact this case is possible only if $\xi > 0$.

Figure 10.2: Put option boundaries



(a) writer's boundary, $r = 0.05$

(b) Price of a game put option, $r = 0.05$

(c) writer's boundary, $r = -0.05$

(d) holder's boundary, $r = -0.05$

(e) Both boundaries, $r = -0.05$, $\lambda = 0.1$

(f) Price of a game put option, $r = -0.05$

Table 10.1: Put option prices, $r = 0.05$

	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
initial price $S_0 = 7$					
$\lambda = 0.001$	3.0000/2.a	3.0245/2.b	3.3163/2.b	3.4402/1.b	3.4402/1.b
$\lambda = 0.01$	3.0000/2.a	3.0193/2.b	3.2915/2.b	3.3599/1.b	3.3599/1.b
$\lambda = 0.1$	3.0000/2.a	3.0000/2.a	3.0445/1.b	3.0445/1.b	3.0445/1.b
initial price $S_0 = 8$					
$\lambda = 0.001$	2.0000/2.a	2.2224/2.b	2.7722/2.b	2.9617/1.b	2.9617/1.b
$\lambda = 0.01$	2.0000/2.a	2.2131/2.b	2.7493/2.b	2.8580/1.b	2.8580/1.b
$\lambda = 0.1$	2.0000/2.a	2.1431/2.b	2.3679/1.b	2.3679/1.b	2.3679/1.b
initial price $S_0 = 9$					
$\lambda = 0.001$	1.0000/2.a	1.5622/2.b	2.3456/2.b	2.5952/1.b	2.5952/1.b
$\lambda = 0.01$	1.0000/2.a	1.5560/2.b	2.3320/2.b	2.4779/1.b	2.4779/1.b
$\lambda = 0.1$	1.0000/2.a	1.5045/2.b	1.8971/1.b	1.8971/1.b	1.8971/1.b
initial price $S_0 = 10$					
$\lambda = 0.001$	0.0010/2.c	1.0000/2.c	2.0000/2.c	2.3060/1.b	2.3060/1.b
$\lambda = 0.01$	0.0010/2.c	1.0000/2.c	2.0000/2.c	2.1810/1.b	2.1810/1.b
$\lambda = 0.1$	0.0010/2.c	1.0000/2.c	1.5559/1.b	1.5559/1.b	1.5559/1.b
initial price $S_0 = 11$					
$\lambda = 0.001$	0.0009/2.d	0.8986/2.d	1.7972/2.d	2.0722/1.b	2.0722/1.b
$\lambda = 0.01$	0.0009/2.d	0.8909/2.d	1.7819/2.d	1.9431/1.b	1.9431/1.b
$\lambda = 0.1$	0.0008/2.d	0.8358/2.d	1.3004/1.b	1.3004/1.b	1.3004/1.b

Table 10.2: Put option prices, $r = -0.05$

	$\eta = 0.001$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$
initial price $S_0 = 7$					
$\lambda = 0.051$	3.0010/1.b	4.0000/1.b	5.0000/1.b	6.0000/1.b	6.8529/1.c
$\lambda = 0.06$	3.0010/1.b	4.0000/1.b	4.9501/1.c	5.5321/2.b.2	5.9839/2.b.2
$\lambda = 0.15$	3.0010/1.b	3.3274/2.b.2	3.6184/2.b.2	3.9151/2.a.2	3.9151/2.a.2
initial price $S_0 = 8$					
$\lambda = 0.051$	2.0010/1.b	3.0000/1.b	4.0000/1.b	5.0000/1.b	5.9827/1.c
$\lambda = 0.06$	2.0010/1.b	3.0000/1.b	4.0000/1.b	4.7835/2.b.2	5.3922/2.b.2
$\lambda = 0.15$	2.0010/1.b	2.5763/2.b.2	3.0559/2.b.2	3.5321 /2.a.2	3.5321 /2.a.2
initial price $S_0 = 9$					
$\lambda = 0.051$	1.0010/1.b	2.0000/1.b	3.0000/1.b	4.0000/1.b	5.000/1.b
$\lambda = 0.06$	1.0010/1.b	2.0000/1.b	3.0000/1.b	3.9413/2.b.2	4.7325/2.b.2
$\lambda = 0.15$	1.0010/1.b	1.8114/2.b.2	2.5253/2.b.2	3.2255/2.a.2	3.2255/2.a.2
initial price $S_0 = 10$					
$\lambda = 0.051$	0.0010/1.b	1.0000/1.b	2.0000/1.b	3.0000/1.b	4.0000/1.b
$\lambda = 0.06$	0.0010/1.b	1.0000/1.b	2.0000/1.b	3.0000/2.b.2	4.0000/2.b.2
$\lambda = 0.15$	0.0010/1.b	1.0000/2.b.2	2.0000/2.b.2	2.9738/2.a.2	2.9738/2.a.2
initial price $S_0 = 11$					
$\lambda = 0.051$	0.0010/1.a	0.9990/1.a	1.9980/1.a	2.9970/1.a	3.9960/1.a
$\lambda = 0.06$	0.0010/1.a	0.9905/1.a	1.9809/1.a	2.9714/2.b.1	3.9619/2.b.1
$\lambda = 0.15$	0.0009/1.a	0.9291/2.b.1	1.8583/2.b.1	2.7631/2.a.2	2.7631/2.a.2

Chapter 11

Perpetual cancellable options with a proportional penalty

based on the paper

Zaevski, Tsvetelin S. "Perpetual game options with a multiplied penalty." Communications in Nonlinear Science and Numerical Simulation 85 (2020): 105248.

Abstract: The purpose of this chapter is to examine a special kind of game options, whose main feature is the presence of an early exercise right for the writer as well as for the holder. The writer has to pay some amount above the usual option payment for this right. Usually, this penalty is presented by a constant amount during the option life. Alternatively, we assume in this chapter it is a proportion of the usual option payoff.

11.1 Motivation and main results

The main novelty of this chapter is the form of the writer's cancellation penalty. We forsake the traditional assumption that the amount which the writer owes for its cancellation right is a constant during the option life. Alternatively, we assume that this cancellation penalty is proportional to the amount that the holder would receive if he decides to exercise. Thus, we introduce a penalty depending on the underlying asset price. In such a way, the writer has to pay a relatively small amount to cancel the contract if the underlying asset price is near the strike. This provides an additional advantage to him/her. Otherwise, the writer has to pay a larger penalty to cancel the contract if the holder could achieve a relatively large profit deciding to exercise the contract immediately. This is extra insurance for the holder in addition to his/her early exercise right. Thus we significantly change the option payment structure and therefore the new game options are qualitatively different from the existing ones.

Our goal in the examination of such options is first to derive the regions in which the immediate exercise is optimal. For different values of the parameters, the form of these regions can be quite different. For the call style options, the writer's exercise region is $(0, A]$ for some constant A larger than the strike K, $K \leq A < \infty$. Note that, unlike the usual game options, the values below the strike are always optimal for the writer. Another difference is that the writer's optimal boundary is finite even in the undiscounted case $\lambda = 0$. The holder's exercise region is of the form (B, ∞), where B is another constant $A \leq B$. This boundary is infinity when $\lambda = 0$. It turns out that both boundaries A and B coincide with the strike when the risk-free rate is negative. This means that the points above the strike are optimal for the holder whereas the rests are optimal for the writer.

The optimal regions stay similar but inverse for the put style options. The writer's exercise region turns into $[B, \infty)$ for some constant B less than the strike, $B \leq K$. The holder's region is $(0, A)$, where $A \leq B$. If the risk-free rate is positive, then $A = B = K$. This means that the points below the strike are optimal for the holder whereas these above are optimal for the writer.

Our approach to deriving the exercise regions is again based on finding the optimal stopping times. Let ζ be a stopping time. We shall denote by $A(\zeta)$ the writer's optimal strategy if the holder follows a strategy ζ. Similarly, we denote by $B(\zeta)$ the holder's optimal strategy assuming that the writer has a strategy ζ. We search for a stopping time such that $\zeta = A(B(\zeta))$. Note that we can restrict our examination to the first hitting times of the underlying asset to some levels. Although we assume that the asset price is driven by a Brownian motion, some results are true for an arbitrary Feller-Markov process. Also, some of the results are true for a finite time horizon. We discuss briefly the case when the maturity is finite.

The chapter is organized as follows. In Section 11.2 we present the base we use later. In Section 11.3 we obtain the optimal regions. In Section

11.4 we derive the fair option price of a game call option paying special attention to the undiscounted case in Section 11.5. The game put options are examined in Section 11.6. Some numerical results are presented in Section 11.7. In Appendices 11.8 we prove that the arising equations for the optimal boundaries have unique solutions.

11.2 Preliminaries

Let us consider the following payoff structures for the call options

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(x-K)^+ \\ N_2(t,x) &= e^{-\lambda t}\eta(x-K)^+ \end{aligned}$$

For the put options, we have

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\ N_2(t,x) &= e^{-\lambda t}\eta(K-x)^+, \end{aligned} \tag{11.1}$$

The time dependence of the option price can be derived again via Proposition 3.3. The optimal strategies and the related optimal regions Υ^b and Υ^s are defined by Definitions 9.1 and 9.2. The continuation region shall be denoted by $\overline{\Upsilon}$.

11.3 Exercise regions

From now on, we assume that $t = 0$. We shall prove series of propositions related to the optimal regions.

11.3.1 Call style options

We can formulate the result for the undiscounted case $\lambda = 0$ having in mind that Proposition 9.1 is proven for arbitrary payoff functions $N_1(t,x) < N_2(t,x)$.

Proposition 11.1. *If the discount factor is zero, $\lambda = 0$, then the holder's exercise region is empty – $\Upsilon^b = \emptyset$.*

Note that all points less the strike are optimal for the writer, because he does not have to pay anything.

Proposition 11.2. *If $x \leq K$, then $x \in \Upsilon^s$.*

The following proposition provides an important feature of the writer's exercise region.

Proposition 11.3. *Suppose that $x \in \Upsilon^s$ and $0 < y < x$. Then $y \in \Upsilon^s$ too.*

Proof: Note that Proposition 11.2 allows us to examine only the case $K < y < x$. Let T be a fixed finite moment, $\zeta \in \mathcal{T}[0,T]$, and $B(\zeta;x)$ be the ζ-holder's optimal strategy. Thus condition (9.7) leads to

$$\mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \left(S_{B(\zeta;x)} - K\right)^+ I_{B(\zeta;x)\leq\zeta} \\ +e^{-(r+\lambda)\zeta} \eta (S_\zeta - K)^+ I_{\zeta<B(\zeta;x)} \end{array} \right] \geq \eta\,(x - K)\,.$$

Using that the discounted asset price is a martingale and the stopping time $\zeta \wedge B(\zeta;y)$ is finite, we obtain

$$\mathbb{E}^y \left[e^{-rB(\zeta;y)} S_{B(\zeta;y)} I_{B(\zeta;y)\leq\zeta} + e^{-r\zeta} S_\zeta I_{\zeta<B(\zeta;y)}\right] = y. \tag{11.2}$$

Using equation (11.2), Lemma 3.1, and the fact that $B(\zeta;y)$ maximizes equation (9.5), we derive

$$
\begin{aligned}
&\mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \big(S_{B(\zeta;y)} - K\big)^+ I_{B(\zeta;y)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \eta (S_\zeta - K)^+ I_{\zeta < B(\zeta;y)} \end{array} \right] - \eta\,(y-K) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \big(S_{B(\zeta;y)} - K\big)^+ I_{B(\zeta;y)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \eta (S_\zeta - K)^+ I_{\zeta < B(\zeta;y)} \end{array} \right] \\
&- \eta \mathbb{E}^y \left[e^{-(r+\lambda)B(\zeta;y)} e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} I_{B(\zeta;y)\le\zeta} + e^{-(r+\lambda)\zeta} e^{\lambda\zeta} S_\zeta I_{\zeta<B(\zeta;y)} \right] + K\eta \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \max \begin{pmatrix} -\left(\eta e^{\lambda B(\zeta;y)} - 1\right) S_{B(\zeta;y)} - K, \\ -\eta e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} \end{pmatrix} I_{B(\zeta;y)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \eta \max \begin{pmatrix} -\left(e^{\lambda\zeta} - 1\right) S_\zeta - K, \\ -e^{\lambda\zeta} S_\zeta \end{pmatrix} I_{\zeta < B(\zeta;x)} \end{array} \right] + K\eta \\
&\ge \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max \begin{pmatrix} -\left(\eta e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} - K, \\ -\eta e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{pmatrix} I_{B(\zeta;x)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \eta \max \begin{pmatrix} -\left(e^{\lambda\zeta} - 1\right) S_\zeta - K, \\ -e^{\lambda\zeta} S_\zeta \end{pmatrix} I_{\zeta < B(\zeta;x)} \end{array} \right] + K\eta \\
&> \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max \begin{pmatrix} -\left(\eta e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} - K, \\ -\eta e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{pmatrix} I_{B(\zeta;x)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \eta \max \begin{pmatrix} -\left(e^{\lambda\zeta} - 1\right) S_\zeta - K, \\ -e^{\lambda\zeta} S_\zeta \end{pmatrix} I_{\zeta < B(\zeta;x)} \end{array} \right] + K\eta \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \big(S_{B(\zeta;x)} - K\big)^+ I_{B(\zeta;x)\le\zeta} \\ + e^{-(r+\lambda)\zeta} \eta (S_\zeta - K)^+ I_{\zeta < B(\zeta;x)} \end{array} \right] - \eta\,(x-K) \ge 0.
\end{aligned}
$$

We finish the proof by taking $T \to \infty$. □

The corresponding proposition for the holder's boundary cannot be proven similarly to Proposition 11.3. We shall use a different approach. First, note that Proposition 9.7 can be rewritten as

Proposition 11.4. *If $\eta = 1$ and $x > K$, then it is optimal for one of the holder or the writer to exercise the option immediately.*

Proposition 11.5. *Suppose that $x \in \Upsilon^b$ and $y > x$. Then $y \in \Upsilon^b$ too.*

Proof: Suppose that the proposition is not true, i.e. there exists $y > x$ such that $y \notin \Upsilon^b$. If the value y is optimal for the writer, then Proposition 11.3 shows that all values below y are also optimal. This can not be true because $x \in \Upsilon^b$ and $x < y$. Therefore $y \in \overline{\Upsilon}$. Note that $\{+\infty\} \in \Upsilon^b$. Hence, a point from the continuation region, say y, is between two points from the holder's exercise region (x and $+\infty$). Therefore, there exist two constants B and C, such that $x < B < y < C$, $\{B, C\} \in \Upsilon^b$, $(B, C) \in \overline{\Upsilon}$, and $y \in (B, C)$. Let the initial point be y and ζ_B and ζ_C be the first hitting times to the values B and C, respectively. Therefore, the holder's optimal strategy is $\zeta_B \wedge \zeta_C$. Also, it is not optimal for the writer to exercise before $\zeta_B \wedge \zeta_C$. Hence,

$$y - K < \mathbb{E}^y \left[e^{-(r+\lambda)(\zeta_B \wedge \zeta_C)} \left(S_{\zeta_B \wedge \zeta_C} - K \right) \right]. \tag{11.3}$$

Let us consider a new game option without penalty, i.e. $\eta = 1$, and the same other parameters. We shall denote by Υ_1^b, Υ_1^s, and $\overline{\Upsilon}_1$ its regions and by $A_1(\cdot)$ and $B_1(\cdot)$ the corresponding optimal stopping times. Suppose that $y \in \Upsilon_1^s$. Proposition 11.3 gives that $x \in \Upsilon_1^s$ since $x < y$. On the other hand, $x \in \Upsilon^b$. This means that it is optimal for the holder to exercise immediately, provided that the writer can cancel the contract by paying a positive penalty. Hence, if the writer's penalty is zero, then it is optimal for the holder to exercise immediately too. Therefore, $x \in \Upsilon_1^b$. This contradicts to $x \in \Upsilon_1^s$ since the sets Υ_1^s and Υ_1^b are disjoint. Hence $y \notin \Upsilon_1^s$. Note that this is true for all points from the interval (B, C).

Suppose now that $y \in \Upsilon_1^b$. It is proven above that there are no writer's optimal points in the interval (B, C). Hence, if $\zeta = \zeta_B \wedge \zeta_C$, then $A_1(\zeta, y) > \zeta$. Therefore,

$$\begin{aligned} y - k &\geq \mathbb{E}^y \left[e^{-(r+\lambda)(\zeta \wedge A_1(\zeta_C; y))} \left(S_{\zeta \wedge A_1(\zeta; y)} - K \right) \right] \\ &= \mathbb{E}^y \left[e^{-(r+\lambda)(\zeta_C \wedge \zeta_B)} \left(S_{\zeta_C \wedge \zeta_B} - K \right) \right], \end{aligned}$$

that contradicts inequality (11.3). Hence $y \notin \Upsilon_1^b$. Therefore $y \in \overline{\Upsilon}_1$. But this contradicts Proposition 11.4 because it states that $\overline{\Upsilon}_1$ is empty when $\eta = 1$. Therefore $y \in \Upsilon^b$. □

Remark 11.1. *Actually, the proof is based on the fact that if a part of the continuation region is between two points of the holder's optimal region, then the option payment structure is the same as the payment structure of a game option with* $\eta = 1$.

The following proposition provides that if the risk-free rate is negative, then the writer's exercise region consists only of the points below the strike.

Proposition 11.6. *If* $r < 0$*, then* $\Upsilon^s = (0, K]$.

Proof: Suppose that x is larger than the strike, $x > K$, $x \in \Upsilon^s$, and it is not an exercise boundary. Proposition 11.5 gives that there exists a constant $K_1 > x$, such that $K_1 \notin \Upsilon^b$ too. Let us define ζ as the first exit time from the strip (K, K_1). Proposition 11.5 shows also that $B(\zeta; x) > \zeta$. Hence,

$$\begin{aligned} &\mathbb{E}^x\left[e^{-(r+\lambda)\zeta}\eta\left(S_\zeta - K\right)I_{\zeta\le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)}\left(S_\zeta - K\right)I_{B(\zeta;x)<\zeta}\right] \\ &\le \mathbb{E}^x\left[e^{-r\zeta}\eta\left(S_\zeta - K\right)\right] < \eta\left(x - K\right), \end{aligned}$$

that leads to $x \notin \Upsilon^s$. We have used above the martingality of the discounted asset price. □

11.3.2 Put style options

Now we shall prove the analogues of the propositions in Section 11.3.1. First, note that all points above the strike are optimal for the writer since he does not have to pay anything.

Proposition 11.7. *If $x \ge K$, then $x \in \Upsilon^s$.*

Proposition 11.8. *Suppose that $x \in \Upsilon^s$ and $x < y$. Then $y \in \Upsilon^s$ too.*

Proof: We need to examine only the case $x < y < K$. Let T be a fixed finite moment, $\zeta \in \mathcal{T}[0,T]$, and $B(\zeta; x)$ be the corresponding ζ-holder's optimal strategy. Condition (9.7) leads to

$$\mathbb{E}^x\left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K - S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta(K - S_\zeta)^+ I_{\zeta<B(\zeta;x)} \end{array}\right] \ge \eta\left(K - x\right).$$

Analogously to the proof of Proposition 11.3, we derive

$$
\begin{aligned}
&\mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K - S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta(K - S_\zeta)^+ I_{\zeta<B(\zeta;y)} \end{array} \right] - \eta\,(K-y) \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)}\left(K - S_{B(\zeta;y)}\right)^+ I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta(K - S_\zeta)^+ I_{\zeta<B(\zeta;y)} \end{array} \right] \\
&+ \eta\mathbb{E}^y \left[e^{-(r+\lambda)B(\zeta;y)}e^{\lambda B(\zeta;y)}S_{B(\zeta;y)}I_{B(\zeta;y)\le\zeta} + e^{-(r+\lambda)\zeta}e^{\lambda\zeta}S_\zeta I_{\zeta<B(\zeta;y)}\right] - K\eta \\
&= \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;y)} \max\left(\begin{array}{c} \left(\eta e^{\lambda B(\zeta;y)} - 1\right) S_{B(\zeta;y)} + K, \\ \eta e^{\lambda B(\zeta;y)} S_{B(\zeta;y)} \end{array} \right) I_{B(\zeta;y)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta \max\left(\begin{array}{c} \left(e^{\lambda\zeta} - 1\right) S_\zeta + K, \\ e^{\lambda\zeta} S_\zeta \end{array} \right) I_{\zeta<B(\zeta;x)} \end{array} \right] - K\eta \\
&\ge \mathbb{E}^y \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max\left(\begin{array}{c} \left(\eta e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} + K, \\ \eta e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{array} \right) I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta \max\left(\begin{array}{c} \left(e^{\lambda\zeta} - 1\right) S_\zeta + K, \\ e^{\lambda\zeta} S_\zeta \end{array} \right) I_{\zeta<B(\zeta;x)} \end{array} \right] - K\eta \\
&> \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)} \max\left(\begin{array}{c} \left(\eta e^{\lambda B(\zeta;x)} - 1\right) S_{B(\zeta;x)} + K, \\ \eta e^{\lambda B(\zeta;x)} S_{B(\zeta;x)} \end{array} \right) I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta \max\left(\begin{array}{c} \left(e^{\lambda\zeta} - 1\right) S_\zeta + K, \\ e^{\lambda\zeta} S_\zeta \end{array} \right) I_{\zeta<B(\zeta;x)} \end{array} \right] - K\eta \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;x)}\left(K - S_{B(\zeta;x)}\right)^+ I_{B(\zeta;x)\le\zeta} \\ +e^{-(r+\lambda)\zeta}\eta(K - S_\zeta)^+ I_{\zeta<B(\zeta;x)} \end{array} \right] - \eta\,(K-x) \ge 0.
\end{aligned}
$$

We finish the proof by taking $T \to \infty$. □

Note that Proposition 11.4 still holds:

Proposition 11.9. *If $\eta = 1$ and $x < K$, then the immediate exercise is optimal for one of the holder or the writer.*

Proof: The proof is very close to the proof of Proposition 10.5 and we omit it. □

The following proposition for the holder's exercise boundary stands.

Proposition 11.10. *Suppose that $x \in \Upsilon^b$ and $0 < y < x$. Then $y \in \Upsilon^b$ too.*

Proof: The proof is identical to the proof of Proposition 11.5. □

Finally, we shall prove the analogue of Proposition 11.6.

Proposition 11.11. *If $r > 0$, then $\Upsilon^s = [K, \infty)$.*

Proof: Suppose that $x < K$, $x \in \Upsilon^s$, and x is not the exercise boundary. Using Proposition 11.10, we see that there exists a constant $K_1 < x$, such that $K_1 \notin \Upsilon^b$. Let ζ be the first exit from the strip (K_1, K). Proposition 11.10 leads also to $B(\zeta; x) > \zeta$. Therefore,

$$\begin{aligned} &\mathbb{E}^x \left[e^{-(r+\lambda)\zeta} \eta (K - S_\zeta) I_{\zeta \le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)} (K - S_\zeta) I_{B(\zeta;x)<\zeta} \right] \\ &\le \mathbb{E}^x \left[e^{-r\zeta} \eta (K - S_\zeta) \right] < \eta (K - x) . \end{aligned}$$

Therefore $x \notin \Upsilon^s$. □

11.3.3 Some notes on the finite maturity case

Although the purpose of this chapter is to study game options without maturity restrictions, most of the proven propositions still hold when the maturity is finite, $T < \infty$. Unfortunately, we can not expect in this case that the exercise boundaries are flat. However, the obtained propositions give us some hints about their form.

Let us consider the call style options. First, Proposition 11.2 states that all points below the strike are optimal for the writer. Also, if the current asset value is larger than the strike and the time to maturity is small enough, then the writer should keep the option till maturity. This means that the writer's exercise boundary is the strike near the maturity. If the penalty coefficient is small enough and the risk-free rate is positive (see Proposition 11.6), we may expect that the writer's exercise boundary is larger than the strike for large enough maturities. However, in both cases, Proposition 11.3 shows that all points below the writer's exercise boundary are optimal for him. Proposition 11.1 shows that it is never optimal for the holder to exercise when $\lambda = 0$. Otherwise, if $\lambda > 0$, then Proposition 11.5 provides that all points above the holder's exercise boundary are optimal for the holder.

The put style options lead to, in some sense, symmetric exercise regions. All points above the strike are optimal for the writer and near the maturity they are the unique such points. Proposition 11.11 indicates that when the risk-free rate is positive, only the points above the strike form the writer's optimal region. Otherwise, if $r < 0$ and the penalty coefficient is small enough, then there exists a critical value for the time to maturity above which the writer's exercise boundary is below the strike. However, regardless of the writer's boundary value, all points above it are optimal for the writer – see Proposition 11.8. Also, Proposition 11.10 hints that there exists a boundary below which early exercising is optimal for the holder.

11.4 Pricing call options

Propositions 11.2, 11.3, and 11.5 prompt us to expect that the exercise regions have the form $\Upsilon^s = (0, A]$ and $\Upsilon^b = (B, \infty)$, $K \le A \le B$. As we shall see later, both boundaries can be equal to the strike. For this, we assume that the writer's optimal region is closed whereas the holder's one is open. Note that the strike is always optimal for the writer and never for the holder since the option payment is zero. Suppose that the initial asset price is between A and B, $A < x \le B$, and ζ_A and ζ_B are the first hitting times of the asset to the boundaries A and B. Using Lemma 3.3 for the Laplace transforms of the first exit time of the Brownian motion from a strip, we derive the price of an arising financial instrument as

$$\begin{aligned} f(A,B,x) &= \mathbb{E}\left[\begin{array}{l} e^{-(r+\lambda)\zeta^B}\left(S_{\zeta^B}-K\right)I_{\zeta^B\le\zeta^A} \\ +e^{-(r+\lambda)\zeta^A}\eta\left(S_{\zeta^A}-K+\eta\right)I_{\zeta^A<\zeta^B} \end{array}\right] \\ &= \eta(A-K)\left(\frac{A}{x}\right)^q\frac{B^p-x^p}{B^p-A^p}+(B-K)\left(\frac{B}{x}\right)^q\frac{x^p-A^p}{B^p-A^p}. \end{aligned} \tag{11.4}$$

We shall fix one of the boundaries A/B and then we shall derive which value of B/A maximizes/minimizes function (11.4). First, suppose that B is fixed and let us change the variables as $a = \frac{A}{B}$, $k = \frac{K}{B}$, and $y = \frac{x}{B}$. Hence, $0 < k < a < 1$. Equation (11.4) can be rewritten as

$$f(A,B,x) = \frac{B}{y^q}\frac{\eta(a-k)a^q(1-y^p)+(1-k)(y^p-a^p)}{1-a^p}. \tag{11.5}$$

We have to find the value of the variable a that minimizes the function

$$g(a) = \frac{\eta(a-k)a^q(1-y^p)+(1-k)(y^p-a^p)}{1-a^p}.$$

Its first derivative is

$$g_a(a) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \left[\begin{array}{c} a^{p+1}\eta(p-q-1) - a^p\eta k(p-q) \\ -a^{p-q}p(1-k) + a(q+1)\eta - q\eta k \end{array} \right]. \tag{11.6}$$

We prove in Appendix 11.A that (11.6) has just one root in the interval $(0,1)$ – we denote it by $a(B)$. Note that it is independent of the variable y, respectively x. It leads to a minimum for option price (11.5). Hence, if the holder's strategy is the first hit to the value B, then the writer's strategy is the first hit to $Ba(B)$.

We proceed analogously for the holder's boundary. Let us fix A and change the variables as $b = \frac{B}{A}$, $k = \frac{K}{A}$, and $y = \frac{x}{A}$. Thus, price (11.4) turns into

$$f(A,B,x) = \frac{A}{y^q} \frac{\eta(1-k)(b^p - y^p) + (b-k)b^q(y^p-1)}{b^p-1}. \tag{11.7}$$

Let the function $g(\cdot)$ be defined as

$$g(b) = \frac{A}{y^q} \frac{\eta(1-k)(b^p - y^p) + (b-k)b^q(y^p-1)}{b^p-1}. \tag{11.8}$$

Its derivative is

$$g_b(b;y) = \frac{y^p-1}{(b^p-1)^2} b^{q-1} \left[\begin{array}{c} -b^{p+1}(p-q-1) + b^p k(p-q) \\ +b^{p-q}p(1-k)\eta - b(q+1) + qk \end{array} \right]. \tag{11.9}$$

We shall examine now the case when discounting indeed exists, i.e. $\lambda > 0$ equivalently to $p > q+1$. The undiscounted case is studied later. It is proven in Appendix 11.B that this derivative has just one root in the interval $(1,\infty)$ when $A > K$, or when $A = K$ and $r > 0$ – we shall denote this root by $b(A)$. It leads to a maximum for function (11.8). Hence, if the writer's strategy is the first hit to the value A, then the holder's strategy is the first hit to $Ab(A)$. Thus we can derive our candidate $\overline{A}$ for the writer's boundary as the solution of the equation

$$b(A)\,a(Ab(A)) = 1. \tag{11.10}$$

The holder's boundary candidate is $\overline{B} = b\left(\overline{A}\right)\overline{A}$. Note that we avoid a non-differentiability in price function (11.4) by changing the payoff $\eta(A-k)^+$ to

$\eta(A-k)$. Hence, it is possible $\overline{A} < K$. Proposition 11.2 states that all points below the strike are optimal for the writer. Thus, the true writer's boundary is

$$A^* = \max\left(K, \overline{A}\right). \tag{11.11}$$

Proposition 11.6 shows that $A^* = K$ when $r \leq 0$. It is proven in Appendix 11.B that if $A^* = K$ and $r \leq 0$, then derivative (11.9) is negative and therefore the holder's exercise region is $\Upsilon^b = (K, \infty)$. Otherwise, if $r > 0$ and $A^* = K$, then the holder's exercise boundary is larger than the strike, and it is obtained as

$$B^* = A^* b\left(A^*\right). \tag{11.12}$$

We summarize the derived results in the following theorem.

Theorem 11.1. *[Theorem 4.1 of Zaevski (2020d)]*
Let the discount rate be positive, $\lambda > 0$.

1. *If $r > 0$, then the optimal boundaries are the defined in equations (11.11) and (11.12) constants A^* and B^*. The constant $\overline{A}$ is the solution of equation (11.10). Thus the game call option price V is*

   (a) *If $x \leq A^*$, then $V = \eta(x-K)^+$.*

   (b) *If $A^* < x < B^*$, then*

   $$V = \eta\left(A^* - K\right)\left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + \left(B^* - K\right)\left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}.$$

   (c) *If $B^* \leq x$, then $V = x - K$.*

2. *If $r \leq 0$, then $\Upsilon^s = (0, K]$ and $\Upsilon^b = (K, \infty)$. If $x \leq K$, then $Y = 0$, and $Y = x - K$, otherwise.*

11.5 Call options without discounting

Suppose that $\lambda = 0$ and therefore $r > 0$. Proposition 11.1 gives that the early exercise is never optimal for the holder. The same conclusion can be deduced using Appendix 11.B. It is proven there that derivative (11.9) is positive in the interval $(1, \infty)$ and thus price function (11.7) has a maximum for $b = \infty$. Suppose that the writer's optimal strategy is the first hit to the barrier $A \geq K$. Let this stopping time be ζ. The price of such can be written as

$$\begin{aligned} f(A) &= \mathbb{E}^x \left[\eta e^{-r\zeta}(S_\zeta - K)^+ I_{\zeta<\infty}\right] + \lim_{T\to\infty} \mathbb{E}^x \left[e^{-rT}(S_T - K)^+ I_{T<\zeta}\right] \\ &= \eta (A - K) \mathbb{E}^x \left[e^{-r\zeta} I_{\zeta<\infty}\right] + \lim_{T\to\infty} C_{DO}(x, A, K), \end{aligned} \tag{11.13}$$

where $C_{DO}(x, A, K)$ is the price of a down-and-out barrier option with the strike K and the barrier A. Using Rubinstein (1991) we can see that

$$\lim_{T\to\infty} C_{DO}(x, A, K) = x \left(1 - \left(\frac{A}{x}\right)^{1+\frac{2r}{\sigma^2}}\right). \tag{11.14}$$

Alternatively, we can derive the limit in formula (11.13) as

$$\lim_{T\to\infty} \mathbb{E}^x \left[e^{-rT}(S_T - K)^+ I_{T<\zeta}\right] = x \lim_{T\to\infty} e^{-\frac{\sigma^2}{2}T} \mathbb{E}^x \left[e^{\sigma B_T}\right] - \lim e^{-rT} \mathbb{Q}(T < \zeta).$$

Obviously, the second limit is zero. We apply the eleventh statement of Theorem 2.10 for

$$\begin{aligned} b_1 &= \frac{\sigma}{2} - \frac{r}{\sigma} \\ b_2 &= \frac{1}{\sigma} \ln \frac{A}{x} \\ k &= -\frac{\sigma^2}{2} \\ \theta &= \sigma \end{aligned} \tag{11.15}$$

to derive the first limit:

$$\lim_{T\to\infty} e^{-\frac{\sigma^2}{2}T} \mathbb{E}^x \left[e^{\sigma B_T}\right] = 1 - \left(\frac{A}{x}\right)^{1+\frac{2r}{\sigma^2}}. \tag{11.16}$$

Using equation (3.15) from Proposition 3.5 we find

$$\mathbb{E}^x\left[e^{-r\zeta}I_{\zeta<\infty}\right]=\left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}. \tag{11.17}$$

Combining equations (11.14) and (11.17), we turn (11.13) into

$$f\left(A\right)=x+\left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}\left(A\left(\eta-1\right)-\eta K\right). \tag{11.18}$$

Calculating the A-derivative of function (11.18) we find

$$f'\left(A\right)=\left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}\left(\left(\eta-1\right)\frac{2r+\sigma^2}{\sigma^2}-\frac{2r\eta K}{\sigma^2 A}\right).$$

Let $\overline{\eta}$ be

$$\overline{\eta}=\frac{2r+\sigma^2}{\sigma^2}. \tag{11.19}$$

Obviously, $f'\left(A\right)$ is an increasing function. If $\overline{\eta}<\eta$, then $f'\left(A\right)>0$ for all $A\geq K$ and therefore function (11.18) achieves its minimum for $A=K$. Hence, the writer's optimal strategy is the first hit to the strip $(0,K]$ and the option price is

$$V=x-K\left(\frac{K}{x}\right)^{\frac{2r}{\sigma^2}} \tag{11.20}$$

when $x>K$, and zero otherwise. Alternatively, if $1<\eta\leq\overline{\eta}$, then the function $f\left(A\right)$ first decreases and then increases. Therefore, the optimal writer's strategy is the first hit to $\overline{A}$ where

$$\overline{A}=\frac{2r\eta K}{\left(\eta-1\right)\left(2r+\sigma^2\right)}. \tag{11.21}$$

Hence, if the initial asset value x is below the barrier $\overline{A}$, then the immediate canceling is optimal, and therefore the option price is $V=\eta(x-K)^+$. Otherwise, if $x>\overline{A}$, then the writer's optimal strategy is the first hit to $\overline{A}$ that leads to the price

$$V=x-\frac{\eta K\sigma^2}{\left(2r+\sigma^2\right)}\left(\frac{\overline{A}}{x}\right)^{\frac{2r}{\sigma^2}}. \tag{11.22}$$

We can summarize these results in the following theorem.

Theorem 11.2 (Theorem 5.1 of Zaevski (2020d)). *If $\lambda = 0$, then $\Upsilon^b = \emptyset$.*

1. *If $\overline{\eta} < \eta$, where $\overline{\eta}$ is defined by (11.19), then the writer's exercise region is $\Upsilon^s = (0, K]$. If $x > K$, then the option price is given by formula (11.20). Otherwise, it is zero.*

2. *If $1 < \eta \leq \overline{\eta}$, then the writer's exercise region is $\Upsilon^s = \left(0, \overline{A}\right]$, where $\overline{A}$ is given by equation (11.21). If $x > \overline{A}$, then the option price is given by equation (11.22). Otherwise, it is $V = \eta(x-K)^+$.*

This result is alternatively derived in Ekström and Villeneuve (2006).

11.6 Pricing put style options

The payments for a put style option are defined by equations (11.1). We shall proceed as in Section 11.4 emphasizing the differences between the call and put style options. Propositions 11.7, 11.8, and 11.10 indicate that the optimal regions are $\Upsilon^s = [B, \infty)$ and $\Upsilon^b = (0, A)$, where $A \leq B \leq K$. Suppose that $A < x < B$. The analogue of price function (11.4) is

$$f(A, B, x) = (A - K) \left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + \eta (B - K) \left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}. \quad (11.23)$$

Suppose that $A < K$ is fixed. After changing the variables as $b = \frac{B}{A}$, $k = \frac{K}{A}$, and $y = \frac{x}{A}$, we derive for the b-derivative of price function (11.23)

$$g_b(b; y) = \frac{y^p - 1}{(b^p - 1)^2} b^{q-1} \left[\begin{array}{c} b^{p+1}\eta (p - q - 1) - b^p \eta k (p - q) \\ +b^{p-q} p (k - 1) + b (q + 1) \eta - q\eta k \end{array} \right]. \quad (11.24)$$

Suppose that $\lambda > 0$ or equivalently $p > q + 1$. We shall discuss the undiscounted case $\lambda = 0$ later. It is proven in Appendix 11.D that the equation

$$b^{p+1}\eta (p - q - 1) - b^p \eta k (p - q) + b^{p-q} p (k - 1) + b (q + 1) \eta - q\eta k = 0$$

has a unique solution larger than one – we denote it by $b(A)$. It minimizes price function (11.23).

Let now the boundary $B \leq K$ be fixed. We change the variables as $a = \frac{A}{B}$, $k = \frac{K}{B}$, and $y = \frac{x}{B}$. Hence, the a-derivative of price function (11.23) is

$$g_a(a) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \left[\begin{array}{c} -a^{p+1}(p-q-1) + a^p k (p-q) \\ -a^{p-q} p (k-1) \eta - a (q+1) + qk \end{array} \right]. \qquad (11.25)$$

It is proven in Appendix 11.C that if $B < K$, then the equation

$$-a^{p+1}(p-q-1) + a^p k (p-q) - a^{p-q} p (k-1) \eta - a (q+1) + qk = 0$$

has a unique solution less than one – we denote it by $a(B)$. It leads to a maximum for price function (11.23). Hence, if the writer's strategy is the first hitting moment to B, then the holder's optimal strategy is the first hit to $Ba(B)$. Thus, our candidate for the writer's boundary $\overline{B}$ is the solution of the equation

$$1 = a(B)\, b(Ba(B)). \qquad (11.26)$$

Proposition 11.7 shows that the true boundary is

$$B^* = \min\left(\overline{B}, K\right). \qquad (11.27)$$

Suppose that $r \geq 0$. Proposition 11.11 gives that $\Upsilon^s = [K, \infty)$ or equivalently $B^* = K$. On the other hand, we prove in Appendix 11.C that if the writer's boundary is the strike and $r \geq 0$, then the derivative (11.25) is positive. Hence, the holder's exercise region is $\Upsilon^b = (0, K)$.

Let us discuss the undiscounted case $\lambda = 0$. We have $r > 0$ since $r + \lambda > 0$. Therefore, the conclusions above are valid. Also, we prove in Appendix 11.D that if $\lambda = 0$, then derivative (11.24) is negative in the whole interval $(1, \infty)$. Hence, the price function (11.23) has minimum for $\overline{B} = \infty$ and therefore $B^* = K$.

Suppose now that $r < 0$. The boundary

$$A^* = B^* a(B) \qquad (11.28)$$

indeed exists and $A^* < B^* \leq K$. We summarize the derived results in the following theorem.

Theorem 11.3 (Theorem 6.1 of Zaevski (2020d)). *1. If $r \geq 0$, then the exercise regions are $\Upsilon^s = [K, \infty)$ and $\Upsilon^b = (0, K)$. If $x < K$, then the option price is $V = K - x$ and it is zero, otherwise.*

2. If $r < 0$, then the exercise regions are $\Upsilon^s = [B^*, \infty)$ and $\Upsilon^b = (0, A^*)$, where B^* and A^* are given by equations (11.27) and (11.28), respectively. The value of $\overline{B}$ is obtained as the solution of equation (11.26). Thus the option price is

(a) $V = K - x$, if $x < A^*$;

(b) $V = (A^* - K)\left(\frac{A^*}{x}\right)^q \frac{B^{*p} - x^p}{B^{*p} - A^{*p}} + \eta (B^* - K)\left(\frac{B^*}{x}\right)^q \frac{x^p - A^{*p}}{B^{*p} - A^{*p}}$, if $x \in [A^*, B^*)$;

(c) $V = \eta(K - x)^+$, if $B^* \leq x$.

11.7 Numerical results

We present in Figure 11.1 the behavior of call options. We assume that the risk-free rate is $r = 0.05$, the volatility is $\sigma = 0.3$, the strike is $K = 5$, and the initial asset price is $x = 8$. We vary the discount factor between 0.01 and 0.1, and the penalty coefficient η between 1.001 and 2. The holder's and writer's boundaries are shown in Figures 11.1a and 11.1b, respectively. The behavior of the option price is presented in Figure 11.1c. We mark by red points the values of η above which the writer's boundary is the strike. Note also that the optimal boundaries and the option prices are independent of η above these critical values. In Figure 11.1d we commonly present the holder's and writer's boundaries for a fixed value of the discount rate, $\lambda = 0.03$. The red point is again the critical value of the penalty coefficient. We have a confirmation of Proposition 11.4. When η tends to one, the boundaries tend to each other.

In Table 11.1 we report some particular prices as well as the corresponding optimal boundaries. We vary the initial asset price as $S \in \{6; 7; 8; 9\}$, the penalty coefficient as $\eta \in \{1.001; 1.2; 1.5; 1.7; 2\}$, and the discount rate as $\lambda \in \{0.001; 0.01; 0.1; 0.15; 0.2\}$. The rest parameters are the same. After the corresponding price, we report the region for the initial asset price – "b" for the holder's region, "s" for the writer's region, and "c" for the continuation region. Proposition 11.1 has a confirmation – when the discount rate is small, $\lambda = 0.001$, the holder's boundary is very high. Also, note that when the writer's boundary is equal to the strike, the optimal boundaries and the option prices are independent of the penalty coefficient η because price function (11.4) is independent of η.

We present the behavior of put style options in Figure 11.2. The initial asset price, the volatility, and the penalty coefficient are as above. We assume that the strike is $K = 10$. We consider $r = -0.05$ because the behavior is more variable for negative rates. We vary the discount rate between 0.051 and 0.1 since $r + \lambda > 0$. The holder's and writer's boundaries are presented in Figures 11.2a and 11.2b, respectively. We mark by red points the values of the penalty η above which the writer's boundary is the strike. The prices are plotted in Figure 11.2c. Both boundaries for $\lambda = 0.08$ are presented in Figure 11.2d. We can see that if η tends to one, both boundaries tend one to other. This confirms Proposition 11.9.

Some particular prices and optimal boundaries for put options are presented in Table 11.2. We vary the initial asset value as $S \in \{5; 6; 7; 8\}$, the discount rate as $\lambda \in \{0.051; 0.06; 0.1; 0.15; 0.2\}$, and the penalty as $\{1.001; 1.2; 1.5; 1.7; 2\}$. We again use values $r = -0.05$, $K = 10$, $\sigma = 0.3$. We also report the region for the initial asset price.

11.8 Conclusions

We have examined in this chapter cancellable options assuming that the penalty that the writer owes for its cancellation right is a proportion of the usual payoff. We have proved some propositions for the shape of the early exercise regions. Using these propositions, we have obtained the optimal strategies for both participants as well as the fair option prices. The obtained results are illustrated by some numerical examples. The results for the shape of the exercise regions, derived in Section 11.3, are valid when the maturity is finite too. The main and very important difference is that the optimal boundaries are not flat due to the existing forced exercise at maturity.

11.A Uniqueness of the solutions, call writer's boundary

We shall prove that derivative (11.6) has only one root in the interval $(0, 1)$. Let the functions $m(a)$, $l(a)$, and $h(a; \eta)$ be defined as

$$m(a) = a^{p+1}(p-q-1) - a^p k(p-q) + a(q+1) - qk$$

$$l(a) = -a^{p-q} p(1-k)$$

$$h(a;\eta) = \eta m(a) + l(a) . \tag{11.29}$$

We have to show that function (11.29) has just one root. We shall examine first the undiscounted case $p = q+1$. We shall omit for simplicity the mark η. Function (11.29) turns into

$$h(a) = -a^p \eta k + ap(\eta - 1 + k) - (p-1) k\eta.$$

Its derivative is

$$h_a(a) = -a^{p-1} p\eta k + p(\eta - 1 + k) .$$

It is decreasing and positive in the whole interval $(0,1)$ since $h_a(1) = p(1-k)(\eta-1) > 0$. Therefore, the function $h(a)$ is increasing and therefore it has just one root because $h(0) = -(p-1)k\eta < 0$ and $h(1) = p(1-k)(\eta-1) > 0$.

Suppose now that $p > q+1$. We shall use the following two lemmas.

Lemma 11.1. *If the function $h(a;1)$ increases, then the functions $m(a)$ and $h(a;\eta)$ increase too.*

Proof: Suppose that the function $h(a;1)$ increases at some point a. The function $m(a)$ has to be increasing at this point since the function $l(a)$ decreases. Thus, the function $h(a;\eta) = (\eta-1)m(a) + h(a;1)$ is a sum of two increasing functions, and therefore it increases too. □

Lemma 11.2. *If $h(\overline{a};1) = h(\overline{a};\eta)$ for some $\overline{a}$, then $h(\overline{a};\eta) = h(\overline{a};1) < 0$.*

Proof: If $h(\overline{a};1) = h(\overline{a};\eta)$, then $m(\overline{a}) = 0$. Hence, $h(a;1) = h(\overline{a};\eta) = l(\overline{a}) < 0$. □

We shall examine now the behavior of the function $h(a;1)$. Its derivatives are

$$h_a(a;1) = a^p(p+1)(p-q-1) - a^{p-1}pk(p-q) - a^{p-q-1}(p-q)p(1-k) + (q+1)$$
$$h_{aa}(a;1) = a^{p-q-2}p\left[\begin{array}{l} a^{q+1}(p+1)(p-q-1) - a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(1-k) \end{array}\right].$$

Let us define the function $n(\cdot)$ as

$$n(a) = a^{q+1}(p+1)(p-q-1) - a^q(p-1)k(p-q) - (p-q-1)(p-q)(1-k).$$

Its derivative is

$$n_a(a) = a^{q-1}\left[a(q+1)(p+1)(p-q-1) - q(p-1)k(p-q)\right]$$

and its positive root is

$$\overline{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}.$$

Suppose first that $\overline{a} \geq 1$ – we entitle this case by (A). The derivative $n_a(a)$ is negative in the whole interval $(0,1)$ and therefore the function $n(a;1)$ is decreasing. Thus the derivative $h_{aa}(a;1)$ is negative since $n(0) < 0$. Hence, the function $h_a(a;1)$ is decreasing and therefore it is positive since $h_a(1;1) = 0$. Thus, the function $h(a;1)$ increases from $-qk$ to 0 in the interval $(0,1)$.

Suppose now that $\overline{a} < 1$ – we denote this case by (B). Thus, the derivative $n_a(a)$ is negative in the interval $(0,\overline{a})$ and positive in the interval $(\overline{a},1)$. Therefore, the function $n(a)$ starts from a negative value, decreases to a minimum, and then increases. If $n(1) \leq 0$, then we are in the case (A). Suppose that $n(1) > 0$. Hence, the derivative $h_a(a;1)$ starts from the positive value $h_a(0;1) = q+1$, decreases to a negative minimum, and then increases to zero staying negative. Thus, it is first positive and then negative. Therefore the function $h(a;1)$ starts from the negative value $-qk$, increases to a positive maximum having a root, and then decreases to 0.

If we are in the case (A), then Lemma 11.1 gives that the function $h(a;\eta)$ increases too. Hence, it has just one root in the interval $(0,1)$ because $h(0) = -qk\eta < 0$ and $h(1;\eta) = p(1-k)(\eta-1) > 0$.

Suppose that the case (B) holds. Let a^* be the maximum of the function $h(a;1)$. Having in mind that $h(a;1)$ increases in the interval $(0, a^*)$ and Lemma 11.1, we conclude that the function $h(a;\eta)$ increases too. Therefore, it has only one root in the interval $(0, a^*)$. Lemma 11.2 shows that the function $h(a;\eta)$ has no roots in the interval $(a^*, 1)$ because $h(a;1)$ is positive and $h(1;\eta) = p(1-k)(\eta-1) > h(1;1) = 0$. Therefore the function $h(a;\eta)$ has just one root.

11.B Uniqueness of the solutions, call holder's boundary

We shall prove that derivative (11.9) has just one root larger than one except when (A) $\{\lambda = 0\}$ or (B) $\{\lambda > 0, k = 1, r <\} 0$. Let the function $h(\cdot)$ be defined as

$$h(b) = -b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q} p(1-k)\eta - b(q+1) + qk. \tag{11.30}$$

Suppose first that $\lambda = 0$ or equivalently $p = q+1$. Thus, the function (11.30) turns into

$$h(b) = b^p k + bp[(1-k)\eta - 1] + (p-1)k \tag{11.31}$$

and its derivative is

$$h_b(b) = p\left[b^{p-1}k + (1-k)\eta - 1\right].$$

It is positive in the whole interval $(1, \infty)$ since $h_b(1) = p(1-k)(\eta-1) \geq 0$. Therefore, function (11.31) is increasing. Hence, the function $h(b)$ is positive when b is larger than one since $h(1) = p(1-k)(\eta-1) \geq 0$.

Suppose now that $p > q+1$. The derivatives of function (11.30) are

$$\begin{aligned} h_b(b) &= -b^p(p+1)(p-q-1) + b^{p-1}pk(p-q) \\ &\quad + b^{p-q-1}(p-q)p(1-k)\eta - (q+1) \\ h_{bb}(b) &= b^{p-q-2}p\left[\begin{array}{l} -b^{q+1}(p+1)(p-q-1) + b^q(p-1)k(p-q) \\ +(p-q-1)(p-q)(1-k)\eta \end{array}\right]. \end{aligned} \tag{11.32}$$

Suppose first, that $k = 1$. The second derivative (11.32) turns into

$$h_{bb}(b) = b^{p-2}p\left[-b(p+1)(p-q-1)+(p-1)(p-q)\right]$$

and it has a root

$$\bar{b} = \frac{(p-1)(p-q)}{(p+1)(p-q-1)}.$$

If the risk-free rate is negative, then Lemma 3.2 gives $2q+1<p$ – this is equivalent to $\bar{b}<1$. Therefore, $h_{bb}(b)$ is negative in the whole interval $(1,\infty)$. Hence, $h_b(b)$ is a decreasing function and thus it is negative since $h_b(1)=0$. Analogously, we conclude that $h(b)$ is negative since $h(1)=0$.

Otherwise, if $r>0$, then $\bar{b}>1$. Hence, $h_{bb}(b)$ is positive in the interval $\left(1,\bar{b}\right)$ and negative in the interval $\left(\bar{b},\infty\right)$. Therefore, the function $h_b(b)$ starts from zero, increases to a positive maximum, and then decreases to minus infinity. Thus it has a root before which it is positive and negative after. Analogously, we conclude that $h(b)$ has just one root larger than one.

Suppose now that $k<1$. Let the function $l(\cdot)$ be defined as

$$\begin{aligned} l(b) = &-b^{q+1}(p+1)(p-q-1)+b^{q}(p-1)k(p-q) \\ &+(p-q-1)(p-q)(1-k)\eta. \end{aligned}$$

Its derivative is

$$l_b(b) = b^{q-1}\left[-b(q+1)(p+1)(p-q-1)+q(p-1)k(p-q)\right]$$

and it has a root

$$\bar{b} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}.$$

Suppose first that $\bar{b}\le 1$. Hence, $l_b(b)\le 0$ for $b\in(1,\infty)$. Therefore, the function $l(b)$ is decreasing and thus it can have at most one root larger than one. Suppose that it has a root. Thus, $l(b)$ is positive before it and negative after. Hence, the derivative $h_b(b)$ starts from the positive value $h_b(1)=p(1-k)(p-q)(\eta-1)$, increases to a maximum, and decreases to minus infinity. Otherwise, if the function $l(b)$ does not have a root, then it is negative in the whole interval $(1,\infty)$. Thus, $h_b(b)$ starts from the positive value and decreases to minus infinity. In both cases, it is first positive and then negative. The same arguments show that the function $h(b)$ exhibits the same behavior and thus it has just one root in the interval $(1, \infty)$.

Suppose now that $\bar{b} > 1$. Thus, $l_b(b) > 0$ for $b \in \left(0, \bar{b}\right)$ and $l_b(b) < 0$ for $b \in \left(\bar{b}, \infty\right)$. Therefore, the function $l(b)$ starts from a positive value $l(1) > l(0) = (p-q-1)(p-q)(1-k)\eta$, increases to a maximum and then decreases to minus infinity. This behavior is already examined above.

11.C Uniqueness of the solutions, put holder's boundary

We shall prove that derivative (11.25) has a unique root in the interval $(0,1)$ except when (A) $\{\lambda = 0, k = 1\}$ or (B) when $\{\lambda > 0, k = 1, r > 0\}$. Let the function $h(\cdot;\cdot)$ be defined as

$$h(a;\eta) = -a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q} p(k-1)\eta - a(q+1) + qk. \tag{11.33}$$

We shall examine first the undiscounted case $\lambda = 0$ or equivalently $p = q+1$. We shall omit the notation η for simplicity. Function (11.33) turns into

$$h(a) = a^p k - ap[(k-1)\eta + 1] + qk.$$

Its derivative is

$$h_a(a) = p\left(a^{p-1}k - (k-1)\eta - 1\right).$$

The derivative $h_a(a)$ is negative in the interval $(0,1)$ since $h_a(1) = -p(k-1)(\eta-1) \leq 0$. Therefore, the function $h(a)$ is decreasing. We have $h(0) = qk > 0$ and $h(1) = -p(k-1)(\eta-1) \leq 0$. Hence, the function $h(a)$ has just one root in the interval $(0,1)$ except when $k = 1$ – it is positive in this case.

Suppose now that $p > q+1$. The derivatives of function (11.33) are

$$\begin{aligned} h_a(a;\eta) &= -a^p(p+1)(p-q-1) + a^{p-1}pk(p-q) \\ &\quad - a^{p-q-1}(p-q)p(k-1)\eta - (q+1) \\ h_{aa}(a;\eta) &= a^{p-q-2}p\left[\begin{array}{l} -a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ -(p-q-1)(p-q)(k-1)\eta \end{array}\right]. \end{aligned}$$

Suppose that $k = 1$. The second derivative turns into

$$h_{aa}(a;\eta) = a^{p-2}p\left[-a(p+1)(p-q-1) + (p-1)(p-q)\right]$$

and it is zero for

$$\overline{a} = \frac{(p-1)(p-q)}{(p+1)(p-q-1)}.$$

If the risk-free rate is positive, then Lemma 3.2 gives $2q+1 > p$ which is equivalent to $\overline{a} > 1$. Hence, the derivative $h_{aa}(a;\eta)$ is positive in the interval $(0,1)$, and therefore the first derivative $h_a(a;\eta)$ is an increasing and negative function since $h_a(1;\eta) = 0$. Thus, the function $h(a;\eta)$ is decreasing and positive since $h(1;\eta) = 0$.

Otherwise, if the risk-free rate is negative, then $\overline{a} < 1$. Therefore, the second derivative $h_{aa}(a;\eta)$ is positive in the interval $(0,\overline{a})$ and negative in the interval $(\overline{a},1)$. Hence, the first derivative $h_a(a;\eta)$ starts from the negative value $h_a(0;\eta) = -(q+1)$, increases to a maximum and then decreases to zero. Thus it has just one root before which it is negative and positive after. Therefore, the function $h(a;\eta)$ starts from the positive value $h(0;\eta) = q$, decreases to a minimum and then increases to zero. Thus it has just one root in the interval $(0,1)$.

Suppose now that $k > 1$. We shall examine firstly the case $\eta = 1$. Note that $h_a(a;\eta) < h_a(a;1)$. Let us define the function $l(\cdot;\cdot)$ as

$$\begin{aligned} l(a) = &-a^{q+1}(p+1)(p-q-1) + a^q(p-1)k(p-q) \\ &-(p-q-1)(p-q)(k-1). \end{aligned}$$

Its derivative is

$$l_a(a) = a^{q-1}\left[-a(q+1)(p+1)(p-q-1) + q(p-1)k(p-q)\right].$$

The positive root of this derivative is

$$\overline{a} = \frac{q(p-1)k(p-q)}{(q+1)(p+1)(p-q-1)}.$$

Suppose that $\overline{a} \geq 1$ and thus $l_a(a) > 0$ for all $a \in (0,1)$. Therefore, the function $l(a)$ is increasing. Hence, it has at most one root less than one. If it does not have a root, then $l(a) < 0$ for all $a \in (0,1)$ since $l(0) < 0$. Thus the derivative $h_a(a;1)$ is decreasing.[1] The derivative $h_a(a;1)$ is negative

[1] This case is possible only when $\eta > 1$.

in the whole interval $(0,1)$ since $h_a(0;1) = -(q+1) < 0$. Otherwise, if the function $l(a)$ has one root, then $l(a)$ has to be negative before it and positive after. Thus, the derivative $h_a(a;1)$ starts from the negative value $h_a(0;1) = -(q+1)$, decreases to a minimum and then increases to zero staying negative. Thus we conclude that in both cases, the derivative $h_a(a;1)$ is negative. The derivative $h_a(a;\eta)$ is negative too since $h_a(a;\eta) < h_a(a;1)$. Therefore, the function $h(a;\eta)$ is decreasing. We conclude that the function $h(a;\eta)$ has just one root in the interval $(0,1)$ because $h(0;\eta) = qk > 0$ and $h(1;\eta) = -p(k-1)(\eta-1) < 0$.

Suppose now that $\overline{a} < 1$. Hence, the derivative $l_a(a)$ is positive in the interval $(0,\overline{a})$ and negative in the interval $(\overline{a},1)$. Therefore, the function $l(a)$ starts from the negative value, increases to a maximum and then decreases. Let us denote by a^* this maximum. If $l(a^*) \leq 0$ or $l(1) > 0$, then we are in the previous case. Suppose that $l(a^*) > 0$ and $l(1) < 0$. Therefore, the derivative $h_a(a;1)$ starts from the negative value $h_a(0;1) = -(q+1)$, decreases to a negative minimum, increases to a positive maximum, and then decreases to zero. Hence, it is first negative and then positive. Thus the function $h(a;1)$ starts from the positive value $h(0;1) = qk$, decreases to a negative minimum, and increases to zero. Note that $h_a(a;\eta) < h_a(a;1)$. This means that when $h(a;1)$ decreases, $h(a;\eta)$ decreases faster than $h(a;1)$. Also if $h(a;1)$ increases, then $h(a;\eta)$ increases slower or decreases. Thus we conclude that the function $h(a;\eta)$ has just one root in the interval $(0,1)$ even though it may be not monotone.

11.D Uniqueness of the solutions, put writer's boundary

We shall prove now that derivative (11.24) has only one root in the interval $(1,\infty)$ except in the undiscounted case. Let the function $h(b;\eta)$ be defined as

$$\begin{aligned} h(b;\eta) &= b^{p+1}\eta(p-q-1) - b^{p}\eta k(p-q) \\ &\quad + b^{p-q}p(k-1) + b(q+1)\eta - q\eta k. \end{aligned} \tag{11.34}$$

Suppose first that $\lambda = 0$ or equivalently $p = q+1$. We shall omit for simplicity the mark η. Thus function (11.34) turns into

$$h\left(b\right)=-b^{p}\eta k+bp\left(k-1+\eta\right)-\left(p-1\right)\eta k. \tag{11.35}$$

Its derivative is

$$h_{b}\left(b\right)=-b^{p-1}p\eta k+p\left(k-1+\eta\right).$$

The function (11.35) is decreasing in the interval $(1,\infty)$ since $h_b(1)=-p(k-1)(\eta-1)<0$. Also, the function $h(b)$ is negative in the interval $(1,\infty)$ because $h(1)=-p(k-1)(\eta-1)<0$.

Suppose now that $p>q+1$. We change the variables as $d=\frac{1}{b}$. Thus, the function (11.34) turns into

$$h\left(d;\eta\right)=\frac{1}{d^{p+1}}\left[\begin{array}{l}-d^{p+1}q\eta k+d^{p}\left(q+1\right)\eta+d^{q+1}p\left(k-1\right)\\-d\eta k\left(p-q\right)+\eta\left(p-q-1\right)\end{array}\right].$$

Let the function $\overline{h}(d;\eta)$ be defined as

$$\begin{aligned}\overline{h}\left(d;\eta\right)=&-d^{p+1}q\eta k+d^{p}\left(q+1\right)\eta+d^{q+1}p\left(k-1\right)\\&-d\eta k\left(p-q\right)+\eta\left(p-q-1\right).\end{aligned}$$

Let us define the functions $m(d)$ and $l(d)$ as

$$m\left(d\right)=-d^{p+1}qk+d^{p}\left(q+1\right)-dk\left(p-q\right)+\left(p-q-1\right)$$

$$l\left(d\right)=d^{q+1}p\left(k-1\right).$$

We can rewrite the function $\overline{h}(d;\eta)$ as

$$\overline{h}\left(d;\eta\right)=\eta m\left(d\right)+l\left(d\right).$$

We shall prove the following two lemmas.

Lemma 11.3. *When the function $\overline{h}(d;1)$ decreases, the functions $m(d)$ and $\overline{h}(d;\eta)$ decrease too.*

Proof: Suppose that the function $\overline{h}(d;1)$ decreases at some point d. The function $m(d)$ has to be decreasing at this point since the function $l(d)$ is increasing. Thus the function $\overline{h}(d;\eta) = (\eta-1)\,m(d) + \overline{h}(d;1)$ is a sum of two decreasing functions and therefore is decreasing too. □

Lemma 11.4. *If $\overline{h}\left(\overline{d};1\right) = \overline{h}\left(\overline{d};\eta\right)$ for some $\overline{d}$, then $\overline{h}\left(\overline{d};\eta\right) = \overline{h}\left(\overline{d};1\right) > 0$.*

Proof: If $\overline{h}\left(\overline{d};1\right) = \overline{h}\left(\overline{d};\eta\right)$, then $m\left(\overline{d}\right) = 0$. Hence, $\overline{h}\left(\overline{d};1\right) = \overline{h}\left(\overline{d};\eta\right) = l\left(\overline{d}\right) > 0$. □

The derivatives of function $\overline{h}(d;1)$ are

$$\begin{aligned}\overline{h}_d(d;1) &= -d^p(p+1)\,qk + d^{p-1}p\,(q+1) \\ &\quad + d^q(q+1)\,p\,(k-1) - k\,(p-q) \\ \overline{h}_{dd}(d;1) &= d^{q-1}p\left[\begin{array}{l} -d^{p-q}(p+1)\,qk + d^{p-q-1}(p-1)(q+1) \\ +q\,(q+1)(k-1)\end{array}\right].\end{aligned}$$

Let the function $n(\cdot)$ be defined as

$$n(d) = -d^{p-q}(p+1)\,qk + d^{p-q-1}(p-1)(q+1) + q\,(q+1)(k-1).$$

Its derivative is

$$n_d(d) = d^{p-q-2}\left[-d\,(p-q)(p+1)\,qk + (p-q-1)(p-1)(q+1)\right].$$

It has a positive root

$$\overline{d} = \frac{(p-q-1)(p-1)(q+1)}{(p-q)(p+1)\,qk}.$$

Suppose first, that $\overline{d} \geq 1$. The derivative $n_d(d)$ is positive in the whole interval $(0,1)$ and therefore the function $\overline{h}_{dd}(d;1)$ is increasing. The first derivative $\overline{h}_d(d;1)$ is an increasing function too since $\overline{h}_{dd}(0^+;1) > 0$. We have $\overline{h}_d(1;1) = 0$ and thus the function $\overline{h}(d;1)$ is decreasing from $(p-q-1)$ to 0. We shall entitle this behavior by (A).

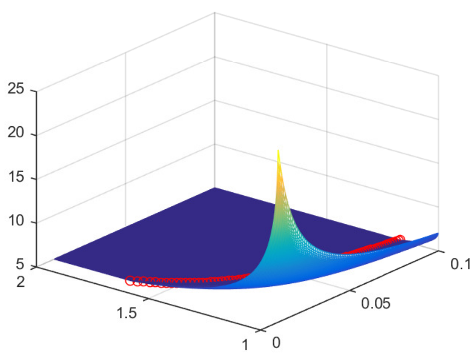
Suppose now, that $\overline{d} < 1$. Hence, the derivative $n_d(d)$ is positive in the interval $\left(0,\overline{d}\right)$ and negative in the interval $\left(\overline{d},1\right)$. Therefore, the function

$\overline{h}_{dd}(d;1)$ starts from zero, increases to a maximum, and then decreases. If $\overline{h}_{dd}(1;1) \geq 0$, then we are in the case (A). If $\overline{h}_{dd}(1;1) < 0$, then the first derivative $\overline{h}_d(d;1)$ starts from the negative value $-k(p-q)$, increases to a positive maximum, and then decreases to zero. Thus we conclude that the function $\overline{h}(d;1)$ starts from the positive value $(p-q-1)$, decreases to a negative minimum, and then increases to zero. We shall denote this case by (B).

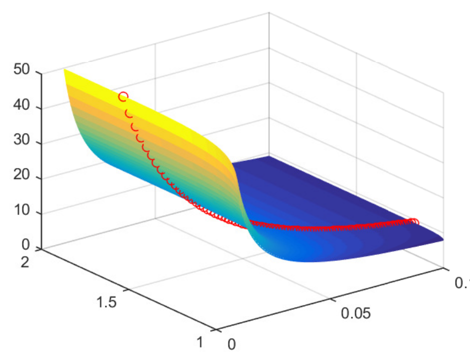
If we are in the case (A), then Lemma 11.3 gives that the function $\overline{h}(a;\eta)$ decreases too. The function $\overline{h}(a;\eta)$ has just one root because $\overline{h}(0;\eta) = \eta(p-q-1) > 0$ and $h(1;\eta) = -p(k-1)(\eta-1) < 0$.

Suppose now that the case (B) holds. Let us denote by d^* the minimum of the function $\overline{h}(d;1)$. Having in mind that $h(d;1)$ decreases in the interval $(0,d^*)$ and Lemma 11.3, we conclude that the function $\overline{h}(d;\eta)$ decreases too. Therefore it has only one root in the interval $(0,d^*)$. Lemma 11.4 shows that the function $\overline{h}(d;\eta)$ has no roots in the interval $(d^*,1)$ because $\overline{h}(d;1)$ is negative and $\overline{h}(1;\eta) = -p(k-1)(\eta-1) < \overline{h}(1;1) = 0$. Therefore, the function $\overline{h}(d;\eta)$ has just one root in the interval $(0,1)$ and thus function (11.34) has a unique root in the interval $(1,\infty)$.

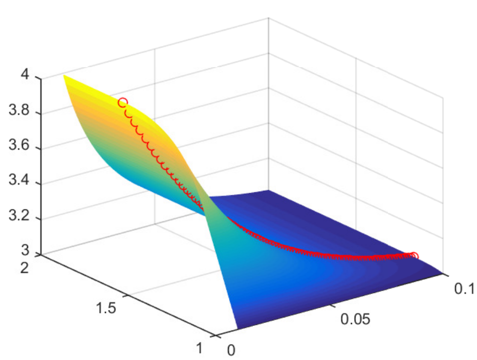
Figure 11.1: Call options



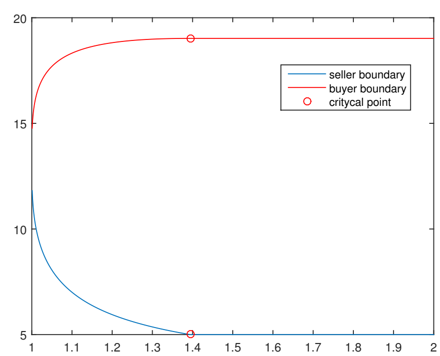
(a) writer's boundary



(b) holder's boundary

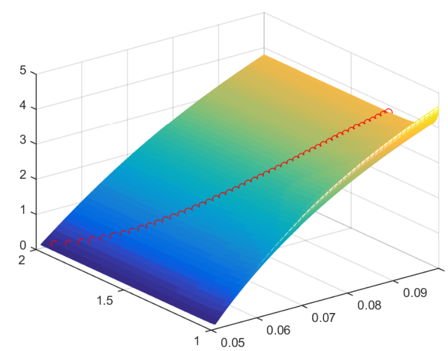


(c) Prices of game call options

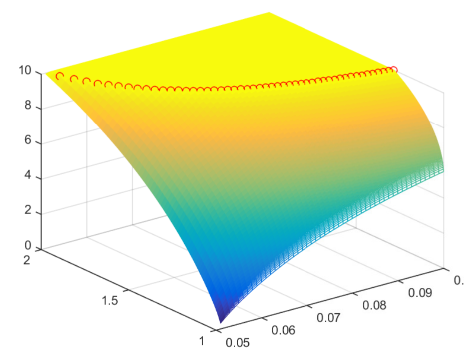


(d) Both boundaries

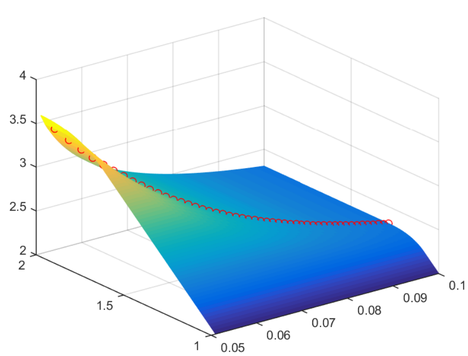
Figure 11.2: Put options



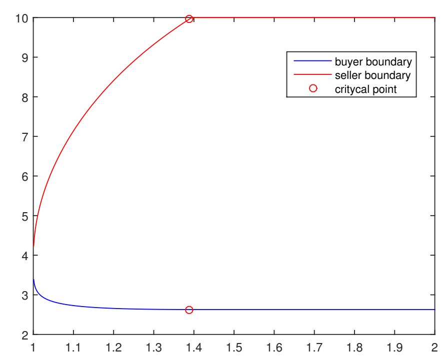
(a) writer's boundary



(b) holder's boundary



(c) Prices of game put option



(d) Both boundaries

Table 11.1: Call option prices and optimal boundaries

	A^*	B^*	$S_0 = 6$	$S_0 = 7$	$S_0 = 8$	$S_0 = 9$
		penalty cefficient $\eta = 1.001$				
$\lambda = 0.001$	157.1248	349.5925	1.0010/s	2.0020/s	3.0030/s	4.0040/s
$\lambda = 0.01$	24.6603	35.1478	1.0010/s	2.0020/s	3.0030/s	4.0040/s
$\lambda = 0.1$	7.0278	7.9334	1.0010/s	2.0020/s	3.0000/b	4.0000/b
$\lambda = 0.15$	6.3311	6.9701	1.0010/s	2.0000/b	3.0000/b	4.0000/b
$\lambda = 0.2$	5.9832	6.4884	1.0010/c	2.0000/b	3.0000/b	4.0000/b
		penalty cefficient $\eta = 1.2$				
$\lambda = 0.001$	13.0937	472.2606	1.2000/s	2.4000/s	3.6000/s	4.8000/s
$\lambda = 0.01$	7.9555	48.4716	1.2000/s	2.4000/s	3.5999/c	4.7561/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b
		penalty cefficient $\eta = 1.5$				
$\lambda = 0.001$	7.2323	475.8458	1.5000/s	3.0000/s	4.4581/c	5.8037/c
$\lambda = 0.01$	5.5373	49.5742	1.4740/c	2.7808/c	3.9641/c	5.0703/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b
		penalty cefficient $\eta = 1.7$				
$\lambda = 0.001$	5.9906	476.3748	1.7000/c	3.2865/c	4.7086/c	6.0231/c
$\lambda = 0.01$	5.0000	49.6448	1.5084/c	2.8092/c	3.9882/c	5.0911/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b
		penalty cefficient $\eta = 2$				
$\lambda = 0.001$	5.0260	476.5680	1.8265/c	3.3929/c	4.8002/c	6.1034/c
$\lambda = 0.01$	5.0000	49.6448	1.5084/c	2.8092/c	3.9882/c	5.0911/c
$\lambda = 0.1$	5.0000	8.9543	1.0740/c	2.0531/c	3.0160/c	4.0000/b
$\lambda = 0.15$	5.0000	7.5978	1.0356/c	2.0082/c	3.0000/b	4.0000/b
$\lambda = 0.2$	5.0000	6.9322	1.0170/c	2.0000/b	3.0000/b	4.0000/b

Table 11.2: Put option prices and optimal boundaries

	A^*	B^*	$S_0 = 5$	$S_0 = 6$	$S_0 = 7$	$S_0 = 8$
penalty cefficient $\eta = 1.001$						
$\lambda = 0.051$	0.1430	0.3182	5.0050/s	4.0040/s	3.0030/s	2.0020/s
$\lambda = 0.06$	1.4226	2.0275	5.0050/s	4.0040/s	3.0030/s	2.0020/s
$\lambda = 0.1$	4.6118	5.4933	5.0020/c	4.0040/s	3.0030/s	2.0020/s
$\lambda = 0.15$	6.3026	7.1146	5.0000/b	4.0000/b	3.0028/c	2.0020/s
$\lambda = 0.2$	7.1735	7.8975	5.0000/b	4.0000/b	3.0000/b	2.0020/s
penalty cefficient $\eta = 1.2$						
$\lambda = 0.051$	0.1059	3.8186	6.0000/s	4.8000/s	3.6000/s	2.4000/s
$\lambda = 0.06$	1.0315	6.2850	5.8511/c	4.7925/c	3.6000/s	2.4000/s
$\lambda = 0.1$	3.8070	9.3291	5.0857/c	4.2015/c	3.2836/c	2.2904/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/b	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c
penalty cefficient $\eta = 1.5$						
$\lambda = 0.051$	0.1051	6.9134	7.0688/c	5.9010/c	4.5000/s	3.0000/s
$\lambda = 0.06$	1.0086	9.0296	6.1262/c	5.2078/c	4.1384/c	2.9056/c
$\lambda = 0.1$	3.7952	10.0000	5.0888/c	4.2081/c	3.2946/c	2.3068/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/b	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c
penalty cefficient $\eta = 1.7$						
$\lambda = 0.051$	0.1050	8.3464	7.2638/c	6.1880/c	4.8972/c	3.3865/c
$\lambda = 0.06$	1.0072	10.0000	6.1444/c	5.2353/c	4.1772/c	2.9579/c
$\lambda = 0.1$	3.7952	10.0000	5.0888/c	4.2081/c	3.2946/c	2.3068/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/c	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c
penalty cefficient $\eta = 2$						
$\lambda = 0.051$	0.1049	9.9484	7.3351/c	6.2931/c	5.0429/c	3.5799/c
$\lambda = 0.06$	1.0072	10.0000	6.1444/c	5.2353/c	4.1772/c	2.9579/c
$\lambda = 0.1$	3.7952	10.0000	5.0888/c	4.2081/c	3.2946/c	2.3068/c
$\lambda = 0.15$	5.5839	10.0000	5.0000/c	4.0086/c	3.0667/c	2.1177/c
$\lambda = 0.2$	6.5809	10.0000	5.0000/b	4.0000/b	3.0069/c	2.0481/c

Chapter 12

Perpetual cancellable options with convertible features

based on the paper

Zaevski, Tsvetelin. "Perpetual cancellable American options with convertible features." Modern Stochastics: Theory and Applications 10.4 (2023): 367-395.

Abstract: The major characteristic of the cancellable American options is the existing writer's right to cancel the contract prematurely paying some penalty amount. The main purpose of this chapter is to introduce and examine a new subclass of such options for which the penalty that the writer owes for this right consists of three parts – a fixed amount, shares of the underlying asset, and a proportion of the usual option payment. We examine the asymptotic case in which the maturity is set to be infinity. We determine the optimal exercise regions for the option's holder and writer and derive the fair option price.

12.1 Motivation and main results

A traditional assumption is that the penalty which the writer owes for his early canceling right is a constant during the option life. We abandon this assumption considering a three component penalty – a proportion of the usual option payment, some shares of the underlying asset, and a fixed amount.

Our study begins exploring the so-called early exercise regions. They consist of these values of the underlying asset that make the immediate exercise optimal for one or the other option's participant. On the contrary, the continuation region consists of the points that give better opportunities for both of the option's holder and writer. The boundaries between the optimal and continuation regions are known as early exercise or optimal boundaries. The facts that (A) the underlying asset is driven by a Markov process, (B) the lapse of maturity, and (C) the discount time dependence in the payment functions show that both exercise boundaries are flat. It turns out that the holder's optimal region for a put-style option has the form $(0, A)$ for some constant A less than the strike, $A < K$. The writer's exercise set is more variable. It can be the interval (B, K) for some constant $B \in (A, K)$, the singleton $\{K\}$ or even the empty set. The optimal regions for the call options are similar but in some sense inverse. The holder's one has the form (B, ∞), whereas the writer's set may again have three forms – the interval (K, A), the singleton $\{K\}$ or the empty set.

The approach we use to derive the exercise boundaries is based on maximizing the future utilities of both of the holder and writer. We assume first that one of the participants exercises when the underlying asset reaches some level and then obtain the optimal value for the other. We derive the equation that this optimal value has to satisfy. In such a way, we look for the early exercise boundaries that suffice both of the option's holder and writer. Once we derive the exercise boundaries, we use some Brownian motion's hitting properties to obtain the fair option prices. We investigate also the impact that the penalty coefficients have. As a rule, as higher they are, as the option is more similar to the corresponding non-cancellable one. It turns out that the smooth fit principle always holds at the holder's boundary, but it is satisfied at the writer's one only when the writer's optimal set is an interval (not a singleton). We present also some numerical results for different values of the penalty components.

We have to mention that the call case in the absence of discounting is specific. As for the classical American options, early exercising is never optimal for the option's holder. This allows us to derive a closed-form formula for the writer's optimal boundary as well as for the fair option price. It turns out that the writer's boundary is finite in the presence of the first or second penalty components (proportion of the usual option payment or shares of the underlying asset). Otherwise, if the penalty consists only of a fixed amount, then the writer's exercise boundary is infinite too.

The plan of the chapter is as follows. In Section 12.2 we provide the base of our study. Section 12.3 presents the results for the call style options, whereas the put options are considered in Section 12.4. Some numerical results are presented in Section 12.5.

12.2 Preliminaries

Let again the function $N_1(t,x)$ present the amount that the writer owes if the holder exercises the option in the moment t at the spot price $S_t = x$. Analogously, the function $N_2(t,x)$ defines the amount that the writer has to pay if he cancels the contract. Suppose that the penalty consists of three parts – the constant $\eta_1 \geq 1$ leads to a proportion of the usual option payment, $\eta_2 \geq 0$ is the number of shares, and $\eta_3 \geq 0$ is a fixed amount. Thus the functions $N_1(t,x)$ and $N_2(t,x)$ are

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(x-K)^+ \\ N_2(t,x) &= e^{-\lambda t}\big(\eta_1 (x-K)^+ + \eta_2 x + \eta_3\big) \end{aligned}$$

or

$$\begin{aligned} N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\ N_2(t,x) &= e^{-\lambda t}\big(\eta_1 (K-x)^+ + \eta_2 x + \eta_3\big). \end{aligned} \tag{12.1}$$

for the call or put style options, respectively. Thus, the strategies $\zeta^b \geq t$ and $\zeta^s \geq t$ for the holder and writer, respectively, lead to the following value function at the point (t,x)

$$M\left(t,x;\zeta^b,\zeta^s\right) = \mathbb{E}^{t,x}\left[e^{-r\left(\zeta^b-t\right)} N_1\left(\zeta^b, S_{\zeta^b}\right) I_{\zeta^b \leq \zeta^s} + e^{-r(\zeta^s-t)} N_2\left(\zeta^s, S_{\zeta^s}\right) I_{\zeta^s<\zeta^b}\right]. \tag{12.2}$$

Hence, the upper and lower value of this game are

$$V^*(t,x) = \inf_{\tau^b}\sup_{\tau^s} M\left(t,x;\tau^b,\tau^s\right); \quad V_*(t,x) = \sup_{\tau^s}\inf_{\tau^b} M\left(t,x;\tau^b,\tau^s\right).$$

We need now the following lemma.

Lemma 12.1. *The expectation of the sup-process, $H_t = \sup_{0\leq u<t} S_u$, is*

$$\begin{cases} \lambda > 0: \; e^{-\lambda t}\left(1-\frac{\sigma^2}{2\lambda}\right)\overline{N}\left(\left(\frac{\lambda}{\sigma}-\frac{\sigma}{2}\right)\sqrt{t}\right)+\left(1+\frac{\sigma^2}{2\lambda}\right)N\left(\left(\frac{\lambda}{\sigma}+\frac{\sigma}{2}\right)\sqrt{t}\right) \\ \lambda = 0: \; 2N\left(\frac{\sigma\sqrt{t}}{2}\right)+\frac{\sigma^2 t}{2}N\left(\frac{\sigma\sqrt{t}}{2}\right)+\sigma\sqrt{t}n\left(\frac{\sigma\sqrt{t}}{2}\right), \end{cases}$$

where $n(\cdot)$, $N(\cdot)$, and $\overline{N}(\cdot)$ are the probability density, the cumulative distribution function, and its complement of the standard normal distribution.

Proof. The lemma can be obtained using the distribution of the sup-Brownian motion with drift μ and a variance coefficient σ^2:

$$\mathbb{Q}(H_t < x) = N\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right) - e^{\frac{2x\mu}{\sigma^2}}\overline{N}\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right). \tag{12.3}$$

See Corollary 10.1 from Ross (2014) for the proof of equation (12.3). □

Having in mind Lemma 12.1, which leads to $\mathbb{E}\left[\sup_{t\in[0,T]} e^{-rt}N_2(t,S_t)=0\right] < \infty$ and $\mathbb{Q}(\lim_{t\to\infty} e^{-rt}N_2(t,S_t)=0)=1$ we see that the conditions of Theorem 2.1 from Ekström and Peskir (2008) are satisfied except when $\lambda = 0$ together with $T = \infty$. This exception is considered separately in Section 12.3.4 for call options; see Proposition 12.12 for the puts (note that $r > 0$ when $\lambda = 0$). Therefore, the defined above problem exhibits a Nash equilibrium, see also Peskir (2009).

The value function can be defined as $V(t,x) = V_*(t,x) \equiv V^*(t,x)$. The optimal regions – Υ^b and Υ^s – and the optimal strategies – ζ^b and ζ^s – for the holder and writer, respectively, are

$$\Upsilon^b = \{(t,x) : V(t,x) = N_1(t,x)\} \text{ and } \Upsilon^s = \{(t,x) : V(t,x) = N_2(t,x)\}$$
$$\zeta^b = \inf\left\{t : S_t \in \Upsilon^b\right\} \text{ and } \zeta^s = \inf\left\{t : S_t \in \Upsilon^s\right\}.$$

We shall denote the continuation region by $\overline{\Upsilon}$. Suppose that ζ is a stopping time. We define the ζ-writer's/holder's optimal strategy – we denote them by $A(\zeta;x)$ and $B(\zeta;x)$ marking the dependence on the initial asset value – as the stopping time which minimizes/maximizes expected payoff (12.2) w.r.t. ζ^s or ζ^b, respectively. This way, we deduce as a corollary the writer's/holder's optimal conditions, namely

$$\begin{aligned}(t,x) \in \Upsilon^s &\to N_2(t,x) \le M(t,x;\zeta,B(\zeta;x)) \ \forall \text{ stopping times } \zeta \\ (t,x) \in \Upsilon^b &\to N_1(t,x) \ge M(t,x;\zeta,A(\zeta;x)) \ \forall \text{ stopping times } \zeta.\end{aligned} \tag{12.4}$$

We need to restrict the writer's optimal set in some marginal cases to keep the generality of the presentation. In fact, we impose that the writer would not cancel the option immediately even if this is optimal, when some future strategy provides the same result. This assumption is not so restrictive from a financial point of view.

Condition 12.1. *Let the option be out-of-the-money, $\lambda = 0$, and $\eta_3 = 0$. Suppose that $V(t,x) = N_2(t,x)$ and there exists a stopping time $\zeta > t$ a.s. such that $N_2(t,x) = M(t,x;\zeta,B(\zeta;x))$. Then $(t,x) \notin \Upsilon^s$.*

12.3 Call options

We assume hereafter $t = 0$ – this is possible due to the Markovian property driving the asset price. We shall prove first a series of propositions for the optimal regions. Using them we shall obtain the optimal boundaries as well as the fair price.

12.3.1 Exercise regions

Proposition 12.1. *If $x < K$, then $x \in \overline{\Upsilon}$.*

Proof. Suppose that $x \notin \overline{\Upsilon}$. Obviously, $x \notin \Upsilon^b$ and thus $x \in \Upsilon^s$. Let $\epsilon > 0$ be some small enough constant and ζ be the smaller between the first hitting to the strike moment and ϵ. Since $e^{-rt}S_t$ is a Q-martingale, we derive

$$\begin{aligned}&N_2(t,x) \equiv \eta_2 x + \eta_3 = \mathbb{E}^x\left[\eta_2 e^{-r\zeta} S_\zeta\right] + \eta_3 \\ &\ge \mathbb{E}^x\left[\eta_2 e^{-(r+\lambda)\zeta} S_\zeta + \eta_3 e^{-(r+\lambda)\zeta}\right] = \mathbb{E}^x\left[e^{-r\zeta} N_2(\tau, S_\zeta)\right].\end{aligned}$$

Note that $B(\zeta;x) > \zeta$, because $S_t \le K$ on every sample path at which $t \le \zeta$ and therefore the exercise before ζ can not be optimal for the holder (he will receive nothing). Thus writer's optimal condition (12.4) can not be true for the stopping time ζ – see also Condition 12.1 for the marginal case $\lambda = 0$, $\eta_3 = 0$. Hence, $x \notin \Upsilon^s$. The contradiction finishes the proof. □

The following restriction on the penalty coefficients appears – if $\eta_3 \geq \eta_1 K$, then early canceling is never optimal for the writer. Hence, the option would be a pure American.

Proposition 12.2. *If $\eta_3 \geq \eta_1 K$, then $\Upsilon^s = \emptyset$.*

Proof. Suppose that $\eta_3 \geq \eta_1 K$ and $x \in \Upsilon^s$. Let us denote by $\overline{\eta}$ the price of the ordinary at-the-money American call without maturity restrictions. Theorem 4.1 leads to $\overline{\eta} < K$. The strike cannot be optimal for the writer because the never canceling strategy, which makes the option pure American, leads to a better financial result for the writer than the immediate cancellation – $\eta_2 K + \eta_3 \geq \eta_2 K + \eta_1 K \geq K > \overline{\eta}$. On the other hand, Proposition 12.1 gives $(0, K) \notin \Upsilon^s$. Hence, x is strictly above the strike, $x > K$. For a positive ϵ and a constant K_1 such that $K < K_1 < x$, we define τ as the lower between the asset's first hit to the value K_1 and ϵ. Note that τ is a finite stopping time and $S_\tau > K$. Also, $S_{B(\tau;\cdot)} > K$, because in the opposite case the holder receives nothing. Let $\zeta = \tau \wedge B(\tau; \cdot)$ – note that it is finite. Hence,

$$
\begin{aligned}
&\eta_1 (x - K) + \eta_2 x + \eta_3 \leq M (x; \tau, B(\tau; x)) = \\
&= \mathbb{E}^x \left[\begin{array}{l} e^{-(r+\lambda)B(\tau;\cdot)} \left(S_{B(\tau;\cdot)} - K \right) I_{B(\tau;\cdot) \leq \tau} \\ + e^{-(r+\lambda)\tau} \left((\eta_1 + \eta_2) S_\tau + (\eta_3 - \eta_1 K) \right) I_{\tau < B(\tau;\cdot)} \end{array} \right] \\
&\leq \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} \left((\eta_1 + \eta_2) S_\zeta + (\eta_3 - \eta_1 K) \right) \right] \\
&= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} (\eta_3 - \eta_1 K) \right] + \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} (\eta_1 + \eta_2) S_\zeta \right] \\
&< (\eta_3 - \eta_1 K) + (\eta_1 + \eta_2) \mathbb{E}^x \left[e^{-r\zeta} S_\zeta \right] = \eta_1 (x - K) + \eta_2 x + \eta_3.
\end{aligned}
\tag{12.5}
$$

The contradiction finishes the proof. □

Hereafter, we assume that $\eta_3 < \eta_1 K$. The following propositions hold.

Proposition 12.3. *Suppose that a larger than the strike constant x is optimal for the writer, $x \in \Upsilon^s$. Let y be another constant such that $K < y < x$. Then $y \in \Upsilon^s$.*

Proof. Let us fix some future moment T and a writer's strategy ζ with values between 0 and T. This leads to the ζ-holder's optimal strategy $B(\zeta; x)$. Let us define the function $f(\cdot)$ as $f(z) = M(0, z; \zeta, B(\zeta; z)) - N_2(0, z)$. Using the martingality of the discounted prices we derive

$$f(z) = \eta_1 K - \eta_3 + \mathbb{E}^z \left[\begin{array}{l} e^{-(r+\lambda)B(\zeta;z)} \max \left(\begin{array}{c} -\left(e^{\lambda B(\zeta;z)}(\eta_1+\eta_2)-1\right) S_{B(\zeta;z)} - K, \\ -e^{\lambda B(\zeta;z)}(\eta_1+\eta_2) S_{B(\zeta;z)} \end{array} \right) I_{B(\zeta;z)\le\zeta} \\ +e^{-(r+\lambda)\zeta} \max \left(\begin{array}{c} -(\eta_1+\eta_2)\left(e^{\lambda\zeta}-1\right) S_\zeta - \eta_1 K + \eta_3, \\ -\left(e^{\lambda\zeta}(\eta_1+\eta_2)-\eta_2\right) S_\zeta + \eta_3 \end{array} \right) I_{\zeta<B(\zeta;z)} \end{array} \right].$$

We can see that this function is decreasing and therefore $f(y) > f(x) > 0$ since $x \in \Upsilon^s$. We finish the proof by taking $T \to \infty$. □

Proposition 12.4. *If $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = 0$, then $\overline{\Upsilon} = (0, K)$.*

Proof. See Propositions 9.7. □

Proposition 12.5. *If $x \in \Upsilon^b$ and $y > x$, then $y \in \Upsilon^b$.*

Proof. The proof is very similar to the proof of Proposition 9.4 and we omit it. □

Proposition 12.6. *If $\lambda = 0$, then the holder's exercise region is empty.*

Proof. Note that Proposition 9.1 does not explore the form of the cancellation payment, but only the fact that $N_1(t,x) < N_2(t,x)$. □

Proposition 12.7. *If $r < 0$, then $\Upsilon^s \equiv \emptyset$ or $\Upsilon^s \equiv \{K\}$.*

Proof. First, Proposition 12.1 states that all points below the strike are not optimal for the writer. Suppose that $K_1 \in \Upsilon^s$ for some $K_1 > K$. Proposition 12.3 gives that the whole strip $(K, K_1) \in \Upsilon^s$. Let $x \in (K, K_1)$ and ζ be the first exit time from the strip (K, K_1). Therefore $B(\zeta; x) > \zeta$. Using the martingality of the discounted asset price and the inequality $\eta_3 < \eta_1 K$, we derive

$$\begin{aligned} &\mathbb{E}^x \left[e^{-(r+\lambda)\zeta} \left(\eta_1 (S_\zeta - K) + \eta_2 S_\zeta + \eta_3\right) I_{\zeta \le B(\zeta;x)} + e^{-(r+\lambda)B(\zeta;x)} (S_\zeta - K) I_{B(\zeta;x)<\zeta} \right] \\ &= \mathbb{E}^x \left[e^{-(r+\lambda)\zeta} \left(\eta_1 (S_\zeta - K) + \eta_2 S_\zeta + \eta_3\right) \right] \\ &\le \mathbb{E}^x \left[e^{-r\zeta} \left(\eta_1 (S_\zeta - K) + \eta_2 S_\zeta + \eta_3\right) \right] < \eta_1 (x - K) + \eta_2 x + \eta_3, \end{aligned} \tag{12.6}$$

which contradicts the writer's optimal condition (12.4). □

12.3.2 Pricing

Propositions 12.1, 12.3, and 12.5 indicate that the holder's exercise region has the form $\Upsilon^b = [B, \infty)$ for some constant $B > K$, whereas the writer's one is the interval $\Upsilon^s = [K, A]$ (A is a constant less than B, $K < A < B$), the singleton $\Upsilon^s = \{K\}$, or the empty set $\Upsilon^s \equiv \emptyset$.

Suppose that the starting point x is between the boundaries A and B, $A < x < B$, and let us denote the option price as $f(A, B, x)$ under these assumptions. In such a way the pricing problem turns into a problem of the first exit of a Brownian motion with drift $\psi = \frac{r}{\sigma} - \frac{\sigma}{2}$ from the strip (A_1, B_1) for

$$A_1 = \frac{\ln A - \ln x}{\sigma} < 0; \quad B_1 = \frac{\ln B - \ln x}{\sigma} > 0.$$

Denoting by ζ^A and ζ^B the first hits to the values A_1 and B_1 and using Lemma 3.3, we derive the option price as

$$\begin{aligned} f(A, B, x) &= M\left(x; \zeta^A, \zeta^B\right) \\ &= (B - K)\, \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^B} I_{\zeta^B \le \zeta^A} \right] \\ &+ ((\eta_1 + \eta_2) A - \eta_1 K + \eta_3)\, \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^A} I_{\zeta^A < \zeta^B} \right] \\ &= ((\eta_1 + \eta_2) A - \eta_1 K + \eta_3) \left(\frac{A}{x}\right)^q \frac{B^p - x^p}{B^p - A^p} + (B - K) \left(\frac{B}{x}\right)^q \frac{x^p - A^p}{B^p - A^p}, \end{aligned} \tag{12.7}$$

where the constants p and q are defined by formulas (3.17). Recall that $p \ge q + 1$ and the equality holds only when $\lambda = 0$. Suppose that $\lambda > 0$. Let us fix the upper boundary B. After the substitution $a = \frac{A}{B}$, $k = \frac{K}{B}$, $\xi = \frac{\eta_3}{B}$, and $y = \frac{x}{B}$, formula (12.7) turns into

$$\begin{aligned} f(A, B, x) &= \frac{B}{y^q} \frac{((\eta_1 + \eta_2) a - \eta_1 k + \xi) a^q (1 - y^p) + (1 - k)(y^p - a^p)}{1 - a^p} \\ &= \frac{B}{y^q} \frac{-a^p (1 - k) + a^{q+1} (\eta_1 + \eta_2)(1 - y^p) - a^q (\eta_1 k - \xi)(1 - y^p) + y^p (1 - k)}{1 - a^p}. \end{aligned} \tag{12.8}$$

We have the order $0 \le \frac{\xi}{\eta_1} < k < a < 1$ because $\eta_3 < \eta_1 K$. Since the option's writer minimizes his financial result, we have to derive for which value of a the function

$$g(a;y) = \frac{-a^p(1-k) + a^{q+1}(\eta_1+\eta_2)(1-y^p) - a^q(\eta_1 k - \xi)(1-y^p) + y^p(1-k)}{1-a^p}$$

is smallest in the interval $(0,1)$. Its derivative is

$$g_a(a) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \left[\begin{array}{l} a^{p+1}(\eta_1+\eta_2)(p-q-1) - a^p(\eta_1 k - \xi)(p-q) \\ -a^{p-q} p(1-k) + a(q+1)(\eta_1+\eta_2) - q(\eta_1 k - \xi) \end{array} \right]. \tag{12.9}$$

We prove in Proposition 12.13 that function (12.9) has a unique root in the interval $(0,1)$ – we denote it by $a(B)$. It leads to the minimum of price function (12.8).

Let us fix now the writer's boundary A. We have to find the value B that maximizes function (12.7) since the option's holder maximizes his financial utility. Let us change the variables as $b = \frac{B}{A}$, $k = \frac{K}{A}$, $\xi = \frac{\eta_3}{A}$, and $y = \frac{x}{A}$. Now the order is $0 \leq \frac{\xi}{\eta_1} < k \leq 1 < b$. Thus price function (12.7) turns into

$$f(A,B,x) = \frac{A}{y^q} \frac{(\eta_1+\eta_2-\eta_1 k+\xi)(b^p-y^p) + (b-k)b^q(y^p-1)}{b^p-1}. \tag{12.10}$$

Hence, we have to derive the maximum of the function

$$g(b) = \frac{(\eta_1+\eta_2-\eta_1 k+\xi)(b^p-y^p) + (b-k)b^q(y^p-1)}{b^p-1}$$
$$= \frac{b^p(\eta_1+\eta_2-\eta_1 k+\xi) + b^{q+1}(y^p-1) - b^q k(y^p-1) - (\eta_1+\eta_2-\eta_1 k+\xi)y^p}{b^p-1}.$$

Its derivative is

$$g_b(b) = \frac{y^p-1}{(b^p-1)^2} b^{q-1} \left[\begin{array}{l} -b^{p+1}(p-q-1) + b^p k(p-q) \\ +b^{p-q} p(\eta_1+\eta_2-\eta_1 k+\xi) - b(q+1) + qk \end{array} \right]. \tag{12.11}$$

We show in Proposition 12.14 that this function has just one root in the interval $(1,\infty)$ except in the marginal case $\eta_2 = \eta_3 = 0$ examined in Chapter 11. We shall denote the root by $b(A)$. Thus pricing function (12.10) has a maximum for $B = b(A)A$.

Let us denote the true boundaries (if they exist) by A^* and B^*. We search the potential writer's optimal boundary as the unique solution $\overline{A}$ of the equation $yb(y)\,a(yb(y)) = y$ or equivalently

$$b(y)\,a(yb(y)) = 1. \tag{12.12}$$

If $\overline{A} \geq K$, then this is the true boundary, i.e. $A^* = \overline{A}$ and $B^* = b\left(\overline{A}\right)\overline{A}$. It may happen $\overline{A} < K$ because we have changed the original payment functions in formula (12.7) from $N_1(t,x) = e^{-\lambda t}(x-K)^+$ and $N_2(t,x) = e^{-\lambda t}\left(\eta_1(x-K)^+ + \eta_2 x + \eta_3\right)$ to $N_1(t,x) = e^{-\lambda t}(x-K)$ and $N_2(t,x) = e^{-\lambda t}(\eta_1(x-K) + \eta_2 x + \eta_3)$, respectively. Note that we are just in this case when $r < 0$ due to Proposition 12.7. Hence, if $\overline{A} < K$, then we have to recognize whether the writer's exercise region is the singleton $\{K\}$ or the empty set. Suppose that the initial asset price is the strike, $x = K$. The writer has the alternatives to cancel immediately or to do nothing. In the first case he has to pay the amount of $\eta_2 K + \eta_3$ whereas in the second one, the option turns into ordinary American. It is shown in Theorem 4.1 that its price is

$$\overline{\eta} = \frac{K}{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} \tag{12.13}$$

for $\gamma = p - q$. We conclude that if $\eta_2 K + \eta_3 \leq \overline{\eta}$, then the writer would prefer to cancel the option immediately, i.e. $\Upsilon^s = \{K\}$ and thus $A^* = K$ and $B^* = b(K)\,K$. Otherwise, if $\eta_2 K + \eta_3 > \overline{\eta}$, then $\Upsilon^s = \emptyset$, which means that the option is ordinary American, A^* does not exist, and $B^* = \frac{\gamma}{\gamma-1}K$ – see Theorem 4.1.

Note at last that if the writer's exercise region is not empty and the asset starts below the strike, $x < K$, then the writer cancels when the asset hits the strike, and this strategy leads to the option price

$$\begin{aligned} &\mathbb{E}^x\left[e^{-(r+\lambda)\tau}\left(\eta_1(S_\tau-K)^+ + \eta_2 K + \eta_3\right)I_{\tau<\infty}\right] \\ &= (\eta_2 K + \eta_3)\,\mathbb{E}^x\left[e^{-(r+\lambda)\tau}I_{\tau<\infty}\right] = (\eta_2 K + \eta_3)\left(\frac{x}{K}\right)^{\gamma} \end{aligned} \tag{12.14}$$

due to Proposition 3.5.

Remark 12.1. *If $\eta_2 = \eta_3 = 0$, then we have an option with a proportional penalty – we refer to Theorem 11.1 and the whole Chapter 11. There all* *points below the strike are considered as writer optimal since the writer owes nothing. On the other hand, these points can be viewed also as belonging to the continuation region since the first hitting to the strike strategy gives the same result – we can apply Condition 12.1. Let us mention, that if* $r \leq 0$*, then* $\Upsilon^s = \{K\}$ *and* $\Upsilon^b = (K, \infty)$*. If* $r > 0$*, then the results are similar to the general case presented below.*

We summarize the derived results in the following theorem.

Theorem 12.1. *[Theorem 3.8 of Zaevski (2023b)]*
Let $\lambda > 0$ *and* $\eta_2 + \eta_3 > 0$*.*

1. *If* $\eta_3 \geq \eta_1 K$*, then* $\Upsilon^s = \emptyset$ *and the option is ordinary American – for more details about these options, see Theorem 4.1.*

2. *If* $\eta_3 < \eta_1 K$ *in addition to* $\eta_2 + \eta_3 > 0$*, then* $\overline{A}$ *is defined as the solution of equation* (12.12) *and the following statements hold:*

   (a) *If* $\overline{A} \geq K$*, then* $A^* = \overline{A}$ *and* $B^* = \overline{A} b\left(\overline{A}\right)$*. The exercise regions for the writer and holder are* $\Upsilon^s = [K, A^*]$ *and* $\Upsilon^b = [B^*, \infty)$*, respectively. The option price* $V(x)$ *is given by:*

      i. *equation* (12.14) *when* $x \leq K$*;*
      ii. $V(x) = (\eta_1 + \eta_2) x - \eta_1 K + \eta_3$ *when* $K < x < A^*$*;*
      iii. $V(x) = x - K$ *when* $x > B^*$*;*
      iv. $V(x)$ *is given by formula* (12.7) *when* $A^* \leq x \leq B^*$*.*

   (b) *If* $\overline{A} < K$ *and* $\eta_2 K + \eta_3 \leq \overline{\eta}$*,* $\overline{\eta}$ *is given by equation* (12.13)*, then* $A^* = K$ *and* $B^* = K b(K)$*. The exercise regions are* $\Upsilon^s = \{K\}$ *and* $\Upsilon^b = \left[\overline{B}, \infty\right)$*. The option price* $V(x)$ *is determined as in the previous case.*

   (c) *If* $\overline{A} < K$ *and* $\eta_2 K + \eta_3 > \overline{\eta}$*, then the option is again ordinary American.*

12.3.3 Smooth fit principle

Let us discuss now the smooth fit principle, i.e. when the derivative of the value function $V(x)$ is continuous at the optimal boundaries. We have that $V'(x) = 1$ for $x \in \Upsilon^b$ and $V'(x) = \eta_1 + \eta_2$ for $x \in \Upsilon^s$. If $x > K$ and $x \in \overline{\Upsilon}$, then we derive $V'(x)$ differentiating formula (12.7):

$$V'(x) = \frac{(B^* - K)\, B^{*q}\, (x^p\, (p-q) + qA^{*p})}{x^{q+1}\, (B^{*p} - A^{*p})} - \frac{((\eta_1 + \eta_2)\, A^* - \eta_1 K + \eta_3)\, A^{*q}\, (x^p\, (p-q) + qB^{*p})}{x^{q+1}\, (B^{*p} - A^{*p})}. \tag{12.15}$$

Let us check the smooth fit at the holder's boundary. Using again the change of variables $b^* = \frac{B^*}{A^*}$, $k = \frac{K}{A^*}$, $\xi = \frac{\eta_3}{A^*}$, and $y = \frac{x}{A^*}$, we derive for derivative (12.15) at the point b^*:

$$V'(b^*) = \frac{(b^* - k)\,(b^{*p}\,(p-q) + q) - ((\eta_1 + \eta_2) - \eta_1 k + \xi)\,(b^{*p}\,(p-q) + qb^{*p})}{b^*\,(b^{*p} - 1)}.$$

We can easily check that $V'(b^*) = 1$, because b^* is the root of function (12.32). Hence, there is a smooth fit at the holder's boundary. Analogously, we can establish the smooth fit at the writer's boundary A^* when $A^* = \overline{A} \geq K$, having in mind that we use the root of function (12.31) to derive the value of $\overline{A}$. Otherwise, if $\overline{A} < K$, then we do not have the smooth fit at the writer's boundary, namely the strike, because it is not B^*-writer optimal. Note that B^* is K-holder optimal which confirms the smooth fit at B^*.

On the other hand, all points below the strike belong to the continuation region. Suppose that the writer's optimal region is not empty. We can not expect a lower smooth fit at the strike because the writer's payoff function $N_2(t, x)$ is not smooth namely at the strike.

We can summarize: we have always a smooth fit at the holder's boundary, but only when $\overline{A} \geq K$ at the writer's one.

12.3.4 Absence of discounting

Suppose now, that $\lambda = 0$ or equivalently $p = q + 1$. We have $r > 0$, because the total discount factor is positive. Proposition 12.6 shows that it is never optimal for the holder to exercise the option. Hence, his boundary is infinitely large. This conclusion is supported by the fact that derivative (12.11) is always positive – this is proven in Proposition 12.14. Thus the holder maximizes his utility for $B = \infty$. Suppose that the writer's optimal boundary is a constant $A \geq K$ and the underlying asset starts above it, $x > A$. Let us denote the first hitting moment of the asset to the level A by

ζ and the price of a down-and-out barrier option with strike K and barrier A by $C_{DO}(x, A, K)$. Therefore, the option price can be presented as the following dependent on A function

$$F(A) = \mathbb{E}^x\left[e^{-r\zeta}\big(\eta_1(S_\zeta - K)^+ + \eta_2 S_\zeta + \eta_3\big) I_{\zeta<\infty}\right] + \lim_{T\to\infty} \mathbb{E}^x\left[e^{-rT}(S_T - K)^+ I_{T<\zeta}\right]$$
$$= ((\eta_1 + \eta_2) A - \eta_1 K + \eta_3) \mathbb{E}^x\left[e^{-r\zeta} I_{\zeta<\infty}\right] + \lim_{T\to\infty} C_{DO}(x, A, K). \quad (12.16)$$

Using Proposition 3.5 and equation (10.45) from Zhang (1997), we obtain

$$\mathbb{E}^x\left[e^{-r\zeta} I_{\zeta<\infty}\right] = \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}}$$
$$\lim_{T\to\infty} C_{DO}(x, A, K) = x\left(1 - \left(\frac{A}{x}\right)^{1+\frac{2r}{\sigma^2}}\right). \quad (12.17)$$

Alternatively, we can derive the limit in (12.16) without using barrier options. Instead, we can apply the eleventh statement of Theorem 2.10 for values (11.15). For more details, see equation (11.16) and its derivation.

Substituting equations (12.17) into (12.16), we derive for the option price

$$F(A) = x + \left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}} (A(\eta_1 + \eta_2 - 1) + \eta_3 - \eta_1 K). \quad (12.18)$$

Its derivative is

$$F'(A) = \frac{1}{A\sigma^2}\left(\frac{A}{x}\right)^{\frac{2r}{\sigma^2}} h(A),$$

where the function $h(A)$ is

$$h(A) = A(\eta_1 + \eta_2 - 1)\left(2r + \sigma^2\right) + 2r(\eta_3 - \eta_1 K). \quad (12.19)$$

It is linear and increasing with the root

$$M = \frac{2r(\eta_1 K - \eta_3)}{(\eta_1 + \eta_2 - 1)(2r + \sigma^2)}. \quad (12.20)$$

Note that it is positive. We have two cases to examine – suppose first that $M \le K$. Thus derivative $F'(A)$ is always positive for $A > K$. Hence, the price function (12.18) is increasing and therefore its minimum is for $A = K$. We have to recognize whether the writer's exercise region is the singleton $\{K\}$ or the empty set. Suppose that the option is at-the-money, i.e. the initial asset price is the strike. The writer has the alternatives to cancel the option immediately or to do nothing. If he chooses the first one, then he has to pay $\eta_2 K + \eta_3$. The second alternative turns the option into European since the holder will never exercise earlier too. Its price is just the initial asset value – we can see this if we take the limit $T \to \infty$ in the Black-Scholes formula. Hence, $\Upsilon \equiv \{K\}$ when $\eta_2 K + \eta_3 \leq K$, and it is the empty set otherwise. Thus, if $\eta_2 K + \eta_3 \leq K$, then the option price (12.18) takes the form

$$x + \left(\frac{K}{x}\right)^{\frac{2r}{\sigma^2}} \left(K\left(\eta_2 - 1\right) + \eta_3\right) \tag{12.21}$$

when $x \geq K$. Note that function (12.21) is increasing since $K\left(\eta_2 - 1\right) + \eta_3 < 0$. Therefore, its minimum is achieved for $x = K$ and it is $\eta_2 K + \eta_3$. When $x < K$ we use Proposition 3.5 to derive the option price as

$$\left(\eta_2 K + \eta_3\right) E^x \left[e^{-r\zeta} I_{\zeta<\infty}\right] = \frac{\left(\eta_2 K + \eta_3\right) x}{K}. \tag{12.22}$$

If $\eta_2 K + \eta_3 > K$, then early exercising is never optimal neither for the writer nor for the holder. Hence, the option turns into a European one and its price is the initial asset value x.

Suppose now that $M > K$ and thus function (12.19) is negative for $A \in (K, M)$ and positive for $A > M$. Therefore, price function (12.18) has a minimum for $A = M$. This means that the exercise boundary is given by formula (12.20) and the writer's exercise region is the interval (K, M). Hence, the option price is given by formula (12.22) when $x < K$. Also, if $x \geq M$, then option price formula (12.18) turns into

$$x\left(1 - \frac{\sigma^2\left(\eta_1 + \eta_2 - 1\right)}{2r}\left(\frac{2r\left(\eta_1 K - \eta_3\right)}{x\left(\eta_1 + \eta_2 - 1\right)\left(2r + \sigma^2\right)}\right)^{\frac{2r}{\sigma^2}+1}\right). \tag{12.23}$$

We can summarize the results above in the following theorem.

Theorem 12.2. *[Theorem 3.9 of Zaevski (2023b)]*
Let $\lambda = 0$.

1. *Suppose that* $M < K$ *– the constant* M *is defined by formula* (12.20)*.*

   (a) *If* $\eta_2 K + \eta_3 > K$*, then early exercising is never optimal for both participants and the option price is* $V(x) = x$*.*

   (b) *If* $\eta_2 K + \eta_3 \leq K$*, then the writer's exercise region is the strike. The option price* $V(x)$ *is given by equation*

      i. (12.21) *when* $x \geq K$*.*

      ii. (12.22) *when* $x < K$*.*

2. *If* $M \geq K$*, then the writer's exercise region is* $\Upsilon^s = [K, M]$*. The price* $V(x)$ *is given by:*

   (a) *statement* (12.22) *when* $x < K$*;*

   (b) *statement* (12.23) *when* $M < x$*;*

   (c) $V(x) = (\eta_1 + \eta_2)x - \eta_1 K + \eta_3$ *when* $K \leq x \leq M$*.*

12.4 Put style options

We turn to the cancellable put options considering payment structures (12.1). We work in a similar manner giving only the differences with the call case. Analogously to Proposition 12.1, we can prove that all points above the strike are not optimal for both participants:

Proposition 12.8. *If* $x > K$ *, then* $x \in \overline{\Upsilon}$*.*

The following restriction for the penalty coefficients stands.

Proposition 12.9. *If* $\eta_2 \geq \eta_1$*, then* $\Upsilon^s \equiv \emptyset$*.*

Proof. Suppose that $\eta_2 \geq \eta_1$ and $x \in \Upsilon^s$. Proposition 12.8 gives that $x \leq K$. Note that the price of the ordinary at-the-money American put, denoted by $\overline{\eta}$, is less than the strike, $\overline{\eta} < K$. The point $x = K$ cannot be optimal for the writer because he has to pay $\eta_2 K + \eta_3 \geq K > \overline{\eta}$ (note that $\eta_2 \geq \eta_1 \geq 1$), i.e. the strategy of never canceling, which leads to a pure American option, is better for him. Hence $x < K$. We continue in the same manner as in Proposition 12.2 turning contradictory inequality (12.5) into

$$\begin{aligned}\eta_1 (K - x) + \eta_2 x + \eta_3 &\leq M (x; \tau, B (\tau; x)) \\ &= E^x \left[\begin{array}{l} e^{-(r+\lambda)B(\tau;\cdot)} \left(K - S_{B(\tau;\cdot)}\right) I_{B(\tau;\cdot)\leq\tau} \\ +e^{-(r+\lambda)\tau} (\eta_1 K + \eta_3 + (\eta_2 - \eta_1) S_\tau) I_{\tau<B(\tau;\cdot)} \end{array} \right] \\ &\leq E^x \left[e^{-(r+\lambda)\zeta} (\eta_1 K + \eta_3 + (\eta_2 - \eta_1) S_\zeta)\right] \\ &< \eta_1 (K - x) + \eta_2 x + \eta_3.\end{aligned}$$

□

We assume hereafter that $\eta_2 < \eta_1$. The following statements describe the shape of the exercise boundaries.

Proposition 12.10. *If $\eta_1 = 1$, $\eta_2 = 0$, and $\eta_3 = 0$, then $\overline{\Upsilon} = (K, \infty)$.*

Proof. See Proposition 10.5. □

Proposition 12.11. *The following two statements hold:*

1. *If $x \in \Upsilon^b$ and $y < x$, then $y \in \Upsilon^b$.*
2. *If $x < K$, $x \in \Upsilon^s$, and $x < y < K$, then $y \in \Upsilon^s$.*

Proof. We refer to Proposition 10.2 for the proof of the first part. Let us turn to the second statement of the proposition. Suppose that a point y from the interval (x, K) is not writer optimal, $y \notin \Upsilon^s$. If $y \in \Upsilon^b$, then the first part of the proposition leads to $x \in \Upsilon^b$ which is impossible. Hence, $y \in \overline{\Upsilon}$. Furthermore, all points between x and y are not optimal for the holder. Suppose also, that all points above y are not optimal for the writer. Therefore, they belong to the continuation region. Let us examine the writer's minimization problem if the initial asset price is $S_0 = y$. Let τ_z be the first hit of the asset to the value z. The writer has to minimize the following term in the interval $z \in (0, y)$

$$\begin{aligned}h (z) &= \mathbb{E}^y \left[e^{-(r+\lambda)\tau_z} (\eta_1 (K - S_{\tau^z}) + \eta_2 S_{\tau^z} + \eta_3)\right] \\ &= (\eta_1 - \eta_2) \mathbb{E}^y \left[e^{-(r+\lambda)\tau_z} \left(\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2} - S_{\tau^z}\right)\right].\end{aligned} \tag{12.24}$$

Function (12.24) is the payment of $(\eta_1 - \eta_2)$ shares of ordinary American put options with strike $\frac{\eta_1 K+\eta_3}{\eta_1-\eta_2}$ – note that $\eta_1 - \eta_2 > 0$. This function first increases to a maximum and then decreases – for the proof see Theorem 4.2. Hence, its minimum is either for $z = 0$ or for $z = y$. The second one contradicts to $y \in \overline{\Upsilon}$, whereas the first one contradicts to $x \in \Upsilon^s$.

Suppose now that some point between y and K is writer optimal. Therefore there exist points $B < C$ such that $\{B, C\} \in \Upsilon^s$ and the interval between them is a part of the continuation region, $(B, C) \in \overline{\Upsilon}$. We can think that $B < y < C$. Let us denote by ζ_B and ζ_C the first hitting moments of the underlying asset to the levels B and C, respectively, and by ζ the less of them, $\zeta = \zeta_B \wedge \zeta_C$. Note that $\zeta < B(\zeta, y)$. Therefore

$$\eta_1 K + \eta_3 - (\eta_1 - \eta_2)\, y > \mathbb{E}^y \left[e^{-(r+\lambda)\zeta} \left(\eta_1 K + \eta_3 - (\eta_1 - \eta_2)\, S_\zeta \right) \right]. \quad (12.25)$$

Let us define a new cancellable option with strike $\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2}$ and without penalty. We shall denote by Υ_1^s, Υ_1^b, and $\overline{\Upsilon}_1$ the corresponding regions and by $A_1(\cdot)$ and $B_1(\cdot)$ the writer's and holder's optimal strategies, respectively. The fact that $x \in \Upsilon^s$ means that the writer prefers to cancel the option immediately provided that he may pay less if the holder exercises. This means that the writer will prefer to stop the contract immediately again if there is no possibility for a lower payment when the holder exercises the option. Hence, $x \in \Upsilon_1^s$ and thus $y \notin \Upsilon_1^b$ because the opposite would contradict the first part of the proposition. Note that the conclusion above is true for all points between B and C. Suppose that $y \in \Upsilon_1^s$ and let us examine the strategy ζ. We have $\zeta < B_1(\zeta, y)$ since $(B, C) \notin \Upsilon_1^b$. Therefore,

$$\begin{aligned}
&(\eta_1 - \eta_2) \left(\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2} - y \right) \\
&\leq (\eta_1 - \eta_2)\, \mathbb{E}^y \left[e^{-(r+\lambda)(\zeta \wedge B_1(\zeta_C; y))} \left(\frac{\eta_1 K + \eta_3}{\eta_1 - \eta_2} - S_{\zeta \wedge B_1(\zeta; y)} \right) \right] \\
&= \mathbb{E}^y \left[e^{-(r+\lambda)\zeta} \left(\eta_1 K + \eta_3 - (\eta_1 - \eta_2)\, S_\zeta \right) \right],
\end{aligned}$$

which contradicts inequality (12.25). Thus $y \in \overline{\Upsilon}_1$, which is impossible due to Proposition 12.10. The last contradiction finishes the proof. □

Proposition 12.12. *If $r > 0$, then $\Upsilon^s \equiv \emptyset$ or $\Upsilon^s \equiv \{K\}$.*

Proof. The proof is similar to the proof of Proposition 12.7. Supposing the opposite, we construct ζ as the first exit from the strip (K_1, K). Assuming $x \in (K_1, K)$ we modify inequality (12.6) to

$$\begin{aligned}
&\mathbb{E}^x\left[e^{-(r+\lambda)\zeta}\left(\eta_1\left(K-S_\zeta\right)+\eta_2 S_\zeta+\eta_3\right)I_{\zeta\le B(\zeta;x)}+e^{-(r+\lambda)B(\zeta;x)}\left(K-S_\zeta\right)I_{B(\zeta;x)<\zeta}\right]\\
&=\mathbb{E}^x\left[e^{-(r+\lambda)\zeta}\left(\eta_1\left(K-S_\zeta\right)+\eta_2 S_\zeta+\eta_3\right)\right]\\
&\le\mathbb{E}^x\left[e^{-r\zeta}\left(\eta_1\left(K-S_\zeta\right)+\eta_2 S_\zeta+\eta_3\right)\right]<\eta_1\left(K-x\right)+\eta_2 x+\eta_3.
\end{aligned}$$

Hence, the point x cannot be optimal for the writer. □

Let us turn to the pricing problem. We shall use an approach similar to those presented in Section 12.3.2 to obtain the equations that the optimal boundaries solve. Propositions 12.8 and 12.11 indicate that the holder's exercise region has the form $\Upsilon^b=(0,A]$ for some constant A, whereas the writer's set has one of the following three forms – $\Upsilon^s=[B,K]$, $\Upsilon^s=\{K\}$, or $\Upsilon^s=\emptyset$.

If $\eta_2=\eta_3=0$ we have an option with multiplied penalty. These options are examined in Chapter 11. Suppose now that $\eta_2+\eta_3>0$ and $A<x<B<K$. Let us denote again by ζ^A and ζ^B the first hitting moments of the underlying asset to the values A and B, respectively. The pricing function of the option can be written as

$$\begin{aligned}
f\left(A,B,x\right)&=\mathbb{E}^x\left[\begin{array}{l} e^{-(r+\lambda)\zeta^B}\left(\eta_1 K-\left(\eta_1-\eta_2\right)S_{\zeta^B}+\eta_3\right)I_{\zeta^B\le\zeta^A}\\ +e^{-(r+\lambda)\zeta^A}\left(K-S_{\zeta^A}\right)I_{\zeta^A<\zeta^B}\end{array}\right]\\
&=\left(K-A\right)\left(\frac{A}{x}\right)^q\frac{B^p-x^p}{B^p-A^p}+\left(\eta_1 K-\left(\eta_1-\eta_2\right)B+\eta_3\right)\left(\frac{B}{x}\right)^q\frac{x^p-A^p}{B^p-A^p}. \qquad (12.26)
\end{aligned}$$

For the meaning of p and q, see equations (3.17). Note that $p\ge q+1$ and the equality is reached when $\lambda=0$. First, let us fix the boundary B. The change of variables $a=\frac{A}{B}$, $k=\frac{K}{B}$, $y=\frac{x}{B}$, and $\xi=\frac{\eta_3}{B}$ leads to the order $0<a<1\le k$. Thus, pricing function (12.26) can be transformed into

$$f\left(A,B,x\right)=\frac{B}{y^q}g\left(a\right)$$

where the function $g\left(a\right)$ is

$$\begin{aligned}
g\left(a\right)&=\frac{\left(k-a\right)a^q\left(1-y^p\right)+\left(\eta_1 k-\eta_1+\eta_2+\xi\right)\left(y^p-a^p\right)}{1-a^p}\\
&=\frac{-a^p\left(\eta_1 k-\eta_1+\eta_2+\xi\right)-a^{q+1}\left(1-y^p\right)+a^q k\left(1-y^p\right)+y^p\left(\eta_1 k-\eta_1+\eta_2+\xi\right)}{1-a^p}.
\end{aligned}$$

It is proven in Appendix 12.A, Proposition 12.16, that its derivative

$$g_a(a;y) = \frac{1-y^p}{(1-a^p)^2} a^{q-1} \left[\begin{array}{l} -a^{p+1}(p-q-1) + a^p k(p-q) \\ -a^{p-q} p(\eta_1 k - \eta_1 + \eta_2 + \xi) - a(q+1) + qk \end{array} \right]$$

has a unique root in the interval $(0,1)$ that leads to the maximum of the price function. We shall denote this root by $a(B)$. Hence, if the writer's strategy is to cancel when the asset reaches the level B, then the holder's strategy is to exercise at the value $Ba(B)$.

Let us fix now the value A in formula (12.26). We change the variables to $b = \frac{B}{A}$, $k = \frac{K}{A}$, $y = \frac{x}{A}$, and $\xi = \frac{\eta_3}{A}$. Therefore, we have to examine $b > 1$. Note that $k > 1$. Price function (12.26) turns into

$$f(A, B, x) = \frac{A}{y^q} g(b)$$

where

$$g(b) = \frac{b^p(k-1) - b^{q+1}(\eta_1 - \eta_2)(y^p - 1) + b^q(\eta_1 k + \xi)(y^p - 1) - (k-1)y^p}{b^p - 1}.$$

Its derivative is

$$g_b(b;y) = \frac{y^p - 1}{(b^p - 1)^2} b^{q-1} \left[\begin{array}{l} b^{p+1}(p-q-1)(\eta_1 - \eta_2) - b^p(\eta_1 k + \xi)(p-q) \\ +b^{p-q} p(k-1) + b(q+1)(\eta_1 - \eta_2) - q(\eta_1 k + \xi) \end{array} \right]. \tag{12.27}$$

Suppose first that $\lambda > 0$ or equivalently $p > q+1$. It is proven in Proposition 12.15 that derivative (12.27) has a unique root larger than one. It leads to the minimum of the price function – we denote it by $b(A)$. Hence, our candidate for the writer's boundary is the solution $\overline{B}$ of the equation $Ba(y)b(ya(y)) = B$ or equivalently

$$a(y)b(ya(y)) = 1.$$

We shall denote the true holder's and writer's boundaries by A^* and B^*. If $\overline{B} \leq K$, then $B^* = \overline{B}$ and $A^* = B^* a(B^*)$. If $\overline{B} > K$, then we need to recognize when $\Upsilon^s = \{K\}$ and when $\Upsilon^s = \emptyset$. Similarly in the call case, we conclude that $\Upsilon^s = \{K\}$ when $\eta_2 K + \eta_3 \leq \overline{\eta}$ and $\Upsilon^s = \emptyset$ when $\eta_2 K + \eta_3 >$

$\overline{\eta}$, where $\overline{\eta}$ is the price of the corresponding non-cancellable at-the-money American option. We derive its value via Theorem 4.2:

$$\overline{\eta} = \frac{K}{q+1}\left(\frac{q}{q+1}\right)^{q}. \tag{12.28}$$

If $\lambda = 0$, then derivative (12.27) is negative for $b > 1$ due to Proposition 12.15 from Appendix 12.A. Therefore $\overline{B} = \infty$, particularly $\overline{B} > K$, and hence the writer's exercise region is either empty or the singleton $\{K\}$. This case is examined above. Note that the same result can be established via Proposition 12.12 – $r > 0$ because $r + \lambda > 0$ and $\lambda = 0$.

Finally, if we suppose that the writer's optimal region is not empty and the asset starts above the strike $x > K$, then the optimal writer's strategy is the first hit to the strike. We use Proposition 3.5 to obtain the option price as

$$\mathbb{E}^{x}\left[e^{-(r+\lambda)\tau}\left(\eta_1\left(S_\tau - K\right)^{+} + \eta_2 S_\tau + \eta_3\right) I_{\tau<\infty}\right] = \left(\eta_2 K + \eta_3\right)\left(\frac{K}{x}\right)^{q}. \tag{12.29}$$

We summarize the derived results in the following theorem.

Theorem 12.3. *[Theorem 4.6 of Zaevski (2023b)]*

We recognize the following cases:

1. *If $\eta_2 = \eta_3 = 0$ we refer to the results of Chapter 11. See also Remark 12.1. Let us mention that if $r \geq 0$, then $\Upsilon^s = \{K\}$ and $\Upsilon^b = (0, K)$. The rest of the results are similar to the general case.*

2. *If $\eta_2 \geq \eta_1$, then the option is ordinary American.*

3. *Suppose now that $\eta_2 < \eta_1$ and $\eta_2 + \eta_3 > 0$.*

   (a) *If $\overline{B} \leq K$, then $B^* = \overline{B}$ and $A^* = B^* a\left(B^*\right)$ – thus the exercise regions for the writer and holder are $\Upsilon^s = [B^*, K]$ and $\Upsilon^b = [0, A^*)$, respectively. The option price $V(x)$ is given by:*

      i. *equation (12.29) when $x \geq K$;*
      ii. *by formula (12.26) when $\overline{A} \leq x \leq \overline{B}$;*
      iii. *by $V(x) = -\left(\eta_1 - \eta_2\right)x + \eta_1 K + \eta_3$ when $\overline{B} < x < K$;*
      iv. *by $V(x) = K - x$ when $x < \overline{A}$.*

(b) *If $K < \overline{B}$ and $\eta_2 K + \eta_3 \leq \overline{\eta}$, $\overline{\eta}$ is given by equation (12.28), then $B^* = K$ and $A^* = Ka(K)$. The exercise regions are $\Upsilon^s = \{K\}$ and $\Upsilon^b = \left(0, \overline{A}\right]$. The option price is determined as in the previous case.*

(c) *If $K < \overline{B}$ and $\eta_2 K + \eta_3 > \overline{\eta}$, then the option is ordinary American.*

Remark 12.2. *Analogously to the results from Section 12.3.3, we always establish a smooth fit at the holder's boundary, but only when $\overline{B} < K$ at the writer's one.*

12.5 Numerical results

Now we discuss some numerical examples based on the theoretical results derived above.

12.5.1 Call options

As we have seen above, the penalty coefficients η_1, η_2, and η_3 influence significantly the option behavior. Roughly said, the option looks more like the corresponding ordinary American call when they are larger. We shall see for which values the cancellable feature has its impact. First, Proposition 12.2 says that η_3 is limited by the inequality $\eta_3 < \eta_1 K$. It turns out that this restriction is too weak. We know that if the writer's optimal set is not empty, then the strike belongs to it. Hence, $\eta_2 K + \eta_3 < \overline{\eta}$, where $\overline{\eta}$ is given by equation (12.13). Obviously, this equation is stronger due to $\overline{\eta} < K$. Otherwise, the assumption $\Upsilon^s = \emptyset$ means that the non-use of the canceling right is the best writer's strategy, particularly better than the immediate exercise. So, the inequality $\eta_2 K + \eta_3 \geq \overline{\eta}$ holds and hence it determines whether the option is ordinary American or cancellable. We can also see that if the number of shares, η_2, is larger than $\frac{1}{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1}$, then the option is ordinary American ($\gamma = p - q$). Otherwise, the value $K\left[\frac{1}{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\gamma-1} - \eta_2\right]$ is critical for the fixed amount η_3 – if it is larger, then the option is ordinary American; otherwise it is a real cancellable option. We have to mention an important fact, the option's essence does not depend on the coefficient η_1 – it influences whether the writer's optimal set is only the strike once we know that the option is real cancellable.

Having in mind the previous restrictions, we examine call options with the following parameters – the risk-free rate $r = 0.05$, the discount factor $\lambda = 0.01$, the volatility $\sigma = 0.3$, the strike $K = \$5$, and the initial asset value $x = \$20$. We vary the penalty parts as $\eta_1 \in (1, 1.1)$, $\eta_2 \in (0, 0.3)$, and $\eta_3 \in (0, 1)$. When we fix some of these penalties, we use the values $\eta_1 = 1.05$, $\eta_2 = 0.2$, and $\eta_3 = 0.5$.

The behavior of the optimal boundaries w.r.t. the three penalty components is presented in Figure 12.1. We can see that the writer's boundary decreases to the strike when the penalty coefficients increase – we mark by red color the critical values. Note that this boundary vanishes when the penalties are large enough. Also, the holder's boundary is an increasing function and it tends to the American optimal boundary. We present the call prices in Figures 12.3a, 12.3b, and 12.3c varying the three different penalty coefficients.

The results for some particular options are reported in Table 12.1. There can be seen the option prices – the second line – as well as the optimal boundaries. The writer's boundary is the first value at the first line, whereas the holder's one is given at the second place. We vary the three parts of the penalty among $\eta_1 \in \{1; 1.05; 1.1; 1.2\}$, $\eta_2 \in \{0.05; 0.1; 0.15, 0.2\}$, and $\eta_3 \in \{0.25; 0.5; 0.75; 1\}$.

12.5.2 Put options

Analogously to the call case, we can see that the inequality $\eta_2 K + \eta_3 < \overline{\eta}$ determines when the option is ordinary American or cancellable – we have a cancellable option when it holds and a pure American otherwise. Note that $\overline{\eta}$ is given by equation (12.28). We conclude that we have a real cancellable option if (A) the number of shares is less than $\frac{1}{q+1}\left(\frac{q}{q+1}\right)^q$ and (B) the fixed amount η_3 is less than $K\left[\frac{1}{q+1}\left(\frac{q}{q+1}\right)^q - \eta_2\right]$. If one of these conditions does not hold, then we have a non-cancellable American option. Let us discuss the role of the penalty coefficient η_1. Proposition 12.9 says that a necessary condition for the option to be real cancellable is $\eta_2 < \eta_1$. On the other hand, inequality $\eta_2 K + \eta_3 < \overline{\eta}$ is stronger since $\overline{\eta} < K$ and therefore $\eta_2 < 1 \leq \eta_1$. Hence, as in the call case, the coefficient η_1 influences whether the writer's optimal set is the strike, but not whether we have a real cancellable option or pure American.

Taking into account these limitations, we consider put options with the following parameters – the risk-free rate $r = -0.03$, the discount factor $\lambda = 0.05$, the volatility $\sigma = 0.3$, the strike $K = \$10$, and the initial asset value $x = \$5$. The penalties are taken as before – $\eta_1 \in (1, 1.1)$, $\eta_2 \in (0, 0.3)$, and $\eta_3 \in (0, 1)$. When we fix some of them, we use the values $\eta_1 = 1.05$, $\eta_2 = 0.2$, and $\eta_3 = 0.5$.

We present the optimal boundaries in Figure 12.2 fixing one of the penalty parts and varying the others. As we expected, the writer's boundary increases w.r.t. the penalties and goes to the strike. The meaning of the red points is preserved – they mark namely the values for which the writer's boundary turns into the strike. Also, we can see that the holder's boundaries are decreasing functions. The resulting price behavior is presented in Figures 12.3d, 12.3e, and 12.3f.

We report some results for option prices and the related exercise boundaries in Table 12.1. The optimal boundaries are placed at the first line – the holder's boundary is first; the writer's one is second. The obtained prices are presented in the second line. The three parts of the penalties are again among $\eta_1 \in \{1; 1.05; 1.1; 1.2\}$, $\eta_2 \in \{0.05; 0.1; 0.15, 0.2\}$, and $\eta_3 \in \{0.25; 0.5; 0.75; 1\}$.

12.A Uniqueness of the solutions

Let p and q be defined as in equations (3.17).

Lemma 12.2. *Let $\eta > 1$ and $k < 1$. The function $\overline{h}(a)$, defined on the interval $(0, 1)$ as*

$$\overline{h}(a) = a^{p+1}\eta(p-q-1) - a^{p}\eta k(p-q) - a^{p-q}p(1-k) + a(q+1)\eta - q\eta k, \tag{12.30}$$

starts from a negative value, increases having a root, and then stays positive.

Proof. See Appendix 11.A. □

Proposition 12.13. *Let $\eta_1 \geq 1$, $\eta_2 \geq 0$, and $0 \leq \frac{\xi}{\eta_1} < k < 1$. The function*

$$\begin{aligned} h(a) = a^{p+1}(\eta_1+\eta_2)(p-q-1) - a^{p}(\eta_1 k - \xi)(p-q) \\ -a^{p-q}p(1-k) + a(q+1)(\eta_1+\eta_2) - q(\eta_1 k - \xi) \end{aligned} \tag{12.31}$$

has a unique root in the interval $(0, 1)$.

Proof. First, note that if $\eta_1 = 1$ and $\eta_2 = 0$, we have an option with a constant penalty. Hence, we can use Appendix 9.A. Suppose now that $\eta_1 + \eta_2 > 1$. We can decompose function (12.31) as

$$\begin{aligned} h(a) &= \overline{h}(a) + \widetilde{h}(a) \\ \widetilde{h}(a) &= (\eta_2 k + \xi)(a^p (p-q) + q), \end{aligned}$$

The function $\overline{h}(a)$is defined as (12.30) for $\eta = \eta_1 + \eta_2$. We finish the proof using the inequality $h(0) = -q(\eta_1 k - \xi) < 0$, Lemma 12.2, and the fact that $\widetilde{h}(a)$ is an increasing positive function. □

Proposition 12.14. *Let $\eta_1 \geq 1$, $\eta_2 \geq 0$, $0 \leq \frac{\xi}{\eta_1} < k \leq 1$, and the function $h(\cdot)$ be defined as*

$$h(b) = -b^{p+1}(p-q-1) + b^p k(p-q) + b^{p-q} p(\eta_1 + \eta_2 - \eta_1 k + \xi) - b(q+1) + qk \tag{12.32}$$

in the interval $b \in [1, \infty)$. The following statements hold:

1. *If $p = q + 1$, then function (12.32) is positive.*
2. *If $p > q + 1$, $k = 1$, $\eta_2 = \eta_3 = 0$, and $r < 0$, then function (12.32) is negative.*
3. *In all other cases, function (12.32) has just one root larger than one.*

Proof. When $\eta_2 = \eta_3 = 0$ we refer to Appendix 11.B. The proof when $\eta_2 + \eta_3 > 0$ is very similar to the the case $\{\eta_1 = 1, \eta_2 = 0,\ \eta_3 > 0\}$ examined in Appendix 9.B – thus we omit it. □

Proposition 12.15. *Let $k > 1$, $\eta_1 > \eta_2 \geq 0$, $\eta_1 \geq 1$, and $\xi \geq 0$.*

1. *If $p > q + 1$, then the function*

$$\begin{aligned} h(b) = {} & b^{p+1}(p-q-1)(\eta_1 - \eta_2) - b^p(\eta_1 k + \xi)(p-q) \\ & + b^{p-q} p(k-1) + b(q+1)(\eta_1 - \eta_2) - q(\eta_1 k + \xi) \end{aligned} \tag{12.33}$$

   has a unique root larger than one.

2. *Otherwise, if $p = q + 1$, then function (12.33) is negative for $b > 1$.*

Proof. We rewrite function (12.33) as

$$h(b) = (\eta_1 - \eta_2)\,\overline{h}(b)$$

for

$$\begin{aligned}\overline{h}(b) &= b^{p+1}(p-q-1) - b^p\left(\overline{k}+\overline{\xi}\right)(p-q) + b^{p-q}p\left(\overline{k}-1\right) + b(q+1) - q\left(\overline{k}+\overline{\xi}\right)\\ \overline{k} &= 1 + \frac{k-1}{\eta_1-\eta_2}\\ \overline{\xi} &= \frac{(k-1)(\eta_1-1)+\eta_2+\xi}{\eta_1-\eta_2}.\end{aligned}$$

Note that $\eta_1 > \eta_2$, $\overline{k} > 1$, and $\overline{\xi} > 0$. The desired result is proven for the function $\overline{h}(b)$ in Appendix 10.B. □

Lemma 12.3. *Let $k \geq 1$ and $\xi \geq 0$. The function $\overline{h}(a)$, defined on the interval $(0,1)$ as*

$$\overline{h}(a) = -a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q}p(k-1+\xi) - a(q+1) + qk, \tag{12.34}$$

starts from a positive value, decreases having a root, and then stays negative.

Proof. See Appendix 10.B. □

Proposition 12.16. *Let $k \geq 1$, $\eta_1 > \eta_2 \geq 0$, $\eta_1 \geq 1$, $\xi \geq 0$, and the function*

$$h(a) = -a^{p+1}(p-q-1) + a^p k(p-q) - a^{p-q}p(\eta_1 k - \eta_1 + \eta_2 + \xi) - a(q+1) + qk \tag{12.35}$$

be defined on the interval $a \in (0,1)$.

1. *Let $\eta_2 + \eta_3 = 0$.*
   (a) *The function (12.35) is positive when $\{p = q+1, k = 1\}$ or $\{p > q+1, k = 1, r \geq 0\}$.*
   (b) *In the rest of the cases, namely when $\{p = q+1, k > 1\}$, $\{p > q+1, k = 1, r < 0\}$, and $\{p > q+1, k > 1\}$, function (12.35) has a unique root.*

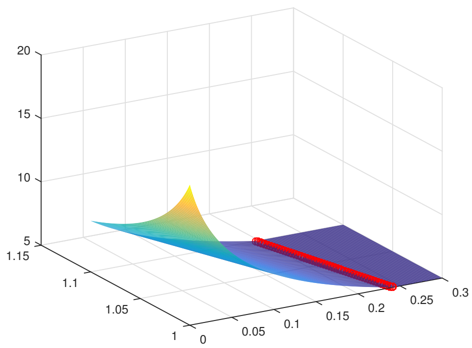
2. *If $\eta_2 + \eta_3 > 0$, then function (12.35) has just one root. Also $h(a)$ is positive before the root and negative after it.*

Proof. The proposition is proven in Appendix 11.C when $\eta_2 + \eta_3 = 0$. Suppose now that $\eta_2 + \eta_3 > 0$. We can decompose $h(a) = \overline{h}(a) + \widetilde{h}(a)$ where function $\overline{h}(a)$ is defined by formula (12.34) and

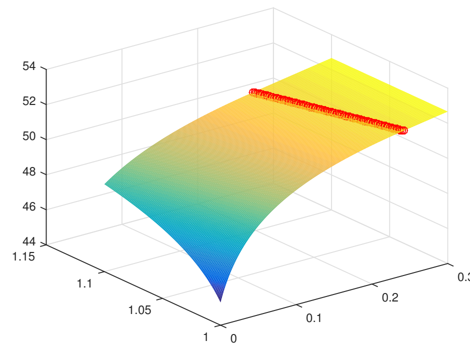
$$\widetilde{h}(a) = -a^{p-q} p\left(\left(k-1\right)\left(\eta_1 - 1\right) + \eta_2\right).$$

We complete the proof using Lemma 12.3 and observing that $\widetilde{h}(a)$ is a decreasing negative function and $h(0) = qk > 0$. □

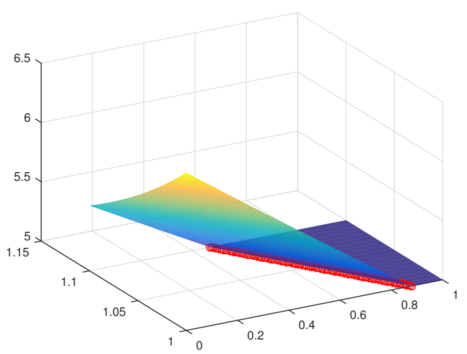
Figure 12.1: Call options boundaries



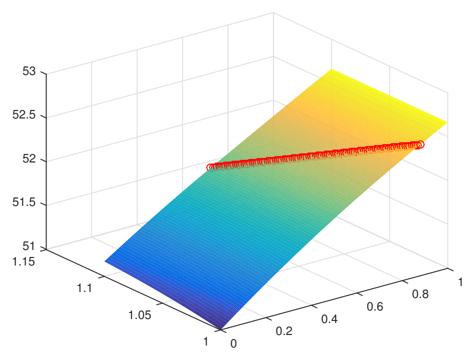
(a) writer's boundary w.r.t η_1 and η_2



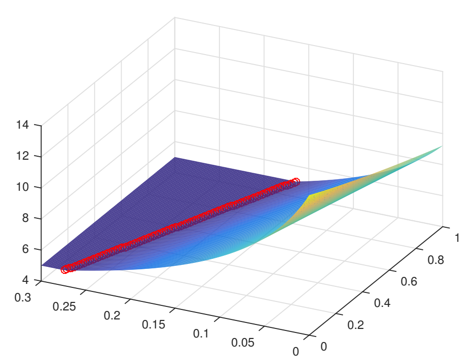
(b) holder's boundary w.r.t η_1 and η_2



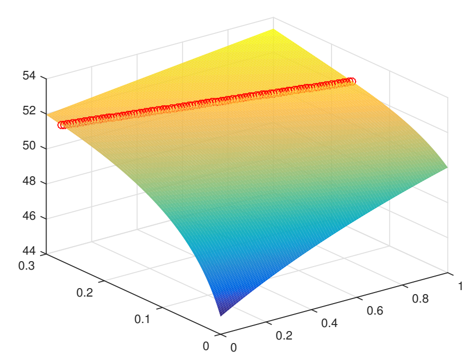
(c) writer's boundary w.r.t η_1 and η_3



(d) holder's boundary w.r.t η_1 and η_3

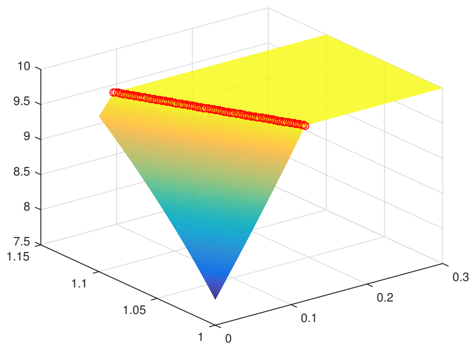


(e) writer's boundary w.r.t η_2 and η_3

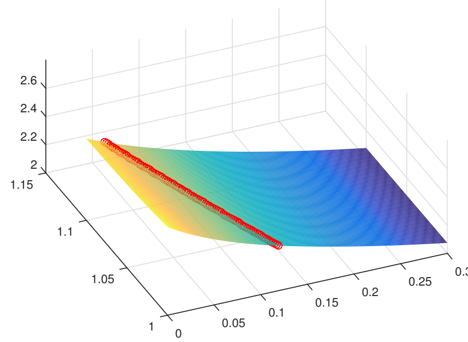


(f) holder's boundary w.r.t η_2 and η_3

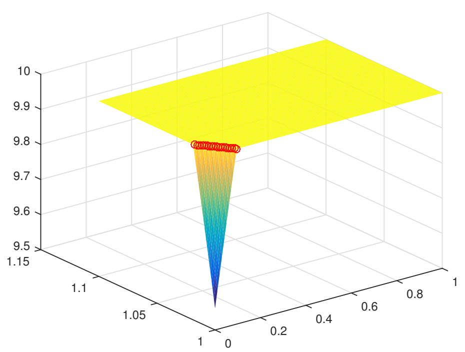
Figure 12.2: Put options boundaries



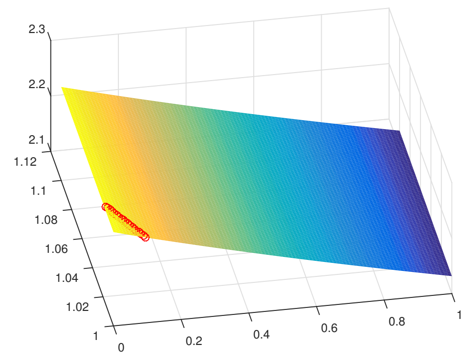
(a) writer's boundary w.r.t η_1 and η_2



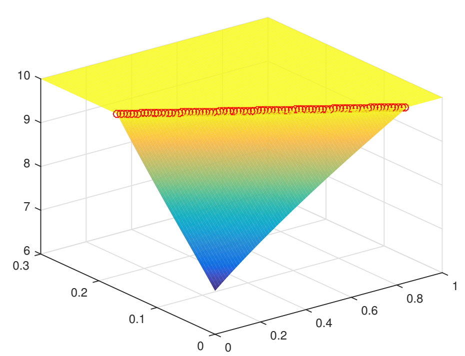
(b) holder's boundary w.r.t η_1 and η_2



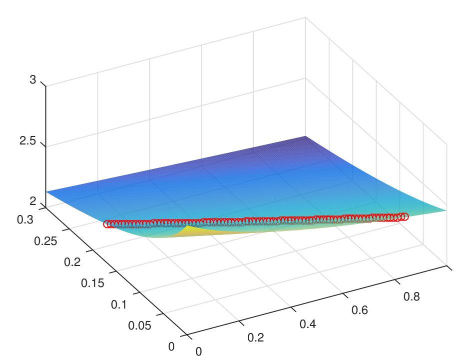
(c) writer's boundary w.r.t η_1 and η_3



(d) holder's boundary w.r.t η_1 and η_3

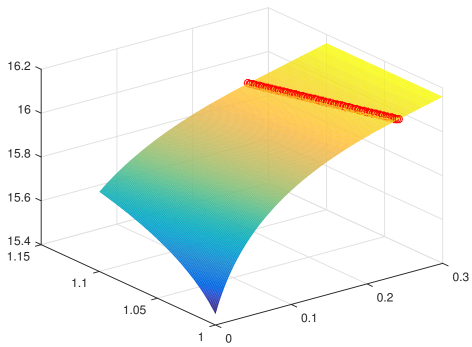


(e) writer's boundary w.r.t η_2 and η_3

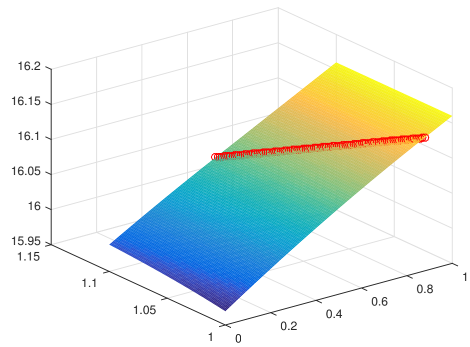


(f) holder's boundary w.r.t η_2 and η_3

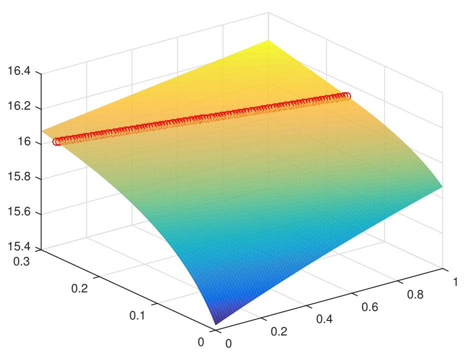
Figure 12.3: Options prices



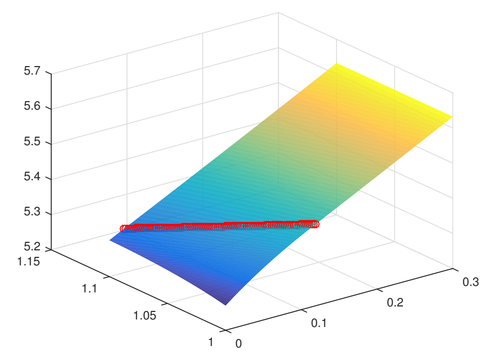
(a) Call prices w.r.t η_1 and η_2



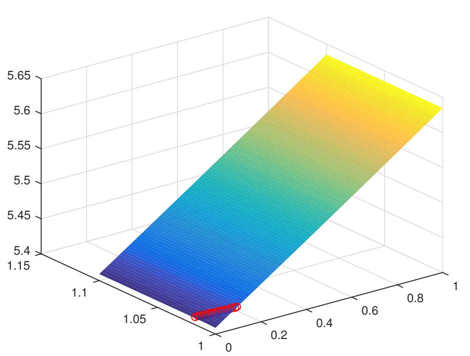
(b) Call prices w.r.t η_1 and η_3



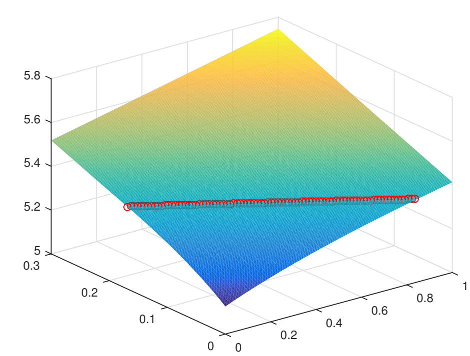
(c) Call prices w.r.t η_2 and η_3



(d) Put prices w.r.t η_1 and η_2



(e) Put prices w.r.t η_1 and η_3



(f) Put prices w.r.t η_2 and η_3

Table 12.1: Call option prices and optimal boundaries

η_2	0.05	0.1	0.15	0.2
		penalty coefficient $\eta_1 = 1$		
$\eta_3 = 0.25$	{10.9835;47.9751} \$15.6691	{8.4011;49.8094} \$15.8449	{6.9288;50.8208} \$15.9507	{5.9432;51.4786} \$16.0229
$\eta_3 = 0.5$	{10.0752;49.0768} \$15.7722	{7.7986;50.5301} \$15.9196	{6.4656;51.3630} \$16.0100	{5.5627; 51.9147} \$16.0723
$\eta_3 = 0.75$	{9.2236;50.0061} \$15.8649	{7.2169;51.1625} \$15.9879	{6.0134;51.8470} \$16.0646	{5.1889;52.3078} \$16.1179
$\eta_3 = 1$	{8.4209;50.7978} \$15.9482	{6.6546;51.7185} \$16.0500	{5.5718;52.2787} \$16.1145	{5; 52.6652} \$16.1602
		penalty coefficient $\eta_1 = 1.05$		
$\eta_3 = 0.25$	{9.0261;48.9852} \$15.7633	{7.4034;50.2132} \$15.8863	{6.3308;50.9953} \$15.9696	{5.5558;51.5439} \$16.0302
$\eta_3 = 0.5$	{8.4011;49.8094} \$15.8449	{6.9288;50.8208} \$15.9507	{5.9432;51.4786} \$16.0229	{5.2260;51.9456} \$16.0759
$\eta_3 = 0.75$	{7.7986;50.5301} \$15.9196	{6.4656;51.3630} \$16.0100	{5.5627;51.9147} \$16.0723	{5.1889;52.3078} \$16.1179
$\eta_3 = 1$	{7.2169;51.1625} \$15.9879	{6.0134;51.8470} \$16.0646	{5.1889;52.3078} \$16.1179	{5;52.6652} \$16.1602
		penalty coefficient $\eta_1 = 1.1$		
$\eta_3 = 0.25$	{7.8900;49.5313} \$15.8169	{6.7256;50.4598} \$15.9122	{5.8903;51.1023} \$15.9813	{5.2554;51.5765} \$16.0339
$\eta_3 = 0.5$	{7.4034;50.2132} \$15.8863	{6.3308;50.9953} \$15.9696	{5.5558;51.5439} \$16.0302	{5;51.9526} \$16.0767
$\eta_3 = 0.75$	{6.9288;50.8208} \$15.9507	{5.9432;51.4786} \$16.0229	{5.2260;51.9456} \$16.0759	{5;52.3120} \$16.1184
$\eta_3 = 1$	{6.4656;51.3630} \$16.0100	{5.5627;51.9147} \$16.0723	{5;52.3120} \$16.1184	{5;52.6652} \$16.1602
		penalty coefficient $\eta_1 = 1.2$		
$\eta_3 = 0.25$	{6.5741;50.0836} \$15.8729	{5.8489;50.7178} \$15.9396	{5.2793;51.2001} \$15.9920	{5;51.5866} \$16.0351
$\eta_3 = 0.5$	{6.2297;50.6170} &15.9288	{5.5504;51.1660} \$15.9882	{5.0151;51.5866} \$16.0350	{5;51.9526} \$16.0767
$\eta_3 = 0.75$	{5.8903;51.1023} &15.9813	{ 5.2554;51.5765} \$16.0339	{5;51.9526} \$16.0767	{5;52.3120} \$16.1184
$\eta_3 = 1$	{5.5558;51.5439} \$16.0302	{5;51.9526} \$16.0767	{5;52.3120} \$16.1184	{5;52.6652} \$16.1602

Table 12.2: Put option prices and optimal boundaries

η_2	0.05	0.1	0.15	0.2
		penalty coefficient $\eta_1 = 1$		
$\eta_3 = 0.25$	{2.5588;7.6681}	{2.4126;8.6126}	{2.3100;9.6050}	{2.2308;10}
	\$5.2703	\$5.3475	\$5.4102	\$5.4650
$\eta_3 = 0.5$	{2.4656;8.7271}	{2.3541;9.6389}	{2.2686;10}	{2.1955; 10}
	\$5.3480	\$15.9196	\$5.4663	\$5.5212
$\eta_3 = 0.75$	{2.4022;9.6654}	{2.3093;10}	{2.2308;10}	{2.1625;10}
	\$5.4120	\$5.4673	\$5.5224	\$5.5778
$\eta_3 = 1$	{2.3534;10}	{2.2686;10}	{2.1955;10}	{2.1315;10}
	\$5.4680	\$5.5233	\$5.5789	\$5.6348
		penalty coefficient $\eta_1 = 1.05$		
$\eta_3 = 0.25$	{2.5247;8.8198}	{2.4022;9.6654}	{ 2.3093;10}	{ 2.2308;10}
	\$5.2946	\$5.3564	\$5.4108	\$5.4650
$\eta_3 = 0.5$	{2.4551;9.6858}	{2.3534;10}	{2.2686;10}	{2.1955;10}
	\$5.3572	\$5.4119	\$5.4663	\$5.5212
$\eta_3 = 0.75$	{2.4015;10}	{2.3093;10}	{2.2308;10}	{2.1625;10}
	\$5.4126	\$5.4673	\$5.5224	\$5.5778
$\eta_3 = 1$	{2.3534;10}	{2.2686;10}	{2.1955;10}	{2.1315;10}
	\$5.4680	\$5.5233	\$5.5789	\$5.6348
		penalty coefficient $\eta_1 = 1.1$		
$\eta_3 = 0.25$	{ 2.5139;9.7012}	{2.4015;10}	{2.3093;10}	{2.2308;10}
	\$5.3032	\$5.3570	\$5.4108	\$5.4650
$\eta_3 = 0.5$	{2.4544;10}	{2.3534;10}	{2.2686;10}	{2.1955;10}
	\$5.3579	\$5.4119	\$5.4663	\$5.5212
$\eta_3 = 0.75$	{2.4015;10}	{2.3093;10}	{2.2308;10}	{2.1625;10}
	\$5.4126	\$5.4673	\$5.5224	\$5.5778
$\eta_3 = 1$	{2.3534;10}	{2.2686;10}	{2.1955;10}	{2.1315;10}
	\$5.4680	\$5.5233	\$5.5789	\$5.6348
		penalty coefficient $\eta_1 = 1.2$		
$\eta_3 = 0.25$	{2.5132;10}	{2.4015;10}	{2.3093;10}	{2.2308;10}
	\$5.3038	\$5.3570	\$5.4108	\$5.4650
$\eta_3 = 0.5$	{2.4544;10}	{2.3534;10}	{2.2686;10}	{2.1955;10}
	&5.3579	\$5.4119	\$5.4663	\$5.5212
$\eta_3 = 0.75$	{2.4015;10}	{2.3093;10}	{2.2308;10}	{2.1625;10}
	&5.4126	\$5.4673	\$5.5224	\$5.5778
$\eta_3 = 1$	{2.3534;10}	{2.2686;10}	{2.1955;10}	{2.1315;10}
	\$5.4680	\$5.5233	\$5.5789	\$5.6348

Chapter 13

Pricing cancellable American put options on the finite time horizon

based on the paper

Zaevski, Tsvetelin S. "Pricing cancellable American put options on the finite time horizon." Journal of Futures Markets 42.7 (2022): 1284-1303.

Abstract: The purpose of this chapter is to present a numerical approach for pricing cancellable American put options, also known as game or Israeli options, on the finite time horizon. These options generalize the concept of American derivatives adding an early exercise right for the option's writer to the existing holder's right. The writer has to pay a penalty amount above the usual option payment to use this right. We first obtain the shape of the optimal regions for both participants. Then we approximate the optimal exercise boundaries maximizing the option's writer and holder financial expectations using some first exit properties of the Brownian motion. We also construct an efficient pricing algorithm based on these boundaries. A semi-closed form formula is derived when the underlying asset starts above the strike.

13.1 Motivation and main results

Options contracts are some of the most tradable derivative instruments in modern financial markets. They depend on an underlying asset and give the option's holder the right to sell or buy it at a pre-agreed price, named strike price, until or at some maturity date. There are two main types of options – European and American. The basic difference between them is the moment they expire. The European contracts can be exercised only at the maturity date. Alternatively, an American option gives its holder the right to choose the expiration date. In such a way, he can capture immediately the payoff when the underlying asset reaches the desirable level. However, the option's holder is in a privileged position due to the essence of this early exercising right. Later, new financial instruments, called cancellable American options, appear to compensate this imbalance. They give the option's writer the right to cancel the option prematurely paying some amount above the usual option payment. This way the writer's financial interest is protected from the possible large market deviations.

Two main questions arise for all American-style derivatives, particularly for the cancellable ones. First, is it optimal to exercise immediately at the actual underlying asset price? This question holds for both option participants when the American option exhibits a cancellable feature. The second important question is what is the fair option price if immediate exercise is not optimal, neither for the writer nor for the holder. We answer these questions in the present chapter for finite maturity put options. The basic novelties can be summarized as follows. First, we examine options written on an asset that continuously pays dividends. It is well-known that this is a realistic assumption and it leads to completely new model features. For example, if there are no dividends, then the cancellable American put turns into an American-style option which expires when the underlying asset hits the strike. This is a one-sided optimal stopping problem. On the contrary, early cancellation may be writer optimal not only at the strike in the presence of dividends which leads to a two-sided problem. Second, we prove a series of propositions that determine the optimal regions for both option's participants. Based on them, we construct a numerical algorithm to approximate the optimal boundaries. In such a way we know at every moment whether the exercise is profitable or not for the option's writer or holder. Also, we create a new relatively fast Monte Carlo algorithm to derive the fair option price when the asset starts between the optimal boundaries. Finally, we derive a semi-closed form formula for the option price when the initial asset price is above the strike.

Our first task is to determine the shape of the optimal regions. It turns out that the holder's one for a fixed moment is an interval $(0, A)$ for some constant A less than the strike, $A < K$. The writer's region may be an interval $(B, K]$, $0 < A < B < K$, the singleton $\{K\}$, or the empty set. Once we know the form of the exercise regions, we use specific American-style derivatives to approximate the actual values of the optimal boundaries. These derivatives require one of the option's holder or writer to exercise when the asset reaches some boundary and give the other the right to exercise earlier. In such a way these instruments allow one of the participants to maximize his financial result assuming that the other has a known strategy. A special role in our examination has a subclass of such derivatives. Their owner has the right to exercise at every moment before the maturity receiving the usual option payment. In addition, if the underlying asset hits the strike when the remaining time to maturity is larger than some previously defined value τ, the derivative expires paying some amount η. We shall call this derivative a (τ, η)-American option. Obviously, if the time to maturity is less than τ, then the (τ, η)-American option coincides with the ordinary American one.

Our results, summarized w.r.t the time to maturity, are as follows. The holder's exercise boundary starts from a point less than or equal to the strike and decreases to its perpetual value. There are three possibilities for the writer's boundary. In all of them, it does not exist for small enough maturities – it is more profitable for the writer to wait for maturity instead of canceling and paying the penalty. This is always true for some very large penalties – thus the option turns into non-cancellable. Otherwise, small enough penalties lead to the strike for the writer's boundary for some medium maturities. It may stay always equal to the strike after the initial period turning the option into a (τ, η)-American put. Alternatively, it may fall below the strike at some moment after which it decreases to its perpetual value.

In all cases, we first approximate the optimal boundaries using an approach similar to those used for the strangle strategies. The main difference is that we have a min-max problem for the cancellable options in contrast to the double-max task for the strangles. This is due to the minimizing purpose of the writer. This way, applying the results of Chapters 2.2 and 2.4, we can immediately tell the continuation region from the optimal points from the holder or the writer. Also, we derive the option price with a relatively high precision.

On the other hand, we can approximate the boundaries at a denser grid. Thus, the cancellable option pricing problem turns into a partial differential equation in a known region. Its solution can be expressed as the expectation related to the first hitting moment of the asset to these boundaries. The continuation region consists of two parts – one between the optimal boundaries and another above the strike. Hence, we have to examine them separately. If the asset starts between the boundaries, then we have a two-sided exit problem. We may solve it by modifying the used in the previous chapters Crank-Nicolson finite difference approach. Alternatively, we create a relatively fast and efficient Monte Carlo algorithm to derive the fair option price. Otherwise, if the initial asset price is above the strike we have a one-sided hitting problem. In this case, we derive a semi-closed form formula that depends on the prices of some ordinary American options. Their maturity is the critical time value at which the writer's boundary appears.

Also, we provide various numerical experiments to validate the derived theoretical results. That way we confirm their applicability and economic importance. The constructed algorithms are applied to different values of the parameters which describe all possible cases for the option features. Thus the accuracy and efficiency of the proposed approach are checked. We summarize the produced numerical results in several figures and tables. The impact of the writer's early canceling right is evaluated through a comparison with the corresponding non-cancellable American option.

The chapter is organized as follows. In Section 13.2 we introduce the model and give the base we use later. In Section 13.3 we obtain the shape of the exercise regions. In Section 13.4 we motivate our approach for deriving the optimal boundaries. The option pricing problem is considered in Section 13.5. Some numerical examples are conferred in Section 13.6. We conclude by Section 13.7.

13.2 Preliminaries

Let $T < \infty$ be the maturity date and $\mathcal{T}_{[t,T]}$ and $\overline{\mathcal{T}}_{[t,T]}$ be the sets of all stopping times with values in $[t, T]$ and $[t, T] \bigcup \{\infty\}$, respectively. They will be associated with the holder's and writer's strategies. The state $\{\infty\}$ means that the writer will not cancel earlier. Note that the elements of the set $\mathcal{T}_{[t,T]}$ are always finite since the option must be exercised, even if this is done on the expiration date T. Let $\eta > 0$ be the penalty. In such a way the payment structure for a cancellable put option is

$$
\begin{aligned}
N_1(t,x) &= e^{-\lambda t}(K-x)^+ \\
N_2(t,x) &= e^{-\lambda t}\left(\eta + (K-x)^+\right),
\end{aligned}
\tag{13.1}
$$

i.e. the writer owes amount of $N_2(t,x)$ if he cancels the option in moment t at price $S_t = x$. Analogously, he is obliged to pay $N_1(t,x)$ if the holder exercises. Proposition 3.3 gives the time-dependence of the option price. The optimal strategies ($A(\zeta)$ and $B(\zeta)$) and the regions (Υ^s and Υ^b) are defined by Definitions 9.1 and 9.2 – the payoffs now are given by formulas (10.1). Recall that $A(\zeta)$ and $B(\zeta)$ are the optimal strategies for one of the option's participants if the other follows the strategy ζ. The way we determine the exercise regions in Definition 9.2 together with the form of payment structure (13.1) leads to another proposition related to the time dependence.

Proposition 13.1. *Let us denote by $\alpha(t,T)$ and $\beta(t,T)$ the holder's and writer's exercise boundaries at the moment t if the maturity is T. Let also $a(\tau)$ and $b(\tau)$ be the corresponding boundaries at the initial moment provided that the time to maturity is τ. Then $\alpha(t,T) = a(T-t)$ and $\beta(t,T) = b(T-t)$.*

Propositions 3.3 and 13.1 have a major importance. First, the future option price can be computed using the initial price function and the time to maturity. Also, the exercise boundaries depend only on the remaining time to maturity, not on the particular values of t and T. These facts allow us to use an alternative parametrization w.r.t. the time to maturity, denoted by τ, instead of the original one. The related sets will be denoted by $\mathcal{T}_\tau$, $\overline{\mathcal{T}}_\tau$, Υ^b_τ, Υ^s_τ, and $\overline{\Upsilon}_\tau$ instead $\mathcal{T}_{[t,T]}$, $\overline{\mathcal{T}}_{[t,T]}$, $\Upsilon^b(t)$, $\Upsilon^s(t)$, and $\overline{\Upsilon}(t)$, respectively. We can see that the relations between them are $\mathcal{T}_{[t,T]} \equiv \mathcal{T}_{T-t}$, $\overline{\mathcal{T}}_{[t,T]} \equiv \overline{\mathcal{T}}_{T-t}$, $\Upsilon^b(t) \equiv \Upsilon^b_{T-t}$, $\Upsilon^s(t) \equiv \Upsilon^s_{T-t}$, and $\overline{\Upsilon}(t) \equiv \overline{\Upsilon}_{T-t}$.

13.3 Exercise regions

Note that Propositions 10.1, 10.2, and 10.3 hold for the finite maturity options as well as for the perpetual ones. Hence, for a fixed moment t, the holder's exercise region is of the form $\Upsilon^b(t) = (0,a)$ for some constant a.

On the contrary, we can expect that the writer's region is the empty set, the singleton $\{K\}$, or an interval with the strike for the upper boundary.

Now we need the following lemma that is very similar to one presented in Kunita and Seko (2004), lemma 3.3.

Lemma 13.1. *The price function of a cancellable option is non-decreasing w.r.t the time to maturity.*

Using Lemma 13.1 we shall prove a proposition that provides the behavior of the holder's and writer's optimal boundaries – $a(\tau)$ and $b(\tau)$, respectively.

Proposition 13.2. *Both boundaries are non-increasing functions w.r.t. the time to maturity.*

Proof: We denote again the option price by $V(\tau, x)$ where τ is the time to maturity and x is the initial asset price. Let ϵ be an arbitrary positive constant and δ be small enough. Using Lemma 13.1 and $a(\tau) + \delta \notin \Upsilon_\tau^b$ we conclude

$$\begin{aligned} V(\tau+\varepsilon, a(\tau)+\delta) &\geq V(\tau, a(\tau)+\delta) \\ &> e^{-\lambda\tau}(K-(a(\tau)+\delta))^+ \\ &\geq e^{-\lambda(\tau+\varepsilon)}(K-(a(\tau)+\delta))^+. \end{aligned}$$

Therefore $a(\tau)+\delta \notin \Upsilon_{\tau+\varepsilon}^b$ too. Hence, $a(\tau)+\delta > a(\tau+\varepsilon)$. Taking $\delta \to 0$ we derive $a(\tau) \geq a(\tau+\varepsilon)$.

Analogously, we use that $b(\tau)-\delta \notin \Upsilon_\tau^s$ to obtain

$$\begin{aligned} V(\tau-\varepsilon, b(\tau)-\delta) &\leq V(\tau, b(\tau)-\delta) \\ &< e^{-\lambda\tau}\left((K-(b(\tau)-\delta))^+ + \eta\right) \\ &\leq e^{-\lambda(\tau-\varepsilon)}\left((K-(b(\tau)-\delta))^+ + \eta\right). \end{aligned}$$

Hence, $b(\tau)-\delta \notin \Upsilon_{\tau-\varepsilon}^s$ and therefore $b(\tau)-\delta < b(\tau-\varepsilon)$ – this leads to $b(\tau) \leq b(\tau-\varepsilon)$. □

We shall examine now the option's behavior near maturity. Let us denote by $V_{am}(\tau, x)$ the price of an American option with the same parameters, but without the writer's canceling right.

Proposition 13.3. *Let τ_1 be the largest value (possibly infinity) for the time to maturity below which the price of the American option $V_{am}(\tau, K)$ is less than the penalty η. Note that τ_1 exists since $V_{am}(\tau, K) \to 0$ for $\tau \to 0$. Then the writer's optimal region Υ_τ^s is empty for every $\tau \le \tau_1$ and vice versa – the set Υ_τ^s is not empty for $\tau > \tau_1$.*

Also, let $\overline{\eta}$ be defined as

$$\overline{\eta} = K \frac{q^q}{(q+1)^{q+1}}, \tag{13.2}$$

where the constant q is defined by formulas (3.17). We have that $\tau_1 < \infty$ if $\eta < \overline{\eta}$ and $\tau_1 = \infty$, otherwise.

Proof: Theorem 4.2 gives that the price of the perpetual American put option $V_{am}(\infty, K)$ is given just by formula (13.2). Hence, if $\eta \ge \overline{\eta}$, then $V_{am}(\tau, K) < \eta$ for all $\tau > 0$ because the price of an American option is non-decreasing w.r.t. the time to maturity. Therefore $\tau_1 = \infty$. In the opposite case $\eta < \overline{\eta}$, we have $\tau_1 < \infty$.

Suppose that there exists $\tau \le \tau_1$ such that $\Upsilon_\tau^s \ne \emptyset$. Hence, there exists some $x \le K$, such that $x \in \Upsilon_\tau^s$. Using Proposition 10.3 we see that $K \in \Upsilon_\tau^s$ too. Suppose that the writer does not cancel earlier. Hence, the option turns into an ordinary American put. We have assumed above that its price is less than the penalty and therefore the strategy to do nothing gives a better result for the writer than the immediate cancellation. Therefore, the set Υ_τ^s is empty for all $\tau \le \tau_1$.

Suppose now that $\Upsilon_\tau^s = \emptyset$ for some $\tau > \tau_1$. Note that this is true for all $t < \tau$ due to Propositions 10.3 and 13.2. Hence, the option is ordinary American. This means that never canceling is the optimal strategy for the writer of an at-the-money option. It leads to the financial result $V_{am}(\tau_1, K)$ that is higher than η due to $\tau > \tau_1$. Therefore, the immediate exercise would be preferable for the writer. The contradiction finishes the proof. □

We shall denote hereafter the price of (τ_1, η)-American put option by $V_{(\tau_1,\eta)}(\tau, x)$. Let $\tau_2 \ge \tau_1$ be this value of the time to maturity, possibly infinity, at which the writer's exercise boundary detaches from the strike. Note that Proposition 13.2 shows that once the writer's boundary falls below the strike, it never turns back. The following proposition stands.

Proposition 13.4. *Let $\tau > \tau_1$. The writer's exercise region consists only of the strike, $\Upsilon_\tau^s = \{K\}$, if $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) \geq -1$. If Υ_τ^s is an interval, then $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) \leq -1$.*

Furthermore, if the writer's optimal boundary for a perpetual cancellable put with the same parameters is equal to the strike, then $\tau_2 = \infty$. Otherwise, $\tau_2 < \infty$.

Proof: If the writer's exercise region is the singleton $\{K\}$, then $(x, K) \in \overline{\Upsilon}_\tau$ for some $x < K$. Therefore, $V_{(\tau_1,\eta)}(\tau, y) < N_2(0, y)$ for all $y \in (x, K)$. Taking the limit $y \to K$, we derive

$$\begin{aligned} \frac{d}{dx}V_{(\tau_1,\eta)}\left(\tau, K^-\right) &= \lim_{y \to K^-} \frac{V_{(\tau_1,\eta)}(\tau, y) - V_{(\tau_1,\eta)}(\tau, K)}{y - K} \\ &\geq \lim_{y \to K^-} \frac{N_2(0, y) - N_2(0, K)}{y - K} \\ &= \frac{\partial N_2(0, K^-)}{\partial x} = -1. \end{aligned} \tag{13.3}$$

Otherwise, suppose that the writer's boundary is below the strike for some τ. Therefore, there exists $x < K$ such that $V(\tau, y) = N_2(0, y)$ for all $y \in (x, K)$. Note that $V_{(\tau_1,\eta)}(\tau, y) \geq V(\tau, y)$. Using similar arguments as in inequality (13.3) we derive $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-) \leq -1$.

We finish the proof by the observation that if the writer's exercise region of the perpetual option is the singleton $\{K\}$, then $\tau_2 = \infty$ and vice versa. □

Remark 13.1. *Note that the value of $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau_1, K^-)$ is larger than -1 due to the smooth fit principle for the non-cancellable American options and the convexity of their price functions. Thus τ_2 is featured as the moment at which $\frac{d}{dx}V_{(\tau_1,\eta)}(\tau, K^-)$ falls below -1.*

If we use the notations with the current moment and the maturity, t and T, then the moment τ_2 is characterized as the lowest t for which

$$\frac{d}{dx}V_{(\tau_1,\eta)}\left(t, T, K^-\right) > -e^{-\lambda t}.$$

We summarize in the following theorem the derived results for the shape of the exercise regions of a cancellable put option.

Theorem 13.1. *[Theorem 3.1 of Zaevski (2022b)] The holder's exercise boundary is a decreasing function starting from the point*

$$\min\left(\frac{r+\lambda}{\lambda}, 1\right) K. \tag{13.4}$$

Note that equation (13.4) gives the optimal boundary of an American option at the maturity – see Proposition 4.5.

The form of the writer's exercise boundary is more complicated. Let B be its perpetual value if it exists. It can be obtained via Theorem 10.1. The following statements characterize the boundary curve:

1. *If B does not exist, equivalently to $\eta \geq \overline{\eta}$ for $\overline{\eta}$ given in equation (13.2), then $\tau_1 = \tau_2 = \infty$ and $\Upsilon^s \equiv \emptyset$.*

2. *If $B = K$, then $\tau_1 < \infty$, but $\tau_2 = \infty$. Also, $\Upsilon^s_\tau \equiv \emptyset$ for $\tau \leq \tau_1$ and $\Upsilon^s_\tau \equiv \{K\}$ otherwise. Note that this is the case when $r \geq 0$ due to Proposition 10.4.*

3. *If $B < K$, then $\tau_1 < \tau_2 < \infty$. Thus the writer's exercise boundary does not exist for τ less than τ_1, it coincides with the strike for $\tau \in (\tau_1, \tau_2)$, and it is a decreasing tending to B function for $\tau \geq \tau_2$.*

In such a way the option is ordinary American when $\tau \in (0, \tau_1]$, it is (τ_1, η)-American for $\tau \in (\tau_1, \tau_2)$, and a real cancellable option for $\tau \in [\tau_2, \infty)$.

13.4 Deriving the exercise boundaries

Let us first define the following European-style derivatives for some functions $0 < a(t) < b(t)$. They expire at the maturity date or when the underlying asset exits the strip $(a(t), b(t))$. The derivatives pay the amount of $N_1(t, a(t))$ or $N_2(t, b(t))$ if the exit happens from the lower or upper boundary, respectively. We shall name these derivatives $(a(t), b(t))$-European options.

We shall construct now an approximation algorithm for both exercise boundaries. Proposition 3.3 allows us to derive their values in some future moment t as the values at the initial moment of an option with a lower maturity $T-t$. Assume that the time to maturity is large enough. Note that the moment τ_1 can be obtained numerically – Proposition 13.3 says that the price of an at-the-money non-cancellable American option maturating after τ_1

time is equal to the penalty. Unfortunately, this is not the case for the second important moment τ_2. Proposition 13.4 states that τ_2 is characterized by the derivative of the price of a (τ_1, η)-American option. Usually, such instruments can be priced only numerically with some precision. This does not allow the limit in the derivative to be calculated with sufficient accuracy. Hence, we shall approximate τ_2 as the largest value of τ for which our algorithm returns the strike for the writer's optimal boundary.

We need to obtain first which case of Theorem 13.1 holds. If the penalty is larger than the critical value $\overline{\eta}$, given in equation (13.2), then the option is ordinary American. Suppose now that $\eta < \overline{\eta}$. We can calculate the writer's exercise boundary of the perpetual cancellable put, B, using Theorem 10.1. It can be equal to or less than the strike. We shall examine separately both cases later.

The next step is to divide the time to maturity interval into $n \geq 2$-sub-intervals, $0 \equiv t_0 < t_1 < ... < t_n \equiv T \equiv \tau$. We can think that $\tau_1 < \tau$, because in the opposite case, the option is non-cancellable. We impose two requirements – τ_1 to be a grid node and the division to be relatively uniform. To do this, we use the following procedure. First, we divide the interval into two parts – $(0, \tau_1)$ and (τ_1, τ). After, that we divide uniformly both intervals into m_1 and m_2 parts, respectively, such that $m_1 + m_2 = n$ and

$$m_1 = \min\left(\max\left(1, \mathit{Round}\left(\frac{\tau_1}{\tau} n\right)\right), n - 1\right). \tag{13.5}$$

We have used above the notation $\mathit{Round}(x)$ for the nearest to x integer. Formulation (13.5) guarantees that $m_1 \geq 1$ and $m_2 \geq 1$, i.e. there is at least one sub-interval before τ_1 as well as after. Also, note that $t_{m_1} = \tau_1$.

We shall approximate the holder's and writer's exercise boundaries by exponents of piecewise linear functions

$$a(t) = \sum_{i=1}^{n} \exp(a_i(t)) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^{n} \exp(a_{1,i} t + a_{2,i}) I_{t \in (t_{i-1}, t_i]}$$
$$b(t) = \sum_{i=1}^{m_1} \exp(b_i(t)) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^{m_1} \exp(b_{1,i} t + b_{2,i}) I_{t \in (t_{i-1}, t_i]},$$

respectively. We assume that $a(t) < b(t) \leq K$ for every t less the maturity. We require continuity at the grid nodes – $a_i(t_i) = a_{i-1}(t_i)$ and $b_i(t_i) = b_{i-1}(t_i)$ – we shall denote by $A_0, A_1, ..., A_n$ and $B_0, B_1, ..., B_{m_1}$ the corresponding values. Theorem 13.1 states that it is never optimal for the writer to cancel if $\tau \leq \tau_1$ and thus the writer's exercise boundary does not exist for $i > m_1$. Let

$$G\left(x, T; a\left(t\right), b\left(t\right)\right) \equiv G\left(x, \tau; \left\{t_0, ..., t_n\right\}, \left\{A_0, ..., A_n\right\}, \left\{B_0, ..., B_{m_1}\right\}\right)$$

be the price of the $\left(a\left(t\right), b\left(t\right)\right)$-European option if the initial asset value is x and the time to maturity is $\tau \equiv T$. We consider $x \in (A_0, B_0)$. Let us denote by ζ_1 and ζ_2 the first hitting moments of the underlying asset to the functions $a\left(t\right)$ and $b\left(t\right)$, respectively, and $\zeta = \zeta_1 \wedge \zeta_2$. Since the asset price is described by the log-normal process (3.4), the stopping times ζ_1 and ζ_2 can be viewed as the Brownian motion's first hits to the piece-wise linear boundaries

$$\begin{aligned} c\left(t\right) &= \sum_{i=1}^{n} c_i\left(t\right) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^{n} \left(c_{1,i} t + c_{2,i}\right) I_{t \in (t_{i-1}, t_i]} \\ d\left(t\right) &= \sum_{i=1}^{m_1} d_i\left(t\right) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^{m_1} \left(d_{1,i} t + d_{2,i}\right) I_{t \in (t_{i-1}, t_i]} \end{aligned}$$

for

$$\begin{aligned} c_{1,i} &= \frac{a_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \ \ i = 1, ..., n \\ c_{2,i} &= \frac{a_{2,i} - \ln\left(x\right)}{\sigma}, \ \ i = 1, ..., n \\ d_{1,i} &= \frac{b_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \ \ i = 1, ..., m_1 \\ d_{2,i} &= \frac{b_{2,i} - \ln\left(x\right)}{\sigma}, \ \ i = 1, ..., m_1. \end{aligned}$$

Thus the $\left(a\left(t\right), b\left(t\right)\right)$-European option price turns into

$$
\begin{aligned}
&V\left(x,T;a\left(t\right),b\left(t\right)\right)=\mathbb{E}^{x}\left[e^{-(r+\lambda)(\zeta_1\wedge T)}(K-S_{\zeta_1\wedge T})^{+}I_{(\zeta_1\wedge T)\leq\zeta_2}\right]\\
&+\mathbb{E}^{x}\left[e^{-(r+\lambda)\zeta_2}\left((K-S_{\zeta_2})^{+}+\eta\right)I_{\zeta_2<(\zeta_1\wedge T)}\right]\\
&=K\sum_{i=1}^{n}\mathbb{E}\left[e^{-(r+\lambda)\zeta_1}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right]-x\sum_{i=1}^{n}e^{\sigma c_{2,i}}\mathbb{E}\left[e^{-\psi_{1,i}\zeta_1}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_1}\right]\\
&+(K+\eta)\sum_{i=1}^{m_1}\mathbb{E}\left[e^{-(r+\lambda)\zeta_2}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right]-x\sum_{i=1}^{m_1}e^{\sigma d_{2,i}}\mathbb{E}\left[e^{-\psi_{2,i}\zeta_2}I_{\zeta\in(t_{i-1},t_i],\zeta=\zeta_2}\right]\\
&+Ke^{-(r+\lambda)T}\mathbb{Q}\left(B_T<k,T\leq\zeta\right)-xe^{-\psi_3 T}\mathbb{E}\left[e^{\sigma B_T}I_{B_T<k,T\leq\zeta}\right],
\end{aligned}
\tag{13.6}
$$

where

$$
\begin{aligned}
\psi_{1,i}&=(r+\lambda)-\left(r-\frac{\sigma^2}{2}\right)-\sigma c_{1,i}=\lambda+\frac{\sigma^2}{2}-\sigma c_{1,i}\\
\psi_{2,i}&=(r+\lambda)-\left(r-\frac{\sigma^2}{2}\right)-\sigma d_{1,i}=\lambda+\frac{\sigma^2}{2}-\sigma d_{1,i}\\
\psi_3&=\lambda+\frac{\sigma^2}{2}\\
k&=\frac{1}{\sigma}\ln\left(\frac{K}{x}\right)-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)T.
\end{aligned}
\tag{13.7}
$$

We use formulas (2.30) to calculate the expectations in the first, second, third, and fourth terms of formula (13.6). The fifth and sixth terms can be obtained from equation (2.31). We use Laplace transform (2.31) taken in the point zero for the probability in the fifth term.

We shall work backwards to derive the values of $A_0, A_1, ..., A_n$ and $B_0, B_1, ..., B_{m_1}$. Formula (13.4) leads to

$$A_n=\min\left(\frac{r+\lambda}{\lambda},1\right)K.$$

Since the option is non-cancellable for $\tau<\tau_1$, we derive the values $A_p, A_{p+1}, ..., A_n$ as in Chapter 4. Also, the writer's exercise boundary in the critical point τ_1 is equal to the strike, and thus $B_{m_1}=K$. Now we have to examine separately the cases when the writer's exercise boundary is always the strike or not.

13.4.1 Writer's exercise boundary equal to the strike

Suppose that the writer's exercise boundary of the perpetual option is equal to the strike. Hence, this is true for the finite maturity case too, and therefore $B_1 = B_2 = ... = B_{m_1} = K$. Thus the problem for pricing a cancellable option turns into a pricing problem of a (τ_1, η)-American put. The price function (13.6) turns into

$$\begin{aligned}
&V\left(x, T; a\left(t\right), b\left(t\right) \equiv K\right) = \mathbb{E}^x\left[e^{-(r+\lambda)(\zeta_1 \wedge T)}(K - S_{\zeta_1 \wedge T})^+ I_{(\zeta_1 \wedge T) \leq \zeta_2}\right] \\
&+ \eta \mathbb{E}^x\left[e^{-(r+\lambda)\zeta_2} I_{\zeta_2 < (\zeta_1 \wedge T)}\right] \\
&= K \sum_{i=1}^{n} \mathbb{E}\left[e^{-(r+\lambda)\zeta_1} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_1}\right] - x \sum_{i=1}^{n} e^{\sigma c_{2,i}} \mathbb{E}\left[e^{-\psi_{1,i}\zeta_1} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_1}\right] \\
&+ \eta \sum_{i=1}^{m_1} \mathbb{E}\left[e^{-(r+\lambda)\zeta_2} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_2}\right] \\
&+ K e^{-(r+\lambda)T} \mathbb{Q}\left(B_T < k, T \leq \zeta\right) - x e^{-\psi_3 T} \mathbb{E}\left[e^{\sigma B_T} I_{B_T < k, T \leq \zeta}\right].
\end{aligned} \tag{13.8}$$

We have to derive the values of the holder's optimal boundary. Suppose that we know the values $A_m, A_{m+1}, ..., A_n$ for some $m \leq m_1$. We want to find the value of A_{m-1} that maximizes the holder's financial result. Let us examine a (τ_1, η)-American option with time to maturity $T - t_{m-1}$. Suppose that the initial asset value is fixed to x. We shall find the value for $A_{m-1} \leq x$, for which the price of an $\left(a\left(t + t_{m-1}\right), b\left(t + t_{m-1}\right)\right)$-European option with maturity $T - t_{m-1}$,

$$V\left(x, T - t_{m-1}; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\} \{A_{m-1}, ..., A_n\}, \{K, ..., K\}\right),$$

is largest. If the true holder's exercise boundary is indeed an exponent of a piecewise linear function, then the value A_{m-1} should be the same for all $x \in \overline{\Upsilon}_{T - t_{m-1}}$. But we cannot expect such a thing and therefore we have to work differently. Let us denote by $\alpha\left(x\right)$ the negative number that maximizes the following $\left(a\left(t + t_{m-1}\right), b\left(t + t_{m-1}\right)\right)$-European option price

$$V\left(x, T - t_{m-1}; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\} \{e^{\alpha} x, ..., A_n\}, \{K, ..., K\}\right).$$

We search for the largest value of x, for which $\alpha\left(x\right) = 0$. In terms of the American derivatives, it approximates the holder's exercise boundary, because it is the largest value of the initial asset price for which the immediate exercise is optimal. Thus namely this value of x is our approximation of A_{m-1}. Although Figure 13.2a does not present the actual case, it is informative on how we derive the holder's exercise boundary. The red point markes the largest x such that $\alpha(x) = 0$, and just it is our approximation of the boundary.

13.4.2 Writer's exercise boundary below the strike

Assume now that the writer's exercise boundary of the corresponding perpetual option is less than the strike. Therefore, the second important moment at which the boundary detaches from the strike is finite, $\tau_2 < \infty$. As we mentioned above we can not derive this moment via Proposition 13.4. We shall build a similar but two-sided algorithm to derive the writer's exercise boundary as well as the holder's.

Suppose that we have derived the values $A_m, A_{m+1}, ..., A_n$ and $B_m, ..., B_{m_1}$ for some $m \leq m_1$. We shall derive first the value B_{m-1}. Let us fix some x below the strike and some β in the interval $[0, \ln K - \ln x]$. For a fixed positive β, we denote by $\alpha(x, \beta)$ the negative value of α that maximizes the price of the following $(a(t + t_{m-1}), b(t + t_{m-1}))$-European option with time to maturity $T - t_{m-1}$

$$V\left(x, T - t_{m-1}; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\} \{e^{\alpha} x, ..., A_n\}, \left\{e^{\beta} x, B_m, ..., B_{m_1}\right\}\right). \tag{13.9}$$

Let $\beta(x)$ be the value of β that minimizes

$$V\left(x, T - t_{m-1}; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\} \left\{e^{\alpha(x,\beta)} x, ..., A_n\right\}, \left\{e^{\beta} x, B_m, ..., B_{m_1}\right\}\right).$$

We search for the lowest x for which $\beta(x) = 0$. Namely, this is our approximation for the writer's boundary at the moment t_{m-1}. In fact, this is the lowest value for the underlying asset for which immediate canceling is optimal for the writer.

We derive analogously the holder's boundary A_{m-1}. Let for a fixed x, $0 < x < K$, and $\alpha \in (-\infty, 0]$, $\beta(x, \alpha)$ minimizes price (13.9) in the interval $[0, \ln K - \ln x]$. Let also $\alpha(x)$ be the value that maximizes

$$V\left(x, T - t_{m-1}; \{0, t_m - t_{m-1}, ..., t_n - t_{m-1}\} \{e^{\alpha} x, ..., A_n\}, \left\{e^{\beta(x,\alpha)} x, B_m, ..., B_p\right\}\right).$$

We search for the highest x for which $\alpha(x) = 0$. As we mentioned above, this is the largest value for the underlying asset which makes the immediate exercise optimal for the option's holder.

In Figure 13.1, we show the way we derive the exercise boundaries. The holder's boundary is the largest x such that $\alpha(x) = 0$. It is presented by the red point in Figure 13.2a – its value is 5.8562. The writer's boundary is the smallest x for which $\beta(x) = 0$ – the red point in Figure 13.2b – the value is 17.8275. The results are based on a three-step algorithm with parameters $\tau = 3$, $r = -0.03$, $\lambda = 0.05$, $\sigma = 0.3$, $k = 20$, and $\eta = 1$.

13.5 Pricing

If the time to maturity is less than the first critical value τ_1, the option turns into an ordinary American put. Suppose now $T \equiv \tau > \tau_1$. We have proved that the continuation region consists of two parts – one between the optimal boundaries, and one above the strike. We shall examine these cases separately.

13.5.1 Initial asset value between the optimal boundaries

Having in mind the already approximated boundaries, we state our fast method:

Fast Pricing Approach 13.1. *We price the option via formulas* (13.6) *and* (13.7) *when the writer's exercise boundary is below the strike. Otherwise, if the boundary is equal to the strike, then we have to use formula* (13.8). *These formulas have to be taken at the point* $x = S_0$.

Alternatively, we shall create an efficient and relatively fast Monte Carlo algorithm for deriving the expectations in function (13.6) if we need a denser grid. It is based on the simulations of the Brownian motion's sample paths and a numerical evaluation of the integrals in expectations (2.28), (2.30) and (2.31), provided that the Brownian motion stays in the strip. Alternatively, we can adapt the Crank-Nicolson finite difference approach presented in Section 3.5. The algorithm is as follows.

1. We generate randomly $n - 1$ numbers using the standard normal distribution. These numbers form the vector $\overline{u}$.

2. Let $m \leq n$ and the vector u consists of the first $m-1$ elements of $\overline{u}$. Let D be a $(m-1) \times (m-1)$ diagonal matrix with elements $\sqrt{\Delta t_i / n}$. Note that the length of the intervals, Δt_i, differs before and after the moment $T - \tau_1$. We calculate the vector x as $x = MDu$, where M is a $(m-1) \times (m-1)$ lower triangle matrix with values one.

3. If $t_{m-1} < T - \tau_1$ we derive the values v as

$$v = v(x_1, ..., x_{m-1}) = \prod_{i=1}^{m-1} I_{c_i < x_i < d_i} \left(1 - \sum_{j=1}^{\infty} q_{ij}(x_{i-1}, x_i)\right).$$

   Otherwise, if $m_1 \leq m-1$, then v is obtained as

$$v = v(x_1, ..., x_{m-1}) = \prod_{i=1}^{p-1} I_{c_i < x_i < d_i} \left(1 - \sum_{j=1}^{\infty} q_{ij}(x_{i-1}, x_i)\right) \times \prod_{i=p}^{m-1} I_{c_i < x_i} \left(1 - \exp\left(-\frac{2(c_{i-1} - x_{i-1})(c_i - x_i)}{\Delta t_i}\right)\right).$$

4. We derive the values w as $w = e^{-\xi t_{m-1}} L_{1,2}(\cdot)$ for equations (2.28) and (2.30) and $w = e^{\xi x_{n-1}} V(\cdot)$ for equation (2.31).

5. We calculate the product $P = vw$.

6. We repeat the procedure above H times and after averaging, we derive the necessary expectations as $\frac{1}{H} \sum_{i=1}^{H} P_i$.

Let us give some comments on the algorithm above. In steps 1 and 2 we simulate the Brownian motion paths. The term v from step 3 pertains to the requirement that the Brownian motion stays in the strip until the moment t_{m-1}. If $t_{m-1} < T - \tau_1$ we have to use formula (2.28). Otherwise, if $t_{m-1} \geq T - \tau_1$, the writer's exercise boundary does not exist after some moment and therefore the strip is open above. In this case, we have to use equation (2.30). It turns out that five iterations are sufficient in the infinite sum except for some extremely large parameter values when ten iterations are necessary. The term w in step 4 has two meanings. If the first exit is before the maturity, w is related to the first exit in the interval $(t_{m-1}, t_m]$. We use L_1 if the exit occurs from the lower (holder's) boundary whereas L_2 is used for the upper (writer's) boundary. Otherwise, if the Brownian motion stays in the strip until maturity, then w pertains to the terminal option payment at the maturity. The fifth term in option price formula (13.6) is obtained for $\theta = 0$ in equation (2.31). The last two steps allow us to derive the corresponding expectations. Note that the terms related to the transition density of the Brownian motion in equations (2.28), (2.30) and (2.31) are incorporated hiddenly when we generate the Brownian motion's sample paths. For some more comments see Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001).

13.5.2 Initial asset value above the strike

Suppose now that $S_0 \equiv x > K$ – pricing turns into a one-sided hitting problem. The option expires when the underlying asset hits the strike if the remaining time to maturity is more than τ_1 and pays the amount of η. If the underlying asset stays above the strike until the moment $\tau - \tau_1$, then the option turns into a non-cancallable American put. Let us denote by ζ the first hitting moment, and by d, a_1, a_2, and $f(y;t)$, the following terms

$$\begin{aligned} a_1 &= -\frac{r}{\sigma} + \frac{\sigma}{2} \\ a_2 &= -\frac{\ln S_0 - \ln K}{\sigma} \\ d &= a_1 (T - \tau_1) + a_2 \\ f(y;t) &= \frac{1}{\sqrt{2\pi t}} \left(1 - \exp\left(-\frac{2a_2 (a_1 t + a_2 - y)}{t}\right)\right) \exp\left(-\frac{y^2}{2t}\right). \end{aligned}$$

Using Propositions 2.5 and 2.3, we derive the following semi-closed formula

$$
\begin{aligned}
V(\tau, S_0) &= \mathbb{E}\left[e^{-(r+\lambda)\zeta}\eta I_{\zeta \le T-\tau_1}\right] \\
&+ e^{-r(T-\tau_1)} \int_d^\infty e^{-\lambda(T-\tau_1)} V_{am}\left(\tau_1, S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau_1)+\sigma y}\right) d\mathbb{Q}\left(B_{T-\tau_1} < y, \zeta > T-\tau_1\right) \\
&= \eta e^{-a_2\left(\sqrt{a_1^2+2(r+\lambda)}+a_1\right)} g\left(T-\tau_1, -\sqrt{a_1^2+2(r+\lambda)}, a_2\right) \\
&+ e^{-(r+\lambda)(T-\tau_1)} \int_d^\infty V_{am}\left(\tau_1, S_0 e^{\left(r-\frac{\sigma^2}{2}\right)(T-\tau_1)+\sigma y}\right) f(y; T-\tau_1)\, dy,
\end{aligned}
$$

where the function $g(\cdot,\cdot,\cdot)$ is given in equation (2.14).

13.6 Numerical results

We present in this section the results of some numerical experiments. The main values we use are – time to maturity $\tau = 3$; risk free rate $r = -0.03$; discount rate $\lambda = 0.05$; volatility $\sigma = 0.3$; strike $K = \$20$; penalty $\eta = \$1$. We shall vary some of them to describe the option's behavior. The Brownian motion's paths are simulated by $n = 200\ 000$ steps. We divide the time interval into 16 sub-intervals. For each node of this grid, we use a procedure with 3-steps to derive the boundaries' approximations.

First, using the results of Chapter 10, we derive the exercise boundaries for the related perpetual option – the writer's optimal boundary is $B = 15.7208$, and the holder's one is $A = 5.2520$. We have to use the presented in Section 13.4.2 algorithm since B is less than the strike. We derive the first critical value τ_1 at which the pure American option price is equal to the penalty – it is $\tau_1 = 0.1604$. Hence, the option is non-cancellable when the time to maturity is less than τ_1. The writer's boundary does not exist for $\tau \in [0, 0.1604]$ and the the holder's one can be obtained using the method of Chapter 4. The given in equation (13.4) holder's boundary at the maturity is currently \$8. It turns out that the second important time to maturity value at which the writer's boundary falls below the strike is $\tau_2 = 1.3459$. Both boundaries for short maturities are presented in Figure 13.2c. Note that the time dependence is presented by the actual time, not by the time to maturity. The writer's boundary is the upper line – it is plotted by a red color. The holder's boundary is the lower blue line. The critical values τ_1 and τ_2 are marked by the red and green points. The long maturity behavior is presented in Figure 13.2d. The blue points are the perpetual values. We can see that the long maturity values tend to the perpetual ones.

The boundaries' surface w.r.t the discount rate, $\lambda \in [0.04, 0.1]$, and for short maturities, $\tau \in [0, 3]$, can be viewed in Figures 13.3a and 13.3c, whereas the long maturities behavior, $\tau \in [3, 20]$, is presented in Figures 13.3b and 13.3d. Note that the time dependence is w.r.t. the time to maturity. Once we approximate the boundaries, we can use the presented in Section 13.5 Monte Carlo method to evaluate the options. We assume that the initial asset price is $S_0 = \$15$. The price behavior for short and long maturities can be viewed in Figures 13.3e and 13.3f. The meaning of the green, red, and blue points is preserved. We can see that the long maturity values tend to the perpetual ones. Some particular prices are presented in Table 13.1. The discount rate is among $\lambda \in \{0.04,\ 0.05,\ 0.06,\ 0.07,\ 0.08,\ 0.09,\ 0.1\}$, the penalty varies as $\eta \in \{\$0.1, \$1,\ \$2,\ \$3,\ \$4,\ \$5\}$, the initial asset value is taken to be $x = \$10$, $x = \$11$, $x = \$12$, $x = \$13$, or $x = \$14$.

In Figure 13.2e, we compare the cancellable and non-cancellable options presenting the difference between their prices. In such a way, we can appreciate the impact of the writer's early canceling right. The initial asset price is varied in the interval $(0, 50)$. The time to maturity is assumed to be $T = 1$, $T = 2$, or $T = 3$. The holder's boundary at these moments are 6.6643, 6.1739, and 5.8559, whereas the writer's ones are 20, 19.2911, and 17.8291. We can see that when $T = 1$ we have a (τ_1, η)-American option, whereas for $T = 2$ and $T = 3$ the option is a real cancellable one (note that the second critical value has been obtained as $\tau_2 = 1.3459$). We can see that the prices of both options tend one to other when $S_0 \to 0$ or $S_0 \to \infty$. When $S_0 \to 0$, the limit is the strike because it is very likely S_t to stay near the zero. Analogously, when the option is deeply out-of-the-money, then $S_t > K$ with a very high probability, and therefore both prices tend to zero. Obviously, we do have not smoothness in the strike. We can see also that the impact of the writer's canceling right is more significant when the underlying asset starts near the strike. Also, this impact is larger for the higher maturities.

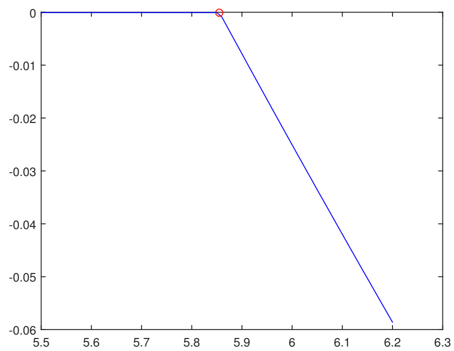
If the risk-free rate is $r = -0.01$, then the perpetual writer's boundary is $B = \$20$. Since it is equal to the strike, the option turns into a (τ_1, η)-American put. Hence, we have to use the algorithm presented in Section 13.4.1. The first critical value is $\tau_1 = 0.1719$. Of course, the second one is the infinity. Both boundaries are presented in Figure 13.2f.

13.7 Conclusions

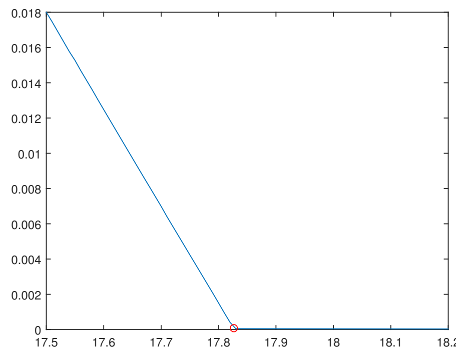
The pricing problem for cancellable American puts option written on a dividend-paying asset has been examined in this chapter. This is done through an additional discount factor. The maturity is assumed to be finite. A series of propositions that characterize the optimal regions are proven. Both early exercise boundaries are approximated numerically maximizing the financial utilities of both option's participants. A Monte Carlo method for option pricing has been constructed when the asset starts between the exercise boundaries. Alternatively, a semi-closed pricing formula has been derived if the initial asset price is above the strike. We have validated the consistency and relevance of the derived results by performing various numerical experiments.

Figures and Tables

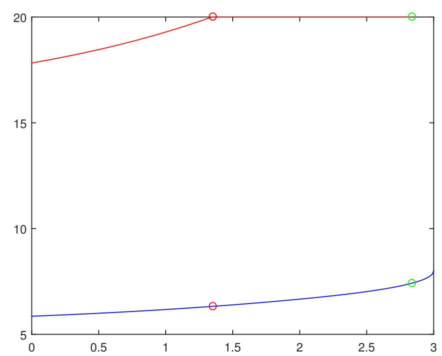
Figure 13.1: Exercise boundaries. (a) and (b) deriving the boundaries – the red points are our approximations; (c), (d), (e), and (f) both boundaries with different parameters – the red upper line is the writer's boundary; the blue line is the holder's one.



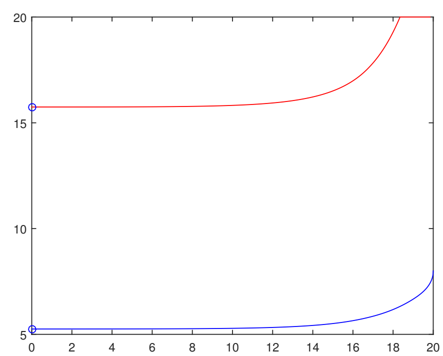
(a) holder's boundary



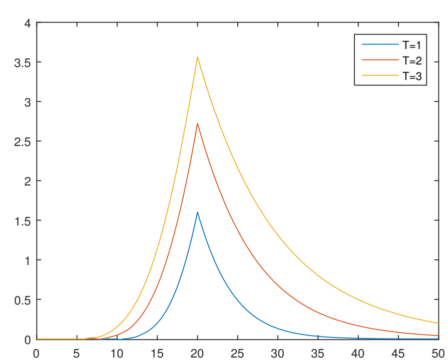
(b) writer's boundary



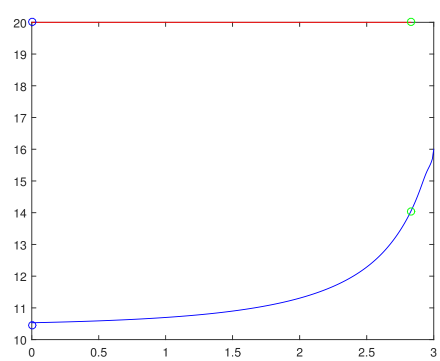
(c) Short maturities, $r = -0.03$



(d) Long maturities, $r = -0.03$

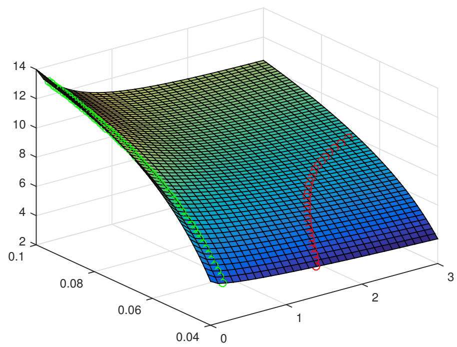


(e) Impact of the writer's exercise right

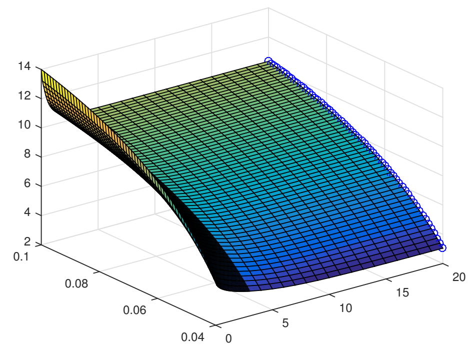


(f) Exercise boundaries, $r = -0.01$

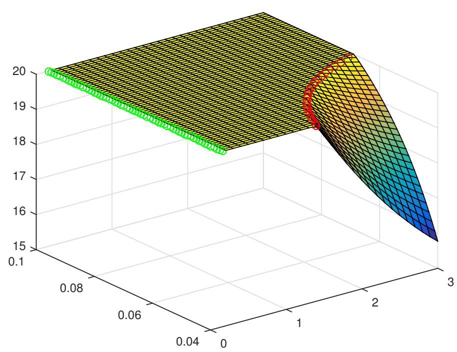
Figure 13.2: Exercise boundaries and put option prices.



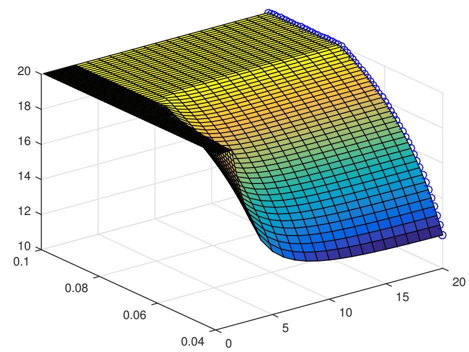
(a) holder's boundary – short maturities



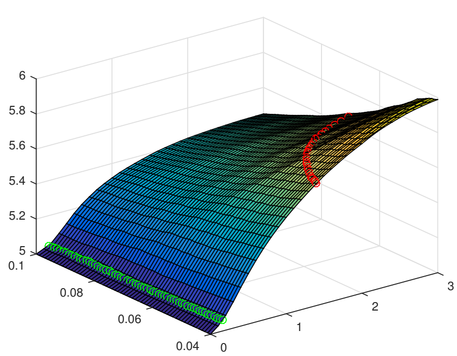
(b) holder's boundary – long maturities



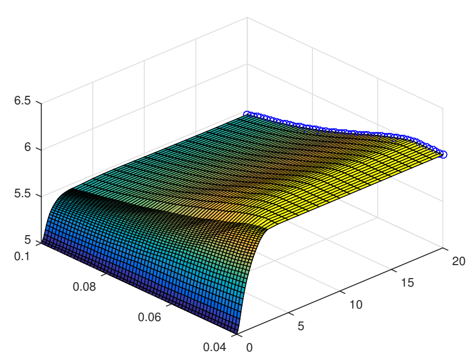
(c) writer's boundary – short maturities



(d) writer's boundary – long maturities



(e) prices – short maturities



(f) prices – long maturities

Table 13.1: Option prices

$S_0 = 10$	$\eta = 0.1$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$
$\lambda = 0.04$	10.1000	10.5802	10.6633	10.7234	10.7776	10.8319
$\lambda = 0.05$	10.0919	10.3330	10.4129	10.4720	10.5255	10.5791
$\lambda = 0.06$	10.0392	10.1818	10.2547	10.3107	10.3628	10.4149
$\lambda = 0.07$	10.0022	10.0772	10.1422	10.1963	10.2468	10.2969
$\lambda = 0.08$	10.0000	10.0201	10.0648	10.1100	10.1551	10.2008
$\lambda = 0.09$	10.0000	10.0011	10.0225	10.0540	10.0904	10.1299
$\lambda = 0.1$	10.0000	10.0000	10.0032	10.0207	10.0470	10.0786
$S_0 = 11$	$\eta = 0.1$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$
$\lambda = 0.04$	9.1000	9.6897	9.8207	9.9196	10.0112	10.1028
$\lambda = 0.05$	9.1000	9.4571	9.5813	9.6791	9.7696	9.8600
$\lambda = 0.06$	9.0745	9.2942	9.4129	9.5076	9.5962	9.6850
$\lambda = 0.07$	9.0251	9.1797	9.2889	9.3794	9.4649	9.5513
$\lambda = 0.08$	9.0022	9.0953	9.1932	9.2802	9.3629	9.4466
$\lambda = 0.09$	9.0000	9.0407	9.1200	9.1976	9.2752	9.3553
$\lambda = 0.1$	9.0000	9.0123	9.0670	9.1331	9.2039	9.2772
$S_0 = 12$	$\eta = 0.1$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$
$\lambda = 0.04$	8.1000	8.7882	8.9794	9.1305	9.2727	9.4150
$\lambda = 0.05$	8.1000	8.5779	8.7590	8.9080	9.0486	9.1893
$\lambda = 0.06$	8.0931	8.4129	8.5878	8.7341	8.8730	9.0119
$\lambda = 0.07$	8.0600	8.2946	8.4610	8.6039	8.7398	8.8758
$\lambda = 0.08$	8.0244	8.2026	8.3575	8.4938	8.6260	8.7590
$\lambda = 0.09$	8.0047	8.1300	8.2732	8.4043	8.5317	8.6599
$\lambda = 0.1$	8.0000	8.0777	8.2019	8.3256	8.4490	8.5738
$S_0 = 13$	$\eta = 0.1$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$
$\lambda = 0.04$	7.1000	7.8711	8.1405	8.3575	8.5648	8.7721
$\lambda = 0.05$	7.1000	7.6968	7.9465	8.1605	8.3656	8.5707
$\lambda = 0.06$	7.1000	7.5370	7.7815	7.9928	8.1957	8.3988
$\lambda = 0.07$	7.0858	7.4167	7.6497	7.8568	8.0568	8.2571
$\lambda = 0.08$	7.0576	7.3222	7.5470	7.7493	7.9455	8.1420
$\lambda = 0.09$	7.0302	7.2466	7.4578	7.6539	7.8464	8.0396
$\lambda = 0.1$	7.0119	7.1820	7.3825	7.5700	7.7563	7.9449
$S_0 = 14$	$\eta = 0.1$	$\eta = 1$	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$
$\lambda = 0.04$	6.1000	6.9388	7.3059	7.6017	7.8879	8.1742
$\lambda = 0.05$	6.1000	6.7984	7.1308	7.4234	7.7070	7.9905
$\lambda = 0.06$	6.1000	6.6584	6.9799	7.2693	7.5503	7.8313
$\lambda = 0.07$	6.0992	6.5399	6.8549	7.1402	7.4182	7.6964
$\lambda = 0.08$	6.0850	6.4485	6.7510	7.0313	7.3057	7.5805
$\lambda = 0.09$	6.0630	6.3720	6.6659	6.9395	7.2091	7.4798
$\lambda = 0.1$	6.0414	6.3088	6.5889	6.8574	7.1229	7.3894

Chapter 14

MATLAB codes

Abstract: The derived in the previous chapters theoretical results are implemented by the use of MATLAB. We provide now some of the most important codes – their total number is larger than two hundred. All of them are available and can be provided upon request. Note that they are not professionally prepared, but are rather for personal use. We discuss also some specifics of the used algorithms.

14.1 Codes related to the usual American options

14.1.1 Main code

Below is presented the MATLAB code for evaluating a put-style derivative that expires at the lower moment between the first hit of the underlying asset to an exponent of a piecewise linear function or maturity.

```
function [pr] =premium_put_1_terminal(r,lambda,sigma
   ,k,x,t,c,N);
% Derive the price of the put-style derivative
   maturating at the boundary c evaluated at the
   time-grid t

% r      -- risk-free rate
% lambda -- discount factor
% sigma  -- volatility
```

```
% k       -- strike
% x       -- initial asset price
% t       -- time-grid
% c       -- boundary values at the time-gridre

if N==1
    t0=t(1); t1=t(2);
    c0=c(1); c1=c(2);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)
       /sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma
       );
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    X=0;

    alfa=sigma^2/2-a*sigma+lambda;
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(alfa, t1,a,b );
    if r<0
        kk=(log(k/x)-(r-sigma^2/2)*t1)/sigma;
        [term1]=exp(-(r+lambda)*t1)*terminal1_1_new(
           t1-t0,X,C0,C1,kk,C1);
        [term2]=exp(-(lambda+sigma^2/2)*t1)*
           terminal1_2_new(sigma,t1-t0,X,C0,C1,kk,C1
           );
    else
        term1=0;
        term2=0;
    end
    pr=k*(l1(1)+term1)-x*(l2(1)+term2);
elseif N==2
    t0=t(1); t1=t(2); t2=t(3);
    c0=c(1); c1=c(2); c2=c(3);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)
       /sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma
       );
    alfa=sigma^2/2-a*sigma+lambda;

```

```
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(alfa, t1,a,b );
    I1=k*l1(1)-x*l2(1);
%
  ---------------------------------------------------------------------
    [ a,b]= linnear( t1,(log(c1/x)-(r-sigma^2/2)*t1)
       /sigma,t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma
       );

    alfa=sigma^2/2-a*sigma+lambda;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    if r<0
        kk=(log(k/x)-(r-sigma^2/2)*t2)/sigma;
        [term1,term2]=terminal2_2(sigma,X,C0, C1,C2,
           t0,t1,t2,kk );
        term1=exp(-(lambda+r)*t2)*term1;
        term2=exp(-(lambda+sigma^2/2)*t2)*term2;
    else
        term1=0;
        term2=0;
    end
    l1(2)=laplace_2(r+lambda,X,C0, C1,C2,t0,t1,t2,a,
       b );
    l2(2)=exp(sigma*b)*laplace_2(alfa,X,C0, C1,C2,t0
       ,t1,t2,a,b );
    I2=k*(l1(2)+term1)-x*(l2(2)+term2);
    pr=I1+I2;
elseif N==3
    t0=t(1); t1=t(2); t2=t(3); t3=t(4);
    c0=c(1); c1=c(2); c2=c(3); c3=c(4);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)
       /sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma
       );

```

```
    alfa=sigma^2/2-a*sigma+lambda;
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(alfa, t1,a,b );
    I1=k*l1(1)-x*l2(1);
   %-----------------------------------------------------------
    [ a,b]= linnear( t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma);
    alfa=sigma^2/2-a*sigma+lambda;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1(2)=laplace_2(r+lambda,X,C0, C1,C2,t0,t1,t2,a,b );
    l2(2)=exp(sigma*b)*laplace_2(alfa,X,C0, C1,C2,t0,t1,t2,a,b );
    I2=k*l1(2)-x*l2(2);
    %----------------------------------------------------------
    [ a,b]= linnear( t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(c3/x)-(r-sigma^2/2)*t3)/sigma);
    alfa=sigma^2/2-a*sigma+lambda;
     C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    C3=(log(c3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    if r<0
        kk=(log(k/x)-(r-sigma^2/2)*t3)/sigma;
        [term1,term2]=terminal3(sigma,X,C0, C1,C2,C3,t0,t1,t2,t3,kk );
        term1=exp(-(lambda+r)*t3)*term1;
        term2=exp(-(lambda+sigma^2/2)*t3)*term2;
```

```
    else
        term1=0;
        term2=0;
    end
    l1(3)=laplace_3(r+lambda,X,C0, C1,C2,C3,t0,t1,t2
       ,t3,a,b );
    l2(3)=exp(sigma*b)*laplace_3(alfa,X,C0, C1,C2,C3
       ,t0,t1,t2,t3,a,b );
    I3=k*(l1(3)+term1)-x*(l2(3)+term2);
    pr=I1+I2+I3;
elseif N==4
    t0=t(1); t1=t(2); t2=t(3); t3=t(4); t4=t(5);
    c0=c(1); c1=c(2); c2=c(3); c3=c(4); c4=c(5);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)
       /sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma
       );
    alfa=sigma^2/2-a*sigma+lambda;
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(alfa, t1,a,b );
    I1=k*l1(1)-x*l2(1);
   %
      -----------------------------------------------------------------
    [ a,b]= linnear( t1,(log(c1/x)-(r-sigma^2/2)*t1)
       /sigma,t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma
       );
    alfa=sigma^2/2-a*sigma+lambda;
     C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1(2)=laplace_2(r+lambda,X,C0, C1,C2,t0,t1,t2,a,
       b );
    l2(2)=exp(sigma*b)*laplace_2(alfa,X,C0, C1,C2,t0
       ,t1,t2,a,b );
    I2=k*l1(2)-x*l2(2);
%
   --------------------------------------------------------------------
```

```
    [ a,b]= linnear( t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(c3/x)-(r-sigma^2/2)*t3)/sigma);
    alfa=sigma^2/2-a*sigma+lambda;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    C3=(log(c3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    l1(3)=laplace_3(r+lambda,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    l2(3)=exp(sigma*b)*laplace_3(alfa,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    I3=k*l1(3)-x*l2(3);
%---------------------------------------------------------------------
    [ a,b]= linnear( t3,(log(c3/x)-(r-sigma^2/2)*t3)/sigma,t4,(log(c4/x)-(r-sigma^2/2)*t4)/sigma);
    alfa=sigma^2/2-a*sigma+lambda;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    C3=(log(c3/x)-(r-sigma^2/2)*t3)/sigma;
    C4=(log(c4/x)-(r-sigma^2/2)*t4)/sigma;
    X=0;
    if r<0
        kk=(log(k/x)-(r-sigma^2/2)*t4)/sigma;
        [term1,term2]=terminal4(sigma,X,C0, C1,C2,C3,C4,t0,t1,t2,t3,t4,kk );
        term1=exp(-(lambda+r)*t4)*term1;
        term2=exp(-(lambda+sigma^2/2)*t4)*term2;
    else
        term1=0;
        term2=0;
    end
```

```
    l1(4)=laplace_4(r+lambda,X,C0, C1,C2,C3,C4,t0,t1,t2,t3,t4,a,b );
    l2(4)=exp(sigma*b)*laplace_4(alfa,X,C0, C1,C2,C3,C4,t0,t1,t2,t3,t4,a,b );
    I4=k*(l1(4)+term1)-x*(l2(4)+term2);
    pr=I1+I2+I3+I4;
end
end
```

14.1.2 Optimal boundary and pricing

We present now the MATLAB codes for deriving the optimal boundary and for deriving the option prices.

```
function [ t,cc] = bondary_put_points_1_terminal_new( r,lambda,sigma,k,t,N);
% Derive the optimal boundary of an American call
% t      -- the time grid
% cc     -- the boundary

% r      -- risk-free rate
% lambda -- discount factor
% sigma  -- volatility
% k      -- strike
% x      -- initial asset price
% t      -- time-grid
% c      -- boundary values at the time-grid
% N      -- control parameter with values 1,2,3, or 4; it has to be equal to the number of the intervals of the time-grid, i.e.   N=length(t)-1 and N=length(c)-1
cc(1)=min((r+lambda)/lambda,1)*k;
if N==1
   h=@(c,x)-premium_put_1_terminal(r,lambda,sigma,k,x,t,[c,min((r+lambda)/lambda,1)*k],N);
else
   [ ~,cc ] = bondary_put_points_1_terminal_new( r,lambda,sigma,k,t(2:end)-t(2),N-1);
```

```
   h=@(c,x)-premium_put_1_terminal(r,lambda,sigma,k,
      x,t,[c,cc],N);
end
x1=0;
x2=min((r+lambda)/lambda,1)*k;
while abs(x1-x2)>10^(-6)
    x=(x1+x2)/2;
    hh=@(c)h(c,x);
    if hh(x-0.000001)<x-k
       x2=x;
    else
       x1=x;
    end
end
cc=[(x1+x2)/2,cc];
end
```

```
function [pr] =prices_terminal_put(r,lambda,sigma,k,
   x,t,cc);
% Monte Carlo method for the price of the American
   put option

% r      -- risk-free rate
% lambda -- discount factor
% sigma  -- volatility
% k      -- strike
% x      -- initial asset price
% t      -- time-grid
% cc     -- boundary values at the time-gridre

T=t(end);
kk=(log(k/x)-(r-sigma^2/2)*T)/sigma;
c=-(log(cc/x)-(r-sigma^2/2)*t)/sigma;
[ a,b]= linnear(t(1:end-1),c(1:end-1),t(2:end),c(2:
   end));
alfa=sigma^2/2+a*sigma+lambda;
beta=exp(-sigma*b);
if r<0
```

```
    [l1,l2]=laplace_MC(t,c,r+lambda,alfa,beta);
    [tr1,tr2]=terminal_MC(sigma,t,-c,kk);
    ttr1=exp(-(lambda+r)*T)*tr1;
    ttr2=exp(-(lambda+sigma^2/2)*T)*tr2;
    pr=k*(ttr1+l1)-x*(ttr2+l2);
else
    [l1,l2]=laplace_MC(t,c,r+lambda,alfa,beta);
    pr=k*l1-x*l2;
end
end
```

```
function [pr1,pr2] =laplace_MC(t,c,alfa1,alfa2,beta)
    ;
%the Laplace transform at the levels alfa1,alfa2,
   and beta of a Brownian
%motion first hit to the boundary c evaluated at the
    time grid t and
%conditioned to occure before t(end)

   n=length(alfa2);
   H=200000;
   uu=normrnd(0,1,n-1,H);
   m=(1:n);
   u=uu(1:n-1,:);
   Dsq=diag(sqrt(t(2:n)-t(1:n-1)));
   M=tril(ones(n-1));
   x=[zeros(H,1),(M*Dsq*u )'];
   [a,b]=linnear(t(m),c(m),t(m+1),c(m+1));
   q1=((ones(H,1)*exp(-alfa1.*t(m)))).*
      laplace_1_num_1(alfa1*ones(1,n),t(m+1)-t(m),a,
      ones(H,1)*c(m)-x(:,m) );
   q2=((ones(H,1)*exp(-alfa2(m).*t(m)))).*
      laplace_1_num_1(alfa2(m),t(m+1)-t(m),a,ones(H
      ,1)*c(m)-x(:,m) );
   vv=(x(:,2:n)<(c(2:n)'*ones(1,H))').*(1-exp(-2*((c
      (1:n-1)'*ones(1,H))'-x(:,1:n-1)).*((c(2:n)'*
      ones(1,H))'-x(:,2:n))./(((t(2:n)-t(1:n-1))'*
      ones(1,H))')));
```

```
    z=[ones(H,1),vv];
    zz1=cumprod(z')'.*q1;
    zz2=cumprod(z')'.*q2;
    pr1=sum(sum(zz1))/H;
    pr2=sum(beta.*sum(zz2))/H;
end
```

```
function [pr1,pr2,ppr1,ppr2] =prob_terminal_num_2(sigma,t,c,kk);
%the Laplace transform at the level sigma of the Brownian conditioned to
%stay above the piece-wise linear boundary c defined on the time-grid t
    m=length(c)-1;
    H=200000;
    uu=normrnd(0,1,m-1,H);
    u=uu(1:m-1,:);
    Dsq=diag(sqrt(t(2:m)-t(1:m-1)));
    M=tril(ones(m-1));
    x=[zeros(H,1),(M*Dsq*u )'];
    q1=terminal1_1_new(t(end)-t(end-1),0,(c(m)'*ones(1,H)'-x(:,m)),(c(m+1)'*ones(1,H))'-x(:,m),kk-x(:,m),(c(m+1)'*ones(1,H))'-x(:,m) );
    q2=(exp(sigma*x(:,m))).*terminal1_2_new(sigma,t(end)-t(end-1),0,(c(m)'*ones(1,H)'-x(:,m)),(c(m+1)'*ones(1,H))'-x(:,m),kk-x(:,m),(c(m+1)'*ones(1,H))'-x(:,m) );
    vv=(x(:,2:m)>(c(2:m)'*ones(1,H))').*(1-exp(-2*((c(1:m-1)'*ones(1,H))'-x(:,1:m-1)).*((c(2:m)'*ones(1,H))'-x(:,2:m))./(((t(2:m)-t(1:m-1))'*ones(1,H))')));
    zz=[vv,ones(H,1)];
    h1=prod(zz').*q1';
    pp1(m)=sum(h1)/H;
    h2=prod(zz').*q2';
    pp2(m)=sum(h2)/H;
    pr1=sum(pp1);
    pr2=sum(pp2);
```

```
end
```

14.1.3 Auxiliary codes

Below are presented the auxiliary codes required for the correct work of the main codes.

```
function [ a,b] = linnear( t1,y1,t2,y2 );
%deriving the linear function between the points (t1,y1) and (t2,y2)
   a=(y1-y2)./(t1-t2);
   b=(y2.*t1-y1.*t2)./(t1-t2);
end
```

```
function LLL = laplace_1(alfa,t1,a,b );
%derive the Laplace transform at the level alfa of a Brownian motion to the linear boundary a*t+b if it happens before t1
a=sign(b)*a;
b=abs(b);
if b==0
     LLL=1;
else
     LLL=exp(-b.*(a+sqrt(a.^2+2*alfa))).*cdf_hiting(-sqrt(a.^2+2*alfa),b, t1 );
end
end
```

```
function L = cdf_hiting( a,b, t );
% derive the CDF of the first hit of the Brownian motion to the linear
% function a*t+b
   c=exp(-2*a.*abs(b)*2.*((-2*a.*abs(b)<709)-0.5));
        % izh=1-normcdf((a*t+b)/sqrt(t))+exp(-2*a.*abs(b)).*normcdf((a*t-b)/sqrt(t));
   L=1-(1-erfz(-(a*t+b)/sqrt(2*t)))/2+c.*(1-erfz(-(a*t-b)/sqrt(2*t)))/2;
end
```

```matlab
function [term1]=terminal1_1_new(t,x0,c0,c1,kk,kk1 );
% Derive the probabilities of the Brownian motion to be in the iterval
% (kk,kk1) if it stays below the linear function determined by the points
% (0,c0) and (t,c1)
%x0 -- the starting point of the Brownian motion; typically x0=0
   L1=-erfc(kk/sqrt(2*t))/2+erfc(kk1/sqrt(2*t))/2;
   c=exp(2*(c0-x0).*(c0-x0-c1)/t);
   L2=c.*(-erfc((kk-2*(c0-x0))/sqrt(2*t))+erfc((kk1-2*(c0-x0))/sqrt(2*t)))/2;
   L1(isnan(L1))=0;
   L2(isnan(L2))=0;
   L1(isinf(L1))=0;
   L2(isinf(L2))=0;
   term1=L1-L2;
end
```

```matlab
function [term2]=terminal1_2_new(sigma,t,x0,c0,c1,kk,kk1 );
% Derive the Laplace transform at the level sigma of the Brownian motion
%restricted to the iterval (kk,kk1) and if it stays below the linear
%function determined by the points (0,c0) and (t,c1)
%x0 -- the starting point of the Brownian motion; typically x0=0
   L3=exp(t*sigma^2/2)*(-erfc((kk-t*sigma)/sqrt(2*t))+erfc((kk1-t*sigma)/sqrt(2*t)))/2;
   L4=exp(t*sigma^2/2+2*(c0-x0)*sigma+2*(c0-x0).*(c0-x0-c1)/t).*(-erfc((kk-t*sigma - 2*(c0-x0))/sqrt(2*t))+erfc((kk1-t*sigma - 2*(c0-x0))/sqrt(2*t)))/2;
   L4(isnan(L4))=0;
   L3(isnan(L3))=0;
   L4(isinf(L4))=0;
```

```
   L3(isinf(L3))=0;
   term2=L3-L4;
end
```

```
function [L,L2] = terminal2_2(sigma,x0,c0, c1,c2,t0,t1,t2,kk );
% Derive the Laplace transforms at of the Brownian motion
%restricted to the iterval (c2,kk) and if it stays below the two-part linear
%function determined by the points (t0,c0), (t1,c1), and (t2,c2)
%x0 -- the starting point of the Brownian motion; typically x0=0
  izz1=@(x1)terminal1_1_new(t2-t1,0,c1-x1,c2-x1,kk-x1,c2-x1);
  izz2=@(x1)exp(sigma*x1).*terminal1_2_new(sigma,t2-t1,0,c1-x1,c2-x1,kk-x1,c2-x1);
  iz2=@(x1)1-exp((-2*(x0-c0)*(x1-c1))/(t1-t0));
  iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi*(t1-t0));
  iz=@(x1)izz1(x1).*iz2(x1).*iz3(x1);
  iiz2=@(x1)izz2(x1).*iz2(x1).*iz3(x1);
  L=integral(iz,c1,100);
  L2=integral(iiz2,c1,100);
end
```

```
function L = laplace_2(alfa,x,c0, c1,c2,t0,t1,t2,a,b);
%derive the Laplace transform at the level alfa of a Brownian motion first
%hit to the  two-part piecewise linear boundary determined by the points
% (t0,c0), (t1,c1), and (t2,c2), if it happens before t2
if b==0
   iz=@(x1)1;
else
```

```
    izz=@(x1)(exp(-alfa*t1)*abs(c1-x1)./(abs(a*t1+b-x1))).* laplace_1(alfa,t2-t1,a,a*t1+b-x1 );
    izz=@(x1)(exp(-alfa*t1))* laplace_1(alfa,t2-t1,a,a*t1+b-x1 );
    iz2=@(x1)1-exp((-2*(x-c0)*(x1-c1))/(t1-t0));
    iz3=@(x1)exp(-(x1-x).^2/(2*(t1-t0)))/sqrt(2*pi*(t1-t0));
end
    iz=@(x1)izz(x1).*iz2(x1).*iz3(x1);
    L=integral(iz,c1,15);
end
```

```
function L = laplace_3(alfa,x,c0, c1,c2,c3,t0,t1,t2,t3,a,b );
%derive the Laplace transform at the level alfa of a Brownian motion first
%hit to the three-part piecewise linear boundary determined by the points
% (t0,c0), (t1,c1), (t2,c2), and (t3,c3) if it happens before t3
  h=@(x1)izz_laplace_3(alfa,x,c0, c1,c2,c3,t0,t1,t2,t3,a,b,x1 );
  L=integral(h,c1,50);
end
```

```
function L =izz_laplace_3(alfa,x,c0, c1,c2,c3,t0,t1,t2,t3,a,b,x1 );
% Alternative variant for integration in the code laplace_3()
  izz=@(x2)exp(-alfa*t2)* laplace_1(alfa,t3-t2,a,a*t2+b-x2 );
  iz2=@(x2)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0))).*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1)));
  iz3=@(x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
  iz=@(x2)izz(x2).*iz2(x2).*iz3(x2);
  L=integral(iz,c2,50,'ArrayValued',true);
```

```
end
```

```
function [L1,L2] = terminal3(sigma,x,c0, c1,c2,c3,t0,t1,t2,t3,kk );
% Derive the Laplace transforms at of the Brownian motion
%restricted to the iterval (c3,kk) and if it stays below the three-part linear
%function determined by the points (t0,c0), (t1,c1), (t2,c2), and (t3,c3)
%x0 -- the starting point of the Brownian motion; typically x0=0
  h1=@(x1)izz_terminal3_1(x,c0, c1,c2,c3,t0,t1,t2,t3,kk,x1 );
  h2=@(x1)izz_terminal3_2(sigma,x,c0, c1,c2,c3,t0,t1,t2,t3,kk,x1 );
  L1=integral(h1,c1,150);
  L2=integral(h2,c1,150);
end
```

```
function L=izz_terminal3_1(x,c0, c1,c2,c3,t0,t1,t2,t3,kk,x1 );
% Alternative variant for integration in the code terminal3()
  izz=@(x2)terminal1_1_new(t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
  iz2=@(x2)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0))).*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1)));
  iz3=@(x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
  iz=@(x2)izz(x2).*iz2(x2).*iz3(x2);
  L=integral(iz,c2,150,'ArrayValued',true);
end
```

```
function L=izz_terminal3_2(sigma,x,c0, c1,c2,c3,t0,t1,t2,t3,kk,x1 );
% Alternative variant for integration in the code terminal3()
```

```
  izz=@(x2)exp(sigma*x2).*terminal1_2_new(sigma,t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
  iz2=@(x2)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0))).*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1)));
  iz3=@(x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
  iz=@(x2)izz(x2).*iz2(x2).*iz3(x2);
  L=integral(iz,c2,150,'ArrayValued',true);
end
```

```
function L= laplace_4(alfa,x,c0, c1,c2,c3,c4,t0,t1,t2,t3,t4,a,b );
%derive the Laplace transform at the level alfa of a Brownian motion first
%hit to the four-part piecewise linear boundary determined by the points
% (t0,c0), (t1,c1), (t2,c2), (t3,c3), and (t4,c4) if it happens before t4
if b==0
   iz=@(x1,x2)1;
else
   izz=@(x1,x2,x3)(exp(-alfa*t3)).* laplace_1(alfa,t4-t3,a,c3-x3 );
   iz2=@(x1,x2,x3)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0))).*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1))).*(1-exp((-2*(x2-c2).*(x3-c3))/(t3-t2)));
   iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
end
iz=@(x1,x2,x3)izz(x1,x2,x3).*iz2(x1,x2,x3).*iz3(x1,x2,x3);
L=integral3(iz,c1,15,c2,15,c3,15);
end
```

```
function [L1,L2] =terminal4(sigma,x,c0, c1,c2,c3,c4,
```

```
    t0,t1,t2,t3,t4,kk );
% Derive the Laplace transforms at of the Brownian motion
%restricted to the iterval (c4,kk) and if it stays below the three-part linear
%function determined by the points (t0,c0), (t1,c1), (t2,c2),(t3,c3), and (t4,c4)
%x0 -- the starting point of the Brownian motion; typically x0=0
  izz1=@(x1,x2,x3)terminal1_1_new(t4-t3,0,c3-x3,c4-x3,kk-x3,c4-x3);
  izz2=@(x1,x2,x3)exp(sigma*x3).*terminal1_2_new(sigma,t4-t3,0,c3-x3,c4-x3,kk-x3,c4-x3);
  iz2=@(x1,x2,x3)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0))).*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1))).*(1-exp((-2*(x2-c2).*(x3-c3))/(t3-t2)));
  iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
  iz1=@(x1,x2,x3)izz1(x1,x2,x3).*iz2(x1,x2,x3).*iz3(x1,x2,x3);
  iz2=@(x1,x2,x3)izz2(x1,x2,x3).*iz2(x1,x2,x3).*iz3(x1,x2,x3);
  L1=integral3(iz1,c1,15,c2,15,c3,15);
  L2=integral3(iz2,c1,15,c2,15,c3,15);
end
```

14.1.4 Comments

We use three main and several auxiliary MATLAB codes to implement the approach for pricing American put options presented in Chapter 4. The first of them, `premium_put_1_terminal.m`, calculates the price of a financial instrument related to the first hit (ζ) to the exponent of a piece-wise linear function consisting of at most four parts. It is based on formula (4.7). The different cases w.r.t. the division's number N are considered separately. For example, if $N = 2$, then the expectations are related to the sets $\zeta \in (0, t_1)$, $\zeta \in [t_1, T)$, and $\zeta \geq T$. This case relies on the part of the code `premium_put_1_terminal.m` between lines 32-61. If $\zeta \in (0, t_1)$ (lines 32-41), then we use the code `laplace_1.m` – it is about formulas (2.4) and (2.15). The Gaussian CDF in these formulas is derived via the code `cdf_hiting.m`. Usually, it uses the integrated in MATLAB function `normcdf.m`. Sometimes, the cumulative error function `erfc.m` is more appropriate. Note that `erf.m` does not provide the desired results. If $\zeta \in (t_1, t_2)$ (lines 42-61), then we use the code `laplace_2.m` – it is based on formulas (2.13) and (2.17). On the other hand, if $\zeta \geq T$, then we need the Laplace transform of the Brownian motion instead of the stopping time – lines 49-57. We use the code `terminal2_2.m`. Note that the terminal value has an impact only when $r < 0$. If $r \geq 0$, then the optimal value at maturity is the strike and thus the option does not pay anything if the asset does not hit the boundary before maturity.

The algorithm for deriving the optimal boundary presented in Section 4.4 is implemented in the code `bondary_put_points_1_terminal_new.m`. It works for divisions based on one, two, three, and four steps in the piece-wise linear function. Larger numbers lead to a very high computational time due to the multiple integration in formulas (2.13) and (2.17). Our experiments show that an appropriate choice is $N = 3$. Also, two-step functions provide admissible results. A proper approach to collect more points from the optimal boundary is to run the code `bondary_put_points_1_terminal_new.m` several times with different divisions.

The fast approach provided as 4.1 can be checked via the following example:

```
>> r=-0.01;
>> lambda=0.03;
>> sigma=.3;
>> k=20;
>> S_0=20;
>> T=3;
>> [ t,boundary] = bondary_put_points_1_terminal_new
   ( r,lambda,sigma,k,[0,T/3,2*T/3,T],3)
price=premium_put_1_terminal(r,lambda,sigma,k,S_0,t,
   boundary,3)

t =

```

```
     0     1     2     3


boundary =

    7.5680    8.3409    9.7192   13.3333


price =

    4.1147
```

If we need to use a denser grid, then we can use one of the numerical methods presented in the dissertation. We implement the Monte-Carlo approach presented in Section 4.4.2 into the MATLAB code `prices_terminal_put.m`. It is based on two axillary codes, namely `laplace_MC.m` and `terminal_MC.m`.

14.2 Cancellabel American put options

14.2.1 Main premium code

We present now the codes for pricing financial instruments studied in Chapter 13. We present also the codes for deriving he optimal boundaries as well as for the presented in Section 13.5 Monte Carlo method.

```
function [pr] =premium_put_1_game_inf_terminal(r,
   lambda,sigma,k,eta,x,t,alpha,beta,N,flag,n);
% Derive the price of the put-style game derivative
   maturating at the boundaries alpha and beta
   evaluated at the time-grid t

% r      -- risk-free rate
% lambda -- di
% c      -- boundary values at the time-gridrescount
    factor
% sigma  -- volatility
% k      -- strike
% eta    -- penalty
% x      -- initial asset price
```

```
% t       -- time-grid
if N==1
    t0=t(1); t1=t(2);
    alpha0=alpha(1); alpha1=alpha(2);
    beta0=beta(1); beta1=beta(2);
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    X=0;
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );

    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2- b1*sigma+lambda;
    l1_lower(1)=laplace_1_finite_put_lower_inf(r+lambda,t1,a1,a2,b1,b2,flag,n);
    l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower_inf(mu_lower, t1,a1,a2,b1,b2,flag,n );
    pr_lower=k*l1_lower(1)-x*l2_lower(1);

   if flag==0
      l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
      l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
      pr_upper=(k+eta)*l1_upper(1)-x*l2_upper(1);
   else
       pr_upper=0;
   end

   if r>=0
```

```
        term1=0;
        term2=0;
    elseif flag==0
         kk=(log(k/x)-(r-sigma^2/2)*t1)/sigma;
         [term1]=exp(-(r+lambda)*t1)*terminal1_1_game
            (a1,a2,b1,b2,t1-t0,n);
         [term2]=exp(-(lambda+sigma^2/2)*t1)*
            terminal1_2_game(sigma,t1-t0,X,A0,A1,B0,
            B1,n);
    else
        kk=(log(k/x)-(r-sigma^2/2)*t1)/sigma;
         [term1]=exp(-(r+lambda)*t1)*terminal1_1_new(
            t1-t0,X,A0,A1,kk,A1);
         [term2]=exp(-(lambda+sigma^2/2)*t1)*
            terminal1_2_new(sigma,t1-t0,X,A0,A1,kk,A1
            );
    end
    pr=pr_lower+pr_upper+k*term1-x*term2;

elseif N==2
    t0=t(1); t1=t(2); t2=t(3);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha(3)
       ;
    beta0=beta(1); beta1=beta(2);  beta2=beta(3);

    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma
       ^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma
       ^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)
       *t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/
       sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    l1_lower(1)=laplace_1_finite_put_lower_inf(r+
       lambda,t1,a1,a2,b1,b2,flag-1,n );
    l2_lower(1)=exp(sigma*a2)*
       laplace_1_finite_put_lower_inf(mu_lower, t1,a1
       ,a2,b1,b2,flag-1,n );
```

```
    if flag<=1
        l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
        l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
        I1_upper=(k+eta)*l1_upper(1)-x*l2_upper(1);
    else
        I1_upper=0;
    end
    I1_lower=k*l1_lower(1)-x*l2_lower(1);
    % ------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    if r<0
        kk=(log(k/x)-(r-sigma^2/2)*t2)/sigma;
        [term1,term2]=terminal2_2_game(sigma,X,A0,A1,A2,B0, B1,B2,t0,t1,t2,kk,flag,n );
        term1=exp(-(lambda+r)*t2)*term1;
        term2=exp(-(lambda+sigma^2/2)*t2)*term2;
    else
        term1=0;
        term2=0;
```

```
    end
    l1_lower(2)=laplace_2_finite_put_lower_inf(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,flag,n );
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower_inf(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,flag,n );
    if flag==0
         l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
         l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
         I2_upper=(k+eta)*l1_upper(2)-x*l2_upper(2);
    else
        I2_upper=0;
    end

    I2_lower=k*l1_lower(2)-x*l2_lower(2);
    pr=I1_lower+I2_lower+I1_upper+I2_upper+k*term1-x*term2;
elseif N==3
    t0=t(1); t1=t(2); t2=t(3); t3=t(4);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha(3); alpha3=alpha(4);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3); beta3=beta(4);
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
```

```
    l1_lower(1)=laplace_1_finite_put_lower_inf(r+lambda,t1,a1,a2,b1,b2,flag-2,n );
    l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower_inf(mu_lower, t1,a1,a2,b1,b2, flag-2,n );
    if flag<=2
        l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
        l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
        I1_upper=(k+eta)*l1_upper(1)-x*l2_upper(1);
    else
        I1_upper=0;
    end
    I1_lower=k*l1_lower(1)-x*l2_lower(1);
    %------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1_lower(2)=laplace_2_finite_put_lower_inf(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,flag-1,n );
```

```
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower_inf(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,flag-1,n );
    if flag<=1
        l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
        l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
        I2_upper=(k+eta)*l1_upper(2)-x*l2_upper(2);
    else
         I2_upper=0;
    end
    I2_lower=k*l1_lower(2)-x*l2_lower(2);
    % ------------------------------------------------------------------
    [ a1,a2]= linnear( t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma );
    [ b1,b2]= linnear( t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(beta3/x)-(r-sigma^2/2)*t3)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    if r<0
```

```
            kk=(log(k/x)-(r-sigma^2/2)*t3)/sigma;
            [term1,term2]=terminal3_game(sigma,X,A0,A1,A2,A3,B0,B1,B2,B3,t0,t1,t2,t3,kk,flag,n );
            term1=exp(-(lambda+r)*t3)*term1;
            term2=exp(-(lambda+sigma^2/2)*t3)*term2;
        else
            term1=0;
            term2=0;
        end
         l1_lower(3)=laplace_3_finite_put_lower_inf(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,flag,n );
         l2_lower(3)=exp(sigma*a2)*laplace_3_finite_put_lower_inf(mu_lower,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,flag,n );
         if flag==0
             l1_upper(3)=laplace_3_finite_put_upper(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
             l2_upper(3)=exp(sigma*b2)*laplace_3_finite_put_upper(mu_upper,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
             I3_upper=(k+eta)*l1_upper(3)-x*l2_upper(3);
         else
             I3_upper=0;
         end
         I3_lower=k*l1_lower(3)-x*l2_lower(3);
        pr=I1_lower+I2_lower+I3_lower+I1_upper+I2_upper+I3_upper+k*term1-x*term2;
elseif N==4
        t0=t(1); t1=t(2); t2=t(3); t3=t(4); t4=t(5);
        alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha(3); alpha3=alpha(4); alpha4=alpha(5);
        beta0=beta(1); beta1=beta(2);  beta2=beta(3); beta3=beta(4); beta4=beta(5);
```

```
[ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma );
[ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );
mu_lower=sigma^2/2-a1*sigma+lambda;
mu_upper=sigma^2/2-b1*sigma+lambda;
l1_lower(1)=laplace_1_finite_put_lower_inf(r+lambda,t1,a1,a2,b1,b2,flag-3,n );
l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower_inf(mu_lower, t1,a1,a2,b1,b2,flag-3,n );
if flag<=3
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
    I1_upper=(k+eta)*l1_upper(1)-x*l2_upper(1);
else
    I1_upper=0;
end
I1_lower=k*l1_lower(1)-x*l2_lower(1);
% ------------------------------------------------------------------------
[ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
[ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
mu_lower=sigma^2/2-a1*sigma+lambda;
mu_upper=sigma^2/2-b1*sigma+lambda;
A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
```

```
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1_lower(2)=laplace_2_finite_put_lower_inf(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,flag-2,n );
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower_inf(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,flag-2,n );
    if flag<=2
        l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
        l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
        I2_upper=(k+eta)*l1_upper(2)-x*l2_upper(2);
    else
        I2_upper=0;
    end
    I2_lower=k*l1_lower(2)-x*l2_lower(2);
    % -------------------------------------------------------------
    [ a1,a2]= linnear( t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma );
    [ b1,b2]= linnear( t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(beta3/x)-(r-sigma^2/2)*t3)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
```

```
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    l1_lower(3)=laplace_3_finite_put_lower_inf(r+
       lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,
       t3,a1,a2,b1,b2,flag-1,n );
    l2_lower(3)=exp(sigma*a2)*
       laplace_3_finite_put_lower_inf(mu_lower,X,A0,
        A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,
       b2,flag-1,n );
    if flag<=1
        l1_upper(3)=laplace_3_finite_put_upper(r+
           lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,
           t2,t3,a1,a2,b1,b2,n );
        l2_upper(3)=exp(sigma*b2)*
           laplace_3_finite_put_upper(mu_upper,X,A0,
            A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,
           b1,b2,n );
        I3_upper=(k+eta)*l1_upper(3)-x*l2_upper(3);
    else
        I3_upper=0;
    end
    I3_lower=k*l1_lower(3)-x*l2_lower(3);
    %
       ------------------------------------------------------------------
    [ a1,a2]= linnear( t3,(log(alpha3/x)-(r-sigma
       ^2/2)*t3)/sigma,t4,(log(alpha4/x)-(r-sigma
       ^2/2)*t4)/sigma );
    [ b1,b2]= linnear( t3,(log(beta3/x)-(r-sigma
       ^2/2)*t3)/sigma,t4,(log(beta4/x)-(r-sigma
       ^2/2)*t4)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
```

```
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    A4=(log(alpha4/x)-(r-sigma^2/2)*t4)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    B4=(log(beta4/x)-(r-sigma^2/2)*t4)/sigma;
    X=0;
    if r<0
        kk=(log(k/x)-(r-sigma^2/2)*t4)/sigma;
        [term1,term2]=terminal4_game(sigma,X,A0, A1,
           A2,A3,A4,B0,B1,B2,B3,B4,t0,t1,t2,t3,t4,kk
           ,flag,n );
        term1=exp(-(lambda+r)*t4)*term1;
        term2=exp(-(lambda+sigma^2/2)*t4)*term2;
    else
        term1=0;
        term2=0;
    end
    l1_lower(4)=laplace_4_finite_put_lower_inf(r+
       lambda,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,
       t1,t2,t3,t4,a1,a2,b1,b2,flag,n );
    l2_lower(4)=exp(sigma*a2)*
       laplace_4_finite_put_lower_inf(mu_lower,X,A0,
        A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,
       a1,a2,b1,b2,flag,n );
    if flag==0
         l1_upper(4)=laplace_4_finite_put_upper(r+
            lambda,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4
            ,t0,t1,t2,t3,t4,a1,a2,b1,b2,n );
         l2_upper(4)=exp(sigma*b2)*
            laplace_4_finite_put_upper(mu_upper,X,A0
            , A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,
            t3,t4,a1,a2,b1,b2,n );
         I4_upper=(k+eta)*l1_upper(4)-x*l2_upper(4);
    else
```

```
            I4_upper=0;
        end
        I4_lower=k*l1_lower(4)-x*l2_lower(4);
        pr=I1_lower+I2_lower+I3_lower+I4_lower +I1_upper+I2_upper+I3_upper+I4_upper+k*term1-x*term2;
end
end
```

```
function [ A,B,price,flag ] = game_eq_putt(r,sigma,lambda,k,eta,x );
%derive the optimal boundaries and the price for the operpetual option
% The output flag indicates the case of Theorem that holds

fun=@(y)(y-func_a_putt(func_b_putt(y,r,sigma,lambda,k,eta,x ),r,sigma,lambda,k,eta,x ))^2;
opt=optimset('MaxIter',10^(200),'TolX',10^(-200),'TolFun',10^(-200),'MaxFunEvals',10^(200));
[A,C]=fminbnd(fun,0,10000,opt);
B=func_b_putt(A,r,sigma,lambda,k,eta,x );
mu=r/(sigma^2)-0.5;
c=sqrt(mu^2+2*(r+lambda)/(sigma^2));
q=c+mu;
p=2*c;
gam=c-mu;
if B>k
    B=k;
    eta1=(k/(q+1))*(q/(q+1))^q;
    if eta>eta1
        [price,flag,A]=american_put(r,sigma,lambda,k,x );
    else
        AA = func_a_putt(k,r,sigma,lambda,k,eta,x );
        A=AA;
        if x>k
            flag='2.b.1';
            price=eta*(k/x)^q;
```

```
        elseif x<AA
            price=k-x;
            flag='2.b.3';
        else
            flag='2.b.2';
            a=log(AA/x);
            b=log(k/x);
            price=Game_price_new_ab_putt( c,eta,mu,k
               ,x,a,b );
        end
    end
elseif x>B & x<=k
    price=k-x+eta;
    flag='1.b';
elseif  x<A
    price=k-x;
    flag='1.d';
elseif x>k
    price=eta*(k/x)^q;
    flag='1.a';
else
    flag='1.c';
   a=log(A/x);
   b=log(B/x);
   price=Game_price_new_ab_putt( c,eta,mu,k,x,a,b );
end
end
```

Auxilary codes

```
function L = laplace_1_finite_put_lower_inf(xi,t,a1,
   a2,b1,b2,flag,n );
%derive the Laplace transform at the level xi of a
   Brownian motion to the
%lower of two linear boundaries defined by the
   points {a1,a2} and {b1,b2}
%at the time grid t
```

```

% flag --  the numebr of the empty positions for the upper boiundary
% n    --  the nuber of the used terms for the infinite sum in formula (2.28)

if flag<=0
    L=exp(a2.*(sqrt(a1.^2+2*xi)-a1)).*P_1_l_num(sqrt(a1.^2+2*xi),a2,b1+sqrt(a1.^2+2*xi)-a1,b2, t, n );
else
    L=laplace_1(xi,t,a1,a2 );
end
L(isnan(izh))=0;
end
```

```
function PP= P_1_l_num(a1,a2,b1,b2, t,n );
% see laplace_1_finite_put_lower_inf.m
d=size(b1);
H=d(1);
t=ones(H,1)*t;
izz=zeros(size(a2));
for j=1:n
     b=exp(-2*(-j*a2+(j-1)*b2).*(-j*a1+(j-1)*b1)).*normcdf((-a1.*t-2*(j-1)*b2+(2*j-1)*a2)./sqrt(t))...
      -exp(-2*(j^2*(a1.*a2+b1.*b2)-j*(j-1)*a2.*b1-j*(j+1)*b2.*a1)).*normcdf((-a1.*t-2*j*b2+(2*j-1)*a2)./sqrt(t))...
      -exp(-2*(-(j-1)*a2+j*b2).*(-(j-1)*a1+j*b1)).*normcdf((a1.*t-2*j*b2+(2*j-1)*a2)./sqrt(t))...
      +exp(-2*(j^2*(a1.*a2+b1.*b2)-j*(j-1)*b2.*a1-j*(j+1)*a2.*b1)).*normcdf((a1.*t+(2*j+1)*a2-2*j*b2)./sqrt(t));
  b(isnan(b))=0;
  izz=izz+b;
end
```

```
PP=izz+normcdf((a1.*t+a2)./sqrt(t));
end
```

```
function L = laplace_1_finite_put_upper(xi,t,a1,a2,b1,b2,n );
%derive the Laplace transform at the level xi of a Brownian motion to the
%lower of two linear boundaries defined by the points {a1,a2} and {b1,b2}
%at the time grid t

% flag --  the numebr of the empty positions for the upper boiundary
% n    --  the nuber of the used terms for the infinite sum in formula (2.28)

L=exp(b2*(sqrt(b1^2+2*xi)-b1)).*P_2_l(a1+sqrt(b1^2+2*xi)-b1,a2,sqrt(b1^2+2*xi),b2, t,n );
end
```

```
function izhod = P_2_l(a1,a2,b1,b2, t,n );
%see laplace_1_finite_put_upper.m

izz=zeros(size(a2));
for j=1:n
    b=exp(-2*(j*b2-(j-1)*a2).*(j*b1-(j-1)*a1)).*normcdf((b1*t+2*(j-1)*a2-(2*j-1)*b2)/sqrt(t))...
      -exp(-2*(j^2*(b1*b2+a1*a2)-j*(j-1)*b2*a1-j*(j+1)*a2*b1)).*normcdf((b1*t+2*j*a2-(2*j-1)*b2)/sqrt(t))...
      -exp(-2*((j-1)*b2-j*a2).*((j-1)*b1-j*a1)).*normcdf((-b1*t+2*j*a2-(2*j-1)*b2)/sqrt(t))...
      +exp(-2*(j^2*(b1*b2+a1*a2)-j*(j-1)*a2*b1-j*(j+1)*b2*a1)).*normcdf((-b1*t-(2*j+1)*b2+2*j*a2)/sqrt(t));
    b(isnan(b))=0;
```

```
    izz=izz+b;
end
izhod=izz+1-normcdf((b1*t+b2)/sqrt(t));
end
```

```
function [term]=terminal1_1_game(a1,a2,b1,b2,T,n );
% related to the option value at maturity
  p1= P_1_l(a1,a2,b1,b2, T,n );
  p2= P_2_l(a1,a2,b1,b2, T,n );
  term=1-p1-p2;
end
```

```
function [term]=terminal1_2_game(theta,T,x0,a0,a1,b0
    ,b1,n );
% related to the option value at maturity

c0=b0-a0;    c1=b1-a1;
s0=1;  s2=1;  s4=1; s1=-1; s3=-1;
z=a1;
lam_j0=0;
xi_j0=0;
term=s0*exp(lam_j0*theta+((lam_j0^2+2*xi_j0)/(2*T)))
    *(normcdf((b1-(theta*T+lam_j0))/(sqrt(T)))-
    normcdf((z-(theta*T+lam_j0))/(sqrt(T))));
for j=1:n
   lam_j1=2*(j*c0+a0-x0);
   xi_j1=-2*(j*c0+a0-x0).*(j*c1+a1);
   lam_j2=2*j*c0;
   xi_j2=-2*j*(j*c0.*c1+c0.*a1-c1.*(a0-x0));
   lam_j3=-2*(j*c0-(b0-x0));
   xi_j3=-2*(j*c0-(b0-x0)).*(j*c1-b1);
   lam_j4=-2*j*c0;
   xi_j4=-2*j*(j*c0.*c1-c0.*b1+c1.*(b0-x0));
   L=s1*exp(lam_j1*theta+((lam_j1.^2+2*xi_j1)/(2*T))
      ).*(normcdf((b1-(theta*T+lam_j1))/(sqrt(T)))-
      normcdf((z-(theta*T+lam_j1))/(sqrt(T))))...
    +s2*exp(lam_j2*theta+((lam_j2.^2+2*xi_j2)/(2*T))
       ).*(normcdf((b1-(theta*T+lam_j2))/(sqrt(T)))-
```

```
        normcdf((z-(theta*T+lam_j2))/(sqrt(T))))...
     +s3*exp(lam_j3*theta+((lam_j3.^2+2*xi_j3)/(2*T))).*(normcdf((b1-(theta*T+lam_j3))/(sqrt(T)))-normcdf((z-(theta*T+lam_j3))/(sqrt(T))))...
     +s4*exp(lam_j4*theta+((lam_j4.^2+2*xi_j4)/(2*T))).*(normcdf((b1-(theta*T+lam_j4))/(sqrt(T)))-normcdf((z-(theta*T+lam_j4))/(sqrt(T))));
    L(isnan(L))=0;
    term=term+L;
end
term=exp(theta^2*T/2)*term;
end
```

```
function [term1]=terminal1_1_new(t,x0,c0,c1,kk,kk1 );
% related to the option value at maturity

L1=normcdf(kk/sqrt(t))-normcdf(kk1/sqrt(t));
L2=exp(2*(c0-x0).*(c0-x0-c1)/t).*(normcdf((kk-2*(c0-x0))/sqrt(t))-normcdf((kk1-2*(c0-x0))/sqrt(t)));
L2(isnan(L2))=0;
term1=L1-L2;
%term1(isnan(term1))=0;
end
```

```
function [term2]=terminal1_2_new(sigma,t,x0,c0,c1,kk,kk1 );
% related to the option value at maturity

L3=exp(t*sigma^2/2)*(normcdf((kk-t*sigma)/sqrt(t))-normcdf((kk1-t*sigma)/sqrt(t)));
L4=exp(t*sigma^2/2+2*(c0-x0)*sigma+2*(c0-x0).*(c0-x0-c1)/t).*(normcdf((kk-t*sigma - 2*(c0-x0))/sqrt(t))-normcdf((kk1-t*sigma - 2*(c0-x0))/sqrt(t)));
L4(isnan(L4))=0;
term2=L3-L4;
end
```

```
function [Y1,Y2] = terminal2_2_game(sigma,x0,c0, c1,
   c2,b0,b1,b2,t0,t1,t2,kk,flag,n );
% related to the option value at maturity

if flag==0
    [ aa1,aa2]= linnear( t1,c1,t2,c2 );
    [ bb1,bb2]= linnear( t1,b1,t2,b2 );
    izz1=@(x1)terminal1_1_game(aa1,c1-x1,bb1,b1-x1,
       t2-t1,n);
    izz2=@(x1)exp(sigma*x1).*terminal1_2_game(sigma,
       t2-t1,0,c1-x1,c2-x1,b1-x1,b2-x1,n);
    iz2=@(x1)(1-qq( x0,x1,c0,c1,b0,b1,t1-t0,n ));
    iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi
       *(t1-t0));
elseif flag==1
    izz1=@(x1)terminal1_1_new(t2-t1,0,c1-x1,c2-x1,kk
       -x1,c2-x1);
    izz2=@(x1)exp(sigma*x1).*terminal1_2_new(sigma,
       t2-t1,0,c1-x1,c2-x1,kk-x1,c2-x1);
    iz2=@(x1)(1-qq( x0,x1,c0,c1,b0,b1,t1-t0,n ));
    iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi
       *(t1-t0));
else
    izz1=@(x1)terminal1_1_new(t2-t1,0,c1-x1,c2-x1,kk
       -x1,c2-x1);
    izz2=@(x1)exp(sigma*x1).*terminal1_2_new(sigma,
       t2-t1,0,c1-x1,c2-x1,kk-x1,c2-x1);
    iz2=@(x1)1-exp((-2*(x0-c0)*(x1-c1))/(t1-t0));
    iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi
       *(t1-t0));
    b1=5;
end
iz=@(x1)izz1(x1).*iz2(x1).*iz3(x1);
iiz2=@(x1)izz2(x1).*iz2(x1).*iz3(x1);
Y1=integral(iz,c1,b1);
Y2=integral(iiz2,c1,b1);
end
```

```
function q = qq( y,z,alpha_min,alpha_i,beta_min,
   beta_i,t,n );
% calculating the infinite sum

gama_min=beta_min-alpha_min;
gama=beta_i-alpha_i;
q=zeros(size(z));
for j=1:n
    b= exp(-2*(j*gama_min+alpha_min-y).*(j*gama+
        alpha_i-z)/t)...
      -exp(-2*j*(j*gama_min*gama+gama_min*(alpha_i-z)
        -gama*(alpha_min-y))/t)...
      +exp(-2*(j*gama_min-(beta_min-y)).*(j*gama-(
        beta_i-z))/t)...
      -exp(-2*j*(j*gama_min*gama-gama_min*(beta_i-z)+
        gama*(beta_min-y))/t);
    b(isnan(b))=0;
    q=q+b;
end
end
```

```
function Y = laplace_2_finite_put_lower_inf(xi,x,
   alpha0, alpha1,alpha2,beta0, beta1,beta2,t0,t1,t2
   ,a1,a2,b1,b2,flag,n );
% related to the option value at the lower boundary
   and the second interval

if flag<=0
     izz=@(x1)(exp(-xi*t1)).*
        laplace_1_finite_put_lower_inf(xi,t2-t1,a1,
        alpha1-x1,b1,beta1-x1,flag,n );
     iz2=@(x1)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,
        t1-t0,n ));
     iz3=@(x1)exp(-(x1-x).^2/(2*(t1-t0)))/sqrt(2*pi*(
        t1-t0));
elseif flag==1
     izz=@(x1)exp(-xi*t1).* laplace_1(xi,t2-t1,a1,a1*
        t1+a2-x1 );
```

```
    iz2=@(x1)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,
       t1-t0,n ));
    iz3=@(x1)exp(-(x1-x).^2/(2*(t1-t0)))/sqrt(2*pi*(
       t1-t0));
else
    izz=@(x1)exp(-xi*t1).* laplace_1(xi,t2-t1,a1,a1*
       t1+a2-x1 );
    iz2=@(x1)1-exp((-2*(x-alpha0)*(x1-alpha1))/(t1-
       t0));
    iz3=@(x1)exp(-(x1-x).^2/(2*(t1-t0)))/sqrt(2*pi*(
       t1-t0));
    beta1=5;
end
iz=@(x1)izz(x1).*iz2(x1).*iz3(x1);
Y=integral(iz,alpha1,beta1);
end
```

```
function Y = laplace_2_finite_put_upper(xi,x,alpha0,
     alpha1,alpha2,beta0, beta1,beta2,t0,t1,t2,a1,a2,
    b1,b2,n );
% related to the option value at the upper boundary
    and the second interval

izz=@(x1)(exp(-xi*t1)).* laplace_1_finite_put_upper(
    xi,t2-t1,a1,alpha1-x1,b1,beta1-x1,n );
iz2=@(x1)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,t1-t0
    ,n ));
iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0)))/sqrt(2*pi*(
    t1-t0));
iz=@(x1)izz(x1).*iz2(x1).*iz3(x1);
Y=integral(iz,alpha1,beta1);
end
```

```
function [ Y1,Y2] = terminal3_game(sigma,x,c0,c1,c2,
    c3,b0,b1,b2,b3,t0,t1,t2,t3,kk,flag,n );
% related to the option value at maturity

if flag==0
```

```
    [ aa1,aa2]= linnear( t2,c2,t3,c3 );
    [ bb1,bb2]= linnear( t2,b2,t3,b3 );
    izzz1=@(x1,x2) terminal1_1_game(aa1,c2-x2,bb1,b2-x2,t3-t2,n);
    izzz2=@(x1,x2)exp(sigma*x2).*terminal1_2_game(sigma,t3-t2,0,c2-x2,c3-x2,b2-x2,b3-x2,n);
    iz2=@(x1,x2)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
                .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ));
    iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));;
elseif flag==1
    izzz1=@(x1,x2) terminal1_1_new(t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
    izzz2=@(x1,x2)exp(sigma*x2).*terminal1_2_new(sigma,t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
    iz2=@(x1,x2)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
                .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ));
    iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
elseif flag==2
     izzz1=@(x1,x2) terminal1_1_new(t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
     izzz2=@(x1,x2)exp(sigma*x2).*terminal1_2_new(sigma,t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
     iz2=@(x1,x2)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
             .*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1)));
     iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
     b2=5;
else
     izzz1=@(x1,x2) terminal1_1_new(t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
```

```
    izzz2=@(x1,x2)exp(sigma*x2).*terminal1_2_new(
       sigma,t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
    iz2=@(x1,x2)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0)))
       .*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1)));
    iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(
       x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi
       *(t2-t1));
    b1=5;
    b2=5;
end
izz1=@(x1,x2)izzz1(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
izz2=@(x1,x2)izzz2(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
Y1=integral2(izz1,c1,b1,c2,b2 );
Y2=integral2(izz2,c1,b1,c2,b2 );
end
```

```
function Y = laplace_3_finite_put_lower_inf(xi,x,
   alpha0, alpha1,alpha2,alpha3,beta0, beta1,beta2,
   beta3,t0,t1,t2,t3,a1,a2,b1,b2,flag,n );
% related to the option value at the lower boundary
   and the third interval

if flag<=0
   izz=@(x1,x2)(exp(-xi*t2)).*
      laplace_1_finite_put_lower_inf(xi,t3-t2,a1,
      alpha2-x2,b1,beta2-x2,flag,n );
   iz2=@(x1,x2)(1-qq( x,x1,alpha0,alpha1,beta0,beta1
      ,t1-t0,n ))...
                .*(1-qq( x1,x2,alpha1,alpha2,beta1,
                   beta2,t2-t1,n ));
   iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2
      -x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(
      t2-t1));
elseif flag==1
    izz=@(x1,x2)exp(-xi*t2).* laplace_1(xi,t3-t2,a1,
       a1*t2+a2-x2 );
    iz2=@(x1,x2)(1-qq( x,x1,alpha0,alpha1,beta0,
       beta1,t1-t0,n ))...
```

```
                  .*(1-qq( x1,x2,alpha1,alpha2,beta1,
                    beta2,t2-t1,n ));
    iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(
       x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi
       *(t2-t1));
elseif flag==2
    izz=@(x1,x2)exp(-xi*t2).* laplace_1(xi,t3-t2,a1,
       a1*t2+a2-x2 );
    iz2=@(x1,x2)(1-qq( x,x1,alpha0,alpha1,beta0,
       beta1,t1-t0,n ))...
                   .*(1-exp((-2*(x1-alpha1).*(x2-alpha2
                     ))/(t2-t1)));
    iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(
       x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi
       *(t2-t1));
    beta2=5;
else
    izz=@(x1,x2)exp(-xi*t2).* laplace_1(xi,t3-t2,a1,
       a1*t2+a2-x2 );
    iz2=@(x1,x2)(1-exp((-2*(x-alpha0)*(x1-alpha1))/(
       t1-t0))).*(1-exp((-2*(x1-alpha1).*(x2-alpha2)
       )/(t2-t1)));
    iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(
       x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi
       *(t2-t1));
    beta1=5;
    beta2=5;
end
iz=@(x1,x2)izz(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
Y=integral2(iz,alpha1,beta1,alpha2,beta2);
end
```

```
function Y = laplace_3_finite_put_upper(xi,x,alpha0,
    alpha1,alpha2,alpha3,...
    beta0, beta1,beta2,beta3,t0,t1,t2,t3,a1,a2,b1,b2
       ,n );
h=@(x1)izz_laplace_3_finite_put_upper(xi,x,alpha0,
   alpha1,alpha2,alpha3,beta0,...
```

```
    beta1,beta2,beta3,t0,t1,t2,t3,a1,a2,b1,b2,n,x1);
Y=integral(h,alpha1,beta1);
end
```

```
function Y =izz_laplace_3_finite_put_upper(xi,x,alpha0, alpha1,alpha2,alpha3,beta0,...
    beta1,beta2,beta3,t0,t1,t2,t3,a1,a2,b1,b2,n,x1);
izz=@(x2)(exp(-xi*t2)).* laplace_1_finite_put_upper(xi,t3-t2,a1,alpha2-x2,b1,beta2-x2,n );
iz2=@(x2)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,t1-t0,n ))...
               .*(1-qq( x1,x2,alpha1,alpha2,beta1,beta2,t2-t1,n ));
iz3=@(x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
iz=@(x2)izz(x2).*iz2(x2).*iz3(x2);
Y=integral(iz,alpha2,beta2,'ArrayValued',true);
end
```

```
function Y = laplace_4_finite_put_lower_inf(xi,x,alpha0, alpha1,alpha2,alpha3,alpha4,beta0, beta1,beta2,beta3,beta4,t0,t1,t2,t3,t4,a1,a2,b1,b2,flag,n );
% related to the option value at the lower boundary and the fourth interval

if flag<=0
     izz=@(x1,x2,x3)(exp(-xi*t3)).*laplace_1_finite_put_lower_inf(xi,t4-t3,a1,alpha3-x3,b1,beta3-x3,flag,n );
     iz2=@(x1,x2,x3)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,t1-t0,n ))...
               .*(1-qq( x1,x2,alpha1,alpha2,beta1,beta2,t2-t1,n ))...
               .*(1-qq( x2,x3,alpha2,alpha3,beta2,beta3,t3-t2,n ));
     iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2)
```

```
        .^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
elseif flag==1
    izz=@(x1,x2,x3)(exp(-xi*t3)).* laplace_1(xi,t4-t3,a1,alpha3-x3 );
    iz2=@(x1,x2,x3)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,t1-t0,n ))...
                .*(1-qq( x1,x2,alpha1,alpha2,beta1,beta2,t2-t1,n ))...
                .*(1-qq( x2,x3,alpha2,alpha3,beta2,beta3,t3-t2,n ));
    iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
elseif flag==2
       izz=@(x1,x2,x3)(exp(-xi*t3)).* laplace_1(xi,t4-t3,a1,alpha3-x3 );
    iz2=@(x1,x2,x3)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,t1-t0,n ))...
                .*(1-qq( x1,x2,alpha1,alpha2,beta1,beta2,t2-t1,n ))...
        .*(1-exp((-2*(x2-alpha2).*(x3-alpha3))/(t3-t2)));
    iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
    beta3=5;
elseif flag==3
     izz=@(x1,x2,x3)(exp(-xi*t3)).* laplace_1(xi,t4-t3,a1,alpha3-x3 );
    iz2=@(x1,x2,x3)(1-qq( x,x1,alpha0,alpha1,beta0,beta1,t1-t0,n ))...
        .*(1-exp((-2*(x1-alpha1).*(x2-alpha2))/(t2-t1))).*(1-exp((-2*(x2-alpha2).*(x3-alpha3))/(t3-t2)));
    iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp
```

```
        (-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
    beta2=5;
    beta3=5;
else
    izz=@(x1,x2,x3)(exp(-xi*t3)).* laplace_1(xi,t4-t3,a1,alpha3-x3 );
    iz2=@(x1,x2,x3)(1-exp((-2*(x-alpha0)*(x1-alpha1))/(t1-t0))).*(1-exp((-2*(x1-alpha1).*(x2-alpha2))/(t2-t1))).*(1-exp((-2*(x2-alpha2).*(x3-alpha3))/(t3-t2)));
    iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
    beta1=5;
    beta2=5;
    beta3=5;
end
Y=integral3(iz,alpha1,beta1,alpha2,beta2,alpha3,beta3);
end
```

```
function L = laplace_1_finite_put_lower_inf(xi,t,a1,a2,b1,b2,flag,n );
%derive the Laplace transform at the level xi of a Brownian motion to the
%lower of the linear boundaries defined by the points {a1,a2} and {b1,b2}
%at the time grid t
if flag<=0
    L=exp(a2.*(sqrt(a1.^2+2*xi)-a1)).*P_1_l(sqrt(a1.^2+2*xi),a2,b1+sqrt(a1.^2+2*xi)-a1,b2, t,n );
else
    L=laplace_1(xi,t,a1,a2 );
end
```

```
L(isnan(L))=0;
end
```

```
function Y = laplace_4_finite_put_upper(xi,x,alpha0,
    alpha1,alpha2,alpha3,alpha4,beta0, beta1,beta2,
    beta3,beta4,t0,t1,t2,t3,t4,a1,a2,b1,b2,n );
% related to the option value at the upper boundary
    and the fourth interval

izz=@(x1,x2,x3)(exp(-xi*t3)).*
    laplace_1_finite_put_upper(xi,t4-t3,a1,alpha3-x3,
    b1,beta3-x3,n );
iz2=@(x1,x2,x3)(1-qq( x,x1,alpha0,alpha1,beta0,beta1
    ,t1-t0,n ))...
                .*(1-qq( x1,x2,alpha1,alpha2,beta1,
                    beta2,t2-t1,n ))...
                .*(1-qq( x2,x3,alpha2,alpha3,beta2,
                    beta3,t3-t2,n ));
iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2
    -x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2))
    )/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
iz=@(x1,x2,x3)izz(x1,x2,x3).*iz2(x1,x2,x3).*iz3(x1,
    x2,x3);
Y=integral3(iz,alpha1,beta1,alpha2,beta2,alpha3,
    beta3);
end
```

14.2.2 Deriving the boundaries

We present now the main codes for deriving the optimal boundaries. For the perpetual case, we present the solver for the optimal boundary equation as well as for the option price.

```
function [ t,alpha,beta,flag] =boundary_put_game( r,
    lambda,sigma,k,eta,T,N,n);
% derive the optimal boundaries and the time grid

[ A,B,~,~ ] = game_eq_putt(r,sigma,lambda,k,eta,k );
```

```
T2 = find_T2(r,lambda,sigma,k,eta);
if T-T2<10^(-10)
   N=N-1;
   t=(0:T/N:T);
   flag=N;
   [~,alpha]= bondary_put_points_1_terminal_new( r,
      lambda,sigma,k,t,N);
   beta=k*ones(1,N+1);
elseif B==k
    [ t , flag] = grid_11( T-T2,T,N );
    alpha= bondary_put_points_game_new_upper_K( r,
       lambda,sigma,k,eta,t,N,flag,n);
    beta=k*ones(1,N+1);
else
     [ t , flag] = grid_11( T-T2,T,N );
     [alpha,beta] = bondary_put_points_game_new( r,
        lambda,sigma,k,eta,t,N,flag,n);
end
```

```
function T2 = find_T2(r,lambda,sigma,k,eta);
%derive the first critical time-value

st=0.001;
f=@(t)(am_cir(k,k,r+lambda,t,st,sigma,0,lambda)-eta)
   ^2;
ff=@(t)(am_cir(k,k,r+lambda,t,st,sigma,0,lambda)-eta
   );
opt=optimset('MaxIter',10^(200),'TolX',10^(-200),'
   TolFun',10^(-200),'MaxFunEvals',10^(200));
T2=fminbnd(f,0,10,opt);
```

```
function [alpha] =
   bondary_put_points_game_new_upper_K( r,lambda,
   sigma,k,eta,t,N,flag,n);
% deriving the lower boundary if the perpetual upper
    one is the strike

if N==flag
```

```
    [ ~,alpha] = bondary_put_points_1_terminal( r,
       lambda,sigma,k,t,N);
else
     [a1]= bondary_put_points_game_new_upper_K( r,
        lambda,sigma,k,eta,t(2:end)-t(2),N-1,flag,n);
    a2=find_lower_game_a_put_upper_K(r,lambda,sigma,
        k,eta,a1,t,N,flag,n);
    alpha=[a2,a1];
end
end
```

The writer's perpetual boundary equal to the strike

We present in this section the necessary codes if the perpetual writer's boundary is equal to the strike.

```
function [V] = find_lower_game_a_put_upper_K(r,
   lambda,sigma,k,eta,alpha,T,N,flag,n);
% deriving the lower boundary if the upper one is
   the strike

A=0;
B=k;
while B-A>0.0001
    xstar=(A+B)/2;
    b_fstar= find_aa_max_put_game_upper_K(r,lambda,
       sigma,k,eta,alpha,T,N,xstar,flag,n);
     if b_fstar
        B=xstar;
    else
        A=xstar;
    end
end
V=(A+B)/2;
end
```

```
function izhod= find_aa_max_put_game_upper_K(r,
   lambda,sigma,k,eta,alpha,T,N,x,flag,n);
```

```
% derive an indicator for the position of x w.r.t. the lower optimal boundary

beta=k*ones(size(alpha));
cost=@(a)-premium_put_1_game_inf_terminal(r,lambda,sigma,k,eta,x,T,[exp(a)*x,alpha],[k,beta],N,flag,n);
if cost(-10^(-4))<x-k
    izhod=1;
else
    izhod=0;
end
end
```

The writer's perpetual boundary below the strike

We present now the necessary codes if the perpetual writer's boundary is below the strike.

```
function [alpha,beta] = bondary_put_points_game_new(r,lambda,sigma,k,eta,t,N,flag,n);
% deriving the lower boundary if the perpetual upper one is below the strike

if N==flag
   [ ~,alpha] = bondary_put_points_1_terminal_new( r,lambda,sigma,k,t,N);
   beta=ones(size(alpha))*k;
else
   [a1,b1]= bondary_put_points_game_new( r,lambda,sigma,k,eta,t(2:end)-t(2),N-1,flag,n);
   a2=find_lower_game_a_put_1(r,lambda,sigma,k,eta,a1,b1,t,N,flag,n);
   b2=find_upper_game_b_put_1(r,lambda,sigma,k,eta,a1,b1,t,N,flag,n);
   alpha=[a2,a1];
   beta=[b2,b1];
end
```

```
end
```

```
function V = find_lower_game_a_put_1(r,lambda,sigma,k,eta,alpha,beta,T,N,flag,n);
% deriving the holder's boundary

A=0;
B=k;
while B-A>0.0001
    xstar=(A+B)/2;
    b_fstar= find_aa_max_put_game(r,lambda,sigma,k,eta,alpha,beta,T,N,xstar,flag,n);
     if b_fstar%<-10^(-4)
        B=xstar;
    else
        A=xstar;
    end
end
V=(A+B)/2;
end
```

```
function izhod= find_aa_max_put_game(r,lambda,sigma,k,eta,alpha,beta,T,N,x,flag,n);
% derive an indicator for the position of x w.r.t. the lower optimal boundary

cost=@(a)-f_b_put_game(a,r,lambda,sigma,k,eta,alpha,beta,T,N,x,flag,n);
if cost(-0.0001)<x-k
    I=1;
else
    I=0;
end
end
```

```
function [V,Y] = f_b_put_game(a,r,lambda,sigma,k,eta,alpha,beta,T,N,x,flag ,n);
%deriving the writer's boundary if the holder's one is known
```

```

h=@(b)premium_put_1_game_inf_terminal(r,lambda,sigma,k,eta,x,T,[exp(a)*x,alpha],[exp(b)*x,beta],N,flag,n);
[Y,V]=fminbnd(h,0,log(k/x));
end
```

```
function V = find_upper_game_b_put_1(r,lambda,sigma,k,eta,alpha,beta,T,N,flag,n);
% deriving the writer's boundary

A=0;
B=k;
while B-A>0.0001
    xstar=(A+B)/2;
    b_fstar= find_bb_max_put_game(r,lambda,sigma,k,eta,alpha,beta,T,N,xstar,flag,n);
    if b_fstar
        A=xstar;
    else
        B=xstar;
    end
end
V=(A+B)/2;
if abs(izhod-k)<0.0001
    V=k;
end
end
```

```
function I= find_aa_max_put_game(r,lambda,sigma,k,eta,alpha,beta,T,N,x,flag,n);
% derive an indicator for the position of x w.r.t. the lower optimal boundary

cost=@(a)-f_b_put_game(a,r,lambda,sigma,k,eta,alpha,beta,T,N,x,flag,n);
if cost(-0.0001)<x-k
    I=1;
```

```
else
    I=0;
end
end
```

```
function [value ] = f_a_put_game(b,r,lambda,sigma,k,
   eta,alpha,beta,T,N,x,flag,n );
%deriving the holder's boundary if the holder's one
   is known

h=@(a)premium_put_1_game_inf_terminal(r,lambda,sigma
   ,k,eta,x,T,[exp(a)*x,alpha],[exp(b)*x,beta],N,
   flag,n);
hh=@(a)(-h(a));
V=fminbnd(hh,-2,0);
value=h(V);
end
```

14.2.3 Monte Carlo pricing method

We present in this section MATLAB code implementing the Monte Carlo pricing algorithm presented in Section 13.5.

```
function [pr] =premium_game_put_MC(r,lambda,sigma,k,
   eta,x,T,nnn,Nh);
% Nh     -- the number of the time nodes
% nnn    -- the nuber of the sum in the infinite sum

H=200000; % H      -- the number of the MC
   simulations
T2 = find_T2(r,lambda,sigma,k,eta);
if T>T2
    [ t  , flag] = grid_11( T-T2,T,Nh );
else
    t=0:T/Nh:T;
    flag=Nh;
end
for ns=1:Nh
```

```
    br=3;
    [~,aaa,bbb] =boundary_put_game( r,lambda,sigma,k
       ,eta,t(end)-t(ns),br,nnn);
    aa(ns)=aaa(1);
    bb(ns)=bbb(1);
end
aa=[aa,min((r+lambda)/lambda,1)*k]
bb=[bb,k]
if x<=aa(1)
    pr=k-x;
elseif x>=bb(1)
    pr=k-x+eta;
else
    kk_l=(log(k/x)-(r-sigma^2/2)*T)/sigma;
    c_l=-(log(aa/x)-(r-sigma^2/2)*t)/sigma;
    d_l=-(log(bb/x)-(r-sigma^2/2)*t)/sigma;
    [ a1_l,a2_l]= linnear(t(1:end-1),c_l(1:end-1),t
       (2:end),c_l(2:end));
    [ b1_l,b2_l]= linnear(t(1:end-1),d_l(1:end-1),t
       (2:end),d_l(2:end));
    alfa_ac_l=sigma^2/2+a1_l*sigma+lambda;
    alfa_bd_l=sigma^2/2+b1_l*sigma+lambda;
    beta_ac_l=exp(-sigma*a2_l);
    beta_bd_l=exp(-sigma*b2_l);
    kk_u=-(log(k/x)-(r-sigma^2/2)*T)/sigma;
    c_u=(log(aa/x)-(r-sigma^2/2)*t)/sigma;
    d_u=(log(bb/x)-(r-sigma^2/2)*t)/sigma;
    [ a1_u,a2_u]= linnear(t(1:end-1),c_u(1:end-1),t
       (2:end),c_u(2:end));
    [ b1_u,b2_u]= linnear(t(1:end-1),d_u(1:end-1),t
       (2:end),d_u(2:end));
    alfa_ac_u=sigma^2/2-a1_u*sigma+lambda;
    alfa_bd_u=sigma^2/2-b1_u*sigma+lambda;
    beta_ac_u=exp(sigma*a2_u);
    beta_bd_u=exp(sigma*b2_u);

    uu=normrnd(0,1,Nh-1,H);
```

```
    [l1_lower,l2_lower]=prob_laplace_num_game_upper(
       t,d_l,c_l,r+lambda,alfa_ac_l,beta_ac_l,flag,
       nnn,uu);
    [l1_upper,l2_upper]=
       prob_laplace_num_game_upper_cut(t,c_u,d_u,r+
       lambda,alfa_bd_u,beta_bd_u,flag,nnn,uu);
    if r<0
        [tr1,tr2]=prob_terminal_num_many(sigma,t,c_u
           ,d_u,kk_l,flag,nnn,uu);
        ttr1=exp(-(lambda+r)*T)*tr1;
        ttr2=exp(-(lambda+sigma^2/2)*T)*tr2;
    else
        ttr1=0;
        ttr2=0;
    end
    pr=k*(ttr1+l1_lower)-x*(ttr2+l2_lower)+(k+eta)*
       l1_upper-x*l2_upper;
end
end
```

```
function [pr1,pr2] =prob_laplace_num_game_upper(t,c,
   d,alfa1,alfa2,beta,flag,nnn,uu);
% MC simulations

n=length(alfa2);
hh=size(uu);
H=hh(2);
m=(1:n);
u=uu(1:n-1,:);
Dsq=diag(sqrt(t(2:n)-t(1:n-1)));
M=tril(ones(n-1));
x=[zeros(H,1),(M*Dsq*u )'];
[b1,b2]=linnear(t(m),d(m),t(m+1),d(m+1));

fl =(max(0,flag-n+m)>0);
q1=((ones(H,1)*exp(-alfa1.*t(m)))).*
   laplace_1_finite_put_upper_inf_num(alfa1*ones(1,n
   ),t(m+1)-t(m),a1,ones(H,1)*c(m)-x(:,m),b1,ones(H
```

```
,1)*d(m)-x(:,m),fl,nnn );
q2=((ones(H,1)*exp(-alfa2(m).*t(m)))).*laplace_1_finite_put_upper_inf_num(alfa2(m),t(m+1)-t(m),a1,ones(H,1)*c(m)-x(:,m),b1,ones(H,1)*d(m)-x(:,m),fl,nnn );
vv1=((x(:,2:n)<(d(2:n)'*ones(1,H))')&(x(:,2:n)>(c(2:n)'*ones(1,H))')).*...
                   (1-qq_num( (x(:,1:n-1))',(x(:,2:n))',c(1:n-1)'*ones(1,H),c(2:n)'*ones(1,H),d(1:n-1)'*ones(1,H),d(2:n)'*ones(1,H),(t(2:n)-t(1:n-1))'*ones(1,H),nnn ))';
vv2=(x(:,2:n)<( d(2:n)'*ones(1,H))').*(1-exp(-2*((d(1:n-1)'*ones(1,H))'-x(:,1:n-1)).*((d(2:n)'*ones(1,H))'-x(:,2:n))./(((t(2:n)-t(1:n-1))'*ones(1,H))')));
ffl=ones(H,1)*fl(1:end-1);
vv=vv1.*not(ffl)+vv2.*ffl;
z=[ones(H,1),vv];
zz1=cumprod(z')'.*q1;
zz2=cumprod(z')'.*q2;
pr1=sum(sum(zz1))/H;
pr2=sum(beta.*sum(zz2))/H;
```

```
function [pr1,pr2] =prob_laplace_num_game_upper_cut(t,c,d,alfa1,alfa2,beta,flag,nnn,uu);
% MC simulations

n=length(alfa2);
hh=size(uu);
H=hh(2);
m=(1:n);
u=uu(1:n-1,:);
Dsq=diag(sqrt(t(2:n)-t(1:n-1)));
M=tril(ones(n-1));
x=[zeros(H,1),(M*Dsq*u )'];
[a1,a2]=linnear(t(m),c(m),t(m+1),c(m+1));
[b1,b2]=linnear(t(m),d(m),t(m+1),d(m+1));
```

```

fl =(max(0,flag-n+m)>0);
q1=((ones(H,1)*exp(-alfa1.*t(m)))).*laplace_1_finite_put_upper_inf_num(alfa1*ones(1,n),t(m+1)-t(m),a1,ones(H,1)*c(m)-x(:,m),b1,ones(H,1)*d(m)-x(:,m),fl,nnn );
q2=((ones(H,1)*exp(-alfa2(m).*t(m)))).*laplace_1_finite_put_upper_inf_num(alfa2(m),t(m+1)-t(m),a1,ones(H,1)*c(m)-x(:,m),b1,ones(H,1)*d(m)-x(:,m),fl,nnn );
vv1=((x(:,2:n)<(d(2:n)'*ones(1,H))')&(x(:,2:n)>(c(2:n)'*ones(1,H))')).*...
                  (1-qq_num( (x(:,1:n-1))',(x(:,2:n))',c(1:n-1)'*ones(1,H),c(2:n)'*ones(1,H),d(1:n-1)'*ones(1,H),d(2:n)'*ones(1,H),(t(2:n)-t(1:n-1))'*ones(1,H),nnn ))';
vv2=(x(:,2:n)>( c(2:n)'*ones(1,H))').*(1-exp(-2*((c(1:n-1)'*ones(1,H))'-x(:,1:n-1)).*((c(2:n)'*ones(1,H))'-x(:,2:n))./(((t(2:n)-t(1:n-1))'*ones(1,H))')));
ffl=ones(H,1)*fl(1:end-1);
vv=vv1.*not(ffl)+vv2.*ffl;
z=[ones(H,1),vv];
zz1=cumprod(z')'.*q1;
zz2=cumprod(z')'.*q2;
pr1=sum(sum(zz1(:,1:end-flag)))/H;
pr2=sum(beta(:,1:end-flag).*sum(zz2(:,1:end-flag)))/H;
end
```

```
function [pr1,pr2] =prob_terminal_num_many(sigma,t,c,d,kk,flag,nnn,uu);
% MC simulations

n=length(c)-1;
hh=size(uu);
H=hh(2);
```

```
m=n;
u=uu(1:m-1,:);
Dsq=diag(sqrt(t(2:m)-t(1:m-1)));
M=tril(ones(m-1));
x=[zeros(H,1),(M*Dsq*u )'];
fl =(max(0,flag-n+(1:n))>0);
q1=terminal1_1_new(t(end)-t(end-1),0,(c(m)'*ones(1,H
    )'-x(:,m)),(c(m+1)'*ones(1,H))'-x(:,m),kk-x(:,m)
    ,(c(m+1)'*ones(1,H))'-x(:,m) );
q2=(exp(sigma*x(:,m))).*terminal1_2_new(sigma,t(end)
    -t(end-1),0,(c(m)'*ones(1,H)'-x(:,m)),(c(m+1)'*
    ones(1,H))'-x(:,m),kk-x(:,m),(c(m+1)'*ones(1,H))
    '-x(:,m) );
vv1=((x(:,2:n)<(d(2:n)'*ones(1,H))')&(x(:,2:n)>(c(2:
    n)'*ones(1,H))')).*...
        (1-qq_num( (x(:,1:n-1))',(x(:,2:n))',c(1:n-1)
            '*ones(1,H),c(2:n)'*ones(1,H),d(1:n-1)'*
            ones(1,H),d(2:n)'*ones(1,H),(t(2:n)-t(1:n
            -1))'*ones(1,H),nnn ))';
vv2=(x(:,2:n)>( c(2:n)'*ones(1,H))').*(1-exp(-2*((c
    (1:n-1)'*ones(1,H))'-x(:,1:n-1)).*((c(2:n)'*ones
    (1,H))'-x(:,2:n))./(((t(2:n)-t(1:n-1))'*ones(1,H)
    )')));
ffl=ones(H,1)*fl(1:end-1);
vv=vv1.*not(ffl)+vv2.*ffl;
zz=[vv,ones(H,1)];
h1=prod(zz').*q1';
h2=prod(zz').*q2';
pp2(m)=sum(h2)/H;
pr1=sum(pp1);
pr2=sum(pp2);
end
```

14.2.4 Comments

The main code `premium_put_1_game_inf_terminal.m` presented in Section 14.2.1 prices a financial instrument that pays the usual option payoff if the asset exits the piece-wise linear strip from the lower boundary and the penalized one if the exit is from above. Some last upper boundaries may be missed. Note that the parts of the piece-wise linear functions are limited to four since the multiple integration is time-consuming. The perpetual boundaries as well as the option price are obtained via the code `game_eq_putt.m`. The output `flag` indicates the case of Theorem 10.1 that holds. The provided in Section 13.4 algorithm for approximating the boundaries is implemented by the codes of Section 14.2.2 – the main of them is `boundary_put_game.m`. If the writer's perpetual boundary is the strike, then it is always the strike after the first critical moment – both cases are considered separately. Note that differently from the time grid for the usual American options, the grid for the cancellable ones is not arbitrary, but it is obtained by the code `grid_11.m` – see the beginning of Section 13.4. The work of our fast approach, provided as 13.1, can be seen below:

```
>>  r=-0.01;
lambda=0.03;
sigma=.3;
k=20;
S_0=13;
T=3;
eta=0.1;
>> [ t,holder,writer,flag] =boundary_put_game( r,
    lambda,sigma,k,.1,T,3,5)
price=premium_put_1_game_inf_terminal(r,lambda,sigma
    ,k,eta,S_0,t,holder,writer,3,flag,5)

t =

          0    1.4994    2.9989    3.0000


holder =

   10.5209   10.7290   13.2493   13.3333


writer =
```

```

   17.5653   17.8482   20.0000   20.0000


flag =

     1


price =

    7.0336
```

The Monte Carlo pricing method from Section 13.5 can be applied to a denser grid. It is implemented in the codes of Section 14.2.3. In the first part of the code `premium_game_put_MC.m`, we derive the necessary boundary values running several times the code `boundary_put_game.m` and then we follow the Monte Carlo approach.

14.3 Strangle derivatives

We present now the MATLAB codes for the American strangles.

14.3.1 The main code

We first present the main MATLAB code for pricing strangle-style instruments.

```
function [pr] =premium_stradle(r,lambda,sigma,C1,C2,
   k1,k2,x,t,alpha,beta,N,n);
% Derive the price of the strangle derivative
   maturating at the boundaries alpha and beta
   evaluated at the time-grid t

d0=(C1*k1+C2*k2)/(C1+C2);
D0=(log(d0/x)-(r-sigma^2/2)*t(end))/sigma;
if N==1
    t0=t(1); t1=t(2);
```

```
    alpha0=alpha(1); alpha1=alpha(2);
    beta0=beta(1); beta1=beta(2);
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    X=0;
    [ a1,a2]= linnear( t0,A0,t1,A1 );
    [ b1,b2]= linnear( t0,B0,t1,B1 );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2- b1*sigma+lambda;
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda,
       t1,a1,a2,b1,b2,n);
    l2_lower(1)=exp(sigma*a2)*
       laplace_1_finite_put_lower(mu_lower, t1,a1,a2
       ,b1,b2,n );
    pr_lower=C1*(k1*l1_lower(1)-x*l2_lower(1));
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,
       t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*
       laplace_1_finite_put_upper(mu_upper, t1,a1,a2
       ,b1,b2,n );
    pr_upper=C2*(-k2*l1_upper(1)+x*l2_upper(1));
if alpha(end)<beta(end)
    if alpha(end)==d0
        z1=D0;
        z2=B1;
        [term1]=exp(-(r+lambda)*t1)*
           terminal1_2_stradle(0,t1-t0,X,A0,A1,B0,B1
           ,z1,z2,n);
        [term2]=exp(-(lambda+sigma^2/2)*t1)*
           terminal1_2_stradle(sigma,t1-t0,X,A0,A1,
           B0,B1,z1,z2,n);
        ttt=C2*(-k2*term1+x*term2);
    elseif beta(end)==d0
        z1=A1;
        z2=D0;
```

```
        [term1]=exp(-(r+lambda)*t1)*
           terminal1_2_stradle(0,t1-t0,X,A0,A1,B0,B1
           ,z1,z2,n);
        [term2]=exp(-(lambda+sigma^2/2)*t1)*
           terminal1_2_stradle(sigma,t1-t0,X,A0,A1,
           B0,B1,z1,z2,n);
        ttt=C1*(k1*term1-x*term2);
    else
        z1=A1;
        z2=D0;
        [term1]=exp(-(r+lambda)*t1)*
           terminal1_2_stradle(0,t1-t0,X,A0,A1,B0,B1
           ,z1,z2,n);
        [term2]=exp(-(lambda+sigma^2/2)*t1)*
           terminal1_2_stradle(sigma,t1-t0,X,A0,A1,
           B0,B1,z1,z2,n);
        ttt1=C1*(k1*term1-x*term2);
        z1=D0;
        z2=B1;
        [term1]=exp(-(r+lambda)*t1)*
           terminal1_2_stradle(0,t1-t0,X,A0,A1,B0,B1
           ,z1,z2,n);
        [term2]=exp(-(lambda+sigma^2/2)*t1)*
           terminal1_2_stradle(sigma,t1-t0,X,A0,A1,
           B0,B1,z1,z2,n);
        ttt2=C2*(-k2*term1+x*term2);
        ttt=ttt1+ttt2;
    end
else
    ttt=0;
end
    pr=pr_lower+pr_upper+ttt;
elseif N==2
    t0=t(1); t1=t(2); t2=t(3);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha
       (3);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3);
```

```
[ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma );
[ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );
mu_lower=sigma^2/2-a1*sigma+lambda;
mu_upper=sigma^2/2-b1*sigma+lambda;
l1_lower(1)=laplace_1_finite_put_lower(r+lambda, t1,a1,a2,b1,b2,n );
l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower(mu_lower, t1,a1,a2,b1,b2,n );
I1_lower=C1*(k1*l1_lower(1)-x*l2_lower(1));
l1_upper(1)=laplace_1_finite_put_upper(r+lambda, t1,a1,a2,b1,b2,n );
l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
I1_upper=C2*(-k2*l1_upper(1)+x*l2_upper(1));
% ----------------------------------------------------------
[ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
[ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
mu_lower=sigma^2/2-a1*sigma+lambda;
mu_upper=sigma^2/2-b1*sigma+lambda;
A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
X=0;
```

```
    if alpha(end)<beta(end)
        if alpha(end)==d0
            z1=D0;
            z2=B2;
            [term1,term2]=terminal2_2_stradle(sigma,X,A0, A1,A2,B0, B1,B2,a1,a2,b1,b2,z1,z2,t0,t1,t2,n );
            term1=exp(-(lambda+r)*t2)*term1;
            term2=exp(-(lambda+sigma^2/2)*t2)*term2;
            ttt=C2*(-k2*term1+x*term2);
         elseif beta(end)==d0
            z1=A2;
            z2=D0;
            [term1,term2]=terminal2_2_stradle(sigma,X,A0, A1,A2,B0, B1,B2,a1,a2,b1,b2,z1,z2,t0,t1,t2,n );
            term1=exp(-(lambda+r)*t2)*term1;
            term2=exp(-(lambda+sigma^2/2)*t2)*term2;
            ttt=C1*(k1*term1-x*term2);
        else
            z1=A2;
            z2=D0;
            [term1,term2]=terminal2_2_stradle(sigma,X,A0, A1,A2,B0, B1,B2,a1,a2,b1,b2,z1,z2,t0,t1,t2,n );
            term1=exp(-(lambda+r)*t2)*term1;
            term2=exp(-(lambda+sigma^2/2)*t2)*term2;
            tt1=C1*(k1*term1-x*term2);
            z1=D0;
            z2=B2;
            [term1,term2]=terminal2_2_stradle(sigma,X,A0, A1,A2,B0, B1,B2,a1,a2,b1,b2,z1,z2,t0,t1,t2,n );
            term1=exp(-(lambda+r)*t2)*term1;
```

```
            term2=exp(-(lambda+sigma^2/2)*t2)*term2;
            tt2=C2*(-k2*term1+x*term2);
            ttt=tt1+tt2;
        end
    else
        ttt=0;
    end
    l1_lower(2)=laplace_2_finite_put_lower(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_lower=C1*(k1*l1_lower(2)-x*l2_lower(2));
    l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_upper=C2*(-k2*l1_upper(2)+x*l2_upper(2));
    pr=I1_lower+I2_lower+I1_upper+I2_upper+ttt;
elseif N==3
    t0=t(1); t1=t(2); t2=t(3); t3=t(4);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha(3); alpha3=alpha(4);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3); beta3=beta(4);
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
```

```
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda, t1,a1,a2,b1,b2,n );
    l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower(mu_lower, t1,a1,a2,b1,b2,n );
    I1_lower=C1*(k1*l1_lower(1)-x*l2_lower(1));
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda, t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
    I1_upper=C2*(-k2*l1_upper(1)+x*l2_upper(1));
%  ------------------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1_lower(2)=laplace_2_finite_put_lower(r+lambda, X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower(mu_lower,X,A0, A1, A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_lower=C1*(k1*l1_lower(2)-x*l2_lower(2));
```

```
    l1_upper(2)=laplace_2_finite_put_upper(r+lambda,
       X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n
       );
    l2_upper(2)=exp(sigma*b2)*
       laplace_2_finite_put_upper(mu_upper,X,A0, A1,
       A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_upper=C2*(-k2*l1_upper(2)+x*l2_upper(2));
  %
     ------------------------------------------------------------
    [ a1,a2]= linnear( t2,(log(alpha2/x)-(r-sigma
       ^2/2)*t2)/sigma,t3,(log(alpha3/x)-(r-sigma
       ^2/2)*t3)/sigma );
    [ b1,b2]= linnear( t2,(log(beta2/x)-(r-sigma
       ^2/2)*t2)/sigma,t3,(log(beta3/x)-(r-sigma
       ^2/2)*t3)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    if alpha(end)<beta(end)
        if alpha(end)==d0
            z1=D0;
            z2=B3;
            [term1,term2]=terminal3_stradle(sigma,X
               ,A0, A1,A2,A3,B0, B1,B2,B3,a1,a2,b1,
               b2,z1,z2,t0,t1,t2,t3,n );
            term1=exp(-(lambda+r)*t3)*term1;
            term2=exp(-(lambda+sigma^2/2)*t3)*term2
               ;
            ttt=C2*(-k2*term1+x*term2);
```

```
            elseif beta(end)==d0
                z1=A3;
                z2=D0;
                [term1,term2]=terminal3_stradle(sigma,X,A0, A1,A2,A3,B0, B1,B2,B3,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,n );
                term1=exp(-(lambda+r)*t3)*term1;
                term2=exp(-(lambda+sigma^2/2)*t3)*term2;
                ttt=C1*(k1*term1-x*term2);
            else
                z1=A3;
                z2=D0;
                [term1,term2]=terminal3_stradle(sigma,X,A0, A1,A2,A3,B0, B1,B2,B3,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,n );
                term1=exp(-(lambda+r)*t3)*term1;
                term2=exp(-(lambda+sigma^2/2)*t3)*term2;
                tt1=C1*(k1*term1-x*term2);
                z1=D0;
                z2=B3;
                [term1,term2]=terminal3_stradle(sigma,X,A0, A1,A2,A3,B0, B1,B2,B3,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,n );
                term1=exp(-(lambda+r)*t3)*term1;
                term2=exp(-(lambda+sigma^2/2)*t3)*term2;
                tt2=C2*(-k2*term1+x*term2);
                ttt=tt1+tt2;
            end
        else
            ttt=0;
        end
         l1_lower(3)=laplace_3_finite_put_lower(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
```

```
      l2_lower(3)=exp(sigma*a2)*laplace_3_finite_put_lower(mu_lower,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
      I3_lower=C1*(k1*l1_lower(3)-x*l2_lower(3));
      l1_upper(3)=laplace_3_finite_put_upper(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
      l2_upper(3)=exp(sigma*b2)*laplace_3_finite_put_upper(mu_upper,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
      I3_upper=C2*(-k2*l1_upper(3)+x*l2_upper(3));
      pr=I1_lower+I2_lower+I3_lower+I1_upper+I2_upper+I3_upper+ttt;
elseif N==4
    t0=t(1); t1=t(2); t2=t(3); t3=t(4); t4=t(5);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha(3); alpha3=alpha(4); alpha4=alpha(5);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3); beta3=beta(4); beta4=beta(5);
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda,t1,a1,a2,b1,b2,n );
    l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower(mu_lower, t1,a1,a2,b1,b2,n );
    I1_lower=C1*(k1*l1_lower(1)-x*l2_lower(1));
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
```

```
    l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
    I1_upper=C2*(-k2*l1_upper(1)+x*l2_upper(1));
    % ------------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1_lower(2)=laplace_2_finite_put_lower(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_lower=C1*(k1*l1_lower(2)-x*l2_lower(2));
    l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_upper=C2*(-k2*l1_upper(2)+x*l2_upper(2));
   % -------------------------------------------------------------------
```

```
    [ a1,a2]= linnear( t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma );
    [ b1,b2]= linnear( t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(beta3/x)-(r-sigma^2/2)*t3)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    l1_lower(3)=laplace_3_finite_put_lower(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
    l2_lower(3)=exp(sigma*a2)*laplace_3_finite_put_lower(mu_lower,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n);
    I3_lower=C1*(k1*l1_lower(3)-x*l2_lower(3));
    l1_upper(3)=laplace_3_finite_put_upper(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n );
    l2_upper(3)=exp(sigma*b2)*laplace_3_finite_put_upper(mu_upper,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n);
    I3_upper=C2*(-k2*l1_upper(3)+x*l2_upper(3));
%-------------------------------------------------------------
```

```
    [ a1,a2]= linnear( t3,(log(alpha3/x)-(r-sigma
       ^2/2)*t3)/sigma,t4,(log(alpha4/x)-(r-sigma
       ^2/2)*t4)/sigma );
    [ b1,b2]= linnear( t3,(log(beta3/x)-(r-sigma
       ^2/2)*t3)/sigma,t4,(log(beta4/x)-(r-sigma
       ^2/2)*t4)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    A4=(log(alpha4/x)-(r-sigma^2/2)*t4)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    B4=(log(beta4/x)-(r-sigma^2/2)*t4)/sigma;
    X=0;
if alpha(end)<beta(end)
        if alpha(end)==d0
             z1=D0;
             z2=B4;
             [term1,term2]=terminal4_stradle(sigma,X
                ,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,a1,
                a2,b1,b2,z1,z2,t0,t1,t2,t3,t4,n );
             term1=exp(-(lambda+r)*t4)*term1;
             term2=exp(-(lambda+sigma^2/2)*t4)*term2
                ;
             ttt=c2*(-k2*term1+x*term2);
         elseif beta(end)==d0
             z1=A4;
             z2=D0;
             [term1,term2]=terminal4_stradle(sigma,X
                ,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,a1,
                a2,b1,b2,z1,z2,t0,t1,t2,t3,t4,n );
             term1=exp(-(lambda+r)*t4)*term1;
```

```
            term2=exp(-(lambda+sigma^2/2)*t4)*term2;
            ttt=C1*(k1*term1-x*term2);
        else
            z1=A4;
            z2=D0;
            [term1,term2]=terminal4_stradle(sigma,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,t4,n );
            term1=exp(-(lambda+r)*t4)*term1;
            term2=exp(-(lambda+sigma^2/2)*t4)*term2;
            tt1=C1*(k1*term1-x*term2);
            z1=D0;
            z2=B4;
            [term1,term2]=terminal4_stradle(sigma,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,t4,n );
            term1=exp(-(lambda+r)*t4)*term1;
            term2=exp(-(lambda+sigma^2/2)*t4)*term2;
            tt2=C2*(-k2*term1+x*term2);
            ttt=tt1+tt2;
        end
    else
        ttt=0;
    end
     l1_lower(4)=laplace_4_finite_put_lower(r+lambda,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,a2,b1,b2,n );
     l2_lower(4)=exp(sigma*a2)*laplace_4_finite_put_lower(mu_lower,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,a2,b1,b2,n );
     I4_lower=C1*(k1*l1_lower(4)-x*l2_lower(4));
     l1_upper(4)=laplace_4_finite_put_upper(r+lambda,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,a2,b1,b2,n );
```

```
      l2_upper(4)=exp(sigma*b2)*
         laplace_4_finite_put_upper(mu_upper,X,A0, A1
         ,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,
         a2,b1,b2,n );
      I4_upper=C2*(-k2*l1_upper(4)+x*l2_upper(4));
      pr=I1_lower+I2_lower+I3_lower+I4_lower +
         I1_upper+I2_upper+I3_upper+I4_upper+ttt;
end
end
```

14.3.2 Auxilary files

We provide now the additional necessary sources for the code `premium_stradle`.

```
function [term]=terminal1_2_stradle(theta,T,x0,a0,a1
   ,b0,b1,z1,z2,n );
% related to the option value at maturity

c0=b0-a0;  c1=b1-a1;
s0=1; s2=1; s4=1; s1=-1; s3=-1;
term=0;
lam_j0=0;
xi_j0=0;
term=s0*exp(lam_j0*theta+((lam_j0^2+2*xi_j0)/(2*T)))
   *(normcdf((z2-(theta*T+lam_j0))/(sqrt(T)))-
   normcdf((z1-(theta*T+lam_j0))/(sqrt(T))));
for j=1:n
   lam_j1=2*(j*c0+a0-x0);
   xi_j1=-2*(j*c0+a0-x0).*(j*c1+a1);
   lam_j2=2*j*c0;
   xi_j2=-2*j*(j*c0.*c1+c0.*a1-c1.*(a0-x0));
   lam_j3=-2*(j*c0-(b0-x0));
   xi_j3=-2*(j*c0-(b0-x0)).*(j*c1-b1);
   lam_j4=-2*j*c0;
   xi_j4=-2*j*(j*c0.*c1-c0.*b1+c1.*(b0-x0));
   L=s1*exp(lam_j1*theta+((lam_j1.^2+2*xi_j1)/(2*T))
      ).*(normcdf((z2-(theta*T+lam_j1))/(sqrt(T)))-
      normcdf((z1-(theta*T+lam_j1))/(sqrt(T)))...
```

```
    +s2*exp(lam_j2*theta+((lam_j2.^2+2*xi_j2)/(2*T))
       ).*(normcdf((z2-(theta*T+lam_j2))/(sqrt(T)))-
       normcdf((z1-(theta*T+lam_j2))/(sqrt(T))))...
    +s3*exp(lam_j3*theta+((lam_j3.^2+2*xi_j3)/(2*T))
       ).*(normcdf((z2-(theta*T+lam_j3))/(sqrt(T)))-
       normcdf((z1-(theta*T+lam_j3))/(sqrt(T))))...
    +s4*exp(lam_j4*theta+((lam_j4.^2+2*xi_j4)/(2*T))
       ).*(normcdf((z2-(theta*T+lam_j4))/(sqrt(T)))-
       normcdf((z1-(theta*T+lam_j4))/(sqrt(T))));
   L(isnan(L))=0;
   term=term+L;
end
term=exp(theta^2*T/2)*term;
end
```

```
function [Y1,Y2] = terminal2_2_stradle(sigma,x0,c0,
   c1,c2,b0,b1,b2,aa1,aa2,bb1,bb2,z1,z2,t0,t1,t2,n )
   ;
% related to the option value at maturity

izz1=@(x1)terminal1_2_stradle(0,t2-t1,0,c1-x1,c2-x1,
   b1-x1,b2-x1,z1-x1,z2-x1,n);
izz2=@(x1)exp(sigma*x1).*terminal1_2_stradle(sigma,
   t2-t1,0,c1-x1,c2-x1,b1-x1,b2-x1,z1-x1,z2-x1,n);
iz2=@(x1)(1-qq( x0,x1,c0,c1,b0,b1,t1-t0,n ));
iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi*(t1-
   t0));
iz=@(x1)izz1(x1).*iz2(x1).*iz3(x1);
iiz2=@(x1)izz2(x1).*iz2(x1).*iz3(x1);
Y1=integral(iz,c1,b1);
Y2=integral(iiz2,c1,b1);
end
```

```
function [Y1,Y2] = terminal3_stradle(sigma,x,c0,c1,
   c2,c3,b0,b1,b2,b3,...
    aa1,aa2,bb1,bb2,z1,z2,t0,t1,t2,t3,n );
% related to the option value at maturity

```

```
h1=@(x1)izz_terminal3_1_straddle(x,c0,c1,c2,c3,b0,b1,b2,b3,aa1,aa2,...
    bb1,bb2,z1,z2,t0,t1,t2,t3,n,x1);
h2=@(x1)izz_terminal3_2_straddle(sigma,x,c0,c1,c2,c3,b0,b1,b2,b3,aa1,...
    aa2,bb1,bb2,z1,z2,t0,t1,t2,t3,n,x1);
Y1=integral(h1,c1,b1);
Y2=integral(h2,c1,b1);
end
```

```
function Y =izz_terminal3_1_straddle(x,c0,c1,c2,c3,b0,b1,b2,b3,aa1,aa2,...
    bb1,bb2,z1,z2,t0,t1,t2,t3,n,x1);
   izz=@(x2) terminal1_2_stradle(0,t3-t2,0,c2-x2,c3-x2,b2-x2,b3-x2,z1-x2,z2-x2,n);
   iz2=@(x2)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
               .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ));
    iz3=@(x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))...
       /sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
iz=@(x2)izz(x2).*iz2(x2).*iz3(x2);
Y=integral(iz,c2,b2,'ArrayValued',true);
end
```

```
function Y =izz_terminal3_2_straddle(sigma,x,c0,c1,c2,c3,b0,b1,b2,...
    b3,aa1,aa2,bb1,bb2,z1,z2,t0,t1,t2,t3,n,x1);
   izz=@(x2)exp(sigma*x2).*terminal1_2_stradle(sigma,t3-t2,0,c2-x2,...
      c3-x2,b2-x2,b3-x2,z1-x2,z2-x2,n);;
   iz2=@(x2)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
               .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ));
    iz3=@(x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))...
       /sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
   iz=@(x2)izz(x2).*iz2(x2).*iz3(x2);
```

```
    Y=integral(iz,c2,b2,'ArrayValued',true);
end
```

```
function [Y1,Y2] =terminal4_game_mult(sigma,x,c0, c1
    ,c2,c3,c4,b0,b1,b2,b3,b4,aa1,aa2,bb1,bb2,z1,z2,t0
    ,t1,t2,t3,t4,n );
% related to the option value at maturity

izz1=@(x1,x2,x3)terminal1_2_stradle(0,t4-t3,0,c3-x3,
    c4-x3,b3-x3,b4-x3,z1-x3,z2-x3,n);
izz2=@(x1,x2,x3)exp(sigma*x3).*terminal1_2_stradle(
    sigma,t4-t3,0,c3-x3,c4-x3,b3-x3,b4-x3,z1-x3,z2-x3
    ,n);
iz2=@(x1,x2,x3)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
          .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ))...
          .*(1-qq( x2,x3,c2,c3,b2,b3,t3-t2,n ));
iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2
    -x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2))
    )/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
iz1=@(x1,x2,x3)izz1(x1,x2,x3).*iz2(x1,x2,x3).*iz3(x1
    ,x2,x3);
iz2=@(x1,x2,x3)izz2(x1,x2,x3).*iz2(x1,x2,x3).*iz3(x1
    ,x2,x3);
Y1=integral3(iz1,c1,b1,c2,b2,c3,b3);
Y2=integral3(iz2,c1,b1,c2,b2,c3,b3);
end
```

```
function [ A,B,price] = straddle_perpetual(r,sigma,
    lambda,k1,k2,c1,c2,x );
% perpetual strangles

fun=@(y)(y-func_a_straddle(func_b_straddle(y,r,sigma
    ,lambda,k1,k2,c1,c2),r,sigma,lambda,k1,k2,c1,c2))
    ^2;
ff=@(y) func_b_straddle(y,r,sigma,lambda,k1,k2,c1,c2
    );
opt=optimset('MaxIter',10^(200),'TolX',10^(-200),'
    TolFun',10^(-200),'MaxFunEvals',10^(200));
```

```
[A,C]=fminbnd(fun,0,1000,opt);
B=func_b_straddle(A,r,sigma,lambda,k1,k2,c1,c2);

mu=r/(sigma^2)-0.5;
c=sqrt(mu^2+2*(r+lambda)/(sigma^2));
q=c+mu;
a=log(A/x);
b=log(B/x);
price=straddle_perpetual_ab( c,mu,k1,k2,c1,c2,x,a,b
    );
end
```

14.3.3 Deriving the boundaries

```
function [alpha,beta,t] = bondary_stradle( r,lambda,
    sigma,c1,c2,k1,k2,t,N,n,nach_a,nach_b);
% optimal boundaries

if N==1
    d0=(c1*k1+c2*k2)/(c1+c2);
    a1=min([k1*(r+lambda)/lambda,k1,d0]);
    b1=max([k2*(r+lambda)/lambda,k2,d0]);
    alpha=find_lower_stradle(r,lambda,sigma,c1,c2,k1,
        k2,a1,b1,t,N,n,nach_a,a1,b1,nach_b);
    beta=find_upper_stradle(r,lambda,sigma,c1,c2,k1,
        k2,a1,b1,t,N,n,b1,nach_b,nach_a,a1);
    alpha=[alpha,a1];
    beta=[beta,b1];
else
    [aa1,bb1]= bondary_stradle( r,lambda,sigma,c1,c2,
        k1,k2,t(2:end)-t(2),N-1,n,nach_a,nach_b);
    a2=find_lower_stradle(r,lambda,sigma,c1,c2,k1,k2,
        aa1,bb1,t,N,n,nach_a,aa1(1),bb1(1),nach_b);
    b2=find_upper_stradle(r,lambda,sigma,c1,c2,k1,k2,
        aa1,bb1,t,N,n,bb1(1),nach_b,nach_a,aa1(1));
    alpha=[a2,aa1];
    beta=[b2,bb1];
```

```
end
end
```

```
function L= find_lower_stradle(r,lambda,sigma,c1,c2,
    k1,k2,alpha,beta,T,N,n,A,B,C,D);
% lower boundary

while B-A>0.0001
    xstar=(A+B)/2;
    b_fstar= find_aa_stradle(r,lambda,sigma,c1,c2,k1
        ,k2,alpha,beta,T,N,xstar,n,C,D);
     if b_fstar
        B=xstar;
    else
        A=xstar;
    end
end
L=(A+B)/2;
end
```

```
function I= find_aa_stradle(r,lambda,sigma,c1,c2,k1,
    k2,alpha,beta,T,N,x,n,C,D);
%indicator for the possition of x w.r.t. the lower
    boundary

cost=@(a)f_b_stradle(a,r,lambda,sigma,c1,c2,k1,k2,
    alpha,beta,T,N,x,n,C,D);
if cost(-10^(-4))<cost(0)
    I=1;
else
    I=0;
end
end
```

```
function [V,Y] = f_b_stradle(a,r,lambda,sigma,c1,c2,
    k1,k2,alpha,beta,T,N,x,n,C,D);
%finding the upper boundary if the lower one is
    known

```

```
h=@(b)-premium_stradle(r,lambda,sigma,c1,c2,k1,k2,x,
   T,[exp(a)*x,alpha],[exp(b)*x,beta],N,n);
l1=log(C/x);
l2=log(D/x);
[Y,V]=fminbnd(h,l1,l2);
end
```

```
function [V] = find_upper_stradle(r,lambda,sigma,c1,
   c2,k1,k2,alpha,beta,T,N,n,A,B,C,D);
% upper boundary

while B-A>0.0001
    xstar=(A+B)/2;
    b_fstar= find_bb_stradle(r,lambda,sigma,c1,c2,k1
       ,k2,alpha,beta,T,N,xstar,n,C,D);
    if b_fstar
        A=xstar;
    else
        B=xstar;
    end
end
V=(A+B)/2;
end
```

```
function [I]=find_bb_stradle(r,lambda,sigma,c1,c2,k1
   ,k2,alpha,beta,T,N,x,n,C,D);
%indicator for the possition of x w.r.t. the upper
   boundary

cost=@(b)f_a_stradle(b,r,lambda,sigma,c1,c2,k1,k2,
   alpha,beta,T,N,x,n,C,D );
if cost(10^(-4))<cost(0)
    I=1;
else
    I=0;
end
end
```

```
function [V] = f_a_stradle(b,r,lambda,sigma,c1,c2,k1
   ,k2,alpha,beta,T,N,x,n,C,D );
%finding the lower boundary if the upper one is
   known

h=@(a)-premium_stradle(r,lambda,sigma,c1,c2,k1,k2,x,
   T,[exp(a)*x,alpha],[exp(b)*x,beta],N,n);
opt=optimset('MaxIter',10^(200),'TolX',10^(-200),'
   TolFun',10^(-200),'MaxFunEvals',10^(200));
l1=log(C/x);
l2=log(D/x);
[~,V]=fminbnd(h,l1,l2);
end
```

14.3.4 Crank-Nicolson finite difference approach

We present now our the used implementation of the Crank-Nicolson finite difference method.

```
function [price] =american_stradle_CN(r,sigma,lambda
   ,k1,k2,c1,c2,S0,T ,t_out,c_low,c_up);
% Crank-Nicolson finite difference approach for the
   American Strangles
%c1     -- lower boundary
%c2     -- upper boundary
%t_out  -- the time grid for the desired result
%c_low  -- the lower boundary at the grid
%c_low  -- the upper boundary at the grid

M=100;   % time nodes for the finite difference
N=5000;  % state nodes
tt=t_out(end:-1:1);
del_t=T/M;
del_x=((c_up(1)+1)-(c_low(1)-1))/N;
x=(c_low(1)-1):del_x:(c_up(1)+1);
t=T:-T/M:0;
ts=t(end:-1:1);
cc_low=spline(tt,c_low,t);
```

```
C_low=cc_low(end:-1:1);
cc_up=spline(tt,c_up,t);
C_up=cc_up(end:-1:1);

G=zeros(1,N+1);
for m=1:M+1
   L_low(m)=find(x>C_low(m),1);
   L_up(m)=find(x>C_up(m),1);
end
for n=1:N+1
   F(n)=exp(-lambda*T)*max([c1*(k1-x(n)),c2*(x(n)-k2),0]);
end
FG=F;
for m=2:M+1
    for n=1:L_low(m)-1
        G(n)=exp(-lambda*t(m))*c1*max(k1-x(n),0);
    end
    for n=L_up(m):N+1
        G(n)=exp(-lambda*t(m))*c2*max(x(n)-k2,0);
    end
    A=zeros(L_up(m)-L_low(m),L_up(m)-L_low(m));
    B=zeros(1,L_up(m)-L_low(m));
    n=L_low(m);
    A(1,1) =(1/del_t-0.5*r*x(n)/del_x+0.5*sigma^2*(x(n))^2/del_x^2+0.5*r);
    A(1,2) =-0.25*sigma^2*(x(n))^2/del_x^2;
    B(1)   =F(n)  *(1/del_t+0.5*r*x(n)/del_x-0.5*sigma^2*(x(n))^2/del_x^2-0.5*r)...
              +F(n-1)*(-0.5*r*x(n)/del_x+0.25*sigma^2*(x(n))^2/del_x^2)...
              +F(n+1)*(0.25*sigma^2*(x(n))^2/del_x^2)...
              -G(n-1)*(0.5*r*x(n)/del_x-0.25*sigma^2*(x(n))^2/del_x^2);
    for n=L_low(m)+1:L_up(m)-2
       A(n-L_low(m)+1,n-L_low(m))  =0.5*r*x(n)/del_x-0.25*sigma^2*(x(n))^2/del_x^2;
```

```
            A(n-L_low(m)+1,n-L_low(m)+1)=1/del_t-0.5*r*x(n)/del_x+0.5*sigma^2*(x(n))^2/del_x^2+0.5*r;
            A(n-L_low(m)+1,n-L_low(m)+2)=-0.25*sigma^2*(x(n))^2/del_x^2;
            B(n-L_low(m)+1)=F(n)  *(1/del_t+0.5*r*x(n)/del_x-0.5*sigma^2*(x(n))^2/del_x^2-0.5*r)...
                        +F(n-1)*(-0.5*r*x(n)/del_x+0.25*sigma^2*(x(n))^2/del_x^2)...
                        +F(n+1)*(0.25*sigma^2*(x(n))^2/del_x^2);
        end
        n=L_up(m)-L_low(m);
        n=L_up(m);
        A(n-L_low(m),n-L_low(m)-1)  =0.5*r*x(n)/del_x-0.25*sigma^2*(x(n))^2/del_x^2;
        A(n-L_low(m),n-L_low(m))=1/del_t-0.5*r*x(n)/del_x+0.5*sigma^2*(x(n))^2/del_x^2+0.5*r;
        B(n-L_low(m))=F(n)  *(1/del_t+0.5*r*x(n)/del_x-0.5*sigma^2*(x(n))^2/del_x^2-0.5*r)...
                        +F(n-1)*(-0.5*r*x(n)/del_x+0.25*sigma^2*(x(n))^2/del_x^2)...
                        +F(n+1)*(0.25*sigma^2*(x(n))^2/del_x^2)...
                       +G(n+1)*(0.25*sigma^2*(x(n))^2/del_x^2);
    GG=linsolve(A,B');
    G(L_low(m):L_up(m)-1)=GG;
    F=G;
    FG=[F;FG];
end
price=spline(x,G,S0);
end
```

14.3.5 Comments

The price of the derivative that expires at the first exit from a piece-wise linear strip is derived via the code `premium_stradle` – see formula (7.34). The perpetual boundaries and the strangle prices are obtained via the code `straddle_perpetual.m` that uses the results of Section 7.4. If the exit is from the upper boundary, then the holder receives the call payoff, otherwise the put one. Also, the terminal condition is included. The number of the division is again restricted to four. The optimal boundaries are derived via the code `bondary_stradle.m` that uses `find_lower_stradle.m` and `find_upper_stradle.m`. They are based on the algorithm presented in Section 7.5. Our fast pricing method 7.1 works as follows

```
 r=-0.01;
lambda=0.03;
sigma=.3;
S_0=19;
T=1;
k1=20;
k2=15;
c1=1;
c2=1.5;
[ A,B ] = straddle_perpetual(r,sigma,lambda,k1,k2,c1
   ,c2,S_0 );
[put_boundary,call_boundary,t] = bondary_stradle( r,
   lambda,sigma,c1,c2,k1,k2,[0,T/3,2*T/3,T],3,5,A,B)
price=premium_stradle(r,lambda,sigma,c1,c2,k1,k2,S_0
   ,t,put_boundary,call_boundary,3,5)

put_boundary =

    8.2934    9.2066   11.8826   13.3333


call_boundary =

   29.7458   27.4712   25.0981   17.0000


```

```
t =

          0    0.3333    0.6667    1.0000


price =

    8.9131
```

The pricing task at a denser grid is solved through the Crank-Nicolson finite difference approach given in Section 3.5 and implemented in the code `american_stradle_CN.m`. Note that it is modified to the strangles through the boundary conditions. The optimal boundaries are derived by several iterations of the code `bondary_stradle.m`.

14.4 Quadratic strangles

Now we present the codes for the quadratic strangles discussed in Chapter 8. We have proved there that some parameters may lead to one-sided optimal stopping problem and others to two-sided tasks. In the first case, these derivatives are studied in a way similar to those for put options – see Chapter 4. The results for the two-sided quadratic strangles are implemented similarly to the approach used for the usual strangles – see Chapter 7. Thus we discuss only the arising differences.

14.4.1 One-sided problems

```
function [pr] =premium_quadratic_one(r,lambda,sigma,
   k,x,t,c,N);
% one-sided

if N==1
  t0=t(1);
  t1=t(2);
  c0=c(1);
  c1=c(2);
  [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)/
     sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma );
```

```
  C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
  C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
  X=0;
  mu=sigma^2/2-a*sigma+lambda;
  eta=sigma^2-2*a*sigma+lambda-r;
  l1(1)=laplace_1(r+lambda,t1,a,b );
  l2(1)=exp(sigma*b)*laplace_1(mu, t1,a,b );
  l3(1)=exp(2*sigma*b)*laplace_1(eta, t1,a,b );
  I1=k^2*l1(1)-2*k*x*l2(1)+x^2*l3(1);
  kk=inf;
  [term1]=exp(-(r+lambda)*t1)*terminal1_1_new(t1-t0,
    X,C0,C1,kk,C1);
  [term2]=exp(-(lambda+sigma^2/2)*t1)*
    terminal1_2_new(sigma,t1-t0,X,C0,C1,kk,C1);
  [term3]=exp(-(lambda+sigma^2-r)*t1)*
    terminal1_2_new(2*sigma,t1-t0,X,C0,C1,kk,C1);
  ttt=k^2*term1-2*k*x*term2+x^2*term3;
  pr=I1+ttt;
elseif N==2
    t0=t(1); t1=t(2); t2=t(3);
    c0=c(1); c1=c(2); c2=c(3);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)
      /sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma
      );
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(mu, t1,a,b );
    l3(1)=exp(2*sigma*b)*laplace_1(eta, t1,a,b );
    I1=k^2*l1(1)-2*k*x*l2(1)+x^2*l3(1);
  %
    --------------------------------------------------------------------
    [ a,b]= linnear( t1,(log(c1/x)-(r-sigma^2/2)*t1)
      /sigma,t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma
      );
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
```

```
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    kk=inf;
    [term1,term2,term3]=terminal2_quadratic(sigma,X,
       C0, C1,C2,t0,t1,t2,kk );
    term1=exp(-(lambda+r)*t2)*term1;
    term2=exp(-(lambda+sigma^2/2)*t2)*term2;
    term3=exp(-(lambda+sigma^2-r)*t2)*term3;
    ttt=max(k^2*term1-2*k*x*term2+x^2*term3,0);

    l1(2)=laplace_2(r+lambda,X,C0, C1,C2,t0,t1,t2,a,
       b );
    l2(2)=exp(sigma*b)*laplace_2(mu,X,C0, C1,C2,t0,
       t1,t2,a,b );
    l3(2)=exp(2*sigma*b)*laplace_2(eta,X,C0, C1,C2,
       t0,t1,t2,a,b );
    I2=max(k^2*l1(2)-2*k*x*l2(2)+x^2*l3(2),0);
    pr=I1+I2+ttt;
elseif N==3
      t0=t(1); t1=t(2); t2=t(3); t3=t(4);
    c0=c(1); c1=c(2); c2=c(3); c3=c(4);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)
       /sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma
       );
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(mu, t1,a,b );
    l3(1)=exp(2*sigma*b)*laplace_1(eta, t1,a,b );
    I1=k^2*l1(1)-2*k*x*l2(1)+x^2*l3(1);
   %
       ------------------------------------------------------------

    [ a,b]= linnear( t1,(log(c1/x)-(r-sigma^2/2)*t1)
       /sigma,t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma
```

```
   );
mu=sigma^2/2-a*sigma+lambda;
eta=sigma^2-2*a*sigma+lambda-r;
C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
X=0;
l1(2)=laplace_2(r+lambda,X,C0, C1,C2,t0,t1,t2,a,b );
l2(2)=exp(sigma*b)*laplace_2(mu,X,C0, C1,C2,t0,t1,t2,a,b );
l3(2)=exp(2*sigma*b)*laplace_2(eta,X,C0, C1,C2,t0,t1,t2,a,b );
I2=k^2*l1(2)-2*k*x*l2(2)+x^2*l3(2);
%
   -----------------------------------------------------------------------

[ a,b]= linnear( t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(c3/x)-(r-sigma^2/2)*t3)/sigma);
mu=sigma^2/2-a*sigma+lambda;
eta=sigma^2-2*a*sigma+lambda-r;
C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
C3=(log(c3/x)-(r-sigma^2/2)*t3)/sigma;
X=0;
kk=inf;
[term1,term2,term3]=terminal3_quadratic(sigma,X,C0, C1,C2,C3,t0,t1,t2,t3,kk );
term1=exp(-(lambda+r)*t3)*term1;
term2=exp(-(lambda+sigma^2/2)*t3)*term2;
term3=exp(-(lambda+sigma^2-r)*t3)*term3;
ttt=k^2*term1-2*k*x*term2+x^2*term3;

l1(3)=laplace_3(r+lambda,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
```

```
    l2(3)=exp(sigma*b)*laplace_3(mu,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    l3(3)=exp(2*sigma*b)*laplace_3(eta,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    I3=k^2*l1(3)-2*k*x*l2(3)+x^2*l3(3);
    pr=I1+I2+I3+ttt;

    elseif N==4
    t0=t(1); t1=t(2); t2=t(3); t3=t(4); t4=t(5);
    c0=c(1); c1=c(2); c2=c(3); c3=c(4); c4=c(5);
    [ a,b]= linnear( t0,(log(c0/x)-(r-sigma^2/2)*t0)/sigma,t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma);
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
    l1(1)=laplace_1(r+lambda,t1,a,b );
    l2(1)=exp(sigma*b)*laplace_1(mu, t1,a,b );
    l3(1)=exp(2*sigma*b)*laplace_1(eta, t1,a,b );
    I1=k^2*l1(1)-2*k*x*l2(1)+x^2*l3(1);
  % ------------------------------------------------------------

    [ a,b]= linnear( t1,(log(c1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma);
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1(2)=laplace_2(r+lambda,X,C0, C1,C2,t0,t1,t2,a,b );
    l2(2)=exp(sigma*b)*laplace_2(mu,X,C0, C1,C2,t0,t1,t2,a,b );
    l3(2)=exp(2*sigma*b)*laplace_2(eta,X,C0, C1,C2,t0,t1,t2,a,b );
```

```
    I2=k^2*l1(2)-2*k*x*l2(2)+x^2*l3(2);
%  -----------------------------------------------------------------
    [ a,b]= linnear( t2,(log(c2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(c3/x)-(r-sigma^2/2)*t3)/sigma);
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    C3=(log(c3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    l1(3)=laplace_3(r+lambda,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    l2(3)=exp(sigma*b)*laplace_3(mu,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    l3(3)=exp(2*sigma*b)*laplace_3(eta,X,C0, C1,C2,C3,t0,t1,t2,t3,a,b );
    I3=k^2*l1(3)-2*k*x*l2(3)+x^2*l3(3);
%  -----------------------------------------------------------------
    [ a,b]= linnear( t3,(log(c3/x)-(r-sigma^2/2)*t3)/sigma,t4,(log(c4/x)-(r-sigma^2/2)*t4)/sigma);
    mu=sigma^2/2-a*sigma+lambda;
    eta=sigma^2-2*a*sigma+lambda-r;
    C0=(log(c0/x)-(r-sigma^2/2)*t0)/sigma;
    C1=(log(c1/x)-(r-sigma^2/2)*t1)/sigma;
    C2=(log(c2/x)-(r-sigma^2/2)*t2)/sigma;
    C3=(log(c3/x)-(r-sigma^2/2)*t3)/sigma;
    C4=(log(c4/x)-(r-sigma^2/2)*t4)/sigma;
    X=0;
    kk=inf;
    [term1,term2,term3]=terminal4_quadratic(sigma,X,C0, C1,C2,C3,C4,t0,t1,t2,t3,t4,kk );
```

```
    term1=exp(-(lambda+r)*t4)*term1;
    term2=exp(-(lambda+sigma^2/2)*t4)*term2;
    term3=exp(-(lambda+sigma^2-r)*t4)*term3;
    ttt=k^2*term1-2*k*x*term2+x^2*term3;
    l1(4)=laplace_4(r+lambda,X,C0, C1,C2,C3,C4,t0,t1
        ,t2,t3,t4,a,b );
    l2(4)=exp(sigma*b)*laplace_4(mu,X,C0, C1,C2,C3,
        C4,t0,t1,t2,t3,t4,a,b );
    l3(4)=exp(2*sigma*b)*laplace_4(eta,X,C0, C1,C2,
        C3,C4,t0,t1,t2,t3,t4,a,b );
    I4=k^2*l1(4)-2*k*x*l2(4)+x^2*l3(4);
    pr=I1+I2+I3+I4+ttt;
end
end
```

```
function [Y1,Y2,Y3] = terminal2_quadratic(sigma,x0,
    c0, c1,c2,t0,t1,t2,kk );
% related to the option value at maturity

izz1=@(x1)terminal1_1_new(t2-t1,0,c1-x1,c2-x1,kk-x1,
    c2-x1);
izz2=@(x1)exp(sigma*x1).*terminal1_2_new(sigma,t2-t1
    ,0,c1-x1,c2-x1,kk-x1,c2-x1);
izz3=@(x1)exp(2*sigma*x1).*terminal1_2_new(2*sigma,
    t2-t1,0,c1-x1,c2-x1,kk-x1,c2-x1);
iz2=@(x1)1-exp((-2*(x0-c0)*(x1-c1))/(t1-t0));
iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi*(t1-
    t0));
iz=@(x1)izz1(x1).*iz2(x1).*iz3(x1);
iiz2=@(x1)izz2(x1).*iz2(x1).*iz3(x1);
iiz3=@(x1)izz3(x1).*iz2(x1).*iz3(x1);
Y1=integral(iz,c1,1000);
Y2=integral(iiz2,c1,1000);
Y3=integral(iiz3,c1,1000);
end
```

```
function [Y1,Y2,Y3] = terminal3_quadratic(sigma,x,c0
    , c1,c2,c3,t0,t1,t2,t3,kk );
```

```
% related to the option value at maturity

izzz1=@(x1,x2) terminal1_1_new(t3-t2,0,c2-x2,c3-x2,
    kk-x2,c3-x2);
izzz2=@(x1,x2)exp(sigma*x2).*terminal1_2_new(sigma,
    t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
izzz3=@(x1,x2)exp(2*sigma*x2).*terminal1_2_new(2*
    sigma,t3-t2,0,c2-x2,c3-x2,kk-x2,c3-x2);
iz2=@(x1,x2)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0)))
    .*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1)));
iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1
    ).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1))
    ;
iiz1=@(x1,x2)izzz1(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
iiz2=@(x1,x2)izzz2(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
iiz3=@(x1,x2)izzz3(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
Y1=integral2(iiz1,c1,1000,c2,1000, 'AbsTol',1e-10,'
    RelTol',1e-5 );
Y2=integral2(iiz2,c1,1000,c2,1000, 'AbsTol',1e-10,'
    RelTol',1e-5  );
Y3=integral2(iiz3,c1,1000,c2,1000, 'AbsTol',1e-10,'
    RelTol',1e-5  );
end
```

```
function [Y1,Y2,Y3] =terminal4_quadratic(sigma,x,c0,
     c1,c2,c3,c4,t0,t1,t2,t3,t4,kk );
% related to the option value at maturity

izz1=@(x1,x2,x3)terminal1_1_new(t4-t3,0,c3-x3,c4-x3,
    kk-x3,c4-x3);
izz2=@(x1,x2,x3)exp(sigma*x3).*terminal1_2_new(sigma
    ,t4-t3,0,c3-x3,c4-x3,kk-x3,c4-x3);
izz3=@(x1,x2,x3)exp(2*sigma*x3).*terminal1_2_new(2*
    sigma,t4-t3,0,c3-x3,c4-x3,kk-x3,c4-x3);
iz2=@(x1,x2,x3)(1-exp((-2*(x-c0)*(x1-c1))/(t1-t0)))
    .*(1-exp((-2*(x1-c1).*(x2-c2))/(t2-t1))).*(1-exp
    ((-2*(x2-c2).*(x3-c3))/(t3-t2)));
iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2
```

```
-x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2))
   )/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
iiz1=@(x1,x2,x3)izz1(x1,x2,x3).*iz2(x1,x2,x3).*iz3(
   x1,x2,x3);
iiz2=@(x1,x2,x3)izz2(x1,x2,x3).*iz2(x1,x2,x3).*iz3(
   x1,x2,x3);
iiz3=@(x1,x2,x3)izz3(x1,x2,x3).*iz2(x1,x2,x3).*iz3(
   x1,x2,x3);
Y1=integral3(iiz1,c1,1000,c2,1000,c3,1000);
Y2=integral3(iiz2,c1,1000,c2,1000,c3,1000);
Y3=integral3(iiz3,c1,1000,c2,1000,c3,1000);
end
```

14.4.2 Two-sided problems

```
function [pr] =premium_strangle_quadratic(r,lambda,
   sigma,k,x,t,alpha,beta,N,n);
% two-side

if N==1
    t0=t(1); t1=t(2);
    alpha0=alpha(1); alpha1=alpha(2);
    beta0=beta(1); beta1=beta(2);
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    X=0;
    [ a1,a2]= linnear( t0,A0,t1,A1 );
    [ b1,b2]= linnear( t0,B0,t1,B1 );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2- b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda,
       t1,a1,a2,b1,b2,n);
    l2_lower(1)=exp(sigma*a2)*
```

```
        laplace_1_finite_put_lower(mu_lower, t1,a1,a2
        ,b1,b2,n );
    l3_lower(1)=exp(2*sigma*a2)*
        laplace_1_finite_put_lower(eta_lower, t1,a1,
        a2,b1,b2,n );
    pr_lower=k^2*l1_lower(1)-2*k*x*l2_lower(1)+x^2*
        l3_lower(1);
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,
        t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*
        laplace_1_finite_put_upper(mu_upper, t1,a1,a2
        ,b1,b2,n );
    l3_upper(1)=exp(2*sigma*b2)*
        laplace_1_finite_put_upper(eta_upper, t1,a1,
        a2,b1,b2,n );
    pr_upper=k^2*l1_upper(1)-2*k*x*l2_upper(1)+x^2*
        l3_upper(1);
    z1=A1;
    z2=B1;
    [term1]=exp(-(r+lambda)*t1)*terminal1_2_stradle
        (0,t1-t0,X,A0,A1,B0,B1,z1,z2,n);
    [term2]=exp(-(lambda+sigma^2/2)*t1)*
        terminal1_2_stradle(sigma,t1-t0,X,A0,A1,B0,B1
        ,z1,z2,n);
    [term3]=exp(-(lambda+sigma^2-r)*t1)*
        terminal1_2_stradle(2*sigma,t1-t0,X,A0,A1,B0,
        B1,z1,z2,n);
    ttt=k^2*term1-2*k*x*term2+x^2*term3;
    pr=pr_lower+pr_upper+ttt;
elseif N==2
    t0=t(1); t1=t(2); t2=t(3);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha
        (3);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3);
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma
        ^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma
        ^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma
```

```
^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda,t1,a1,a2,b1,b2,n );
    l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower(mu_lower, t1,a1,a2,b1,b2,n );
    l3_lower(1)=exp(2*sigma*a2)*laplace_1_finite_put_lower(eta_lower, t1,a1,a2,b1,b2,n );
    I1_lower=k^2*l1_lower(1)-2*k*x*l2_lower(1)+x^2*l3_lower(1);
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
    l3_upper(1)=exp(2*sigma*b2)*laplace_1_finite_put_upper(eta_upper, t1,a1,a2,b1,b2,n );
    I1_upper=k^2*l1_upper(1)-2*k*x*l2_upper(1)+x^2*l3_upper(1);
   % --------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
```

```
eta_upper=sigma^2-2*b1*sigma+lambda-r;
A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
X=0;
z1=A2;
z2=B2;
[term1,term2,term3]=
   terminal2_3_stradle_quadratic(sigma,X,A0, A1,
   A2,B0, B1,B2,a1,a2,b1,b2,z1,z2,t0,t1,t2,n );
term1=exp(-(lambda+r)*t2)*term1;
term2=exp(-(lambda+sigma^2/2)*t2)*term2;
term3=exp(-(lambda+sigma^2-r)*t2)*term3;
ttt=k^2*term1-2*k*x*term2+x^2*term3;
l1_lower(2)=laplace_2_finite_put_lower(r+lambda,
   X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n
   );
l2_lower(2)=exp(sigma*a2)*
   laplace_2_finite_put_lower(mu_lower,X,A0, A1,
   A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
l3_lower(2)=exp(2*sigma*a2)*
   laplace_2_finite_put_lower(eta_lower,X,A0, A1
   ,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
I2_lower=k^2*l1_lower(2)-2*k*x*l2_lower(2)+x^2*
   l3_lower(2);
l1_upper(2)=laplace_2_finite_put_upper(r+lambda,
   X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n
   );
l2_upper(2)=exp(sigma*b2)*
   laplace_2_finite_put_upper(mu_upper,X,A0, A1,
   A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
l3_upper(2)=exp(2*sigma*b2)*
   laplace_2_finite_put_upper(eta_upper,X,A0, A1
   ,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
I2_upper=k^2*l1_upper(2)-2*k*x*l2_upper(2)+x^2*
```

```
        l3_upper(2);;
    pr=I1_lower+I2_lower+I1_upper+I2_upper+ttt;
elseif N==3
    t0=t(1); t1=t(2); t2=t(3); t3=t(4);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha
        (3); alpha3=alpha(4);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3);
        beta3=beta(4);
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma
        ^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma
        ^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma
        ^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma
        ^2/2)*t1)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda,
        t1,a1,a2,b1,b2,n );
    l2_lower(1)=exp(sigma*a2)*
        laplace_1_finite_put_lower(mu_lower, t1,a1,a2
        ,b1,b2,n );
    l3_lower(1)=exp(2*sigma*a2)*
        laplace_1_finite_put_lower(eta_lower, t1,a1,
        a2,b1,b2,n );
    I1_lower=k^2*l1_lower(1)-2*k*x*l2_lower(1)+x^2*
        l3_lower(1);
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,
        t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*
        laplace_1_finite_put_upper(mu_upper, t1,a1,a2
        ,b1,b2,n );
    l3_upper(1)=exp(2*sigma*b2)*
        laplace_1_finite_put_upper(eta_upper, t1,a1,
        a2,b1,b2,n );
    I1_upper=k^2*l1_upper(1)-2*k*x*l2_upper(1)+x^2*
        l3_upper(1);
```

```
% ---------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1_lower(2)=laplace_2_finite_put_lower(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    l3_lower(2)=exp(2*sigma*a2)*laplace_2_finite_put_lower(eta_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_lower=k^2*l1_lower(2)-2*k*x*l2_lower(2)+x^2*l3_lower(2);
    l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    l3_upper(2)=exp(2*sigma*b2)*
```

```
    laplace_2_finite_put_upper(eta_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_upper=k^2*l1_upper(2)-2*k*x*l2_upper(2)+x^2*l3_upper(2);
%-----------------------------------------------------------------------
    [ a1,a2]= linnear( t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma );
    [ b1,b2]= linnear( t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(beta3/x)-(r-sigma^2/2)*t3)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    z1=A3;
    z2=B3;
    [term1,term2,term3]=terminal3_3_stradle_quadratic(sigma,X,A0, A1,A2,A3,B0, B1,B2,B3,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,n );
    term1=exp(-(lambda+r)*t3)*term1;
    term2=exp(-(lambda+sigma^2/2)*t3)*term2;
    term3=exp(-(lambda+sigma^2-r)*t3)*term3;
    ttt=k^2*term1-2*k*x*term2+x^2*term3;
    l1_lower(3)=laplace_3_finite_put_lower(r+lambda,X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2
```

```
        ,b1,b2,n );
    l2_lower(3)=exp(sigma*a2)*
        laplace_3_finite_put_lower(mu_lower,X,A0, A1,
        A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
        );
    l3_lower(3)=exp(2*sigma*a2)*
        laplace_3_finite_put_lower(eta_lower,X,A0, A1
        ,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
         );
    I3_lower=k^2*l1_lower(3)-2*k*x*l2_lower(3)+x^2*
        l3_lower(3);
    l1_upper(3)=laplace_3_finite_put_upper(r+lambda,
        X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2
        ,b1,b2,n );
    l2_upper(3)=exp(sigma*b2)*
        laplace_3_finite_put_upper(mu_upper,X,A0, A1,
        A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
        );
    l3_upper(3)=exp(2*sigma*b2)*
        laplace_3_finite_put_upper(eta_upper,X,A0, A1
        ,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
         );
    I3_upper=k^2*l1_upper(3)-2*k*x*l2_upper(3)+x^2*
        l3_upper(3);
    pr=I1_lower+I2_lower+I3_lower+I1_upper+I2_upper+
        I3_upper+ttt;
elseif N==4
    t0=t(1); t1=t(2); t2=t(3); t3=t(4); t4=t(5);
    alpha0=alpha(1); alpha1=alpha(2); alpha2=alpha
        (3); alpha3=alpha(4); alpha4=alpha(5);
    beta0=beta(1); beta1=beta(2);  beta2=beta(3);
        beta3=beta(4); beta4=beta(5);
    [ a1,a2]= linnear( t0,(log(alpha0/x)-(r-sigma
        ^2/2)*t0)/sigma,t1,(log(alpha1/x)-(r-sigma
        ^2/2)*t1)/sigma );
    [ b1,b2]= linnear( t0,(log(beta0/x)-(r-sigma
        ^2/2)*t0)/sigma,t1,(log(beta1/x)-(r-sigma
        ^2/2)*t1)/sigma );
```

```
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    l1_lower(1)=laplace_1_finite_put_lower(r+lambda,t1,a1,a2,b1,b2,n );
    l2_lower(1)=exp(sigma*a2)*laplace_1_finite_put_lower(mu_lower, t1,a1,a2,b1,b2,n );
    l3_lower(1)=exp(2*sigma*a2)*laplace_1_finite_put_lower(eta_lower, t1,a1,a2,b1,b2,n );
    I1_lower=k^2*l1_lower(1)-2*k*x*l2_lower(1)+x^2*l3_lower(1);
    l1_upper(1)=laplace_1_finite_put_upper(r+lambda,t1,a1,a2,b1,b2,n );
    l2_upper(1)=exp(sigma*b2)*laplace_1_finite_put_upper(mu_upper, t1,a1,a2,b1,b2,n );
    l3_upper(1)=exp(2*sigma*b2)*laplace_1_finite_put_upper(eta_upper, t1,a1,a2,b1,b2,n );
    I1_upper=k^2*l1_upper(1)-2*k*x*l2_upper(1)+x^2*l3_upper(1);
%  --------------------------------------------------------------
    [ a1,a2]= linnear( t1,(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma );
    [ b1,b2]= linnear( t1,(log(beta1/x)-(r-sigma^2/2)*t1)/sigma,t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
```

```
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    X=0;
    l1_lower(2)=laplace_2_finite_put_lower(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_lower(2)=exp(sigma*a2)*laplace_2_finite_put_lower(mu_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    l3_lower(2)=exp(2*sigma*a2)*laplace_2_finite_put_lower(eta_lower,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_lower=k^2*l1_lower(2)-2*k*x*l2_lower(2)+x^2*l3_lower(2);
    l1_upper(2)=laplace_2_finite_put_upper(r+lambda,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n);
    l2_upper(2)=exp(sigma*b2)*laplace_2_finite_put_upper(mu_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    l3_upper(2)=exp(2*sigma*b2)*laplace_2_finite_put_upper(eta_upper,X,A0, A1,A2,B0, B1,B2,t0,t1,t2,a1,a2,b1,b2,n );
    I2_upper=k^2*l1_upper(2)-2*k*x*l2_upper(2)+x^2*l3_upper(2);
%  ----------------------------------------------------------------------
    [ a1,a2]= linnear( t2,(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma );
    [ b1,b2]= linnear( t2,(log(beta2/x)-(r-sigma^2/2)*t2)/sigma,t3,(log(beta3/x)-(r-sigma^2/2)*t3)/sigma );
    mu_lower=sigma^2/2-a1*sigma+lambda;
```

```
    mu_upper=sigma^2/2-b1*sigma+lambda;
    eta_lower=sigma^2-2*a1*sigma+lambda-r;
    eta_upper=sigma^2-2*b1*sigma+lambda-r;
    A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
    A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
    A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
    A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
    B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
    B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
    B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
    B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
    X=0;
    l1_lower(3)=laplace_3_finite_put_lower(r+lambda,
       X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2
       ,b1,b2,n );
    l2_lower(3)=exp(sigma*a2)*
       laplace_3_finite_put_lower(mu_lower,X,A0, A1,
       A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
       );
    l3_lower(3)=exp(2*sigma*a2)*
       laplace_3_finite_put_lower(eta_lower,X,A0, A1
       ,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
        );
    I3_lower=k^2*l1_lower(3)-2*k*x*l2_lower(3)+x^2*
       l3_lower(3);
    l1_upper(3)=laplace_3_finite_put_upper(r+lambda,
       X,A0, A1,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2
       ,b1,b2,n );
    l2_upper(3)=exp(sigma*b2)*
       laplace_3_finite_put_upper(mu_upper,X,A0, A1,
       A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
       );
    l3_upper(3)=exp(2*sigma*b2)*
       laplace_3_finite_put_upper(eta_upper,X,A0, A1
       ,A2,A3,B0, B1,B2,B3,t0,t1,t2,t3,a1,a2,b1,b2,n
        );
    I3_upper=k^2*l1_upper(3)-2*k*x*l2_upper(3)+x^2*
       l3_upper(3);
```

```
% ----------------------------------------------------------------
[ a1,a2]= linnear( t3,(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma,t4,(log(alpha4/x)-(r-sigma^2/2)*t4)/sigma );
[ b1,b2]= linnear( t3,(log(beta3/x)-(r-sigma^2/2)*t3)/sigma,t4,(log(beta4/x)-(r-sigma^2/2)*t4)/sigma );
mu_lower=sigma^2/2-a1*sigma+lambda;
mu_upper=sigma^2/2-b1*sigma+lambda;
eta_lower=sigma^2-2*a1*sigma+lambda-r;
eta_upper=sigma^2-2*b1*sigma+lambda-r;
A0=(log(alpha0/x)-(r-sigma^2/2)*t0)/sigma;
A1=(log(alpha1/x)-(r-sigma^2/2)*t1)/sigma;
A2=(log(alpha2/x)-(r-sigma^2/2)*t2)/sigma;
A3=(log(alpha3/x)-(r-sigma^2/2)*t3)/sigma;
A4=(log(alpha4/x)-(r-sigma^2/2)*t4)/sigma;
B0=(log(beta0/x)-(r-sigma^2/2)*t0)/sigma;
B1=(log(beta1/x)-(r-sigma^2/2)*t1)/sigma;
B2=(log(beta2/x)-(r-sigma^2/2)*t2)/sigma;
B3=(log(beta3/x)-(r-sigma^2/2)*t3)/sigma;
B4=(log(beta4/x)-(r-sigma^2/2)*t4)/sigma;
X=0;
z1=A4;
z2=B4;
[term1,term2,term3]=terminal4_3_stradle_quadratic(sigma,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,a1,a2,b1,b2,z1,z2,t0,t1,t2,t3,t4,n );
term1=exp(-(lambda+r)*t4)*term1;
term2=exp(-(lambda+sigma^2/2)*t4)*term2;
term3=exp(-(lambda+sigma^2-r)*t4)*term3;
ttt=k^2*term1-2*k*x*term2+x^2*term3;
l1_lower(4)=laplace_4_finite_put_lower(r+lambda,X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,a2,b1,b2,n );
l2_lower(4)=exp(sigma*a2)*
```

```
        laplace_4_finite_put_lower(mu_lower,X,A0, A1,
        A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,a2
        ,b1,b2,n );
    l3_lower(4)=exp(2*sigma*a2)*
        laplace_4_finite_put_lower(eta_lower,X,A0, A1
        ,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,
        a2,b1,b2,n );
    I4_lower=k^2*l1_lower(4)-2*k*x*l2_lower(4)+x^2*
        l3_lower(4);
    l1_upper(4)=laplace_4_finite_put_upper(r+lambda,
        X,A0, A1,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3
        ,t4,a1,a2,b1,b2,n );
    l2_upper(4)=exp(sigma*b2)*
        laplace_4_finite_put_upper(mu_upper,X,A0, A1,
        A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,a2
        ,b1,b2,n );
    l3_upper(4)=exp(2*sigma*b2)*
        laplace_4_finite_put_upper(eta_upper,X,A0, A1
        ,A2,A3,A4,B0, B1,B2,B3,B4,t0,t1,t2,t3,t4,a1,
        a2,b1,b2,n );
    I4_upper=k^2*l1_upper(4)-2*k*x*l2_upper(4)+x^2*
        l3_upper(4);
    pr=I1_lower+I2_lower+I3_lower+I4_lower +I1_upper
        +I2_upper+I3_upper+I4_upper+ttt;
end
end
```

```
function [Y1,Y2,Y3] = terminal2_3_stradle_quadratic(
    sigma,x0,c0,c1,c2,b0,b1,b2,aa1,aa2,bb1,bb2,z1,z2,
    t0,t1,t2,n );
% related to the option value at maturity

izz1=@(x1)terminal1_2_stradle(0,t2-t1,0,c1-x1,c2-x1,
    b1-x1,b2-x1,z1-x1,z2-x1,n);
izz2=@(x1)exp(sigma*x1).*terminal1_2_stradle(sigma,
    t2-t1,0,c1-x1,c2-x1,b1-x1,b2-x1,z1-x1,z2-x1,n);
izz3=@(x1)exp(2*sigma*x1).*terminal1_2_stradle(2*
    sigma,t2-t1,0,c1-x1,c2-x1,b1-x1,b2-x1,z1-x1,z2-x1
```

```
,n);
iz2=@(x1)(1-qq( x0,x1,c0,c1,b0,b1,t1-t0,n ));
iz3=@(x1)exp(-(x1-x0).^2/(2*(t1-t0)))/sqrt(2*pi*(t1-t0));
iz=@(x1)izz1(x1).*iz2(x1).*iz3(x1);
iiz2=@(x1)izz2(x1).*iz2(x1).*iz3(x1);
iiz3=@(x1)izz3(x1).*iz2(x1).*iz3(x1);
Y1=integral(iz,c1,b1);
Y2=integral(iiz2,c1,b1);
Y3=integral(iiz3,c1,b1);
end
```

```
function [Y1,Y2,Y3] = terminal3_3_stradle_quadratic(sigma,x,c0,c1,c2,c3,b0,b1,b2,b3,aa1,aa2,bb1,bb2,z1,z2,t0,t1,t2,t3,n );
% related to the option value at maturity

izzz1=@(x1,x2) terminal1_2_stradle(0,t3-t2,0,c2-x2,c3-x2,b2-x2,b3-x2,z1-x2,z2-x2,n);
izzz2=@(x1,x2)exp(sigma*x2).*terminal1_2_stradle(sigma,t3-t2,0,c2-x2,c3-x2,b2-x2,b3-x2,z1-x2,z2-x2,n);
izzz3=@(x1,x2)exp(2*sigma*x2).*terminal1_2_stradle(2*sigma,t3-t2,0,c2-x2,c3-x2,b2-x2,b3-x2,z1-x2,z2-x2,n);
iz2=@(x1,x2)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
            .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ));
iz3=@(x1,x2)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2-x1).^2/(2*(t2-t1)))/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1));
izz1=@(x1,x2)izzz1(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
izz2=@(x1,x2)izzz2(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
izz3=@(x1,x2)izzz3(x1,x2).*iz2(x1,x2).*iz3(x1,x2);
Y1=integral2(izz1,c1,b1,c2,b2 );
Y2=integral2(izz2,c1,b1,c2,b2 );
Y3=integral2(izz3,c1,b1,c2,b2 );
end
```

```
function [Y1,Y2,Y3] =terminal4_3_stradle_quadratic(
   sigma,x,c0, c1,c2,c3,c4,b0,b1,b2,b3,b4,aa1,aa2,
   bb1,bb2,z1,z2,t0,t1,t2,t3,t4,n );
% related to the option value at maturity

izz1=@(x1,x2,x3)terminal1_2_stradle(0,t4-t3,0,c3-x3,
   c4-x3,b3-x3,b4-x3,z1-x3,z2-x3,n);
izz2=@(x1,x2,x3)exp(sigma*x3).*terminal1_2_stradle(
   sigma,t4-t3,0,c3-x3,c4-x3,b3-x3,b4-x3,z1-x3,z2-x3
   ,n);
izz3=@(x1,x2,x3)exp(2*sigma*x3).*terminal1_2_stradle
   (2*sigma,t4-t3,0,c3-x3,c4-x3,b3-x3,b4-x3,z1-x3,z2
   -x3,n);
iz2=@(x1,x2,x3)(1-qq( x,x1,c0,c1,b0,b1,t1-t0,n ))...
             .*(1-qq( x1,x2,c1,c2,b1,b2,t2-t1,n ))...
             .*(1-qq( x2,x3,c2,c3,b2,b3,t3-t2,n ));
iz3=@(x1,x2,x3)exp(-(x1-x).^2/(2*(t1-t0))).*exp(-(x2
   -x1).^2/(2*(t2-t1))).*exp(-(x3-x2).^2/(2*(t3-t2))
   )/sqrt(2*pi*(t1-t0)*2*pi*(t2-t1)*2*pi*(t3-t2));
iiz1=@(x1,x2,x3)izz1(x1,x2,x3).*iz2(x1,x2,x3).*iz3(
   x1,x2,x3);
iiz2=@(x1,x2,x3)izz2(x1,x2,x3).*iz2(x1,x2,x3).*iz3(
   x1,x2,x3);
iiz3=@(x1,x2,x3)izz3(x1,x2,x3).*iz2(x1,x2,x3).*iz3(
   x1,x2,x3);
Y1=integral3(iiz1,c1,b1,c2,b2,c3,b3);
Y2=integral3(iiz2,c1,b1,c2,b2,c3,b3);
Y3=integral3(iiz3,c1,b1,c2,b2,c3,b3);
end
```

14.4.3 Perpetual quadratic strangles

```
function [price,A]=perpetual_quadratic_one ( r,
   lambda,sigma,k,x );
%one-sided

mu=r/(sigma^2)-0.5;
```

```
c=sqrt(mu^2+2*(r+lambda)/(sigma^2));
p=2*c;
q=c+mu;
if lambda<r+sigma^2
    A=0;
    price=inf;
else
    A=k*q/(2*(q+1));
    price=(k^2*A^q -2*k*A^(q+1))/(x^q) + x^2;
end
end
```

```
function [A,B,price] = perpetual_quadratic_two (r,sigma,lambda,k,S0 );
%two-sided

mu=r/(sigma^2)-0.5;
c=sqrt(mu^2+2*(r+lambda)/(sigma^2));
p=2*c;
q=c+mu;
fun1=@(a)((p-q-1)*(1-a^(q+1))-sqrt((p-q-1)^2*(1-a^(q+1))^2 - (p-q)*(1-a^(q))* (p-q-2)*(1-a^(q+2))    ))/((p-q)*(1-a^(q)));
fun2=@(a)a*((q+1)*(1-a^(p-q-1))+sqrt((q+1)^2*(1-a^(p-q-1))^2 - (q)*(1-a^(p-q))* (q+2)*(1-a^(p-q-2))  ))/((q)*(1-a^(p-q)));
fun=@(a)(fun1(a)-fun2(a))^2;
opt=optimset('MaxIter',10^(200),'TolX',10^(-200),'TolFun',10^(-200),'MaxFunEvals',10^(200));
aa=fminbnd(fun,0,1,opt);
x=fun1(aa);
A=aa*k/x;
B=k/x;
a=log(A/S0);
b=log(B/S0);
if S0>A & S0<B
   price=strangle_quadratic_perpetual_two_ab( c,mu,k,S0,a,b );
```

```
else
   price=(S0-k).^2;
end
end
```

```
function [ t,cc] = bondary_quadratic_one( r,lambda,sigma,k,t,N,x1);
%boundary of the quadratic strangle -- one sided case
tt(1)=t(end);
if abs(lambda-(r+sigma^2))<10^(-15)
       C=k*(r+lambda)/(2*lambda);
else
        C=k*(lambda-sqrt(r^2+sigma^2*(r+lambda)))/(lambda-r-sigma^2);
end
if N==1
        cc=C;
else
       [ ~,cc ] = bondary_quadratic_one( r,lambda,sigma,k,t(2:end)-t(2),N-1,x1);
end
h=@(c,x)-premium_quadratic_one(r,lambda,sigma,k,x,t,[c,cc],N);
x2=cc(1);
while abs(x1-x2)>10^(-6)
     x=(x1+x2)/2;
     hh=@(c)h(c,x);
if hh(x*exp(-10^(-6)))<-(x-k)^2
     x2=x;
else
     x1=x;
end
end
cc=[(x1+x2)/2,cc];
end
```

```
function [alpha,beta,t] = bondary_strangle_quadratic
```

```
   ( r,lambda,sigma,k,t,N,n,nach_a,nach_b);
if N==1
    a1=k*(lambda-sqrt(r^2+sigma^2*(r+lambda)))/(
       lambda-r-sigma^2);
    b1=k*(lambda+sqrt(r^2+sigma^2*(r+lambda)))/(
       lambda-r-sigma^2);
    alpha=find_lower_strangle_quadratic(r,lambda,
       sigma,k,a1,b1,t,N,n,nach_a,a1,b1,nach_b);
    beta=find_upper_strangle_quadratic(r,lambda,
       sigma,k,a1,b1,t,N,n,b1,nach_b,nach_a,a1);
    alpha=[alpha,a1];
    beta=[beta,b1];
else
   [aa1,bb1]= bondary_strangle_quadratic( r,lambda,
      sigma,k,t(2:end)-t(2),N-1,n,nach_a,nach_b);
   a2=find_lower_strangle_quadratic(r,lambda,sigma,k
      ,aa1,bb1,t,N,n,nach_a,aa1(1),bb1(1),nach_b);
   b2=find_upper_strangle_quadratic(r,lambda,sigma,k
      ,aa1,bb1,t,N,n,bb1(1),nach_b,nach_a,aa1(1));
   alpha=[a2,aa1];
   beta=[b2,bb1];
end
end
```

14.4.4 Comments

The perpetual boundaries and the option prices can be derived via the codes `perpetual_quadratic_one.m` and `perpetual_quadratic_two.m`. The first one is if we have a one-sided hitting problem – this case holds if $\lambda \leq r + \sigma^2$. The second code is for the case $\lambda > r + \sigma^2$ – we have a two-sided exit task. The algorithms for the whole boundaries are similar to those for the American put option (for the one-sided problems, $\lambda \leq r+\sigma^2$) or the American strangles (two-sided tasks, $\lambda > r + \sigma^2$). Note again that they are for piece-wise linear functions with up to four nodes.The fast method for a one-sided instrument is:

```
 r=-0.01;
lambda=0.03;
```

```
sigma=.3;
S_0=19;
T=1;
k=20;
[~,x1]=perpetual_quadratic_one ( r,lambda,sigma,k,k)
    ;
[ t,boundary] = bondary_quadratic_one( r,lambda,
    sigma,k,[0,T/3,2*T/3,T],3,x1)
premium_quadratic_one(r,lambda,sigma,k,S_0,t,
    boundary,3)

t =

         0    0.3333    0.6667    1.0000


boundary =

    4.4858    4.6470    4.8694    5.4356


ans =

  34.0498 - 0.0000i
```

An example for a two-sided quadratic strangle can be seen below:

```
r=-0.05;
lambda=0.06;
sigma=.3;
S_0=19;
T=1;
k=20;
[A,B] = root_strangle_quadratic(r,sigma,lambda,k,k )
    ;
[lower_boundary,upper_boundary] =
    bondary_strangle_quadratic( r,lambda,sigma,k,[0,T
    /3,2*T/3,T],3,5,A,B)
price=premium_strangle_quadratic(r,lambda,sigma,k,
```

```
   S_0,t,lower_boundary,upper_boundary,3,5)

lower_boundary =

    1.4248    1.4680    1.5290    1.6905


upper_boundary =

  141.4095  137.1360  131.4736  118.3095


price =

   34.1306
```

If we want to estimate on a denser grid, then we use the code for the Crank-Nicolson finite difference approach – `american_stradle_CN.m`. We have to modify the boundary constraints in it. Note that if we have a one-sided task, then the set at which the BVP holds is open above. We solve this problem by introducing a large enough auxiliary boundary at which we approximate the American option price by the European one – see Proposition 8.8. Note that some parameter values may lead to the third case of Proposition 2.5. It is implemented in the code `laplace_1.m` and uses `cdf_hitting.m`. Thus we have to change in this code `normcdf.m` to `erfz.m`. It is about the analytic continuation of the error function `erf.m` – see Godfrey (2024).

Chapter 15

Concluding remarks and further works

In this dissertation, we have established a novel fast, approach for pricing American-style financial instruments. This task was solved through the following steps. We first obtain the shape of the optimal regions and the related boundaries. Next, we consider options without maturity constraints deriving the closed-form formulas for the option prices as well as for the optimal boundaries. After this, we examine the finite maturities approximating the optimal boundaries on a relatively rare grid. Thus we received a very fast algorithm with a high accuracy – errors in the fourth sign after the decimal point. For denser grids, we constructed several numerical methods based on Monte Carlo simulations and finite difference schemes. To do all this, we have proved several results for the first hitting moments of a Brownian motion. Two kinds of stopping times have been examined – the first hit to a piecewise linear boundary and the first exit from a strip formed by two such functions. Also, some limits of these Laplace transforms have been considered.

The above-mentioned methodology was first applied to the classical American options as well as to a modification named capped options. These instruments led to a one-sided hitting problem. It turned out that we can generalize significantly the set of these derivatives since we have used a method based on the infinitesimal generators. Something more, we have established a sufficient criterion for recognizing whether the payoff leads to a one-sided task – put- or call-style. As a particular example, we have suggested a new class of instruments that can be viewed as a generalization of the futures contracts.

We continued our investigation with the so-called straddles and strangles which are hybrid strategies between a put and a call option. We have made this without any restrictions for the strikes as well as for the put and call weights. These instruments led to a task for a first exit from a strip. Furthermore, we have established a criterion which guarantees that a given payoff leads to a such two-sided problem. As an example, we have defined and examined a new class of derivatives – we named them quadratic strangles. It is interesting to note that some parameter values lead to two-sided exit problems, while others lead to one-sided put-style tasks. These instruments are interesting for the investors that prefer to hedge strongly the risky positions that are deeply far-from-the-money.

Next, we have examined the cancellable options for which the writer has the right to stop the contract prematurely. These instruments led to two-sided stopping problems too – one of the boundary for the holder's exercise right and another for the writer's one. We have studied separately the options with and without maturity constraints. We have generalized these derivatives assuming that the writer's penalty consists of three parts – a proportion of the usual option payoff, some shares of the underlying asset, and a fixed amount. It turns out that this generalization is not trivial – it seems that this way we can enclose the cancellable options in a way to guarantee a put-call duality similar to one existing for the classical options. This is a task for further work.

Last but not least, we have prepared many MATLAB codes for the studied derivatives to check and validate the derived theoretical results. We have provided the main of them.

The findings of this dissertation can be extended in several directions. First, the ever-changing financial realities lead to an increasing interest in novel instruments against different risks. As we mentioned above, the proposed technique represents a significant generalization and thus it allows the building of various new derivatives. In addition, many new strategies can be considered through the so-developed approach.

On the other hand, it is well-observed in all financial markets that the Gaussian assumptions of the Black and Scholes (1973) model are not consistent with the realities. For this, many authors turn to different alternatives – Lévy processes (we refer to Rachev and Mittnik (2000), Boyarchenko and Levendorskiĭ (2002), Bianchi et al. (2008), Cont and Tankov (2004), Rachev et al. (2005), etc.), stochastic volatilities (Heston (1993), Bates (1996), Zaevski et al. (2014), etc.) and other more general dynamics (Zaevski and Kounchev (2018) and Zaevski et al. (2019)). The approach, used in this dissertation and based on the infinitesimal differential operators, can be further applied to the above-mentioned models since they are based on other Feller-Markov stochastic processes.

The derivatives studied in this dissertation are powerful instruments against different financial risks. In this light, the results can be viewed as another method for analyzing market uncertainty. This is a very important but difficult task since everyone knows what the risk is but there is not a consensus on how to measure it. Different novel techniques are proposed for solving this problem – for example, the block-chains methods, Popchev and Taneva (2018) and Popchev et al. (2021a). For other interesting results, we refer to Denchev (1996), Denchev and Gumnerov (2006), Rachev et al. (2008), Popchev and Radeva (2019), Popchev et al. (2021b), Popchev et al. (2021c), and Zaevski and Nedeltchev (2023).

Last but not least, the personal MATLAB codes prepared for the needs of this dissertation can be extended and prepared as a MATLAB package for analyzing and evaluating the American-style financial instruments.

Chapter 16

Scientific Contributions

In this dissertation, we develop a novel approach for evaluating American-style instruments written on an underlying asset modeled by a log-normal diffusion process. Their main feature is the early exercise right that the holder may use at any time before maturity. Thus Black-Scholes equation (3.3) turns into a free-boundary differential task. The traditional approach for examining these problems is based on several integral equations. However, their numerical solving needs relatively much computation time. Alternatively, the approach we suggest is based on several first-hitting and exiting properties of the Brownian motion. Let us denote by ζ the first hit of a Brownian motion to a piecewise linear function or the exit of such strip. Let also T be a terminal date and θ , σ, and k be constants. We interested in the terms $\mathbb{E}\left[e^{-\theta\zeta} I_{\zeta<T}\right]$ and $\mathbb{E}\left[e^{\sigma B_T} I_{\zeta\geq T}\right]$. The desired results are provided in Chapter 2. Also, we prove in this chapter several results for some important limits of the form $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta}\right]$ for the first hit to a linear boundary. In addition, we derive the necessary results for the limit $\lim_{T\to\infty} e^{kT}\mathbb{E}\left[e^{\theta B_T} I_{T<\zeta, B_T>z(T)}\right]$, where $z\left(t\right)$ is another linear function.

Using these results for the Brownian motion's stopping times, we approximate the optimal boundaries by maximizing the holder's financial utility. Thus we convert the free boundary task into a boundary value problem. Many numerical methods are available for their solving. We create a relatively fast Monte Carlo algorithm for calculated expectation (3.2) that gives the price. Alternatively, we adapt several finite difference schemes to the arising differential task (3.3). It turns out that the Crank-Nicolson method is relatively faster and more accurate. If the financial contracts are not restricted by maturity constraints, then the optimal boundaries are flat due to the Markov property of the stochastic processes that drive the underlying assets. This allows us to derive closed-form formulas for the boundaries and the fair price using the method of maximizing the holder's financial result. This approach is applied to the traditional American options in Chapter 4. Their modification named capped options is studied in Chapter 5. The main characteristic they exhibit is the cap level above which a call option cannot be exercised (below for the puts). Some closed and semi-closed form formulas for the prices are obtained.

Chapter 6 considers some American style instruments with generalized payoffs – the main restriction we impose is a twice differentiability. We obtain sufficient conditions that turn the pricing of such derivatives into one-sided hitting problems. The method is based on the infinitesimal generators. Roughly said, the condition is satisfied if this differential operator applied to the payoff divides the state space into two connected subsets – the first one contains the positive values whereas the other consists of the negative ones. If the hit is below, then the derivative is related to the put options. On the contrary, if the hit is above, then we have a call-style contract. Our method is applied to these derivatives paying special attention to the power payoffs. Although the differentiable payoffs are considered, the presented method can be applied to the traditional options too. Note that their payoffs are not differentiable at the strike – $(x-K)^+$ or $(K-x)^+$. The goal is to approximate them by twice differentiable functions.

We examine in Chapter 7 the so-called American strangle strategies. They appears as a combination between call and put options – the payoff is $\max\{C_1(x-K_1), C_2(K_2-x)\}$. The traditional assumption is that the put strike is lower than the call one, $K_1 \leq K_2$. Our approach allows us to abandon this restriction. In addition, we consider different impacts of the call and put features through the number of shares C_1 and C_2. It turns out that the time-state space can be divided into three connected parts. If the asset price is in the lowest one, then it is optimal for the holder to exercise as a put. The upper one contains the points that make the exercise as a call optimal. The middle set makes keeping the option alive preferable. Thus a two-sided optimal exit problem arises. Closed-form formulas for the perpetual options are obtained. It is important to mention that if the underlying asset does not pay dividends (equivalent to a model without additional discounting), then early exercising as a call is never optimal – a phenomenon that holds for many other American call-style derivatives including the usual options. It is interesting to note that despite this, the call feature has its impact – it appears through the number of the call shares but not via the call strike.

In Chapter 8 we investigate which payoffs lead to a similar two-sided optimal stopping problem – we obtain a necessary condition. It says that this is the case when the infinitesimal generator applied to the payoff divides the state space $\mathbb{R}^+$ into three intervals – the generator is positive in the middle one and negative in the rest. To illustrate our model, we introduce and examine the so-called quadratic strangles with payoffs $(x-K)^2$. These instruments can be useful for investors who prefer to hedge strongly the far-from-the-money positions and weakly the near-the-money ones. It turns out that these derivatives may lead to one-sided hitting problems as well as two-sided ones depending on the position of the discount rate λ w.r.t. the constant $r+\sigma^2$ – both cases are investigated separately.

The rest of the dissertation is devoted to the so-called cancellable American options, also known as game or Israeli options. In addition to the holder's right to exercise prematurely, the cancellable ones provide to their writer the right to execute the contract paying some amount above the usual payoff. As a rule, these instruments lead to two-sided exit problems. Different from the strangles that maximize two-dimensional problems, the game options are related to finding a saddle point in the space of the stopping times. The call and put options are explored under our approach in Chapters 9 and 10. We prove that the optimal boundaries solve a two-dimensional non-linear system that achieves a unique solution. If we have a call-style option, then the holder's exercise set consists of all points above some level whereas the writer's one may be an interval with a left end-point equal to the strike, the singleton $\{K\}$, or even the empty set. In the last case, the option turns into ordinary American. For all other points, keeping the option is better than the immediate exercise for both participants. The results for the put options are similar but in some sense inverse –the holder's region consists of all points below some boundary, whereas the writer's one may be an interval $(B,K]$, the singleton $\{K\}$, or the empty set. In Chapter 11 we investigate options whose writer's penalty is a proportion of the usual payoff. The results for the optimal regions are similar. The main difference is that all points below the strike are optimal for cancellable calls. The same is true for the points above the strike for the puts. On the other hand, under the assumption that the holder would exercise later if this provides the same financial result, these points can be viewed as part of the continuation region. An interesting result is that both optimal boundaries for a put option coincide with the strike when $r\geq 0$. The same is true for the call options when $r\leq 0$ and $\lambda>0$. Finally, we define in Chapter 12 a new class of cancellable options introducing some convertible features. The penalty that the writer owes for his early canceling right is composed of three parts – a proportion of the usual payoff, some shares of the underlying asset, and a fixed amount. We derive the related results for the optimal boundaries and corresponding regions. It seems that this generalization is not trivial, but it closes the set of cancellable options in some sense. This investigation is left for further work. Roughly said, as large penalties as the option is close to the usual American one. The cancellable (put) options under a finite maturity horizon are studied in Chapter 13. The used in this dissertation approach is adapted to these instruments. The main difference is that the holder maximizes his profit, but the writer minimizes the financial result. It turns out that there are two critical values for the time to maturity $0 \leq \tau_1 \leq \tau_2 \leq \infty$. For small enough maturities, $\tau \leq \tau_1$, the option is ordinary American. If $\tau \in (\tau_1, \tau_2]$, then the writer's optimal boundary is the strike. Note that the case $\tau_1 = \infty$ is possible. Finally, if $\tau > \tau_2$, then the option is real cancellable – the writer's optimal region is an interval $(K, A(\tau))$ for call options and $(B(\tau), K)$ for the puts.

At last but not least, we present in Chapter 14 some selected MATLAB codes for pricing the considered financial instruments. They implement the constructed algorithms based on the derived theoretical results.

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