Report by Prof. Dr. Tony Pantev

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on the applications submitted for the position of Associate Professor for the needs of the Institute of Mathematics and Informatics, Bulgarian Academy of Sciences announced in State Newspaper No. 65 on February 8, 2024.

Area of higher education: *4. Natural Sciences, Mathematics and Informatics*,

Professional field: *4.5 Mathematics*,

Scientific speciality: *"Geometry and Topology" (Homogeneous spaces and Geometric Invariant Theory)*

I am an external member of the scientific panel for this search. The panel was convened by order No 347/1.10.2024 of the Director of the Institute of Mathematics and Informatics Corresponding Member of BAS, Prof. D.Sc. Peter Boyvalenkov. The only application for the position was submitted by Dr. Valdemar Vassilev Tsanov, a Staff Researcher at IMI-BAS. As a member of the scientific panel, I have received from Dr. Tsanov all the administrative and scientific documents required by the Act on the Development of the Academic Staff in the Republic of Bulgaria (ADASRB), the Rules for its implementation, and the Rules on the terms and conditions for awarding of academic degrees and occupying of academic positions at the Bulgarian Academy of Sciences.

1 Biographical data

Dr. Valdemar Tsanov was born in Sofia on December 16, 1980. From 1999 to 2004, he was enrolled in the bachelor program at the Faculty of Mathematics and Informatics (FMI) of Sofia University "St. Kliment Ohridski" (SU) majoring in Pure Mathematics. In 2002-2003 he was an exchange bachelor student at the Université de Nantes under the European Union's Erasmus Program. After receiving his B.S. from Sofia University in 2004 he enrolled as a masters student at the FMI of SU and in 2006 he graduated with a M.S. degree in Mathematics, (specialty "Mathematical Physics"). In 2007 he was accepted as a doctoral student at the Department of Mathematics of Queen's University, Canada. He defended his Ph.D. in Mathematics in 2011. From 2011 to 2014 he was a postdoctoral fellow at Ruhr University Bochum, and from 2014 to 2016 he held a postdoctoral position at the University of Göttingen. From 2016-2018 he was a senior researcher and project leader at the University of Göttingen. After that, in the period 2018-2021, he worked as a staff researcher at Ruhr University Bochum, and in 2022 worked as a staff researcher at Jacobs University Bremen. From 2023 until now, he has been working as a staff researcher at the ICMS of IMI-BAS.

2 Scientific work

Dr. Tsanov work covers a variety of topics and addresses deep questions in algebraic geometry, asymptotic representation theory, and low dimensional topology. He is a leading expert on the global geometry of homogeneous manifolds, with numerous original contributions resolving classical open questions, and developing powerful novel techniques for studying the subtle geometry and symmetries of projective embeddings of flag manifolds. He has also made significant advances in infinite dimensional representation theory and in the three dimensional geometrization program. These are all works on topics of current interest, which are on the forefront of their respective fields, and have far reaching applications to a number of important subjects, including singularity theory, enumerative geometry, canonical bases, algebraic matroids, quantum knot invariants, and the geometry of modular forms.

The works submitted to the competition fall roughly into five general streams, which I will briefly describe next:

(i) *Constructive geometrization of knot complements.* A very appealing part of Tsanov's research program revolves around the problem of using techniques from analytic function theory and algebraic geometry to produce explicit differential geometric structures on link complements. In his single author paper [1], Tsanov uses fractional weight automorphic forms to construct the geometric $\widetilde{SL}_2(\mathbb{R})$ structure on the complement of a torus knot in the three sphere. A structure like this is known to exist by the general theory of three manifolds through the the work of Raymond-Vasquez, however an explicit description of the structure is only classically known for the link of a standard cusp singularity. In [1] Tsanov performs the necessary analysis of non-cocompact lattices in triangle subgroups of $\mathbb{P}SL_2(\mathbb{R})$ and describes the rings of automorphic forms which identify the complement of every torus knot in S^3 with the quotient of $\widetilde{SL}_2(\mathbb{R})$ by a finite index subgroup of the preimage of the associated triangle group in $\widetilde{SL}_2(\mathbb{R})$. This work of Tsanov has a lot of potential. It not only resolves the problem of geometrization in the case of torus knots but also sets the stage for developing the constructive geometrization in non-homogeneous settings by studying the behavior of the requisite rings of automorphic forms under standard algebro geometric modifications of links such as the passage to universal abelian covers, plumbing, and splicing.

(ii) *Representation theory of infinite dimensional Lie algebras.* Another major theme of Tsanov's current research activity is the study of the extra algebraic data needed to understand the structure theory of representations of infinite dimensional Lie algebras. In a landmark paper from last year (paper [10] in the list of submitted papers), Penkov and Tsanov construct, for any non-negative integer t, a universal abelian tensor category $\boldsymbol{T}_t$ which is generated by two objects with finite filtrations of length $t+1$, and with a pairing of the two objects valued in the monoidal unit. The category $\boldsymbol{T}_t$ is constructed as a subcategory in the category of representations of a Mackey Lie algebra associated to a diagonalizable pairing between two complex vector spaces of dimension equal to the infinite cardinal $\aleph_t$. The category $\boldsymbol{T}_t$ is built by an explicit but quite delicate completion process of a simple core subcategory in the category of representations of the Mackey Lie algebra and requires a very careful control of the homological properties of the generating objects. This is a highly technical but conceptually illuminating construction as it reveals hidden structural properties of the representations of the Mackey Lie algebra. In particular Penkov and Tsanov discover that the simple objects of the core are enumerates by Young diagrams, and give an explicit expression for the dimensions of their Ext spaces in terms of Littlewood-Richardson numbers. The construction and computations strengthen and generalize a series of previous works of Chirvasity-Penkov, Penkov-Serganova, and Sam-Snowden, where the authors were only able to deal with special cases of the universality question. Clearly the new factor that lead to a successful solution of the universality problem in [10] is the enlistment of Tsanov on the project team. Tsanov's formidable expertise in representation theory is evident in the many non-trivial homological calculations in the paper, and his patented blend of conceptual and concrete thinking is present throughout the paper.

(iii) *Secant varieties, moment maps, and invariants.* A substantial portion of Dr. Tsanov recent work focuses on questions unraveling the intricacies in the global geometry of homogeneous spaces. In the papers [2], [4], [6], and [9], from the list of works submitted to the competition, he uses sophisticated tools from Geometric Invariant Theory, symplectic geometry, representation theory, and orbital degeneration theory, to obtain constraints on the numerical complexity of rings of invariants of projective closed orbits of linear representations. He also gives a geometric method for bounding degrees of invariants from below based on a very interesting a combinatorial structure theory of moment map images of secant varieties.

The landscape covered in the papers [2], [4] and in the solo works [6], and [9] is quite rich not only in results, but also in ideas, insights, new methodology, and unexpected applications to physical chemistry and quantum information theory. An important result here is Tsanov's divisibility theorem describing divisors of degrees of generators of invariant polynomials on irreducible representations in terms of an explicit root/weight relation. Despite questions of this type being classical and popular, prior to Tsanov's work only a few partial results existed in this direction. This was mostly due to the challenging nature of the problems, and the ad hoc nature of the techniques used. Tsanov's papers [6] and [9] changed the playing ground completely. He discovered a conceptual and optimal combinatorial condition on the set of weights of a given irreducible representation, which is easy to check and provides computable lower bounds and divisors of degrees of invariants. Tsanov's result strengthens and vastly generalizes previous works by Sjamaar, Welhau, and Wildberger and opens new avenues for studying invariants of representations. Other highlights from this part of Tsanov's research program include his original approach to the classical question of describing moment polytopes of representations of compact groups, his in-depth study of the moment images of secant varieties and osculating spaces of orbits, as well as the applications to the classification of quantum systems whose spectral polytopes are obtained from doubly excited states, and to the state reconstruction problem in quantum entanglement.

(iv) *Relative invariant theory and Mori dream spaces.* Another very fruitful research direction pursued by Dr. Tsanov is his spectacular work on the relative invariant theory of reductive groups. The problem of classifying and computing relative invariants is a cornerstone problem in representation theory, algebraic combinatorics, and differential geometry. In its most basic and interesting form this problem asks for computing the invariants in an irreducible representation of a reductive group G with respect to the action of a reductive subgroup $\widehat{G} \subset G$. This problem has many important special cases, e.g. the branching problem for representations, and the Schur projector problem for decompositions of tensor products of irreducible representations.

Tsanov's papers [5] and [7], written jointly with H. Sepannën, provide an original nonlinear approach to the problem. They study the the action of the reductive subgroup $\widehat{G}$ on the complete flag variety X of the ambient group G from the perspective of wall crossing in variations of GIT quotients. This allows Tsanov to give a description of the associated unstable locus and to obtain a beautiful combinatorial formula for its dimension. Tsanov leverages this detailed analysis to obtain a complete and algorithmically computable finite collection of inequalities which defines the associated Littlewood-Richardson cone. This extends and explicates previous partial and theoretical results by Berenstein-Sjamaar, Belkale-Kumar, and Ressayre-Richmond. The description given by Sepannën-Tsanov has many computational and theoretical advantages. In contrast to the previously known recipes, Tsanov's list of defining inequalities, does not require any Lie algebra cohomology calculations. The accessible and explicit nature of the list readily allows for a dynamical analysis of the multiplicity spaces of invariants, and leads to a recursive classification of the $\widehat{G}$-equivariant ample classes on X. This allows Tsanov to classify the $\widehat{G}$-equivariant ample and moveable cones and to give some deep applications to the birational geometry of moduli spaces. In particular, in [7], Tsanov gives sufficient conditions for the existence of equivariant polarizations with an unstable locus of codimension at least two which give rise to geometric GIT quotients. He also shows that the resulting quotients are always Mori dream spaces, and that the Mori chambers of their pseudoeffective cone correspond to the GIT-chambers of the $\widehat{G}$-equivariant ample cone of X. Finally he proves that all rational contractions of these quotients to normal projective varieties are induced by GIT from variation of the linearizations of the action of $\widehat{G}$ on X. These construction and classification results provide an exciting new bridge between the minimal model program and geometric representation theory.

(v) *Rank functions and degrees of invariants.* A fundamental invariant of a non-degenerate smooth projective variety $X \subset \mathbb{P}(V)$ is its rank function. This function assigns to each point p of $\mathbb{P}(V)$ the minimal dimension of a secant or an osculating space for X which contains p. Understanding rank functions of projective embeddings is extremely important in the study of low codimension subvarieties, the theory of higher projective normality, in Hartshorne liaison theory, and in Zak's theory of Segre varieties. The study of rank functions has a long history with a lot of interesting results and Tsanov currently has some of the strongest general results on the topic. In particular, Tsanov's papers [3], [6], and [9] submitted to the competition, mark the first significant progress in our understanding of rank functions since the foundational results of Landsberg-Manivel from fifteen years ago. The paper [3] (joint with Petukhov) and the paper [9] contain a characterization and a complete classification of all equivariantly embedded homogeneous projective varieties with a lower semicontinuous rank function. Using the taxonomy of his classification, and the structure theory of Landsberg-Manivel, Tsanov settles several open questions in global projective geometry. For instance, he shows that a homogeneous equivariantly embedded variety has a lower semicontinuous rank function if and only if it is subcominuscule or is a hyperplane section of a subcominuscule variety in its minimal embedding. As a consequence he shows that the ideal of the r-th secant variety of a homogeneous X with a lower semicontinuous rank function, is generated in degree $r+1$ by the $(r-1)$-th prolongation of the generating set of the ideal of X. These are timely and highly original results. I expect they will jump-start novel research that will invigorate this classical area.

3 Grants and awards

Dr. Tsanov has participated as a postdoctoral researcher in the DFG-SFB/TR12 project at Rhur University Bochum and in the DFG Priority Program 1388 at the University of Göttingen. He was the leader of the DFG-AZ:TS/352/1-1 project at the University of Göttingen and he was also a staff researcher and team member in the DFG-AZ/PE980/8-1 project at Jacobs University Bremen and in the DFG-CRC/TRR191 project at Rhur University Bochum. Currently Dr. Tsanov is a staff researcher at the ICMS of IMI-BAS supported by grant DO1-67/05.05.2022 and a staff researcher in the DFG-AZ/PE980/9-1 project at Constructor University Bremen.

4 Personal impressions

I have known Dr. Tsanov since his time as a Ph.D. student at Queen's University and from the very beginning I have been extremely impressed by his talent and work ethic. In the past ten years I have had the opportunity to observe him speak in both informal and formal professional events and I am always amazed by the ease with which he establishes rapport with the audience. Tsanov gives inspiring lectures and is very generous with his insight and guidance in an instructional environment. In addition to being an idea generator, a mathematical toolbox builder, and an excellent problem solver, Tsanov is very particular in matters of scientific exposition. His papers are a pleasure to read, and their ripple effects are clearly visible across algebraic geometry, algebraic combinatorics, and representation theory.

5 Conclusion

The papers submitted by Dr. Valdemar Tsanov to this competition unequivocally demonstrate that he is a true leader in representation theory and geometric invariant theory. His work has had and will continue to have a trasformative impact on global projective geometry, singularity theory, quiver gauge theory, and geometrization in low dimensional topology. Tsanov's scientific achievements are more than sufficient to satisfy the requirements for an appointment as an Associate Professor at any high profile academic institution.

Based on all this, I strongly recommend to the scientific panel to approve the candidacy of Dr. Tsanov and to propose to the Scientific Council of IMI-BAS to appoint Dr. Valdemar Vassilev Tsanov as an Associate Professor in the **area of higher education:** 4. Natural sciences, mathematics and informatics; **professional field:** 4.5 Mathematics; **scientific specialty:** Geometry and Topology (Homogeneous spaces and Geometric Invariant Theory).