Fostering Mathematical Problemsolving Abilities by Textbooks?

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1. Why problemorientation?

There are many reasons to place more emphasis on problemorientation in mathematics instruction:

- the latest results from TIMSS and PISA (cf. e.g. BAUMERT et al. 1997, 2000, 2001),
- educational reasons (cf. e. g. KLAFKI 1985),
- modern learning theories (esp. constructivism and connectionism; cf. VON GLASERSFELD 1991),
- epistemological reasons (cf. Popper 1994),
- increasing challenges by more complex problems from modern life,
- reasons from the nature of mathematics and its history.

More details on possible reasons for more problemorientation in mathematics instruction are outlined, e. g., in ZIMMERMANN 1997.

2. Possible obstacles and the role of the teacher

There is no doubt that the teacher is the most important participant in the "classroom game": he sets educational and mathematical systems of values – and these might be in conflict with a problemoriented approach to some extent. Some important *obstacles to implement a problemsolving approach* into normal classroom teaching could be seen in the following opinions of many teachers (cf. ZIMMERMANN 1997):

- *Concentration on routine techniques:* They are much more important for real life than so called real problems.
- *Time*: Many teachers would like to treat more real problems in their classes if they would have enough time. But normally the syllabus is so overloaded that it is already hard to "cover all that stuff".
- *Examinations*: The teacher has to prepare his students for the tests and for the final examinations. Normally there are no real problems in these examinations. So it is a waste of time to deal with real problems.
- *Problems are only for the gifted*: Real problems are too difficult and they are not appropriate for normal students. Therefore they should be given to gifted or interested students only - maybe in special classes.
- "*We do it already*": One of the results of TIMSS is as follows: many mathematics teachers who participated in this study, e. g. from the US, are very content with their instruction style. They think that they keep quite close to the "standards" of the NCTM 2000, where problemsolving is a main focus. But comprehensive videotaped lessons demonstrate that this self-estimation cannot be proved by reality. There is a similar discrepancy in Germany.
- "I know what works and what will not work": After decades of teaching there might have been developed an equilibrium between all driving forces in the classroom. Why should I change? Resistance concerning change could be seen, e. g., in results form studies on teacher beliefs carried out by PEHKONEN 2001.

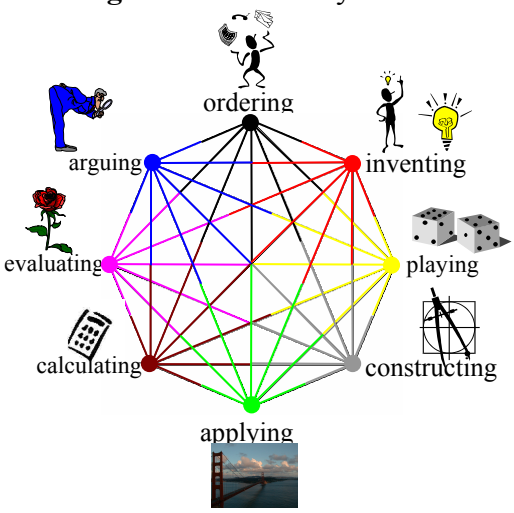
Obstacles like the aforementioned ones, of course, cannot be redressed by a textbook only because of the "*paper is very patient*"-principle: a creative teacher might cope with a very traditional textbook in a very open way and a traditional teacher might teach the stuff from an innovative textbook in a very conventional way.

What might be the consequences? Obviously it is not necessary to write a better textbook for a good teacher, but nevertheless, it might give some additional hints even for a "master teacher".
The normal teacher needs enough "well-known" areas, covered by an innovative book, too. E. g. there should be offered enough exercises so that the pupils can practice routine techniques, which are necessary for them anyway! Furthermore, there should be very well-known tasks, with additional suggestions for opening and enlarging such classical task to a problem field – as far as there is enough time, courage and willingness within all participants of the class.
If there is the main focus on learning routine techniques by heart at the very beginning of mathematics learning, one cannot expect the children to develop problemsolving abilities and flexibility in a satisfactory way later on because a specific way of learning and thinking is installed in their heads in this way. Therefore, it is very important to offer to children an appropriate mixture of non-routine problems as well as situations, where learning of routine techniques is important. As a consequence of this insight, additional help should be given to the novice teacher, e. g. by in-service-courses, by initiating or provoking discussions and reflections about different teaching practice.

3. The leading ideas of the textbook- series

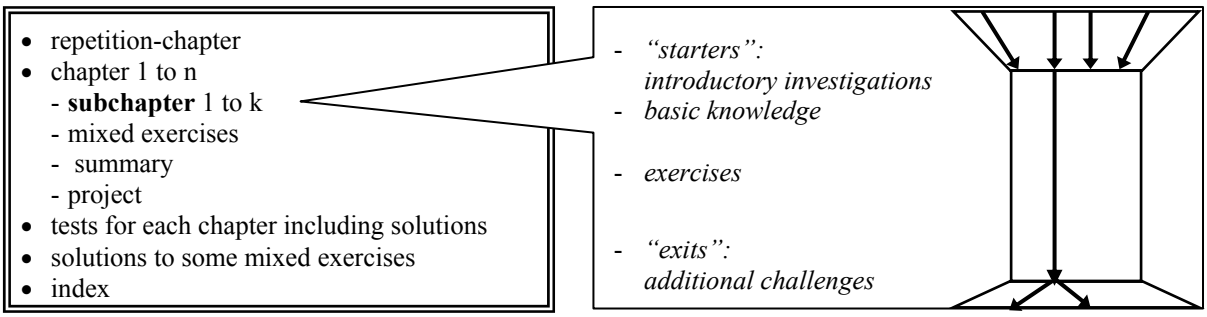
So we came to the conclusion, that sound learning of mathematics is based on productive activities of the students mainly. Therefore the main focus of our new textbook-series "MatheNetz" (MathNet; cf. CUKROWICZ/ZIMMERMANN 2000, 2001, 2002) has to be on this aspect. The corresponding *leading ideas* for the arrangement and stimulation of mathematical activities of the pupils are as follows (cf. CUKROWICZ/ZIMMERMANN et al. 2001):

- **Problemorientation**: see part 1.
- **Orientation on processes and methods**: Problems with a different degree of openness and variability should be offered to the pupils, to give them the opportunity to develop their individual ways for solutions and to communicate them to their classmates. Very often thinking methods and heuristics might help in problemsolving, which proved already to be successful during several thousands of years. Such heuristics are mainly "taught" in an implicit way in our books by offering appropriate problems. Later on there are given also opportunities, to make conscious such thinking methods by reflecting on different ways of solving problems.
- **Orientation on different teaching- and learningstyles:** We try to take into account different boundary conditions of classroom settings as well as different teaching- and learningstyles of the teachers res. pupils by offering a broad range of different learning situations. These might enable the teacher to use the book in very different ways.
- **Orientation on well-tried routines:** In addition to offering comprehensive opportunities for exploring, conjecturing and arguing there are given opportunities as well for the development of a sound basic knowledge, the formation of routines and automation of important techniques by comprehensive exercise material.
- **Orientation on connections and connecting thinking:** There are many connections between activities like those described just before and many further activities, which will be described later on as well as relations to the content to be learned.
  By careful and comprehensive studies of history of mathematics we came to the conclusion that – besides heuristical methods for ***inventing*** already mentioned above (like working backwards, analogizing, change of the mode of representation etc.) there were



many other activities, which proved again and again to help to reveal or produce new mathematical knowledge. Therefore, we selected material for our textbook with respect to that aspect, too, that it might require or help to initiate or foster activities as follows:
In all known cultures it can be found at the very beginning that documentation of quantities and manipulation was of major interest. This lead to the first important mathematical activity: ***calculating***. Problems, e. g., from astronomy and agriculture are until our days (cf. spaceindustry and ecology) very important domains to ***apply*** mathematics res. to develop new mathematical models. ***Constructing*** is not only in geometry but also in architecture – which was taken as a part of mathematics for a long time - the most important activity. ***Arguing***, esp. proving is at the core of modern mathematics and belongs to the more challenging mathematical activities. Of course, this activity is also related to inventing methods, we dealt with already above. The tension to bring new knowledge, a set of new theorems or clusters of solved problems in a systematic ***order***, in the upper grades first approaches to axiomatization, might lead – practiced in appropriate situations and at appropriate time – to a deeper understanding and to more insight into theoretical interrelations. One has to take into account, too, that the striving for religious cognition and related systems of ***values*** generated also new problems and their solutions and produced in this way also new mathematical knowledge during history of mathematics.
The same holds for an approach to mathematics by ***playing*** and the development of recreational mathematics. In this way very oft new branches of mathematics were created like stochastic and game-theory.
These different activities (which are very often important not only in mathematics) are connected and interrelated in many ways, which is represented in the diagram above. This corresponds to the general goal to achieve a high degree of flexibility in thinking, to foster connecting thinking and mastering many routine activities as well. These are essential goals of our new textbook.
Therefore we place less emphasis upon problems with one and only one solution and upon such problems which cannot be augmented and developed to a comprehensive problem-field, as well as upon "single-lane-methods".
I want to emphasize, that I learned much about the stimulation of pupils' thought-processes by problemfields and open-ended problems in the "Hamburg-Modell" (cf. e. g. WAGNER/ZIMMERMANN 1986), where KARL KIEßWETTER is the main generator. I could make the experience in my own teaching and in other normal classes (in Finland as well!) that many of his ideas can be used (and worked!) even in classrooms with lower achievers.

4. The structure of the textbook



Explanations:

- **Repetition-chapter**: In this chapter the mathematical contents from previous grades are summarized. One criterion for the selection of a specific content and for the comprehensiveness of its treatment is included in the question, to what extent the respective content is a prerequisite for the successful work with this very volume.

- **Chapter 1 to n**: The respective chapters are structured in the same way into subchapters, excluding the repetition-chapter and the tests.
  * The characteristic main structure for the **subchapters 1 to k**, into which the content is elaborated, is as follows:
    - At the beginning of each subchapter there are offered two or three, sometimes up to seven "starters" in form of *introductory investigations*. More or less open-ended problems are presented here, which should give an opportunity to the pupils to come to the new main questions, content, methods and concepts of the subchapter by own productive activity mainly. Normally, *one* introductory investigation should do for this end. If this should be not the case, hints are given.
    - In the *basic knowledge* the main content, concepts and methods are presented in a concentrated way. The basic knowledge includes the appropriate and usual terminology in addition to the preceding more or less colloquial language of the pupils during the discussion of an introductory investigation. The understanding and the associative memory of the pupil should be supported by examples and visualizations. Important methods might be applied by the pupils in a better way by giving to them a representative example with complete solution(s). A more demonstrative style of presentation is used taking into account the fact that this part was not developed to give material for the first acquaintance of the pupils with a new content, but should be used by them for repetition or to look up something, mainly.
    - By the following comprehensive *exercises* indispensable tools are given to develop automation and necessary routines when applying methods. This part contains as well enough exercises in the sense of the operatoric principle of Piaget. Furthermore, exercises are offered, too, to generalize and modify problems.
    - At the end of each subchapter some "exits" in form of *additional challenges* are presented, which are normally more open-ended. Pupils have the opportunity to extent and explore themes they became acquainted in this chapter, e. g., in a playful way. They can learn more about possible applications, they can carry out their own research or they may learn more about possible historical origins of the leading ideas of this chapter. The problems are often structured in a way, that student might be motivated to climb up step by step to more challenging situations.
  * The content of the whole chapter should be covered by *mixed exercises*, which are presented after the subchapters. Additional connections to foregoing material are taken into account as well.
  * The **summary** contains the most important themes from the basic knowledge of the subchapters.
  * Material for a **project** is presented at the end of the chapter, giving opportunities to explore a more encompassing and comprehensive theme. The main emphasis is placed upon determination of goals by the pupils as well as opportunities for exploration, cooperation and presentation of results.
- Two **tests** for each chapter are at the end of the book, followed by solutions so that the pupils can check their solutions. They are designed in such a way that each can be done within 45 minutes normally. Sometimes parallel tests are offered; sometimes one test is more challenging than the other one.
- Finally, there are **solutions** to some **mixed exercises**, giving pupils the opportunity to control their knowledge and abilities and to complete their learning if it is necessary, thus leaving more responsibility for the learning process to the pupil.

Some more specialities:
Icons at the margin suggest special forms of cooperative work, the use of additional tools (like calculators or PCs) or other aspects of the specific task (historic relations, additional difficulty, mental geometry, additional inquiries, repetition).

At the bottom of many pages tasks are presented, which require repetition of previous contents, so referring to the leading idea of connectedness in an explicit way.

5. How to use the textbook

The textbook can be used in many different ways, depending on the time available, the abilities of the pupils, the educational goals or the specific content focus.

The **repetition-chapter** might be used as a resource for learners, parents or remedial teachers to get a quick overview about the necessary knowledge the pupils should have to work with this book in a reasonable way. Connected with the index at the end of the book, this chapter can be used also as a "reference book" for mathematical concepts, methods and theorems. The pupils have the opportunity, too, to learn content, they missed formerly, fresh up their memory and control their abilities. Teachers might check with aid of this chapter, whether their pupils learned the necessary content formerly and they can react better to knowledge-gaps, occurring suddenly in a concrete teaching situation. But this chapter is not appropriate to learn a content which is completely new to the whole class.

When choosing and creating the **"starters"** we placed strong emphasis upon a very broad range of variability. So they differ e. g. by

- their length and their degree of operationalization,
- their degree of openness,
- the sort or imbedding of mathematics (e. g., pure, applied or historical context),
- the degree of difficulty,
- the time, necessary to treat this investigation,
- the sort of activity, triggered by this investigation (drawing, calculating, manipulating, writing, discussing, playing etc.),
- the workingstyle and the necessary or possible additional tools (partner- or groupwork; the use of computers, more comprehensive inquiries).

In this way the teacher has the opportunity to adjust his teaching approach to the needs of the specific learning group. In addition to the differentiation, caused by more open ended problems more or less automatically, the teacher might aim on differentiation directly by letting the pupils (or group of pupils) work on different starters.

In case that there is a lack of time or because of other reasons the teacher might chose also a more direct starter or more controlled and less autonomous work of the pupils.

After some time of active and more or less independent productive work of the pupils they will normally discuss their results. The outcome of such work and discussions should be fixed as **basic knowledge**, e. g. on the blackboard or in their exercisebooks.

Normally, one starter should already enable the pupils to come to the main parts of the basic knowledge. In case that this is not possible, appropriate hints are given.

Of course, an investigation might be approached also rather directly in a well experienced way. Very often there are opportunities, to split up a more extended starter into different parts. The first part might be tackled, e. g., by an introductory discussion in the whole class, the next part might be done by the students in single or group work and the final part could be given as homework.

Starters, which were not used as an introduction to the new content, might be used later on as material for additional exercises.

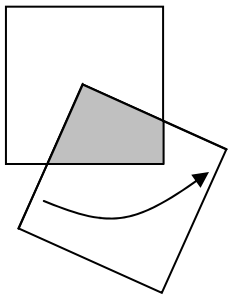
The repetition-problems at the bottom of the page should be tackled in regular manner by the pupils to memorize former topics and to connect it to the new one.

This textbook can be also used as rich collection of problems. There are many opportunities to extent classical tasks to more comprehensive problemfields. So there is given the opportunity to connect the reinforcement of basic knowledge and the improvement of problemsolving abilities. In this way a possible contribution might be given to an even better culture of mathematics teaching.

6. Examples

No. 1 By this first example we want to demonstrate how the more or less boring congruence theorems might be motivated and reinvented by pupils and in which way quite different heuristics like "examine special cases", "working backwards", "intuitive infinitesimal methods" "analogies", "generalizations" and "variation" of a problem might work together in this context (cf. MN8, p. 148):

Two congruent squares are located in such a way, that a corner of one square is placed into the center of the other one (cf. the picture). What is the size of the area, both squares have in common?



a) How does the size of the shaded area changes if you rotate the second square?
b) Instead of two squares, take two congruent rectangles, one side two times as long as the other side. Investigate the area now, the two rectangles have in common? Look for conjectures and try to prove them. What can you find out with other geometrical figures?
c) Investigate the problems from a) and b) by using dynamical geometry-software.

Remarks and some possible solutions:

For most of the pupils the "congruence theorems" (SSS; ASA; SAS and SSA) are more or less obvious principles under which conditions a unique triangles can be constructed. By posing some open ended problems these conditions can be figured out by some exploratory work of the pupils. These are offered in MN8, Ch. 5.1. In this connection these "theorems" are for pupils nothing else but rules for construction.

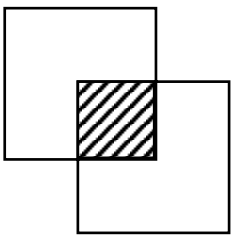
Taking such experience into account it is a very tough issue for the pupils (and demands very hard educational work for the teacher!) to understand, that there is the necessity for a proof. It demands a very abstract point of view for seventhgraders to show and to understand, that, under certain well known conditions, for given two triangles there exists always a congruence transformation, which maps the first triangle onto the second one!

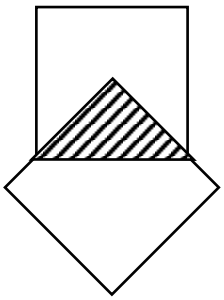
Therefore we looked for an alternative way of motivation.

The very reason of existence of the congruence theorems is their use as an aid or tool for proving some other non trivial and non obvious theorems like, e. g., the concurrence theorems or the problem formulated above.

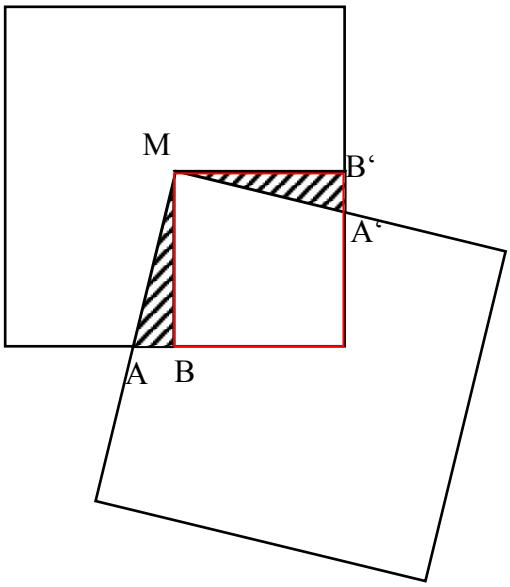
The attempt to solve it might lead not only to the conjecture of SAS but might be connected also with the (at least implicit) use of important heuristic strategies:

a) Let us, for example, "*examine special cases*" (see right). That might trigger the following conjecture:
   The shaded area is always ¼ of the area of the hole square.
   Idea for a proof:
   We take the left special case as a starting point. Turn the lower square around the center M of the upper one a little bit to the left. One might come to the conjecture, that the triangle MA'B' has the same area as the triangle MAB.
   We know, that the angle B'MA' has the same size as BMA: it is the rotation angle.





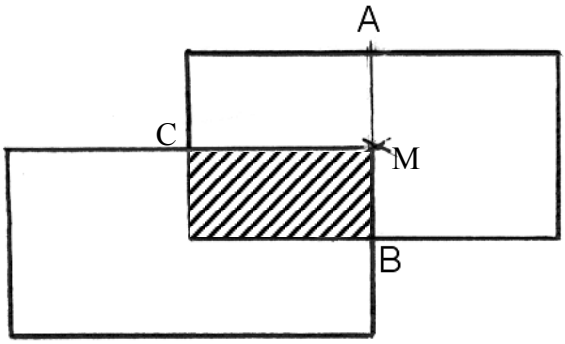
Furthermore, the size of MB' is equal to the size of MB, namely half as long as the side of the square. Finally, both triangles have a right angle in B res. B'. Would this already do to prove that the size of all other corresponding sides and angles of both shaded triangles are equal? In this way ASA is "triggered" and one might feel motivated to prove it. If this conjecture would be true, it would follow that both triangles would have the same area. Therefore, the overlapping area of both squares would be always ¼ of the area of a whole square.



So one has to prove SAS now.

In such context this theorem (and the other ones) is not a "construction-method for triangles", but an economical way, to identify two triangles, of which existence is assumed, as congruent with a minimum of effort.

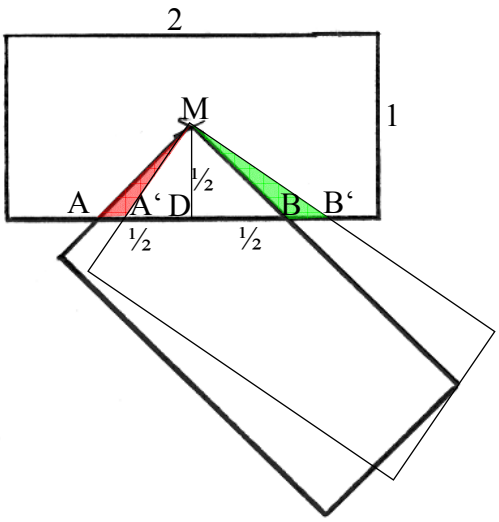
b) In the following position the area covered by both rectangles is a quarter of the area of the whole rectangle, but for other positions the shaded are cannot be always the same like in a):



If you turn the lower rectangle a little bit to the *right*, the size of the shaded area will *decrease* because the rectangular triangle "to be added" at the right side $\overline{MB}$ of the lower rectangle has a smaller basis and altitude than the triangle with side $\overline{MC}$ to the upper left which you have to take away.

Analogously: If you turn the lower rectangle a little bit to the *left*, the size of the shaded area will *increase* because the rectangular triangle "to be added" at the upper side $\overline{MB}$ of the lower rectangle has a larger basis and altitude than that triangle with side $\overline{MB}$ which you have to take away.

In a position where $\overline{AM} = \overline{MB}$ (cf. picture to the right) the size of the overlapping area of both rectangles is a *minimum*.



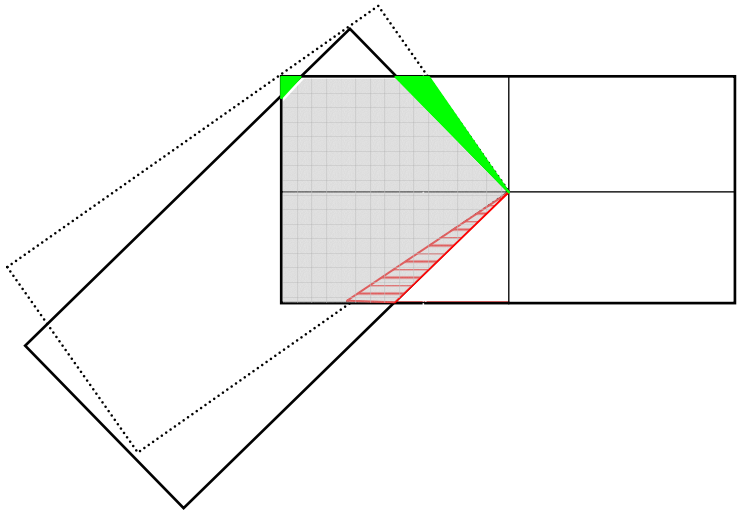
The reason can be given again by the same type of infinitesimal argument like above:

If you rotate the lower rectangle in the drawn position a little bit to the right, the green triangle which has to be "added" is larger than the red one which has to be "subtracted", because BB'>AA'. By symmetry it is clear that the situation is the same if you rotate a little bit to the left.

Both triangles are not congruent, in spite of

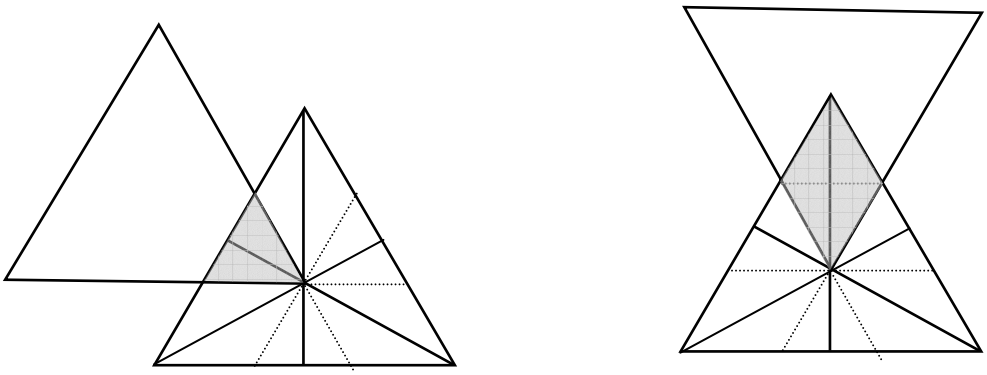
the fact, that they have the length of one side (AM and BM) and the size of two angles in common, but AA'M does not correspond to MBB'. In this way another necessary condition for the theorem ASA can be discovered by the pupils.

It is more difficult to find a *maximum* of the overlapping area. The situation to the right seems to be the solution we look for. But one has to be aware, that infinitesimal small rotations up or down do not reveal whether the shaded area is decreasing.



It might be another interesting investigation for the pupils to find a rectangle with maximum and minimum solutions in the drawn positions (e.g., length of the sides of the rectangle 1 and 1.5 would do)

If we take instead of a square other regular polygons, the situation is a little bit simpler.
Let us take two *equilateral triangles*.
Here we have very clear extreme positions for the triangles. Because we have only one type of concurrent lines, there is a well defined center. We draw parallel lines to all sides through this point as an additional help to determine the size of the shaded areas:



So one can "see", that in case of a *minimum* the size of the area is 1/9, in case of a *maximum* the size of the area is 2/9 of the size of the area of the large triangle.

One can generalize this problem, e. g., by thinking of an arbitrary convex plane figure with an angle rotating around its vortex in its interior.

No. 2: The next example is taken from algebra (MN 9, exercise 16, p. 48):

a) Solve the following system of linear equations:

I)	II)	III)	IV)
$x+2y=1$	$x+2y+3z=1$	$x+2y+3z+4u=1$	$x+2y+3z+4u+5v=1$
$2x+2y=1$	$2x+2y+3z=1$	$2x+2y+3z+4u=1$	$2x+2y+3z+4u+5v=1$
	$3x+3y+3z=1$	$3x+3y+3z+4u=1$	$3x+3y+3z+4u+5v=1$
		$4x+4y+4z+4u=1$	$4x+4y+4z+4u+5v=1$
			$5x+5y+5z+5u+5v=1$

b) What is the solution of the corresponding system of equation with n variables and n equations? If suitable and available you might get some help for your conjectures by a computeralgebrasystem (CAS). Try to prove your conjectures.

c) Replace the number 1 on the right side of the equations by 2; 3; ...; n (or by the consecutive numbers 1; 2; 3; .... ; n). What kind of solution you get now? Prove your conjecture. Look for other variations of the introductory problem.

d) Create your own systems of linear equations with similar patterns (think of figured numbers!) so that it has simple solutions!

e) Given now a sequence of possible solutions as x=1; x=1 and y=2; x=1 and y=2 and z=3; x=1 and y=2 and z=3 and u=4; ..... Create an appropriate sequence of systems of linear equations which has these numbers as solutions.

Remarks:

This problemfield was constructed by using well known properties of figured numbers (think of the representation of squares of a number by the sum of odd numbers).

The degree of difficulty and the degree of openness is increased step by step.

In a) nearly every pupil should have the opportunity to find a solution for at least one system of equations. By the generated pattern of solutions he should have a good chance to come to a conjecture for a general solution.

In b) the opportunity is given to train conjecturing (also without CAS!) and proving.

In c) pupils have a simple and first opportunity to vary the problem.

In d) the pupils can extent the type of variations of the problem. Depending on their experience they can be inspired by patterns of figured numbers like those of triangular numbers to create other coefficient matrices of the system of linear equations. Other possibilities might be triggered e. g. by the Pascal-triangle, the harmonic triangle or the triangle of Farey-fractions (cf. e. g. MatheNetz 7, p. 153). Conjectures might be supported by CAS.

Part e) was constructed by using the operatoric principle: now solutions are given and possible suitable problems are to be created. Here are given many possibilities to play with the problem.

In this way heuristical thinking occurs mainly in the textbooks. It has been incorporated in several other aspects into this textbook-series as well:

Many topics are presented in a historic-genetic way, e. g. like chapter 1 in MatheNetz 7, where we start with classical problems for the “rule of three” and end with the function-concept. This way offers the opportunity to start with every-day-problems and let the pupils generate the function-concept as a general tool to help to solve larger classes of problems. Other textbooks go the other way around, which is orientated more towards the development of concepts and to the systematic of the modern subject(from the general case to the specific situation).

Furthermore there is given also the opportunity – after some 9 years of implicit experience in using and creating heuristical strategies - increasing step by step conscious reflection of such strategies, which might give an additional reinforcement of problemsolving abilities (chapter 8 in MatheNetz 9).

7. Prospects

As a consequence of the aforementioned considerations, e.g., the following research questions might be useful:

- How to operationalize goals on mathematical problem solving and to what extent it is possible and reasonable?
- How textbooks are used in mathematics instruction?
- How textbooks should be used in mathematics instruction?
- To what extent teaching (esp. the creation of productive learning environments) can be supported by textbooks?
- Are methods useful - like those presented in 5. - to improve problem solving performance? Under what conditions this might be possible? What other types of presentation in textbooks might be useful?

This might lead to a research program as follows:

- Determination of common goals.
- Analysis of use of textbooks in mathematics instruction.
- Translation of parts of textbooks and tests in other languages.
- Determination of different modes of use of textbooks in mixed abilities classes.
- Use of multimedia-tools in teacher training.
- Development of appropriate testing instruments to check the effectiveness.

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