"negative times negative gives positive, negative times positive gives negative"
Diophant from Alexandria, III AD

"We should not forget that zero and negative numbers were the last to be accepted"
Garett Birkhoff [1]

An introduction to the arithmetic of approximate numbers

Svetoslav Markov, IMI–BAS

Approximate numbers are ordered pairs of (real) numbers and error bounds. Error bounds, briefly *errors* are represented by nonnegative numbers. To compute with approximate numbers one should know how to perform arithmetic operations over errors and how to compare them. To model computations with errors one should suitably define and study arithmetic operations and order relations over the set of nonnegative (real) numbers. In this work we discuss the algebraic properties of nonnegative numbers starting from familiar properties of real numbers. Then we discuss the algebraic properties of approximate numbers, which are considered as (real) numbers with errors.

1 Introduction

Denote $\mathbb{R}$ the set of real numbers. We start by recalling the familiar system $(\mathbb{R}, +, \leq)$ involving the arithmetic operation addition "+" and the order relation preceding (following)"$\leq$".

Real numbers are usually presented by their sign and modulus, e. g.: $2, -2, \pi, -\pi, 3.14, -3.14$, etc. More generally, a real number $a \in \mathbb{R}$ is presented either as $a = +A$ or $a = -A$, where $A = |a| \geq 0$ is modulus of a.

Thus a real number $a \in \mathbb{R}$ is presented as an ordered pair comprising the modulus of a denoted $A = |a| \in \mathbb{R}^+$ and the sign of a denoted $\alpha = \sigma(a) = \{+, a \geq 0; -, a < 0\} \in \Lambda = \{+, -\}$. Here $\mathbb{R}^+ = \{a \in \mathbb{R} \mid a \geq 0\}$ is the set of nonnegative real numbers. We shall write $a = (A; \alpha) \in \mathbb{R}^+ \otimes \Lambda = \{(X; \xi) \mid X \in \mathbb{R}^+, \xi \in \Lambda\}$.

Note that the set of pairs $\mathbb{R}^+ \otimes \Lambda$ admits both elements $(0; +)$ and $(0; -)$, which correspond to the single element $0 \in \mathbb{R}$. Assuming $-0 = +0$, that is $(0; +) = (0; -)$, we can write: $\mathbb{R} \equiv \mathbb{R}^+ \otimes \Lambda$ in the sense that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^+ \otimes \Lambda$.

In the sequel we denote the elements of $\mathbb{R}$ by lower-case letters $a, b, c, ...$, and the elements of $\mathbb{R}^+$ by upper-case letters, $A, B, C, ...$

Properties of addition. We start from the assumption that addition and order in a $\mathbb{R}^+$ are familiar (as restrictions from $\mathbb{R}$).

Problem 1. Describe the arihmetic operation addition "+" in $\mathbb{R}^+ \otimes \Lambda$, so that $(\mathbb{R}, +) \cong (\mathbb{R}^+ \otimes \Lambda, +)$.

Remark. In school this is done verbally, e. g. "Adding Real Numbers with Opposite Signs. Step 1: Take the difference of the absolute values. Step 2: Attach the sign of the number that has the larger absolute value, etc." [21]. We want a symbolic formulation.

Let $a = (A; \alpha),\ b = (B; \beta) \in \mathbb{R}$. In the case $\alpha = \beta$ we have

$$a + b = (A; \alpha) + (B; \beta) = (C; \gamma),$$

where $C = A + B$, $\gamma = \alpha = \beta$, that is

$$(A; \alpha) + (B; \beta) = (A + B; \alpha),\ \ \alpha = \beta.$$

Here "$A + B$" is the operation addition in $\mathbb{R}^+$.

Inner addition of errors. To examine the case $\alpha \neq \beta$ we need the binary operation "inner addition" in $\mathbb{R}^+$ (briefly: n-addition, i-addition?):

$$A +^- B = \begin{cases} Y|_{B+Y=A} & \text{if } \ B \leq A; \\ X|_{A+X=B} & \text{if } \ B > A. \end{cases} \tag{1}$$

In words, $A +^- B$ is the solution of $B + Y = A$ or the solution of $A + X = B$ depending on which one exists; note that if both solutions exist, they coincide.

Remark. In familiar terms the operation inner addition in $(\mathbb{R}^+, +)$ is written as $A +^- B = |A - B|$. However, strictly speaking we have no right to write $|A - B|$ in $\mathbb{R}^+$, as we have not defined operation subtraction $A - B$. (In fact we cannot define subtraction in $\mathbb{R}^+$ as inverse operation to addition, see [13], [14].)

Denote $\gamma(a, b)$ the sign of the argument with larger module. Symbolically, we define a mapping $\gamma : \mathbb{R}^2 \longrightarrow \Lambda$ as follows:

$$\gamma(a, b) = \gamma((A; \alpha), (B; \beta)) = \begin{cases} \alpha & \text{if } B \leq A, \\ \beta & \text{if } B > A. \end{cases} \tag{2}$$

An equivalent expression for (2) is

$$\gamma(a, b) = \begin{cases} \sigma(a) & \text{if } |b| \leq |a|, \\ \sigma(b) & \text{if } |b| > |a|. \end{cases}$$

Note $A +^- B$ is the distance between the numbers A, B.

Remark. We assumed the order relation "$\leq$" in $\mathbb{R}^+$ to be familiar as restriction of "$\leq$" in $\mathbb{R}$. Note addition induces the order relation "$\leq$" in $\mathbb{R}^+$. Namely, for $A, B \in \mathbb{R}^+$ by definition $A \leq B \Longleftrightarrow \exists X : A + X = B$. Therefore we have the right to use "$\leq$" as we have done in (1) and (2).

It is easy to see now that in the case $\alpha \neq \beta$ the sum $(C; \gamma) = (A; \alpha) + (B; \beta)$ is given by $C = A +^- B$, $\gamma = \gamma(a, b)$. Summarizing, we have

$$a + b = (A; \alpha) + (B; \beta) = \begin{cases} (A + B; \alpha), & \text{if } \alpha = \beta; \\ (A +^- B; \gamma(a, b)), & \text{if } \alpha \neq \beta, \end{cases}$$

which can be compactly written as

$$(A; \alpha) + (B; \beta) = (A +^{\alpha\beta} B; \gamma), \ \gamma = \gamma(a, b). \tag{3}$$

In (3) we assume that for $\alpha, \beta \in \Lambda$ a binary boolean operation "$\cdot$" is defined by $\alpha \cdot \beta = \alpha\beta = \{+, \alpha = \beta; \ -; \alpha \neq \beta\}$.

Remark. It is rather interesting that we perform the operations (1) and (2), resp. (3), in our minds every time when we add two (real) numbers (and we do a bit more work when they have different signs). This procedure we have learned at school and it seems simple, but looks somewhat complicated when described strictly in detail.

Problem 2. Describe order relation preceding (following)"$\leq$" in $\mathbb{R}^+ \otimes \Lambda$, so that $(\mathbb{R}, +) \cong (\mathbb{R}^+ \otimes \Lambda, +)$. (Left for exercise.)

Algebraic properties of $(\mathbb{R}, +)$. Let us first recall the algebraic properties of $(\mathbb{R}, +)$. The system $(\mathbb{R}, +)$ is an additive group, that is

i) "+" is a closed (total) operation,

ii) "+" is associative: $(a + b) + c = a + (b + c)$ [9],

iii) there is an identity (neutral, null) element 0, such that $a + 0 = a$ for all a;

iv) there exists an additive inverse (opposite) element $-a$, such that $a + (-a) = 0$ for all a [13], [14].

Property iv) induces an operation subtraction $a - b = a + (-b)$ in $\mathbb{R}$.

Thus we can fully write $(\mathbb{R}, +) = (\mathbb{R}, +, 0, -, \leq)$ in order to mark that system $(\mathbb{R}, +)$ possesses also null, opposite (subtraction) and order.

We recall that:

— property i) defines a magma (groupoid) [17],

— properties i)–ii) define a semigroup [8],

— properties i)–iii) define a monoid [16],

— properties i)–iv) define a group [10].

Note that a group always obeys the following property:

— v) "cancellation law": $a + x = b + x \Longrightarrow a = b$ [7].

An algebraic system may satisfy also property:

— vi) commutative law $a + b = b + a$ [20]; then it is called commutative (abelian) system.

Clearly, $(\mathbb{R}, +)$ satisfies all enlisted properties i)– vi) and thus is a commutative (abelian) group.

Note also that the commutative group $(\mathbb{R}, +)$ satisfies the property:

— vii) divisibility, which in the additive case means that equation $a+x = b$ has an unique solution for all $a, b \in \mathbb{R}$.

Algebraic properties of $(\mathbb{R}^+, +)$. Consider now the algebraic properties of the system of errors $(\mathbb{R}^+, +, \leq)$. Properties i)–iii) and vi) are satisfied so $(\mathbb{R}^+, +)$ is a commutative monoid, but property iv) fails; indeed, equation $A + X = 0$ has no solution when $A \geq 0$. However, instead of property iv), the (weaker) cancellation property v) $A+X=B+X \Longrightarrow A=B$ holds true [6]. As we noted every group is cancellative, but a cancellative monoid may not be a group, as is the case with $(\mathbb{R}^+,+)$.

As mentioned, property iv) fails that is there is no additive inverse (opposite) in the cancellative monoid $(\mathbb{R}^+,+)$. Divisibility property vii) does not hold as well, as $A+X=B$ does not possess a solution in general; in algebraic language this means that $(\mathbb{R}^+,+)$ is not a quasigroup [11]. However, there is an operation inner addition "$+^-$" defined by (1) which allows solving equations of the form $A+X=B$ in certain cases. MNamely, using inner addition "$+^-$" we can solve equation $A+X=B$ when $A \le B$ or equation $B+X=A$ when $A \ge B$. Thus inner addition "$+^-$" plays a role in $(\mathbb{R}^+,+)$ that is analogous to the role of subtraction in the group $(\mathbb{R},+)$. We may call this property "almost (week) divisibility (subtractability)".

Q. "Week" instead of "almost" ?

Up to now we have seen that system $(\mathbb{R}^+,+)$ possesses null, inner addition and order, so we can fully write $(\mathbb{R}^+,+)=(\mathbb{R}^+,+,0,+^- \le)$. Clearly, the enumerated algebraic properties in $(\mathbb{R}^+,+)$ induce certain manipulation rules, such as the rule based on formula (1), saying that the solution X of equation $A+X=B$ when $A \le B$ is $X=A+^-B$. We may call the system of algebraic properties and arithmetic rules in $(\mathbb{R}^+,+)$ "error arithmetic". Thus error arithmetic is a set of rules needed to compute with errors (error bounds of approximate numbers).

We shall next extend the error arithmetic rules by focusing attention on the operation inner addition.

Algebraic properties of $(\mathbb{R}^+,+^-)$. Consider the algebraic properties of the system $(\mathbb{R}^+,+^-)$ in some detail. We have:

— i) inner addition "$+^-$" is a closed (total) operation. Hence $(\mathbb{R}^+,+^-)$ is a magma [17];

— associativity property ii) $(A+^-B)+^-C = A+^-(B+^-C)$ fails, indeed, e. g. $(7+^-5)+^-3 \neq 7+^-(5+^-3)$, as $(7+^-5)+^-3=1$ and $7+^-(5+^-3)=5$;

— property iii) existence of an identity (neutral, null) element, such that $A+^-0=A$ for all $A \in \mathbb{R}^+$, holds true; hence $(\mathbb{R}^+,+^-)$ is an unital magma [17], [18];

— property iv) for existence of an inverse (opposite) element holds true as well. Indeed, the inverse of A is the element A itself, since $A+^-A=0$

for all A. (This is why we may call the operation "$+^-$" inner subtraction as well.)

— property v) "cancellation law": $A +^- X = B +^- X \Longrightarrow A = B$ fails, e. g. take $A = 1, B = 5, X = 3$. Then $1 +^- 3 = 5 +^- 3$, but $1 \neq 3$;

— property vi) commutative law $A +^- B = B +^- A$ holds true.

System $(\mathbb{R}^+, +^-)$ satisfies properties i), iii) and vi) so it is commutative unital magma.

As we mentioned associativity fails in $(\mathbb{R}^+, +^-)$. In principle associativity means that every three elements can be "summed" up in any order and produce the same result. However, we may notice that associativity holds true under the requirement that the element present in both brackets is the largest one. We call this property "almost associativity" ("weak associativity"?).

Almost associativity rule. Let $A, B, C \in \mathbb{R}^+$ be such that $B \geq A, B \geq C$. Then $(A +^- B) +^- C = A +^- (B +^- C)$.

Almost-associativity is a practically important, as it says that summing up three elements does not depend on the order of summation unless we start summation always from the largest element.

We mentioned that cancellation law $A +^- X = B +^- X \Longrightarrow A = B$ fails in $(\mathbb{R}^+, +^-)$. However, cancellation is "almost" valid, which means the following. If we consider relation $A +^- X = B +^- X$ as equation for X, then it has a unique solution. Namely X is the midpoint between A and B, in usual terms the arithmetic mean $X = (A + B)/2$. So, we have $A +^- X = B +^- X \Longrightarrow A = B$ unless X is the arithmetic mean of A and B. We formulate this as a separate property.

Almost (week) cancellation. Let $A, B \in \mathbb{R}^+$. Equation $A +^- X = B +^- X$ is satisfied for $X = (A + B)/2$. If $X \neq (A + B)/2$, then $A +^- X = B +^- X \Longrightarrow A = B$.

Summarizing we obtain:

Proposition 1. The set of errors with inner addition (subtraction), that is $(\mathbb{R}^+, +^-)$, is an almost-associative and almost-cancellative commutative unital magma.

As mentioned, divisibility [11] does not hold in the cancellative monoid $(\mathbb{R}^+, +)$, it does not hold in $(\mathbb{R}^+, +^-)$ either. This means that we cannot solve directly equations $A + X = B$ and $A +^- X = B$. However, an analogous property is present, to be called "almost-divisibility", which we consider next.

The system $(\mathbb{R}^+, +, 0, +^-, \leq)$. The following proposition shows that both additions "+" and "$+^-$"are closely related. In fact the operation addition "+" induces inner addition "$+^-$" in the monoid $(\mathbb{R}^+, +)$, in a way analogous to the way addition "+" induces negation/subtraction "$-$" in the group $(\mathbb{R}, +)$.

Proposition 2. i) For $A, B \in \mathbb{R}^+$, such that $A \leq B$, the unique solution of $A + X = B$ is $X = B +^- A$. ii) Equation $A +^- X = B$ has a solution $X = A + B$ for $A, B \in \mathbb{R}^+$. If $A, B \in \mathbb{R}^+$ are such that $A \geq B > 0$, then equation $A +^- X = B$ has one more solution $X = A +^- B$.

We thus see that: i) solution of $A+X = B$ is generally not possible unless we do not assume inner addition available, and ii) solution of $A +^- X = B$ is generally not possible unless we do not assume usual addition available. However, in the light of Proposition 2 solution of both $A + X = B$ and $A +^- X = B$ becomes possible, which is a kind of "weak divisibility".

The main algebraic properties of systems $(\mathbb{R}, +)$, $(\mathbb{R}^+, +)$, $(\mathbb{R}^+, +^-)$ are summarized in Table 1.

Table 1

Axiom/System	$(\mathbb{R}, +)$	$(\mathbb{R}^+, +)$	$(\mathbb{R}^+, +^-)$
Closure	Yes	Yes	Yes
Associativity	Yes	Yes	A
Indentity	Yes	Yes	Yes
Inverse	Yes	No	Yes
Cancellation	Yes	Yes	A
Commutativity	Yes	Yes	Yes
Divisibility	Yes	A	A

Table 1. Summary of the algebraic properties of the group $(\mathbb{R}, +)$, the monoid $(\mathbb{R}, +)$ and the loop $(\mathbb{R}, +^-)$. The letter "A" stands for "almost".

2 The extended additive error system

From Table 1 we see that (outer) and inner addition complement each other. For example, addition "+" has no inverse in $\mathbb{R}^+$, whereas *inner addition* "$+^-$"

is invertible. Similar complement is observed with respect to associativity, cancellation and divisibility.

Other examples of complementary rules are the "mixed" ("hybrid"?) associative-like properties of the system $(\mathbb{R}^+, +, 0, +^-, \leq)$.

Define the mapping $\phi : \mathbf{I}\mathbb{R}^2 \to \{+, -\}$ by

$$\phi(A, B) = \begin{cases} +, & \text{if } A \geq B; \\ -, & \text{otherwise.} \end{cases}$$

Proposition 3. Let $A, B, C \in \mathbb{R}^+$. Then

$$(A + B) +^- C = A +^{\phi(B,C)} (B +^- C). \tag{4}$$

Example. Check rule (4) for $(A, B, C) = (1, 1, 1), (1, 3, 2), (1, 2, 3), (4, 6, 3)$.

For some applications the two operations for addition $+, +^-$ can be considered as one operation in two modes (directions). We shall use below the notation "$+^\theta$", wherein $\theta \in \{+, -\}$, and refer to "$+^\theta$" as "directed addition". For $\theta = +$ the operation "$+^\theta$" is the standard (positively directed) addition, "$+$", whereas for $\theta = -$, "$+^\theta$" is the nonstandard (negatively directed) addition, "$+^-$". The directed addition "$+^\theta$" can be expressed:

$$A +^\theta B = \min(A, B) \theta \max(A, B).$$

Associative-like rules for algebraic transformations.

Conditional associativity of directed addition. Directed addition is conditionally associative in the following sense:

Proposition 4. For each triple $A, B, C \in \mathbb{R}^+$ and each pair $\theta_1, \theta_2 \in \{+, -\}$, there exist a pair $\theta_3, \theta_4 \in \{+, -\}$, such that

$$(A +^{\theta_1} B) +^{\theta_2} C = A +^{\theta_3} (B +^{\theta_4} C). \tag{5}$$

Proof. Formula (5) generalizes (4). It can be directly checked that for $A, B, C \in \mathbb{R}^+$ we have

$$\begin{aligned} (A + B) +^- C &= A +^{\phi(B,C)} (B +^- C); \\ (A +^- B) + C &= \begin{cases} A +^{-\phi(B,C)} (B +^- C), & A \geq B, \\ A +^- (B + C), & A < B; \end{cases} \\ (A +^- B) +^- C &= \begin{cases} A +^{-\phi(B,C)} (B +^- C), & A < B, \\ A +^- (B + C), & A \geq B. \end{cases} \end{aligned}$$

From the above formulae we see that θ_3, θ_4 are simple functions of the errors $A, B, C \in \mathbb{R}^+$ and $\theta_1, \theta_2 \in \{+, -\}$ and can be effectively computed. This proves (5). $\square$

Example. For $A, B \in \mathbb{R}^+$, $(A + B) +^- A = B$. Indeed, using (4) we obtain: $(A + B) +^- A = (B + A) +^- A = (B + A) +^- A = (B + A) +^- A = B +^{\phi(A,A)} (A +^- A) = B +^{\phi(A,A)} +0 = B$.

Remark. The conditionally associative rules are useful as they give specific conditions under which "replacement of brackets" can be performed; thereby these conditions are easily programmable. Outer addition is commutative and associative but has no inverse, whereas inner addition is commutative, not associative and has inverse. Considering outer and inner addition together, as a"directed" operation in two different modes; we can say that this directed operation is conditionally associative. Thus both modes complement each other.

In the calculus for interval functions [3] important role play associative-like rules involving four elements.

Associative-like rules involving four elements. For $A, B, C, D \in \mathbb{R}^+$ define $\varphi : \mathbf{I}\mathbb{R}^4 \to \{+, -\}$ as

$$\varphi(A, B, C, D) = \phi(A, B)\phi(C, D).$$

Proposition 5 [3]. For $A, B, C, D \in \mathbb{R}^+$ denote

$$\gamma = \varphi(A, C, B, D) = \phi(A, C)\phi(B, D),$$

$$\delta = \varphi(A, B, C, D) = \phi(A, B)\phi(C, D),$$

then we have

$$\begin{aligned}
(A + B) +^- (C + D) &= (A +^- C) +^{\gamma} (B +^- D); \\
(A +^- B) + (C +^- D) &= \begin{cases} (A +^- C) +^{-\gamma} (B +^- D), & \text{if } \delta < 0; \\ (A + C) +^- (B + D), & \text{if } \delta \geq 0; \end{cases} \\
(A +^- B) +^- (C +^- D) &= \begin{cases} (A +^- C) +^{-\gamma} (B +^- D), & \text{if } \delta \geq 0; \\ (A + C) +^- (B + D), & \text{if } \delta < 0. \end{cases}
\end{aligned}$$

Example. The above relations may be specified in particular cases, e. g. when we know ranges for the arguments. Take for example $A \in [3, 4], B \in$

$[1,2], C \in [4,5], D \in [6,7]$. Since $\gamma = -, \delta = +$, Proposition 6 obtains the form:

$$\begin{aligned}
(A + B) +^- (C + D) &= (A +^- C) +^- (B +^- D), \\
(A +^- B) + (C +^- D) &= (A + C) +^- (B + D), \\
(A +^- B) +^- (C +^- D) &= (A +^- C) + (B +^- D).
\end{aligned}$$

Note in the last relation in Proposition 5

$$(A +^- B) +^- (C +^- D) = (A +^- C) +^{-\gamma} (B +^- D) \text{ if } \delta \geq 0,$$

the condition $\delta \geq 0$ is not as resrictive as it looks like, due to commutativity of "$+^-$", allowing us to write (if necessary) $(B +^- A)$ instead of $(A +^- B)$ and/or $(D +^- C)$ instead of $(C +^- D)$.

In the sequel we use the brief notation for the ordered additive group $(\mathbb{R}, +) = (\mathbb{R}, +, 0, -, \leq)$ and for the extended monoid $(\mathbb{R}^+, +) = (\mathbb{R}^+, +, 0, +^-, \leq\ )$.

Order isotonicity. Addition in $(\mathbb{R}, +) = (\mathbb{R}, +, 0, -, \leq)$ is isotone w. r. t. orfer relation "$\leq$".

Outer addition. We have for $X, X_1, C \in \mathbb{R}^+$ $X \leq X_1 \implies X + C \leq X_1 + C$. As a concequence we have

$$X \leq X_1\ ,\ Y \leq Y_1 \implies X \star Y \leq X_1 + Y_1.$$

Inverse isotonicity of addition. If $A, B, C \in \mathbb{R}^+$, then

$$C + A \leq C + B \Longrightarrow A \leq B,$$

in particular $C + A = C + B \Longrightarrow A = B$ (cancellation law).

Conditional inclusion isotonicity w.r.t. inner addition. Let $X, X_1, Y, Y_1 \in \mathbb{R}^+$. Assuming $X \geq X_1, Y \leq Y_1$, we obtain

$$\begin{aligned}
&\text{if } X \leq Y, \text{ then } X +^- Y \leq X_1 +^- Y_1, \\
&\text{if } \ X_1 \geq Y_1, \text{ then } X +^- Y \geq X_1 +^- Y_1
\end{aligned}$$

3 The linear and quasiliear spaces

All said above can be generalized for n-vectors (n-tuples): $(\mathbb{R}^n, +) = (\mathbb{R}^n, +, 0, -, \leq$) and $(\mathbb{R}^{+n}, +) = (\mathbb{R}^{+n}, +, 0, +^-, \leq)$ noticing that then the order relation is no more total but partial.

We introduce multiplication by scalars from the field $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. The vector space $(\mathbb{R}^n, +, \mathbb{R}, \cdot) = (\mathbb{R}^n, +, 0, -, \mathbb{R}, \cdot, \leq\ )$ models the space of midpoints main values of approximate numbers, whereas the space $(\mathbb{R}^{+^n}, +, \mathbb{R}, *) = (\mathbb{R}^{+^n}, +, 0, +^-, \mathbb{R}, *, \leq\ )$ models the system of errors (radii) of approximate numbers.

The above can be extended to the vector space $(\mathbb{R}^n, +, \mathbb{R}, \cdot, \leq)$, where now $\mathbb{R}^n$ is the space of real vectors $a = (a_1, a_2, ..., a_n)$ and $\mathbb{R}$ is the real field of scalars.

Generalization of previous definitions like $a = (A; \alpha),\ A = (A_1, A_2, ..., A_n) \in (\mathbb{R}^+)^n, \alpha \in \Lambda^n$, etc are obvious.

Multiplication by scalars is presented by

$$c \cdot a = c \cdot (A; \alpha) = (|c| \cdot A; \sigma(c)\alpha).$$

The above shows that multiplication by scalars induces a "quasivector" multiplication by scalars "$*$" in the "error space" $((\mathbb{R}^+)^n, +, +^-, \mathbb{R}, *, \leq)$ given by

$$c * A = |c| \cdot A,\ c \in \mathbb{R},\ A \in (\mathbb{R}^+)^n.$$

We can further extend the system $(\mathbb{R}^+, +, +^-)$ introducing multiplication by scalars, arriving at the space $(\mathbb{R}^+, +, +^-, \mathbb{R}, *)$. More generally, in the n-dimensional case $(\mathbb{R}^{+^n}, +, +^-, \mathbb{R}, *)$ multiplication by scalars is defined by:

$$\gamma * A = |\gamma| A,\ \gamma \in \mathbb{R},\ A \in \mathbb{R}^{+^n}.$$

The first space $(\mathbb{R}^n, +, \mathbb{R}, \cdot)$ is the well-known n-dimensional vector space. The second space $(\mathbb{R}^{+^n}, +, \mathbb{R}, *)$ satisfies the axioms for $(\mathbb{R}^{+^n}, +) = (\mathbb{R}^{+^n}, +, 0, +^-, \leq\ )$ together with the axioms involving multiplication by scalars "$*$". As we know $(\mathbb{R}^{+^n}, +, \mathbb{R}, *)$ is a quasilinear space in the sense of the following definition:

Definition. An algebraic system $(Q, +, \mathbb{R}, *)$ is a *quasilinear space (of* *monoid structure, over* $\mathbb{R}$), if for all $A, B, C \in Q, \quad \alpha, \beta, \gamma \in \mathbb{R}$:

$$\begin{aligned}
(A+B)+C &= A+(B+C), && (6)\\
\exists\, 0 \in Q : A+0 &= A, && (7)\\
A+B &= B+A, && (8)\\
A+C = B+C &\Longrightarrow A=B, && (9)\\
\alpha*(\beta*C) &= (\alpha\beta)*C, && (10)\\
1*A &= A, && (11)\\
\gamma*(A+B) &= \gamma*A+\gamma*B, && (12)\\
(\alpha+\beta)*C &= \alpha*C+\beta*C, \quad \text{if } \alpha\beta \geq 0. && (13)
\end{aligned}$$

As a consequence the following quasididtributive law takes place:

Proposition 8 (Quasidistributive law). For $A \in \mathbf{I}\mathbb{R}$, $p, q \in \mathbb{R}$ and "$*$" multiplication by scalars

$$(p+q)*A = p*A +^{\sigma(p)\sigma(q)} q*A, \tag{14}$$

Important note. For $a, b \in (\mathbb{R}^n, +, \mathbb{R}, \cdot)$:
$a \leq b$ does not imply $\gamma \cdot a \leq \gamma \cdot b$, $\gamma \in \mathbb{R}$.

However, for $A, B \in (\mathbb{R}^{+n}, +, \mathbb{R}, *)$:
$A \leq B$ implies $\gamma * A \leq \gamma * B$, $\gamma \in \mathbb{R}$.

The order relation $\leq$ in the space of errors is called inclusion and is also denoted $\subseteq$. Why?

Approximate numbers. Consider now intervals in MR-form $a = (a', b'')$, $a' \in \mathbb{R}$ (midpoint), $a'' \in \mathbb{R}^+$ (radius, error bound), i. e. $a = (a', b'') \in \mathbb{R} \otimes \mathbb{R}^+$.

As we know the binary arithmetic operations in $(\mathbb{R}, +, \leq)$ are addition "$a+b$" and subtraction "$a-b$", whereas in $(\mathbb{R}^+, +, +^-, \leq)$ are addition "$A+B$" and inner addition "$A +^- B$".

The space $\mathbb{R} = (\mathbb{R}, +, \leq)$ can be identified with the space of midpoints and the space $\mathbb{R}^+ = (\mathbb{R}^+, +, +^-, \leq)$ with the space of radii (errors) or with the space of symmetric intervals. Intervals are pairs of the form $(a'; a'') \in \mathbf{I}\mathbb{R} = \mathbb{R} \otimes (\mathbb{R}^+)$. Thus we can write down the induced binary arithmetic operations for intervals (approximate numbers) by combining the binary arithmetic operations in the space of midpoints $\mathbb{R}$ and those in the space of radii $\mathbb{R}^+$, obtaining thus the following four combinations:

$$
\begin{aligned}
(a';a'') + (b';b'') &= (a' + b'; a'' + b''), && (15)\\
(a';a'') +^- (b';b'') &= (a' + b'; a''| +^- b''), && (16)\\
(a';a'') \neg\, (b';b'') &= (a' - b'; a'' + b''), && (17)\\
(a';a'') -^- (b';b'') &= (a' - b'; a'' +^- b''). && (18)
\end{aligned}
$$

Operations (15)–(18) are the well-known interval arithmetic operations in mid-rad form:

$$
\begin{aligned}
(a';a'') + (b';b'') &= (a' + b'; a'' + b''),\\
(a';a'') +^- (b';b'') &= (a' + b'; |a'' - b''|),\\
(a';a'') \neg\, (b';b'') &= (a' - b'; a'' + b''),\\
(a';a'') -^- (b';b'') &= (a' - b'; |a'' - b''|).
\end{aligned}
$$

Operations (15)–(17) are the standard interval operations, whereas (16)–(18) are the inner interval operations.

Correspondingly, in the interval space $\mathbf{I}\mathbb{R}^n = \mathbb{R}^n \otimes (\mathbb{R}^+)^n$ we have

$$c * (a';\ a'') = (ca'; |c|a'').$$

The IEEE P1788 Standard on interval arithmetic. Late 2008, at SCAN 2008 in El Paso, TX, an effort to standardize interval computations was started by a working group of the IEEE Microprocessor Standards Committee, titled the Interval Arithmetic Working Group of the IEEE P1788 Standard. Paper [2] describes the goals of this effort, the history of the working group, and how it relates to the IEEE 754 Standard. It gives a brief overview of the policies and procedures for constructing the standard, and its expected structure. It also presents some of the questions the group may have to solve in the future.

Conclusions. The approach used in the present work shows that:

i) the mid-rad presentation of intervals — in the aspect of approximate numbers — is a natural form;

ii) the inner operation for addition is naturally induced;

iii) the abstract theory of approximate numbers presents a practically oriented material with definite instructive/didactive quality that can be used for student projects.

iv) the materals can be focused on historical aspects of mathematics as well.

References

[1] Birkhoff, G.: The Role of Order in Computing, in: Moore, R. (ed.), Reliability in Computing, Academic Press, 1988, 357–378.

[2] Edmonson, William, Melquiond, Guillaume, IEEE Interval Standard Working Group - P1788: Current Status, Proc. 19th IEEE Symposium on Computer Arithmetic, 2009. ARITH 2009, 231 234.

[3] Markov, S., Calculus for interval functions of a real variable. Computing 22, 325–337 (1979).

[4] `http://en.wikipedia.org/wiki/List_of_algebraic_structures`

[5] `http://math.chapman.edu/cgi-bin/structures`

[6] `http://en.wikipedia.org/wiki/Cancellation_property`

[7] `http://en.wikipedia.org/wiki/Cancellative_semigroup`

[8] `http://en.wikipedia.org/wiki/Semigroup`

[9] `http://en.wikipedia.org/wiki/Associativity`

[10] `http://en.wikipedia.org/wiki/Group_(mathematics)`

[11] `http://en.wikipedia.org/wiki/Quasigroup`

[12] `http://en.wikipedia.org/wiki/Identity_element`

[13] `http://en.wikipedia.org/wiki/Inverse_element`

[14] `http://en.wikipedia.org/wiki/Additive_inverse`

[15] `http://en.wikipedia.org/wiki/Loop_(algebra)`

[16] `http://en.wikipedia.org/wiki/Monoid`

[17] `http://en.wikipedia.org/wiki/Magma_(algebra)`

[18] `http://en.wikipedia.org/wiki/Unital`

[19] `http://en.wikipedia.org/wiki/Semigroup_with_involution`

[20] http://en.wikipedia.org/wiki/Commutativity

[21] http://www.wtamu.edu/academic/anns/mps/math/mathlab/beg_algebra/beg_alg_tut5