Provoking Student's Sense-of-Mathematics

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The multiple choice competitions could be a platform to extend the range of the mathematics curriculum of compulsory secondary education, e.g. in building student's mathematical synthetic competence. This competence is closely related to the technical concept sense of mathematics. An approach to cultivate and upgrade further the student's sense of mathematics is to include special kind of tasks in multiple choice competitions – we call them monster problems. The article gives an idea of how this is made in the Chernorizetz Hrabar Tournament in Bulgaria. The

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1. EVOLUTIONARY MISSION OF MULTIPLE CHOICE COMPETITIONS

According to some evolutionary rules species in the nature develop special abilities and we label a part of them as senses. Successful design of species in a particular environment includes appropriate set of senses that gives individuals better chances to survive. Any sense is physiologically provided by evolutionary originated organs. And all this works with a small exception: humanity. Some times ago humanity declines to follow evolution and takes its own way. But in its hurry humanity leaves no space to evolution to build up special organs for new senses it needs. The function of the evolution was taken by medicine and didactics. The role of the medicine is out of our interest. The role of the didactics is to provide opportunities for social success – this is what we are going to discuss in the paper.

The modern informational environment calls the need of new human senses to be successful individual. The information avalanche strikes the fragile mind of the human beings and disorients them sometimes to dramatic confusions. Individuals who can manage the information properly can make the right choice in complex situations which gives them better chance for social success. Our stand point is that one significant role of the modern didactics is to create appropriate instruments for building up student's ability to deal with information (technically: a kind of *sense*) and to develop relevant technology for putting into practice these instruments. Thus we approach the topic: an instrument to develop students' abilities in information management is specially designed math tasks and an appropriate technology is the extracurricular math activities like *multiple choice competitions* (briefly MCC).

2. WHICH ARE THE CANDIDATES FOR THE LABEL MONSTER PROBLEM

In this article we classify a task as monster problem (briefly MP) if it is included in a multiple-choice competition theme and contains relatively large amount of information, usually given in different kind of quantities. How terrible the monster is depends on the target group and some additional characteristics like the data structure and the type of the answers that can play the role of a problem decomposition (more details about what problem decomposition means could be seen in (Lazarov, 2006); there is also an explanation of the difference between the concepts of task and problem which we adopt here).

Example 1

There are two circles to accelerate elementary particles in the Middle Hadron Collider: small, with a length of 5 km, and large, 15 kilometers in length. Energy services 1 km from the large circle is 20% more than that required to maintain the same parameters at 1 km from the small circle. In the small circle a proton was accelerated to 60% of the speed of light and the proton made 3 laps of the circle. Then the proton entered the big circle and was accelerated to 75% of the speed of light, as did 4 laps with that speed. About what energy (in gWh) was necessary for these 4 laps in the big circle, if 1.2 gWh were consumed for three laps in the small circle. It is assumed that the increase in energy, required to maintain the speed of a proton from v_1 to $v_2 > v_1$, is $\left(\frac{v_2}{v_1}\right)^2$ times.

A) 6 B) 7 C) 8 D) 9 E) 10

Solution and comments

Energy required to maintain 60 % of the speed of light in a lap of the big circle is $\frac{15}{5} \cdot 1.2$ that of a lap of the small circle. Considering the number of laps, the energy becomes $\frac{15}{5} \cdot 1.2 \cdot \frac{4}{3}$. The increase in speed leads to the multiplier $\left(\frac{v_2}{v_1}\right)^2 = \left(\frac{75}{60}\right)^2 = \frac{5^2}{4^2}$.

Hence, the desired energy is

$$1.2 \cdot \frac{15}{5} \cdot 1.2 \cdot \frac{4}{3} \cdot \frac{5^2}{4^2} = \frac{6}{5} \cdot \frac{15}{5} \cdot \frac{6}{5} \cdot \frac{4}{3} \cdot \frac{5^2}{4^2} = 9 \text{ gWh}.$$

Answer. D)

This problem appears in both themes for 7-8 (No 22 of 25) and 9-10 graders (No 15 of 30) of the 18th (2008) issue of Chernorizetz Hrabar Tournament (ChH).

There is relatively large amount of heterogeneous information in this task, so we can classify it as a *monster*. In fact, the task is nothing special. But taking into account the very limited time in a multiple-choice competition and the target group the problem becomes quite hard for 7-8 graders and still hard enough for the 9-10 graders. The challenge in this problem is to manipulate different data types: length, energy, speed. Student is expected to connect the quantities given, to compose (one or more) expressions and to perform clever calculations. Let us note that there are no tasks close to this in modern Bulgarian secondary school textbooks. However, the situation was different in the past.

3. THE ORIGIN OF *MONSTER PROBLEMS*

Tasks with a lot of data appear in everyday life quite often. The earliest problems of this type in Bulgarian sources known to the author are dated from the early 20th century. The textbook *Smetanka* (Violino Primo & Todorov, 1925) presented by Plamen Mateev at the Workshop *Didactical Modeling*, contains the following task.

Purchase price and selling price.

The shop *Bratsko delo* was delivered

75 kg fish for 3,600 levs (*lev* is the Bulgarian currency)

54 kg caviar for 5,724 levs

93 kg olives for 3,534 levs

74 l olive oil for 4,218 levs

40 kg soap for 1,440 levs.

There were additional expenses of 9 levs for every 100 levs of the price paid for this shipping. What should be the price per unit (kilogram or liter) of any good if *Bratsko delo* plans to have 10 levs profit only per 100 levs of investment?

Покупна и продажна цена.

Кооперация „Братско дѣло“ получи отъ Бургасъ въ магазина си:

75 кгр. хамсия за 3,600 лв.

54 кгр. хайверъ за 5,724 лв.

93 кгр. маслини за 3,534 лв.

74 литра дърв. масло за 4,218 лв.

40 кгр. сапунъ за 1,440 лв.

Освенъ това по тази доставка кооперацията направила общи разноски по 9 лв. на всѣки 100 лв. отъ покупната цена.

Да се опредѣли единичната продажна цена, сиречъ, по колко лв. „Братски трудъ“ трѣбва да продава всѣки килограмъ или литъръ отъ горнитѣ стоки, като се знае, че тая кооперация си опредѣля само 10 лв. печалба на всѣки 100 лв., дадени отъ нея.

The target group is 4th graders (children at the age of 10-11) and meets the competences needed for household management of the early 20th century in Bulgaria.

It is important to note that in this time the 4th grade was the final educational degree for the largest fraction of Bulgarian children. However, this problem is not a monster problem – it is not included in a MCC. Such feature is important for the topic of the paper: our didactic perspective takes into account the limited time of MCC.

The first MP (as far as we know) is the following one, composed under an idea by Lewis Carroll (Kerroll, 1991).

Example 2

Three gold-diggers work for 6 hours per day and for 10 days they managed to cradle 288 grams of gold. On the average, they found 2 grams of gold in 1 liter of sand. How many hours per day do two other gold-diggers work if in 9 days they cradle 180 grams of gold, work twice as hard as the first three gold-diggers and find on average 3 grams of gold per liter of sand?

A) 2 h 5 min B) 3 h 50 min C) 4 h 30 min D) 6 h 15 min
E) none of these

Comment

This is the task 21 from the theme of the 5th issue (1997) of Chernorizetz Hrabar Tournament (ChH). The participants were highly motivated 9-11 graders. Among them were all members of the extended Bulgarian team for IMO 1998. The statistics of the answers given confirms the classification as MP:

A) 53 (8.2%) B) 12 C) 19 D) 17 E) 34 not answered 510.

Let us note that the score system of ChH does not encourage students in guessing, so we assume the number of the correct answers is close to the number of students who solved the problem. The solution requires no special mathematics (Lazarov at al., 2004). Thus, the explanation for this poor performance may be the unexpected task, but (most probably) may be the undeveloped sense of mathematics of the majority of the most capable Bulgarian students.

4. THE *SENSE OF MATHEMATICS* MISSED OUT DEFINITION

The sense of mathematics (as the sense of humor) is strongly individual and perhaps has congenital bedrock on which it could be potentially cultivated and consecutively upgraded (in contrast to the sense of humor that could be either genuine or imitated, or none). We do not step on the slippery way to define explicitly what exactly the sense of mathematics is – one view point is briefly described in (Lazarov, 2010). Further we give more nuances in our understanding: the sense of mathematics allows an individual to discover connections, to estimate values of variables and to make decisions based on both deduction and common sense. The illustrative examples that follow will give a sharper image of what we mean about.

5. THE MP DESIGN

The design of monster problems is tricky business. On one hand, a situation close to reality is needed for the MP base; the natural form of any particular quantity should be kept. On the other hand, the data used should be mutually compatible. In some MP these requirements are fulfilled automatically from the nature of the data – it is necessary only to pick out appropriate numbers to allow fast calculations (this meets the spirit of multiple choice competitions). But sometimes data are heterogeneous and the problem designer is challenged to combine and to modify information to calibrate a MP. We are going to start a series of examples taken from themes of Chernorizetz Hrabar Tournament to give an idea of how it works.

Example 3

Professor Solomonovski typed his report for a scientific congress on 12 pages; each of them has 30 rows and on every row 60 characters can be fitted in. Doctor Pitagorov rewrote Solomonovski's report as a text file. It took 10 pages, with 40 rows on each page and 72 positions for characters. He also added a bibliography text.

How many characters does the added text have, if Prof. Solomonovski filled up 80% of the available space with characters, while Dr Pitagorov filled up 90%?

A) 9600 B) 8640 C) 7776 D) 6912 E) none of these

Solution and comments

The number of characters in the report of the professor is

$$12 \cdot 30 \cdot 60 \cdot 0.8 = 17280 .$$

The number of characters set in the computer is

$$10 \cdot 40 \cdot 72 \cdot 0.9 = 25920 .$$

The signs added are

$$25920 - 17280 = 8640 .$$

Answer. B).

The problem appears in both themes for 7-8 and 9-10 graders of 12^{th} (2003) issue of ChH (Lazarov, B. & I. Kortezov, 2006). It was 13^{th} (of 25) in difficulty in the theme for 7-8 grades. Statistics for 7-8 grades:

A) 8 **B) 108** (45.76%) C) 11 D) 0 E) 24 not answered 85.

The data structure in this puppy-monster allows students only to follow the text and to perform the calculations properly. In fact, the problem is not a proper MP. However, it could be easy starter for building student's sense of mathematics.

Now, let us see what the main course is.

Example 4

The hay on a 4-decare meadow is enough to feed exactly 3 horses for 8 months. What is the largest number of donkeys that could be fed for 6 months with the hay of a 15-decare meadow if it has 20% higher nutritional value and 3 donkeys eat as much hay as 2 horses?

A) 12 B) 24 C) 27 D) 36 E) none of these

Solution and comments

Let us denote by *x* the maximum number of donkeys. We could compose the proportion

$$\frac{x}{3} = \frac{8}{6} \cdot \frac{15}{4} \cdot \frac{6}{5} \cdot \frac{3}{2}$$

taking into account the relevant factors: $\frac{x}{3}$ for the animals; $\frac{8}{6}$ for the months; $\frac{15}{4}$ for the meadow area; $\frac{6}{5}$ for the nutritional value; $\frac{3}{2}$ for the appetite.

Answer. C).

The problem appears in both themes for 7-8 and 9-10 graders of the 13th (2004) issue of ChH (Lazarov & Kortezov, 2006). This MP requires quick orientation in 5 types of data – student's sense of mathematics is challenged to combine related data, i.e. to constitute a kind of classification. After grouping the relevant quantities, the answer could be found without any additional efforts.

Statistics for 7-8 grades:

A) 38 B) 40 C) 51 (18.75%) D) 14 E) 9 not answered 120.

The statistics for 9-10 grades is similar.

ChH tournament is traditional. While preparing for this competition, students are expected to get familiar with the main ideas from the previous issues. This is the reason why in the latest issues of ChH really terrible monsters appeared. To give student's sense of mathematics some more chance the task in the next example is tuned via decomposition: the data are structured.

Example 5

Three adult passengers paid for second-class one-way tickets for a slow train from A to B the total amount of €24. A mom with 2 children bought first-class return tickets for a fast train on the route A-C-A. How many Euros did the mom pay if:

(1) a first-class ticket costs 20% more than a second-class ticket;

(2) children travel at 75% discount;

(3) a ticket for a fast train is 1,25 times more expensive than the ticket for a slow train;

(4) one saves 10% by buying a return ticket instead of two one-way tickets for the same trip;

(5) the distance AC is 50% longer than AB and the ticket's price is proportional to the distance?

A) 90 B) 81 C) 65 D) 72 E) none of these

Solution and comments

One-way second-class adult ticket A→B for a slow train costs €8. Taking into account the conditions (1), (3), (4), (5) and the doubled distance A→C we find that the mom's ticket A↔C costs

$$8 \cdot \frac{6}{5} \cdot \frac{5}{4} \cdot \frac{9}{10} \cdot \frac{3}{2} \cdot 2 = 32.4 Euro.$$

Now by (2) we conclude that the tickets for the two children cost

$$2 \cdot 32.4 \cdot 0.75 = 32.4 \cdot \frac{3}{2} = 48.6 Euro.$$

Answer. B)

The problem appears in both themes for 7-8 and 9-10 graders of 2006 issue of the Chernorizetz Hrabar Tournament (ChH) (Lazarov & Kortezov, 2006). There is a large amount of data, so we have a monster problem. However, the structured information allows the students to follow conditions one after another and to perform consecutively the appropriate operations without some special analysis of the data connections. Such approach serves at least two goals:

- without any decomposition the problem looks insurmountable
- problem decomposition helps technically for a clear formulation of the task.

Reinforced by the problem decomposition, the sense of mathematics should suggest to the student to turn all percents and decimals into fractions and to skip condition (2) during the first pass through the conditions list.

Statistics for 7-8 grades:

A) 29 (15.26%) B) 11 C) 14 D) 22 E) 24 not answered 90.

The mathematics in MPs that appear in ChH up to 2009 is restricted mainly to proportions. The only small deviation from this direction was the MP in the Example 5, where two proportions are connected with one addition. Such a restriction makes MPs even more students' friendly which does not correspond to the image of a real monster. In searching new thrill, the following MP was proposed at the 18th (2009) issue of ChH.

Example 6

The way European feels the air temperature of c° C when there is a *v* km/h wind, could be calculated by the formula

$$T_e = \left(13 + \frac{3}{5}c - 12\varphi(v) + \frac{2}{5}c \cdot \varphi(v)\right)^\circ C.$$

The way American feels the air temperature of f° F when the wind blows at *w* mph, could be calculated by the formula

$$T_a = \left(36 + \frac{3}{5}f - 36\varphi(w) + \frac{2}{5}f \cdot \varphi(w)\right)^\circ F.$$

How an American felt the temperature if two hours ago in the same place an European felt -10° C under wind of 20 km/h and during these two hours the air temperature dropped by 3° C and the wind speed raised by 12 km/h?

Take one mile equal to 1.6 km and $\varphi(x) \approx 0.6 + 0.05x$; keep in mind that y° F are $\frac{5}{9}(y-32)^\circ$ C.

A) below 7° F B) between 7° F and 14° F

C) between 14° F and 21° F D) between 21° F and 28° F

D) above 28° F

Solution and comments

We start with the observation that the wind speed for the American is 20 mph. Now we find $\varphi(20) = 1.6 = \frac{8}{5}$ which we need for both the European and American formulas. From the first formula we find the air temperature on which the European was exposed:

$$-10 = 13 + \frac{3}{5}c - 12 \cdot \frac{8}{5} + \frac{2}{5}c \cdot \frac{8}{5} \Rightarrow c \approx -3.$$

Hence the virtual temperature for the American was -6° C, i.e. about 21° F. Applying the second formula, we obtain

$$T_a = 36 + \frac{3}{5} \cdot 21 - 36 \cdot \frac{8}{5} + \frac{2}{5} \cdot 21 \cdot \frac{8}{5} = 36 - 31.6 \approx 4.4^{\circ}\text{F}.$$

Answer A).

This problem was aimed at the 9-10 graders as a hard one but it was rejected by the Problem Committee as not suitable for the multiple choice competition. The assumption was that the calculations are too heavy to be performed without a calculator in a limited time. However, it is worth discussing the didactic potential of the problem.

Dealing with fractions and taking appropriate approximations, one can get the answer relatively easy. The clue for such approach is the structure of the answers – the intervals given in the answers are large enough to leave space for rather rough approximations. So the main challenge is to establish proper connections between any particular data given and to choose the formula that concerns such data. The author's view point is that all these steps could be done during a MCC. Our opinion is that in this case the students' sense of mathematics was underestimated. Let us remind that ChH offers MPs for more than decade and the majority of the participants expect at least one MP in the theme. The sense of mathematics was systematically build up across the ChH themes during the last years to the level that allows the most advanced students to attack successfully problems of this type. Moreover, the sense of mathematics is helped by the problem decomposition given in answers and formulas.

The chance to surprise the ChH participants with a new monster species was missed this time. However, the inclusion of formulas in the MP design contains didactical potential that will be exploited in the future.

6. CONCLUDING REMARKS

Our hope is that the fuzzy idea of what the *sense of mathematics* is in our perspective takes clearer shape after considering the examples we gave. In conclusion we can say that student's sense of mathematics refers to the individual capacity in several areas:

- to analyze and to organize quantities given and to discover how they interact
- to find and to specify connection types between quantities, and to mathematize the connections into math relations and formulas
- to manage calculations in performable way that includes the form of number presentation and application of reasonable accuracy.

Emphasize in the building of student's sense of mathematics is put on data management, clever fast calculations, evaluation of the results and no sophisticated mathematics. Let us note that there are other parts of the ChH themes dedicated to special math branches in particular and advanced mathematics competence (Lazarov, 2010). The MP in the tournament theme strikes as a data avalanche and the sense of mathematics should help students to escape as fast as possible to the right direction. In general, high mathematical achievements indicate well developed sense of mathematics, but not vice versa. Our stand point is that the sense of mathematics is necessary for the majority of the advanced students whose plans for the future do not include a job in theoretical mathematics. The MCC is the math forum for these students. We believe the MP is one possible instrument in building student's sense of mathematics and an open problem is to find other instruments.

One can say that 'the life itself is a monster problem' in the global MCC in which we all participate. To perform better in it, the sense of mathematics is expected to form student's

- confidence in data management
- synthetic evaluative view on the output of a particular solution
- ability to make decisions in a very limited time.

The synthetic competence we consider as the main goal of the secondary school math education may vary according to the job of the individual. The sense of mathematics is job independent and serves both professional and everyday life challenges.

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APPENDIX

We offer one more problem from the Smetanka (Violino Primo & Todorov, 1925) with the belief that the genuine math sources of the past are rich in nutrition for the modern didactics.

Sliven Railway Station

On 2 March this year (nineteen-twenty-*n*) the Sliven Railway Station sold 6 tickets 3rd class to Sofia, 6 tickets 2nd class to Sofia, 3 tickets 3rd class to Varna, 9 tickets 3rd class to Tarnovo, 6 tickets 3rd class to G.Oryahoviza, 4 tickets 2nd class to Chirpan, 18 tickets 3rd class to Burgas, 9 tickets 3rd class to St.Zagora, 24 tickets 3rd class to Plovdiv, 7 tickets 1st class to N.Zagora, 3 tickets 2nd class to T.Pazardzik. How much money was the income of the Sliven Railway Station on the 2 March?

Tickets price table of the Sliven Railway Station

Destination	3rd class	2nd class	1st class
Zimnica	18	33	49
Karnobat	40	73	110
Aytos	53	104	154
Burgas	73	135	218
Kazanluk	95	190	204
Tarnovo	135	267	399
G.Oryahoviza	168	333	488
Varna	284	566	747
Sofia	239	467	695
Yambol	29	53	80
N.Zagora	53	104	154
S.Zagora	69	135	201
Chirpan	95	190	284
Plovdiv	132	258	383
T.Pazardzik	162	317	409

Ако имате да умножите 520 съ 230, вие можете да награвите това:

520		520		520
×230	или	×230	или	×230
000		15,600		15,600
1,560		+1,040		+104
+1,040		119,600		119,600
119,600				

Разгледайте и тритѣ начина, обяснете втория и третия и посочете най-удобния.

Умножете по третия начинъ:

3,800×40	960×700
72,000×60	1,200×600
8,500×30	2,700×30
720×800	9,600×40

На Сливенската гара.

Таблица за ценитѣ на пжтническитѣ билети:

Отъ Сливенъ до	3 кл.	2 кл.	1 кл.
Зимница	18	33	49
Карнобатъ	40	73	110
Айтосъ	53	104	154
Бургасъ	73	135	218
Казанлъкъ	95	190	204
Търново	135	257	399
Г. Орѣховица	168	333	488
Варна	284	566	747
София	239	467	695
Ямболъ	29	53	80
Н. Загора	53	104	154
Ст. Загора	69	135	201
Чирпанъ	95	190	284
Пловдивъ	132	258	383
Т. Пазарджикъ	162	317	409

1. На 2 Мартъ т. г. Сливенската гара издала 8 билета III кл. за София, 6 билета II кл. за София, 3 билета III кл. за Варна, 9 билета III кл. за Търново, 6 билета III кл. за Г. Орѣховица, 4 билета II кл. за Чирпанъ, 18 билета III кл. за Бургасъ, 9 билета III кл. за Ст. Загора, 24 билета III кл. за Пловдивъ, 7 билета I кл. за Н. Загора, 3 билета II кл. за Т. Пазарджикъ. Колко лева сж постжпали отъ билети на 1 Мартъ?

2. На 2 Мартъ сжщата гара издала: 4 билета I кл. за София, 9 билета II кл. и 16 III кл. за София, 6 — III кл. за Бургасъ, 7 и II кл. и 9 — III кл. за Бургасъ, 5 — II кл. и 12 — III кл. за Пловдивъ, 17 — III кл. за Ямболъ, 11 — II кл. за Н. Загора, 8 II кл. и 6 — III кл. за Ст. Загора. Колко лева сж постжпили отъ билети на 2 Мартъ?

3. Презъ цѣлиятъ м. мартъ сжщата гара издала:

За	Зимница	124 бил. III класа	— II кл.	— I кл.	
„	Карнобатъ	240 „ „ „	36 „ „	17 „ „	
„	Айтосъ	129 „ „ „	40 „ „	3 „ „	
„	Бургасъ	230 „ „ „	72 „ „	14 „ „	
„	Казанлъкъ	63 „ „ „	9 „ „	7 „ „	
„	Търново	29 „ „ „	12 „ „	9 „ „	
„	Г. Орѣхов	17 „ „ „	6 „ „	5 „ „	
„	Варна	89 „ „ „	37 „ „	16 „ „	
„	София	180 „ „ „	36 „ „	13 „ „	
„	Ямболъ	387 „ „ „	195 „ „	30 „ „	
„	Н. Загора	205 „ „ „	53 „ „	7 „ „	
„	Ст Загора	93 „ „ „	28 „ „	10 „ „	
„	Чирпанъ	30 „ „ „	10 „ „	6 „ „	
„	Пловдивъ	83 „ „ „	39 „ „	15 „ „	
„	Т. Пазардж.	20 „ „ „	7 „ „	3 „ „	

Колко лева сж постжпили презъ цѣлия месецъ мартъ?

За земледѣлското производство презъ 1921 год.

Презъ 1921 год. въ България срѣдно на всѣки хектаръ земя, засѣта съ пшеница, се падатъ 3,427 лв. засѣта съ царевица — 1,923 лв. засѣта съ ечемикъ — 2398 лв., засѣта съ тютюнъ 14,912 лв.