PROBLEMS FOR STUDENTS ABOUT TRIANGLES SIMILAR WITH THE EXTOUCH TRIANGLE

Sava Grozdev[*]***, Hiroshi Okumura***[**]***, Deko Dekov***[***]***,***
[*]VUZF University of Finance, Business and Entrepreneurship, Sofia, Bulgaria,
[**]Maebashi Gunma, 371-0123, Japan,
[***]Zahari Knjazheski 81, 6000 Stara Zagora, Bulgaria,
[*]sava.grozdev@gmail.com, [**]hokmr@protonmail.com,
[***]ddekov@ddekov.eu, http://www.ddekov.eu/

Abstract: *We present problems about triangles similar (but not homothetic) with the Extouch triangle. The problems are discovered by the computer program "Discoverer", created by the authors.*

Keywords: *Extouch triangle, triangle geometry, computer discovered mathematics, Euclidean geometry, "Discoverer".*

Introduction

The Extouch triangle is the triangle formed by the points of tangency of a triangle ABC with its excircles. See Extouch triangle in [Weisstein, 2018].

The Extouch triangle is the Cevian triangle of the Nagel point (See the Nagel point in [Weisstein, 2018]), and the pedal triangle of the Bevan point. (See the Bevan point in [Weisstein, 2018]).

The computer program "Discoverer" [Grozdev & Dekov, 2015], [Grozdev et al, 2017-A], created by the authors, has discovered many theorems about triangles similar (but not homothetic) with the Extouch triangle. Here we present a few of these theorems as problems for students.

Given triangle ABC, we denote the side lengths as follows: $a = BC, b = CA$ and $c = AB$.

The reader may find additional theorems and problems, discovered by the "Discoverer" in a number of papers by the authors. See e.g. [Grozdev et al, 2017-B].

Problems about Similar Triangles

In this paper we denote by $PaPbPc$ the Extouch triangle and by $QaQbQc$ a triangle similar (but not homothetic) with the Extouch triangle. If triangles $PaPbPc$ and $QaQbQc$ are similar, then the ratio of the similarity is

$$k = \frac{PbPc}{QbQc} = \frac{PcPa}{QcQa} = \frac{PaPb}{QaQb}.$$

References for Problem 1:

- Bevan point in [Weisstein, 2018],
- Circumcevian triangle in [Weisstein, 2018].

Problem 1. *The Extouch triangle is similar with the Circumcevian triangle of the Bevan point. The ratio of the similarity is*

$$k = \frac{\sqrt{E_1 E_2 E_3}}{2abc(b+c-a)(c+a-b)(a+b-c)},$$

where

$$\begin{aligned} E_1 &= a^4+b^4+c^4+4a^2bc-2c^2a^2-2b^2a^2-2b^2c^2, \\ E_2 &= a^4+b^4+c^4+4ab^2c-2c^2a^2-2b^2a^2-2b^2c^2, \\ E_3 &= a^4+b^4+c^4+4abc^2-2c^2a^2-2b^2a^2-2b^2c^2. \end{aligned}$$

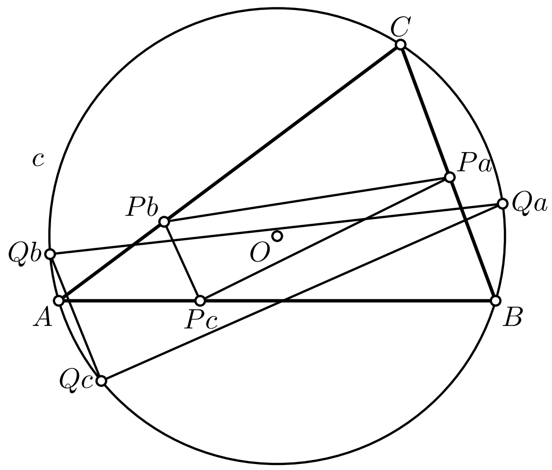

Figure 1:

Figure 1 illustrates Problem 1. In figure 1,

- $PaPbPc$ is the Extouch triangle,
- O is the circumcenter,
- c is the circumcircle,
- $QaQbQc$ is the Circumcevian triangle of the Bevan point.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

Proof. We will use barycentric coordinates. See [Grozdev & Dekov, 2016].

The Nagel point $Na = (u, v, w)$ has barycentric coordinates $u = b+c-a, v = c+a-b, w = a+b-c$, and the Cevain traingle of the Nagel point, that is, the Extouch triangle, $PaPbPc$, has barycentric coordinates

$$Pa = (0, v, w), Pb = (u, 0, w), Pc = (u, v, 0).$$

The Bevan point Be has barycentric coordinates

$$\begin{aligned} u_1 &= a(a^3+a^2(b+c)-(b-c)^2(b+c)-a(b+c)^2), \\ v_1 &= b(b^3+b^2(c+a)-(c-a)^2(c+a)-b(c+a)^2), \\ w_1 &= c(c^3+c^2(a+b)-(a-b)^2(a+b)-c(a+b)^2). \end{aligned}$$

and the Circumcevian triangle of the Bevan point $QaQbQc$ has barycentric coordinates

$$\begin{aligned} Qa &= (-a^2v_1w_1, v_1(c^2v_1+b^2w_1), w_1(c^2v_1+b^2w_1)), \\ Qb &= (u_1(a^2w_1+c^2u_1), -b^2w_1u_1, w_1(a^2w_1+c^2u_1)), \\ Qc &= (u_1(b^2u_1+a^2v_1), v_1(b^2u_1+a^2v_1), -c^2u_1v_1). \end{aligned}$$

By using the distance formula (9) in [Grozdev & Dekov, 2016], we calculate the lengths of the following segments:

$$PbPc = \frac{\sqrt{E_1}}{2\sqrt{bc}}, \quad PcPa = \frac{\sqrt{E_2}}{2\sqrt{ca}}, \quad PaPb = \frac{\sqrt{E_3}}{2\sqrt{ab}},$$

$$QbQc = \frac{a\sqrt{bc}(b+c-a)(c+a-b)(a+b-c)}{\sqrt{E_2E_3}},$$

$$QcQa = \frac{b\sqrt{ca}(b+c-a)(c+a-b)(a+b-c)}{\sqrt{E_3E_1}},$$

$$QaQb = \frac{c\sqrt{ab}(b+c-a)(c+a-b)(a+b-c)}{\sqrt{E_1E_2}}.$$

Hence

$$k = \frac{PbPc}{QbQc} = \frac{PcPa}{QcQa} = \frac{PaPb}{QaQb} = \frac{\sqrt{E_1E_2E_3}}{2abc(b+c-a)(c+a-b)(a+b-c)}.$$

This completes the proof. □

References for Problem 2:

- Mittenpunkt in [Weisstein, 2018],
- Isogonal conjugate in [Weisstein, 2018],
- Circum-Anticevian triangle in [Douillet, 2018].

Problem 2. *The Extouch triangle is similar with the Circum-Anticevian triangle of the Isogonal conjugate of the Mittenpunkt. The ratio of the similarity is as in Problem 1.*

Figure 2 illustrates Problem 2. In figure 2,

- $PaPbPc$ is the Extouch triangle,

- O is the circumcenter,
- c is the circumcircle,
- $MaMbMc$ is the Anticevian triangle of the Isogonal Conjugate of the Mittenpunkt,
- $QaQbQc$ is the Circum-Anticevian triangle of the Isogonal Conjugate of the Mittenpunkt.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

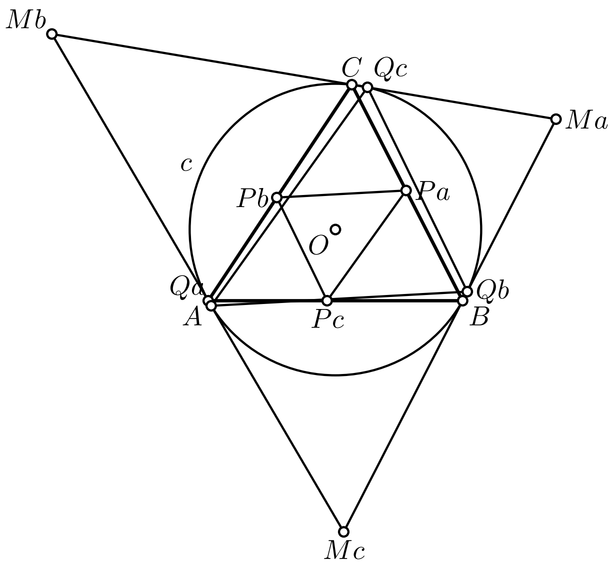

Figure 2:

References for Problem 3:

- Inversion in [Weisstein, 2018],
- Excentral triangle in [Weisstein, 2018].

Problem 3. *Denote by*

- Q *the inversion point of the Incircle wrt the Circumcircle,*
- $MaMbMc$ *the Excentral triangle,*
- Qa *the reflection of point* Q *in the Line* $MbMc$*,*
- Qb *the reflection of point* Q *in the Line* $McMa$*,*
- Qc *the reflection of point* Q *in the Line* $MaMb$*.*

Then the Extouch triangle is similar with triangle $QaQbQc$. The ratio of the similarity is

$$k = \frac{\sqrt{E}}{2\sqrt{abc}},$$

where

$$E = a^3 + b^3 + c^3 + 3abc - a^2b - b^2a - c^2a - ca^2 - cb^2 - bc^2.$$

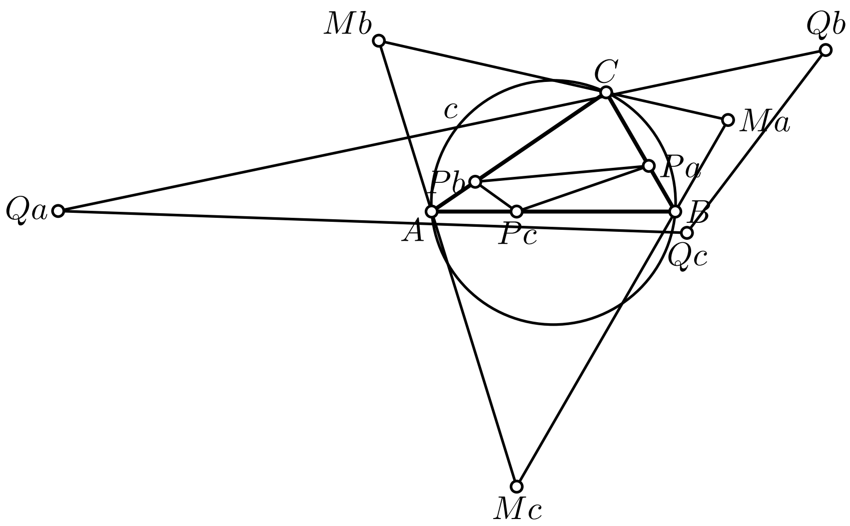

Figure 3:

Figure 3 illustrates Problem 3. In figure 3,

- $PaPbPc$ is the Extouch triangle,
- c is the circumcircle,
- $MaMbMc$ is the Excentral triangle,
- $QaQbQc$ is the Triangle of reflections of point Q in the sidelines of triangle $MaMbMc$.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

References for Problem 4:

- Bevan Point in [Weisstein, 2018],
- Antipedal triangle in [Weisstein, 2018].

Problem 4. *Denote by*

- Be *the Bevan point,*
- $MaMbMc$ *the Antipedal triangle of the Bevan point,*

- *Qa the reflection of the Incenter in the Line $MbMc$,*
- *Qb the reflection of the Incenter in the Line $McMa$,*
- *Qc the reflection of the Incenter in the Line $MaMb$.*

Then the Extouch triangle is similar with triangle $QaQbQc$. The ratio of the similarity is

$$k = \frac{\sqrt{E_1E_2E_3}}{4abc(b+c-a)(c+a-b)(a+b-c)}$$

where E_1, E_2 and E_3 are as in Problem 1.

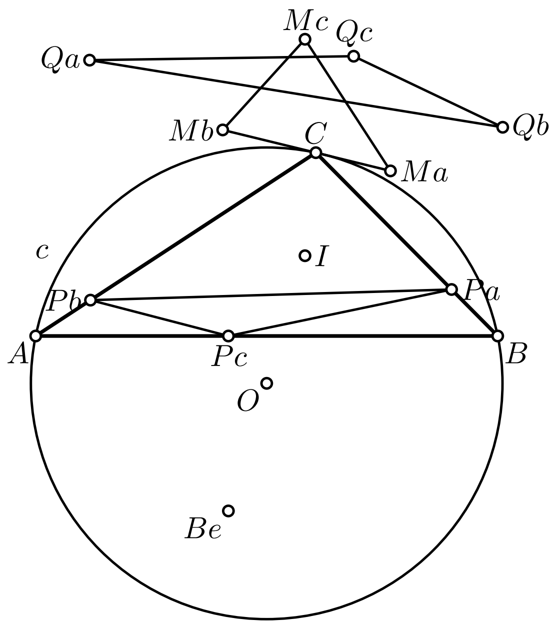

Figure 4:

Figure 4 illustrates Problem 4. In figure 4,

- $PaPbPc$ is the Extouch triangle,
- O is the Circumcenter,
- c is the circumcircle,
- I is the Incenter,
- Be is the Bevan point,
- $MaMbMc$ is the Antipedal triangle of the Bevan point,
- $QaQbQc$ is the Triangle of reflections of the Incenter in the sidelines of triangle $MaMbMc$.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

References for Problem 5:

- Inversion in [Weisstein, 2018],
- Pedal triangle in [Weisstein, 2018].

Problem 5. *Denote by*

- $MaMbMc$ *the Pedal triangle of the inverse point of the Incenter wrt the Circumcircle,*
- Qa *the midpoint of points* A *and* Ma*,*
- Qb *the midpoint of points* B *and* Mb*,*
- Qc *the midpoint of points* C *and* Mc*.*

Then the Extouch triangle is similar with triangle $QaQbQc$*. The ratio of the similarity is*

$$k = \frac{2\sqrt{E}}{\sqrt{abc}},$$

where E *is as in Problem 3.*

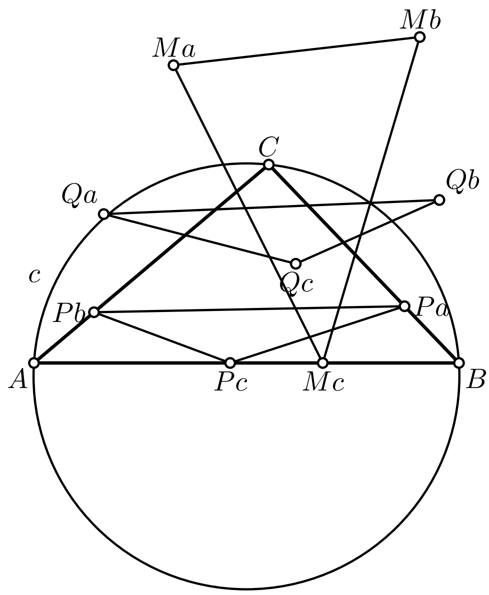

Figure 5:

Figure 5 illustrates Problem 5. In figure 5,

- $PaPbPc$ is the Extouch triangle,

- c is the circumcircle,
- $MaMbMc$ is the Pedal triangle of the inverse point of the Incenter wrt the Circumcircle,
- Qa the midpoint of points A and Ma,
- Qb the midpoint of points B and Mb,
- Qc the midpoint of points C and Mc.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

References for Problem 6:

- Perspector in [Weisstein, 2018],
- Hexyl triangle in [Weisstein, 2018],
- Circumcevian triangle in [Weisstein, 2018.

Problem 6. *Denote by*

- $MaMbMc$ *the Circumcevian Triangle of the Perspector of Triangle ABC and the Hexyl Triangle,*
- Qa *the midpoint of points* A *and* Ma*,*
- Qb *the midpoint of points* B *and* Mb*,*
- Qc *the midpoint of points* C *and* Mc*.*

Then the Extouch triangle is similar with triangle $QaQbQc$*. The ratio of the similarity is*

$$k = \frac{\sqrt{E_1E_2E_3}}{(a+b+c)\sqrt{abc}\sqrt{E_4}},$$

where E_1, E_2 *and* E_3 *are as in Problem 1. Find* E_4*.*

Figure 6 illustrates Problem 6. In figure 6,

- $PaPbPc$ is the Extouch triangle,
- $MaMbMc$ the Circumcevian Triangle of the Perspector of Triangle ABC and the Hexyl Triangle,
- Qa the midpoint of points A and Ma,
- Qb the midpoint of points B and Mb,

- Qc the midpoint of points C and Mc.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

References for Problem 7:

- Perspector in [Weisstein, 2018],
- Hexyl triangle in [Weisstein, 2018],
- Nine-Point Center in [Weisstein, 2018].

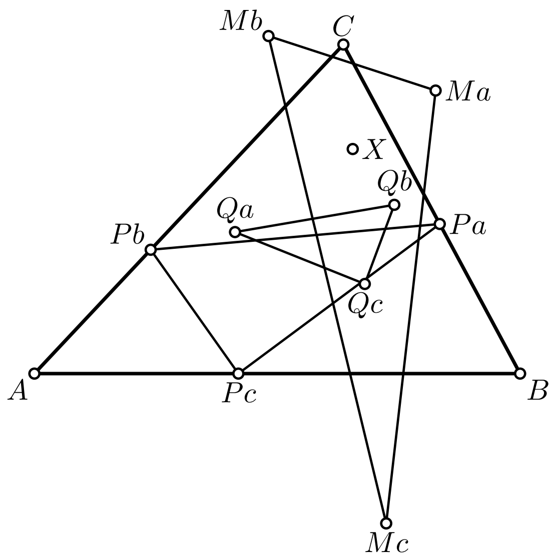

Figure 6:

Problem 7. *Denote by*

- X *the Perspector of Triangle ABC and the Hexyl Triangle,*
- Qa *the Nine-Point Center of triangle* XBC*,*
- Qb *the Nine-Point Center of triangle* AXC*,*
- Qc *the Nine-Point Center of triangle* ABX*.*

Then the Extouch triangle is similar with triangle $QaQbQc$*. The ratio of the similarity is*

$$k = \frac{(b+c-a)(c+a-b)(a+b-c)}{\sqrt{abc}\sqrt{E}},$$

where E *is as in Problem 3.*

Figure 7 illustrates Problem 7. In figure 7,

- $PaPbPc$ is the Extouch triangle,
- X is the Perspector of Triangle ABC and the Hexyl Triangle,
- Qa is the Nine-Point Center of triangle XBC,
- Qb is the Nine-Point Center of triangle AXC,
- Qc is the Nine-Point Center of triangle ABX.

Then triangles $PaPbPc$ and $QaQbQc$ are similar.

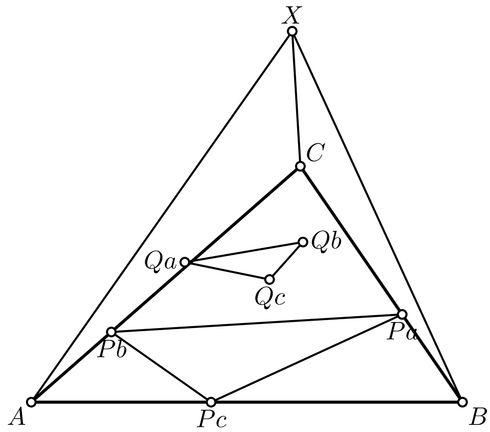

Figure 7:

References for Problem 8:

- Inversion in [Weisstein, 2018],
- Pedal Triangle in [Weisstein, 2018],
- de Longchamps Point in [Weisstein, 2018].

Problem 8. *Denote by*

- X *the Inverse of the Incenter wrt the Circumcircle,*
- $MaMbMc$ *the Pedal triangle of the Inverse of the Incenter wrt the Circumcircle,*
- Qa *is the de Longchamps point of triangle* $AMbMc$*,*
- Qb *is the de Longchamps point of triangle* $MaBMc$*,*
- Qc *is the de Longchamps point of triangle* $MaMbC$*.*

Then the Extouch triangle is similar with triangle $QaQbQc$*. Find the ratio of the similarity.*

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