Parabolic envelopes – laying bridge to kinematics

for advanced X-&-↑ graders

by **Borislav Lazarov**

Definitions. An **envelope** of a family of curves is a curve that touches each curve of the family at some point. We say that **line touches parabola** if the line and the parabola have just one common point.

Comment. Any parabola is the envelope of the perpendicular bisectors of the segments having one end in the focus of the parabola and the other end on its directrix **[1]**.

Dynamic challenge. Construct (e.g. by GeoGebra) two circles K and k with radii 9 and 4 and centered in points at distance 1. Take an arbitrary point A on K and lay the tangents from A to k. Let B and C be the intersection points of tangents and K. Explore the envelope of the triangles ABC when A moves along K **following [2]**.

Comment. Dessislava Dimkova made a dinanic construction that illustrates the above phenomena in a general case (with ellipse) following instructions by Petar Kenderov.

Problem 1. (**Dynamized Ponselet Theorem**) Given two segments R and r, such that $R > 2r$. Construct two circles K and k with radii R and r respectively and centered at points $\sqrt{R(R-2r)}$ apart each other. Take an arbitrary point A on K and lay the tangents from A to k. Let B and C be the intersection points of tangents and K. Explore the envelope of the triangles ABC when A moves along K.

Deductive challenge. Justify the preceding result.

The next problem presents envelope of families of parabola depending on one parameter.

Problem 2. (**A parabolic envelope of parabola family**) Determine the set of points in the plane that are not lying on any of the parabola

$$y = x^2 - 2px + 2p^2 - 3,\ p \in \mathbf{R}.$$

Solution. Let $X(x;y)$ be a point that does not lie on any of the parabola. Then

$$y \neq x^2 - 2px + 2p^2 - 3 \quad \forall x \in \mathbf{R},$$

i.e. the quadratic equation with respect to p

$$2p^2 - 2xp + x^2 - y - 3 = 0$$

has no real roots. Thus, in order to determine the desired set of points, we have to inquire the discriminant of the above quadratic equation in terms of $(x;y)$ considered as parameters:

$$\frac{1}{4}D = x^2 - 2\cdot(x^2 - y - 3).$$

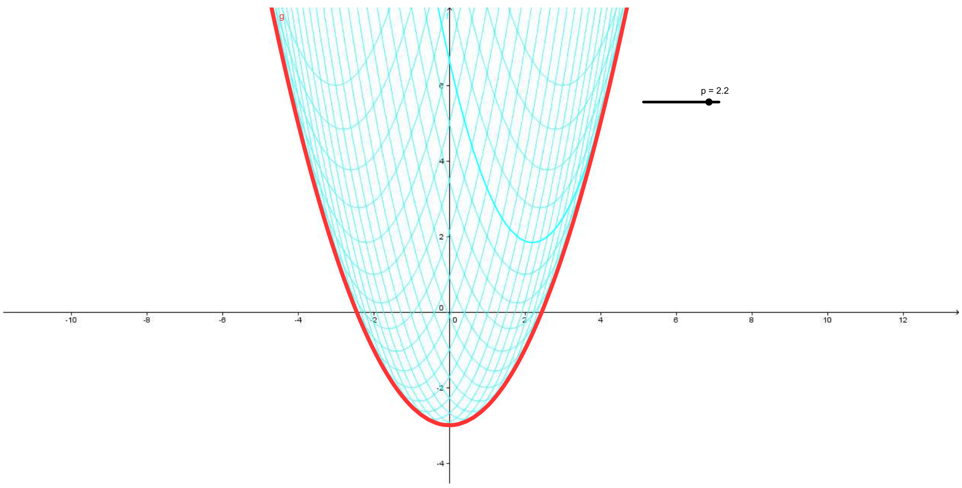
The inequality $D < 0$ is equivalent to

$$y < \frac{1}{2}x^2 - 3,$$

which is the condition for $X(x;y)$ to be a point not lying on any of the parabolas: the desired set consists of the points below (i.e. outward) the parabola $y = \frac{1}{2}x^2 - 3$.

Definition. Two parabola touch each other if they have just one common point.

Challenge. Justify that the parabola $y = \frac{1}{2}x^2 - 3$ touches any of the given parabola $y = x^2 - 2px + 2p^2 - 3,\ p \in \mathbf{R}$, i.e. the system $\left|\begin{array}{l} y = \frac{1}{2}x^2 - 3 \\ y = x^2 - 2px + 2p^2 - 3 \end{array}\right.$ has one and only one solution for any $p \in \mathbf{R}$.

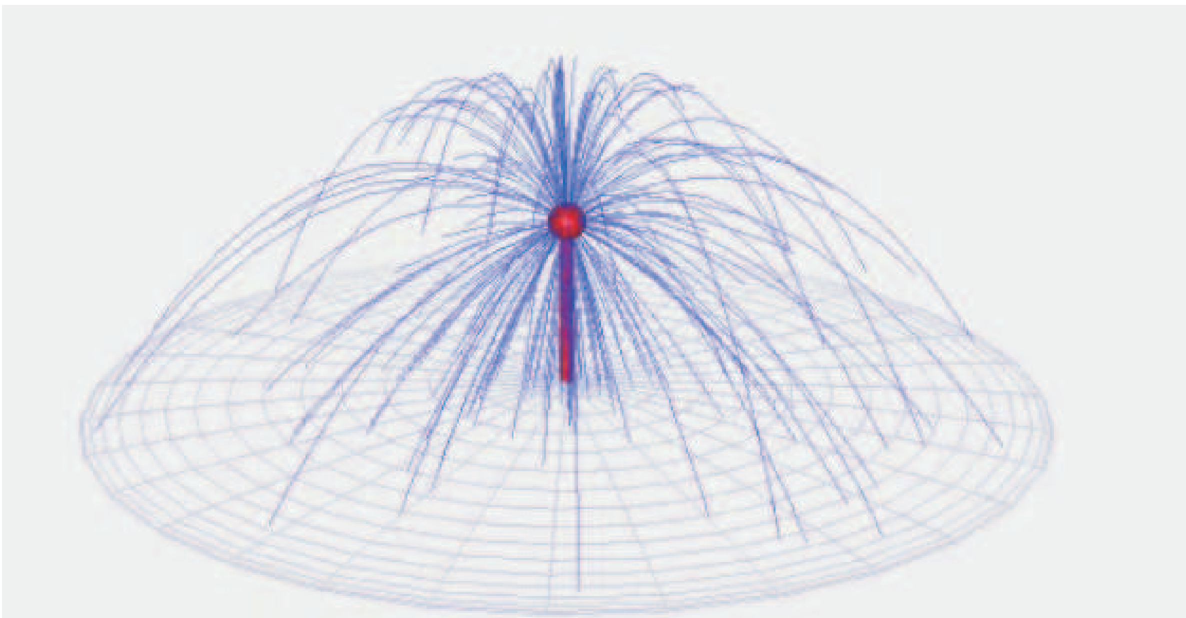


Dynamic drill. Visualize the envelope by GeoGebra.

Comment. The parabola $y = \frac{1}{2}x^2 - 3$ touches any of the given parabola, i.e. it is the envelope of the family of the parabola given.

The next problem presents envelope of another one-parameter family of parabola. The target group is advanced 10th-and-↑ grade students in mathematics but problems require some knowledge in kinematics too.

Problem 3. (**The misty sprinkler**) Water dust is produced by a device with point-tiny size that sprincles water particles uniformely in any direction producing any particle constant initial speed. Determine the shape of the cloud neglecting the air resistance.



Solution. Let us take the origin of the (Cartesian) coordinate system at the source of the dust. Consider a slice of the cloud by the vertical plane through the origine and which contains the abscissa. Let a water particle W leaves the source in the moment $t = 0$ having velocity in this moment

$$\overrightarrow{v} = (q\cos\varphi; q\sin\varphi),\ \varphi \in [0; 2\pi).$$

The location of W in the moment $t > 0$ is $(x(t); y(t))$ where $\left| \begin{array}{l} x(t) = q\cos\varphi t \\ y(t) = q\sin\varphi t - \frac{1}{2}gt^2. \end{array} \right.$ The shape of the cloud we can determine applaying the aproach from the problem 1. From the system we find

$$y = -\frac{g}{2q^2\cos^2\varphi}x^2 + \operatorname{tg}\varphi x.$$

Let $p = \operatorname{tg}\varphi$. Then

$$y = -\frac{g}{2q^2}(p^2+1)x^2 + px, \quad p \in (-\infty; +\infty).$$

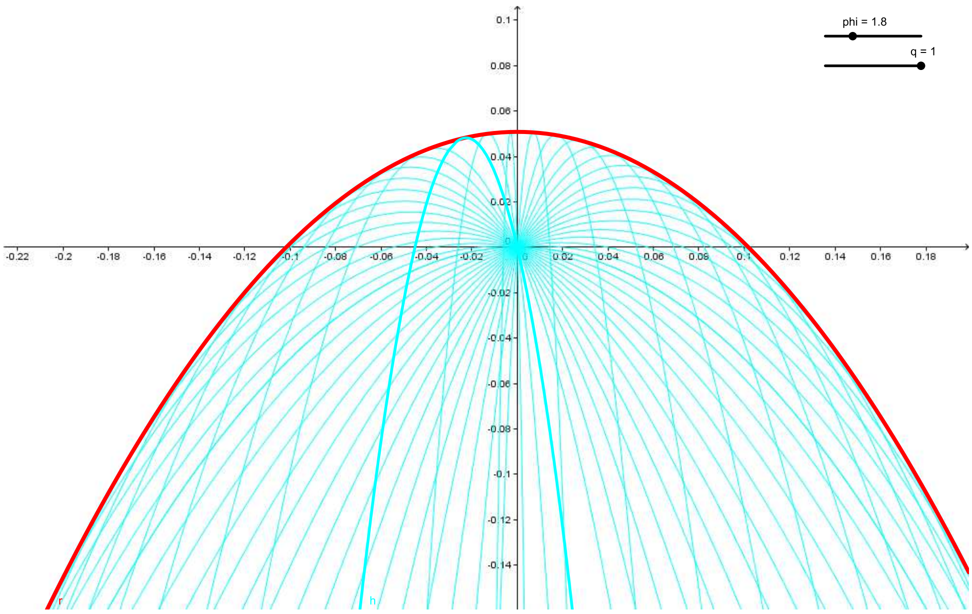
As in the preceding problem we consider the above equation as quadratic with respect to p

$$(gx^2)p^2 + 2q^2xp + (gx^2 + 2q^2y) = 0.$$

To avoid any wet trajectory a point $Q(x;y)$ should not satisfy the above equation for any p, i.e. the discriminant $D = 4q^4x^2 - 4gx^2(gx^2 + 2q^2y)$ must be negative. The inequality $D < 0$ is equivalent to

$$y > \frac{q^2}{2g} - \frac{g}{2q^2}x^2,$$

which is the condition for $Q(x;y)$ to be a dry point: the shape of the cloud is formed by the parabola $y = \frac{q^2}{2g} - \frac{g}{2q^2}x^2$ under revolution about the vertical axis.

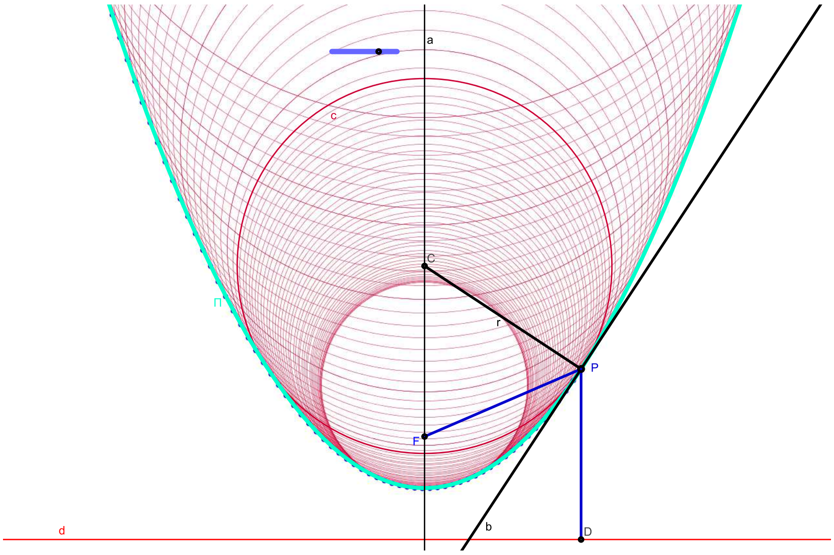


Dynamic drill – Parabola as an envelope of circles. Let C be the intersection point of the axis a and the perpendicular n through P to b. The circle $c(C;CP)$ (c is centered at C and has radius $r = CP$) touches b, i.e. c and Π have a common tangent. In this case we can say that c *touches* Π and vice versa.

Comment. Note that any circle c has two common points with Π. Hence the 'definition' *a circle touches a parabola if the both curves have exactly one common point* does not work. It should be modified in a way that takes into account the configuration of the curves in a *neighbourhood* of their common point.

The parabola Π is the envelope of the family of circles c that pass through P and are centered at the common point of a and the perpendicular through P to b. This envelope could be seen in our dynamic construction by turning on the trace option for c.

Comment. Any of the circles c approximates the parabola near the point of tangency but it is not the best approximation that is given by another circle. The radius of this special circle is called *radius of curvature*.



Challenge. **(12-th graders)** Prove that any circle c touches the parabola Π. You can follow the path:

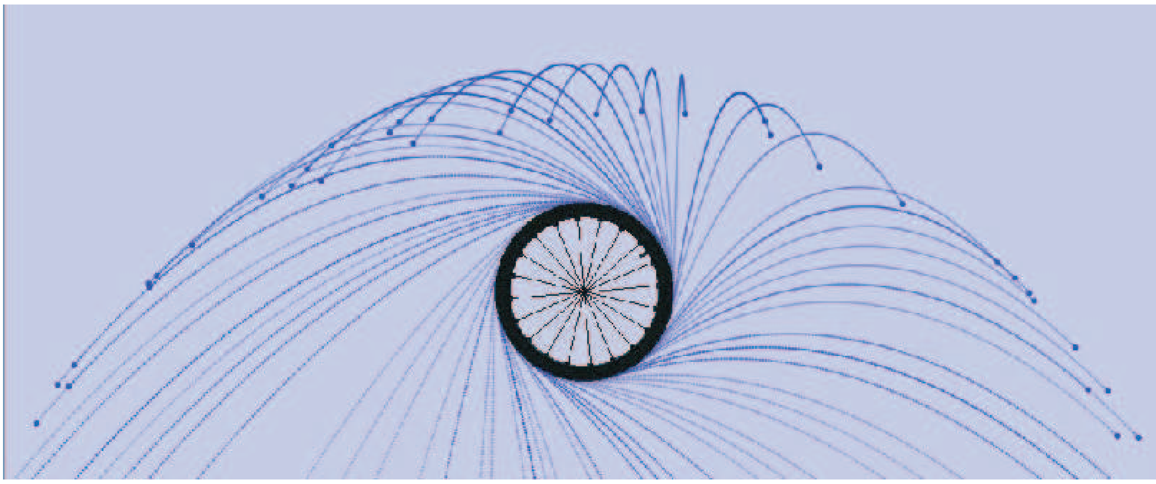
1) Consider the abscissa of D as parameter.

2) Find the coordinates of C.

3) Determine the equation of c.

4) Calculate the derivative of the functions that generate c and Π respectively (at their common point).

Problem 4. (**The wet wheel**) A wet wheel rotates in a vertical plane sprinkling the neighbour area. Determine the dry area in the plane. **[3]**



Solution. Let us take the origin of the (Cartesian) coordinate system at the center of the wheel and let the abscissa be horizontal. Let R be the radius of the wheel and q be the magnitude of the linear velocity of an arbitrary point on the outer circle of the wheel. If there were no gravity force the wet area in any moment $t > 0$ will be the circle centered at the origin and having radius

$$r^2(t) = R^2 + (qt)^2.$$

Taking into account the gravity force, all these circles are falling with acceleration g. So the equation of the 'falling circle' is

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = r^2(t).$$

The envelope of the above family of circles is the boundary between the wet and the dry area in the plane. Let $(X; Y)$ be the coordinates of a boundary point. Then

$$X^2 + \left(Y + \frac{gt^2}{2}\right)^2 = R^2 + q^2t^2.$$

Rewriting the above equation in the form

$$X^2 = -\frac{g^2t^4}{4} + (q^2 - gY)t^2 + R^2 - Y^2$$

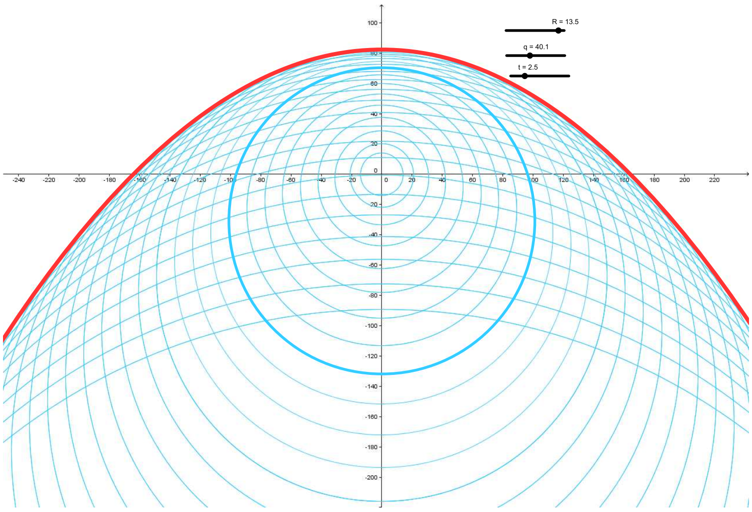
we can consider it as a quadratic with respect to t^2. Since the point $(X;Y)$ belongs to the boudary, it is the ultimate among the 'wet' points with the same ordinate. The maximum of the quadratic function is

$$X^2 = R^2 + \frac{q^4}{g^2} - \frac{2q^2Y}{g},$$

thus

$$Y = -\frac{g}{2q^2}X^2 + \frac{gR^2}{2q^2} + \frac{q^2}{2g}$$

is the equation of the boudary between the wet and the dry area – the envelope of the family of the parabolic trajectoris of the water particles as well as the envelope of the 'falling circles'.



Challenge. Visualize the situation by GeoGebra presenting the wet area as an envelope of parabolas.

The dynamic constructions are done by Albena Vassileva and Dessislava Dimkova.
The pictures of the misty sprinkler and the wet wheel are taken from video clips done by Pavel Boychev via his product ELICA.

Didactics

• Starting points:

The envelope of one-parameter family of smooth curves is rather familiar thing in university mathematics (e.g. the ordinary differential equations). Sometimes the envelope is called discriminant of the family (remember the way we obtain the envelope in problem 1). Presenting the idea in secondary school is a didactical challenge at least in two aspects: to find motivation for it and to find appropriate educational resources.

Our believe is that the kinematics application is reasonable motivation to study envelopes. Resources we use are quadratic function in the frame of the Bulgarian 10th grade curriculum. Modern dynamic softwares allow to explore the constructions in dynamic style and to present them in attractive way.

The ballance between deductive style and inquiery based learning is done by rythmic alternation of calculation and proofs on one hand and dynamic challenges and illustrations on the other hand. The ready-made dynamic applications allow to examine the envelopes from the perspective of initial data.

• Goals: An inquiry based introduction to the fundamental concept of envelope to be organized. To present a variety of one-parameter family of curves that determine the same envelope. To study in experimental and theoretical way some complex real life shapes. The ICT activities to be reinforced by paper-and-pencil techniques, as result to provide students the opportunity to manifest their synthetic abilities in deduction, mathematics, ICT and kinematics.

• Style motivation: The tempered formal definitions-and-problems layout was prefered leaving space for intuition and immagination. The style is close to that in applied science.

• Target group: advanced secondary school students.

• Time consumption: 2×45 minutes, which is the normal time for a single extracurricular activity;

• Knowledge requirements:

Teachers: advanced knowledge in quadratic function, basics of kinematics, GeoGebra initial skills.

Students: skills in quadratic function, basics of kinematics, 2D coordinates, quadratic systems, GeoGebra initial skills.

• Software: Dynamic software, i.e. GeoGebra, GEONExT

• Accommodation (with respect to the Bulgarian curriculum):

lowest: 10-th grade: simultaneously with the school topics about quadratic function;

highest: 12-th grade: simultaneously with the school topics of *analytical geometry*

also anywhere in-between

• Possible upgrade:

12-th grade: dinamic introduction of tangent line, general idea of tangent curves, parametric equations of a curve

• Recommendations

* It could be appropriate to precede this unit with [1].

* Problems 3 and 4 could be decomposed in the following manner:

1) Modeling the phenomenon.
2) Design of a dynamic construction of the model.
3) Examine the dynamic construction.
4) Proof of the results.

Such decomposition allows assessing different student's activities (and competences) separately, i.e. to design a kind of assessment spectrum.

Citations

[1] Lazarov, B. Introducing parabola – a dynamic inquiry based approach.
http://www.math.bas.bg/omi/docs/Parabola/Parabola.html (actual in Apr 2010)

[2] Солтан, В., С. Мейдман. Тождества и неравенства в треугольнике. Изд. Штиница. Кишинев, 1982. (front cover)

[3] Бутиков, Е., А.Быков, А.Кондратьев. Физика в задачах. Издательство Ленинградского университета, Ленинград, 1974.