Introducing Parabola – A Dynamic Inquiry-based Approach

for IX-&-↑ graders

by **Borislav Lazarov**

Prologue

From WIKIPEDIA: The name **parabola** (in geometry, B.L.) is due to Apollonius ... The focus–directrix property of the parabola (which we adopt as definition) ... is due to Pappus.
From SCHWARTZMAN: The term parabola comes from Greek para=*alongside, nearby, right up to* and -bola from the verb ballein=*to cast, to throw.*

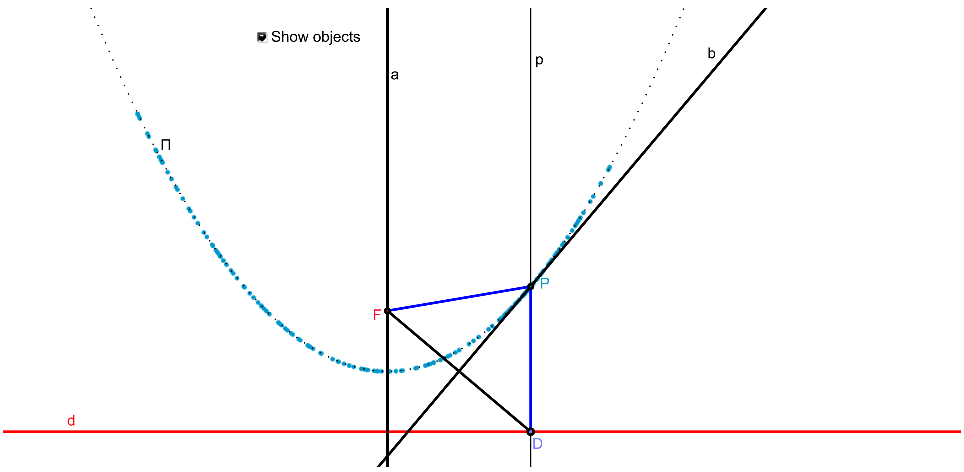
Definition. Given a point F and a line d not passing trough F. The locus Π of the points P that are equidistant to F and d is a curve called *parabola*. The line d is called *directrix* and the point F is called *focus* of the parabola Π. The perpendicular line a to d through F is called *axis* of the parabola and the point of Π that lies on a is called *vertex* of the parabola.

Challenge. Given an arbitrary segment m. Construct a point M at distance m to both d and F, by ruler and compass.

Dynamic construction

1) Take an arbitrary point D on d and erect the perpendicular p to d through D.
2) Draw the perpendicular bisector b of FD.
3) Label by P the intersection point of p and b.
4) Moving D along d we generate points from the parabola Π (the continuous trace of P is the whole parabola).

Proof. Since P lies on b (we write $P \in b$), it is equidistant to F and D, i.e. $PF = PD$. Since $P \in p$, the distance from P to d equals PD. Thus P is equidistant to F and d.∇



Convention. Further we will use by default the notations from the above construction.

Challenge. Let l be a line parallel to d and such that F is between l and d. Let L be the foot of the perpendicular through P to l. Explore $FP + PL$ when P moves along Π, staying between d and l. Prove that $FP + PL$ does not depend on P in this case.

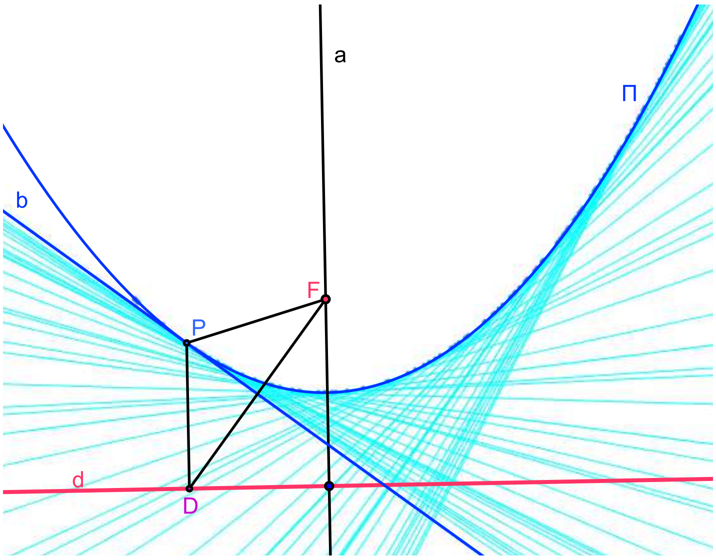
Properties

Pr1. There is no other common point of Π and b than P.

Proof. Suppose the contrary. Let P' be another common point of Π and b. Since $P' \in b$, then $P'F = P'D$. Consider the perpendicular p' to d through P' whose foot on d is D' and let b' be the segment bisector of FD'. Since $P' \in \Pi$, it is equidistant to F and d, which gives $P'F = P'D'$ and hence $P' \in b'$. Therefore,

$P'D = P'F = P'D'$, i.e. P' is equidistant to D and D'. This is a contradiction because P' must lie on the segment bisector of DD', which is parallel to p'.∇

Comment. Pr1 leads to another representation of our parabola Π as the *envelope* of the family of lines b that are the perpendicular bisectors of the segments FD when D moves along d. This envelope could be seen in our dynamic construction by turning on the trace option for b.



Definition. A line that has only one common point with a parabola is called *tangent* to this parabola.

Pr2. The perpendicular bisector b is a tangent line to Π.

Comment. There is only one tangent line through any point of the parabola (*point of tangency*). Our construction guaranties the existence of a tangent through any point of the parabola but here we are not able to prove its uniqueness.

Convention. Further, when we talk about tangent line to the parabola we will mean the one constructed above.

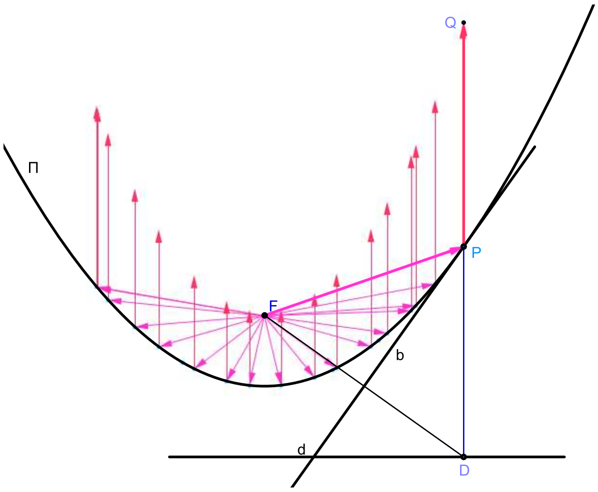
Pr3. The lines PF and PD meet the tangent b at equal angles.

Proof. Since $\triangle FDP$ is isosceles one with $PF = PD$, then the perpendicular bisector b of FD is the angle bisector of $\sphericalangle FPD$.∇

Pr4. A light ray coming from F is reflected by the tangent to the parabola at the tangency point P and turns into a ray that is parallel to the axis of Π. (Keep in mind the *Law of Reflection*: the angle of reflection equals the angle of incidence.)

Proof. Let $PQ^{\rightarrow}$ be the ray of reflection of the ray $FP^{\rightarrow}$. Since the angle between PF and b equals the angle between PQ and b, then (as Pr3 says) it equals the angle between PD and d. Hence, the last two angles are vertically-opposite, which gives that $PQ^{\rightarrow}$ and $PD^{\rightarrow}$ are opposite rays. Thus $PQ \perp d$ (so does the axis a).∇

Comment. The *reflecting curve* reflects an incident ray at a point in the same way as the tangent line at the same point would reflect such a ray. This means that Pr4 could be reformulated in the following manner:
A light ray coming from the focus is reflected by the parabola and turns into a ray that is parallel to the axis of the parabola.
The above statement is known as *the reflective property of parabola*. In 3D case the *reflecting surface* takes the role of the curve and the *tangent plane* takes the role of the tangent line.

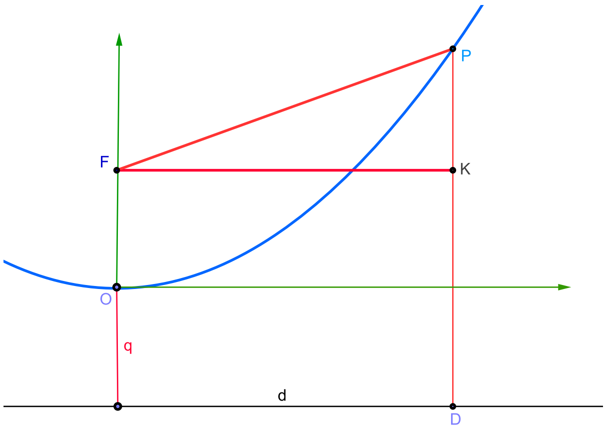


Challenge. What will happen with incident rays that are parallel to the axis of the parabola after reflection by this parabola?

Geometry-and-Linguistics Challenge. Try to spell-out the name *parabola* from the reflective-property perspective.

Equation of the parabola

Coordinatization. Let the vertex of Π be the origin of the (Cartesian) coordinate system whose y-axis lies on a. Let the coordinates of P be $(x; y)$. If the distance from F to d is $2q$, then $F(0; q)$. The distance from P to d is $y + q$ and from the definition of Π we get $PF = y + q$.



Working out the equation. Let $K(x; q)$. From the Pythagorean theorem for $\triangle FKP$ we obtain consecutively

$$FK^2 + KP^2 = PF^2 \Longrightarrow$$
$$x^2 + (y - q)^2 = (y + q)^2 \Longrightarrow$$
$$x^2 + y^2 - 2qy + q^2 = y^2 + 2qy + q^2 \Longrightarrow$$
$$y = \frac{1}{4q}x^2.$$

The last equation is the canonical equation of the parabola.

Challenge. How does the canonical equation of the parabola change when the origin O is taken:

a) in the projection of F onto d;

b) in F?

Challenge. (12-th grade) Prove that any line b is below the parabola. (In fact, this gives us the reason to state that Π is the envelope of $\{b : P \,\mathcal{Z}\, \Pi\}$).

Doubling the cube

From WIKIPEDIA: The problem owes its name to a story concerning the citizens of Delos, who consulted the oracle at Delphi in order to learn how to defeat a plague sent by Apollo. (According to some sources, however, it were the citizens of Athens who consulted the oracle at Delphi.) The oracle responded that they must double the size of the altar to Apollo, which was in the shape of a cube.

Problem. Given an arbitrary segment m. Construct a segment $n = \sqrt[3]{2}m$ by ruler and compass.

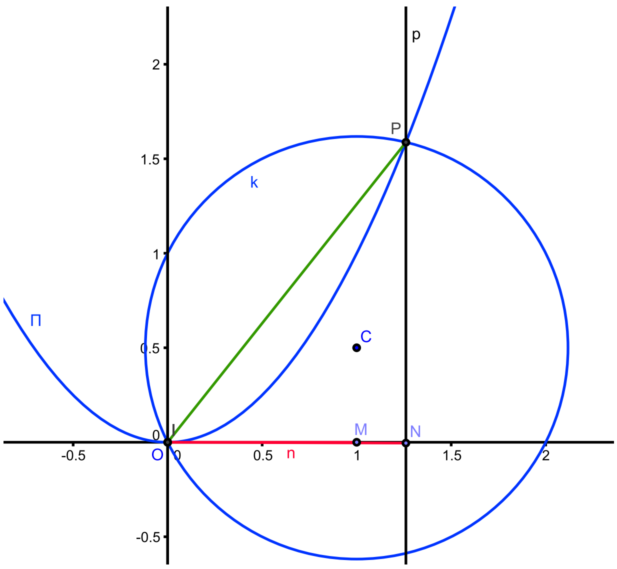
Comment. This problem has no solution. The result was obtained by Galois as a corollary of the fundamental group theory created by him.

Non-dynamic construction. Consider the Cartesian coordinates with unit m in length. Given is the parabola Π with canonical equation $y = x^2$, i.e. F and d are at a distance $\frac{1}{4}$ from the origin O and $d \| Ox$.

1) Draw the circle k centered at $C(1; 0.5)$ and passing through O.

2) Mark the point $P = \Pi \times k$ and draw the line p through P, perpendicular to the abscissa.

3) Mark the projection N of P on the abscissa and draw the line p through P, perpendicular to the abscissa.



Challenge.

a) Calculate that the radius CO of k equals $\frac{\sqrt{5}}{2}$.

b) Prove that for any point $X(x; y)$ of k the equation $(x-1)^2 + (y-0.5)^2 = 1.25$ holds.

c) Solve the system $\left| \begin{array}{l} y = x^2 \\ (x-1)^2 + (y-0.5)^2 = 1.25 \end{array} \right.$

d) If $(x^*; y^*)$ is the solution of the above system for which $x^* > 0$, $y^* > 0$ and $N(x^*; 0)$, calculate the cube of $n = ON$.

Moral. Our non-dynamic construction leads to a construction of the segment $n = \sqrt[3]{2}m$ by ruler and compass. This construction cannot omit the getting of Π with its connectedness. In fact, our dynamic construction gives as many points of Π as we wish, but not all the points of Π. By solving the above challenge we do construct the segment $n = \sqrt[3]{2}m$ by ruler and compass, but we do need the whole parabola Π in advance.

Here endeth the lesson
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The dynamic constructions revision and html format is done by Dessislava Dimkova,
the text revision is done by Albena Vassileva.

Didactics

• Starting points:

Parabola is a geometrical object with astonishing characteristics. But secondary school mathematics presents parabola mainly as the graph of the quadratic function, which does not allow to establish any properties of the parabola. This etude aims at correcting such a misleading view point, as well as at giving a reason why parabola is so important in our everyday live.

The neglect of the geometrical nature of the parabola in secondary school comes possibly from the limited traditional resources. Modern dynamic softwares allow to overcome some restrictions, like time-limitation and visualization. However, the range of the dynamic approach should be clearly specified.

Miscellaneous techniques have been preferred, giving the students some space for exploration but also requiring a tempered conservative deductive style. The comments are meant to expand the main stream with facts beyond the school-students range of reasoning but of significant importance for science and practice.

Challenges are accommodated uniformly. They are within the capacity of the target group. Historical and linguistics spices are added to make the matter tasty.

• Goals: A geometric definition of the parabola to be given and some of its properties important for the applied physics to be established. The concept of envelope to be worked out. The synthetic abilities of the students to be challenged in an inquiry based style and some elements of synthetic competence to be build.

• Style motivation: A *parable* is a brief, succinct story, in prose or verse, that illustrates a moral or religious lesson. Some scholars of the New Testament apply the term *parable* only to the parables of Jesus, though this is not a common restriction of the term. WikipediA

• Target group: advanced secondary school students.

• Time consumption: 2×45 minutes, which is the normal time for a single extracurricular activity;

• Knowledge requirements:

Teachers: advanced secondary-school geometry, GeoGebra initial skills,

Students: loci, Pythagorean theorem, 2D coordinates, reflection, quadratic systems, algebraic inequalities, ruler-and-compass axioms (for dynamic drills – GeoGebra initial skills)

• Software: Dynamic software, i.e. GeoGebra, GEONExT

• Accommodation (with respect to the Bulgarian curriculum):

lowest: 9-th grade: simultaneously with the school topics for *loci*, if it follows the topics listed above

highest: 12-th grade: simultaneously with the school topics of *analytical geometry*

also anywhere in-between

• Recommendations

* It could be appropriate to refresh the following loci:

1) Points equidistant from two given points.

2) Points equidistant from two given lines (consider both cases: lines to be parallel and concurrent).

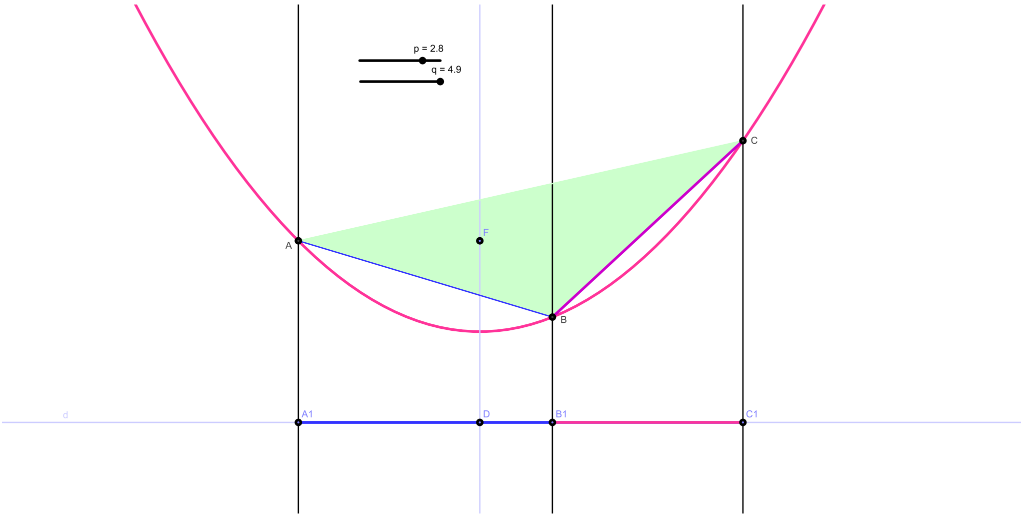
3) Points that lie at a given distance from a given line.

• Possible upgrade:

9-th grade: tool-design to draw a parabola in dynamic mode based on the second challenge; exploring software macro-operators (if any) to get parabola;

10-th grade: exploring the shape of the parabola with respect to the parameters of the quadratic function, studying the trace of a vector that is double the vector $\overrightarrow{DP}$;

12-th grade: proof of the uniqueness of the tangent, area of inscribed triangles in parabola depends only on the projections of the sides onto directrix but not on the location of the triangle along the parabola.



A direction to go further

The focus-directrix definition of the parabola can be written in the following manner. Given a point F and a line d. The locus Π of the points P for which the ratio $e = \dfrac{PF}{PD}$ equals 1 determines parabola.

Definition. Given a point F, a line d and a positive number e. The locus of the points P for which the ratio $\dfrac{PF}{PD}$ equals e is a curve called:

- ellipse when $e < 1$;
- hyperbola when $e > 1$.

The number e is called eccentricity of the curve.

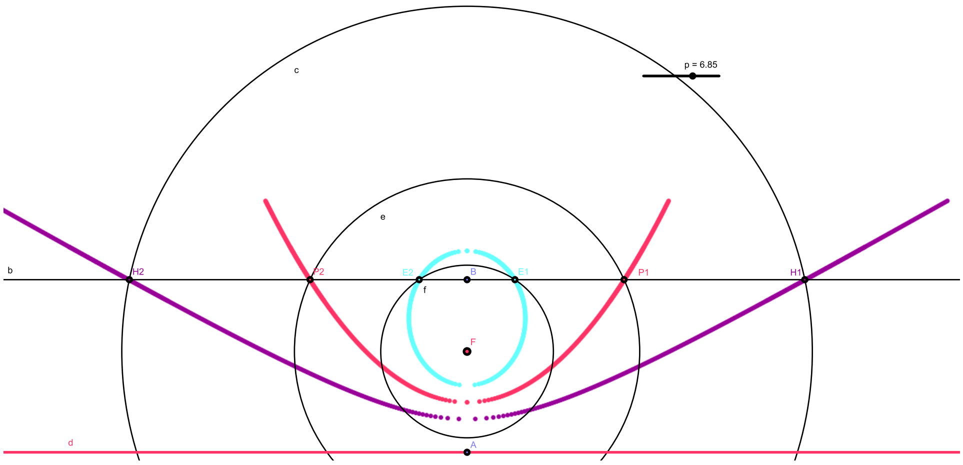
Challenge. Given a point F, a line d, a positive rational number e and a segment m. Construct a point M at distance em to F and at distance m to d by ruler and compass.

Comment. The usual solution of the above challenge requires to construct two familiar loci:

1) Points that lie at distance m from d.

2) Points that lie at distance em from F.

Mega challenge for educators. Design an integrated dynamic inquiry based introduction of parabola, ellipse and hyperbola.



Citations

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S. Schwartzman in http://www.maa.org/editorial/knot/parabola.html (actual in Dec 2009)
http://en.wikipedia.org/wiki/Doubling_the_cube (actual in Dec 2009)