A Review of Four High-School Mathematics Programs

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1. Introduction

Recently I was commissioned by Strategic Teaching to review four high-school mathematics programs for the Washington State Board of Education (SBE).[i] These programs were identified by the Office of the Superintendent of Public Instruction (OSPI) as matching well to the content in Washington's standards. The review is aimed at the *mathematical soundness* of the programs and is to be used by the SBE to inform the OSPI's curriculum recommendations. The programs examined are: Core-Plus Courses 1, 2, and 3; Discovering Algebra, Geometry, and Advanced Algebra; Holt Algebra 1, Algebra 2, and Geometry; and Glencoe Algebra 1, Algebra 2, and Geometry. The examination focused on two topics in algebra, *forms of linear functions and equations* and *forms of quadratic functions and equations*, and one topic in geometry, *parallel lines and the Triangle Sum Theorem*. These topics were chosen because they are viewed as central to the high school curriculum. The examination was to ensure they are coherently developed, completely covered, mathematically correct, and provide students a solid foundation for further study in mathematics.

The algebraic concepts and skills associated with linear functions are crucial for the rest of the study of algebra and beyond. Appropriate definitions and justifications for concepts like *coefficient* and *slope* provide the basis for understanding linear functions and equations. These issues were carefully examined, as were the presence of all forms of linear functions and equations, how these are connected to each other, and the opportunities given to the students to apply them to solve problems. Two particular Washington standards were used as a reference point:

A2.3.2. A1.4.B *Write and graph an equation for a line given the slope and the y-intercept, the slope and a point on the line, or two points on the line, and translate between forms of linear equations.*

B2.3.2. A1.1.B *Solve problems that can be represented by linear functions, equations, and inequalities.*

The ability to put quadratic functions in vertex form allows students to use symmetry and to find the maximum or the minimum of the function. This opens up a new world of problems the student can solve, namely max/min problems. The approach to max/min problems is examined for both the basic algebra and the conceptual development, which includes a coherent definition of a quadratic function and how the line of symmetry is explained and justified. Two Washington standards were used as guideposts:

C2.3.2. *Translate between the standard form of a quadratic function, the vertex form, and the factored form; graph and interpret the meaning of each form.*

A1.1.D *Solve problems that can be represented by quadratic functions, equations, and inequalities.*

The development and application of the Triangle Sum Theorem (that the sum of the angles of a triangle is 180 degrees) was examined. This includes the postulate and the many concepts and relations that lead up to the proof of the theorem. For example, the theorem depends on a solid understanding of parallel lines, the lines that cross them, and the angles associated with them all. In particular, the examination focused on the coherence and logical progression of the material leading up to the theorem. The Washington standard that guided the evaluation was:

G.3.A *Know, explain, and apply basic postulates and theorems about triangles and the special lines, line segments, and rays associated with a triangle.*

The task was to examine the *mathematical soundness* of the programs in relation to the aforementioned five standards. Pedagogy, per se, was not considered. I used the following criteria for *mathematical soundness*:

1. Mathematical justification
   - *Are central theorems stated and proved?*
   - *Are solution methods to problems, conditions, and relations justified?*
   - *Does the program develop norms for mathematical justification, where students gradually learn that empirical observations do not constitute justifications, though they can be a source for forming conjectures?*
2. Symbolism and structure
   - *Does the program emphasize algebraic manipulations and reasoning in general terms?*
   - *Is there an explicit attempt to help students organize what they have learned into a coherent logical structure?*
   - *Does the program attend to crucial elements of deductive reasoning, such as "existence" and "uniqueness," "necessary condition" and "sufficient condition," and the distinction among "definition," "theorem," and "postulate?"*
3. Language
   - *Is the language used clear and accurate?*

The issue of structure is particularly critical in the case of geometry. It is perhaps the only place in high-school mathematics where a (relatively) complete and rigorous mathematical structure can be taught. Deductive geometry can be treated in numerous ways and in different levels of rigor. My examination is based on the view that an adequate level of rigor is necessary and possible in high-school. Deciding what constitutes an "adequate level of rigor" is crucial. In dealing with this question, I used *Euclid's Elements* as a framework. In a program consistent with this framework, subtle concepts and axioms, such those related to "betweeness" and "separation," are dealt with intuitively, but the progression from definitions and axioms to theorems and from one theorem to the next is coherent, logical, and exhibits a clear mathematical structure. Furthermore, such a program would sequence its instructional unit so that *neutral* *geometry*—a geometry without the Parallel Postulate—precedes *Euclidean geometry*—a geometry with the Parallel Postulate.

My examination of the programs focused mostly on these three issues—mathematical justification, symbolism and structure, and language—but I also attended to two other aspects of the programs: the problems assigned for practice and internalization and the way new concepts are introduced. For this, I used the following two criteria.

4. Problems assigned
    - *Does the text include a relatively large number of nontrivial multi-step holistic problems?*
5. Introduction of new concepts
    - *Are new concepts intellectually motivated?*

A *holistic problem* refers to a problem where one must figure out from the problem statement the elements needed for its solution—it does not include hints or cues as to what is needed to solve it. A non-holistic problem, on the other hand, is one which is broken down into small parts, each of which attends to one or two isolated elements. Often each of such parts is a one-step problem. The programs were examined as to whether their instructional units include a relatively large number of non-trivial, multi-step holistic problems.

Generally speaking, *intellectual motivation*, or *intellectual need*, refers to problematic situations through which new concepts and ideas emerge in mathematics.[ii] In this review, I focused on whether problems used to introduce a new concept demonstrate the intellectual benefit of the concept at the time of its introduction. For example, some texts introduce the idea of using equations to solve word problems through trivial, one-step addition or multiplication word problems. This approach is contrived, and is unlikely to intellectually motivate this idea since students can easily solve such problems with tools already available to them.[iii]

Some might argue that these two criteria belong to pedagogy, not mathematics, and so they should not be used to assess the *mathematical soundness* of the programs. My view is that the two criteria are both pedagogical and mathematical. Concerning the first criterion, while there definitely is a place in a textbook for non-holistic problems, it is essential that the text includes a relatively large number of holistic, non-trivial problems, because it is the latter kind that students will encounter in college mathematics and beyond, in professional careers where mathematics is used.

Concerning the second criterion, contrived solutions are alien to mathematical practice. New ideas in mathematics are not created to solve trivial, known problems; rather, new ideas in mathematics are created to tackle new, non-trivial problems or to solve old problems more efficiently. Using trivial problems to introduce new ideas would likely leave the student and teacher confused as to why one would apply a more general line of reasoning when a simpler one will do. Once a new idea is created out of a need to solve problems not solvable, or not easily solvable, with current tools, it is advantageous and necessary to examine the applicability of the idea to known problems. When students realize that old problems are part of a larger family of problems—that all are solvable by the same new tool—their understanding of mathematical structure and mathematical efficiency is advanced. Algebraic representation of word problems and the systematic way of solving equations is an example of such a tool in algebra; and the standard multiplication and division algorithms are an example of such a tool in arithmetic.

My examination, thus, focused on whether each of the above five standards was addressed thoroughly and the *mathematical soundness* of the program in relation to these standards. To make the report self contained, most of the issues discussed in the report are accompanied with direct quotes from the texts. [The quotes appear with a smaller font in order to make them distinguishable from the rest of the narrative.] References to the exact locations in the text are also provided. Both the Student's Edition and the Teacher's Edition served as the source material for the examination; unless it is indicated otherwise, the page numbers refer to the Student's Edition.

The examination of a particular standard focused on units that seemed most relevant to that standard. Occasionally, however, it was necessary to review content outside these core units, specifically when a particular concept in the core unit required an understanding of fundamental concepts from other units. Core units from different programs that are relevant to a particular standard usually differ in the material they cover. For example, some units on linear functions include the concept of absolute value, others do not; some deal with particular applications, others do not. Also, corresponding units differ in the sequencing of their material. For these reasons, the report is not uniform across the four programs with respect to the material covered. The report is, however, uniform in the criteria applied to examine the programs.

The report is organized around five sections. The first four sections correspond to the four programs—Core Plus, Discovering, Glencoe, and Holt—in this alphabetical order. Each section is comprised of four subsections. The first three subsections examine, respectively, the three aforementioned sets of standards: *A1.4.B, A1.1.B* for linear functions; *A2.3.2, A1.1.D* for quadratic functions; and *G.3.A* for geometry. The fourth subsection is a summary of the overall mathematical soundness of the respective program. Each section is completely independent of the rest. The fifth, and last, section discusses the overall conclusions drawn from the examination.

1. Core Plus

Core-Plus' content presentation is unusual in that its instructional units, from the start to the end, are composites of problems. With a few exceptions, the problems are word problems involving "real-life" situations, and typically they consist of a sequence of tasks about such a situation. To review the program, it was necessary to go through all the problems in the core units and their corresponding materials in the Teacher's Edition; in some cases it was also necessary to also examine the distributions of the problems over different aspects of a particular standard.

The decisive majority of the algebra problems are about physical situations or particular functions (where the coefficients are numbers). An adequate number of problems connect the slope and intercept with contextual or geometric meanings. However, problems dealing with general forms of equations and inequalities are rare. The advantage of general algebraic approaches over the other approaches (the use of tables, graphs, and calculators, for example) in logical deduction is not clear. Except for the quadratic formula, fundamental theorems on linear functions and quadratic functions are not justified. This includes the theorems: A line in the plane is represented by a linear equation, and the graph of a linear equation is a line; The graph of $y = ax^2 + bx + c$ is symmetric; The shape of the graph of $y = ax^2 + bx + c$ is determined by the coefficients, a, b, and c. The latter theorem is used without proof to solve quadratic inequalities. The theorem that any quadratic function with roots can be expressed in factored form is neither stated nor proved. The translation from the standard form to the factored form is done in special cases, mostly where the roots are integers. The distribution of the algebra problems across different elements of the standard about translating among different forms of the linear function is not uniform; some elements receive more attention than others.

Like in the algebra texts, the geometry text does not lead to a clear logical structure of the material taught, and there is no clear development of a demarcation line between empirical reasoning and deductive reasoning. Furthermore, while some problems are essential to the development of a geometric structure, others are not. However, all the problems in the text seem to be of equal status. For a teacher to help students discern the essential mathematical progression, he or she must identify all the critical problems—many of which appear in the homework sections—and know in advance what the intended structure is. Missing one or two of these problems would result in an incomplete or deficient structure.

Important theorems in geometry are not justified. Moreover, with the way the material is sequenced, some of these theorems cannot be justified. Also, due to the program's choice of starting with parallel lines rather than congruence, there is loss of an opportunity to convey a critical mathematical lesson about the role of postulates in the development of mathematical structures—that a whole constellation of theorems can be proved without the use of the parallel postulates. This lesson—a landmark in the historical development of mathematics—can and should be within the grasp of high-school students.

The mathematical language used throughout the program is accurate and concepts and ideas usually emerge from non-contrived problems. However, holistic problems, where one needs to figure out from the problem statement what elements are needed to solve the problem, are rare both in algebra and geometry. The program excels in providing ample experience in solving application problems and in ensuring that students understand the meanings of the different parts of the modeling functions. The program also excels in its mission to contextualize the mathematics taught. However, it falls short on conveying the abstract nature of mathematics and the holistic nature of the mathematical problems students are likely to encounter in the future.

2. Discovering

4.1 Algebra: linear functions, equations, and inequalities

A1.4.B *Write and graph an equation for a line given the slope and the y-intercept, the slope and a point on the line, or two points on the line, and translate between forms of linear equations.*

A1.1.B *Solve problems that can be represented by linear functions, equations, and inequalities.*

The Discovering Algebra book devotes two chapters to linear equations (Chapters 3 and 4), a total of 115 pages. There are numerous examples and activities throughout the chapters. Almost 60 pages into these chapters and the slope formula—the mathematical meaning of the coefficient *b* in $y = a + bx$—is yet to be presented. When the formula is eventually presented in Section 4.3, it is not justified. The text presents the slope formula (p. 218) but no mathematical justification is given as to why for any two points on a line with coordinates (x_1, y_1) and (x_2, y_2) the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is fixed and is equal *b*. Sixteen pages later, the Point-Slope Form, $y = y_1 + b(x - x_1)$, is presented (p. 235). But this form, which could have been derived easily from $\frac{y - y_1}{x - x_1} = b$, is generalized from a particular case (pp. 234-5) but is not justified.

In these 115 pages, the text does not justify two fundamental theorems on linear functions: that a line in the plane is represented by a linear equation, and that the graph of a linear equation is a line. In different places and in different contexts these theorems are demonstrated empirically.

There are numerous problems and activities on linear function, equations, and inequalities (the latter is in Chapter 5). A common approach throughout the text is to present the problems and material through small steps in the form of sequences of tasks, what I labeled in the introduction as *non-holistic* problems. This presentation is not mathematical in two respects. First, it makes it difficult to discern the underlying ideas of the content taught. The multitudes of activities and prescribed steps mask the big ideas underlying linear functions, equations, and inequality. The difficulties in discerning a structure is on two levels: local, within a particular lesson, and global, across an instructional unit. Second, consistently the text generalizes from empirical observations without attention to mathematical structure and justifications. This empirical-without-proof approach is not unique to the unit on linear function; rather, it is prevalent throughout all the units that have been examined. I have already mentioned this approach in dealing with the theorems about a line in the plane and its equation. The following is another example, and more examples will be given as the report unfolds.

The "Investigation" on Page 159 presents three figures of composite equilateral triangles made from toothpicks. Students are asked to continue building this pattern of figures out of actual toothpicks and determine the perimeter of each figure. In essence, the solution provided by the text amounts to generating a list of perimeters and recognizing from this list alone the rule for finding the perimeter. In doing so, one is freed from the need to recognize the underlying mathematical structure of the rule—that, for example, a reason the perimeter of a given figure (the n^{th} term in the sequence) is one toothpick greater than the perimeter of its predecessor (the *(n-1)*th term in the sequence) is that the n^{th} figure is constructed by adding 2 toothpicks to the *(n-1)*th figure but only one of these two toothpicks is a side of the new figure. This mathematical reasoning is lost when the focus is on the outcome values of the perimeters rather on understanding the structure of the pattern. This approach is reinforced in Problem 2 (p. 161).

Many of the activities require the use of calculator. Some of these activities are good; others are not. Here are some examples: The text starts the chapter on linear equations with recursive sequences. The first example to illustrate the definition of *recursive sequence* is about a 25-story building. It includes a table with two rows—the first row gives floor number and the second the height (in ft) of that floor, with some of the values in the table missing (p. 158). The task is to find each of these missing values. The solution provided in the text uses calculator:

> Press -4 (ENTER) to start your number sequence. Press +13 (ENTER). … How high up is the 10th floor? Count the number of times you press ENTER until you reach 10. Which floor is at a height of 217 ft? Keep counting until you see that value on your calculator screen.

The use of calculator in this case would likely free the student from the need to infer logically that any n^{th} term in the sequence can be determined from the first term and the recursion rule, and that given any term in the sequence, one can logically infer its location in the sequence by knowing the first value of the sequence. Section 3.2, on the other hand, makes good use of calculator. It nicely demonstrates in calculator language the symbolic representation of the relationship between two neighboring items in a recursive sequence: "To use the rules to get the next term[s] in the [two] sequence[s], press {Ans(1)+1, Ans(2)+13}" (p. 165). This is a precursor to the linear relation: $a_n = a_{n-1} + d$. The authors plot on the calculator screen position versus value of terms in the sequence, and draw the students' attention to critical geometric features of the graph: "The points appear to be in a line" and later "… to get the next point on the graph from any given point, move right 1 unit on the *x*-axis and up 13 units on the *y*-axis."

Although the text makes repeated use of the calculator "recursive routine" (e.g., Ans(1)+1, Ans(2)+13, etc.), it does not connect this routine to its mathematical form: $a_n = a_{n-1} + d$, at least not in these chapters, and the relation between the "recursive routine" and the equation, $y = a + bx$, which is treated in Section 3.4, is vague at best.

2.2 Forms of Quadratic Functions

A2.3.A *Translate between the standard form of a quadratic function, the vertex form, and the factored form; graph and interpret the meaning of each form.*

A2.1.C *Solve problems that can be represented by quadratic functions, equations, and inequalities.*

Quadratic function is dealt with in two different books: Algebra (Lessons 9.1-9.7) and Advanced Algebra (Lessons 7.1-7.4). In both books crucial mathematical theorems appear without justification. Since the treatments are similar, I will focus on the Advanced Algebra book, since it is where a more rigorous approach is expected.

Important ideas about quadratic function are generalized from empirical observations and remain without proof; the following are examples:

Example 1: On Page 371, the text states:

> You now know three different forms of a quadratic function

General $y = ax^2 + bx + c$

Vertex form $y = a(x-h)^2 + k$

Factored form $y = a(x-r_1)(x-r_2)$

Equivalency among these three forms is demonstrated empirically but it is not justified mathematically.

Example 2: The text asserts the (h,k) (in the vertex form) is the vertex of the parabola:

This form, $y = a(x-h)^2 + k$, is called the **vertex form** of a quadratic function because it identifies the vertex, (h,k) … (p. 368)

This assertion is not justified.

Example 3: In an earlier section (Lesson 4.4, p. 194) the text states:

Parabolas always have a line of symmetry that passes through the vertex________.

This statement is not proved. Nor does the text prove the fact that the line $x = h$ is a line of symmetry of $y = a(x-h)^2 + k$. Even a simpler assertion (on the same page) that the line of symmetry of $y = x^2$ is the line $x = 0$ is not proved. This is surprising since the text devotes a lot of space to transformations of graphs. A proof of this simple assertion together with the translation of a function is all what is needed to establish the line of symmetry for quadratic functions.

Example 4: The text asserts (p. 378):

If you know the x-intercept of a parabola, then you can write the quadratic function in factored form, $y = a(x-r_1)(x-r_2)$.

This assertion is not justified. The text devotes a lesson (Lesson 7.8) to division of polynomials, which is a natural place to prove this assertion. However—surprisingly and disappointedly—the text does not revisit this important assertion to prove it.

Example 5: The text uses the equivalency between the general form and the vertex form, which has not been established, to derive the Quadratic Formula)p. 386). On the other hand, this formula is derived correctly in the Algebra book (pp. 531-2), with the omission of the condition, $a \neq 0$, in Step 3.

In addition to these five inadequacies in dealing with quadratic functions, the text does not address explicitly some mathematically important properties of the parabola. Specifically, the vertex form provides critical information about the relationship between the sign of a (the coefficient of x^2) and the form of the parabola, as well as the location of the extreme value (max or min). These derivations are absent from the text.

Quadratic inequalities are not treated in the text.

I will conclude with an additional example to further demonstrate how this program promotes empirical reasoning rather than deductive reasoning: The first example on Quadratics (p. 361) is the following problem:

Find a polynomial function that models the relationship between the number of sides and the number of diagonals of a polygon. Use the function to find the number of diagonals of a dodecagon (a 12-sided polygon).

The solution provided sketches polygons with an increasing number of sides: a triangle, quadrilateral, pentagon, and hexagon. A list is then made that depicts the number of sides and the corresponding number of diagonals in each polygon. These values are obtained by simple counting. Following this, the authors take the second differences in the sequence of the number of diagonals, which turned out to be constant. From this, the text asserts without proof that if the second differences of a pattern are constant, the pattern is quadratic, and so the pattern observed must be of the form: $y = ax^2 + bx + c$ (x is the number of sides and y is the number of diagonals). Following this, the text determines the coefficients a, b, c by solving a linear system of three equations with three unknowns and find the function sought to be: $y = 0.5x^2 - 1.5x$.

As we have seen in the previous section, here too students learn to approach problems empirically. The pattern observed with a limited number of cases is assumed to continue for all cases. Furthermore, a theorem about the relation between the second differences of a pattern and the form of the pattern is only illustrated empirically. Of course there is a place for this beautiful problem, but the approach should be deductive. It is all right to approach a problem empirically, but it is incumbent on the text to push for justifications. This example demonstrates how the text promotes empirical reasoning even in a place where a proof is *readily* available and is within the grasp for high-school students. Here is such a proof: In a polygon with x vertices, if we draw all the possible diagonals emanating from each vertex, we get $x(x-3)$ diagonals. Note, however, in this way each diagonal is counted twice; hence, the number of the diagonals in a polygon with x vertices is: $x(x-3)/2 = 0.5x^2 - 1.5x$.

2.3 Geometry

G.3.A *Know, explain, and apply basic postulates and theorems about triangles and the special lines, line segments, and rays associated with a triangle.*

Judging the Discovering geometry text from the viewpoint I indicated in the introduction to this report, I found this program mathematically inadequate: The text is about empirical observations of geometric facts; it has little or nothing to do with deductive geometry.

As was mentioned, the Triangle Sum Theorem (The sum of the angles for a triangle is 180 degrees) was selected to examine Standard *G.3.A*. This theorem, which the text calls "Triangle Sum Conjecture," is presented on Page 201. Since most of the assertions in the text appear as conjectures—some are "proved," and some are not—it is impossible to judge the soundness of the proof provided. In particular, there is no clear logical development in the text from which I could judge whether the proof of the "Triangle Sum Conjecture" (p. 202) is complete:

On Page 201, an auxiliary line $\overleftrightarrow{EC}$ parallel to $\overline{AB}$ (in triangle ABC) is constructed. Two fundamental ideas about the constructability and uniqueness of such a line are not addressed in the proof. The constructability requires the Exterior Angle Theorem ("An exterior angle of a triangle is greater than either remote interior angle of the triangle."), which is independent of the Parallel Postulate. I was unable to find this theorem in the book. The uniqueness of the parallel line is needed for the claim that the alternate angles are equal ($m\angle 1 = m\angle 4$ and $m\angle 3 = m\angle 5$, p. 202). In the text, however, these equalities are derived from the Alternate Interior Angles

Conjecture. But is this conjecture a postulate or a theorem? Just because it was not proved in the text, it does not mean it is a postulate; numerous other conjectures are stated without proof.

For the sake of completeness, I should mention that later, on Page 218, there appears the Triangle Exterior Angle Conjecture: The measure of an exterior angle of a triangle _____. The blank is to be filled with the statement "is equal to the sum of the measures of the remote interior angles." (Teacher's Edition, p. 218). While one can infer the Exterior Angle Theorem from the Triangle Exterior Angle Conjecture, structurally, the former is independent of the latter: the Exterior Angle Theorem is independent of the Parallel Postulate, whereas the Triangle Exterior Angle Conjecture requires it.

All assertions appear in the form of conjectures; a few are called properties (e.g., the "Parallel Slope Property" and the "Perpendicular Slope Property" on Page 167). Most of these conjectures are not proved. It is difficult, if not impossible, to systematically differentiate which of the conjectures are postulates and which are theorems. It is difficult to learn from this text what a mathematical definition is or to distinguish between a necessary condition and sufficient condition. Thus, for example, the assignment on Page 64 to "write a good definition" of different terms and the classification of quadrilaterals on Page 83 are intuitive at best.

The last chapter (Chapter 13) is labeled "Geometry as a Mathematical System." After almost 700 pages of mostly empirical geometry, the book attempts to introduce a mathematical system. Unfortunately, the development of the material in this chapter is too brief to do so. Assertions in this chapter rely on assertions in the previous 12 chapters, which, as was claimed earlier, are not developed within a logical structure.

The Corresponding Angles Postulate in this chapter (p. 697) is a theorem, not a postulate: The first part of the "Postulate" ("If two parallel lines are cut by a transversal, then the corresponding angles are congruent.") is a theorem based on the Parallel Postulate (p. 696), the Angle Duplication Postulate (p. 696), and the Exterior Angle Theorem. The converse of the "Postulate" ("If two coplanar lines are cut by a transversal forming congruent corresponding angles, then the lines are parallel.") is based on the Vertical Angle Theorem and the Exterior Angle Theorem.

2.4 Summary

The text does not justify fundamental theorems on linear and quadratic functions. In different places and in different contexts these theorems are demonstrated empirically. A common approach throughout the text is to present the problems and material through *non-holistic* problems, which mask the big ideas intended for students to learn. Consistently the text generalizes from empirical observations without attention to mathematical structure and justifications. There is nothing wrong with beginning with particular cases to understand something and make a conjecture about it. In many cases it is advantageous to do so and sometimes even necessary. However, students need to learn the difference between a conjecture generated from particular cases and an assertion that has been proved deductively. Unfortunately, the demarcation line between empirical reasoning and deductive reasoning is very vague in this program.

The approach the program applies to geometry is similar to that applies to algebra. It, too, amounts to empirical observations of geometric facts; it has little or nothing to do with deductive geometry. There is definitely a need for intuitive treatment of geometry in any textbook, especially one intended for high-school students. But the experiential geometry presented in the first 700 pages of the book is not utilized to develop geometry as deductive system. Most, if not all, assertions appear in the form of conjectures and most of the conjectures are not proved. It is difficult, if not impossible, to systematically differentiate which of the conjectures are postulates and which are theorems. It is difficult to learn from this text what a mathematical definition is or to distinguish between a necessary condition and sufficient condition.

3. Glenco

3.1 Forms of Linear Functions and Equations

A1.4.B *Write and graph an equation for a line given the slope and the y-intercept, the slope and a point on the line, or two points on the line, and translate between forms of linear equations.*

A1.1.B *Solve problems that can be represented by linear functions, equations, and inequalities.*

The material in Glenco that is relevant to Standards *A1.4.B* and *A1.1.B* appears in Algebra 1 (Chapters 2-5) and Algebra 2 (Chapters 1-2). In each of these chapters, the mathematical integrity of the content taught is compromised.

Important theorems on linear functions are not proved. Relevant to the above two standards are two fundamental theorems: A line in the plane is represented by a linear equation and the graph of a linear equation is a line. Neither of these theorems is proved.

Critical mathematical ideas, some of which are needed for these theorems, are turned into prescribed rules, and clumsy, imprecise, and even wrong statements are not uncommon. The following comprehensive set of examples, all from Algebra 1, supports these claims:

Example 1: On Page 171, we find:

> A positive rate of change indicates an increase over time. A negative rate of change indicates that a quantity is decreasing.

And on the same Page we find:

> A rate of change is constant for a function when the rate of change is the same between any pair of points on the graph of the function. Linear functions have a constant rate of change.

No justification is given to these shoddily formulated, yet important, ideas. Also, note the asymmetry between the two statements: while "positive rate of change" refers to change over time, "negative rate of change" refers to a decreasing quantity, without indicating what quantity is referred to. Furthermore, nowhere in the text is there a definition for "rate of change of a function."

Example 2: On Page 173, we find:

> The slope m of a non-vertical line through any two points, (x_1, y_1) and (x_2, y_2), can be found as follows: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

And on Page 237-8, we find:

> Parallel lines have the same slope.
> The slopes of perpendicular lines are opposite reciprocals.

No justification is given as to why the above ratio is constant for any choice of points (x_1, y_1) and (x_2, y_2). Likewise, the last two theorems are stated without justification.

Example 3: There are two definitions for absolute value in the text, one geometric ($|a|$ is the distance of the number a from zero) and one algebraic ($|x| = x$ if $x \ge 0$, and $|x| = -x$ if $x < 0$, Algebra 1, p. 262). There is no explanation about the connection between the two definitions. Worse, no use is made of the algebraic definition to solve equations and graph functions involving absolute values. Immediately following this definition, the text presents an example (Example 3) of how to graph the function $f(x) = |x-4|$. One would expect that in this example the algebraic definition of absolute value would be used to conclude that this function consists of two piecewise linear functions (the term "piecewise-linear function" appears two pages earlier, on Page 261), and so, to graph the function one would need no more than four points. Instead, no reference to the definition is made and the function is graphed by making a table consisting of eight points $(x, f(x))$.

Example 4: On Page 161, we find:

> The solution or root of an equation is any value that makes the equation true. A linear equation has at most one root. You can find the root of an equation by graphing its related function. To write the related function for an equation, replace 0 with $f(x)$.

Following this, the text shows how to "solve an equation with one root" and how to "solve an equation with no solution." For the latter, the text offers two methods. Method 1, labeled "solve algebraically," is illustrated by the equation, $3x + 7 = 3x + 1$. The equation is manipulated, leading to $6 = 0$. Following this, it is stated:

> The related function is $f(x) = 6$. The root of a linear equation is the value of x when $f(x) = 0$. Since $f(x)$ is always equal to 6, this equation has no solution.

Students get this convoluted explanation as to why the equation $3x + 7 = 3x + 1$ has no solution, rather than a straightforward, logical justification that utilizes what has been taught earlier. By now, from the treatment in the first half of Chapter 1 (Algebra 1), the students (should) know that for any choice of x, $3x$ is a number, and that if two different numbers, say, a and b, are added to a number c, then $c + a$ is different from $c + b$. These two facts is all what is needed to conclude that there is no x for which $3x + 7 = 3x + 1$. Of course, this straightforward justification cannot be given in cases when the equation is more complex. For this, one would need the concept of *equivalent equations.* On Page 81, this concept is described, but strangely it is not used here to explain why the result $6 = 0$ entails that the original equation has no solution.

Careless or wrong formulations are prevalent in the text. Many statements in the text are hard to understand and many do not make sense. Here are a few examples.

Example 5: On Page 104, we find under "Key Concept [of] Absolute Value Equations:"

> Words: When solving equations that involve absolute values, there are two cases to consider:
>
> Case 1: The expression inside the absolute value symbol is positive.

Case 2: The expression inside the absolute value symbol is negative.

Symbol: For any real numbers a and b, if $|a| = b$, then $a = b$ or $a = -b$

To begin with, when solving equations that involve absolute values, there are three cases to consider, not two. The third case is that the expression inside the absolute value symbol is zero. Second, the Symbol Statement is not a translation of the Words Statement. The Symbol Statement is implied from the definition of absolute value, which is not shown.

Example 6: On Page 153, we find three descriptions for "linear equation:"

(1) "A linear equation is an equation that forms a line when it is graphed";

(2) "Linear equations are often written in the form $Ax + By = C$"; and

(3) "The standard form of a linear equation is $Ax + By = C$, where $A \geq 0$, A and B are not both zero, and A, B, and C are integers with greatest common factor of 1."

To begin with, the third description is wrong: $\sqrt{2}x + 3y = 1$ is a linear equation in the standard form and yet A is not an integer. Second, it is true that any linear equation with *rational coefficients* can be brought to the form described in (3). However, nowhere in the text is there an explanation as to how and why this is the case. Third, and particularly critical, is that either (1) or (2) is a definition, but not both. Once one is chosen as a definition, the second must be a theorem. This critical fact is entirely absent from the text. To clarify, I am not saying that the text should have shown the logical equivalency between these definitions—that if we accept one as a definition, then the other is a theorem. What I am saying is that it is expected from a mathematical text to deal with this important topic by stating a clear definition of *linear function* (preferably the definition in (2) but with more precise formulation) and then prove that the graph of a linear equation is a line. The text fails to address this significant relation.

Example 7: On Page 172, we find:

For the rate of change to be linear, the change in x-value must be constant and the change in y-value must be constant.

It is difficult to comprehend this statement. I assume that by linear rate of change it is meant that $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is constant for all x_1 and x_2 in the domain of f. If so, the condition "the change in x-value must be constant and the change in y-value must be constant" makes no sense.

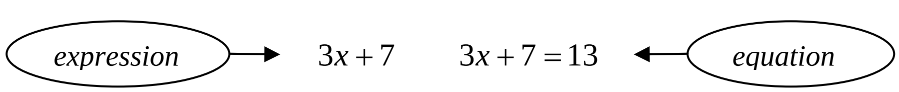
Example 8: On Page 214, we find a "definition" of "slope-intercept form":

An equation of the form $y = mx + b$, where m is the slope and b is the y-intercept, is the slope-intercept form.

The clause "where m is the slope and b is the y-intercept" does not belong to the definition. That m is the slope and b is the y-intercept is a theorem to be proved.

Example 9: On Page 31, we find the following statement:

A mathematical statement that contains two algebraic expressions and a symbol to compare them is an open sentence. A sentence that contains an equal sign, =, is an equation.



And later on Page 83:

"To solve an equation means to find the value of the variable that makes the equation true.

A student reading first statement may/should ask: What is the difference between "sentence" and "open sentence?" Is an equation an open sentence? Is a number an expression? There is no guide for the student or the teacher to answer these questions. Assume that the student concludes from this statement the correct answers: that an equation is an open sentence and a number is an expression. Then, he or she will also conclude that 13=13, 3>4, and 17<25 are all open sentences, for they are "expressions [with] a symbol to compare them." Obviously, this is not true: these are sentences but not open sentences. Furthermore, since an equation is an open sentence, the phrase "an equation is true" is not up to the standard of proper mathematical language.

Example 10: On Page 103, we find the following statement:

Expressions with absolute values define an upper and lower range in which a value must lie.

I cannot make sense of this sentence. What is meant by upper or lower range? Doesn't any algebraic expression define a range? If this is what is meant, what is the purpose of this statement? Likewise, the purpose of the sentence that follows, "Expression involving absolute value can be evaluated using the given value for the variable," is not clear: isn't it the case that any expression, not only those involving absolute value, can be evaluated using a given value for the variable in the expression?

Throughout the text, new concepts are introduced through contrived problems and solution approach used for these problems is alien to mathematical practice. Consider the following examples:

Example 11: Chapter 2 of Algebra 1, entitled Linear Equations, is about "writing equations." The text's general approach to translating a word problem into an equation is by literal translation of sentences in the problem. Earlier, on Page 6, the text prescribes key words for the translation; for example, "more than" corresponds to "addition;" "less than" to "subtraction;" and "of" (when a fraction is involved) to "multiplication." The use of key words is explicitly prescribed. On Page 206, for example, we find the text's "strategies for reading math problems:" To "identify relevant facts [in the problem statement], … look for keywords to solve the problem." A mathematically mature person—and the goal of any mathematics text should be to help students become such persons—would make sense of a word problem, build a coherent image of the situation described in the problem, and accordingly represent the problem algebraically. The approach adopted in this text is not consistent with this mathematical way of thinking.

Example 12: The use of "equation" in the start of the Chapter 2 on writing equations—when students are first introduced to this very important topic—is contrived. The first problem introduced (p. 75) is a simple division problem: if the distance around a track is 2.5 miles and one completes 500 miles driving around the track, the number of laps made around the track is simply $500 \div 2.5 = 200$. Instead, the students are told to go through the following steps:

Words	**The length of each lap** times **the number of laps** is **the length of the race**				
Variable	Let *l* represent the number of laps in the race				
Equation	**2.5**	×	***l***	=	**500**

Similarly, example 2 on Page 76 is a simple division problem that can be solved by dividing 180,000 into 45,000. Rather, the text directs the students to set a variable *d* and, like the above example, translates a verbal statement into the equation, $45,000 \bullet d = 180,000$. Further, the introduction of this critical material is not through problems that require the use of equations. Rather, it is through the translation of statements for the sake of translation, not for the sake of solving problems.

2.2 Forms of Quadratic Functions

A2.3.A *Translate between the standard form of a quadratic function, the vertex form, and the factored form; graph and interpret the meaning of each form.*

A2.1.C *Solve problems that can be represented by quadratic functions, equations, and inequalities.*

The material in Glenco that is relevant to Standards *A2.3.A* and *A2.1.C* appears in Algebra 1 (Chapters 8 and 9) and Algebra 2 (Chapter 5). Here too, like in the case of linear functions, the content presentation is mathematically unsound.

Important theorems on quadratic functions are not proved. Relevant to the above two standards are fundamental theorems about equivalencies among the three forms of quadratic functions—the standard form, the vertex form, and the factored form—and the relationships between the coefficients of the quadratic $ax^2 + bx + c$ and the shape of its graph (i.e., concavity, line of symmetry, and extrema). Neither of these theorems is proved. The quadratic formula is given on Page 558 without justification. The proof appears in Algebra 2 (p. 296) with the omission of a significant condition: that the coefficient a of x^2 must be non-zero (in order to divide by a). This might be just a typo. However, this is not an isolated case. Likewise, factorization of quadratic functions (p. 485) appears, from some mysterious reason, under the title "Arithmetic Sequence." While this, too, might be a typo, it adds to the overall carelessness of this text. Careless and even wrong statements are not uncommon in this text, and critical mathematical ideas, some of which are needed for the above theorems, are turned into prescribed rules. The compromising of mathematics in the chapters on quadratic function is as severe as in the chapter on linear functions. Here are some examples:

Example 1: On Page 493 (Algebra 1) we find:

> To factor trinomials of the form $ax^2 + bx + c$, find two integers, m and p, with a sum of b and a product of ac. Then write $ax^2 + bx + c$ as $ax^2 + mx + px + c$, and factor by grouping.

This rule was generalized from a particular case as follows:

> You can also use the method of factoring by grouping to solve this expression [$2x^2 + 5x + 3$].
>
> Step 1 Apply the pattern $2x^2 + 5x + 3 = 2x^2 + mx + px + 3$
>
> Step 2 Find two numbers that have a product $2 \cdot 3$ or 6 and sum of 5.

What does it mean to solve an expression? In what sense is $2x^2 + 5x + 3 = 2x^2 + mx + px + 3$ a pattern? Beyond these careless formulations, there is no explanation as to why the product of m and p should be 6. The factorization shown earlier on Page 485 deals with a different structure—that of $x^2 + bx + c = (x + m)(x + p)$. Rather than showing algebraically how the factorization of $ax^2 + bx + c$ can be done, the authors turn it into a rule with no justification. The

500 pages preceding this page should have been sufficient to provide the necessary mathematical basis to complete the factorization in a few steps such as: If $ac \neq 0$ and there exist m and p for which $b = m + p$ and $ac = mp$, then $ax^2 + bx + c = ax^2 + (m + p)x + c = ax^2 + mx + px + c$ $= (ax^2 + mx) + (\frac{ac}{m}x + c) = mx(\frac{a}{m}x + 1) + c(\frac{a}{m}x + 1) = (mx + c)(\frac{a}{m}x + 1)$.

Example 2: On Page 525 (Algebra 1) we find:

Standard form	$f(x) = x^2$
Type of Graph	Parabola
Axis of Symmetry	$x = -b/2a$
y-intercept:	c

When $a > 0$, the graph of $ax^2 + bx + c$ opens upward. The lowest point on the graph is the minimum.

When $a < 0$, the, the graph of $ax^2 + bx + c$ opens downward. The highest point on the graph is the maximum. The maximum or minimum is the vertex.

More rules related to these facts are prescribed on Pages 528, with no mathematical justification.

Example 3: On Page 296 (Algebra 2) we find a table that includes four rules for the number of roots of the equation $ax^2 + bx + c = 0$ (where a, b, c are real numbers a $a \neq 0$) and for the kinds of these roots (rational, irrational, or complex). The following are the first two rules:

Rule 1: If $b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square, then the equation $ax^2 + bx + c = 0$ has two rational roots.

Rule 2: If $b^2 - 4ac > 0$ and $b^2 - 4ac$ is not a perfect square, then the equation $ax^2 + bx + c = 0$ has two irrational roots.

The assertions about the number of roots are important; they are used in graphing quadratic functions and solving quadratic equation. Neither of these rules is justified. The assertions about the kinds of roots—rational or irrational—are wrong. For the first rule, take the equation $x^2 + \sqrt{13}x + 1 = 0$. The discriminant is a perfect square ($b^2 - 4ac = 13 - 4 = 9 = 3^2$) and the roots are irrational ($\frac{-\sqrt{13} \pm 3}{2}$). For the second rule, simply take any equation with rational roots, say, $x^2 + 3x + 2 = 0$, and multiply it by any irrational number, say, $\sqrt{2}$. The discriminant of the equation $\sqrt{2}x^2 + 3\sqrt{2}x + 2\sqrt{2} = 0$ is 2, which is not a perfect square, but the roots of the equation are rational numbers, -1 and -2. Furthermore, even if the text stated different, correct rules it is not clear what functions such rules would have in the mathematical context of this text.

Example 4: On Page 493 (Algebra 1) we find:

At amusement parks around the country, the paths of riders can be modeled by the expression $16t^2 - 5t + 120$. Factoring this expression can help the ride operators determine how long a rider rides on the initial swing.

What is exactly being modeled? What are the quantitative relations that are being modeled by the expression $16t^2 - 5t + 120$? And how does factorization help determining the initial swing time?

Example 5: On Page 505

> To find about how long it takes an object to hit the ground if it is dropped from an initial height of h_0 feet, you would need to solve the equation $h = 16t^2 + h_0$.

There is no mention of the initial velocity. Further, the equation to be solved is $0 = -16t^2 + h_0$, not $h = 16t^2 + h_0$.

3.3 Geometry

G.3.A *Know, explain, and apply basic postulates and theorems about triangles and the special lines, line segments, and rays associated with a triangle.*

The material in Glenco that is relevant to Standards *G.3.A* appears in Geometry (Chapters 3 and 4). The treatment of this material in this text is almost identical to that in Holt, and so the report below is similar to the report for Holt.

Section 3-1 (p. 171) starts with definitions and illustrations of basic terms such as parallel lines, parallel planes, and angle pairs formed by a transversal. Section 3-2 (p. 178) starts with the Corresponding Angles Postulate: "If two parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent". This is followed by practice problems on how the postulate is used to compute different angles.

Sections 3-3 and 3-5 (pp. 186-204) digress to analytic geometry about lines and slopes. We find here the following two *postulates* (yes, they are called postulates in this text):

> Two nonvertical lines have the same slope if and only if they are parallel. All vertical lines are parallel
> Two nonvertical lines are perpendicular if and only if the product of their slopes is -1. Vertical and horizontal lines are perpendicular.

This is a pure misuse of the concept of *postulate*. Both of these assertions are theorems. The first could be proved by using material already presented in the text; specifically, by using systems of equations, it can be (easily) shown that two linear equations, $y = m_1x + b_1$ and $y = m_2x + b_2$, intersect if and only if $m_1 \neq m_2$. The second assertion can be proved by congruence, a concept which appears later in the text. Even so, the assertion should not been called a postulate. The text could have indicated that the theorem will be proved later or just wait until congruence is done and then prove it.

Section 3-5 (p. 205) starts with the Converse of the Corresponding Angles Postulated: "If two lines are cut by a transversal so that corresponding angles are congruent, then the two lines are parallel". (The important condition that lines are coplanar is missing.). Following this, the construction of a parallel line to a given line through a point is shown. This construction involves the duplication of an angle, and is followed by the Parallel Postulate (p. 206): "If given a line and a point not on the line, then there exists exactly one line through the point that is parallel to the given line." This postulate is preceded by the following statement:

> The construction establishes that there is *at least* one line through C that is parallel to $\overleftrightarrow{AB}$. The following postulate [the above Parallel Postulate] guarantees that this line is the *only* one.

So the construction of the parallel lines is established by the duplication of an angle, which, in turn, is established by congruence. Congruence, however, does not appear until later in Chapter 4. The end result is that the Parallel Postulate is not a postulate but a theorem derivable (through a simple proof by contradiction) from the Corresponding Angles Postulate. The concern about this presentation is the lack of mathematical accuracy and the lack of attention to the distinction between a “postulate” and a “theorem.”

Lastly, the sequencing of the geometric topics in the text, while mathematically legitimate, deprives the students from the opportunity to deal with important mathematical ideas. I have already mentioned the sequencing of the three postulates in Section 3-2 and 3-3. While there is no circular reasoning in the text, one would expect that when an angle is duplicated to construct parallel lines students and teachers are alerted that this construction requires justification, and that such a justification will appear later in the text. The same concern applies to the construction of perpendicular lines (p. 213). Uniqueness of perpendicularity is critically needed for the definition of “distance from a point to a line” (p. 213), and it can be proved from the Triangle Sum Theorem, which appears later (p. 244). However, the text presents the uniqueness of perpendicularity as a postulate (p. 213)—another misuse of the term *postulate*.

3.4 Summary

In each of the chapters on linear functions and quadratic functions, the mathematical integrity of the content taught is consistently compromised. This manifests itself in different ways. First, with one exception, none of the important theorems about linear functions and quadratic functions is proved. Second, solutions to problems and assertions about conditions and relations appear in the form of prescribed rules, and in some cases the rules are wrong. Third, there is an abundance of imprecise, even wrong, descriptions of concepts and problems. Fourth, many problems are contrived, and non-trivial, holistic problems are rare. Solutions to problems also are often contrived and alien to mathematical practice. Some of the material in Algebra 2 is a repetition of that in Algebra 1, with no improvement on any of these points.

The development that leads up to the proof the Triangle Sum Theorem (Section 4-2) does not include circular reasoning. However, there is repeated misuse of the concept of *postulate* and some important theorems are stated without proof. In addition, this development is interrupted by two sections on analytic geometry, with theorems that are either incorrectly labeled as postulates or appear without proof. In the process of developing a deductive structure for synthetic geometry, the students are introduced to a “foreign object” which does not belong to the development of this structure.

4. Holt

4.1 Forms of Linear Functions and Equations

A1.4.B *Write and graph an equation for a line given the slope and the y-intercept, the slope and a point on the line, or two points on the line, and translate between forms of linear equations.*

A1.1.B *Solve problems that can be represented by linear functions, equations, and inequalities.*

The material in Holt that is relevant to Standards *A1.4.B* and *A1.1.B* appears in Algebra 1 (Chapters 5 and 6).

There is no clear definition of linear function in the text. The text states several properties of linear functions, but never justifies relationships among them or how these properties are entailed from a definition of linear function. In Section 5-1, Page 296, we find the statement*:*

> A function whose graph forms a straight line is called a *linear function*.

Since this is the first characterization of linear function, we may assume it is the text's definition. On Page 298, we find another characterization of *linear equation:*

> A *linear equation* is any equation that can be written in the *standard form* $ax + by = c$.

No justification is given as to why a function whose graph is a line has this form. On Page 299, shortly after the presentation of the first definition of linear function, it says:

> You can sometimes identify a linear function by looking in a table or a list of ordered pairs. In a linear function, a constant change in *x* corresponds to a constant change in *y*.

Again, there is no mention here or elsewhere in the text as to how this property is entailed from the text's (initial) definition of linear function. Instead, the property is followed by tables of ordered pairs and graphs whose purpose is to demonstrate that when the change is constant the corresponding order pairs lie on a straight line, and when the change is not constant the corresponding pairs do not lie on a straight line. This would have been adequate if it were followed with a justification as to why in a linear function a constant change in x corresponds to a constant change in y .

On Page 311, the text defines slope ("slope=change in *y*/change in *x*") and later, on Page 320, it states:

> If (x_1, y_1) and (x_2, y_2) are any two different points on a line, the slope of the line is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

The critical question as to whether m is constant for any choice of (x_1, y_1) and (x_2, y_2) satisfying the equation $y = mx + b$ is never justified, nor is it raised.

The slope-intercept form appears on Page 335. The simple relationship between this form and the standard form is not established. Nowhere in the text is it shown how these two forms are equivalent. The change of forms is done in particular cases, where the coefficients are particular numbers—in most cases the coefficients are integers. For example, the point-slope form is given on Page 342. Even in this simple case where the form can be derived straightforwardly from the equation $m = \frac{y - y_1}{x - x_1}$, the text resorts to a special case ($3 = \frac{y - 1}{x - 2}$) and generalizes from it the form $y - y_1 = m(x - x_1)$.

Consistently, the text avoids dealing with abstract concepts and mathematical justifications. Sadly, the text seems to welcome non-mathematical behaviors by students. On Page 304, a blurb tells how an (actual) student finds the intercepts of a linear function:

> I use the "cover-up" method to find intercepts. … If I have $4x - 3y = 12$: First, I cover $4x$ and solve the equation I can still see. $-3y = 12 \quad y = -4$. [The *x*-intercept is found in a similar way by covering the $-3y$.]

This non-mathematical reasoning would have been tolerable if it were followed with an explanation of the student's method. Astonishingly, even the Teacher's Edition does not alert the teacher to counter this method with correct explanation. On the next page (p. 305), the intercepts of the line $2x-4y=8$ are computed correctly by substituting zero for *x* to find the *y*-intercept, and zero for *y* to find the *x*-intercept. However, no connection is made between this correct method and the student's method.

Finally, two other important theorems are stated but no justification is given. These are:

Two different nonvertical lines are parallel if and only if they have the same slope. (p. 349)

Two different nonvertical lines are perpendicular if and only if the product of their slopes is -1. (p. 351).

4.1 Forms of Quadratic Functions

A2.3.A *Translate between the standard form of a quadratic function, the vertex form, and the factored form; graph and interpret the meaning of each form.*

A2.1.C *Solve problems that can be represented by quadratic functions, equations, and inequalities*.

The material in Holt that is relevant to Standards *A2.3.A* and *A2.1.C* appears in Algebra 2 (Chapters 5). As in the case of linear functions, the text's approach to quadratics is mathematically inadequate. Its treatment consists mainly of rules without mathematical justifications. Students who learn to solve the problems in the text will do so with little or no understanding of what they are doing. In addition, the text consistently avoids dealing with general forms of functions and equations and deals, instead, with particular cases.

The definition of *quadratic function* is in Lesson 5-1 (Algebra 2, p. 315):

A quadratic function is a function that can be written in the form $f(x)=a(x-h)^2+k$ ($a\neq 0$).

On Page 323, it is shown that the function $f(x)=x^2$ is even ($f(x)=f(-x)$) and so it is symmetric about the y-axis. On Page 316, the text indicates:

You can also graph quadratic functions by applying transformations to the parent function $f(x)=x^2$. Transforming quadratic functions is similar to transforming linear function (Lesson 2-4).

Following this, the text shows how to apply transformation of translations, stretching, and compressing on the parent function $f(x)=x^2$ to graph functions of the form $f(x)=a(x-h)^2+k$. Potentially, this is all what is needed to establish symmetry. Unfortunately, Lesson 2-4 on transformations of graph, the basis for the lesson on symmetry (Lesson 5-1) is devoid of mathematical reasoning. Here are a few examples of the rules stated without sufficient mathematical explanation (pp. 134-135):

$f(x)\rightarrow f(x-h)$, $h>0$ moves right $h<0$ moves left

$f(x)\rightarrow f(x)+k$, $k>0$ moves up $k<0$ moves down

$f(x)\rightarrow f(\frac{1}{b}x)$, $b>1$ stretches away from the y-axis. $0<|b|<1$ compresses forward the y-axis

$f(x)\rightarrow a\cdot f(x)$, $a>1$ stretches away from the x-axis. $0<|a|<1$ compresses forward the x-axis

One would expect that the text would make use of its material on functions (Chapter 4 in Algebra 1 and Chapter 1 in Algebra 2) to provide meaningful explanations to these rules. Unfortunately, this is not done.

Properties needed to graph quadratic functions are stated on Page 324 without justification:

> For $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$, the parabola has these properties:
> The parabola opens upward if $a > 0$ and downward if $a < 0$.
> The axis of symmetry is the vertical line $x = -b/2a$ [see discussion above on this issue].
> The vertex is the point $(-b/2a, f(-b/2a))$.

The solution of quadratic equations by factoring is shown only through particular cases. For example, on Page 334, the equation $x^2 - 8x + 12 = 0$ is transformed into the equation $(x-2)(x-6) = 0$, telling the students "Find factors of 12 that add to -8" without any explanation. The theorem on factorization of quadratic functions is neither proved nor stated.

The translations between the vertex form and standard form of the quadratic function is done in general terms (p. 342 and p. 356)—a rare phenomenon in this text.

Quadratic inequalities (with one variable) appear in Lesson 5-7 (Algebra 2, p. 366). A textbook for which mathematical reasoning is a central objective would use quadratic inequalities to apply all what is learned about quadratic functions and quadratic equations. For example, the graph of a quadratic function together with information about its zeros provides the basis for solving the corresponding quadratic inequality. The factorization of a quadratic function into a product of linear functions can be used to solve the corresponding quadratic inequality by considering different cases and applying logical rules involving "or" and "and." None of this is done in the text. Instead, the text presents an example (p. 368) of how to solve a quadratic inequality by applying three ready-made steps: To solve the inequality $x^2 - 4x + 1 > 0$, do the following:

> Step 1: Write the related equation $x^2 - 4x + 1 = 0$
> Step 2: Solve the equation by factorization: $(x-5)(x+1)=0$, $x = 5$ or $x = -1$ [and accordingly] divide the number line into three intervals: $x < 1$, $-1 < x < 5$, and $x > 5$.
> Step 3: Test an x-value in each interval. [The values $x = -2$, $x = 0$, and $x = 6$ are tried in the corresponding intervals, from which the solution to the inequality is determined]

Students who manage to remember these three steps, along the numerous mechanical procedures in the text, may solve the assigned problems correctly, but will do so mindlessly, without any adequate understanding of what they are doing. The concern is not that the text divided the task into steps; rather, the concern is the lack of reasoning, of a mathematical rationale for the steps. For example: Why is the number-line partitioned into three intervals? Why is testing isolated values in each interval sufficient to determine the solution set of the inequality?

4.3 Geometry

G.3.A *Know, explain, and apply basic postulates and theorems about triangles and the special lines,* *line segments, and rays associated with a triangle*.

The material in Holt that is relevant to Standards G.3.A appears in Geometry (Chapters 3 and 4). Section 3-1 (p. 146) starts with definitions and illustrations of basic terms such as parallel lines, parallel planes, and angle pairs formed by a transversal. Section 3-2 (p. 155) starts with the Corresponding Angles Postulate ("If two parallel lines are cut by a transversal, then the pairs of corresponding angles are congruent," p. 155). This is followed by practice problems on how the postulate is used to compute different angles.

Section 3-3 (p. 163) starts with the Converse of the Corresponding Angles Postulate ("If two coplanar lines are cut by a transversal so that a pair of corresponding angles are congruent, then the two lines are parallel," p. 162). This postulate is then followed, almost immediately, by the Parallel Postulate ("Through a point *P* not on line *l*, there is exactly one line parallel to l," p. 163). Following this we find the following statement:

> The Converse of the Corresponding Angles Postulate is used to construct parallel lines. The Parallel Postulate guarantees that for any line *l*, you can always construct a parallel line through a point that is not on *l*.

On the same page (p. 163), the text shows how to construct the parallel line by constructing congruent angles, and in the Teacher's Edition it says explicitly: "we constructed congruent corresponding angles." Congruence, however, appears later, in Chapter 4. So there is a confusion here as to what guarantees the construction of the parallel lines: *congruence* or the *Parallel Postulate*? Assuming the former, since it says so explicitly in the Teacher's addition, there are two structural problems: The first problem is that the existence of a parallel line through a point *P* that is not on *l* is guaranteed by the Converse of the Corresponding Angles Postulate, not by the Parallel Postulate; the latter guarantees that such a line is unique. The second problem is that the Parallel Postulate—that there is a unique parallel—is not a postulate here since it is derivable (through a simple proof by contradiction) from the preceding Corresponding Angles Postulate.

The first, obvious concern in this presentation is the lack of mathematical accuracy. The second concern is that in this presentation the text avoids dealing with the important ideas of "existence" versus "uniqueness" and the distinction between a "postulate" and a "theorem." Another—though minor—issue is that the statement "The Converse of the Corresponding Angles Postulate is used to construct parallel lines" is not completely accurate. The construction of parallel lines involves the duplication of an angle, and the Converse of the Corresponding Angles Postulate guarantees that this construction results in parallel lines.

Lastly, the sequencing of the geometric topics in the text, while mathematically legitimate, deprives the students of the opportunity to deal with important mathematical ideas. I have already mentioned the sequencing of the three postulates in Section 3-2 and 3-3. While there is no circular reasoning here, one would expect that when an angle is duplicated to construct parallel lines (p. 163) the students and teachers are alerted that this construction requires justification, and that such a justification will appear later in the text. The same concern applies to the construction of a perpendicular bisector of a segment (p. 172).

Uniqueness of perpendicularity is critically needed for the definition of "distance from a point to a line" (p. 172) but it is neither proved nor stated.

The development that leads up to the proof of the Triangle Sum Theorem (Section 4-2) is interrupted by two sections on analytic geometry (Sections 3-5 and 3-6). In the process of developing a deductive structure for synthetic geometry, the students are introduced to a "foreign object" which does not belong to the development of this structure. Furthermore, these sections (3-5 and 3-6) include important theorems without proofs. The first theorem is the Parallel Lines Theorem:

> In a coordinate plane, two nonvertical lines are parallel if and only if they have the same slope. Any two vertical lines are parallel.

This theorem could be proved by using material already presented in the text; specifically, by using the material on systems of equations, it can be (easily) shown that two linear equations $y = m_1 x + b_1$ and $y = m_2 x + b_2$ intersect if and only if $m_1 \neq m_2$. The second theorem without proof is the Perpendicular Lines Theorem:

> In a coordinate plane, two nonvertical lines are perpendicular if and only if the product of their slopes is -1. Vertical and horizontal lines are perpendicular." (p. 184)

This theorem can be proved by using congruence, so the text could have said so and proved later, or waited one more chapter to present it with proof.

4.4 Summary

Overall, the text's approach to linear functions and quadratic functions is mathematically unsound. It amounts to a set of rules without mathematical justifications. In this approach students are deprived from learning to reason logically. Students who learn to solve the problems in the text will do so with little or no understanding of what they are doing. Furthermore, the text consistently avoids dealing with general forms of functions and equations, and resorts, instead, to generalize from particular cases. In addition, the text seems to encourage non-mathematical behaviors by students. Many of the facts presented in Algebra 1 are repeated in Algebra 2 in the same manner, without proper justification and without refinement toward more abstract treatments. The material in Algebra 2, where one would expect a more advanced treatment of the content taught, remains mathematically inadequate.

The development of the material on parallel lines that leads up to the proof of the Triangle Sum Theorem does not include apparent circular reasoning. The sequencing of the material, however, results in the loss of important mathematical ideas.

5. Conclusions

The examination focused on five standards related to *forms of linear functions and equations*, *forms of quadratic functions and equations*, and *parallel lines and the Triangle Sum Theorem*. The body of the instructional material was examined with respect to three main criteria: (a) *mathematical justification*, (b) *symbolism and structure*, and (c) *language*. As can be seen in the chart below, none of the programs was found mathematically sound on the first two criteria. The ✓ in Holt on these criteria in geometry is better characterized as the least mathematically unsound. With the exception of the Glenco Program, which severely misuses mathematical language in algebra and does not adequately distinguish between theorems and postulates in geometry, the language used by the rest of the programs is mathematically sound.

	Mathematical Justification		Symbolism and Structure		Language	
	Algebra	Geometry	Algebra	Geometry	Algebra	Geometry
Core-Plus	—	—	—	—	+	+
Discovering	—	—	—	—	+	+
Glencoe	—	—	—	—	—	✓
Holt	—	✓	—	✓	+	+
Key						
+	Mathematically sound					
✓	Mathematical soundness meets minimum standard					
—	Mathematically unsound					

Two additional aspects of the programs were examined: (d) *the problems used for practice and internalization of the material taught* and (e) *the way new concepts are introduced*. Regarding aspect (d), the examination addressed only the presence of a relatively large number of non-trivial, multi-step holistic problems, and regarding aspect (e), the examination addressed the nature of the problems or activities used to mathematically necessitate new concepts. None of the programs was found satisfactory on aspect (d), and only Core-Plus and Discovering were found satisfactory on aspect (e).

Although the number of topics examined is limited, these topics are central to high school mathematics and beyond. The four programs failed to convey critical mathematical concepts and ideas that should and can be within reach for high school students.

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[i] The four programs were examined independently by two reviewers. The second reviewer was Professor Steffen Wilson. His review can be found in http://www.math.jhu.edu/~wsw/ED/wahighschoolwsw.pdf.

[ii] Elsewhere I defined this notion more precisely and discussed its different forms in mathematical practice (see Harel, 1998, 2008).

[iii] To make this point clearer, it is worth presenting an alternative approach—one that is more likely to intellectually necessitate algebraic tools to solve word problems. In this alternative approach, students first learn to solve non-trivial word problems with their current arithmetic tools. For example, they can reason about problems of the following kind directly, without any explicit use of variables.

> Towns A and B are 280 miles apart. At 12:00 PM, a car leaves A toward B, and a truck leaves B toward A. The car drives at 80 m/h and the truck at 60 m/h. When will they meet?

Students can do so by, for example, reasoning as follow:

> After 1 hour, the car drives 80 miles and truck 60 miles. Together they drive 140 miles. In 2 hour, the car drives 160 miles and the truck 120 miles. Together they drive 280 miles. Therefore, they will meet at 2:00 PM.

Through this kind of reasoning, students develop the habit of building coherent images for the problems—a habit they often lack. These problems can then be gradually modified—in context, as well as in quantities—so as to make them harder to solve with arithmetic tools alone, whereby necessitating the use of algebraic tools. For example, varying the distance between the two towns through the sequence of numbers, 420, 350, 245, and 309, results in a new sequence of problems with increasing degree of difficulty. Students still can solve these problems with their arithmetic tools but the problems become harder as the relationship between the given distance and the quantity 140 (the sum of the two given speeds) becomes less obvious. For example, for the case where the distance is 245 miles, the time that takes until the two vehicles meet must be between 1 and 2 hours, and so one might search through the values, 1 hour and 15 minutes ($80 \cdot \frac{75}{60} + 60 \cdot \frac{75}{60} \stackrel{?}{=} 245$), 1 hour and 30 minutes ($80 \cdot \frac{90}{60} + 60 \cdot \frac{90}{60} \stackrel{?}{=} 245$), 1 hour and 45 minutes ($80 \cdot \frac{105}{60} + 60 \cdot \frac{105}{60} \stackrel{?}{=} 245$), and find that the last value is the time sought for. This activity of varying the time needed can give rise to the concept of variable (or unknown) and, in turn, to the equation, $80 \cdot x + 60 \cdot x = 140$. Granted, this is not the only approach to intellectually necessitate the use of algebraic tools for solving word problems. However, whatever approach is used, it is critical to give students ample opportunities to repeatedly reason about the problems with their current arithmetic tools and to gradually lead them to incorporate new, algebraic tools. The goal is for students learn to build coherent mental representations for the quantities involved in the problem and to intellectually necessitating the use of equations to represent these relationships. An added value of this approach is the development of computational fluency with numbers, especially fractions.