# The Controllability Problem for Differential-Algebraic Dynamical Systems 

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Consider the control system

$$
\begin{equation*}
A_{0} \dot{x}=A x+B u, t>0, \tag{1}
\end{equation*}
$$

with the initial condition $x(0)=x_{o}$, where $x \in R^{n}$, и $u \in R^{r}, A_{0}, A$, and are constant matrices of appropriate sizes, $x_{0} \in R^{n}$, and $\operatorname{det} A_{0}=0$.

Let system (1) satisfy the consistency condition [1]. Then its solution can be represented in the form

$$
\begin{gather*}
x(t)=e^{A_{0}^{d} A t} A_{0} A_{0}^{d} q+\int_{0}^{t} e^{A_{0}^{d} A(t-s)} A_{0}^{d} B u(s) d s+\left(E_{n}-A_{0} A_{0}^{d}\right) \sum_{i=0}^{k-1}(-1)^{i}\left(A_{0} A^{d}\right)^{i} A^{d} B u^{(i)}(t),  \tag{2}\\
x(0)=x_{0}=A_{0} A_{0}^{d} q+\left(E_{n}-A_{0} A_{0}^{d}\right) \sum_{i=0}^{k-1}(-1)^{i}\left(A_{0} A^{d}\right)^{i} A^{d} B u^{(i)}(0), \tag{3}
\end{gather*}
$$

where $A_{0}^{d}$ and $A^{d}$ are the Drazin inverses of $A_{o}$ and $A$, respectively, the number $k$ is the index of the matrix $A_{0}, q \in R^{n}$, and $u^{(i)}(0) \in R^{r}, i=0, \ldots, k-1$. Here we assume that the control $u(t), t \geq 0$, is a sufficiently smooth $r$-dimensional vector function.

In this case, one can show that the solution (2) of system (1) is the output of the system

$$
\begin{equation*}
\dot{Y}=\hat{A} Y+\hat{B} v, x=C Y \tag{4}
\end{equation*}
$$

with the initial condition $Y(0)=Y_{0}=\left(q, u^{(i)}(0), i=1, \ldots, k\right)$, where $Y=\left(y, u^{1}, \ldots, u^{k}\right)$, $v=u^{(k)}$, and $\hat{A}=\left(\hat{A}_{p q}\right)$ and ${ }^{\wedge}=\left(\hat{B}_{p 1}\right), p=1, \ldots, k+1, q=1, \ldots, k+1$, are block matrices, moreover, $\hat{A}_{11}=A_{0}^{d} A, \hat{A}_{12}=A_{0}^{d} B, \hat{A}_{23}=\hat{A}_{34}=\ldots=\hat{A}_{k, k+1}=E_{r}, \hat{A}_{i j}=0$ for all remaining indices $p$ and $q, \hat{B}_{k+1,1}=E_{r}, \hat{B}=0, i=1, \ldots, k$, and

$$
C=\left[A_{0} A_{0}^{d},\left(E_{n}-A_{0} A_{0}^{d}\right) A^{d} B, \ldots,(-1)^{k-1}\left(E_{n}-A_{0} A_{0}^{d}\right)\left(A_{0} A^{d}\right)^{k-1} A^{d} B\right] .
$$

We set

$$
\begin{gathered}
\Omega_{0}=\left\{z \in R^{n} \mid z=A_{0} A_{0}^{d} q+\left(E_{n}-A_{0} A_{0}^{d}\right) \sum_{i=0}^{k-1}(-1)^{i}\left(A_{0} A^{d}\right)^{i} A^{d} B u^{(i)}(0),\right. \\
\left.q \in R^{n}, u^{(i)}(0) \in R^{r}, i=\overline{0, k-1}\right\} .
\end{gathered}
$$

Let $H$ be a constant $n \times n$ matrix.
Definition 1. System (1) is said to be $H$-controllable if for each $x_{0} \in \Omega_{0}$, there exists a time $t_{1}<+\infty$ and a smooth control $u(t), t \in\left[0, t_{1}\right]$, such that $x(0)=x_{0}$ and $H x\left(t_{1}\right)=0$.

Definition 2. System (1) is said to be completely $H$ - controllable if for each $x_{0} \in \Omega_{0}$, there exists a time $t_{1}<+\infty$ and a smooth control $u(t), t \geq 0$, such that the solution $x(t)$, $t \geq 0$, of system (1) satisfies the conditions $x(0)=x_{0}$ and $H x(t) \equiv 0, t \geq t_{1}$.

The following theorems hold.
Theorem 1. System (1) is $H$-controllable if and only if
$\operatorname{rank}\left(H A_{0} A_{0}^{d},(-1)^{j} H\left(E_{n}-A_{0} A_{0}^{d}\right)\left(A_{0} A^{d}\right)^{j} A^{d} B, j=\overline{0, k-1} ; H\left(A_{0}^{d} A\right)^{i} A_{0}^{d} B, i=\overline{0, n-1}\right)=$

$$
=\operatorname{rank}\left((-1)^{j} H\left(E_{n}-A_{0} A_{0}^{d}\right)\left(A_{0} A^{d}\right)^{j} B, j=\overline{0, k-1} ; H\left(A_{0}^{d} A\right)^{i} A_{0}^{d} B, i=\overline{0, n-1}\right) .
$$

Theorem 2. System (1) is completely $H$-controllable if and only if

$$
\operatorname{rank}\left(L, \bar{H}\left(A_{0}^{d} A\right)^{i} A_{0}^{d} B, i=\overline{0, k-1}\right)=\operatorname{rank}(L, \bar{H})
$$

where

\[

\]

The proofs of Theorems 1 and 2 readily follow from the representation of system (1) in the form (4) with the use of the results in [2].

Consider systems (1) and (4). We introduce the correspondences

$$
\begin{equation*}
x(t) \rightarrow X_{t} ; u(t) \rightarrow U_{t} ; y(t) \rightarrow Y_{t} ; u^{i}(t) \rightarrow U_{t}^{i} ; p \rightarrow \Delta \tag{5}
\end{equation*}
$$

Here $X_{t}, Y_{t}$, and $U_{t}, U_{t}^{i}$ - are matrices of sizes $n \times r$ and $r \times r ; p \equiv \frac{d}{d t}$ is the differentiation operator, and $\Delta$ is the shift operator ( $\Delta^{i} X_{t}=X_{t+i} ; \Delta^{i} U_{t}=U_{t+i} ; \Delta^{i} U_{t}^{j}=U_{t+i}^{j}$ ).

By using Eq. (4) and the correspondences (5), we pass to the recursion relations

$$
\begin{align*}
& Y_{t+1}=A_{0}^{d} A Y_{t}+A_{0}^{d} B U_{t}^{1}, U_{t+1}^{i}=U_{t}^{i+1}, i=\overline{1, k-1}, \\
& U_{t+1}^{k}=U_{t+k}, X_{t}=C\left[Y_{t}, U_{t}^{1}, \ldots, U_{t}^{k}\right], t \geq 0 \tag{6}
\end{align*}
$$

under the condition that $Y_{t} \equiv 0, t=0, \ldots, k-1, U_{t} \equiv 0, t \neq k$, and $U_{k}=E_{r}$. Relations (6) are referred to as determining equations for the control system (1). By $X_{t}^{*}$ we denote the solution of the determining equations (6) with $Y_{1}=E_{n}$ and $U_{t} \equiv 0, t \geq 0$. Then Theorems 1 and 2 can be stated in terms of solutions of the determining equations.

Theorem 1'. System (1) is $H$-controllable if and only if

$$
\operatorname{rank}\left(H X_{1}^{*}, H X_{i}, i=\overline{1, n+k}\right)=\operatorname{rank}\left(H X_{i}, i=\overline{1, n+k}\right)
$$

Theorem 2'. System (1) is completely $H$-controllable if and only if

$$
\operatorname{rank}\left(L, \bar{H} X_{i}, i=\overline{k+1, n+k}\right)=\operatorname{rank}(L, \bar{H})
$$

where

$$
L=\left[\begin{array}{lll}
H X_{k} & \ldots & H X_{1} \\
\ldots & \ldots & \ldots \\
H X_{n+2 k-1} & \cdots & H X_{n+k}
\end{array}\right], \quad \bar{H}=\left[\begin{array}{c}
H X_{1}^{*} \\
\vdots \\
H X_{n+k}^{*}
\end{array}\right] .
$$

The obtained results can be transferred for the discret systems and for the systems with delay.

For example, consider the control system of the form

$$
\begin{equation*}
\dot{x}(t)=A x(t)+B_{1} u(t)+B_{2} u(t-h)+\int_{0}^{h} B(s) u(t-s) d s, t \geq 0 \tag{7}
\end{equation*}
$$

with the initial condition

$$
x_{0}=x(0), u_{0}(\cdot)=\{u(t)=\varphi(t), t \in[-h, 0)\}
$$

where $x \in R^{n}, u \in R^{r} ; A, B_{1}, B_{2}$ - constant matrices of appropriate size; $B(t)-(n \times r)$ -matrix-function; $x_{0}$ - preassigned $n$-vector; $\varphi(t)$ - piecewise continuous $n$-vector-function.

Definition 3. System (7) is said comolete $H$-controllable it for any vector $x_{0} \in R^{n}$, and any piecewise continuous function $\varphi(t), t \in[-h, 0)$, there exists a time $t_{1}, 0<t_{1}<+\infty$, and control $u(t), t \in\left[0, t_{1}-h\right), u(t) \equiv 0, t \geq t_{1}-h$, such that the trajectory $x(t), t \geq 0$, of the system (7) has the property $x(0)=x_{0}, H x(t) \equiv 0, t \geq t_{1}$.

Let's assume that in any time $t, t \geq 0$, the state $x(t)$ of the system (7) is inadmissible, but the output

$$
\begin{equation*}
y(t)=H x(t), t \geq 0, y_{0}=y(0) \tag{8}
\end{equation*}
$$

is admissible and the control $u(t), t \geq 0$, is choused from the condition that the output of the system (7) is known.

Definition 4. System (7) is said complete controllable with respect to the output (8), if for any vector $m$-vector $y_{0}$ and any piecewise continuous function $\varphi(t), t \in[-h, 0)$, there exists a time $t_{1}, 0<t_{1}<+\infty$, and a control $u(t), t \in[0, t-h), u(t) \equiv 0, t \geq t_{1}-h$, such that the $y(t) \equiv 0 \quad t \geq t_{1}$.

Theorem 3. System (7) is complete $H$-controllable if and only if

$$
\begin{equation*}
\operatorname{rank}\left\{\bar{H} A^{i} B_{h}, i=\overline{0, k-1}\right\}=\operatorname{rank} \bar{H} \tag{9}
\end{equation*}
$$

where $B_{h}=B_{1}+e^{-A h} B_{2}+\int_{0}^{h} e^{-A s} B(s) d s, \bar{H}=\left[\begin{array}{c}\frac{H A^{i}}{0, k-1}\end{array}\right], k$ - is the degree of the minimal polynomial of the matrix $A$.

Theorem 4. System (7) (it is assumed, that $B(s) \equiv 0, s \in[0, h]$,) is complete controllable withe respect to the output (8) it and only if

$$
\operatorname{rank}\left\{\bar{H} A^{i} B_{h}^{*}, i=\overline{0, k-1}\right\}=\operatorname{rank}\left\{\bar{H}, H_{B_{1}}, H_{B_{2}}, \bar{H} A^{i} B_{h}^{*}, i=\overline{0, k-1}\right\}
$$

где $H_{B_{i}}=\left[\begin{array}{cccc}0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & H B_{i} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & H B_{i} & \ldots & H A^{k-3} B_{i} \\ H B_{i} & H A B_{i} & \ldots & H A^{k-2} B_{i}\end{array}\right], i=1,2 ; B_{h}^{*}=B_{1}+e^{-A h} B_{2}$.

## REFERENCES

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