Constructive Methods in Geometric Modeling

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Bézier curves in Geometric Modeling

One basic tool in geometric modeling are Bézier curves:

Let $b_0, ..., b_n$ be vectors either in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $t \in \mathbb{R}$: we interprete b_k as a constant curve, i.e. that for $k = 0, ..., n$ and $t \in [0, 1]$

$$b_k^0(t) := b_k.$$

Then we define curves $b_k^1(t)$ for $t \in [0, 1]$ for $k = 0, ..., n-1$ by

$$b_k^1(t) = (1-t)\, b_k^0(t) + t b_{k+1}^0(t)$$

and inductively for $k = 0, ..., n - r$

$$b_k^r(t) = (1-t)\, b_k^{r-1}(t) + t b_{k+1}^{r-1}(t)$$

In the last step one has only one index, namely $k = 0$.

The curve

$$b_0^n(t) = (1-t)\, b_0^{n-1}(t) + t b_1^{n-1}(t)$$

is called the *Bézier curve.*

The Bézier curve has the following properties

$$b_0^n(0) = b_0 \text{ and } b_0^n(1) = b_n$$

$$b_0^n(t) \in \text{convex hull of } b_0, ..., b_n.$$

The polygon formed by $b_0, ..., b_n$ is called the *Bézier polygon or control polygon.*

An explicit form for the Bézier curve is:

$$b_0^n(t) = \sum_{k=0}^{n} b_k p_{n,k}(t) \tag{1}$$

where

$$p_{n,k}(t) := \binom{n}{k} t^k (1-t)^{n-k}$$

are the Bernstein polynomials. Note that

$$p_{n,k} \text{ has a zero of order } k \text{ at } t = 0$$

$$p_{n,k} \text{ has a zero of order } n-k \text{ at } t = 1.$$

One important generalization of Bézier curves are *B-spline curves* where one replaces the Bernstein polynomial $p_{n,k}$ in (1) by basic splines N_k^n of degree n.

Bernstein bases in extended Chebyshev systems

We want to discuss the concept of a Bernstein basis in a more general setting.

Definition 1. Let $C^n[a,b]$ be the set of all n times continuously differentiable functions $f:[a,b]\to\mathbb{C}$. The function f has a **zero of order** $k\le n$ at a point $x_0\in[a,b]$ if

$$0=f(x_0)=...=f^{(k-1)}(x_0).$$

Definition 2. A linear subspace U of dimension $n+1$ of $C^n[a,b]$ is called an **extended Chebyshev system** over $[a,b]$ if each non-zero function $f\in U$ has at most n zeros.

The following spaces are extended Chebyshev system over the interval $[a,b]$:

1. The linear span of $1,...,x^n$

2. The linear span of $e^{\lambda_0 x},...,e^{\lambda_n x}$ for $\lambda_0,...,\lambda_n$ real pairwise distinct.

3. The linear span of $\cos x,\sin x,1,...,x^{n-2}$ for $[a,b]\subset[0,2\pi)$.

All examples are special cases of so-called exponential polynomials:

Definition 3. Let $\lambda_0, ..., \lambda_n$ be complex numbers and consider the differential operator

$$L_{(\lambda_0,...,\lambda_n)} := \left(\frac{d}{dx} - \lambda_0 \right) .... \left(\frac{d}{dx} - \lambda_n \right)$$

Then each function f in the space

$$E_{(\lambda_0,...,\lambda_n)} = \left\{ f \in C^n [a, b] : L_{(\lambda_0,...,\lambda_n)} f = 0 \right\}$$

is called an **exponential polynomal.**

Proposition 1. *If $\lambda_0, ..., \lambda_n$ are real then $E_{(\lambda_0,...,\lambda_n)}$ is an extended Chebyshev system for any interval.*

Definition 4. A system $p_{n,k} \in E_{(\lambda_0,...,\lambda_n)}, k = 0, ..., n$ is called a **Bernstein basis** over the interval $[a, b]$ if it is a basis with the property that each

$$p_{n,k} \text{ has a zero of order } k \text{ at } t = a,$$

$$p_{n,k} \text{ has a zero of order } n - k \text{ at } t = b.$$

Proposition 2. *Let $\lambda_0, ..., \lambda_n$ be real. Then there exists a Bernstein basis $p_{(\lambda_0,...,\lambda_n),k}$, $k = 0, ..., n+1$.*

Proof. It is convenient to use the following notation:

$$q_k := p_{(\lambda_0,...,\lambda_n),n-k},$$

so q_k has $n-k$ zeros in a and k zeros in b. For $k=0$ set

$$q_0(x) := [\lambda_0, ..., \lambda_n] e^{(x-a)z} = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{(x-a)z}}{(z-\lambda_0)...(z-\lambda_n)} dz.$$

Next put

$$q_1 := q_0^{(1)} - \alpha_0 q_0$$

for $\alpha_0 = q_0^{(1)}(b) / q_0(b)$ which has clearly a zero of order $n-1$ in a and a zero of order 1 in b.

For $k \geq 2$ we define q_k recursively by

$$q_k := q_{k-1}^{(1)} - (\alpha_{k-1} - \alpha_{k-2}) \cdot q_{k-1} - \beta_k q_{k-2}$$

with suitable coefficients $\alpha_{k-1} - \alpha_{k-2}$ and β_k defined $\beta_k := q_{k-1}^{(k-1)}(b) / q_{k-2}^{(k-2)}(b)$ and

$$\alpha_{k-1} = \frac{q_{k-1}^{(k)}(b)}{q_{k-1}^{(k-1)}(b)}.$$

Proposition 3. *Let* $\lambda_0, ..., \lambda_n$ *be real. Then there exists a* **unique** *Bernstein basis* $p_{(\lambda_0,...,\lambda_n),k}$, $k = 0, ..., n+1$ *satisfying the condition*

$$k! \lim_{x \to a, x > a} \frac{p_{(\lambda_0,...,\lambda_n),k}(x)}{(x-a)^k} = p^{(k)}_{(\lambda_0,...,\lambda_n),k}(a) = 1.$$

It follows that

$$p_{n,k}(x) > 0 \text{ for all } x \in [0,1].$$

Theorem 4. *(Carnicer, Mainar, Pena 2004, Mazure 2005) Assume that* $1 \in E_{(\lambda_0,...,\lambda_n)}$*. Then the Bernstein basis is "normalized", i.e. there exist* **positive numbers** d_k

$$1 = \sum_{k=0}^{n} d_k p_{(\lambda_0,...,\lambda_n),k}(x).$$

Properties of the Bernstein operator

The Bernstein operator over the unit interval $[0,1]$ is defined by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

The operator B_n has the following properties:

1. $B_n f$ is a polynomial of degree $\leq n$,
2. $B_n 1 = 1$ and $B_n x = x$,
3. B_n is a positive operator,
4. $B_n f$ converges uniformly to f for each $f \in C[0,1]$,
5. For the computation of $B_n f$ one needs only the data $f\left(\frac{k}{n}\right)$ for $k = 0, ..., n$.

Question: Let $\lambda_0, ..., \lambda_n$ be real. Can one define a "Bernstein operator" $\tilde{B}_n : C[0,1] \to C[0,1]$ such that

1. $\tilde{B}_n f$ is an exponential polynomial for $(\lambda_0, ..., \lambda_n)$,
2. $\tilde{B}_n e^{\lambda_0 x} = e^{\lambda_0 x}$ and $\tilde{B}_n e^{\lambda_1 x} = e^{\lambda_1 x}$,
3. $\tilde{B}_n$ is a positive operator,
4. $\tilde{B}_n f$ converges uniformly to f for each $f \in C[0,1]$,
5. For the computation of $\tilde{B}_n f$ one needs only the data $f(t_k)$ for $k = 0, ..., n$, for some $t_0, ..., t_n \in [0,1]$.

Construction of the Bernstein operator

Ansatz: We want to define the Bernstein operator B_n via a formula of the type

$$B_{(\lambda_0,\ldots,\lambda_n)}f(x)=\sum_{k=0}^{n}\alpha_k f(t_k)\,p_{n,k}$$

where $\alpha_0,\ldots,\alpha_n\geq 0$ and $t_0,\ldots,t_n\in[a,b]$ are to be defined independent of f.

Theorem 5. *Let $\lambda_0\neq\lambda_1$ and $\lambda_0,\ldots,\lambda_n\in\mathbb{R}$. Then there exist unique positive coefficients $\alpha_0,\ldots,\alpha_n$ and unique points $a=t_0\leq\ldots\leq t_n=b$, such that the operator*

$$B_{(\lambda_0,\ldots,\lambda_n)}f(x)=\sum_{k=0}^{n}\alpha_k f(t_k)\,p_{(\lambda_0,\ldots,\lambda_n),k}(x)$$

satisfies

$$B_{(\lambda_0,\ldots,\lambda_n)}\left(e^{\lambda_0 x}\right)=e^{\lambda_0 x}\ \textit{and}\ B_{(\lambda_0,\ldots,\lambda_n)}\left(e^{\lambda_1 x}\right)=e^{\lambda_1 x}.$$

Idea of proof. Let $\beta_0, ..., \beta_n$ and $\gamma_0, ..., \gamma_n$ be the unique coefficients such that

$$e^{\lambda_0(x-a)} = \sum_{k=0}^{n} \beta_k p_{(\lambda_0,...,\lambda_n),k}(x),$$

$$e^{\lambda_1(x-a)} = \sum_{k=0}^{n} \gamma_k p_{(\lambda_0,...,\lambda_n),k}(x).$$

Suppose that there exists an operator as described in the theorem. Then $B_{(\lambda_0,...,\lambda_n)}\left(e^{\lambda_0 x}\right) = e^{\lambda_0 x}$ implies

$$\sum_{k=0}^{n} e^{\lambda_0(t_k-a)} \alpha_k p_{(\lambda_0,...,\lambda_n),k}(x) = \sum_{k=0}^{n} \beta_k p_{(\lambda_0,...,\lambda_n),k}(x).$$

Since $p_{(\lambda_0,...,\lambda_n),k}$ is a base we infer that

$$e^{\lambda_0(t_k-a)}\alpha_k = \beta_k.$$

Similarly, $B_{(\lambda_0,...,\lambda_n)}\left(e^{\lambda_1 x}\right) = e^{\lambda_1 x}$ implies

$$e^{\lambda_1(t_k-a)}\alpha_k = \gamma_k.$$

Then t_k satisfies the equation

$$e^{(\lambda_0-\lambda_1)t_k} = \frac{\beta_k}{\gamma_k} e^{(\lambda_0-\lambda_1)a}.$$

and α_k

$$\alpha_k = e^{-\lambda_0(t_k-a)}\beta_k.$$

In particular, the coefficients α_k and the nodes t_k are uniquely determined by these equations. The points t_k are defined by the equation

$$e^{(\lambda_0-\lambda_1)(t_k-t_{k-1})} = \lim_{x\to b} \frac{p_{(\lambda_0,\lambda_2,...,\lambda_n),k-1}(x)}{p_{(\lambda_1,...,\lambda_n),k-1}(x)},$$

Convergence of the Bernstein operator

Question: Under which conditions at $\lambda_0, ..., \lambda_n$ does the Bernstein operator

$$B_{(\lambda_0,...,\lambda_n)}$$

converge to the identity operator?

Fact: We need conditions!

Consider the case of the Müntz polynomials

$$1, x^{\lambda_1}, x^{\lambda_2}, ..., x^{\lambda_n}$$

defined on $[a, b]$ where

$$\lambda_0 = 0 < \lambda_1 < \lambda_2 < ...$$

is a strictly increasing sequence of non-negative numbers such that $\lambda_n \to \infty$.

The system can be transformed via $x = e^t$ to

$$1, e^{\lambda_1 t}, e^{\lambda_2 t}, ...e^{\lambda_n t}$$

which are exponential polynomials on the interval $[\ln a, \ln b]$.

If $B_{(\lambda_0,...,\lambda_n)}(f)$ converges to f it follows that the space of all Müntz polynomials is dense. This implies the condition

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

Definition 5. Let $a(n,k), k = 0, ..., n$, be non-zero numbers for each $n \in \mathbb{N}$. We say that $a(n,k)$ converges uniformly to a number c if for each each $\varepsilon > 0$ there exists a natural number n_0 such that $|a(n,k) - c| < \varepsilon$ for all $n \geq n_0$ and for all $k = 0, ..., n$.

Theorem 6. *Assume that $\lambda_0, \lambda_1, \lambda_2$ are real distinct numbers. Define the following sequences $a(n,k), n \in \mathbb{N}$, and $b(n,k), n \in \mathbb{N}$, by*

$$a(n,k) = \lim_{x \to b} \frac{p_{(\lambda_0, \lambda_2, ..., \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, ..., \lambda_n), k}(x)},$$

$$b(n,k) = \lim_{x \to b} \frac{p_{(\lambda_0, \lambda_1, \lambda_3, ..., \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, ..., \lambda_n), k}(x)}.$$

Assume that $a(n,k)$ and $b(n,k)$ converge uniformly to 1 and that

$$\frac{1 - a(n,k)}{1 - b(n,k)}$$

converge uniformly to

$$\frac{\lambda_1 - \lambda_0}{\lambda_2 - \lambda_0}.$$

Then the Bernstein operator $B_{(\lambda_0, ..., \lambda_n)}$ converges to the identity operator.

Multivariate Bernstein operators

Definition 6. Let U be an open subset in $\mathbb{R}^d$. A function $f : U \to \mathbb{C}$ is polyharmonic of order p if

$$\Delta^p f(x) = 0 \text{ for all } x \in U$$

where $\Delta := \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator.

Open problem:

Let $B := \left\{x \in \mathbb{R}^d : |x| \leq 1\right\}$ be the unit ball. Construct an operator

$$B_N : C(B) \to C(B)$$

such that

1. $B_N f$ is a polyharmonic function of order $\leq N$,
2. $B_N h = h$ for all harmonic functions $h \in C(B)$,
3. B_N is a positive operator,
4. $B_N f$ converges uniformly to f for each $f \in C(B)$
5. For the computation of $B_N f$ one needs only the data

$$f\left(\frac{j}{N}\theta\right)$$

for all $\theta \in \mathbb{S}^{d-1} = \left\{x \in \mathbb{R}^d : |x| = 1\right\}$ and for all $j = 0, ..., N$.

Ansatz: Let $Y_{k,l}(\theta)$, $l = 1, ..., a_k$ be a basis of the spherical harmonics of degree k.

Consider the Laplace-Fourier series of a function $f \in C(B)$

$$f(r\theta) = \sum_{k=0}^{\infty} \sum_{l=1}^{a_k} f_{k,l}(r) Y_{k,l}(\theta).$$

If f is a polynomial then for any $f_{k,l}(r)$ there exists a polynomial $p_{k,l}(t)$ with

$$f_{k,l}(r) = r^k p_{k,l}\left(r^2\right)$$

and

$$\deg p_{k,l} \leq \inf\left\{p : \Delta^{p+1} f = 0\right\}.$$

In other words: $f_{k,l}(r)$ is a linear combination of

$$r^k, r^{k+2}, ..., r^{k+2p}.$$

Via the transformation $r = e^v$ this is equal to the system of **exponential polynomials**

$$e^{kv}, e^{k+2v}, ..., e^{(k+2p)v}.$$